AUTUMN MEETING 2022
BRAZILIAN PHYSICAL SOCIETY
APRIL 10-14, 2022

SBF
SOCIEDADE BRASILEIRA DE FÍSICA

Majorana fermions in Condensed Matter systems

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Fundação de Amparo à Pesquisa do Estado de Minas Gerais

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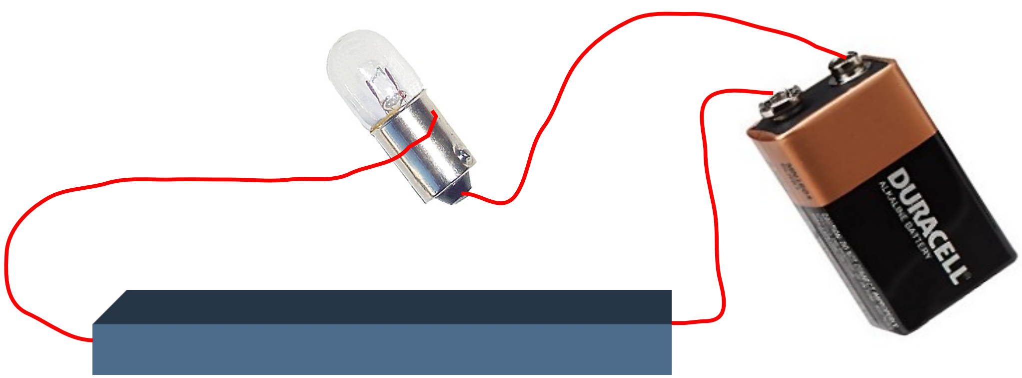
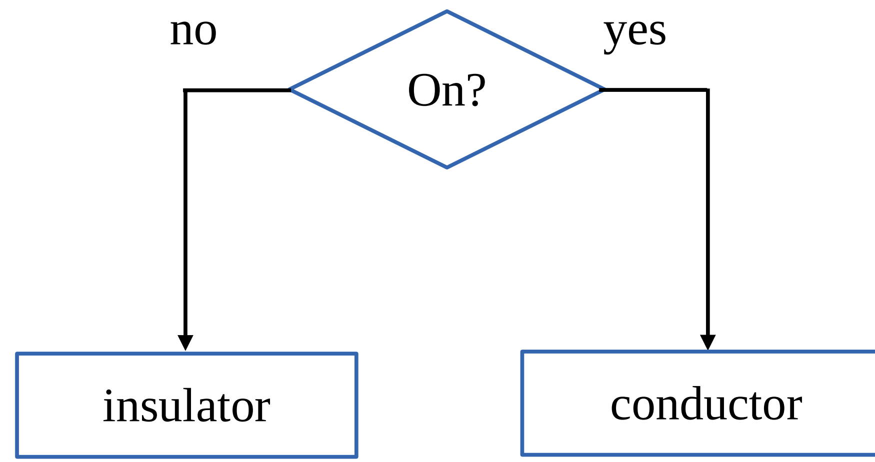
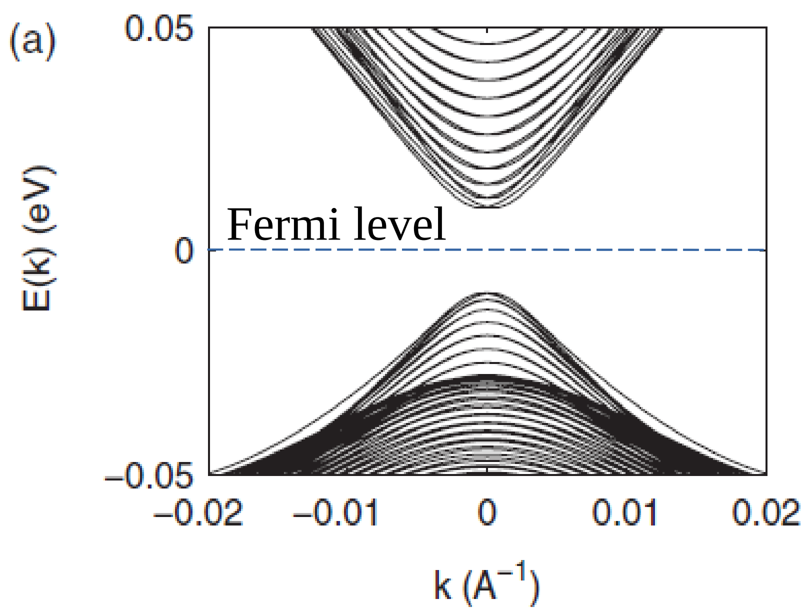
Insulators vs Conductors

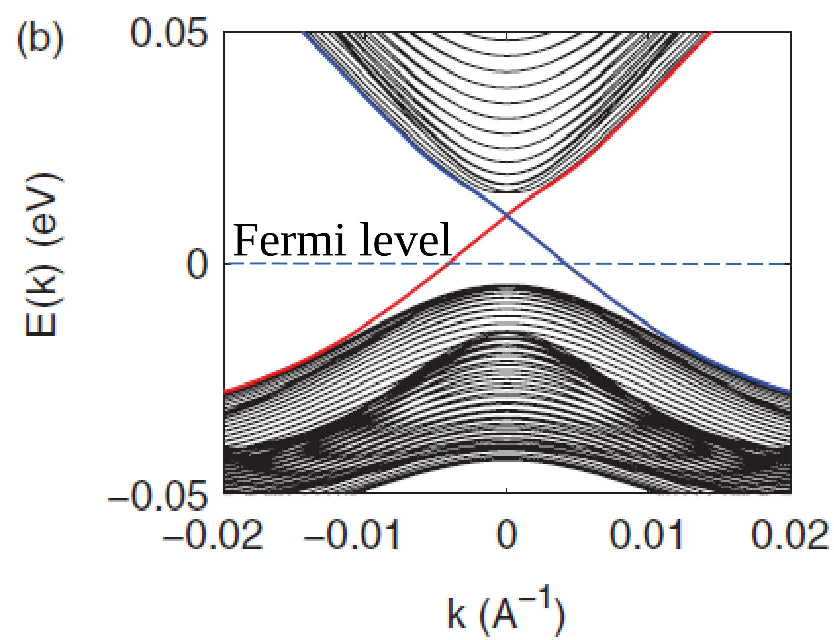
Classification of solids regarding their electrical conductivity





Band structure?





What is a topological material?

Topological insulator

A topological insulator is a material that is gaped in its bulk and gapless on the edges

Edge states are helical states

Topological superconductor

A topological superconductor is a material that is gaped in its bulk and gapless on the edges

Edge states are Majorana bound states

Edge states are protected by symmetry!

Program

Part I: On Topological Phases in Condensed Matter Systems (by Tobias)

Part II: On Majoran Fermions in Condensed Matter Systems (Edson)

Part I

On Topological Phases in Condensed Matter Systems

Tobias Micklitz

Part II

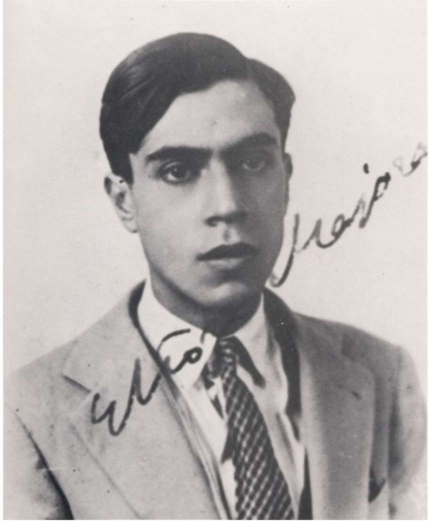
On Majorana Fermions in Condensed Matter Systems

Edson Vernek

What is a Majorana fermion?

$$\gamma = \gamma^{\dagger}$$

It is a particle that is its own antiparticle



Ettore Majorana

Never found though ...

In condensed Matter "Majorana fermions" are just excitations with peculiar properties!
Appear in Topological materials!

Majorana fermions in condensed matter

Basic Theory

1) N. Read and D. Green, PRB (2000).

2) A. Y. Kitaev, Physics-Uspekhi, (2001).

3) L. Fu and C. L. Kane, PRL (2008).

Main proposals

1) Lutchyn -Sau-Das Sarma, PRL (2010).

2) Y. Oreg, *et al.,* PRL (2010).

3) M. Sato et al., PRL (2009).

4) I. Fulga, *et al*., New J. Phys. (2013).

Some experiments

1) Mourik *et al,* Science (2012).

2) M. T. Deng, *et al.*, Nano Lett. (2012).

3) A. Das, *et al.,* Nat. Phys. (2012).

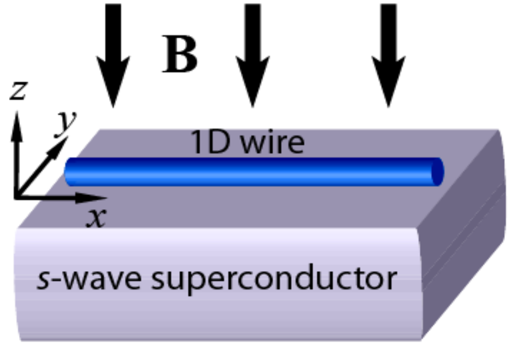
4) E. J. H. Lee *et al*., PRL (2012).

5) O. H. Churchill *et al.*, Phys. Rev. B 87, 241401(R) (2013).

6) H. Zhang, et al., Nature (2018) (RETRACTED).

