Chapter 5
Order Relations and Structures

5.1 Partially Ordered Sets

- A relation R on a set A is called a partial order if R is reflexive, antisymmetric, and transitive.
- The set A with the partial order R is called a partially ordered set, or poset, denoted by (A, R).

5.1 Partially Ordered Sets (cont)

- E.g.
  - Let A be a collection of subsets of a subset of a set S. The relation $\subseteq$ of set inclusion is a partial order on A, so $(A, \subseteq)$ is a poset.
  - Let Z^+ be the set of positive integers. The usual relations $\leq$ (less than or equal to) and $\geq$ (greater or equal to) are partial orders on Z^+, but the relations $<$ (less than) and $>$ (greater than) are not partial order since they are not reflexive.

5.1 Partially Ordered Sets (cont)

- The relation of divisibility ($a\ R\ b$ if and only if $a|b$) is a partial order on Z^+ but R is not partial order on Z since it is not antisymmetric, for example $-2|2$ and $2|2$ but $-2 \neq 2$.

5.1 Partially Ordered Sets (cont)

- Let R be a partial order on a set A, then the inverse relation R^{-1} is also a partial order. The poset (A, R^{-1}) is called the dual of the poset (A, R), and the partial order R^{-1} is called the dual of the partial order R.
- The most familiar partial orders are the relations $\leq$ or $\geq$ on Z and R.
- In general, a partial order relation on a set often use the symbols $\leq$ or $\geq$ for R (relation R). Do not mistake this to familiar relation $\leq$ on Z (integers) or R (real numbers).

5.1 Partially Ordered Sets (cont)

- Symbols such as $\leq_1$, $\leq'$, $\geq_1$, $\geq'$ can be used to denote partial orders.
-

Poset	Dual Poset
$(A, \leq)$	$(A, \geq)$
$(A, \leq_1)$	$(A, \geq_1)$
$(B, \leq')$	$(B, \geq')$

5.1 Partially Ordered Sets (cont)

- If $(A, \leq)$ is a poset, the elements a and b of A are said to be comparable if

  $a \leq b$ or $b \leq a$.

  - Consider $(A, \leq) = (Z^+, |)$

    2 and 6 are comparable since $2 \leq 6$ or $2|6$.

    2 and 7 are not comparable since $2 \nmid 7$ and $7 \nmid 2$.

5.1 Partially Ordered Sets (cont)

- If every pair of elements in a poset A is comparable, then A is a linearly ordered set, and the partial order is called a linear order. We also say that A is a chain.
  - $(Z^+, \leq)$ is linearly ordered poset.
  - $(A, |)$ where $A = \{1, 3, 6, 9, 12\}$ is not a linearly ordered poset.

5.1 Partially Ordered Sets (cont)

- Theorem 1

  If $(A, \leq)$ and $(B, \leq)$ are posets, then $(A \times B, \leq)$ is a poset, with partial order $\leq$ defined by

  $(a, b) \leq (a', b')$ if $a \leq a'$ in A and $b \leq b'$ in B.

  - The symbol $\leq$ is being used to denote three distinct partial orders.
  - The partial order $\leq$ defined on the Cartesian product $A \times B$ is called the product partial order.

5.1 Partially Ordered Sets (cont)

- Let $A = \{1, 3, 5\}$, $B = \{2, 4, 8\}$, and $\leq_A$ means "less than or equal to", | means "divides", then $(A, \leq_A)$ and $(B, |)$ are posets.

  Hence, $(A \times B, \leq)$ is also a poset since

  $1 \leq_A 3$, $2|4$, $(1, 2) \leq (3, 4)$, also

  $3 \leq_A 5$, $4|8$, $(3, 4) \leq (5, 8)$.

  Hence $(1, 2) \leq (3, 4)$, $(3, 4) \leq (5, 8)$

  $\Rightarrow (1, 2) \leq (5, 8)$ because $1 \leq_A 5$, $2|8$.

5.1 Partially Ordered Sets (cont)

- $(A, \leq_A)$ and $(B, |)$ are linearly ordered but not $(A \times B, \leq)$ since some elements are not comparable. For examples,

  $(1, 4) \not\leq (3, 2)$ since $4 \nmid 2$ even $1 \leq_A 3$;

  $(3, 2) \not\leq (1, 4)$ since $3 \not\leq_A 1$ even $2|4$.

  So, A and B are linearly ordered $\not\Rightarrow$ $A \times B$ linearly ordered.

- If $(A, \leq)$ is a poset, we say $a < b$ if $a \leq b$ but $a \neq b$.

5.1 Partially Ordered Sets (cont)

- Suppose that $(A, \leq)$ and $(B, \leq)$ are posets, we define $(A \times B, \prec)$ as

  $(a, b) \prec (a', b')$ if $a < a'$ or if $a = a'$ and $b \leq b'$.

