Mathematics for Political Science

Exercise 1: Foundations & Algebra

August 17th, 2020

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1. For the coffeeshop data from lecture, classify each quantitative variable as:
   (a) dichotomous, discrete, or continuous
   (b) categorical, ordinal, interval, or ratio

2. Using the data in the table below:
   (a) Find output for the functions $f(x)$ and $g(x)$.
   (b) Show the functions equivalent to $f(g(x))$ and $g(f(x))$.
   (c) Find the output for these functions.

x	$f(x) = (3-x)^2$	$g(x) = 2x^3 - 4$	$f(g(x))$	$g(f(x))$
2				
4				
5				
1				
0				
1				

3. Express each of the following complex functions as two simpler functions, one nested inside the other:
   (a) $4(8x-2)^3$
   (b) $\frac{1}{3x-2}$

4. Explain why there is no "largest number" on the interval $(0,1)$.

5. Plot (roughly) the following functions in coordinate space (use separate graphs):
   (a) $f(x) = 2x^2$ for $x \in [0,10]$
   (b) $f(x) = e^x$ for $x \in [0,4]$
   (c) $f(x) = \frac{1}{x}$ for $x \in (0,\infty)$
   (d) $f(x) = x^3 - x^2 + 1$ for $x \in [0,1]$
   (e) $f(x) = \begin{cases} -(x^2) & \text{for } x \in [-2,0) \\ \frac{1}{2} & \text{for } x = 0 \\ x^3 + 1 & \text{for } x \in (0,2] \end{cases}$

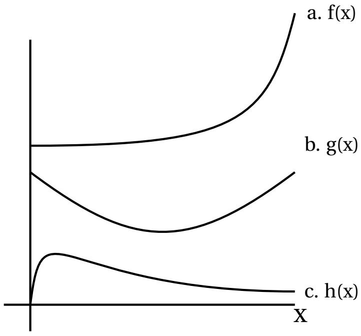
6. (Gill 1.14 [adapted]) The following data are U.S. Census Bureau estimates of population over a 5 year period.

Date	Total U.S. Population
July 1, 2004	293,655,404
July 1, 2003	290,788,976
July 1, 2002	287,941,220
July 1, 2001	285,102,075
July 1, 2000	282,192,162

Characterize the growth in terms of an approximate parametric expression. Graphing may help (optional).

7. Explain in words the difference between a concave and convex function. Draw one of each to illustrate.

8. Characterize the functions below as monotonic or non-monotonic.



9. Given the data below, find:

   (a) $\sum_{1}^{10} \frac{m_i}{2}$

   (b) $\prod_{1}^{6} (m_i - 5)$

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
3	1	8	3	2	7	4	5	4	2

10. (Gill 1.1 [adapted]) Simplify the following expressions as much as possible (if any simplification is possible):

a. $(-x^4y^2)^2$	b. $9(3^0)$	c. $(2a^2)(4a^4)$
d. $\frac{x^4}{x^3}$	e. $y^3 + y^4 + y^5$	f. $\frac{\frac{2a}{7b}}{\frac{11b}{5a}}$
g. $\ln(\frac{e^4}{3})$	h. $\log_8 1$	

11. (Gill 1.2 [adapted]) Simplify the following expressions by expanding the polynomials and grouping like terms:

    (a) $(a+b)^2 + (a-b)^2 + 2(a+b)(a-b) - 3a^2$

    (b) $3p(q+p)^2 - pq + 4x(q+2p)^2$

12. Suppose the vote totals a candidate will receive are given by the equation:

$$V = b + 8s^{\frac{1}{2}}$$

Where V is the number of votes, b is the candidate's number of baseline loyal supports, and s is the amount of money they spend on the campaign.

(a) If candidate A has loyalists $b_A = 20,000$ and spends $s_A = \$1,000,000$, and candidate B has loyalists $b_B = 25,000$ and spends $s_B = \$250,000$, which one will win the election?

(b) Approximately how would the losing candidate have had to spend to pull even? How much additional spending is that?

13. (Gill 1.12) Suppose we are trying to put together a Congressional committee that has representation from four national regions. Potential members are drawn from a pool with 7 from the northeast, 6 from the south, 4 from the Midwest, and 6 from the far west. How many ways can you choose a committee that has 3 members from each region for a total of 12?

14. Solve the following equations for x:

    (a) $12x + 2 = 18x$

    (b) $-6 - 4x = -3 - 8x$

15. Express α in terms of the other unknown variables:

    (a) $3\alpha - 8\theta = \alpha + 2\beta$

    (b) $\alpha x + \alpha y = \alpha x^2 + \alpha y^2 + 4$

16. (Gill 1.6) Solve the following inequalities so that the variable is the only term on the left-hand side:

    (a) $x - 3 < 2x + 15$

    (b) $11 - \frac{4}{3}t > 3$

    (c) $\frac{5}{6}y + 3(y-1) \le \frac{11}{6}(1-y) + 2y$

17. Find the values of x where $f(x) = 0$ using factorization:

    (a) $x^2 + 5x - 14$

    (b) $x^2 - 8x + 16$

    (c) $3x^2 + 9x - 30$

18. Solve the following equations for x using the quadratic formula:

    (a) $18x^2 + 10x = 3 - 15x$

    (b) $20x^2 + 2x - 3 = 5 + 20x - 15x^2$

19. Solve the following systems of equations for a and b using the "direct substitution" approach:

    (a) $b + 5a = 2$
    $7b - 6a = 14$

    (b) $3(a+b) + 7a = 8(b-1) + 33$
    $-3a + 4(1-b) = 4(1-a) - 15$

20. Solve the following systems of equations for c and d using the "elimination" approach:

    (a) $3c + 4d = 13$
    $2c + 5d = 4$

    (b) $c + 4d + 36 = 10d - 3c$
    $2(c+1) + 2(d+1) = 6$

21. Solve this system of equations for x and y in terms of α:

$$2x + y = 10\alpha + 5$$
$$3x + 3y = 18\alpha + 9$$

22. Solve this system of equations for q, r, and s:

$$2q + 4r + s = 1$$
$$4(q+1) + 7(1-r) = 2s + 16$$
$$8q + 4r - 2s = 5q + 19r + 4s$$

23. Calculate the dot product of the vectors below.

    (a) $[3,4,1,7,0] \cdot [5,2,2,0,3]$

    (b) $[4,1,3] \cdot [0,7,5]$

24. (Gill 3.9) For the following matrix, calculate $\mathbf{X}^n$ for $n = 2,3,4,5$. Write a rule for calculating higher values of n.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

25. Using the matrix below, show the identities of multiplication and addition for matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

26. Perform the following matrix multiplications, or explain why they are not possible:

    (a) $\begin{bmatrix} 4 & 5 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 8 & 1 \\ 0 & 9 \\ 6 & 4 \end{bmatrix}$

    (b) $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

    (c) $\begin{bmatrix} \alpha & \beta & \gamma \\ \delta & \epsilon & \eta \end{bmatrix} \begin{bmatrix} \lambda & \sigma \end{bmatrix}$

27. Multiply the matrices below to show that order matters for matrix multiplication:

    a. $\begin{bmatrix} 4 & 7 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 0 \\ 5 \end{bmatrix} \begin{bmatrix} 4 & 7 & 1 \end{bmatrix}$

    b. $\begin{bmatrix} 4 & 8 \\ 1 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 9 & 6 & 3 \\ 1 & 5 & 3 \end{bmatrix} \qquad \begin{bmatrix} 9 & 6 & 3 \\ 1 & 5 & 3 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 1 & 6 \\ 2 & 2 \end{bmatrix}$

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