Math Camp Lesson 3 (Day 1)

Calculus

UW–Madison Political Science

September 1, 2021

Agenda

Day 1

- Limits
- Derivatives

Day 2

- More derivatives
- Integrals
- Applications

Overview

Calculus evaluates the behavior of functions:

- Limits
- Rate of change
- Change in the rate of change
- Area of the region they are defined on

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Concepts from calculus are used widely in the applied math relevant for political science. Some examples:

- Finding the fit line with the minimal distance between predicted and observed data

- Calculating the probability density in regions of continuous distributions

- Solving for the choice that maximizes a decision maker's utility

Limits

The limit of a function is the value it approaches at particular inputs or as inputs change

Limits: Example 1

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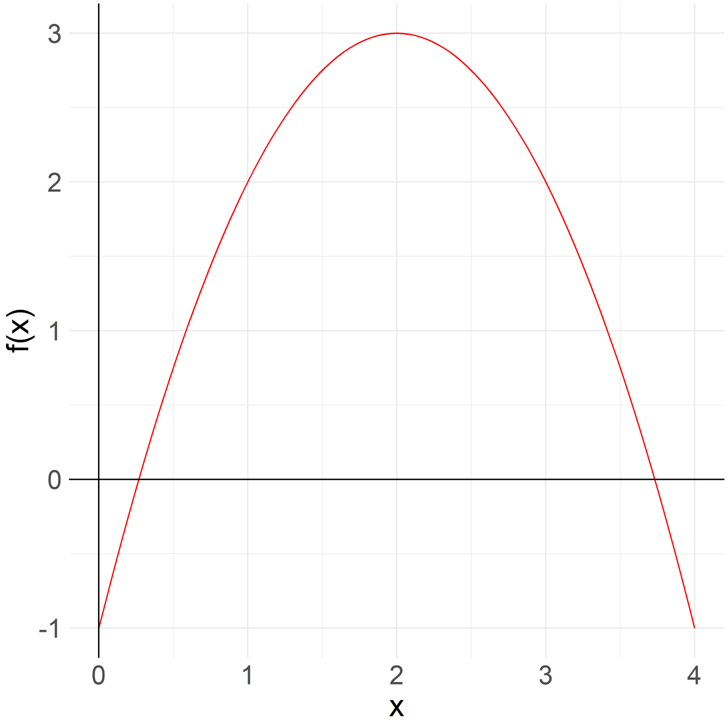
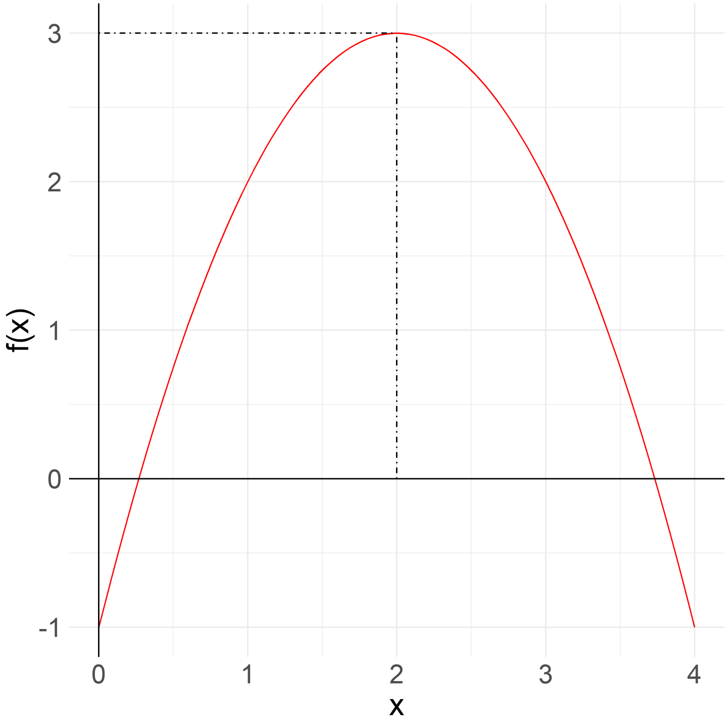


Let's consider the simple function $f(x) = y = 3 - (x-2)^2$, plotted to the left. What is the limit of $f(x)$ as x *approaches* 2?

Limits: Example 1



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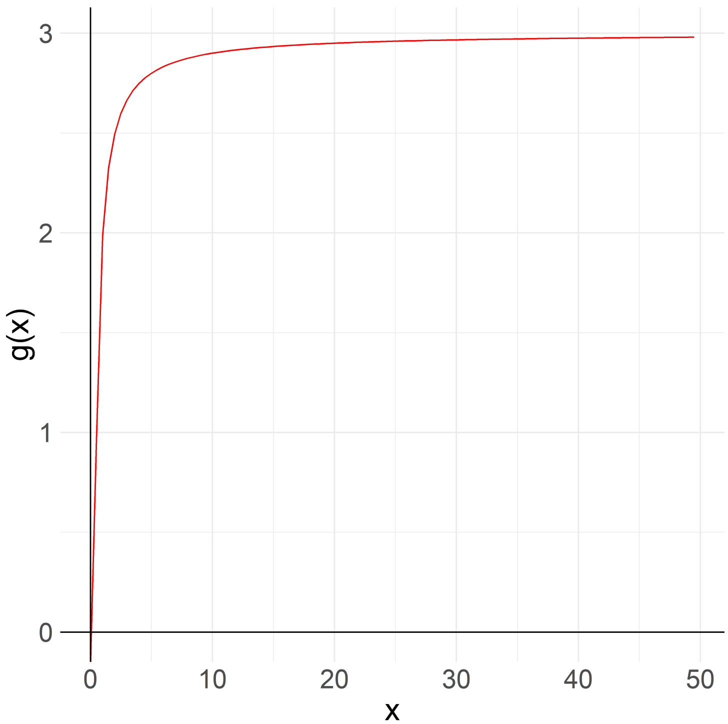
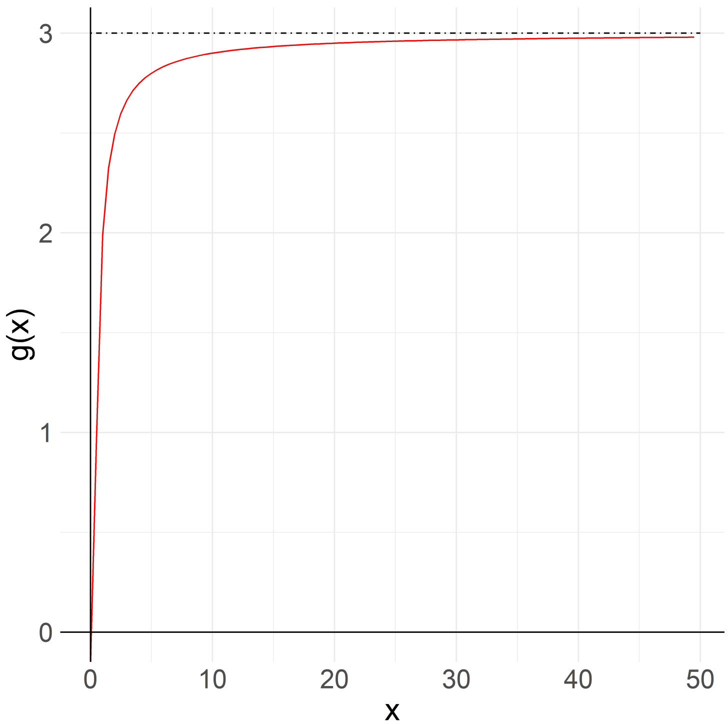
As x approaches 2, $f(x)$ or y approaches $f(2) = 3$.

