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Math 87 HW 6
Due 3/28/21

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1. Using the information from last week's assignment, we have the following graph to describe the problem. Let p_1 = soup, p_2 = salad, p_3 = sandwich, p_4 = tacos.

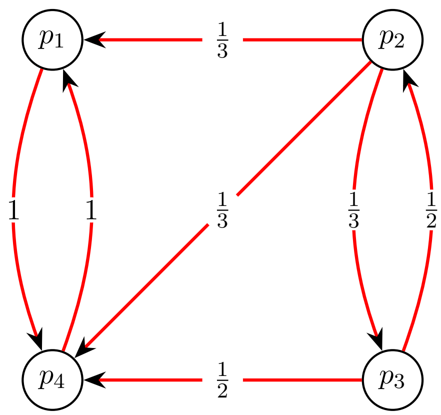

Figure 1: A FSM of the problem described

(a) The lunch choice problem described **is a Markov process** because no matter how the process arrived at its present state, the possible future states are fixed, i.e. the probability of transitioning to any particular state is dependent only on the current state in this example.

(b) The transition matrix for this problem represents the probability of moving to state j given that you are currently at state i. It is important to note that the sum of the entries of the first column is 1. The same holds for second, third, and fourth columns.

$$T = T_{ji} = prob(j|i) = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}$$

(c) The power iteration on a sample initial vector $v_0 = [1, 1, 1, 1]$ does not converge. The output vector from this is $v = [1.80000000e + 00, 3.71157923e - 39, 2.47438615e - 39, 2.20000000e + 00]$, and this is after 100 iterations, which was the maximum number of iterations given. However, running the power iteration code with the transition matrix T above and with an initial vector of $[1, 0, 0, 1]$, we see that the power method converges under 2 iterations. I chose this initial guess because after many iterations the function was close to $[0.82, 0, 0, 1]$. Regardless, the power iteration must converge for any of the the initial vectors given, and therefore the power iteration on the transition matrix **does not converge**.

(d) Checking the two criteria of strongly connected and aperiodicity, the problem appears to be connected at least in some sense, but is not strongly connected because there exists no edges that connect to some nodes, e.g. the nodes p_1 and p_4 only are connected between themselves with total probability and are not connected to the other two edges in the graph. A transition diagram is said to be aperiodic if no integer $n > 1$ divides the length of each cycle. By this definition, this example is said to be periodic (not aperiodic).

Therefore, **the problem does not meet the requirements** for the Perron-Frobenius Theorem.

(e) Our conclusion that the requirements for the Perron-Frobenius Theorem are not met aligns with our answer from (c), i.e. the power iteration does not converge.

(f) The eigenvalues for this matrix are $= -1, 1$, or the absolute value of 1. This reaffirms our answer in c, because this would result in an infinite iteration, as the power iteration would not converge.

(g) An edited transition matrix and graph are below

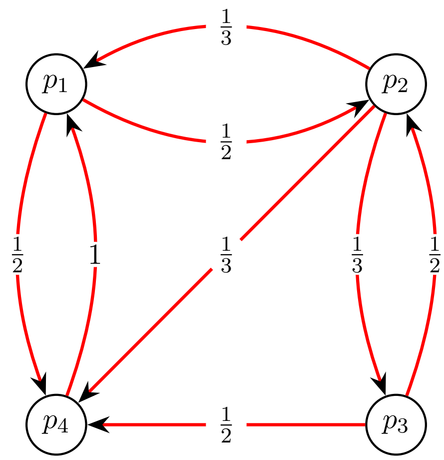

Figure 2: The edited FSM graph

$$T' = T'_{ji} = \begin{bmatrix} p'_11 & p_{12} & p_{13} & p_{14} \\ p'_21 & p_{22} & p_{23} & p_{24} \\ p'_31 & p_{32} & p_{33} & p_{34} \\ p'_41 & p_{42} & p_{43} & p_{44} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}$$

Now that we have made this edit, note that the output vector of the power iteration is the probability vector, which tells us that the edited lunch problem is aperiodic, hence "fixing" the problem with the Perron-Frobenius theorem. As such, we now have a new vector from the power iteration: $[0.31, 0.23, 0.38, 0.08]$. While the power method goes to infinity in part (c), the new model will converge in the edited example because we have since made the problem periodic and strongly connected, as there is now a way to get to and from any edge.

2. (a) Assuming that each fish only gave birth to one other fish over the course of the five years, then 80 percent of young fish live to be old fish. The matrix $A\mathbf{p}$ can be expressed in the following way

$$\begin{pmatrix} 0.6 & 0.5 \\ 0.8 & 0 \end{pmatrix} \begin{pmatrix} p_y \\ p_o \end{pmatrix} = \begin{pmatrix} 0.6p_y + 0.5p_o \\ 0.8p_y \end{pmatrix}$$

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and hence it becomes clearer to see more systematically that the population of young fish that make it to become old fish is 80 percent of the young population at any given time.

(b) We can find the reproduction rate for young fish and old fish from the matrix A and the associated vector $\mathbf{p}$. Thus, the reproduction rate for young fish is 60 percent and 50 percent for old fish.

(c) Using the matrix A given and the `linalg` program, we can find that the eigenvector corresponding to the eigenvalue 1 (the largest eigenvalue of the two) is

$$\begin{bmatrix} 0.78 & 0.62 \end{bmatrix}$$

with the first entry in the vector corresponding to the young fish and the second corresponding to the old fish.