  This ordering is called lexicographic, or "dictionary" order.
  - The ordering of the elements in the first coordinate dominates, except in case of "ties", when attention passes to the second coordinate.

5.1 Partially Ordered Sets (cont)

- If $(A, \leq)$ and $(B, \leq)$ are linearly ordered sets, then the lexicographic order $\prec$ on $A \times B$ is also a linear order.
- From previous example,

  $(1, 4) \prec (3, 2)$ since $1 \leq 3$,

  $(1, 4) \prec (1, 8)$ since $1 = 1$ and $4 | 8$.

5.1 Partially Ordered Sets (cont)

- Lexicographic ordering is easily extended to Cartesian products $A_1 \times A_2 \times \ldots \times A_n$ as follows:

  $a_1 < a_1'$ or

  $a_1 = a_1'$ and $a_2 < a_2'$ or

  $a_1 = a_1'$, $a_2 = a_2'$, and $a_3 < a_3'$ or …

  $a_1 = a_1'$, $a_2 = a_2'$, …, $a_{n-1} = a_{n-1}'$ and $a_n = a_n'$.

  Thus the first coordinate dominates except in equality, in which case we consider the second coordinate. If equality holds again, pass to the next coordinate, and so on.

5.1 Partially Ordered Sets (cont)

- Let $S = \{a, b, \ldots, z\}$ be the ordinary alphabet, linearly ordered in the usual way, ($a \le b$, $b \le c$, …, $y \le z$).

$$S^n = S \times S \times \ldots \times S \text{ (n factors)}$$

can be identified with the set of all words having length n.

Then $park \prec part$, $help \prec hind$, $jump \prec mump$.

5.1 Partially Ordered Sets (cont)

- If S is a poset, we can extend lexicographic order to S^* in the following way.

  If $x = a_1a_2\ldots a_n$ and $y = b_1b_2\ldots b_k$ are in S^* with $n \leq k$, we say that $x \prec y$ if $(a_1a_2\ldots a_n) \prec (b_1b_2\ldots b_n)$ in S^n under lexicographic ordering of S^n.

  For example, $park \prec part \Rightarrow park \prec partition$

  $help \prec helping$, $park \prec parking$

5.1 Partially Ordered Sets (cont)

- Theorem 2

  The digraph of a partial order has no cycle of length greater than 1.

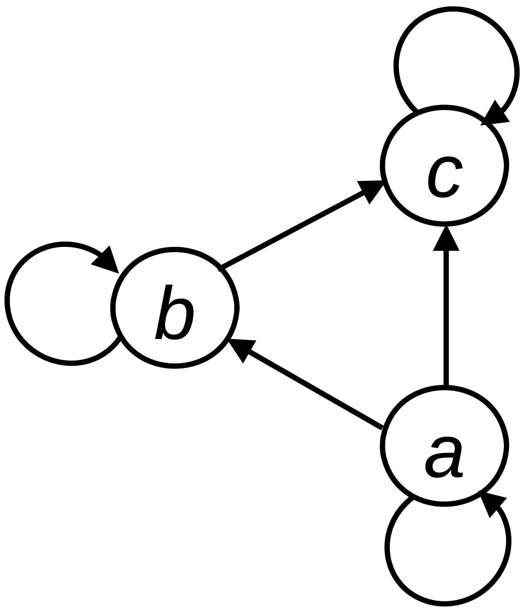
5.2 Hasse Diagram

- A simplification of digraph obtained by:

1. omitting all cycles of length 1;
2. omitting all edges that are implied by the transitive property;
3. drawing all the edges slanting upwards so that the arrow need not be drawn;
4. representing vertices by dots instead of circles.

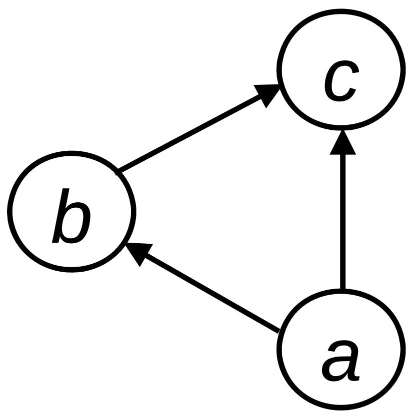
5.2 Hasse Diagram (cont)

- Consider the digraph given:



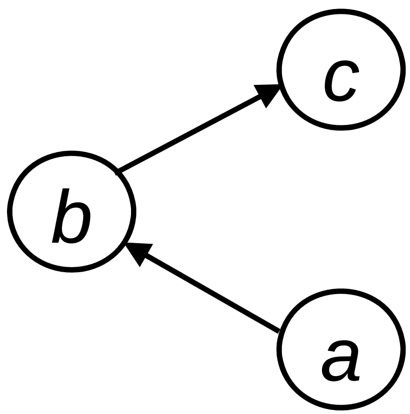
5.2 Hasse Diagram (cont)

