Marcus
Math 87 HW 4
Due 3/14/21

1.

$$\begin{aligned}
\text{Maximize} \quad & 50x_1 + 6x_2 + 35x_3 + 60x_4 \\
\text{subject to} \quad & \\
& 24x_1 + 76x_2 + 43x_3 + 754x_4 \leq 800 \\
& 755x_1 + 27x_2 + 33x_3 + 67x_4 \leq 850
\end{aligned}$$

Using the branch and bound algorithm, we first find the optimal solution to the relaxed LP. These respective matrices are

$$\mathrm{A} = \begin{bmatrix} 755 & 27 & 43 & 754 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 800 \\ 850 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 50 \\ 6 \\ 35 \\ 60 \end{bmatrix}$$

This provides the solution (A) of 143.13, with $\mathbf{x} = [0.996\ 0\ 1\ 0.97]$. We now have to branch off x_1 and x_4, and repeat this same process, pruning both integer values and values that are infeasible, in accordance with the algorithm.

Branching on (A) with $x_1 = 0$, we find an optimal solution to the linear program of (A) with the additional equality constraint $x_1 = 0$. The optimal value is $v_B = 95.24$ and an optimal solution is $\mathbf{x}_B = [1\ 0.0395\ 1\ 1]$. This will be called solution (B).

Branching on (A) with $x_1 = 1$, we find an optimal solution to the linear program of (A) with the additional equality constraint $x_1 = 1$. The optimal value is $v_C = 140.52$ and an optimal solution is $\mathbf{x}_C = [1\ 0\ 1\ 0.925]$. This will be called solution (C).

Branching on (A) with $x_4 = 0$, we find an optimal solution to the linear program of (A) with the additional equality constraint $x_4 = 0$. The optimal value is $v_D = 91$ and an optimal solution is $\mathbf{x}_D = [1\ 1\ 1\ 0]$. This will be called solution (D). This is an integer solution and hence pruned.

Branching on (A) with $x_4 = 1$, we find an optimal solution to the linear program of (A) with the additional equality constraint $x_4 = 1$. The optimal value is $v_D = 127.907$ and an optimal solution is $\mathbf{x}_E = [1\ 0\ 0.512\ 1]$. This will be called solution (E).

We now repeat the steps above for each feasible, non-integer solution.

Branching on (B) with $x_2 = 0$, we find an optimal solution to the linear program of (B) with the additional equality constraint $x_2 = 0$. The optimal value is $v_F = 95.00$

and an optimal solution is $\mathbf{x}_F = [0\ 0\ 1\ 1]$. This will be called solution (F). This is an integer solution and hence pruned.

Branching on (B) with $x_2 = 1$, we find an optimal solution to the linear program of (B) with the additional equality constraint $x_2 = 1$. The optimal value is $v_G = 95.19$ and an optimal solution is $\mathbf{x}_G = [0\ 1\ 1\ 0.903]$. This will be called solution (G).

Branching on (G) with $x_4 = 0$, we find an optimal solution to the linear program of (G) with the additional equality constraint $x_2 = 1$. The optimal value is $v_H = 35.00$ and an optimal solution is $\mathbf{x}_H = [0\ 0\ 1\ 0]$. This will be called solution (H). This solution is infeasible and hence pruned.

Branching on (G) with $x_4 = 1$, we find an optimal solution to the linear program of (G) with the additional equality constraint $x_4 = 1$. The optimal value is $v_I = 95.00$ and an optimal solution is $\mathbf{x}_I = [0\ 0\ 1\ 1]$. This will be called solution (I). This is an integer solution and hence pruned.

We now have the upper most branch of our LP complete. Like before, we repeat.

Branching on (C) with $x_4 = 0$, we find an optimal solution to the linear program of (C) with the additional equality constraint $x_4 = 0$. The optimal value is $v_J = 91.00$ and an optimal solution is $\mathbf{x}_J = [1\ 1\ 1\ 0]$. This will be called solution (J). This is an integer solution and hence pruned.