Limits: Example 2



Consider a less simple function, $g(x) = y = 3 - \frac{1}{x}$, plotted to the left. What is the limit of $g(x)$ as x *approaches* ∞?

Limits: Example 2



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As x approaches ∞, $g(x)$ approaches 3. How do we know?

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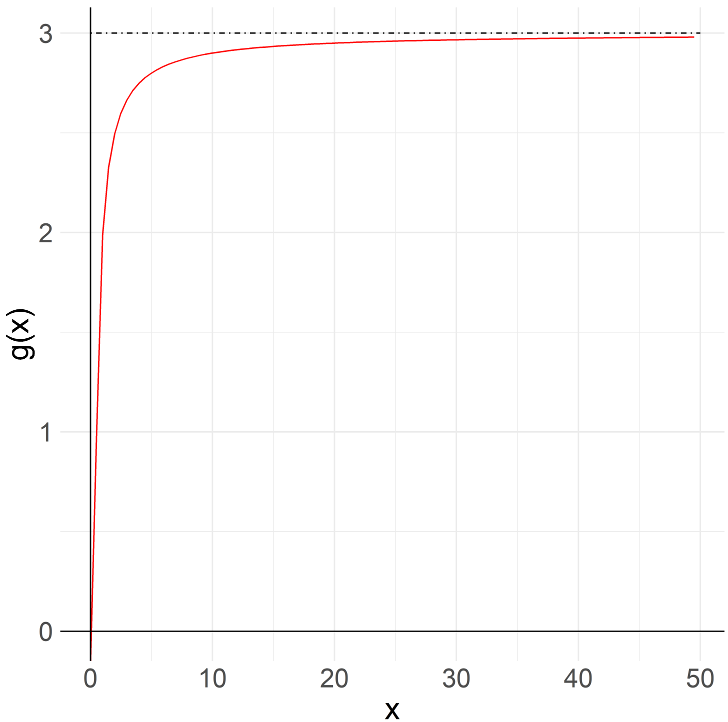

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As x approaches ∞, $g(x)$ approaches 3. How do we know?

As x gets larger, $\frac{1}{x}$ gets smaller and smaller.

$$\left(\frac{1}{2} > \frac{1}{20} > \frac{1}{200} \ldots\right)$$

Limits: formal definition

Limits are expressed as:

$$\lim_{x \to c} f(x) = L$$

Which should be read as: "As x approaches c, the limit of $f(x)$ is L"

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Many times, you will see this expression written as

$$\lim_{x \to c^-} f(x) = L$$

or

$$\lim_{x \to c^+} f(x) = L$$

A negative sign $(-)$ implies "As x approaches c *from the left*"

A positive sign $(+)$ implies "As x approaches c *from the right*"

Tips for taking limits

- Simplify as much as possible.
- Separate out the limits into distinct elements.
- Move constants outside the limit operator.
- Watch out for components that . . .
    - . . . grow very large or very small
    - . . . become zero
    - Are these components in the numerator or denominator of a fraction?
- If you can, evaluate the function at the limit.
- For functions that are well-behaved, the limit as x approaches a finite point is generally the value of the function at that point (if it exists).

Finding limits: Example 1

Consider $\lim_{x \to 2} x^2 - 3x + 1$

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$$
\begin{aligned}
\lim_{x \to 2} x^2 - 3x + 1 &= \lim_{x \to 2} x^2 - 3 \lim_{x \to 2} x + \lim_{x \to 2} 1 \\
&= 2^2 - 3(2) + 1 \\
&= -1
\end{aligned}
$$

Finding limits: Example 2

Now, let's consider $\lim_{x \to \infty} \dfrac{4x^4 + 7x^2 + 8}{3x^4}$

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Now, let's consider $\lim\limits_{x\to\infty} \dfrac{4x^4+7x^2+8}{3x^4}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x^4+7x^2+8}{3x^4} &= \lim_{x\to\infty} \frac{4x^4}{3x^4} + \lim_{x\to\infty} \frac{7x^2}{3x^4} + \lim_{x\to\infty} \frac{8}{3x^4} \\
&= \lim_{x\to\infty} \frac{4}{3} + \lim_{x\to\infty} \frac{7}{3x^2} + \lim_{x\to\infty} \frac{8}{3x^4} \\
&= \frac{4}{3} + 0 + 0 \\
&= \frac{4}{3}
\end{aligned}$$

Why does $\lim_{x\to\infty} \frac{7}{3x^2} = 0$? As $x \to \infty$, $3x^2 \to \infty$, and $\frac{7}{\infty} \to 0$.

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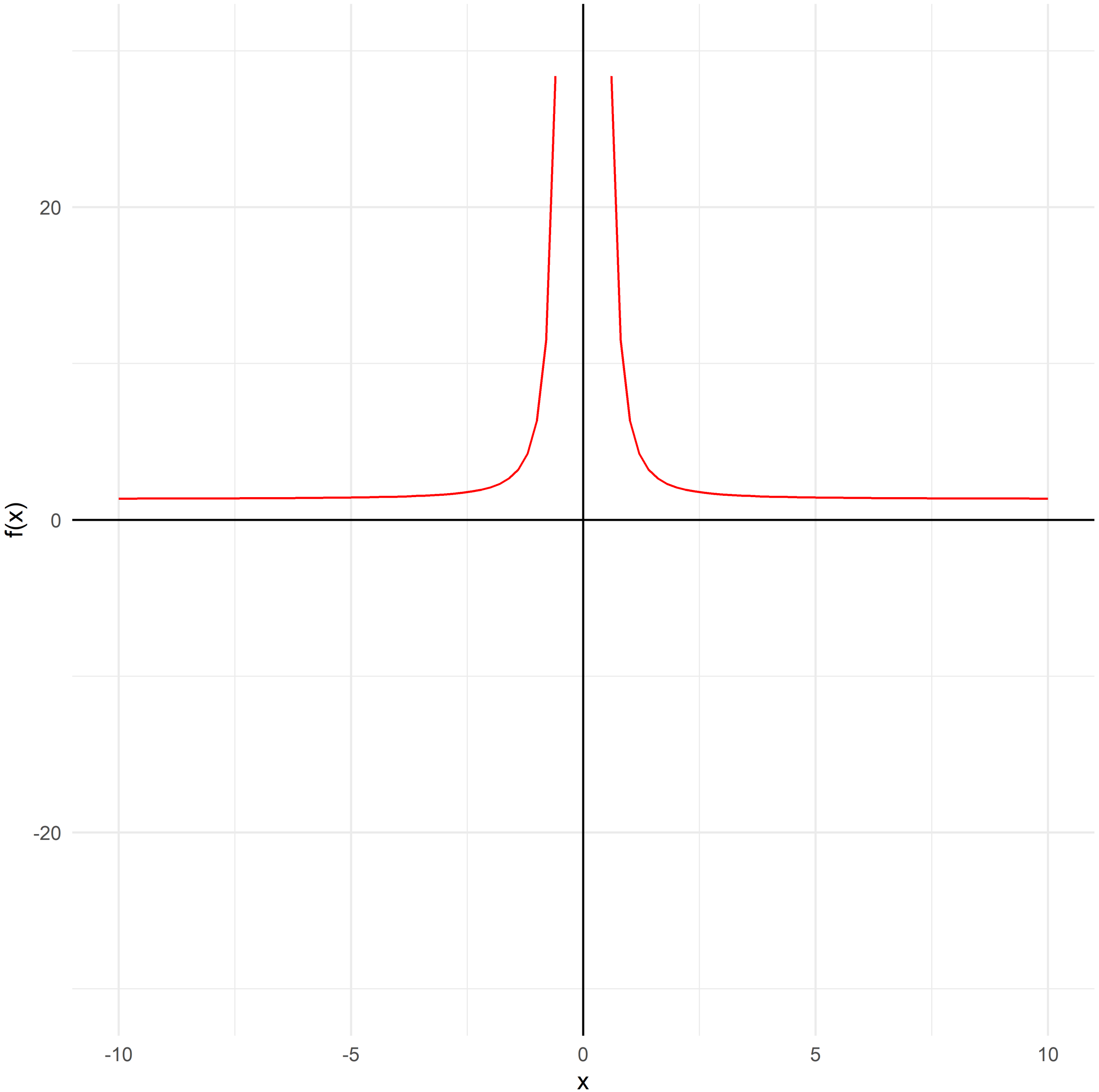
\end{aligned}
$$