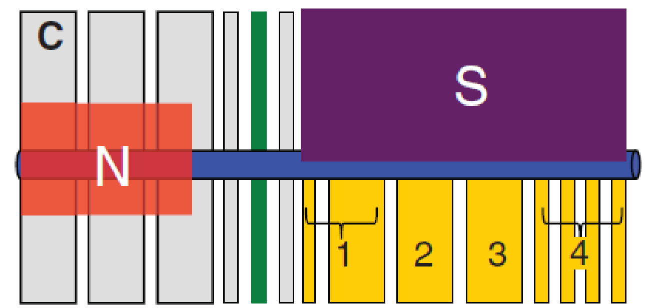
7) S. Vaitiekėnas, Science, (2020).

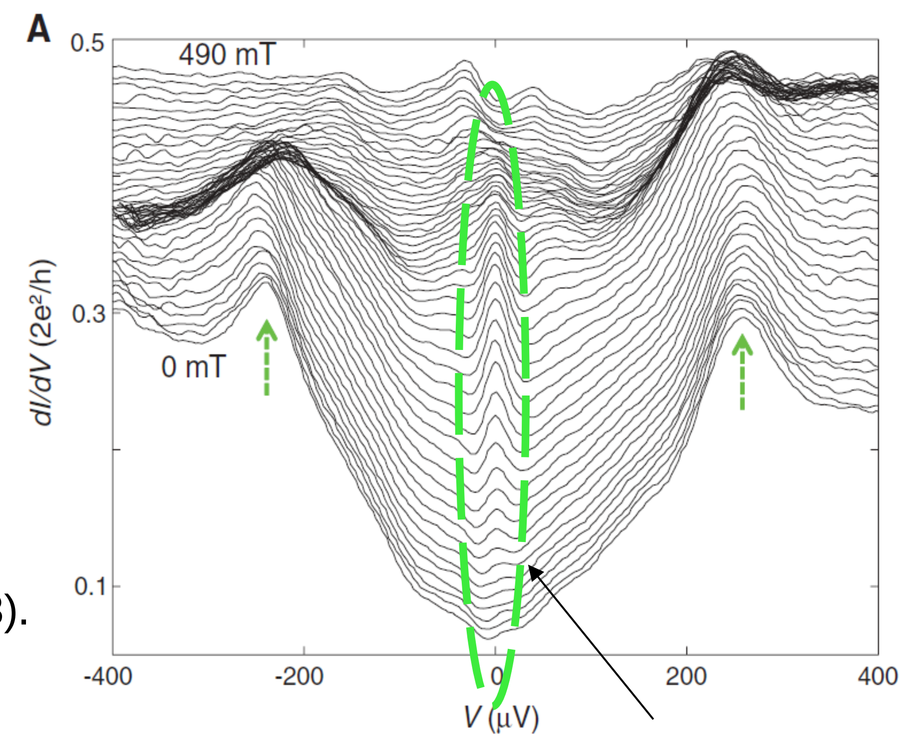
8) S. Frolov AND V. Mourik, arXiv:2203.17060 (2022).



Alicea, 2012.

Mourik *et al,* (2012)





Zero-energy peaks.

Majorana Fermions?

Basic ideas

$\gamma = \gamma^{\dagger}$, How?

Electrons and hole in a metal: $\begin{cases} \text{electron} & -e \\ \text{hole} & +e \end{cases}$

Charge is the problem!

Superconductor:
(Bogoliubov quasiparticles)

$$\begin{cases} d = uc_{\uparrow}^{\dagger} + vc_{\downarrow} \\ d^{\dagger} = u^{*}c_{\downarrow} + v^{*}c_{\downarrow}^{\dagger} \end{cases}$$

if $u = v^{*}$
the charge zero,
Good!

if $u = v^{*}$ $\Rightarrow$ $d = uc_{\uparrow}^{\dagger} + u^{*}c_{\downarrow} \neq d^{\dagger} = uc_{\downarrow}^{\dagger} + u^{*}c_{\uparrow}$

Here the problem is the spin!

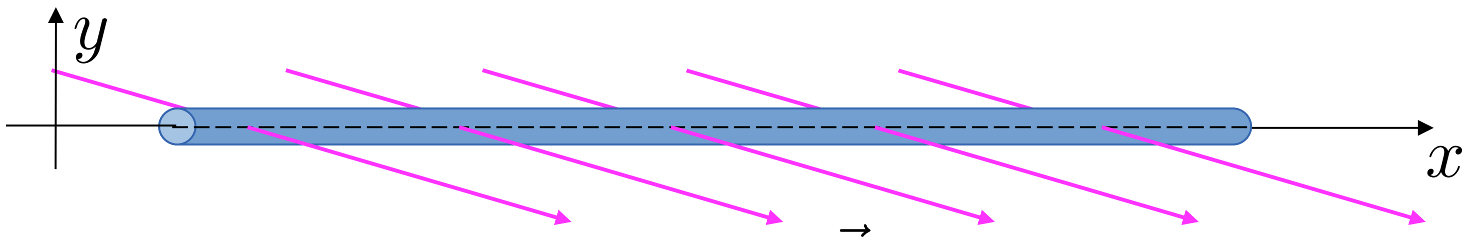
$\Downarrow$

chargless and spinless quasi-particles are required!

It was realized that spin-orbit+magnetic field + superconductivity provides a natural route to the Majoran Fermions

Spin-orbit in 1 and 2D systems

Rashba Spin-orbit in 1D



$$H_{\mathrm{R}} = \int dk_x \psi^{\dagger}_{k_x} \mathcal{H}_R(k_x) \psi_{k_x}$$

$$\mathcal{H}_R(k_x) = \left(\frac{k_x^2}{2m} - \mu \right) \mathbf{I} - \alpha \sigma_y k_x \qquad \psi_{k_x} = \begin{pmatrix} \psi_{k_x \uparrow} \\ \psi_{k_x \downarrow} \end{pmatrix}$$

Magnetic field

$$\mathcal{H}_Z(k_x) = V_Z \sigma_z \quad \Rightarrow \quad \mathcal{H}_0(k_x) = \left(\frac{k_x^2}{2m} - \mu \right) \mathbf{I} - \alpha \sigma_y k_x + V_Z \sigma_Z$$

Note that

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{H}_R(k) + \mathcal{H}_Z = \mathbf{h}(k) \cdot \sigma \qquad \sigma = (\sigma_x, \sigma_y, \sigma_z)$$

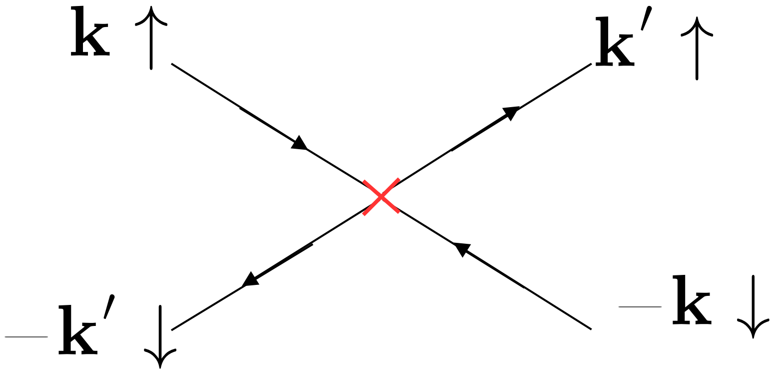
$\mathbf{h}(k)$ Plays the hole of a k-dependent “magnetic field”

Superconductivity

The BCS theory: mean field approximation

Coulomb interaction (mediated by electron-phonon interaction)

$$H_U = \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}$$



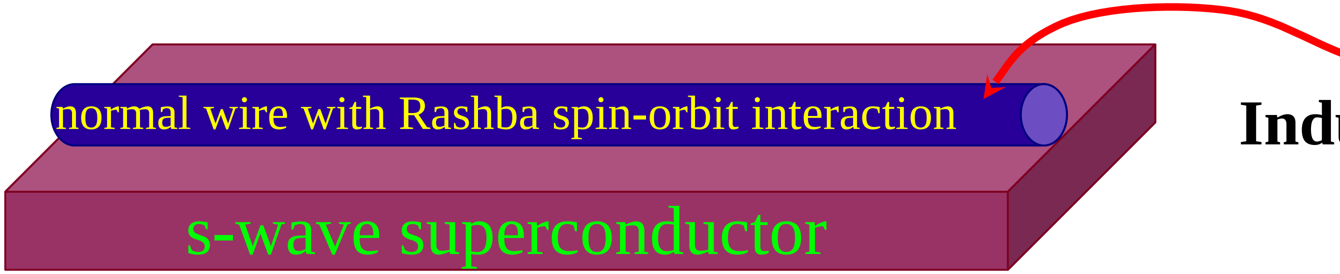
Mean field

$$H_U = \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \left[\langle c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} \rangle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle \right]$$

$$H_{SC} = \sum_{\mathbf{k}} \left[\Delta_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} + \Delta^*_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right]$$

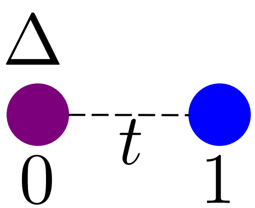
$$\Delta_{\mathbf{k}} = \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle c^\dagger_{\mathbf{k}'\uparrow} c^\dagger_{-\mathbf{k}'\downarrow} \rangle$$ is the mean-field pairing potential

Proximity effect



Induced superconductivity

Toy model



Superconductor

Normal

Hamiltonian

$$H = H_0 + H_1 + H_T$$

$$H_0 = \sum_\sigma \epsilon_1 c^\dagger_{0\sigma} c_{0\sigma} + \Delta \left(c^\dagger_{0\uparrow} c^\dagger_{0} + c_0 c_{0\uparrow} \right)$$

$$H_1 = \sum_\sigma \epsilon_1 c^\dagger_{1\sigma} c_{1\sigma} \qquad H_T = t \sum_\sigma \left(c^\dagger_{1\sigma} c_{0\sigma} + c^\dagger_{0\sigma} c_{1\sigma} \right)$$