Branching on (C) with $x_4 = 1$, we find an optimal solution to the linear program of (C) with the additional equality constraint $x_4 = 1$. The optimal value is $v_K = 91.00$ and an optimal solution is $\mathbf{x}_K = [1\ 0\ 0.512\ 1]$. This will be called solution (K).

Branching on (K) with $x_3 = 0$, we find an optimal solution to the linear program of (K) with the additional equality constraint $x_3 = 1$. The optimal value is $v_L = 56.00$ and an optimal solution is $\mathbf{x}_L = [1\ 1\ 0\ 0]$. This will be called solution (L). This is an integer solution and hence pruned.

Branching on (K) with $x_2 = 0$, we find an optimal solution to the linear program of (K) with the additional equality constraint $x_2 = 0$. The optimal value is $v_M = 111.74$ and an optimal solution is $\mathbf{x}_M = [1\ 0.289\ 0\ 1]$. This will be called solution (M).

Branching on (M) with $x_4 = 1$, we find an optimal solution to the linear program of (M) with the additional equality constraint $x_4 = 1$. The optimal value is $v_N = 50.00$ and an optimal solution is $\mathbf{x}_N = [1\ 0\ 0\ 0]$. This will be called solution (N). This is an integer solution and hence pruned.

Branching on (M) with $x_4 = 1$, we find an optimal solution to the linear program of (M) with the additional equality constraint $x_4 = 1$. The optimal value is $v_O = 0$ and an optimal solution is $\mathbf{x}_O = [0\ 0\ 0\ 0]$. This will be called solution (O). This is infeasible and hence pruned.

Since (D) is pruned from the get go, we are left with the final branch to perform the algorithm.

Branching on (E) with $x_3 = 0$, we find an optimal solution to the linear program of (E) with the additional equality constraint $x_3 = 0$. The optimal value is $v_P = 111.74$ and an optimal solution is $\mathbf{x}_P = [1\ 0.289\ 0\ 1]$. This will be called solution (P).

Branching on (E) with $x_3 = 1$, we find an optimal solution to the linear program of (E) with the additional equality constraint $x_3 = 1$. The optimal value is $v_Q = 101.25$ and an optimal solution is $\mathbf{x}_Q = [0.125\ 0\ 1\ 1]$. This will be called solution (Q).

Branching on (P) with $x_2 = 0$, we find an optimal solution to the linear program of (P) with the additional equality constraint $x_2 = 0$. The optimal value is $v_R = 110.00$ and an optimal solution is $\mathbf{x}_R = [1\ 0\ 0\ 1]$. This will be called solution (R). This is an integer solution and hence pruned.

Branching on (P) with $x_2 = 1$, we find an optimal solution to the linear program of (P) with the additional equality constraint $x_2 = 1$. The optimal value is $v_S = 101.22$ and an optimal solution is $\mathbf{x}_S = [0.00446\ 1\ 1\ 1]$. This will be called solution (S).

Branching on (Q) with $x_1 = 0$, we find an optimal solution to the linear program of (Q) with the additional equality constraint $x_1 = 0$. The optimal value is $v_T = 95.24$ and an optimal solution is $\mathbf{x}_T = [0\ 0.0394\ 1\ 1]$. This will be called solution (T).

Branching on (Q) with $x_1 = 1$, we find an optimal solution to the linear program of (Q) with the additional equality constraint $x_1 = 1$. The optimal value is $v_U = 0$ and an optimal solution is $\mathbf{x}_T = [0\ 0\ 1\ 1]$. This will be called solution (U). This solution is infeasible and hence pruned.

Branching on (T) with $x_2 = 0$, we find an optimal solution to the linear program of (T) with the additional equality constraint $x_2 = 0$. The optimal value is $v_V = 95.00$ and an optimal solution is $\mathbf{x}_V = [0\ 0\ 1\ 1]$. This will be called solution (V). This is an integer solution and hence pruned.

Branching on (T) with $x_2 = 0$, we find an optimal solution to the linear program of (T) with the additional equality constraint $x_2 = 0$. The optimal value is $v_W = 0$ and an optimal solution is $\mathbf{x}_W = [0\ 0\ 0\ 0]$. This will be called solution (W). This solution is infeasible and hence pruned.

