Chapter 4 Functions

4.1 Introduction

4.2 Properties of Functions

4.3 Inverse Function

4.4 Composite Function

4.5 Functions for Computer Science

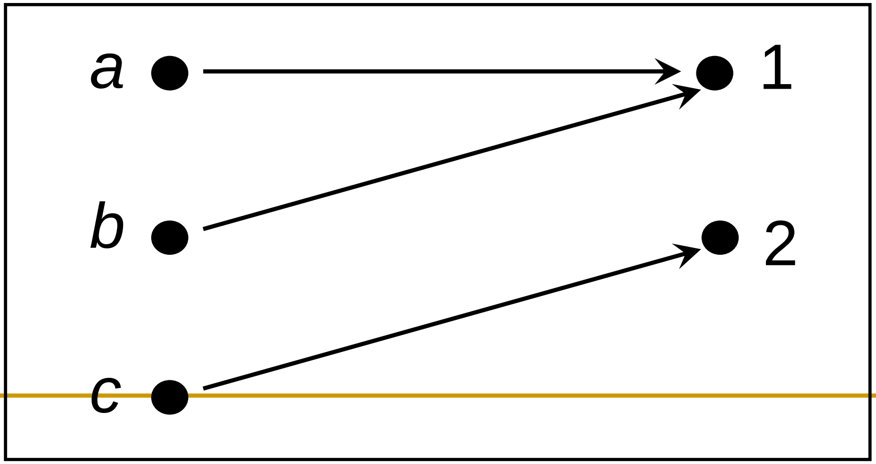
4.6 Permutations

4.1 Introduction

- Functions are binary relations in which further restrictions are imposed on the pairs which can occur.
- A function from a set *A* to a set *B* is a binary relation in which every element of *A* is associated with a uniquely specified element of *B*.
- Functions are also called as mappings or transformation.

4.1 Introduction (cont)

- In digraph terms a function is a relation such that there is precisely one arc leaving every element of *A*.
- E.g. The digraph representing the function from {*a*, *b*, *c*} to {1, 2} containing the pairs (*a*, 1), (*b*, 1) and (*c*, 2).

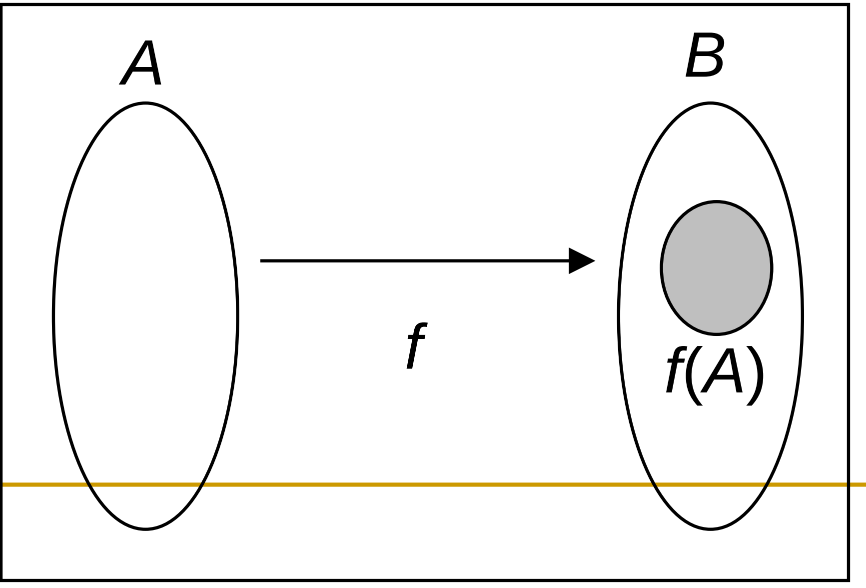


4.1 Introduction (cont)

- Let f be a function from a set A to a set B.
  - For each $a \in A$, there exists a uniquely determined $y \in B$ with $(x, y) \in f$, write as $y = f(x)$, and refer to $f(x)$ as the image of x under f.
  - Write as $f : A \to B$ to indicate that f is a function which transform, or maps, each element of A to a uniquely determined element of B.
  - A is called the domain of f, and B is the codomain of f.

