Marcus Rose
Math 87 HW 3
Due 2/28/21

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1. (a) We begin by establishing a set of variables as follows:
    - Let w represent the amount of wheat per acre
    - Let c represent the amount of corn per acre

    Thus, our goal is to maximize the following function given the following constraints:

$$\begin{aligned} \max f(w,c) = 200w + 300c \quad \text{s.t.} \\ 3w + 2c \leq 100 \\ 2w + 4c \leq 120 \\ w + c \leq 45 \end{aligned}$$

    In matrix form, the primal linear program has the goal of maximizing $(\mathbf{c}, A, \mathbf{b})$ for

$$\mathbf{c} = \begin{bmatrix} 200 & 300 \end{bmatrix}$$
$$A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix}$$
$$\mathbf{b} = \begin{bmatrix} 45 \\ 100 \\ 120 \end{bmatrix}$$

    i.e. the objective function is given by $\mathbf{c} \cdot \begin{bmatrix} w \\ c \end{bmatrix} = 200w + 300c$ where $\begin{bmatrix} w \\ c \end{bmatrix} \geq \mathbf{0}$.

    Hence, solving the associated linear program using the `linprog` function in the `scipy` library, we arrive at a profit maximization point of $\begin{bmatrix} 20 & 20 \end{bmatrix}$, i.e. 20 units of wheat and 20 units of corn.

   (b) The dual linear program is given by the parameters $(\mathbf{b}^T, A^T, \mathbf{c}^T)$. We label the variables of the dual linear program using the two resource constraints:

$$\mathbf{y} = \begin{bmatrix} y_l \\ y_w \\ y_f \end{bmatrix}$$

    where y_l denotes the per acre price of land, y_w denotes the unit price of labor (or workers), and y_f denotes the unit price of fertilizer.

    Hence, the objective function for the dual system is given by

$$\mathbf{b}^T \cdot \begin{bmatrix} y_l \\ y_w \\ y_f \end{bmatrix} = 45y_l + 100y_w + 120y_f$$

and the inequality constraints are given by

$$A^T \cdot \begin{bmatrix} y_l \\ y_w \\ y_f \end{bmatrix} \geq \mathbf{w}^T = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$$

Solving the associated linear program using the `linprog` function in the `scipy` library, we arrive at a profit maximization point of $\begin{bmatrix} 3.686405941\text{e}{-12} & 2.50000000\text{e}{+1} & 6.25000000\text{e}{+01} \end{bmatrix}$. An optimal solution to the dual linear system is

$$\mathbf{y}^* = \begin{bmatrix} y_l \\ y_w \\ y_f \end{bmatrix} = \begin{bmatrix} 3.686405941\text{e}{-12} \\ 25 \\ 62.5 \end{bmatrix}$$

or 25 units of wheat and 62.5 units of corn (as the additional land for the optimization is negligible but still present at a value close to 0).

(c) In the primal problem, the objective function is a linear combination of n variables. There are m constraints, each of which places an upper bound on a linear combination of the n variables. The goal is to maximize the value of the objective function subject to the constraints. A solution is a vector of n values that achieves the maximum value for the objective function. Conversely in the dual problem, the objective function is a linear combination of the m values that are the limits in the m constraints from the primal problem. There are n dual constraints, and each puts a lower bound on a linear combination of m dual variables. The dual vector multiplies the constraints that determine the positions of the constraints in the primal. Varying the dual vector in the dual problem is simply editing the upper bounds in the primal problem.

2.

$$\begin{aligned} \text{Maximize} \quad & 11x_1 + 5x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \leq 7 \\ & 10x_1 + 4x_2 \leq 40 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(a) Note that if the above function is the linear program in maximized standard form, the dual linear program is the minimization function as follows:

$$\begin{aligned} \min f(y_1, y_2) = 7y_1 + 40y_2 \quad \text{s.t.} \\ y_1 + 10y_2 \geq 11 \\ y_1 + 4y_2 \geq 5 \end{aligned}$$