Thus, out of all of the optimal solutions (A) through (W) described above, the optimal solution that maximizes the function with the given integer constraints is node R with optimal value 110 and vector $\mathbf{x} = [1\ 0\ 0\ 1]$. The graph Figure 1 below demonstrates visually the workings of the algorithm. Note that there are some duplicates during the branch and bound that could have been eliminated.

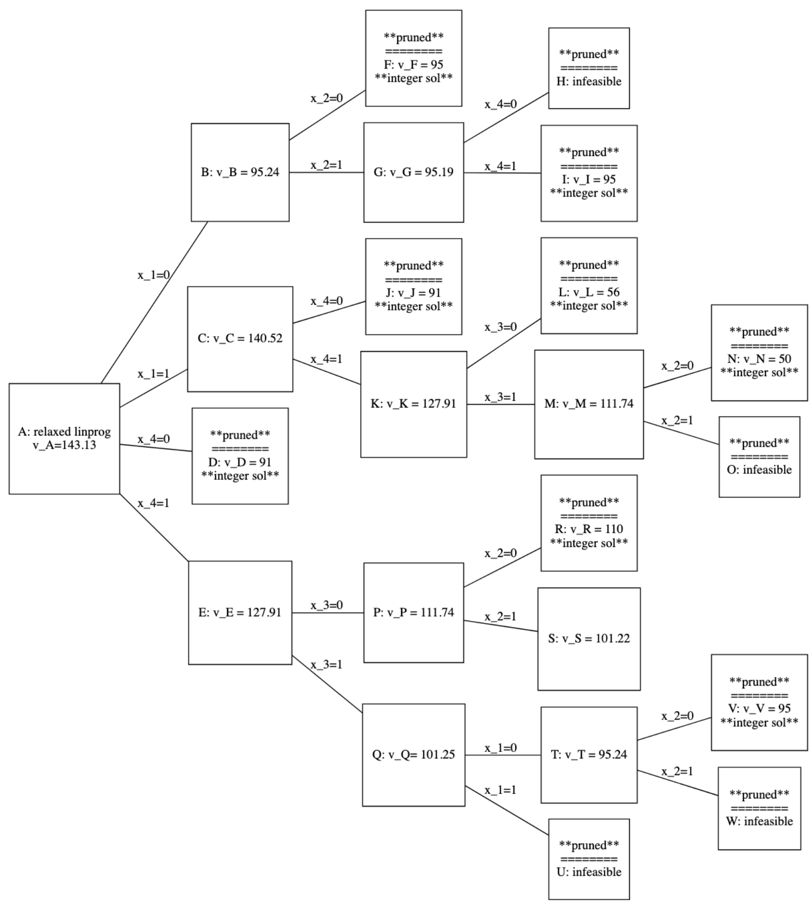

Figure 1

2. (a) A directed graph of the program example is the following

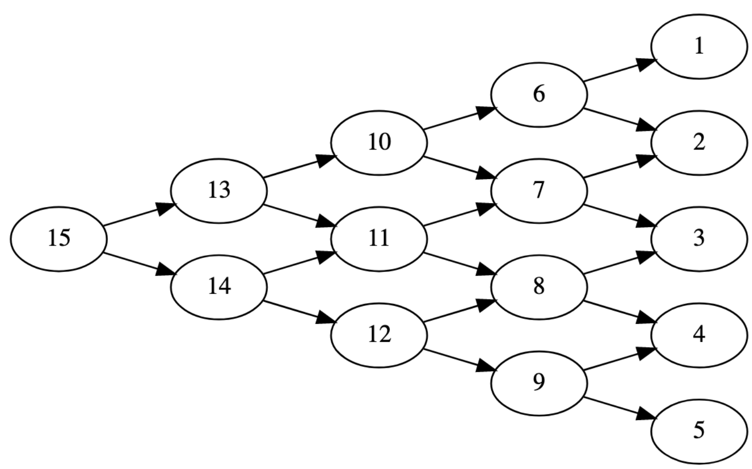

Figure 2

(b) A closed subset of the directed graph from part (a) is the following. Note, rather than circling the closed subset, the closed subset is indicate in the color red.