Step 1: Delete all cycles of length 1



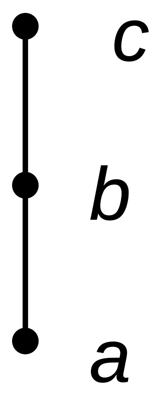
5.2 Hasse Diagram (cont)

Step 2: Eliminate all edges that are implied by the transitive property



5.2 Hasse Diagram (cont)

Step 3: Hasse diagram obtained

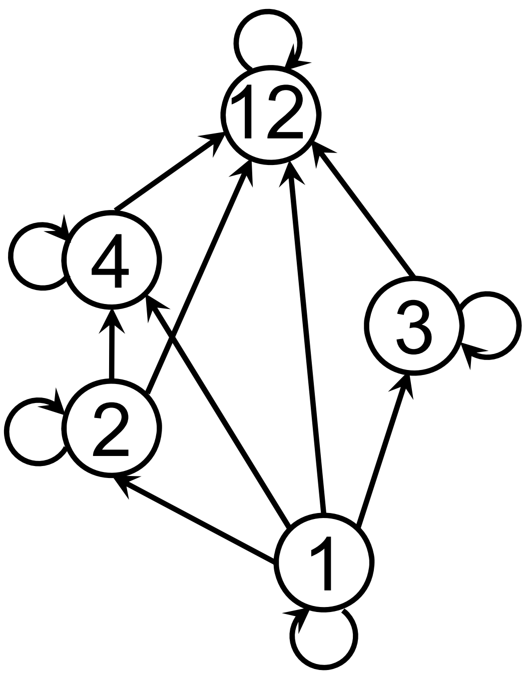


E.g.1

Let $A = \{1, 2, 3, 4, 12\}$. Consider the partial order of divisibility on A. That is, if a and $b \in A$, $a \leq b$ if and only if $a|b$. Draw the Hasse diagram for the poset $(A, \leq)$.

$$
\begin{aligned}
R = \{&(1, 1), (2, 2), (3, 3), (4, 4), (12, 12),\\
&(1, 2), (1, 3), (1, 4), (1, 12), (2, 4),\\
&(2, 12), (3, 12), (4, 12)\}
\end{aligned}
$$

R = {(1, 1), (2, 2), (3, 3), (4, 4), (12, 12), (1, 2), (1, 3), (1, 4), (1, 12), (2, 4), (2, 12), (3, 12), (4, 12)}



E.g.2

Let $a \leq b$ if and only if $a|b$ and $a \geq b$ if and only if a is a multiple of b or $b|a$. Draw the Hasse diagrams of $(A, \leq)$ and $(A, \geq)$ for

i. $A = \{1, 2, 4, 8, 16\}$,

$(A, \leq)$:

$R = \{(1, 1), (1, 2), (1, 4), (1, 8), (1, 16), (2, 2), (2, 4), (2, 8), (2, 16), (4, 4), (4, 8), (4, 16), (8, 8), (8, 16), (16, 16)\}$

$(A, \geq)$:

$R = \{(1, 1), (2, 1), (2, 2), (4, 1), (4, 2), (4, 4), (8, 1), (8, 2), (8, 4), (8, 8), (16, 1), (16, 2), (16, 4), (16, 8), (16, 16)\}$

E.g.2 (cont)

ii. $A = \{2, 3, 4, 5, 15, 60\}$.

$(A, \leq)$:

$R = \{(2, 2), (2, 4), (2, 60), (3, 3), (3, 15),$
$(3, 60), (4, 4), (4, 60), (5, 5), (5, 15),$
$(5, 60), (15, 15), (15, 60), (60, 60)\}$

$(A, \geq)$:

$R = \{(2, 2), (4, 2), (4, 4), (3, 3), (5, 5), (15, 3),$
$(15, 5), (15, 15), (60, 2), (60, 3), (60, 4),$
$(60, 5), (60, 15), (60, 60)\}$

Notes:

- E.g.2 (i) is a finite linearly ordered set.
- If (A, $\leq$) is a poset and (A, $\geq$) is the dual poset, then the Hasse diagram of (A, $\geq$) is just the Hasse diagram of (A, $\leq$) turned upside down.

E.g.3

Let D_n denotes the set of positive divisor of n. Draw the Hasse diagrams of the posets $(D_{24}, |)$ and $(D_{30}, |)$.