4.1 Introduction (cont)

- The range of f is the set of images of all the elements of A under f, denoted by $f(A)$. Hence, $f(A) = \{f(x) : x \in A\}$.
- The Venn diagram provides a useful diagrammatic illustration of a function from a set A to a set B.



E.g.1

For each of the following relations between the sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Determine whether the relation gives a function from A to B.

i. $\{(a, 1), (a, 2), (b, 3), (c, 2)\}$

ii. $\{(a, 1), (b, 2), (c, 1)\}$

iii. $\{(a, 1), (c, 2)\}$

E.g.2

Which of the following relations are functions?

i. The relation x is a brother or sister of y on the set P of people.

ii. The relation on Z given by

$$\{(x, x^2) : x \in Z\}.$$

iii. The relation on R given by

$$\{(x, y) : x = y^2\}.$$

E.g.2

Which of the following relations are functions?

i.The relation x is a brother or sister of y on the set P of people.

E.g.2

Which of the following relations are functions?

ii. The relation on Z given by

$\{(x, x^2) : x \in Z\}$.

E.g.2

Which of the following relations are functions?

iii. The relation on R given by

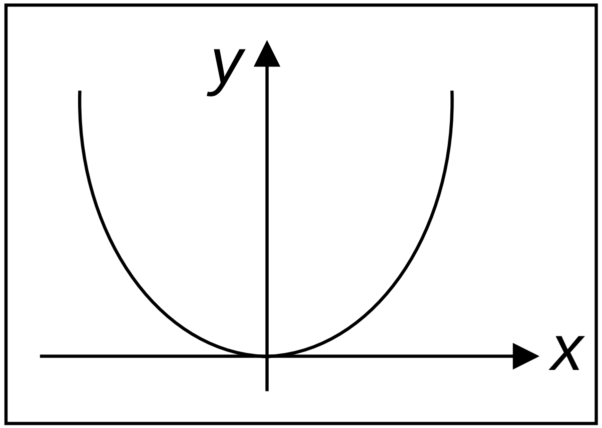
$\{(x, y) : x = y^2\}$.

4.1 Introduction (cont)

- When dealing with a function $f : A \to B$ where A and B are infinite sets of numbers, we can use the more traditional mathematical idea of graphing a function to give a geometric picture of the function.

4.1 Introduction (cont)

- The graph of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$:



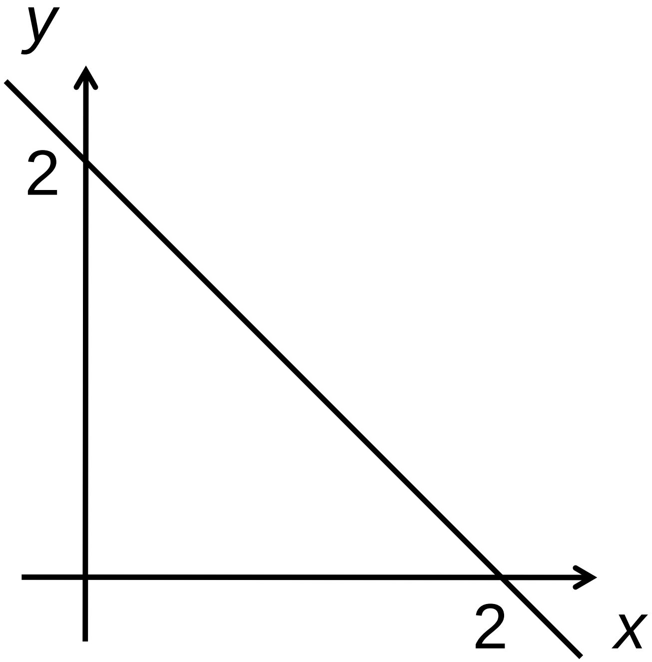
- The horizontal axis (x-axis) denotes the domain R and the vertical axis (y-axis) denotes the codomain R.
- The curve consists of those points (x, y) in Cartesian product R $\times$ R for which $y = f(x)$.