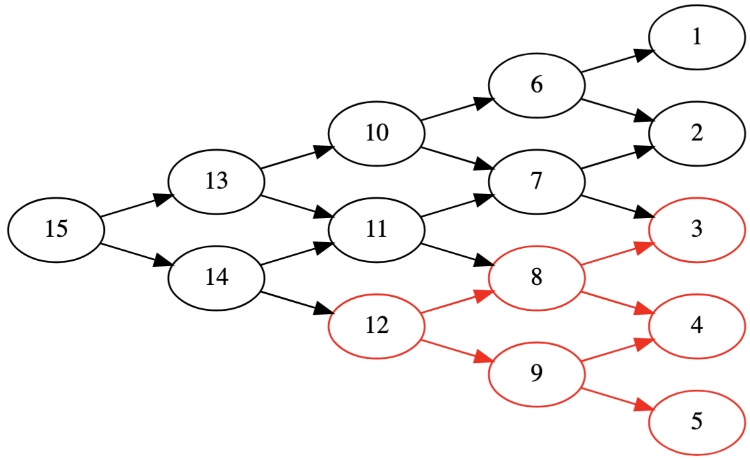

Figure 3

(c) We establish the variables s, and t, each representing outflows and inflows of profit. In order to make each profit value from the mine positive, the "negative" values will flow from their source node to the terminal node t, which corresponds to the blocks that have negative values.

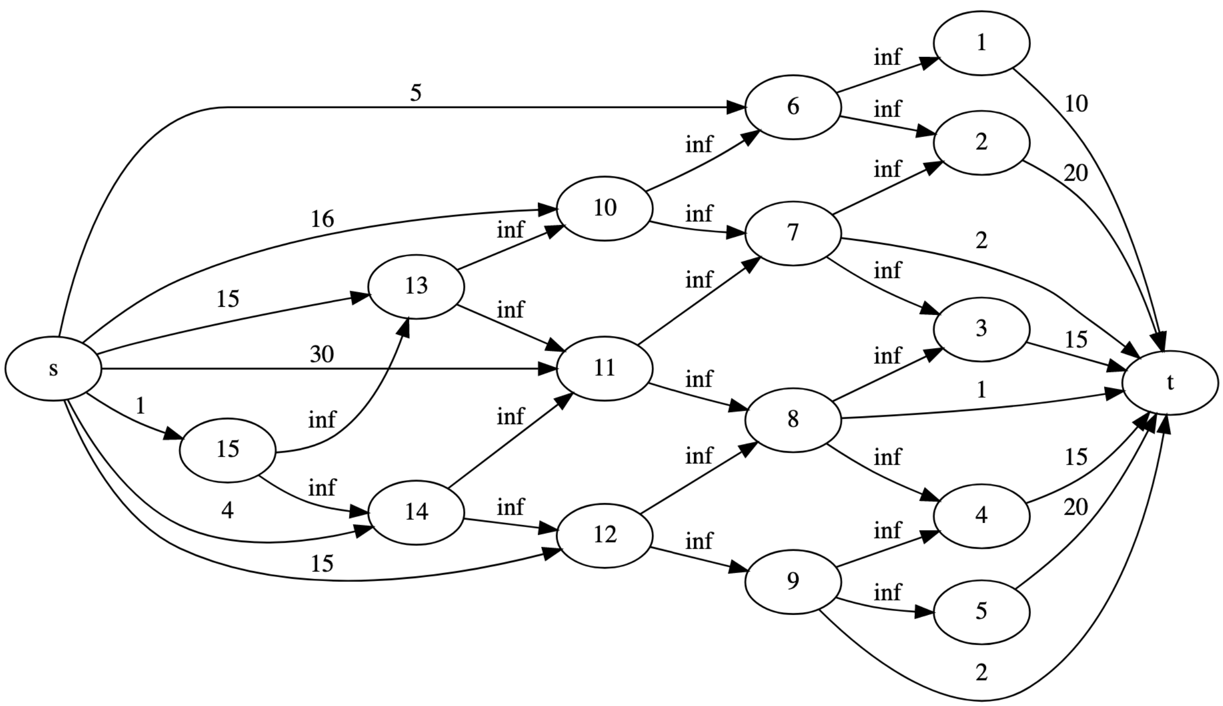

Figure 4

(d) The finite cuts are simply those in which you are not cutting an infinite edge (denoted by inf). Thus, the finite cuts are just those leaving the source node s and entering the terminal node t.