$$= \infty$$

As x approaches 0, the function retains some positive value in the numerator while the denominator *positively* approaches 0 $\rightarrow$ you are dividing by a smaller and smaller number $\rightarrow$ the entire term is getting larger and approaches ∞.



20
0
-20
f(x)
-10
-5
0
5
10
x

Exercises

Find the following limits:

$\lim_{x \to 4} x^2 - 6x + 4$

$\lim_{x \to 4} \frac{x^2}{3x-2}$

$\lim_{x \to \infty} \frac{3x-4}{x+3}$

Derivatives

The derivative of a function is its rate of change in the output as the value of its input changes.

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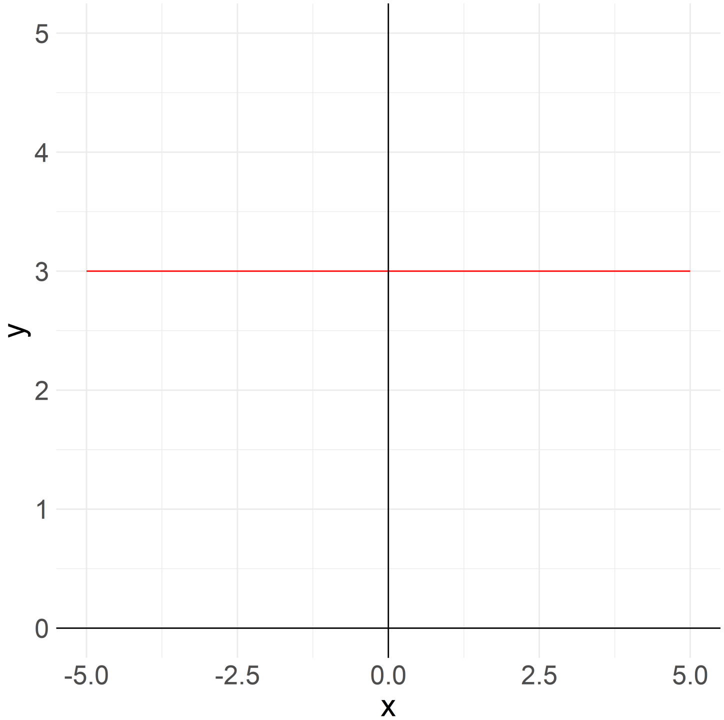
The derivative of a function is its rate of change in the output as the value of its input changes.

It is the instantaneous slope of the line ("rise-over-run") at any given point:

$$\frac{\Delta f(x)}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

where Δ signifies "change"

Slopes



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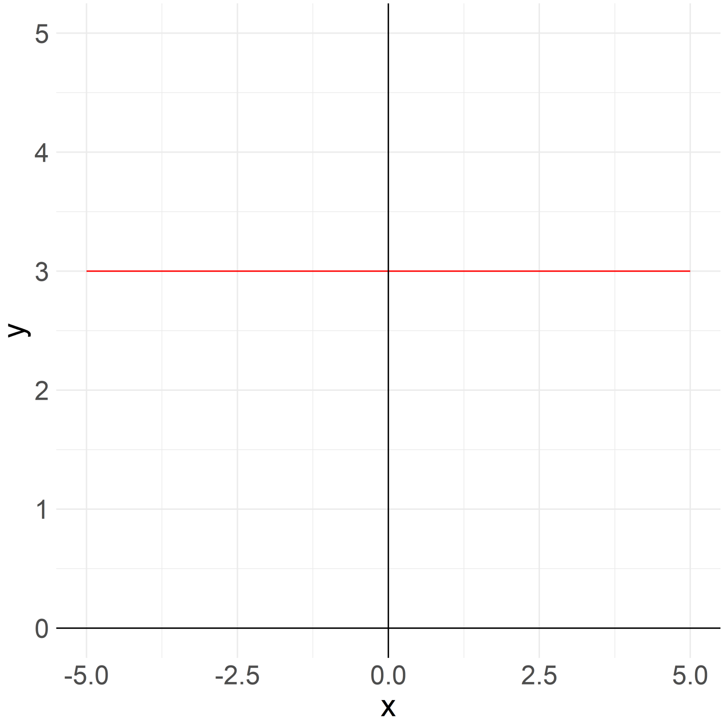
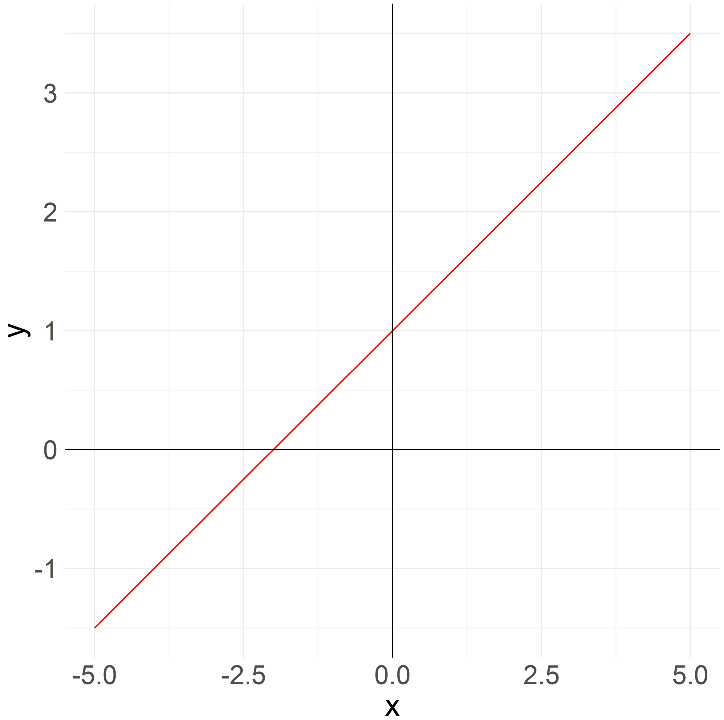
Slopes



Consider the function $y = 3$, plotted to the left. What is its "slope"?