$D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

$(D_{24}, |)$:

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 8), (1, 12), (1, 24), (2, 2), (2, 4), (2, 6), (2, 8), (2, 12), (2, 24), (3, 3), (3, 6), (3, 12), (3, 24), (4, 4), (4, 8), (4, 12), (4, 24), (6, 6), (6, 12), (6, 24), (8, 8), (8, 24), (12, 12), (12, 24), (24, 24)\}$

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

$(D_{30}, |)$:

$R = \{(1, 1), (1, 2), (1, 3), (1, 5), (1, 6), (1, 10), (1, 15), (1, 30), (2, 2), (2, 6), (2, 10), (2, 30), (3, 3), (3, 6), (3, 15), (3, 30), (5, 5), (5, 10), (5, 15), (5, 30), (6, 6), (6, 30), (10, 10), (10, 30), (15, 15), (15, 30), (30, 30)\}$

5.2 Hasse Diagram (cont)

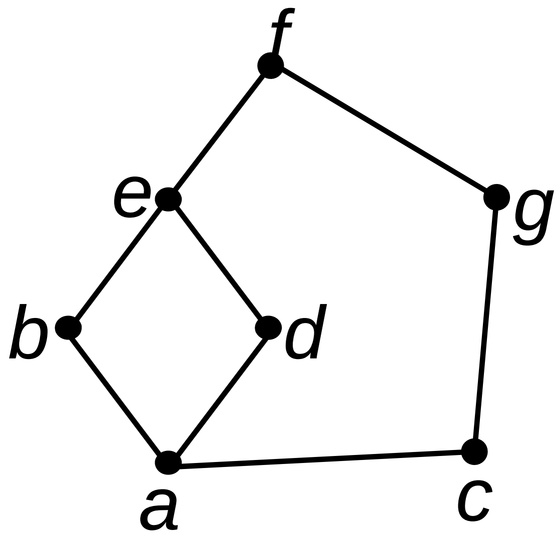
- If A is a poset with partial order $\leq$, sometimes need to find a linear order $\prec$ for the set A that will merely be an extension of the given partial order in the sense that if $a \leq b$, then $a \prec b$.
- The process of constructing a linear order such as $\prec$ is called a topological sorting.

5.2 Hasse Diagram (cont)

- The problem might arise when we have to enter a finite poset A into a computer.
  - The elements of A must be entered in some order, and we might want them entered so that the partial order is preserved.
  - If $a \leq b$, then a is entered before b.
  - A topological sorting $\prec$ will give an order of entry of the elements that meets this condition.

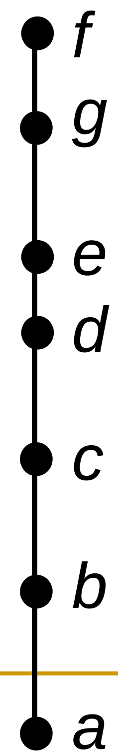
5.2 Hasse Diagram (cont)

- E.g. Refer to the following Hasse diagram.



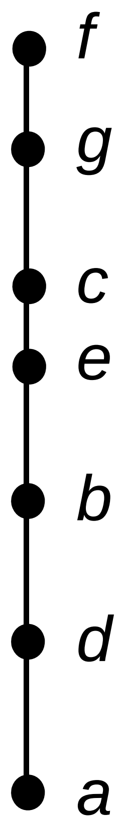
5.2 Hasse Diagram (cont)

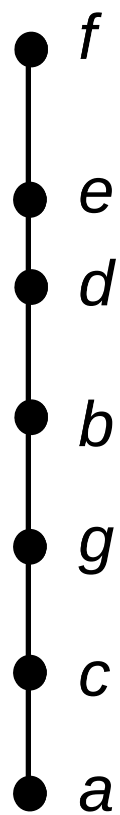
The partial order $\prec$ whose Hasse diagram shown below is clearly a linear order, i.e. every pair in $\leq$ is also in the order $\prec$, so $\prec$ is a topological sorting of the partial order $\leq$.



5.2 Hasse Diagram (cont)

- Below are two other solutions to this problem.





5.2 Hasse Diagram (cont)

- Let $(A, \leq)$ and $(A', \leq')$ be posets and let $f : A \to A'$ be a one-to-one corresponding between A and A'. The function f is called an isomorphism from $(A, \leq)$ to $(A', \leq')$ if, for any a and b in A,

  $a \leq b$ if and only if $f(a) \leq' f(b)$.

5.2 Hasse Diagram (cont)

- If $f : A \rightarrow A'$ is an isomorphism, then $(A, \leq)$ and $(A', \leq')$ are isomorphic posets.
  - Let A be the set Z^+ of positive integers, and let $\leq$ be the usual partial order on A. Let A' be the set of positive even integers, and let $\leq'$ be the usual partial order on A'. Then the function $f : A \rightarrow A'$ is given by $f(a) = 2a$.

    Since f is one-to-one, onto, and everywhere defined, f is one-to-one corresponding.

    Also, $f(a) = 2a$, $f(b) = 2b$,

    so $a \leq b$ if and only if $f(a) \leq' f(b)$.