E.g.3

Sketch the graph of the function $f : \mathrm{R} \to \mathrm{R}$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.

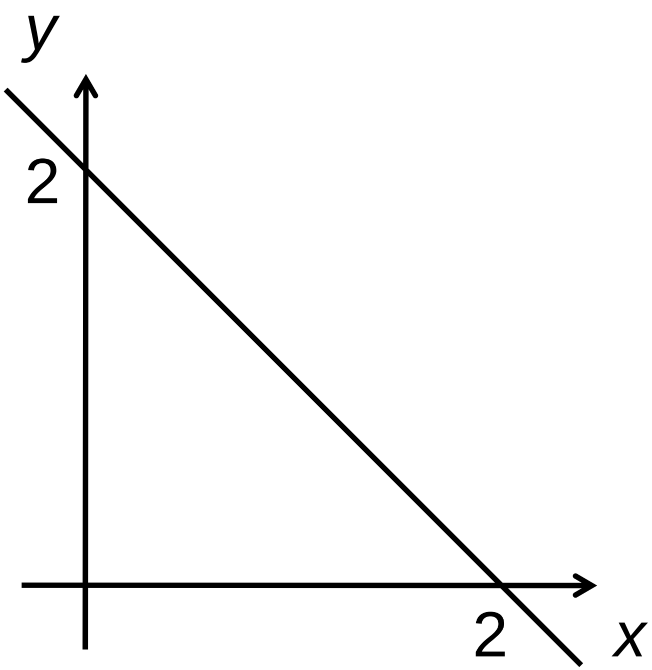
E.g.3

Sketch the graph of the function $f : \mathrm{R} \to \mathrm{R}$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.



E.g.3

Sketch the graph of the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2 - x$. Find the image of x under f when $x = 2$ and $x = 3$.



$x = 2, f(2) = 0$

$x = 3, f(3) = -1$

4.2 Properties of Functions

- Let $f : A \to B$ be a function.
  - f is an injective (or one-to-one) function if

    $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ for all $a_1, a_2 \in A$;

    or $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ i.e. different inputs give different outputs.
  - f is everywhere defined if $\mathrm{Dom}(f) = A$.
  - f is a surjective (or onto) function if the range of f coincides with the codomain of f, i.e. each element in the codomain is a value of the function;

    or for every $b \in B$, there exists an $a \in A$ with $b = f(a)$.

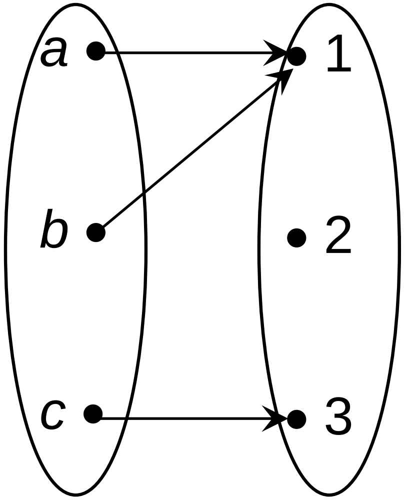
4.2 Properties of Functions (cont)

- *f* is a bijective function if *f* is both injective and surjective.
- *f* is a one-to-one correspondence between *A* and *B* if *f* is everywhere defined, injective, and surjective.

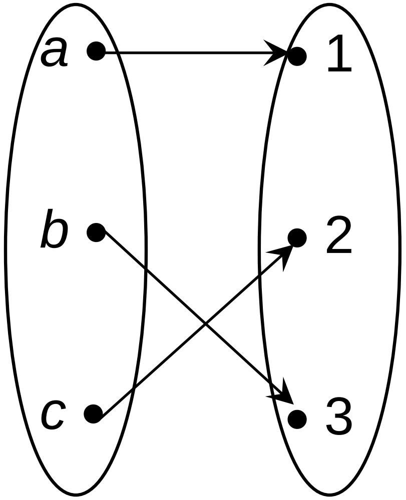
E.g.4

Decide which of the functions below is injective or surjective. Which are bijective?

i.