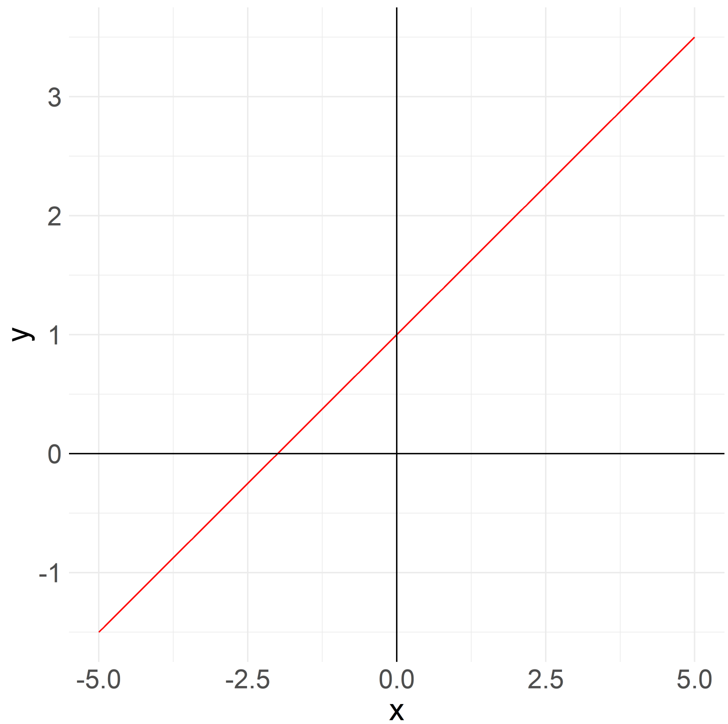
Its slope or $\frac{\Delta f(x)}{\Delta x} = 0$ because there is no "rise".

Slopes



Consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its "slope"?

Slopes

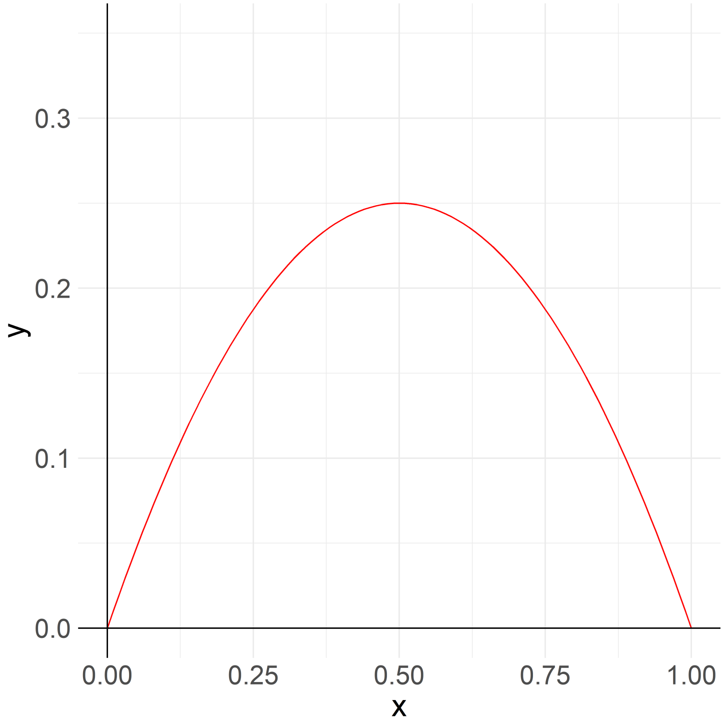


Consider a less simple function, $y = \frac{1}{2}x + 1$, plotted to the left. What is its "slope"?

Its slope or $\frac{\Delta f(x)}{\Delta x} = \frac{1}{2}$.

[Recall: $y = mx + b$ from Day 1]

Slopes



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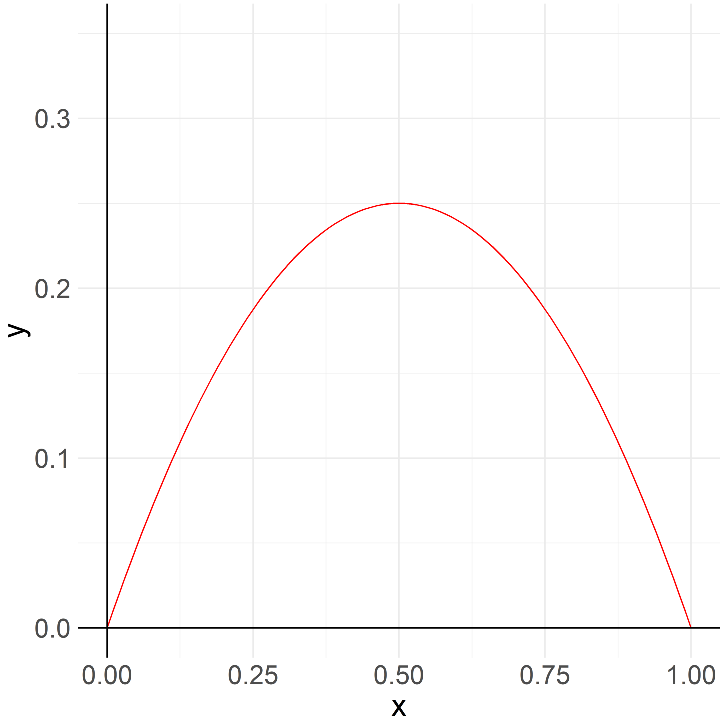
Slopes



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What does "slope" mean in this case, and how do we calculate it?

Derivatives as limits

We can approximate the slope at a certain point by:

- picking a point nearby on the line...
- ...and finding the slope of the straight line connecting these two points

