Corregidum to "Novel whitening approaches in functional settings"

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1. Due to production errors, Equation 3 in pp. 3 is written as

$$\langle f,g\rangle_{\mathbb{M}} = \sum_{j=1}^{\infty} \lambda_j^{-1} \langle f,\gamma_j\rangle \langle g,\gamma_j\rangle = \left\langle \Gamma^{1/2\dagger} f, \Gamma^{1/2\dagger} g \right\rangle f, \quad g \in \mathbb{M},$$

   while it was originally written as

$$\langle f,g\rangle_{\mathbb{M}} = \sum_{j=1}^{\infty} \lambda_j^{-1} \langle f,\gamma_j\rangle \langle g,\gamma_j\rangle = \left\langle \Gamma^{1/2\dagger} f, \Gamma^{1/2\dagger} g \right\rangle \quad f,g \in \mathbb{M}.$$

2. Definition 1 states: "The whitening operator Ψ transforms a functional variable X into a new element $\mathrm{X} = \Psi(X-\mu)$ with zero mean and covariance operator being exactly the identity inside H." The notion of "inside" might be somewhat ambiguous, suggesting that whitening is restricted to a closed subspace, as noted later. However, this restriction is not strictly necessary since in some cases the closure is the entire Hilbert space.

3. In §3, the statement reads, "Then, we can use the inner product (3) to construct a space of isotropic functions (i.e., their covariance operator satisfies the identity), so that the space ends up having a certain Gaussian appearance..." The term *space of isotropic functions* is unclear in the current context. Since our whitening operators are mappings defined through elements over T, this does not necessarily imply that the realizations of $\mathbb{X}$ are on the unit sphere $S = \{f \in \mathbb{M} \mid \|f\|^2 = 1\}$. The isotropy property would be satisfied when whitening the basis expansion coefficients in the direction of its transpose, assuming dependencies in a secondary domain exist. Therefore, one could use the following instead: "....to construct *a space of whitened functions* (i.e., their covariance operator satisfies the identity), so that the space ends up having a certain Gaussian appearance..." .

4. Considering that we work with elements that are already in the range space of Γ, a whitening transformation takes an element from this space to $\mathbb{M}$. Thus, $\Gamma^{1/2\dagger} : \mathrm{ran}(\Gamma) \to \mathbb{M}$ instead of $\Gamma^{1/2\dagger} : \mathbb{M} \to \mathbb{M}$. In the case of correlation-based whitening operators, we need to assume X is in $\mathrm{ran}(\mathrm{span}(e_k))$; see comment bellow on the operator $\mathcal{R}$.

5. In §4, the sentence "As 2tr $(\Gamma_{X\mathbb{X}})$ is the only dependence between the original and the whitened variable, the minimization problem can be reduced to the maximization of tr $(\Gamma_{X\mathbb{X}})$." reads also as ".. is the only *dependent term*...".

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6. Note that, in §4 the term $\mathrm{tr}(\Gamma_{\mathbb{X}})$ in the quadratic distances diverges (the trace of $\Gamma_{\mathbb{X}}$ is an infinite sum of ones). However, $\mathrm{tr}(\Gamma_{\mathbb{X}})$ is not accounted for in the proof. In order for these distances to converge, one has to consider regularization or finite space dependency. Furthermore, the operator $\Gamma_{X\mathbb{X}}$ coincides with $\Gamma^{1/2}$ if $\Psi \equiv \Gamma^{1/2\dagger}$. Note we only know that this operator belongs to the class of Hilbert Schmidt operators (from the trace property of the autocovariance operator), but this fact does not necessarily imply $\Gamma^{1/2}$ has finite trace. Hence, we further assume that under mild conditions, $\mathrm{tr}(\Gamma^{1/2}) < \infty$ is satisfied.

7. In the Technical proofs (first paragraph), due to abuse of notation, in the sentence "Note that Condition 1 cannot be reached when $\langle X, \gamma_j\rangle^2 = \lambda_j$, or for $c_j \to c > 0$, $\langle X, \gamma_j\rangle^2 = \lambda_j c_j$...", X stands for a deterministic function.

8. In the technical proof of Proposition 1, it is stated that "The operator $P_{\overline{\mathrm{ran}}(\Gamma^{1/2})}$ is compact...". However, this characterization of the projection operator as compact appears to be mistaken since projection operators are typically not compact in infinite-dimensional spaces.

CHARACTERIZATION OF $\mathcal{R} = \mathcal{V}^{1/2\dagger}\Gamma\mathcal{V}^{1/2\dagger}$

Suppose X is expanded as $X = \sum_{k=1}^{\infty}\langle X, e_k\rangle e_k$ and note the following:

$$\mathcal{V} = \sum_{k=1}^{\infty} \mathcal{P}_{e_k}\Gamma\mathcal{P}_{e_k} = \sum_{k=1}^{\infty}\langle \Gamma e_k, e_k\rangle \mathcal{P}_{e_k} = \sum_{k=1}^{\infty} E(\langle X, e_k\rangle^2)(e_k \otimes e_k) = \sum_{k=1}^{\infty}\eta_k(e_k \otimes e_k). \tag{1}$$

Now, consider the operator

$$\mathcal{R} \equiv E\left(\sum_{j=1}^{\infty} \frac{\langle X, e_j\rangle}{\eta_j^{1/2}} e_j \otimes \frac{\langle X, e_j\rangle}{\eta_j^{1/2}} e_j\right). \tag{2}$$

Observe that $\mathcal{V}^{1/2\dagger}\Gamma\mathcal{V}^{1/2\dagger} = \sum_{j=1}^{\infty}\eta_j^{-1/2}E(\mathcal{P}_{e_j}X \otimes \mathcal{P}_{e_j}X)\eta_j^{-1/2}$, where $\mathcal{V}^{\dagger}$ is the Moore-Penrose inverse of $\mathcal{V} = \sum_{j=1}^{\infty}\eta_j\mathcal{P}_{e_j}$. This shows that $\mathcal{V}^{1/2\dagger}\Gamma\mathcal{V}^{1/2\dagger}$ is equivalent to the operator $\mathcal{R}$, as defined in 2. Note this operator bears resemblance to the classical correlation matrix in the multivariate setting.

REFERENCES

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