Definition (Shannon Entropy)**.** Let $(\mathcal{X}, \mathcal{A})$ be a measurable space and let μ be a σ-finite measure on it. Let q be a probability measure on $(\mathcal{X}, \mathcal{A})$ such that $q \ll \mu$, and let $r := \frac{dq}{d\mu}$ be its Radon–Nikodym derivative. The (Shannon/differential) entropy of q relative to μ is defined by

$$H_\mu(q) \ := \ -\int_{\mathcal{X}} \log r(x)\, q(dx) \ = \ -\int_{\mathcal{X}} r(x) \log r(x)\, \mu(dx).$$

Lemma (Donsker–Varadhan Identity)**.** Let λ be a σ-finite reference measure on some space $\mathcal{X}$, and let $f : \mathcal{X} \to \mathbb{R}$ be measurable. The Donsker–Varadhan identity says

$$\log \int_{\mathcal{X}} \exp\bigl(f(y)\bigr)\, \lambda(dy) = \sup_{q \ll \lambda} \Bigl\{ \int_{\mathcal{X}} f(y)\, q(dy) + H_\lambda(q) \Bigr\}.$$

Remark. We represent the log-likelihood as

$$\log p_\theta(x) = \log \int \exp\bigl(-E(x, z; \theta)\bigr)\, \mu(dz) - \log \iint \exp\bigl(-E(x, z; \theta)\bigr)\, (\nu \otimes \mu)(dx\, dz).$$

- **Applying the DV identity on μ.** We have

  $$\log \int \exp\bigl(-E(x, z; \theta)\bigr)\, \mu(dz) = \sup_{q(\cdot|x) \ll \mu} \Bigl\{ \mathbb{E}_{q(z|x)}\bigl[-E(x, z; \theta)\bigr] + H_\mu(q(z|x)) \Bigr\}.$$

  For a dataset $\{x_i : i \in [N]\}$, writing $q_i := q(\cdot \mid x_i)$, we obtain

  $$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \log \int \exp\bigl(-E(x_i, z; \theta)\bigr)\, \mu(dz) &= \sup_{\{q_i \ll \mu\}} \frac{1}{N} \sum_{i=1}^N \Bigl\{ \mathbb{E}_{q_i}\bigl[-E(x_i, z; \theta)\bigr] + H_\mu(q_i) \Bigr\} \\ &=: \sup_{\{q_i\}} A(\{q_i\}; \theta). \end{aligned}$$

- **Applying the DV identity on $\nu \otimes \mu$.** We have

  $$\begin{aligned} \log \iint \exp\bigl(-E(x, z; \theta)\bigr)\, (\nu \otimes \mu)(dx\, dz) &= \sup_{\tilde{q} \ll \nu \otimes \mu} \Bigl\{ \mathbb{E}_{\tilde{q}}\bigl[-E(x, z; \theta)\bigr] + H_{\nu \otimes \mu}(\tilde{q}) \Bigr\} \\ &=: \sup_{\tilde{q} \ll \nu \otimes \mu} B(\tilde{q}; \theta). \end{aligned}$$

Putting it all together, the average log-likelihood becomes

$$\begin{aligned} \frac{1}{N} \sum_{i=1}^N \log p_\theta(x_i) &= \sup_{\{q_i\}} A(\{q_i\}; \theta) - \sup_{\tilde{q}} B(\tilde{q}; \theta) \\ &= \sup_{\{q_i\} \subset Q} \inf_{\tilde{q} \in \tilde{Q}} F(\{q_i\}, \tilde{q}, \theta), \end{aligned}$$

where Q and $\tilde{Q}$ denote the corresponding admissible classes of probability measures defined in the main text.