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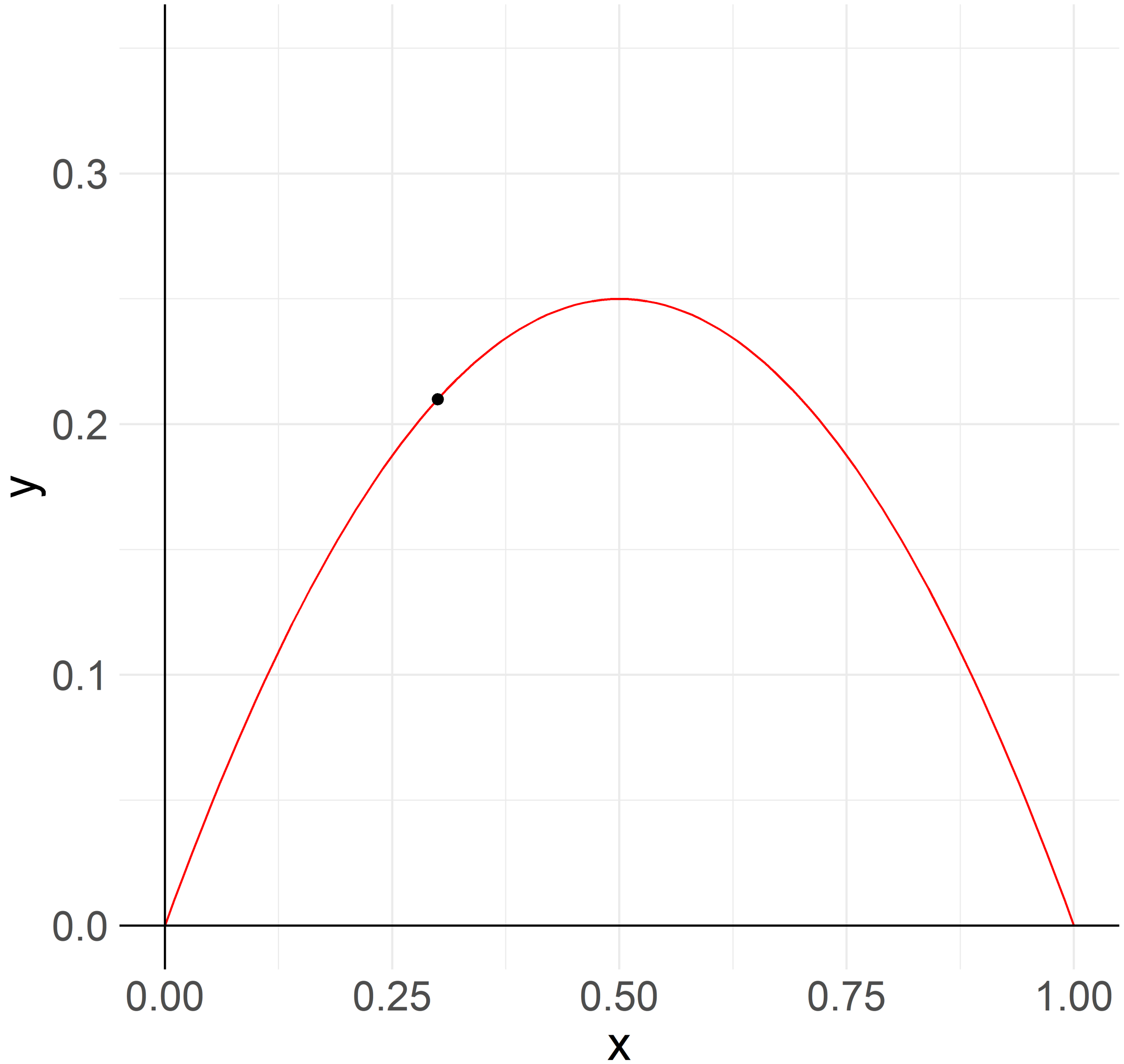
Let's look at some examples.


0.3
0.2
0.1
0.0
y
0.00
0.25
0.50
0.75
1.00
x


When the interval is wide, it is not a good approximation
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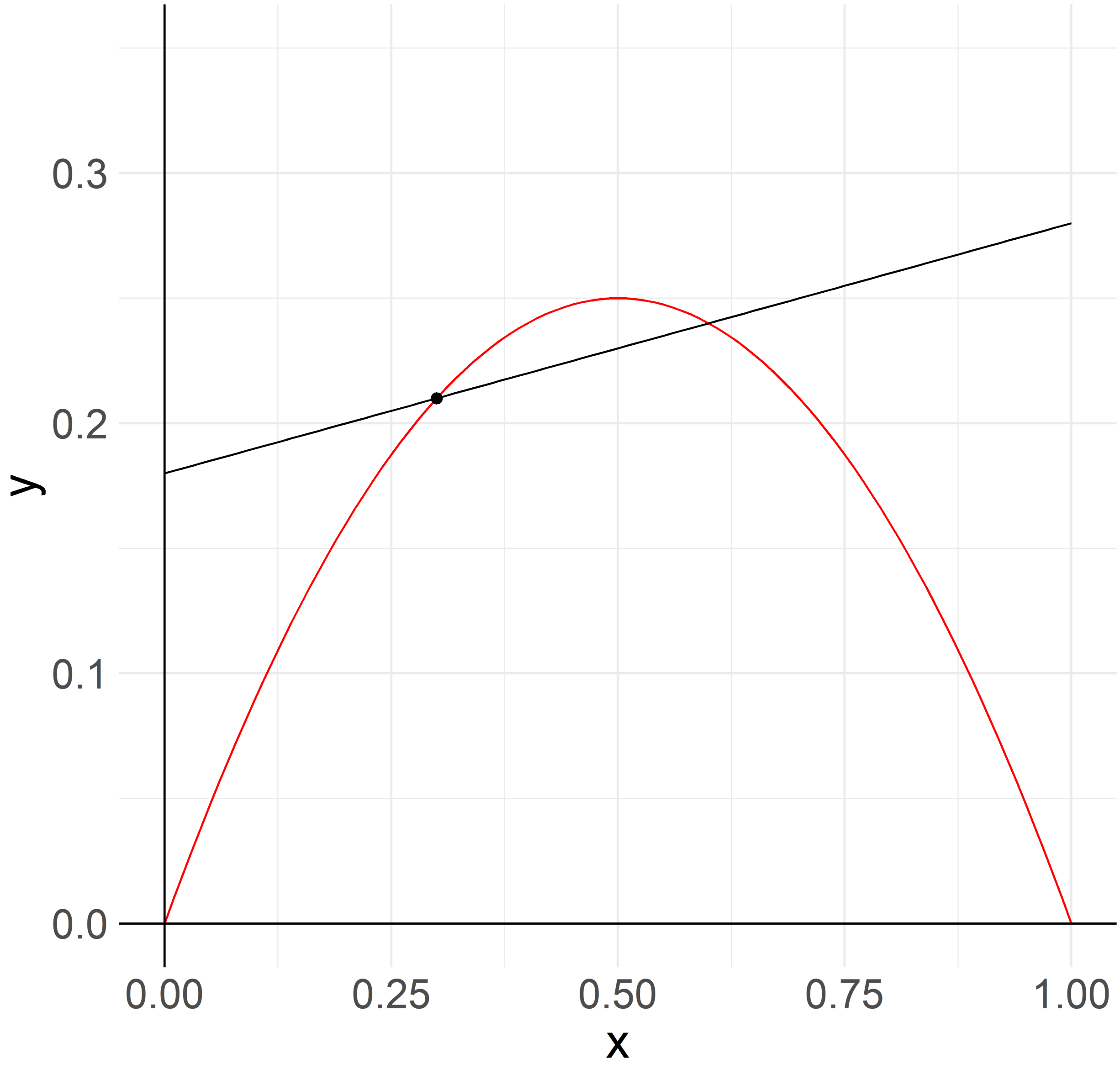