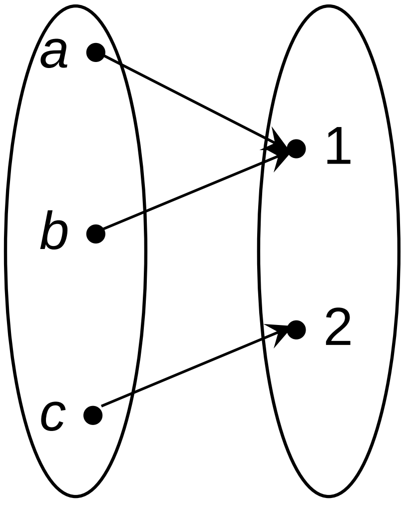


ii.

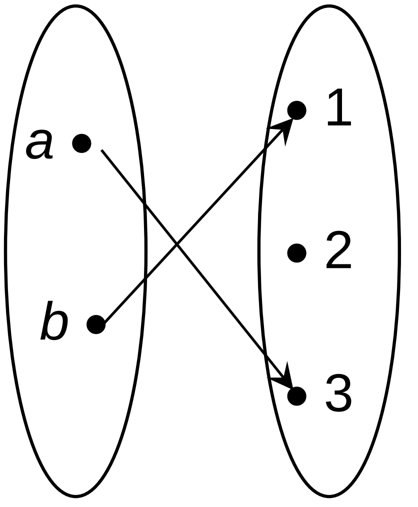


E.g.4 (cont)

iii.



iv.



E.g.5

Let $A = \{1, 2, 3\}$, $B = \{x, y, z\}$, $C = \{\pi, e\}$, and $D = \{\theta, \alpha, \beta, \gamma\}$. Determine whether each of the following function is one-to-one, onto, and everywhere defined.

i. $f : A \to B$; $f = \{(1, y), (2, z), (3, x)\}$

ii. $g: A \to D$; $g = \{(1, \alpha), (2, \theta), (3, \gamma)\}$

iii. $h: B \to C$; $h = \{(x, e), (y, e), (z, \pi)\}$

iv. $k : D \to B$; $k = \{(\theta, x), (\alpha, y), (\beta, x)\}$.

i. $f : A \rightarrow B$; $f = \{(1, y), (2, z), (3, x)\}$

ii. $g: A \to D$; $g = \{(1, \alpha), (2, \theta), (3, \gamma)\}$

iii. $h: B \to C$; $h = \{(x, e), (y, e), (z, \pi)\}$

E.g.6

Show that the function $h : Z \to Z$ given by $h(x) = x^2$ is neither injective nor surjective.

E.g.7

Show that the function $k : R \to R$ given by $k(x) = 4x + 3$ is bijective.

4.3 Inverse Function

- For any invertible function $f : A \to B$, the inverse function $f^{-1}: B \to A$ is the inverse relation, i.e.

  if $f(a) = b$ then $f^{-1}(b) = a$.

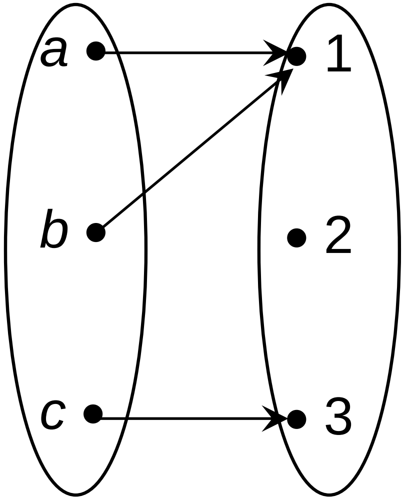
- Theorem

  A function f is invertible if and only if f is a bijective function.

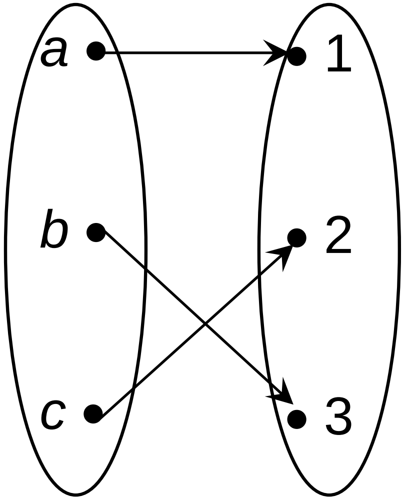
E.g.8

Determine which of the functions in E.g.4 is invertible.

i.