    Thus f is an isomorphism.

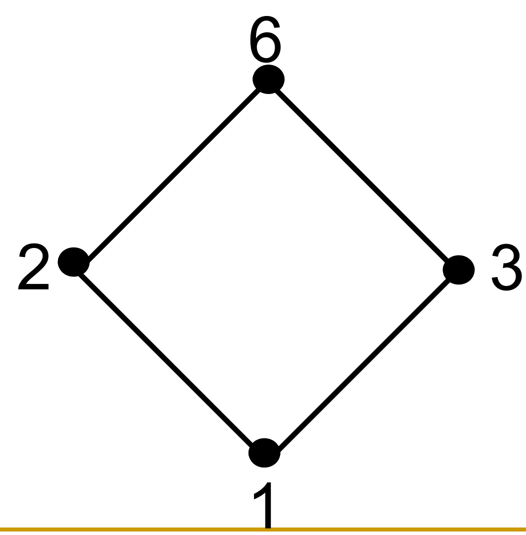
5.2 Hasse Diagram (cont)

- Theorem 1 Principle of Correspondence

  If the elements of *B* have any property relating to one another or to other elements of *A*, and if this property can be defined entirely in terms of the relation $\leq$, then the elements of *B*' must possess exactly the same property, defined in terms of $\leq$'.

5.2 Hasse Diagram (cont)

- Two finite isomorphic posets must have the same Hasse diagrams.
  - Let $A = \{1, 2, 3, 6\}$ and let $\leq$ be the relation $|$ (divides). The Hasse diagram for $(A, \leq)$ is given as follows:



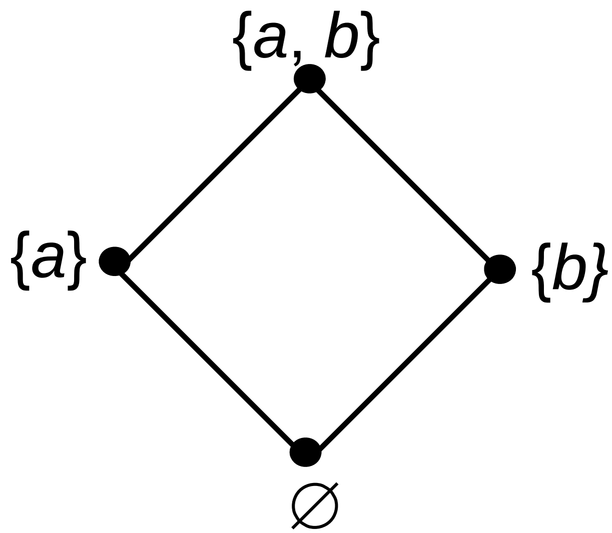
5.2 Hasse Diagram (cont)

Let A' = $\wp(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$, and let $\leq'$ be set containment, $\subseteq$.

If $f : A \to A'$ is defined by $f(1) = \varnothing$, $f(2) = \{a\}$, $f(3) = \{b\}$, $f(6) = \{a, b\}$, then f is one-to-one corresponding.

Since $x|y$ if and only if $f(x) \subseteq f(y)$, f is order preserving. And if each label $a \in A$ of the Hasse diagram is replaced by $f(a)$ and the Hasse diagram for $(A', \leq')$ is obtained.

5.2 Hasse Diagram (cont)



Thus the function *f* is an isomorphism.

5.3 Extremal Elements of Partially Ordered Sets

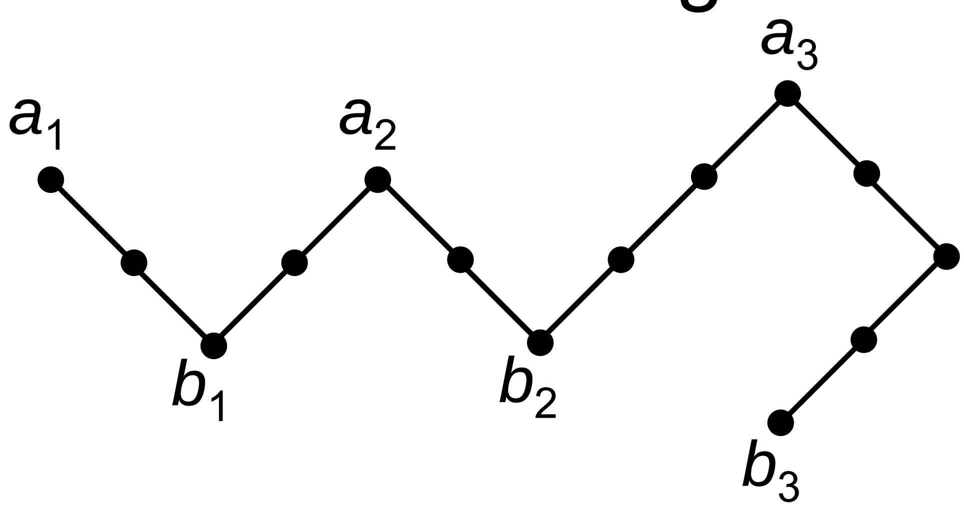
- Consider a poset $(A, \leq)$.
  - An element $a \in A$ is called a maximal element of A if there is no element c in A such that $a < c$.
  - An element $b \in A$ is called a minimal element of A if there is no element c in A such that $c < b$.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- If $(A, \leq)$ is a poset and $(A, \geq)$ is its dual poset,
  - an element $a \in A$ is a maximal element of $(A, \geq) \Leftrightarrow a$ is a minimal element of $(A, \leq)$.
  - an element $a \in A$ is a minimal element of $(A, \geq) \Leftrightarrow a$ is a maximal element of $(A, \leq)$.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- Consider the following Hasse diagram.