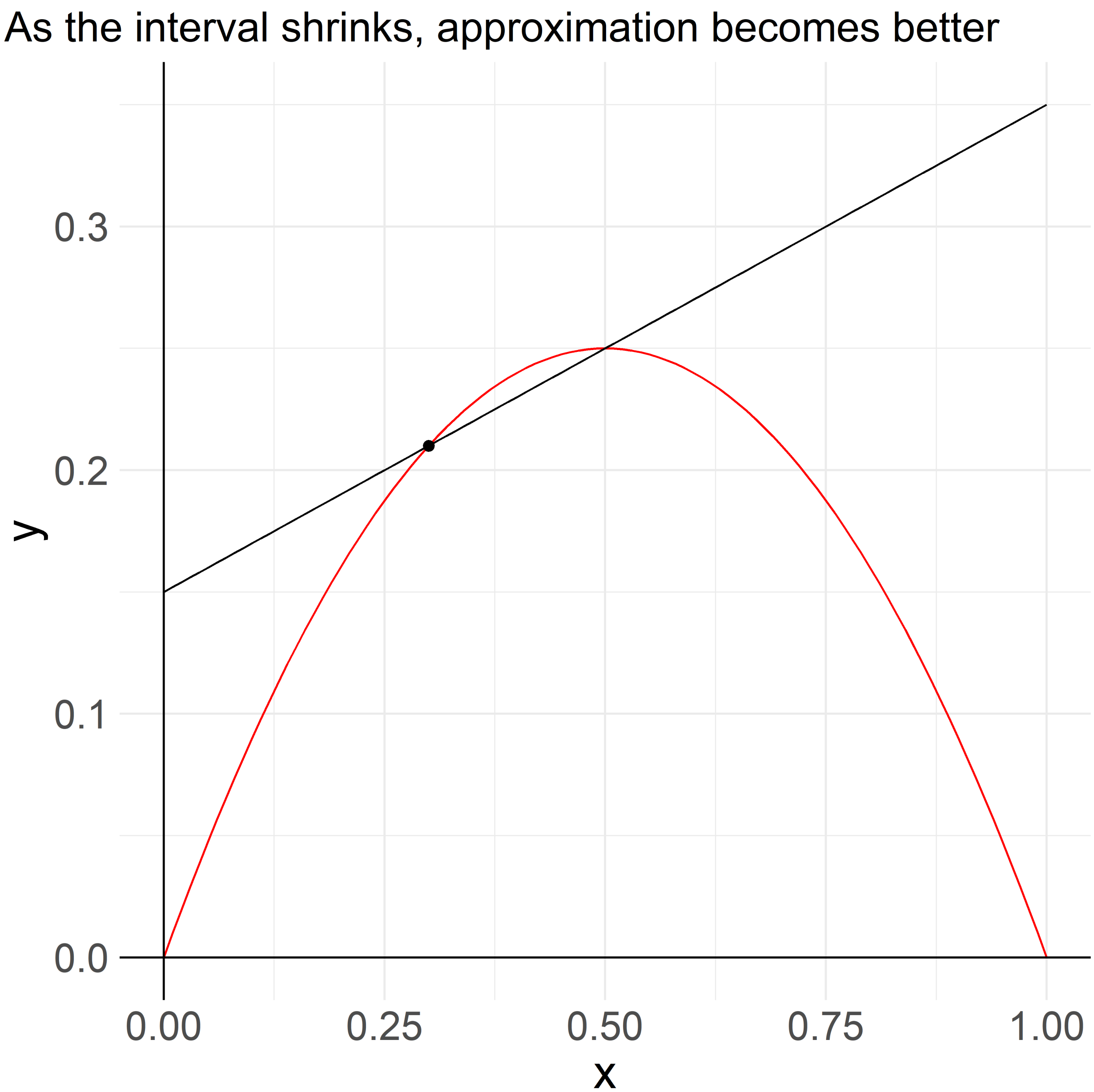
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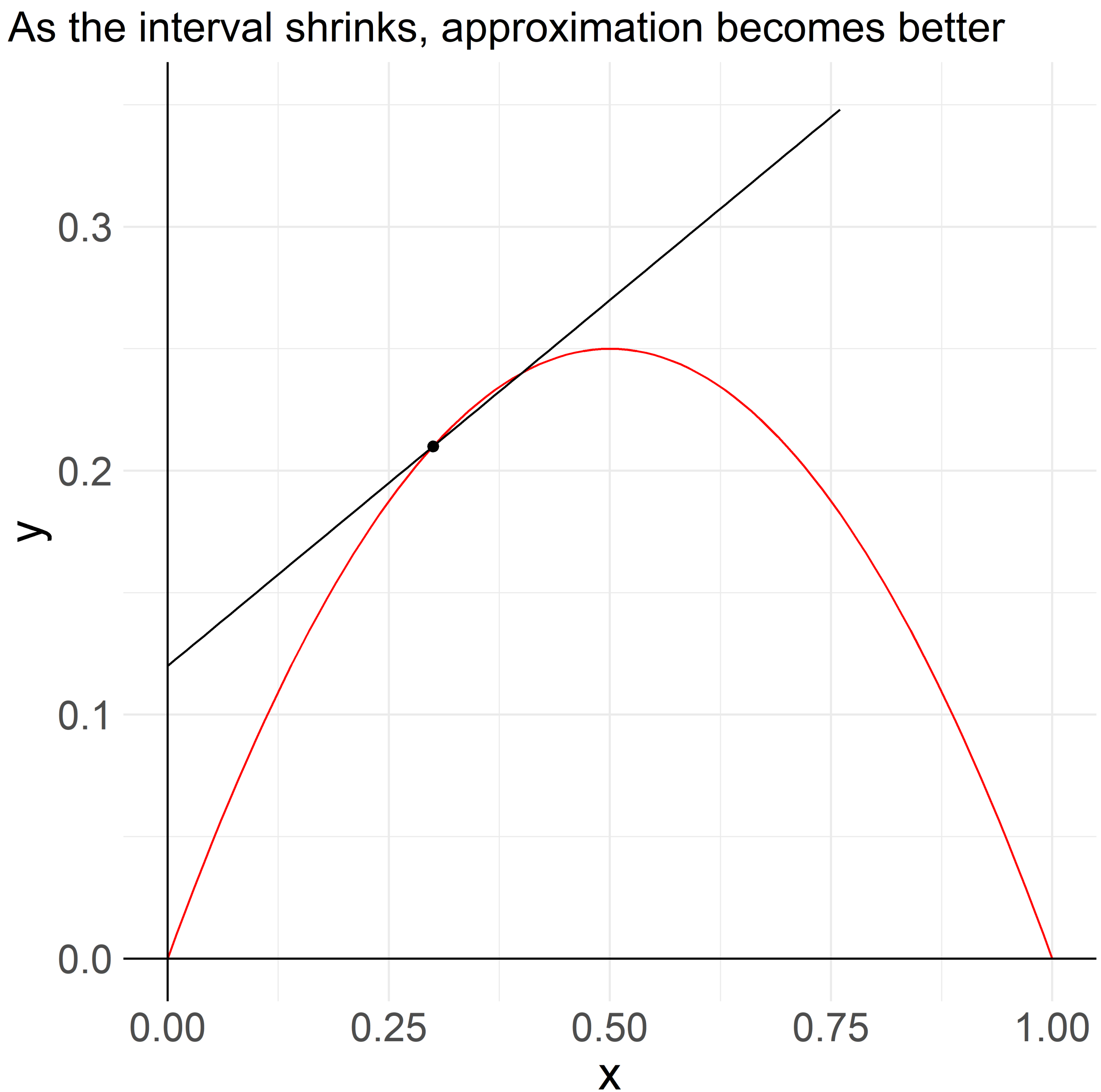
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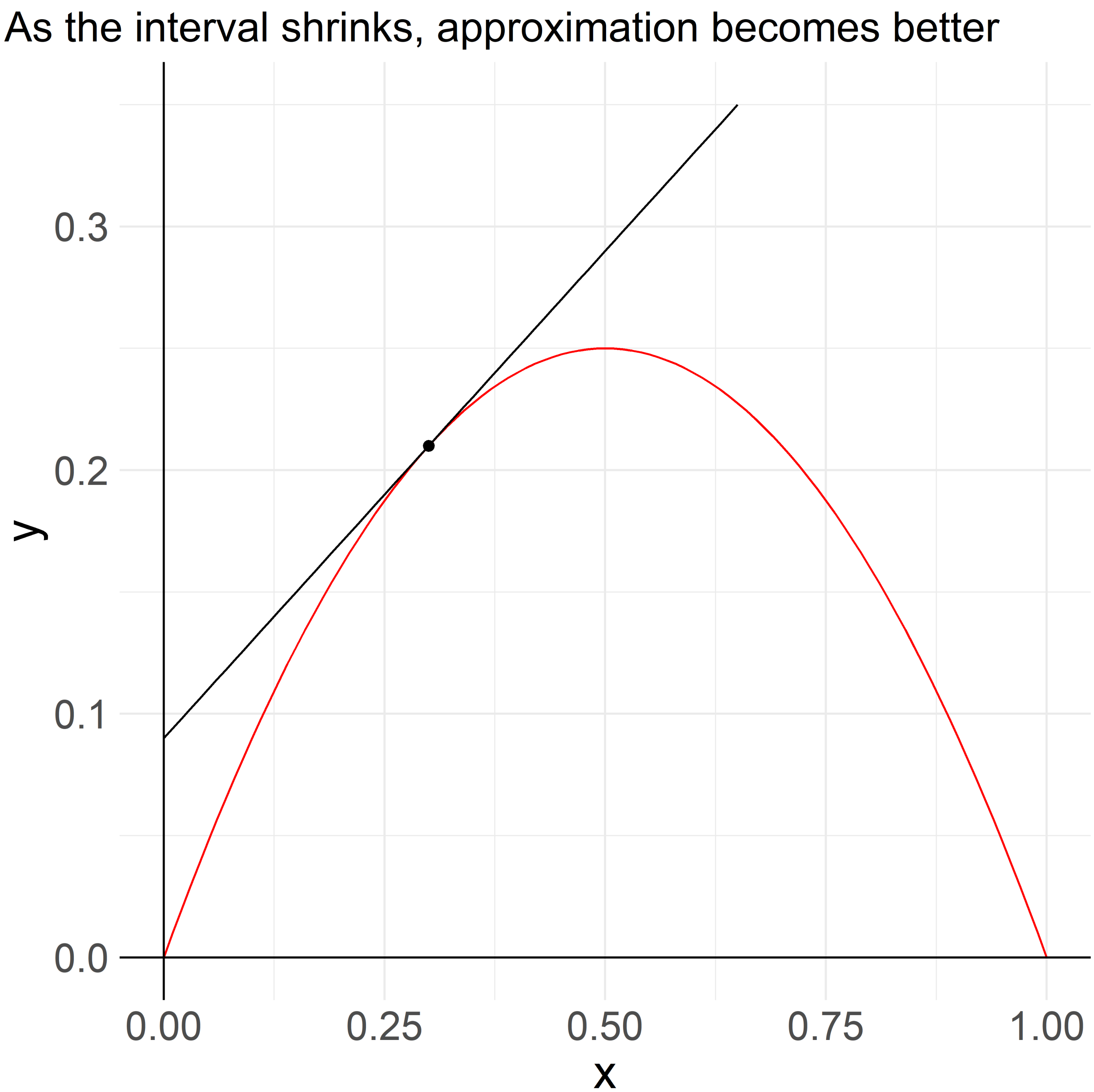