Nambu representation

$$\hat{C} = \begin{pmatrix} c_{0\uparrow} \\ c^\dagger_{0\downarrow} \\ c_{1\uparrow} \\ c^\dagger_{1\downarrow} \end{pmatrix} \Rightarrow \hat{C}^\dagger = \begin{pmatrix} c^\dagger_{0\uparrow} & c_{0\downarrow} & c^\dagger_{1\uparrow} & c_{1\downarrow} \end{pmatrix}$$

Proximity effect

Nambu representation

$$H = \hat{C}^\dagger \mathcal{H} \hat{C} + E_0 \quad \text{where} \quad E_0 = \varepsilon_0 + \epsilon_1$$

$$\mathcal{H} = \left(\begin{array}{cc|cc} \varepsilon_0 & \Delta & t & 0 \\ \Delta & -\varepsilon_0 & 0 & t \\ \hline t & 0 & \varepsilon_1 & 0 \\ 0 & -t & 0 & -\varepsilon_1 \end{array}\right) \equiv \left(\begin{array}{c|c} \mathcal{H}_{00} & \mathcal{H}_{01} \\ \hline \mathcal{H}_{10} & \mathcal{H}_{11} \end{array}\right)$$

Löwdin perturbation theory

$$\hat{\Psi} \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix} \Rightarrow \mathcal{H}\hat{\Psi} = E\hat{\Psi}$$

$$\mathcal{H}_{00}\Psi_0 + \mathcal{H}_{01}\Psi_1 = E\Psi_0$$

$$\mathcal{H}_{10}\Psi_0 + \mathcal{H}_{11}\Psi_1 = E\Psi_1$$

Upon eliminating Ψ_0

$$\mathcal{H}_{10}\left[(E - \mathcal{H}_{00})^{-1} + \mathcal{H}_{11}\right]\Psi_1 = E\Psi_1$$

Schrödinger-like (not quite!) equation

$$\tilde{\mathcal{H}}_{11}\Psi_1 = E\Psi_1$$

$$\tilde{\mathcal{H}} = \mathcal{H}_{11} + (E - \mathcal{H}_{00})^{-1}$$

Só far exact!

$$\tilde{\mathcal{H}} = \mathcal{H}_{11} + (E - \mathcal{H}_{00})^{-1} = \mathcal{H}_{11} + \underbrace{\frac{\mathcal{H}_{10}}{E}\left[1 + \left(\frac{\mathcal{H}_{00}}{E}\right) + \cdots\right]\mathcal{H}_{01}}_{\text{Taylor expansion}}$$

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \cdots$$

$$\mathcal{H}^{(0)} = \mathcal{H}_{11} = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & -\varepsilon_1 \end{pmatrix} \qquad \mathcal{H}^{(1)} = \frac{\mathcal{H}_{10}\mathcal{H}_{01}}{E} = \frac{t}{E}\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

$$\mathcal{H}^{(2)} = \frac{\mathcal{H}_{10}\mathcal{H}_{00}\mathcal{H}_{01}}{E^2} = \frac{t^2}{E^2}\begin{pmatrix} \varepsilon_0 & -\Delta \\ -\Delta & -\varepsilon_0 \end{pmatrix}$$

$$\tilde{\mathcal{H}} \approx \begin{pmatrix} \varepsilon_1 + \varepsilon_0 + (t^2/E)(1 + \varepsilon_0/E) & -t^2\Delta/E^2 \\ -t^2\Delta/E^2 & -\varepsilon_1 - \varepsilon_0 + (t^2/E)(1 + \varepsilon_0/E) \end{pmatrix}$$

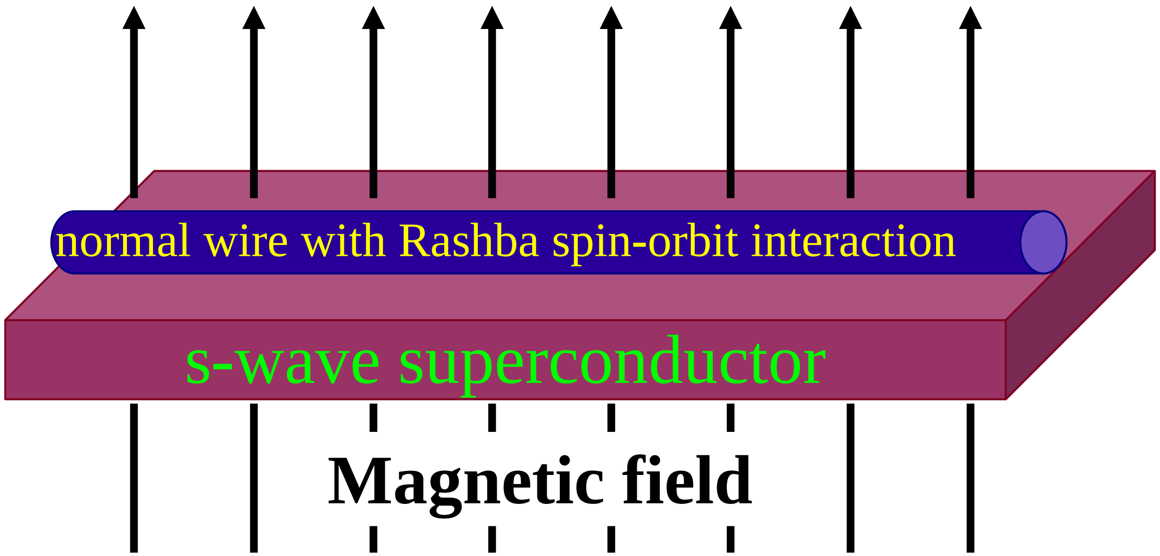
$\tilde{\Delta} = t^2\Delta/E$ Effective induced pairing potential

Validity (quick analysis):

$$E \sim E_0 + E_1 \approx \varepsilon_1 + \sqrt{\varepsilon_0^2 + \Delta^2}$$ Typical value

The truncation is expected to bevalid if $\sqrt{\varepsilon_0 + \Delta^2} \ll \varepsilon_1$

Superconductivity+Rashba+Zeeman



Induced by proximity

$$\mathcal{H}(k) = \left(\frac{k^2}{2m} - \mu\right)\mathbf{I} - \alpha\sigma_y k + V_Z\sigma_Z + H_{SC} = \mathcal{H}_0 + \mathcal{H}_{SC}$$

$$\mathcal{H}_0 = \left(\frac{k^2}{2m} - \mu\right)\mathbf{I} - \alpha\sigma_y k + V_Z\sigma_Z = \begin{pmatrix} \frac{k^2}{2m} - \mu + V_Z & i\alpha k \\ -i\alpha k & \frac{k^2}{2m} - \mu - V_Z \end{pmatrix}$$

Diagonalizing $\mathcal{H}_0$ we obtain $\mathcal{H}_0(k)\psi_{k\pm} = \varepsilon_{k\pm}\psi_{k\pm}$

$$\varepsilon_{k\pm} = \frac{k^2}{2m} - \mu \pm \sqrt{V_Z^2 + \alpha^2 k^2} \qquad \begin{pmatrix} \psi_{k\uparrow} \\ \psi_{k\downarrow} \end{pmatrix} = \begin{pmatrix} a^*_{k+} & a^*_{k-} \\ b^*_{k+} & b^*_{k-} \end{pmatrix} \begin{pmatrix} \psi_{k+} \\ \psi_{k-} \end{pmatrix}$$

$$\mathcal{H}_{SC} = \frac{1}{2}\Big[\Delta_{++}(k)\psi^\dagger_{k+}\psi^\dagger_{-k+} + \Delta_{--}(k)\psi^\dagger_{k-}\psi^\dagger_{-k-} + \Delta_{+-}(k)\psi^\dagger_{k+}\psi^\dagger_{-k-}$$

$$+\Delta^*_{++}(k)\psi_{-k+}\psi_{k+} + \Delta^*_{--}(k)\psi_{-k-}\psi_{k-} + \Delta^*_{+-}(k)\psi_{-k-}\psi_{k+}\Big]$$

Four component spinor $\mathcal{C}^\dagger_k = [\psi^\dagger_{k-}, \psi_{-k-}, \psi^\dagger_{k+}, \psi_{-k+}]$ (Nambu formalism)