(b) The follow is a graph of the feasible region

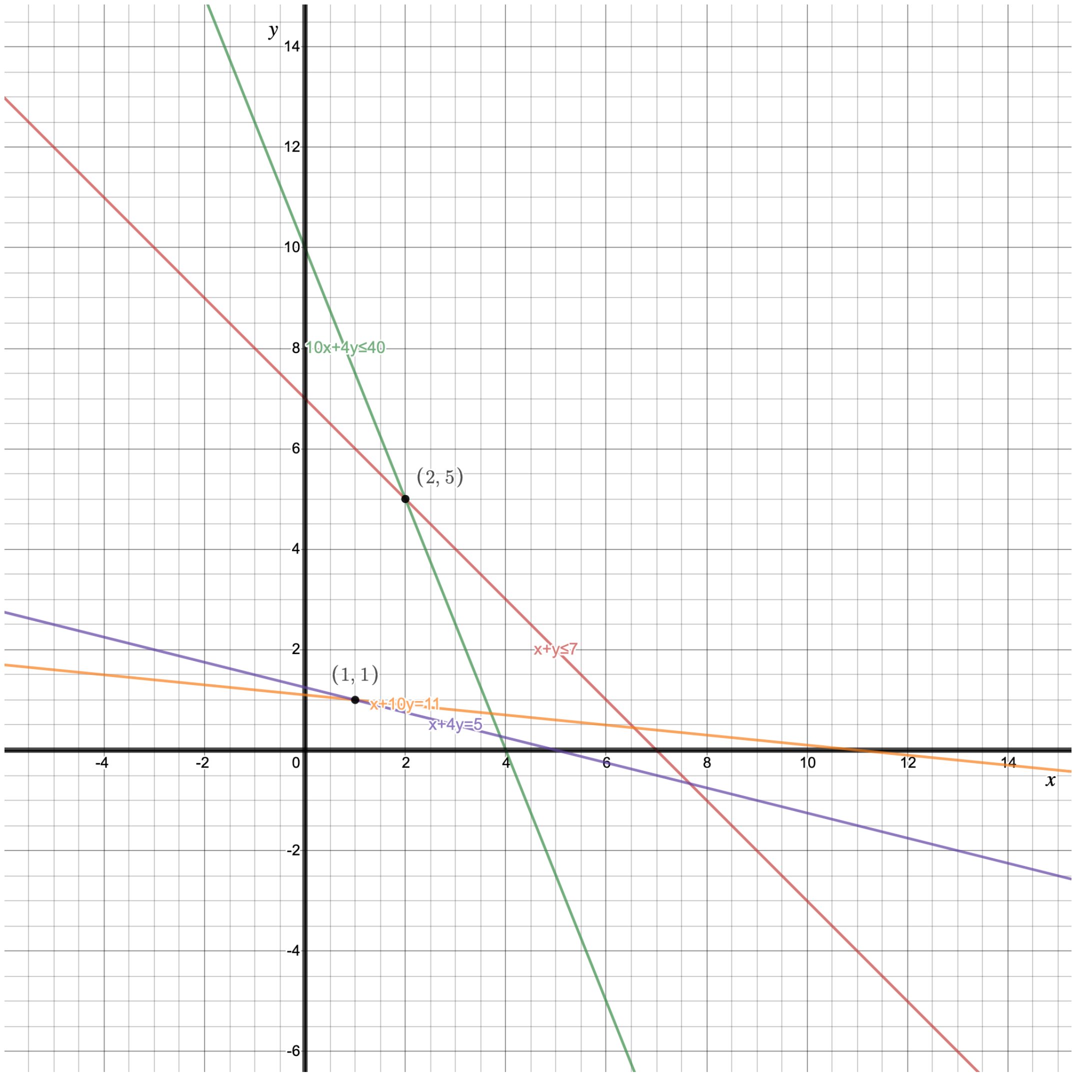

Figure 1: Note that x and y correspond to x_1 and x_2, respectively

Solving the associated linear program using the feasible region, we find the solution of the primal linear program to be $\begin{bmatrix}2 & 5\end{bmatrix}$. Likewise, we find the solution to the dual linear program to be $\begin{bmatrix}1 & 1\end{bmatrix}$. Both have been verified using `scipy`.

The strong duality theorem states that if $\mathbf{x}^*$ is an optimal solution for $\mathcal{L}$ and if $\mathbf{y}^*$ is an optimal solution for $\mathcal{L}'$, then

$$\mathbf{c}\mathbf{x}^* = \mathbf{b}^T\mathbf{y}^*.$$

Hence, the matrix equation becomes

$$\begin{bmatrix}11\\5\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}7 & 40\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$
$$22 + 25 = 40 + 7$$
$$47 = 47$$

The theory of complementary slackness states that if $\mathbf{x}$ is a feasible point for $\mathcal{L}$ and $\mathbf{y}$ a feasible point for $\mathcal{L}'$. Then, $\mathbf{x}$ is optimal for $\mathcal{L}$ and $\mathbf{y}$ is optimal for $\mathcal{L}'$ if and only if

$$(\mathbf{b} - A\mathbf{x})^T \cdot \mathbf{y} = 0 \quad \text{and} \quad (\mathbf{y}^T A - \mathbf{c}) \cdot \mathbf{x} = 0.$$

Hence, we can verify this with the following

$$(\begin{bmatrix} 7 \\ 40 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 10 & 4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix})^T = 0$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 10 & 4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 11 \\ 5 \end{bmatrix} = 0$$

For any values $\mathbf{y}$ and $\mathbf{x}$, because the above values equal zero, complementary slackness is verified.

(c) Performing the sensitivity analysis, if we increase the right hand of the first constraint, i.e. $x_1 + x_2 \leq 7$, to 8, 9, and 11, we can note that smaller increases in the constraint do not affect the change of the optimal solution as much as larger increases. As such, increasing the first constraint by 1 and 2 do actually not change the dual solution; however, an increase of 4 in the first constraint will change the solution, as the dual price lemma states that a perturbation this large will cause the constraint to be out of range. Another way to think of this is if the right-hand side (where $b = 1$, 2 and 4 for each set increase) of the first primal constraint is changed to obtain a new problem with constraints $Ax \geq b + \Delta b$, only the objective function of the dual changes to $\lambda(b + \Delta b)$. Thus, λ' is still feasible under the dual LP and provides a lower bound $\lambda'(b+\Delta b)$ on the optimal value of the primal problem when right hand side constraints are edited. Simply put, the constraint change increases the optimal value $\lambda * b$ of the original problem by at least $\lambda * \Delta b$, but the change cannot be too large or the LP will not have an accurate solution.