ACM40290: Assignment 4

Lee Kai Yang (23205838)
kai.y.lee@ucdconnect.ie

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Question 1. Consider numerically integrating the initial value problem

$$\frac{dy}{dt} = f(t, y), \qquad y(0) = y_0$$

using the Leapfrog scheme given by

$$\frac{Y^{k+1} - Y^{k-1}}{2h} = f\left(t_k, Y^k\right)$$

where Y^k is the numerical approximation to the solution $y(t_k)$ with constant time-step sizes h.

Problem 1a. Show that the numerical solution arising from this scheme has a local truncation error of $O(h^3)$.

Solution
Rearranging the scheme:

$$Y^{k+1} = Y^{k-1} + 2hf\left(t_k, Y^k\right)$$

The local error would be:

$$\text{local error} = y(t_{k+1}) - Y^{k+1}$$

where $y(t_{k+1})$ is the exact solution and Y^{k+1} is the numerical approximation. Then we Taylor expand the term $y(t_{k+1})$ to the third-order since the Leapfrog scheme is a second-order method:

$$\begin{aligned} y(t_{k+1}) &= y(t_k + h) \\ &= y(t_k) + hy'(t_k, y(t_k)) + \frac{h^2}{2} y''(t_k, y(t_k)) + \frac{h^3}{6} y'''(\theta) \\ &= y(t_k) + hf(t_k, y(t_k)) + \frac{h^2}{2} f'(t_k, y(t_k)) + \frac{h^3}{6} f''(\theta) \end{aligned}$$

Substitute it back to the local error:

$$\begin{aligned} \text{local error} &= y(t_k) + hf(t_k, y(t_k)) + \frac{h^2}{2} f'(t_k, y(t_k)) + \frac{h^3}{6} f''(\theta) - \left[Y^{k-1} + 2hf\left(t_k, Y^k\right)\right] \\ &= \underbrace{y(t_k) - Y^{k-1} + hf(t_k, y(t_k)) - 2hf\left(t_k, Y^k\right) + \frac{h^2}{2} f'(t_k, y(t_k))}_{\text{error propagated from } t_k} + \underbrace{\frac{h^3}{6} f''(\theta)}_{\text{local truncation error}} \end{aligned}$$

Note that since local error is between one subsequent step hence $\theta \in (t_k, t_k + h)$. The local truncation error is $O(h^3)$ because h^3 is the highest order term in the expression $\frac{h^3}{6} f''(\theta)$.

Problem 1b. What is the first value of k, for numerical solution Y^k, that you can calculate with this method? Given $Y^0 = y_0$, suggest how you might find any additional (intermediate) solution values required.

Solution
To use the Leapfrog scheme, we need to have Y_0, Y_1 and Y_2 because of the term Y^k, Y^{k-1} and Y^{k+1}. Hence, the first value of k that can be calculated in this method is $k = 1$.

$$\text{when } k = 1, \quad \frac{Y^2 - Y^0}{2} h = f(t_1, Y^1)$$

Given $Y^0 = y_0$, to find Y_1 and Y_2 we can use the Forward Euler's method:

$$Y^{n+1} = Y^n + hf(Y^n)$$

This will form $Y^1 = Y^0 + hf(t_0, Y^0)$ and $Y^2 = Y^1 + hf(t_1, Y^1)$.

Question 2. For the ordinary differential equation

$$\frac{\mathrm{dy}}{\mathrm{dt}} = f(t)$$

we wish to use an integration scheme given by

$$y_{j+2} = y_{j+1} + \frac{h}{2}(3f_{j+1} - f_j)$$

where h is the time-step. Take $y_{j+2} = y(t_j + 2h)$, $y_{j+1} = y(t_j + h)$, $f_j = f(t_j)$ and $f_{j+1} = f(t_j + h)$.

Problem 2a. Show that the local error is $O(h^3)$ and global error for $N \approx 1/h$ steps is $O(h^2)$.