  - The elements a_1, a_2, and a_3 are maximal elements of A, and the elements b_1, b_2, and b_3 are minimal elements.
  - Since there is no line between b_2 and b_3, neither $b_2 \leq b_3$ nor $b_3 \leq b_2$.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- Let A be the poset of nonnegative real numbers with the usual partial order $\leq$. Then 0 is a minimal element and there are no maximal elements of A.
- The poset Z with the usual partial order $\leq$ has no maximal elements and has no minimal elements.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- Theorem 1

  Let A be a finite nonempty poset with partial order $\leq$. Then A has at least one maximal element and at least one minimal element.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- By using the concept of a minimal element, we can give an algorithm for finding a topological sorting of a given finite poset $(A, \leq)$.
  - If $a \in A$ and $B = A - \{a\}$, then B is also a poset under the restriction of $\leq$ to $B \times B$.
  - Assume a linear array name SORT that produced is ordered by increasing index, that is SORT[1] $\prec$ SORT[2] $\prec \ldots$
  - The relation $\prec$ on A defined in this way is a topological sorting of $(A, \leq)$

5.3 Extremal Elements of Partially Ordered Sets (cont)

- Algorithm for finding a topological sorting of a finite poset $(A, \leq)$:

  Step 1: Choose a minimal element of A.

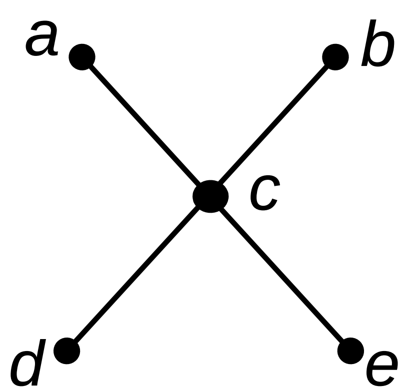
  Step 2: Make a next entry of SORT and replace A with $A - \{a\}$.

  Step 3: Repeat steps 1 and 2 until $A = \{\ \}$.

  End of algorithm.

5.3 Extremal Elements of Partially Ordered Sets (cont)

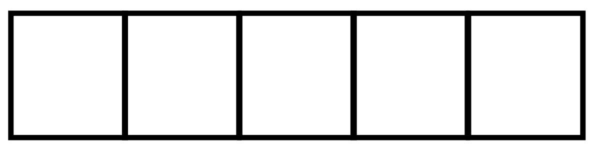
- Let $A = \{a, b, c, d, e\}$, and let the Hasse diagram of a partial order $\leq$ on A be as shown below.



5.3 Extremal Elements of Partially Ordered Sets (cont)

A minimal element of this poset is the vertex labelled *d* (could also have chosen *e*). Put *d* in SORT [1] and show the following Hasse diagram of $A - \{d\}$.

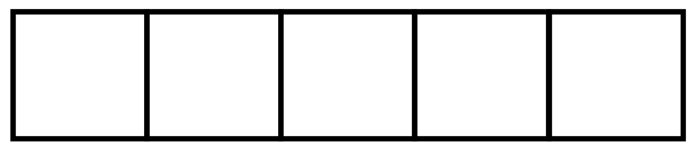
SORT



5.3 Extremal Elements of Partially Ordered Sets (cont)

A minimal element of the new *A* is *e*, so *e* becomes SORT [2], and *A* – {*e*} is shown below.

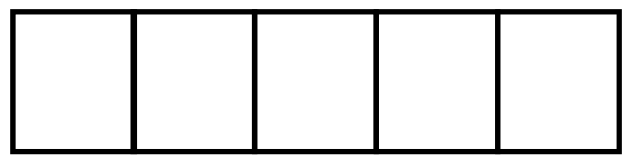
SORT



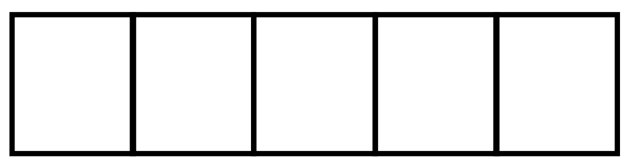
5.3 Extremal Elements of Partially Ordered Sets (cont)

The process continues until we have exhausted *A* and filled SORT.