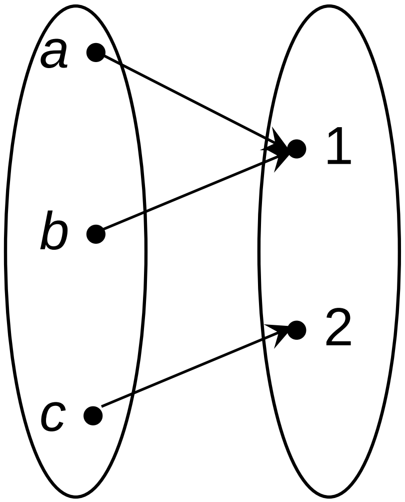


ii.

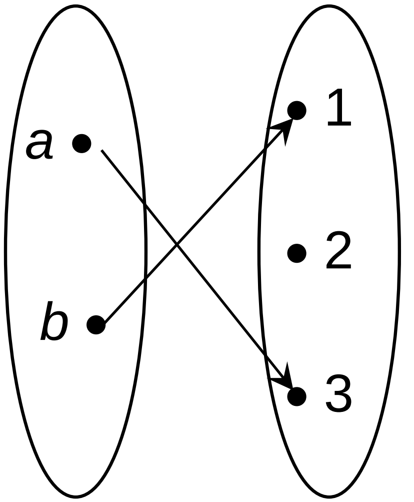


E.g.8 (cont)

iii.



iv.



E.g.9

Let $A = \{x : x \in \mathrm{R} \text{ and } x \neq 1\}$ and $f : A \to A$ be given by $f(x) = \dfrac{x}{x-1}$.

Show that f is bijective and determine the inverse function.

Let $f(x_1) = f(x_2)$

$\Rightarrow$

4.4 Composite Function

- If $f : A \to B$ and $g : B \to C$ are functions then the composite relation $g \circ f$ between A and C consists of pairs of the form (a, c) where, for some $b \in B$, $(a, b) \in f$ and $(b, c) \in g$.
- $c = g(f(a))$ is uniquely determined by a and so the composition of f and g is a function.
- $g \circ f : A \to C$ is a function given by

  $(g \circ f)(x) = g(f(x))$.

E.g.10

Consider the function $f : \mathrm{R} \to \mathrm{R}$ given by $f(x) = x^2$ and $g : \mathrm{R} \to \mathrm{R}$ given by $g(x) = 4x + 3$. Calculate

i. $g \circ f \quad =$

ii. $f \circ g \quad =$

E.g.10

iii. $f \circ f$;

iv. $g \circ g$.

4.5 Functions for Computer Science

- Let A be a subset of the universal set $U = \{u_1, u_2, u_3, \ldots, u_n\}$. The characteristics function of A is defined as a function from U to $\{0, 1\}$ by the following:
  - $f_A(u_i) = \begin{cases} 1 & \text{if } u_i \in A \\ 0 & \text{if } u_i \notin A \end{cases}$
  - If $A = \{4, 7, 9\}$ and $U = \{1, 2, 3, \ldots, 10\}$, then $f_A(2) = 0$, $f_A(4) = 1$, $f_A(7) = 1$, and $f_A(12)$ is undefined.
  - f_A is everywhere defined and onto, but not one-to-one.

4.5 Functions for Computer Science (cont)

- A Family of mod-n functions, one for each positive integer n, $f_n(m) = m \pmod{n}$, is a function from the nonnegative integers to the set $\{0, 1, 2, 3, ..., n - 1\}$.
  - For a fixed n, any nonnegative integer z can be written as $z = kn + r$ with $0 \le r < n$.

    Then $f_n(z) = r$, which can also written as $z \equiv r \pmod{n}$.
  - Each member of the mod function family is everywhere defined and onto, but not one-to-one.

