Lemma (Donsker-Varadhan Identity)**.** Let λ be a reference measure on some space $\mathcal{X}$, and let $f : \mathcal{X} \to \mathbb{R}$ be measurable. The Donsker-Varadhan identity says:

$$\log \int \exp(f(y))\lambda(dy) = \sup_{q \ll \lambda} \Big\{ \int f(y)q(dy) - D_{\mathrm{KL}}(q \| \lambda) \Big\},$$

where the supremum is over all measures q absolutely continuous w.r.t. λ ($q \ll \lambda$).

Remark. We represent the log likelihood as

$$\log p_\theta(x) = \log \int \exp(-E(x,z;\theta))dz - \log \iint \exp(-E(x,z;\theta))dxdz.$$

- Applying the DV identity on μ: we have

  $$\log \int \exp(-E_\theta(x,z))\mu(dz) = \sup_{q(z|x) \ll \mu} \Big\{ \mathbb{E}_{q(z|x)}[-E_\theta(x,z) - D_{\mathrm{KL}}(q(z|x) \| \mu)] \Big\}.$$

  For a dataset $\{x_i : i \in [N]\}$, we have

  $$\frac{1}{N}\sum_{i=1}^{N} \log \int \exp(-E_\theta(x_i,z))\mu(dz) = \sup_{q_i \ll \mu} \frac{1}{N} \sum_{i \in [N]} \{\mathbb{E}_{q_i}[-\mathbb{E}_\theta(x_i,z)] - D_{\mathrm{KL}}(q_i \| \mu)\} =: \sup_{\{q_i\}} A(\{q_i\};\theta)$$

  where $q_i := q(\cdot|x_i)$.

- Applying the DV identity on $\mu \otimes \nu$: we have

  $$\begin{aligned}\log Z(\theta) &= \log \iint \exp(-E_\theta(x,z))d\mu \otimes \nu(x,z) \\ &= \sup_{\tilde{q} \ll \nu \otimes \mu} \{\mathbb{E}_{\tilde{q}}[-E - \theta(x,z)] - D_{\mathrm{KL}}(\tilde{q} \| \mu \otimes \nu)\} =: \sup_{\tilde{q} \ll \nu \otimes \mu} B(\tilde{q};\theta).\end{aligned}$$

Putting it all together, the average log likelihood becomes

$$\frac{1}{N} \sum_{i \in [N]} \log p_\theta(x_i) = \sup_{q_i} A(\{q_i\};\theta) - \sup_{\tilde{q}} B(\tilde{q};\theta) = \sup_{\theta, q^i \in Q} \inf_{\tilde{q} \in \tilde{Q}} F(\bar{q}, \tilde{q}, \theta).$$