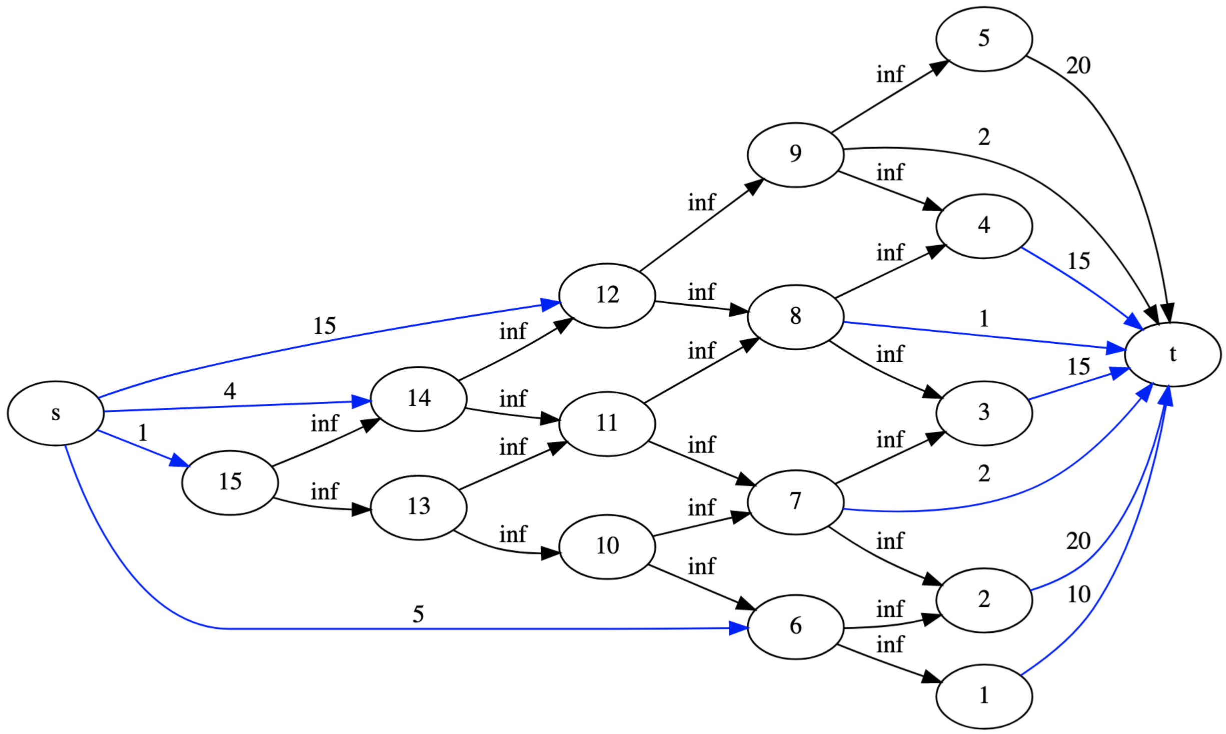

Figure 5: Used for parts (d) and (e)

(e) It is important to note that finding the min cut can be found by solving the dual LP on the flow described. Creating this LP, we note that there are 7 variables leaving the source node, 20 variables that represent flows within source and terminal nodes, and 8 variables that represent the flows into the terminal node (these can be counted in part c, noting that you cannot cut through an infinite edge). The min cut is indicated in blue in part d. Hence, compared to the primal LP with 15 variables, the dual LP will have 35 variables corresponding to dual matrix equation.

Because we are told that the profit is the difference between the total of all the weights on edges from s and the min cut, we can just subtract and find profit. Thus, note the min cut in the graph above and we can use this to determine our total profit through the following subtraction. $(1+4+15+10+20+15+1+15+5)-(1+4+15+10+20+2+15+1+15) = 3$, which is the profit.