Solution
Taylor expand the term y_{j+2}:

$$\begin{aligned} y_{j+2} &= y(t_j + 2h) \\ &= y(t_j) + 2hy'(t_j) + 2h^2 y''(t_j) + \frac{4}{3}h^3 y'''(\theta) \\ &= y(t_j) + 2hf(t_j) + 2h^2 f'(t_j) + \frac{4}{3}h^3 f''(\theta) \end{aligned}$$

Substituting it into the local error:

$$
\begin{aligned}
\text{local error} &= y_{j+2} - y_{j+1} - \frac{h}{2}(3f_{j+1} - f_j) \\
&= y(t_j + 2h) - y(t_j + h) - \frac{h}{2}(3f(t_j + h) - f(t_j)) \\
&= y(t_j) + 2hf(t_j) + 2h^2 f'(t_j) + \frac{4}{3}h^3 f''(\theta) - y(t_j + h) - \frac{h}{2}(3f(t_j + h) - f(t_j)) \\
&= \underbrace{y(t_j) - y(t_j + h) + \frac{5h}{2}f(t_j) - \frac{3h}{2}f(t_j + h) + 2h^2 f'(t_j)}_{\text{error propagated from } t_j} + \underbrace{\frac{4}{3}h^3 f''(\theta)}_{\text{local truncation error}}
\end{aligned}
$$

Note that since local error is between one subsequent step hence $\theta \in (t_j, t_j + h)$. The local error is $O(h^3)$ because h^3 is the highest order term in the expression $\frac{4}{3}h^3 f''(\theta)$.

Since local error is $\frac{4}{3}h^3 f''(\theta)$ then global error must be $N \cdot \frac{4}{3}h^3 f''(\theta)$ for $\theta \in (t_0, t_N)$ hence:

$$
\begin{aligned}
\text{global error} &\approx N \cdot \frac{4}{3}h^3 f''(\theta) \\
&\approx \frac{1}{h}\frac{4}{3}h^3 f''(\theta) \\
&\approx \frac{4}{3}h^2 f''(\theta)
\end{aligned}
$$

Therefore, the global error is $O(h^2)$ because the highest order term in the expression $\frac{4}{3}h^2 f''(\theta)$ is h^2.

Problem 2b. The scheme requires two prior points to calculate the next one. If given a single initial point y_0, explain in two sentences how might you generate a y_1 so that you can begin the numerical method at y_2.

Solution

Given the initial point y_0, we can use a single-step method such as Forward Euler's method to get a good approximation of y_1 with a step size of h:

$$y_1 = y_0 + hf(t_0)$$

Then this value of y_1 can then be used in conjunction with the initial point y_0 to begin the numerical method at y_2.

Question 3. For the IVP,

$$\frac{\mathrm{du}}{\mathrm{dt}} = \lambda u \qquad\qquad u(0) = u_0$$

consider the numerical scheme

$$U^{n+1} = U^n + h\lambda\left(\frac{U^n}{2} + \frac{U^{n+1}}{2}\right)$$

with $U^0 = u_0, U^n \approx u(t_n)$ for $n > 0,$ and h the time step.

Problem 3a. Rewrite the above scheme as

$$U^{n+1} = g(h, \lambda)U^n$$

for a function g of h and λ to be determined.

Solution

$$\begin{aligned}
U^{n+1} &= U^n + h\lambda\frac{U^n}{2} + h\lambda\frac{U^{n+1}}{2} \\
U^{n+1} - h\lambda\frac{U^{n+1}}{2} &= U^n + h\lambda\frac{U^n}{2} \\
U^{n+1}\left(1 - \frac{h\lambda}{2}\right) &= U^n\left(1 + \frac{h\lambda}{2}\right) \\
U^{n+1} &= \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}U^n
\end{aligned}$$

Hence,

$$g(h, \lambda) = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}$$

Problem 3b. Determine the range of values of h for which the solution is stable. Note: when considering the step $|\cdot| < 1$ you may then need to write $h\lambda$ as a complex number, i.e., $h\lambda = a + ib$, to determine the condition in full (λ may be complex, but it is easier here to use $h\lambda$ as a single variable).

Solution

For the solution to be stable, $\left|\frac{1+\frac{h\lambda}{2}}{1-\frac{h\lambda}{2}}\right| < 1$ must be true.