SORT



SORT



5.3 Extremal Elements of Partially Ordered Sets (cont)

The completed array SORT and the Hasse diagram of the poset corresponding to SORT is shown below. This is a topological sorting of $(A, \leq)$.

SORT

5.3 Extremal Elements of Partially Ordered Sets (cont)

- An element $a \in A$ is called a greatest element of A if $x \leq a$ for all $x \in A$.
- An element $a \in A$ is called a least element of A if $a \leq x$ for all $x \in A$.
- An element a of $(A, \leq)$ is a greatest (or least) element $\Leftrightarrow$ it is a least (or greatest) element of $(A, \geq)$.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- Let A be the poset of nonnegative real numbers with the usual partial order $\leq$. Then 0 is the least element and there is no greatest element.
- Let $S = \{a, b, c\}$ and the power set $A = \wp(S)$. Consider the poset $(A, \subseteq)$.

$A = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

The empty set is a least element of A and the set S is a greatest element of A.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- The poset Z with usual partial order $\leq$ has neither a least nor greatest element.
- Theorem 2

  A poset has at most one greatest element and at most one least element.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- The greatest element of a poset, if exists, is denoted by *I* and is often called the unit element. The least element, if exists, is denoted by *0* and is often called the zero element.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- Consider a poset A and a subset B of A.
  - An element $a \in A$ is called an upper bound of B of $b \leq a$ for all $b \in B$.
  - An element $a \in A$ is called a lower bound of B of $a \leq b$ for all $b \in B$.
- A subset B of a poset may or may not have upper or lower bounds (in A). Moreover, an upper or lower bound of B may or may not belong to B itself.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- Let A be a poset and B is a subset of A.
  - An element $a \in A$ is called a least upper bound of B, (LUB(B)), if a is an upper bound of B and $a \leq a'$, whenever a' is an upper bound of B.

    Thus a = LUB(B) if $b \leq a$ for all $b \in B$, and if whenever $a' \in A$ is also an upper bound of B, then $a \leq a'$.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- An element $a \in A$ is called a greatest lower bound of B, (GLB(B)), if a is a lower bound of B and $a' \leq a$, whenever a' is a lower bound of B.

  Thus a = GLB(B) if $a \leq b$ for all $b \in B$, and if whenever $a' \in A$ is also a lower bound of B, then $a' \leq a$.

5.3 Extremal Elements of Partially Ordered Sets (cont)

- Upper bounds in $(A, \leq)$ correspond to lower bounds in $(A, \geq)$ (for the same set of elements), and lower bounds in $(A, \leq)$ correspond to upper bounds in $(A, \geq)$. Similar statements hold for greatest lower bounds and least upper bounds.
- Theorem 3

  Let $(A, \leq)$ be a poset. Then a subset B of A has at most one LUB and at most one GLB.

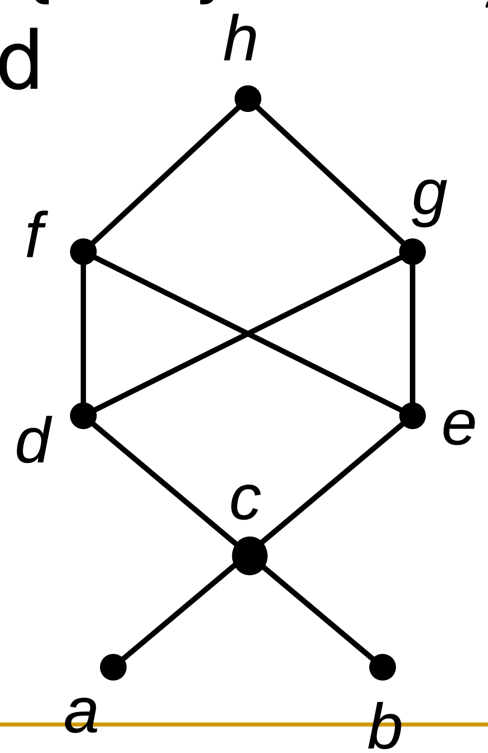
5.3 Extremal Elements of Partially Ordered Sets (cont)

- In a finite poset A, as viewed from the Hasse diagram of A. Let $B = \{b_1, b_2, \ldots, b_r\}$. If a = LUB(B), then a is the first vertex that can be reached from $b_1, b_2, \ldots, b_r$ by upward paths. Similarly if a = GLB(B), then a is the first vertex that can be reached from $b_1, b_2, \ldots, b_r$ by downward paths.