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As we reduce the interval size to 0, this line converges in the limit to the line that lies tangent to the curve at that point.

Recall that $f(x) = x - x^2$ and consider a very small interval $\epsilon \ldots$

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Derivatives as limits

As we reduce the interval size to 0, this line converges in the limit to the line that lies tangent to the curve at that point.

Recall that $f(x) = x - x^2$ and consider a very small interval ϵ:

$$\begin{aligned}
\frac{\Delta f(x)}{\Delta x} &= \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{(x+\epsilon) - x} \\
&= \lim_{\epsilon \to 0} \frac{[(x+\epsilon) - (x+\epsilon)^2] - [(x) - (x)^2]}{(x+\epsilon) - x} \\
&= \lim_{\epsilon \to 0} \frac{[x + \epsilon - (x^2 + 2\epsilon x + \epsilon^2)] - [(x) - (x)^2]}{(x+\epsilon) - x} \\
&= \lim_{\epsilon \to 0} \frac{x + \epsilon - x^2 - 2\epsilon x - \epsilon^2 - x + x^2}{x + \epsilon - x}
\end{aligned}$$

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&= \lim_{\epsilon \to 0} 1 - 2x - \epsilon
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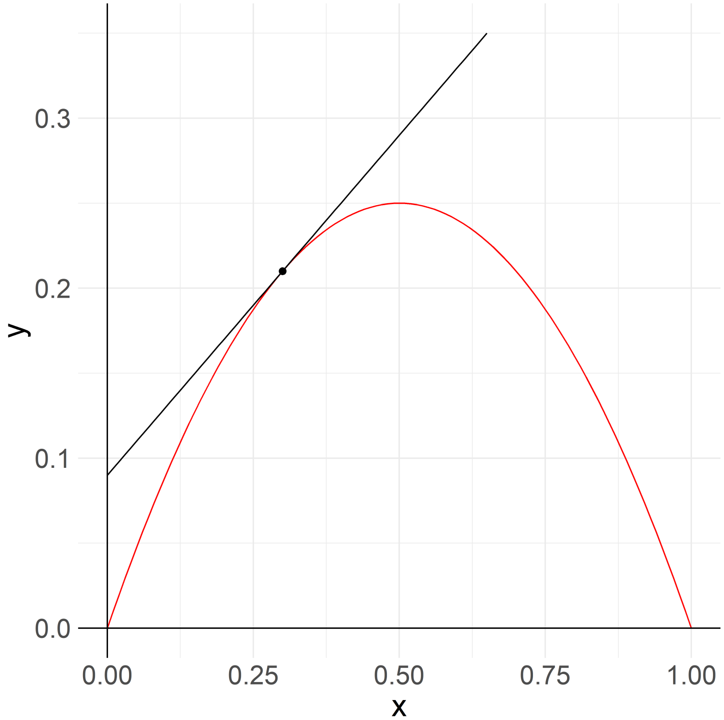
This is the *general formula* for the derivative of the function $f(x) = x - x^2$; we can apply it to any point in the function's domain.

Derivatives as limits



Using this formula, the slope of the curve at $x = .3$ (the point from the previous examples) is exactly:

$$\begin{aligned} \text{slope} &= 1 - 2(.3) \\ &= 0.4 \end{aligned}$$

Or, if we want to find the point at which the slope is 0 (rate of change is 0):

$$\begin{aligned} 0 &= 1 - 2x \\ 2x &= 1 \\ x &= 0.5 \end{aligned}$$

Recap: derivatives

To sum up, the derivative is equivalent to:

$$\frac{d[f(x)]}{dx} = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{(x+\epsilon) - x}$$

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Using this approach, we can find:

- A general equation for the slope at any point
- The exact value of the slope at a given point
- The point that has a particular slope

Derivative notation

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Second derivative: $\frac{d^2[f(x)]}{dx^2}$, $\frac{d^2}{dx^2}[f(x)]$, $f''(x)$, or $f^2(x)$

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In this course, we'll be using $f'(x)$ and $f''(x)$, but feel free to use whatever notation makes sense to you!