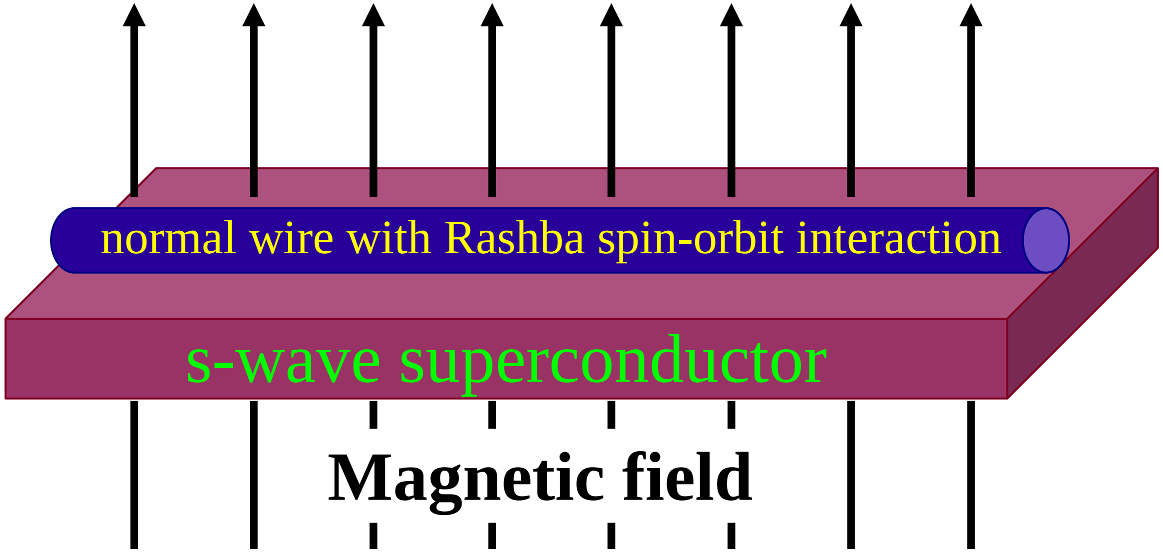
$$\mathcal{H}_{SC}(k) = \begin{bmatrix} \varepsilon_{k-} & \Delta_{--}(k) & 0 & 0 \\ \Delta^*_{--}(k) & -\varepsilon_{k-} & \Delta^*_{+-}(k) & 0 \\ 0 & \Delta_{+-}(k) & \varepsilon_{k+} & \Delta_{++}(k) \\ 0 & 0 & \Delta^*_{++}(k) & -\varepsilon_{k+} \end{bmatrix}.$$

$$\underbrace{\Delta_{++}(k) = \frac{-i\Delta\alpha k}{\sqrt{V_Z^2 + \alpha^2 k^2}}, \qquad \Delta_{--}(k) = \frac{-i\Delta\alpha k}{\sqrt{V_Z^2 + \alpha^2 k^2}}}_{\text{intra-band p-wave pairing}}$$

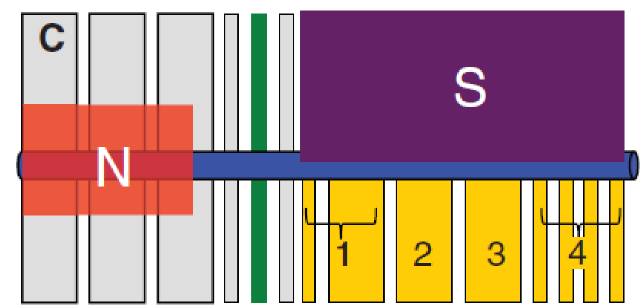
intra-band p-wave pairing $\begin{cases} \Delta_{++}(-k) = -\Delta_{++}(k) \\ \Delta_{--}(-k) = -\Delta_{--}(k) \end{cases}$

$$\Delta_{+-}(k) = \frac{\Delta V_Z}{\sqrt{V_Z^2 + \alpha^2 k^2}}$$ s-wave inter-band pairing $\Delta_{+-}(-k) = \Delta_{+-}(k)$

Band structure: infinite wire

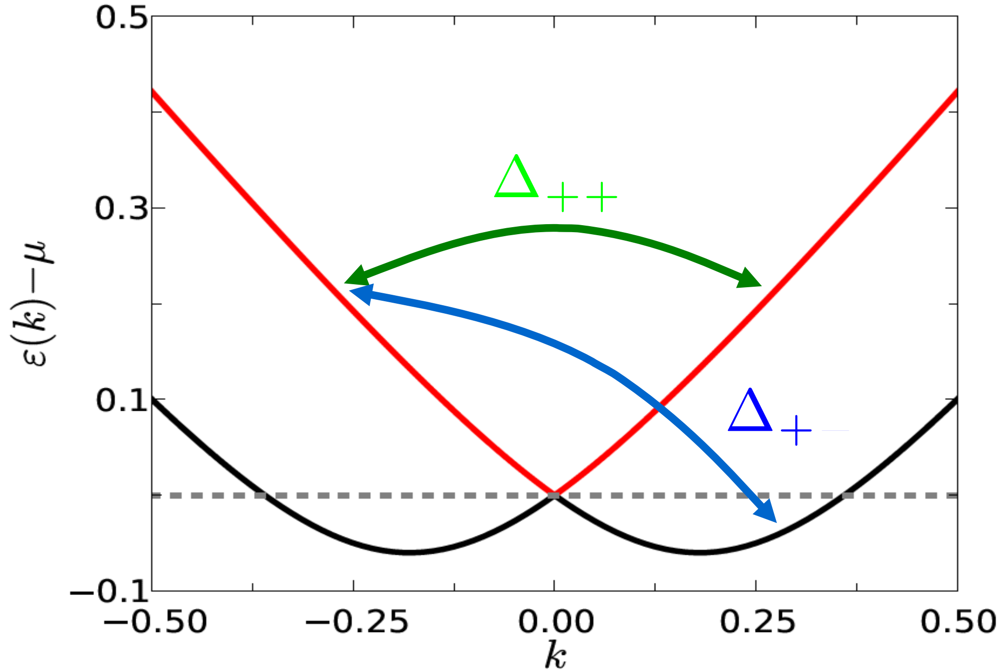


Mourik *et al*, (2012)

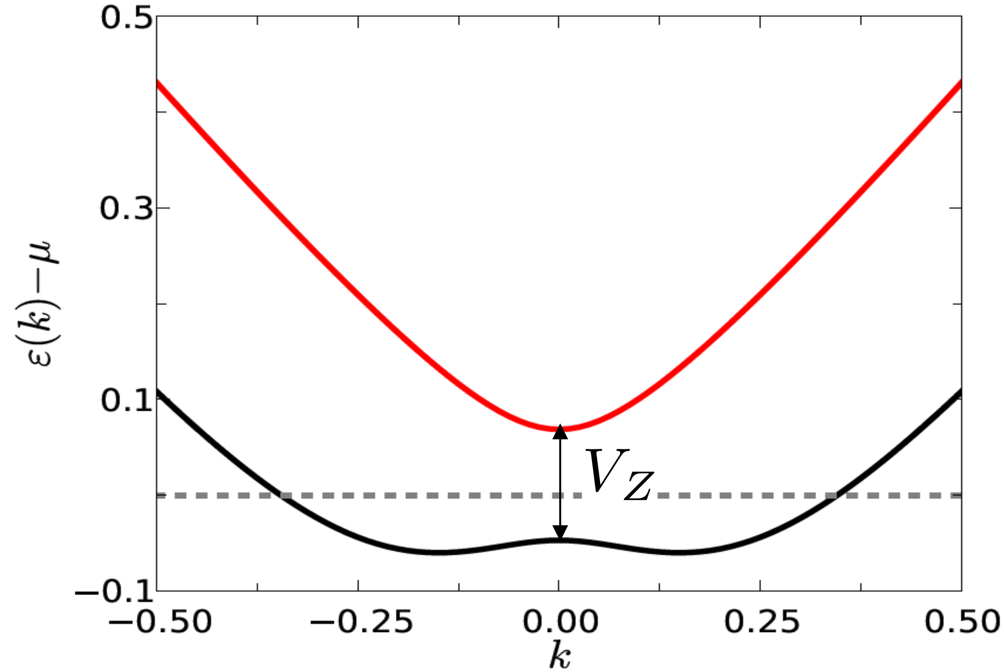


Rashba bands $\varepsilon_{k\pm} = \frac{k^2}{2m} - \mu \pm \sqrt{V_Z^2 + \alpha^2 k^2}$

Effect of the magnetic SC pairing



Effect of the magnetic field



From 4x4 to 2x2 matrix

Löwdin perturbation theory

4x4

$$\mathcal{H}_{SC}(k) = \left[\begin{array}{cc|cc} \varepsilon_{k-} & \Delta_{--}(k) & 0 & 0 \\ \Delta^*_{--}(k) & -\varepsilon_{k-} & \Delta^*_{+-}(k) & 0 \\ \hline 0 & \Delta_{+-}(k) & \varepsilon_{k+} & \Delta_{++}(k) \\ 0 & 0 & \Delta^*_{++}(k) & -\varepsilon_{k+} \end{array}\right] = \left[\begin{array}{c|c} H_P & H_{PQ} \\ \hline H_{QP} & H_Q \end{array}\right],$$

2x2

$$\tilde{\mathcal{H}}(k) = H_P + H_{PQ}\,(E - H_Q)^{-1}\,H_{QP}$$

$$= H_P + \frac{H_{PQ}}{E}\left[1 + \frac{H_Q}{E} + \left(\frac{H_Q}{E}\right)^2 + \cdots\right] H_{QP}$$

$$\tilde{\mathcal{H}}(k) = \begin{bmatrix} \varepsilon_{k-} & \Delta_{--}(k) \\ \Delta^*_{--}(k) & -\varepsilon_{k-} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{|\Delta_{+-}|^2}{E} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{|\Delta_{+-}|^2\,{}_{+}}{E^2} \end{bmatrix} + \cdots$$

For $\Delta \ll |V_Z - \mu|$

$$\tilde{\mathcal{H}}(k) \approx \begin{bmatrix} \varepsilon_{k-} & \Delta_{--}(k) \\ \Delta^*_{--}(k) & -\varepsilon_{k-} \end{bmatrix}$$

This is the 2x2 Bogoliubov-de Gennes Hamiltonian for the 1D topological supercunductor

Trivial vs topological phases

BdG Hamiltonian

$$\mathcal{H}_k = \begin{pmatrix} \varepsilon_k & \tilde{\Delta}_k^* \\ \tilde{\Delta}_k & -\varepsilon_k \end{pmatrix} = \begin{pmatrix} \varepsilon_k & \mathrm{Re}\,[\tilde{\Delta}_k] - i\mathrm{Im}\,[\tilde{\Delta}_k] \\ \mathrm{Re}\,[\tilde{\Delta}_k] + i\mathrm{Im}\,[\tilde{\Delta}_k] & -\varepsilon_k \end{pmatrix}$$

Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{H}_k = \mathrm{Re}\,[\tilde{\Delta}_k]\sigma_x + \mathrm{Im}\,[\tilde{\Delta}_k]\sigma_y + \varepsilon_k\sigma_z$$

Compacting the notation:

$$\begin{cases} \mathbf{h}(k) = (\mathrm{Re}\,[\tilde{\Delta}_k], \mathrm{Im}\,[\tilde{\Delta}_k], \varepsilon_k) \\ \sigma = (\sigma_x, \sigma_y, \sigma_z) \end{cases}$$

$$h_x(k) = \mathrm{Re}\,[\tilde{\Delta}_k],\; h_y(k) = \mathrm{Im}\,[\tilde{\Delta}_k],\; h_z(k) = \varepsilon_k$$

$$\Rightarrow \quad \mathcal{H}_k = \mathbf{h}(k)\cdot\sigma$$

This is formally equivalent to a particle in a k-dependent magnetic field

Trivial vs topological phases

$$\tilde{\Delta}(k) = \frac{i\Delta\alpha k}{\sqrt{V_Z^2 + \alpha^2 k^2}}$$ For Δ Real, $\mathrm{Re}\tilde{\Delta}(k) = 0$

$$\mathbf{h}(k) = (0, \mathrm{Im}\ [\tilde{\Delta}_k], \varepsilon_k)$$

$$h_y(k) = \frac{\mathrm{Re}[\Delta]\alpha k}{\sqrt{V_Z^2 + \alpha^2 k^2}}$$ $h_y(-k) = -h_y(k)$ (p-wave)

$$h_z(k) = \frac{k^2}{2m} - \mu - \sqrt{V_Z^2 + \alpha^2 k^2}$$

$$\hat{\mathbf{h}}(k) = \frac{\mathbf{h}(k)}{|\mathbf{h}(k)|}$$

$$\hat{\mathbf{h}}(k) = \frac{\mathbf{h}(k)}{|\mathbf{h}(k)|}$$ $$\hat{\mathbf{h}}(0) = \frac{-\mu - V_Z}{|-\mu - V_Z|}\hat{z}$$ $$\hat{\mathbf{h}}(\infty) = \hat{z}$$

$$\hat{\mathbf{h}}(0) = \frac{-\mu - V_Z}{|-\mu - V_Z|}\hat{z} \quad \hat{\mathbf{h}}(\infty) = \hat{z}$$

Topological invariant

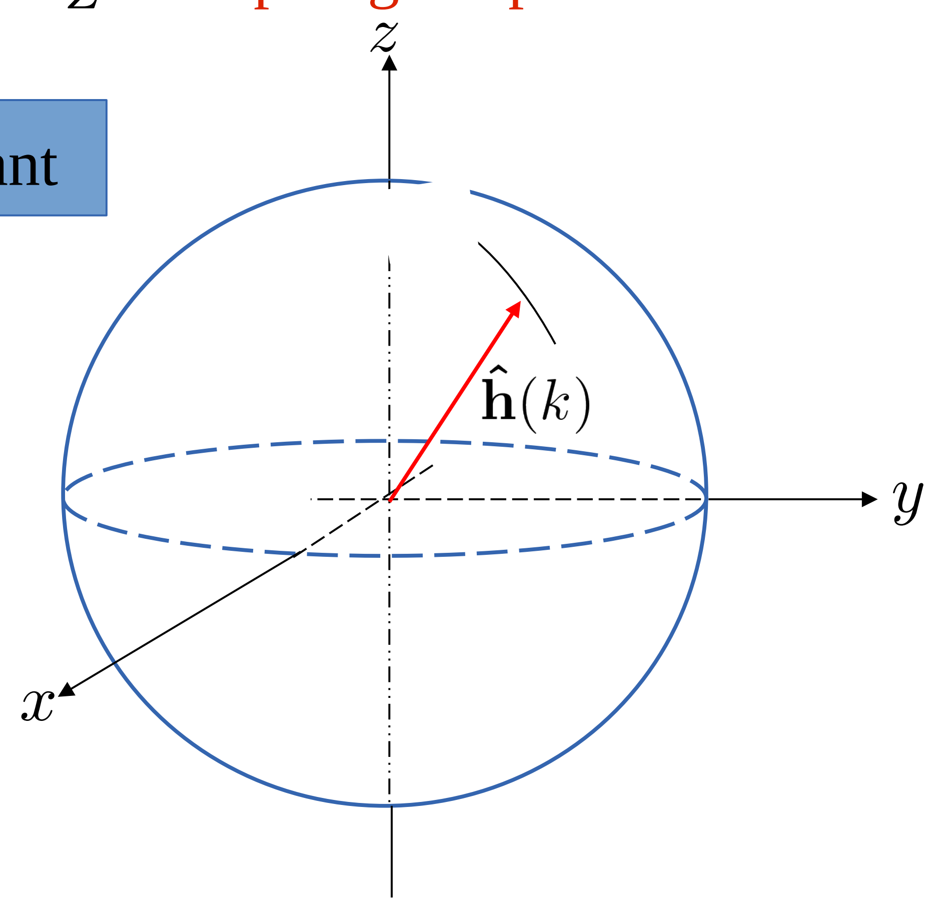
$$(-1)^{\nu} = \hat{\mathbf{h}}(0)\cdot\hat{\mathbf{h}}(\infty) = \frac{-\mu - V_Z}{|-\mu - V_Z|} = \begin{cases} 1 \text{ if } \mu < -V_Z \\ -1 \text{ if } \mu > -V_Z \end{cases}$$

$$\begin{cases} \nu = 0 \mod 2 \text{ if } \mu < -V_Z & \text{Trivial phase} \\ \nu = 1 \mod 2 \text{ if } \mu > -V_Z & \text{Topological phase} \end{cases}$$

ν is the so-called $\mathbb{Z}_2$ topological invariant

Chern number

$$C = \frac{1}{4\pi}\int_{occ} d^2k\,\hat{\mathbf{h}}\cdot\left(\frac{\partial\hat{\mathbf{h}}}{\partial k_x}\times\frac{\partial\hat{\mathbf{h}}}{\partial k_y}\right)$$



Connection with the Kitaev model

Discrete version of the Hamiltonian

For small k we make $\alpha k \ll V_Z$ (Low-energy regime)

$$\tilde{\Delta}(k) = \frac{i\Delta\alpha k}{\sqrt{V_Z^2 + \alpha^2 k^2}} \rightarrow \frac{i\Delta\alpha k}{V_Z} = i\Delta_0 k$$

$V_Z > 0$, for simplicity

$$h_z(k) = \frac{k^2}{2m} - \mu - \sqrt{V_Z^2 + \alpha^2 k^2} \rightarrow \frac{k^2}{2m} - \mu - V_Z$$

The discretized version (tight-binding)

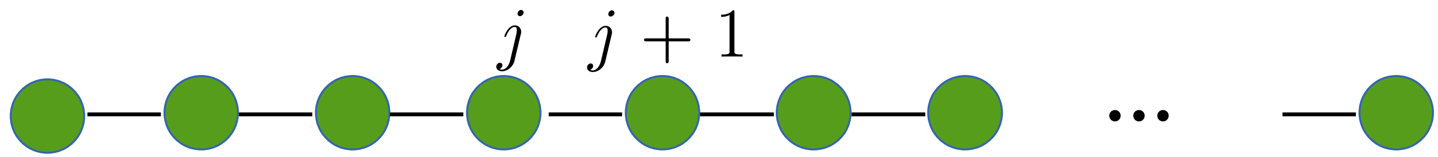
$$k \rightarrow \sin ka \qquad \frac{k^2}{2m} \rightarrow t \cos ka$$

Looks like the Kitaev model

$$\mathcal{H}_k = \begin{pmatrix} -\mu - V_z - t\cos ka & \Delta_0^* \sin(ka) \\ \Delta_0 \sin(ka) & -(-\mu - V_z - t\cos ka) \end{pmatrix}$$

Kitaev toy model

Chain of spinless fermions



$$H = \mu \sum_j c_j^\dagger c_j - \frac{1}{2} \sum_j \left(t c_j^\dagger c_{j+1} + \Delta e^{i\phi} c_j c_{j+1} + H.c. \right)$$

chemical potential

hopping

pairing potential

ϕ is the superconducting phase

Features: Superconducting gap
Topological phase
Majorana edge modes

Main question: How Majorana modes appear in the Kitaev model?