4.5 Functions for Computer Science (cont)

- The floor function, which is defined for rational numbers as $f(q) = \lfloor q \rfloor$ is the largest integer less than or equal to q.
  - E.g. $\lfloor 1.5 \rfloor = 1$, $\lfloor -3 \rfloor = -3$, $\lfloor -2.7 \rfloor = -3$
- The ceiling function, which is defined for rational numbers as $c(q) = \lceil q \rceil$ is the smallest integer greater than or equal to q.
  - E.g. $\lceil 1.5 \rceil = 2$, $\lceil -3 \rceil = -3$, $\lceil -2.7 \rceil = -2$

4.5 Functions for Computer Science (cont)

- Many common algebraic functions are used in computer science, often with domains restricted to subsets of integers.
  - Any polynomial with integer coefficients, p, can be used to define a function on Z as follows:

    If $p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ and $z \in Z$, then $f(z)$ is the value of p evaluated at z.

4.5 Functions for Computer Science (cont)

- Let $A = B = Z^{+}$ and let $f : A \to B$ be defined by $f(z) = 2^{z}$, which is called as base 2 exponential function. Other bases can be used to define similar functions.
- Let $A = B = R$ and let $f_n : A \to B$ be defined for each positive integer $n > 1$ as $f_n(x) = \log_n x$, the logarithm to the base n of x. Usually bases 2 and 10 are used.

4.5 Functions for Computer Science (cont)

- The domains and codomains in the function need not be sets of numbers.
  - Let A be a finite set and define $l: A \to Z$ as $l(w)$ is the length of the string w.
  - Let B be a finite subset of the universal set U and define $pow(B)$ to be the power set of B. Then pow is a function from V, the power set of U, to the power set of V.

4.5 Functions for Computer Science (cont)

- Let $A = B =$ the set of 2×2 matrices with real number entries and let $t(\mathbf{M}) = \mathbf{M}^T$, the transpose of **M**. Then t is everywhere defined, onto, and one-to-one.
- For elements of $Z^+ \times Z^+$, $g(z_1, z_2)$ is defined to be GCD (z_1, z_2) and $m(z_1, z_2)$ to be LCM (z_1, z_2). Then g and m are a function from $Z^+ \times Z^+$ to Z^+.

4.5 Functions for Computer Science (cont)

- A Boolean function is a set from A to B, where B = {True, False}.

  Let $P(x)$: x is even and $Q(y)$: y is odd.

  Then P and Q are functions from Z to B.

  The predicate $R(x, y)$: x is even or y is odd is a Boolean function of two variables from Z $\times$ Z to B.

4.6 Permutations

- A bijection from a set *A* to itself.
  - E.g. Let $A = \{1, 2, 3\}$. Then all the permutations of *A* are

$$I_A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

4.6 Permutations (cont)

- Theorem 1

  If $A = \{a_1, a_2, \ldots, a_n\}$ is a set containing n elements, then there are $n!$ permutations on A.

- The composition of two permutations, $p_1 \circ p_2$ is another permutation, usually referred to as the product of these permutations.

4.6 Permutations (cont)

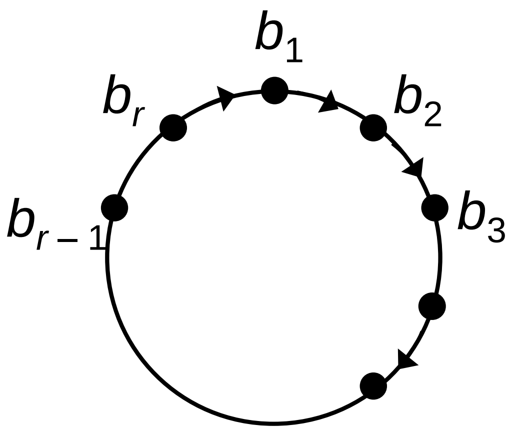
- Let $b_1, b_2, \ldots, b_r$ be r distinct elements of the set $A = \{a_1, a_2, \ldots, a_n\}$. The permutation $p: A \to A$ defined by

$$
\begin{aligned}
p(b_1) &= b_2 \\
p(b_2) &= b_3 \\
&\vdots \\
p(b_{r-1}) &= b_r \\
p(b_r) &= b_1 \\
p(x) &= x, \text{ if } x \in A,\ x \notin \{b_1, b_2, \ldots, b_r\}
\end{aligned}
$$

is called a cyclic permutation of length r, or a cycle of length r, and denoted by $(b_1, b_2, \ldots, b_r)$.