E.g.4

Consider the poset $A = \{a, b, c, d, e, f, g, h\}$, whose Hasse diagram is shown.

Let $B_1 = \{a, b\}$ and $B_2 = \{c, d, e\}$ be subsets of A. Find



E.g.4 (cont)

i. upper and lower bounds of B_1 and B_2;

ii. all least upper bounds and greatest lower bounds of B_1 and B_2.

E.g.4 (cont)

i. upper and lower bounds of B_1 and B_2;

upper bounds of B_1 =

lower bounds of B_1 =

upper bounds of B_2 =

lower bounds of B_2 =

E.g.4 (cont)

ii. all least upper bounds and greatest lower bounds of B_1 and B_2.

LUB of B_1 =

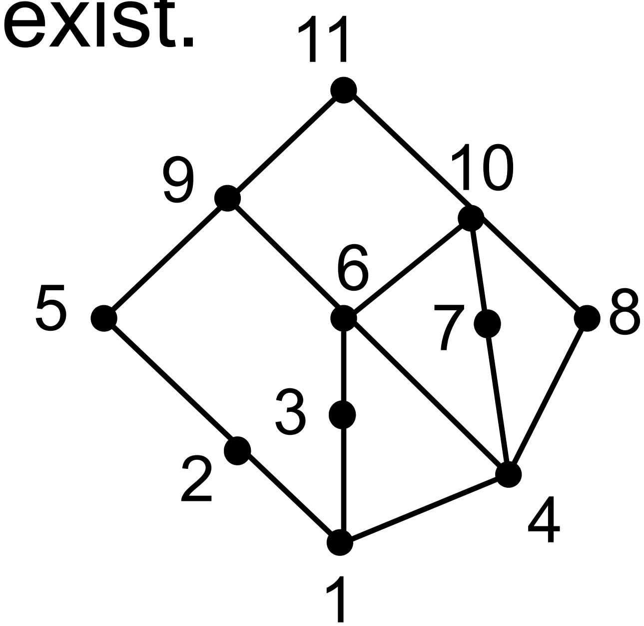
GLB of B_1 =

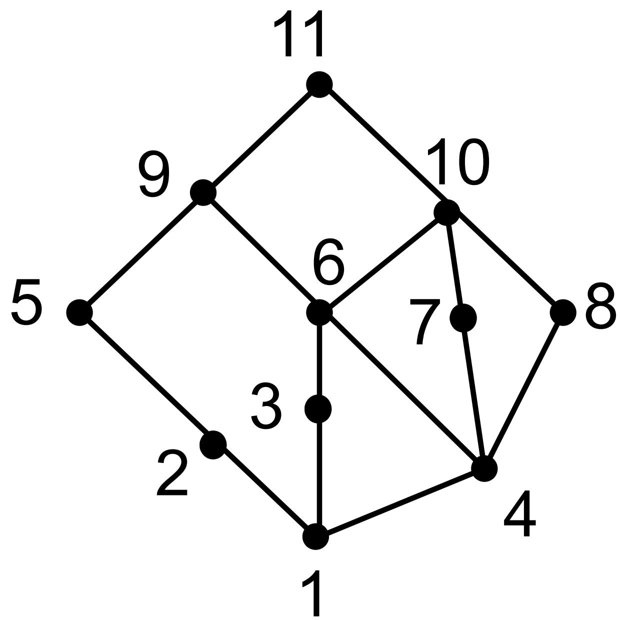
LUB of B_2 =

GLB of B_2 =

E.g.5

Let A = {1, 2, 3, 4, 5, …, 11} be the poset whose Hasse diagram is shown below. Find the LUB and GLB of B = {6, 7, 10}, if they exist.





Exploring all upward paths from 6, 7, and 10

$\Rightarrow$ LUB (B) =

Exploring all downward paths from 6, 7, and 10 $\Rightarrow$ GLB (B) =

5.3 Extremal Elements of Partially Ordered Sets (cont)

- Theorem 4

  Suppose that (A, $\leq$) and (A', $\leq'$) are isomorphic posets under the isomorphism $f : A \to A'$.

a) If a is a maximal (minimal) element of (A, $\leq$), then $f(a)$ is a maximal (minimal) element of (A', $\leq'$).

5.3 Extremal Elements of Partially Ordered Sets (cont)

b) If a is a greatest (least) element of $(A, \leq)$, then $f(a)$ is a greatest (least) element of $(A', \leq')$.

c) If a is an upper bound (lower bound, least upper bound, greatest lower bound) of a subset B of A, then $f(a)$ is an upper bound (lower bound, least upper bound, greatest lower bound) for the subset $f(B)$ of A'.

5.3 Extremal Elements of Partially Ordered Sets (cont)

d) If every subset of (A, $\leq$) has a LUB (GLB), then every subset of (A', $\leq'$) has a LUB (GLB).