Majorana end modes

$$H = \mu \sum_j c_j^\dagger c_j - \frac{1}{2} \sum_j \left(t c_j^\dagger c_{j+1} + \Delta e^{i\phi} c_j c_{j+1} + H.c. \right)$$

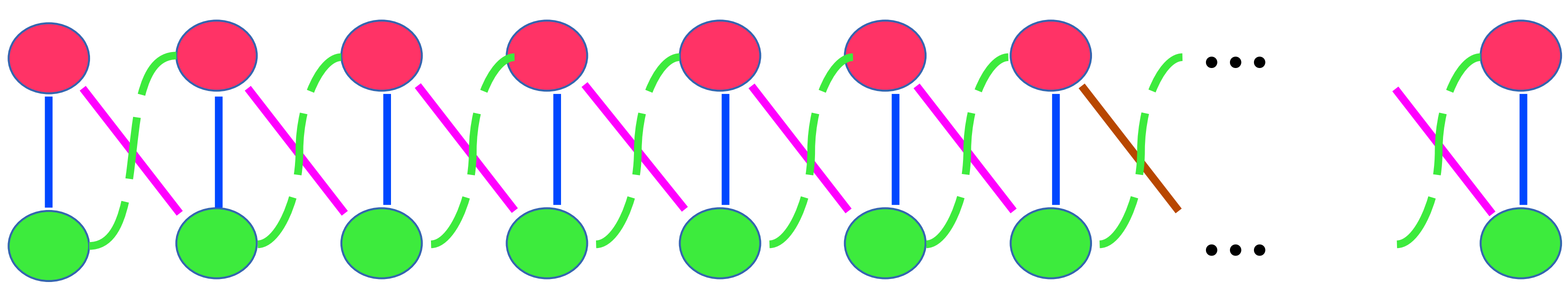
- Introduce the Majorana operators

$$\begin{cases} c_j = \dfrac{e^{-i\phi/2}}{2} \left(\gamma_{Bj} + i\gamma_{Aj} \right) \\ c_j^\dagger = \dfrac{e^{i\phi/2}}{2} \left(\gamma_{Bj} - i\gamma_{Aj} \right) \end{cases}$$

Just a replacement

$$H = -\frac{\mu}{2} \sum_{j=1}^{N} (1 + i\gamma_{Bj}\gamma_{Aj}) - \frac{i}{4} \sum_{j=1}^{N-1} \left[(\Delta + t)\gamma_{Bj}\gamma_{Aj+1} + (\Delta - t)\gamma_{Aj}\gamma_{Bj+1} \right].$$

Majorana chain representation A B

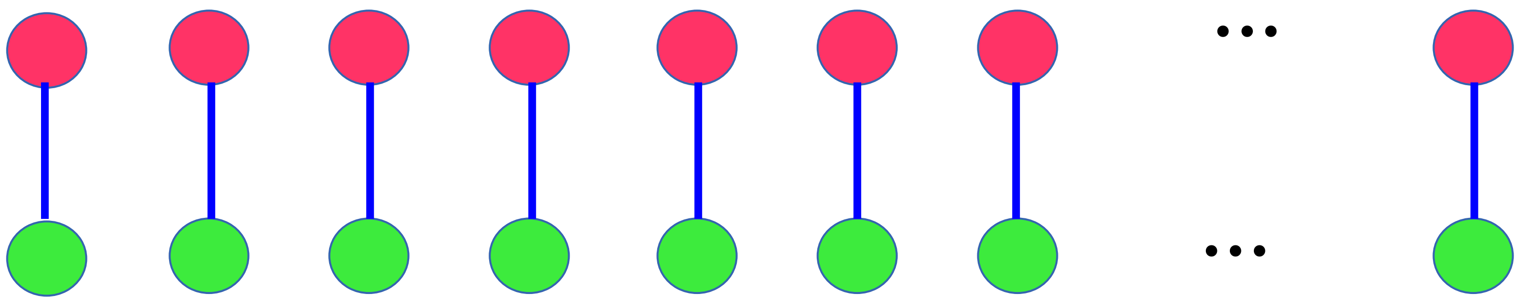


Majorana end modes

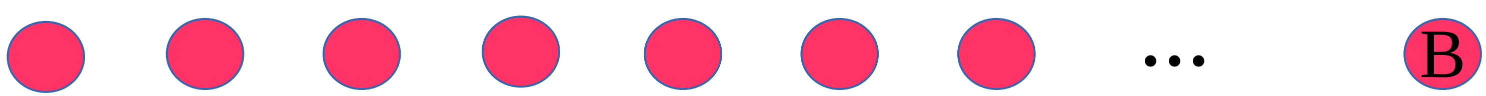
Trivial phase: $\mu < 0,\ \Delta = t = 0$ (for instance)

$$H = -\frac{\mu}{2}\sum_{j=1}^{N}(1 + i\gamma_{Bj}\gamma_{Aj}) - \frac{i}{4}\sum_{j=1}^{N-1}\left[\cancelto{0}{(\Delta + t)}\gamma_{Bj}\gamma_{Aj+1} + \cancelto{0}{(\Delta - t)}\gamma_{Aj}\gamma_{Bj+1}\right].$$

Infinite chain → gapped



Regular fermion representation



This is the atomic limit of a tight binding chain

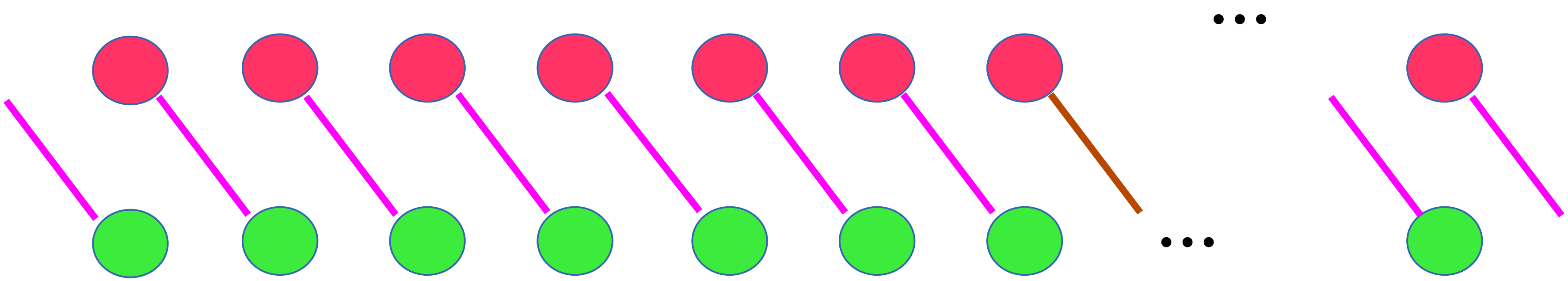
Electron simply cannot hop between the sites → trivial insulator

Majorana end modes

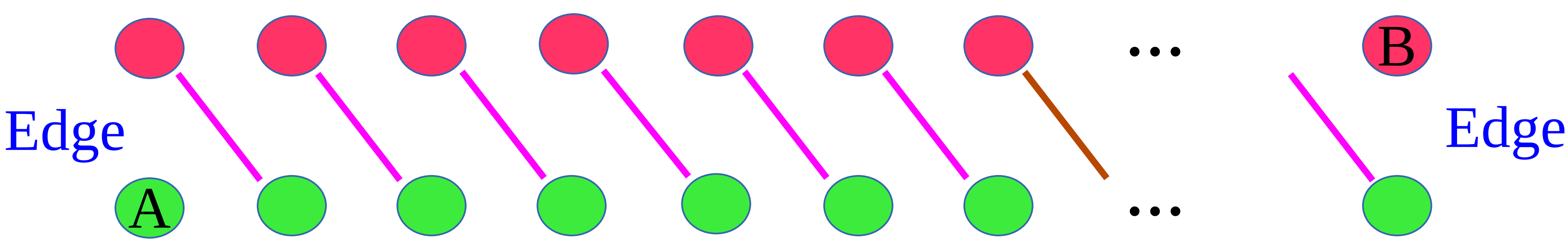
Topological phase: $\underline{\mu = 0,\ \Delta = t}$

$$H = -\cancelto{0}{\frac{\mu}{2}} \sum_{j=1}^{N} (1 + i\gamma_{Bj}\gamma_{Aj}) - \frac{i}{4}\sum_{j=1}^{N-1} \left[(\Delta + t)\gamma_{Bj}\gamma_{Aj+1} + \cancelto{0}{(\Delta - t)}\gamma_{Aj}\gamma_{Bj+1}\right]$$

Infinite chain $\rightarrow$ gapped



Finite chain $\rightarrow$ bound states



Majorana end modes

Topological phase: $\mu = 0, \; \Delta = t$

$$H = -\frac{i}{4}\sum_{j=1}^{N-1}\left[(\Delta + t)\gamma_{Bj}\gamma_{Aj+1}\right].$$

Note that γ_{A1} and γ_{BN} do not appear in the Hamiltonian. This mean that there is a zero-energy state.

Let us make the following transformation within the A1 and BN Majorana subspaces.

$$f^{\dagger} = \frac{1}{2}\left(\gamma_{BN} - i\gamma_{A1}\right) \quad \text{and} \quad f = \frac{1}{2}\left(\gamma_{BN} + i\gamma_{A1}\right)$$

These operators creates and annihilates non-local conventional Fermions. They live in the Fock space $\{|0\rangle, \; |1\rangle\}$. We can then write the "edge" Hamiltonian

$$H_{\text{“edge”}} = \begin{pmatrix} \varepsilon_1 = 0 & 0 \\ 0 & \varepsilon_2 = 0 \end{pmatrix}$$

Since this block Hamiltonian is completely decoupled from the rest, the diagonal elements are eigenenergies of the system

We have seen

- What a Majorana Fermion is;
- What a Majorana bound state is;
- How MBS appear in condensed matter;
- The main ingredient to obtain MBS;
- The Mechanism of the proximity effect;
- Effective Hamiltonian & Kitaev’s toy model;
- How non-local fermions appear in topological wires.

Thank you!