Let $h\lambda = a + ib$, we can write

$$\begin{aligned}
\left|\frac{1 + \frac{a+ib}{2}}{1 - \frac{a+ib}{2}}\right| &< 1 \\
\left|\frac{\frac{2+a+ib}{2}}{\frac{2-a-ib}{2}}\right| &< 1 \\
\left|\frac{2 + a + ib}{2 - a - ib}\right| &< 1 \\
\frac{\sqrt{(2+a)^2 + b^2}}{\sqrt{(2-a)^2 + b^2}} &< 1 \\
\frac{(2+a)^2 + b^2}{(2-a)^2 + b^2} &< 1 \\
(2+a)^2 + b^2 &< (2-a)^2 + b^2 \\
4 + a^2 + 4a + b^2 &< 4 + a^2 - 4a + b^2
\end{aligned}$$

$$\begin{aligned}
4a &< -4a \\
8a &< 0 \\
a &< 0
\end{aligned}$$

Since a is the real part of $h\lambda$, therefore we can say that for the solution to be stable, the real part of $h\lambda$ must be negative (lesser than 0). Visualising it on the cartesian plane, it looks as follows:

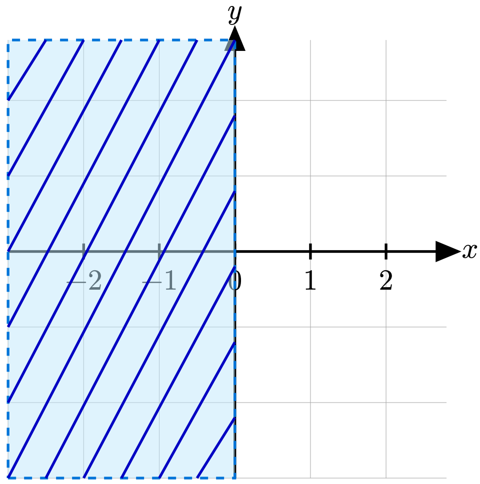

Figure 1: Range of stable solution

where the blue region indicate the region where the solution is stable.

Question 4. Consider the linear system

$$y'' - (\lambda - 1)y' - \lambda y = 0, \quad y(0) = 1, \quad y'(0) = -\lambda - 2$$

for $\lambda < 0$. Determine which of $\lambda = -2$ or $\lambda = -45$ will produce a stiff system.

Solution

First we linearise the system:

Let $v = y'$ and $v' = y''$, then we will have

$$\frac{\mathrm{dy}}{\mathrm{dt}} = y' = v$$

$$\begin{aligned} v' - (\lambda - 1)v - \lambda y = 0 \\ v' &= (\lambda - 1)v + \lambda y \end{aligned}$$

Now we express it in matrix form:

Let $Y = \binom{y}{v}$, then

$$\begin{aligned} \frac{\mathrm{dY}}{\mathrm{dt}} &= A\binom{y}{v} \\ &= \begin{pmatrix} 0 & 1 \\ \lambda & \lambda - 1 \end{pmatrix}\binom{y}{v} \end{aligned}$$

Find the eigenvalues of A when $\lambda = -2$:

$$\det(\lambda I - A) = 0$$
$$\det\left(\lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}0 & 1\\-2 & -3\end{pmatrix}\right) = 0$$
$$\det\begin{pmatrix}\lambda & -1\\2 & \lambda+3\end{pmatrix} = 0$$
$$\lambda^2 + 3\lambda + 2 = 0$$
$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4(1)(2)}}{2} = \frac{-3 \pm \sqrt{1}}{2}$$
$$\lambda_+ = -1, \quad \lambda_- = -2$$

Find the eigenvalues of A when $\lambda = -45$:

$$\det(\lambda I - A) = 0$$
$$\det\left(\lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}0 & 1\\-45 & -46\end{pmatrix}\right) = 0$$
$$\det\begin{pmatrix}\lambda & -1\\45 & \lambda+46\end{pmatrix} = 0$$
$$\lambda^2 + 46\lambda + 45 = 0$$
$$\lambda = \frac{-46 \pm \sqrt{46^2 - 4(1)(45)}}{2} = \frac{-46 \pm \sqrt{1936}}{2} = \frac{-46 \pm 44}{2}$$
$$\lambda_+ = -1, \quad \lambda_- = -45$$

Since stiffness arises when $\max|\lambda_j| \gg \min|\lambda_j|$, $\lambda = -45$ will produce a stiff system because $45 \gg 1$.