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Math 87 HW 8
Due 4/21/21

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1. (a) The probability x that there is a tied vote corresponds to the probability that exactly six of the jurors vote guilty and the rest (6) vote not-guilty is given by $P(X = 6)$, where X corresponds to the Binomial random variable that describes the inconclusive decision. Thus, the probability of a tie is

$$P(X_{12} = 6) = \binom{12}{6} 0.7^6 (0.3)^6 = 0.07925.$$

(b) Let us establish a new random variable Y that corresponds to the probability that the judge votes correctly, i.e. $P(Y = 0.8)$. This random variable is derived from the inconclusive vote, and as such Y will only occur if X has happened. Thus,

$$P(Y|X) = P(X)P(Y) = 0.07925 \times 0.8 = 0.06340.$$

(c) Let us establish a new random variable Z that corresponds to the probability that at least 7 jury members vote the same way, i.e. $P(Z \leq 7)$. A verdict is reached if at least 7 jury members vote the same way or if tie is broken under the circumstances of random variable Y. Thus, the probability that a correct verdict is reached is

$$P(Z_{correct}) = P(Z)P(Y) = 0.9456.$$

(d) For the jury to reach an incorrect verdict (which I will call $P(A)$, note that $P(Z_{correct})$ cannot be satisfied. Thus,

$$P(A) = 1 - P(Z_{correct}) = 0.05445.$$

(e) Simulating the wrongful conviction random variable A 2 million times, we find that 108900 people are wrongfully convicted cumulatively throughout the trials.

(f)
```python
import numpy as np
from numpy.random import default_rng
rng=default_rng()

# In this problem, p is taken to be the the likelihood of a juror to vote
# correctly.

# takes in p of correct vote
def wrongful_convict(p):
  # Binomial calculation... wrote my own for this
  def binomial(n,m):
    def combo(n,m):
      return np.math.factorial(n) / (np.math.factorial(m) * np.math.
  factorial(n-m))
    return combo(n,m)*p**m*(1-p)**(n-m)

  if n == 6:
    print("\n".join([f"m = {m} -- P(X={m+1}) = {0.8*binomial(12,m):.8f}"
  for m in range(13)]))
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  else:
    qq = list(map(binomial(12,m),range(M)))
    qq = qq + [1-sum(qq,0)]

    print (rng.choice(list(range(M+1)),p=qq))

```

Listing 1: Code

(g) Testing the likelihoods for the respective values, we get the following results indicated in the table below. With more attention, we can see that as the rate at which the jury votes correctly increases by a constant rate (50% to 60% to 70%, and so on), the probability of wrongful conviction dramatically decreases with increasing magnitude per constant 10% increase in correct-verdict likelihood. This implies that maximizing voting correctly will have an even more dramatic impact in lowering the probability of a wrongful conviction as that likelihood increases. However, as I will explain in the part (h), this finding is not always sound.

(h) A key flaw in the assumptions of this problem has to do with the rate or likelihood of a juror and judge to vote correctly. Due to the nature of the law, it is hard to apply a set-in-stone probability of voting correctly or incorrectly to a diverse set of legal cases. That is, if we were to assign a random variable for each legal case that was used to make these assumptions given in the problem, each of these random variables would be independent, and would have have no impact on each other. Thus, saying a judge votes correctly 80 percent of the time or that 90% of people on trial are guilty but not convicted and 10% are innocent but convicted can only be applied to a single case (or random variable) and not an aggregate set of independent legal cases treated dependently.

Likelihood	Probability of Wrongful Conviction	Expected Number of Wrongful Convicts
50%	4.32	86465
60%	1.94	38706
70%	0.54	10890
80%	0.07	1400
90%	0.0015	39

2. (a) Under the given conditions, we can see that $\binom{n}{j} = \binom{n}{n-j}$ via combinatorics: every j-subset of an n-set is associated with an $(n-j)$-subset, i.e. if the n-set is A and if $B \subseteq A$ has size j, then the complement of B has size nj. This establishes a one-to-one correspondence between sets of size n and sets of size $n-j$, so the numbers of each are the same. The following equalities demonstrate why this is the case:

$$(x+y)^n = \sum_{n=0}^{n} \binom{n}{n-j} x^{n-j} y^j$$

$$(y+x)^n = \sum_{n=0}^{n} \binom{n}{j} y^{n-j} x^j$$

(b)

$$(x+y)^n = (x+y)(x+y)^{n-1} = x(x+y)^{n-1} + y(x+y)^{n-1} \tag{1}$$

$$(x+y)^n = \sum_{n=0}^{j} \binom{n}{j} x^{n-j} y^j \tag{2}$$

$$x(x{+}y)^{n-1} = [\sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-1-j} y^j]x = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-1+1-j} y^j = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-j} y^j$$

$$y(x+y)^{n-1} = [\sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-1-j} y^j]y = \sum_{j=0}^{n-1} \binom{n-1}{j} x^{n-1-j} y^{j+1} \tag{3}$$

$$= \sum_{j=0}^{n-1} \binom{n-1}{j-1} x^{n-1-(j-1)} y^{(j-1)+1} = \sum_{j=0}^{n} \binom{n-1}{j-1} x^{n-1} y^j$$

$$\sum_{n=0}^{j} \binom{n}{j} x^{n-j} y^j = \sum_{n=0}^{j} \binom{n-1}{j} x^{n-j} y^j + \sum_{n=0}^{j} \binom{n-1}{j-1} x^{n-1} y^j \tag{4}$$

$$\implies \binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}, 1 \leq j \leq n-1$$

3. Using the Poisson distribution to model customer arrival X, the probability that m customers arrive over the course of 1 day is given by

$$P(X = m) = \frac{p^m \cdot e^{-p}}{m!}.$$

We know that

$$E(X_{\text{poisson}}) = \sum_{m=7}^{12} \cdot P(X_{\text{poisson}} = m) = p = 42.$$

Therefore,

(a) $P(X = 30) = 0.0108$

(b) $P(40 \leq X \leq 45) = P(X \leq 40)P(45 \geq X) = 0.354$

(c) $P(X \geq 55) = 0.222$