Band analysis

Band structure (bulk)

- Assume infinite chain – or periodic boundary conditions
- Take Fourier transform

$$c_k = \frac{1}{\sqrt{\mathcal{N}}} \sum_j e^{ikx_j} c_j \quad \Longleftrightarrow \quad c_j = \frac{1}{\sqrt{\mathcal{N}}} \sum_k e^{-ikx_j} c_k$$

$\mathcal{N} \rightarrow \#$ of sites

After some straightforward manipulations …

$$H = \frac{1}{2} \sum_k \left\{ -2\left[\mu + t\cos(ka)\right] c_k^\dagger c_k \right\}$$

$\leftarrow$ kinetic term

pairing term $\rightarrow$

$$+\frac{1}{2} \sum_k \left\{ i\Delta e^{i\phi} \sin(ka) c_k c_{-k} - i\Delta e^{-i\phi} \sin(ka) c_{-k}^\dagger c_k^\dagger \right\}$$

Bogoliubov-de Gennes form

Two-component spinor: $C_k^\dagger = \left[c_k^\dagger, c_{-k}\right]$

2x2 Matrix

(p-type)

$$H = \frac{1}{2} \sum_k C_k^\dagger \mathcal{H}_k C_k$$

With $\rightarrow$

$$\mathcal{H}_k = \begin{pmatrix} \varepsilon_k & \tilde{\Delta}_k^* \\ \tilde{\Delta}_k & -\varepsilon_k \end{pmatrix}$$

Band structure (bulk)

$\Delta = 0$: Duplicated bands: redundancy

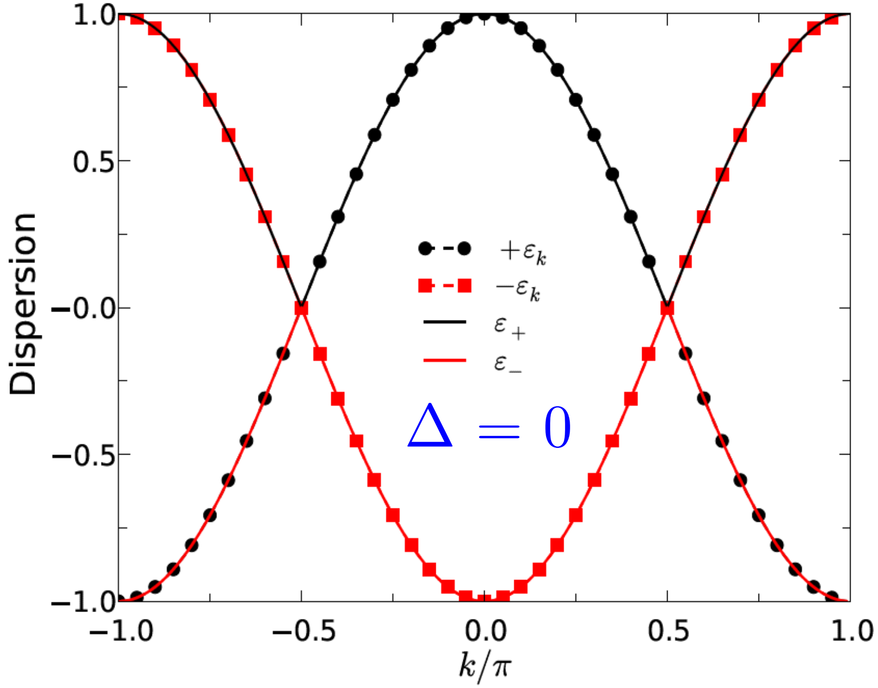
$$\mathcal{H}_k = \begin{pmatrix} \varepsilon_k & 0 \\ 0 & -\varepsilon_k \end{pmatrix}$$

eigenvalues

$$\varepsilon_\pm = \pm|\varepsilon_k|$$

$$\varepsilon_k = -[\mu + t\cos(ka)]$$

$a = 1 \quad \mu = 0 \quad t = 1$

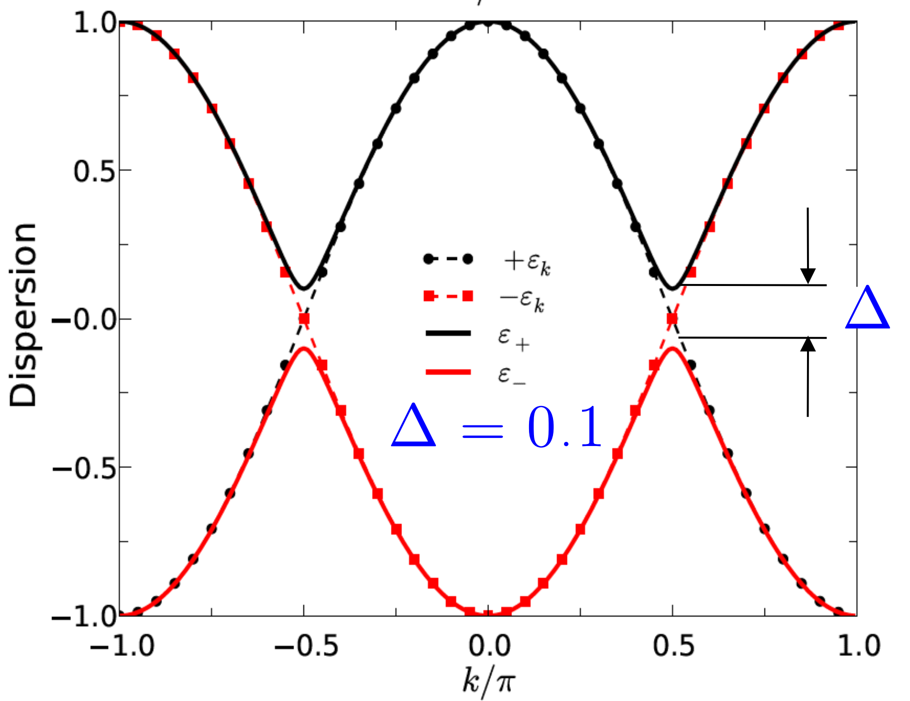


$\Delta \neq 0$: bands are coupled

$$\mathcal{H}_k = \begin{pmatrix} \varepsilon_k & \tilde{\Delta}_k^* \\ \tilde{\Delta}_k & -\varepsilon_k \end{pmatrix}$$

eigenvalues $\qquad \varepsilon_\pm = \pm\sqrt{\varepsilon_k^2 + |\tilde{\Delta}_k|^2}$

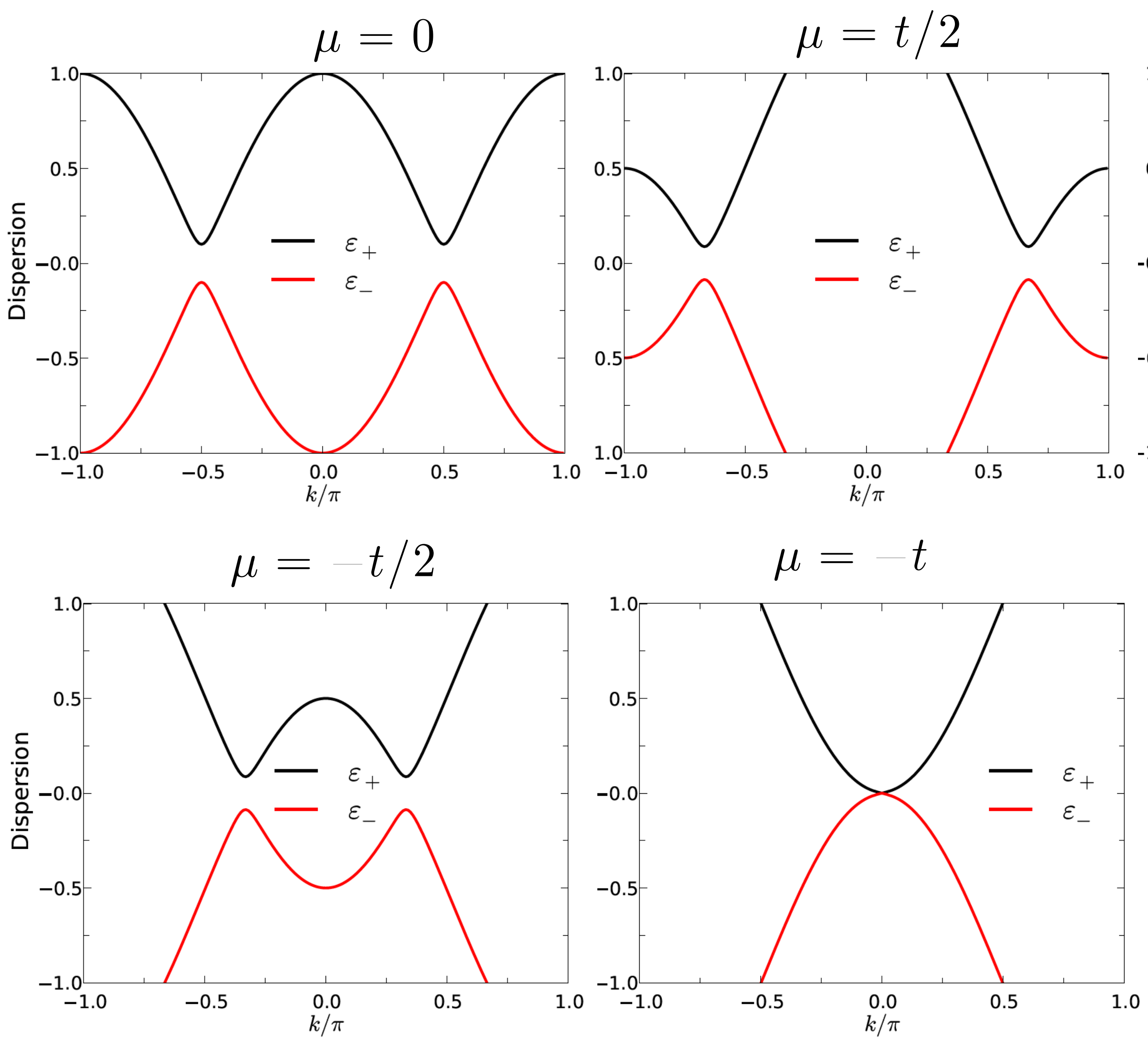
bands are coupled $\rightarrow$ a gap is open



While the BdG Formalism seems to be useless in free electron systems, it proves to be very useful for the superconducting case!