4.6 Permutations (cont)

- The cyclic permutation p can be written by starting with any b_i, $1 \leq i \leq r$, and moving in a clockwise direction.



- E.g. Let $A = \{1, 3, 5\}$. The cycle $(1, 3, 5)$ denotes the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$.

4.6 Permutations (cont)

- The notation for a cycle does not indicate the number of elements in the set A. Thus (3, 2, 1, 4) could be a permutation of the set {1, 2, 3, 4} or of {1, 2, 3, 4, 5, 6, 7, 8}.
- The cycle on a set A is of length 1 if and only if it is the identity permutation, I_A.

4.6 Permutations (cont)

- Two cycles of a set A are said to be disjoint if no element of A appears in both cycles.
  - E.g. Let $A = \{1, 2, 3, 4, 5, 6\}$. Then the cycles $(1, 2, 5)$ and $(3, 4, 6)$ are disjoint whereas $(1, 2, 5)$ and $(2, 4, 6)$ are not.
- Theorem 2

  A permutation of a finite set that is not the identity or a cycle can be written as a product of disjoint cycles of length ≥ 2.

4.6 Permutations (cont)

- When a permutation is written as a product of disjoint cycles, the product is unique except for the order of the cycles.

E.g.11

Let A = {1, 2, 3} and $p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

Find

i. p_1^{-1};

ii. $p_3 \circ p_2$.

E.g.12

Let $A = \{1, 2, 3, 4, 5, 6\}$. Compute

i. $(4, 1, 3, 5) \circ (5, 6, 3)$

ii. $(5, 6, 3) \circ (4, 1, 3, 5)$.

E.g.13

Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 6 & 5 & 2 & 1 & 8 & 7 \end{pmatrix}$

of the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disjoint cycles.

4.6.1 Even and Odd Permutations

- A cycle of length 2 is called a transposition, that is $p = (a_i, a_j)$, where $p(a_i) = a_j$ and $p(a_j) = a_i$.
- If $p = (a_i, a_j)$ is a transposition of A, then $p \circ p = I_A$, the identity permutation of A.
- Every cycle can be written as a product of transpositions. In fact
  $(b_1, b_2, \ldots, b_{k+1})$
  $= (b_1, b_{k+1}) \circ (b_1, b_k) \circ \ldots \circ (b_1, b_3) \circ (b_1, b_2)$.
  - $(1, 2, 3, 4, 5) = (1, 5) \circ (1, 4) \circ (1, 3) \circ (1, 2)$

4.6.1 Even and Odd Permutations (cont)

- Corollary 1

  Every permutation of a finite set with at least two elements can be written as a product of transpositions which need not be disjoint.

- Every cycle can be written as a product of transpositions in many different ways.
  - $(1, 2, 3) = (1, 3) \circ (1, 2) = (2, 1) \circ (2, 3)$

    $= (1, 3) \circ (3, 1) \circ (1, 3) \circ (1, 2) \circ (3, 2) \circ (2, 3)$

4.6.1 Even and Odd Permutations (cont)

- Theorem 3

  A permutation of a finite set can be written as a product of an even number of transpositions, then it can never be written as a product of an odd number of transpositions, and conversely.

4.6.1 Even and Odd Permutations (cont)

- A permutation of a finite set is called even if it can be written as a product of an even number of transpositions, and it is called odd if it can be written as a product of an odd number of transpositions.
  - The product of two even permutations is even.
  - The product of two odd permutations is even.
  - The product of an even and an odd permutation is odd.

4.6.1 Even and Odd Permutations (cont)

- Theorem 4

  Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite set with n elements, $n \geq 2$. There are $\frac{n!}{2}$ even permutations and $\frac{n!}{2}$ odd permutations.

E.g.14

Write the permutation in E.g.13 as a product of transpositions and determine whether it is an even or an odd permutation.

E.g.15

Determine whether the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$$

is even or odd.