Cautionary notes on derivatives

A few assumptions in using this approach to find the slope:

- The function is continuous (no gaps or jumps).
- The derivative exists (the limit of the slope is the same from the left as it is from the right), or *no sharp corners*.

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For usual political science applications, these are fine assumptions. But it is important to state them explicitly and be aware of them.

It gets easier

Fortunately (for humans), it is not necessary to use this whole formula with ϵ and to take limits to find derivatives.

There are a few rules that allow us to easily find the derivatives of most functions.

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A special case: the *derivative of a constant is zero*.

E.g., if $f(x) = 3$, $f'(x) = 0$

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Let $h(x) = 7x^{\frac{1}{2}}$, then:

$$\begin{aligned} h'(x) &= \left(\frac{1}{2}\right) 7x^{\frac{1}{2}-1} \\ &= \frac{7}{2}x^{-\frac{1}{2}} \end{aligned}$$

Exercises

Find the following derivatives and calculate the instantaneous slope of the curves at the point $x = 2$:

$f(x) = \frac{1}{4}x^4$

$g(x) = \frac{2}{x^3}$ [Hint: How else can we express fractions?]

$h(x) = 4x^{\frac{5}{2}}$

$j(x) = \sqrt[3]{x}$ [Hint: How else can we express roots?]

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$$\begin{aligned} f'(x) + g'(x) &= (4x^2)' + (5x^3)' \\ &= (2)4x^{2-1} + (3)5x^{3-1} \\ &= 8x + 15x^2 \end{aligned}$$

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Let $f(x) = x$ and $g(x) = x^2$. What is $(f(x) - g(x))'$?

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$$(f(x) + g(x))' = f'(x) + g'(x)$$

For example, let $f(x) = 4x^2$ and $g(x) = 5x^3$. What is $(f(x) + g(x))'$?

$$\begin{aligned} f'(x) + g'(x) &= (4x^2)' + (5x^3)' \\ &= (2)4x^{2-1} + (3)5x^{3-1} \\ &= 8x + 15x^2 \end{aligned}$$

Let $f(x) = x$ and $g(x) = x^2$. What is $(f(x) - g(x))'$?

$$\begin{aligned} f'(x) - g'(x) &= (x)' - (x^2)' \\ &= (1)x^{1-1} - (2)x^{2-1} \\ &= 1 - 2x \end{aligned}$$

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What is the rate of change when x is equal to 1?

$$\begin{aligned} h'(1) &= 25x^4 - 30x^2 + 12x \\ h'(1) &= 25(1)^4 - 30(1)^2 + 12(1) \\ h'(1) &= 7 \end{aligned}$$

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This is **the product rule**.

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&= (2x)(x^3-4x) + (x^2+1)(3x^2-4) \\
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$$\begin{aligned}
\left(\frac{f(x)}{g(x)}\right)' &= \left(\frac{x^2+1}{x^3-4x}\right)' \\
&= \frac{(x^2+1)'(x^3-4x) - (x^2+1)(x^3-4x)'}{(x^3-4x)^2} \\
&= \frac{(2x)(x^3-4x) - (x^2+1)(3x^2-4)}{(x^3-4x)^2} \\
&= \frac{2x^4 - 8x^2 - (3x^4 - 4x^2 + 3x^2 - 4)}{(x^3-4x)^2} \\
&= \frac{-x^4 - 7x^2 + 4}{(x^3-4x)^2}
\end{aligned}$$

Exercises

Find the derivatives of the following expressions:

$$(3x^2 - 4x + 2)(x^3 - x^2 + x - 1)$$

$$\frac{4x+1}{3x^2-2}$$

Derivatives of nested functions

Let's say $h(x) = f(g(x))$.

The derivative of one function nested inside another is:

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or the derivative of the outside with respect to the inside *times* the derivative of the inside function.

This is referred to as **the chain rule**.

Extremely useful as a way to find derivatives of complex functions: you can treat them as nested chains of functions.

The chain rule: examples

Let $h(x) = 6(3x^2 + 2)^4$. This can be thought of as two nested functions, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $h(x) = f(g(x))$.

What is $h'(x)$?

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What is $h'(x)$?

$$
\begin{aligned}
h(x)' = (f(g(x)))' &= (6(3x^2 + 2)^4)' \\
&= (4)6(3x^2 + 2)^{4-1}(3x^2 + 2)' \\
&= 24(3x^2 + 2)^3(6x) \\
&= 144x(3x^2 + 2)^3
\end{aligned}
$$

The chain rule: examples

Let $k(x) = 3(6x^4)^2 + 2$. This can be thought of as the same two functions nested in the *reverse order*, such that $g(x) = 3x^2 + 2$ and $f(x) = 6x^4$, and $k(x) = g(f(x))$.

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What is $k'(x)$?

$$
\begin{aligned}
k(x)' = (g(f(x)))' &= (3(6x^4)^2 + 2)' \\
&= (3(6x^4)^2)' + (2)' \\
&= (2)3(6x^4)^{2-1}(6x^4)' + 0 \\
&= (2)3(6x^4)^{2-1}(24x^{4-1}) \\
&= (2)3(6x^4)(24x^3) \\
&= 864x^7
\end{aligned}
$$

Exercises

Express the functions below as the nested result of two simpler functions and use the chain rule to find the derivative:

$(3x - 1)^4$

$2(x^4 + x^3) + 7$

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$$\begin{aligned}(log_e(x))' = (ln(x))' &= \frac{1}{ln(e)x} \\ &= \frac{1}{x}\end{aligned}$$

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Let $g(x) = ln(3x^2 + 4)$. What is $g'(x)$?

(using the chain rule):

$$\begin{aligned} g'(x) = (ln(3x^2 + 4))' &= \frac{1}{3x^2 + 4} \times (3x^2 + 4)' \\ &= \frac{6x}{3x^2 + 4} \end{aligned}$$

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$$\begin{aligned}(e^x)' &= ln(e)e^x \\ &= 1 \times e^x \\ &= e^x\end{aligned}$$

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$$\begin{aligned} g'(x) = (2^{3x})' &= ln(2) \times 2^{3x} \times (3x)' \\ &= 3ln(2) \times 2^{3x} \end{aligned}$$

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Recap: why we're doing this

All these rules help us to:

- Find general formulas for the derivatives of various functions
- Find the slopes of such functions at any given point
- Find when the function is increasing or decreasing, and how fast
- Explore the behavior of functions in various other ways, e.g., look for maxima and minima (more on that tomorrow)

Bonus: L'Hôpital's Rule

Most often you won't need to calculate limits by hand. But if you do, there is one useful trick. It is especially helpful if both the numerator and the denominator are 0, ∞, or $-\infty$ because in that case, we cannot evaluate the limit.

Consider the limit: $\lim_{x \to 2} \frac{x^2+x-6}{x^2-4}$

$$\frac{2^2+2-6}{2^2-4} = \frac{0}{0}$$

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$\frac{2^2+2-6}{2^2-4} = \frac{0}{0}$

What can we do? Take the derivative first!

L'Hôpital's rule says: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

Then, $\lim_{x\to 2} \frac{x^2+x-6}{x^2-4} = \lim_{x\to 2} \frac{(x^2+x-6)'}{(x^2-4)'} = \lim_{x\to 2} \frac{2x+1}{2x} = 1.25$

Can use this even if you can evaluate the limit using the "original" function, as taking the derivative simplifies the function $\to$ less work to do

End Day 3