Band structure (bulk)

The system is gapped for all $\Delta > 0$ and $\mu \neq \pm t$



Gapless excitations

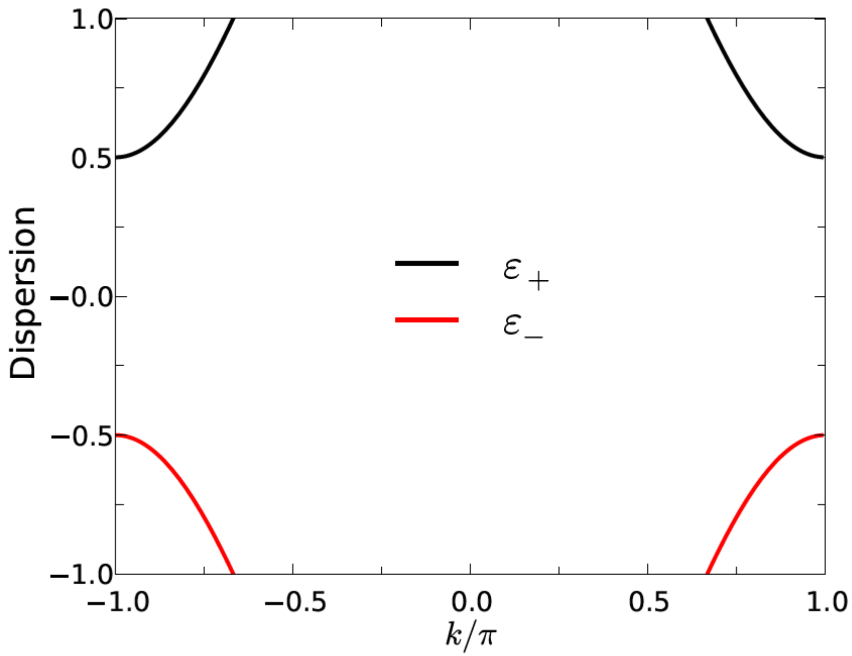
$$\sqrt{\varepsilon_k^2 + |\tilde{\Delta}_k|^2} = 0$$

Only for $\mu = \pm t$

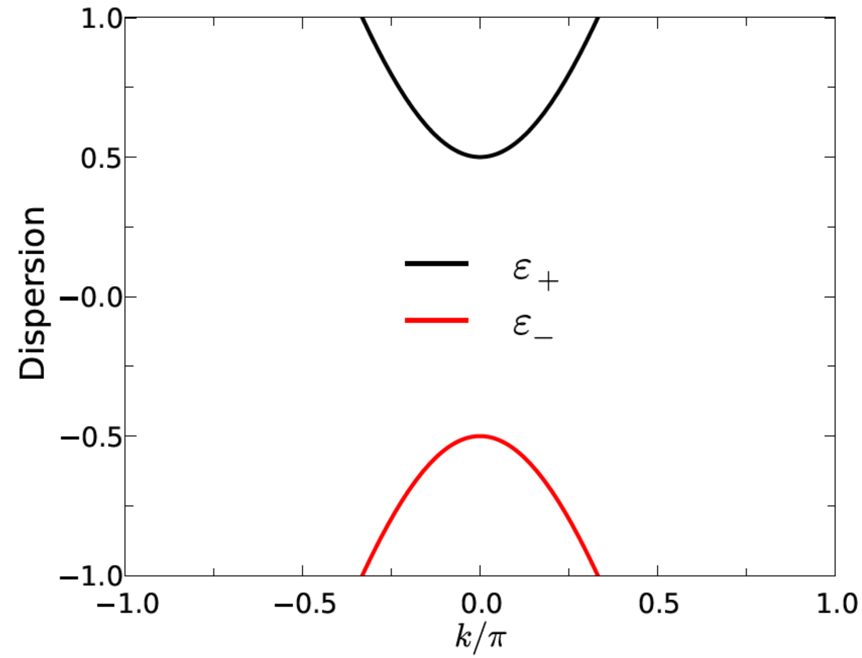
Band structure (bulk)

The system is gapped for all $\Delta > 0$ and $\mu \neq \pm t$

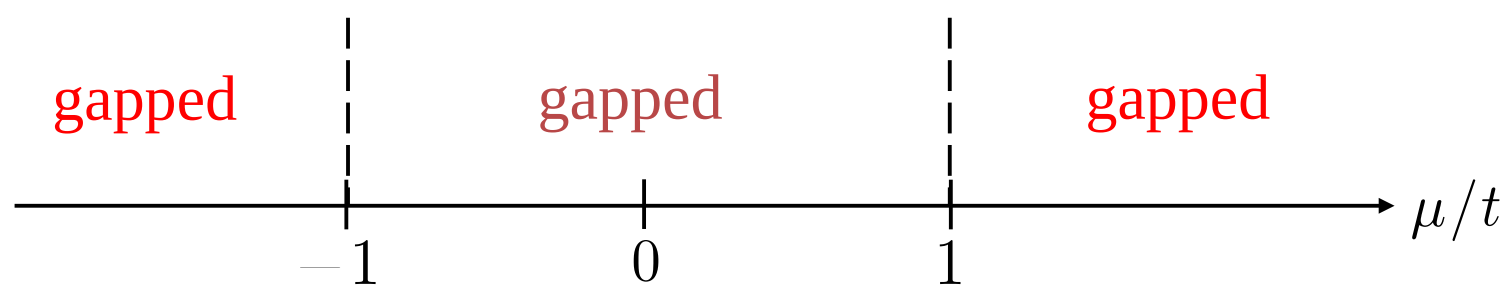
$\mu = 3t/2$



$\mu = -3t/2$



Summary:

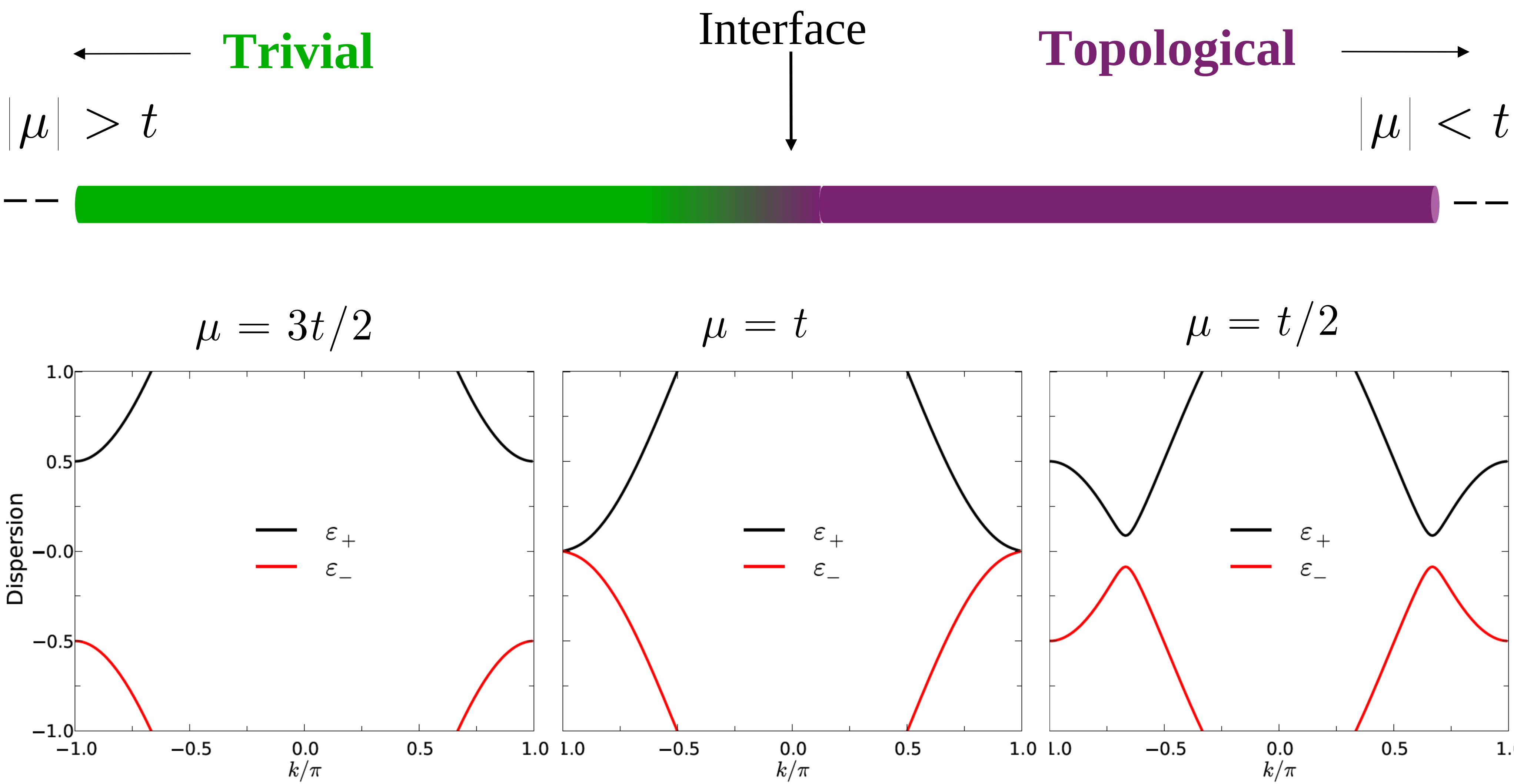


Central question: What is the difference between these gapped phases?

or

How these phases can be classified?

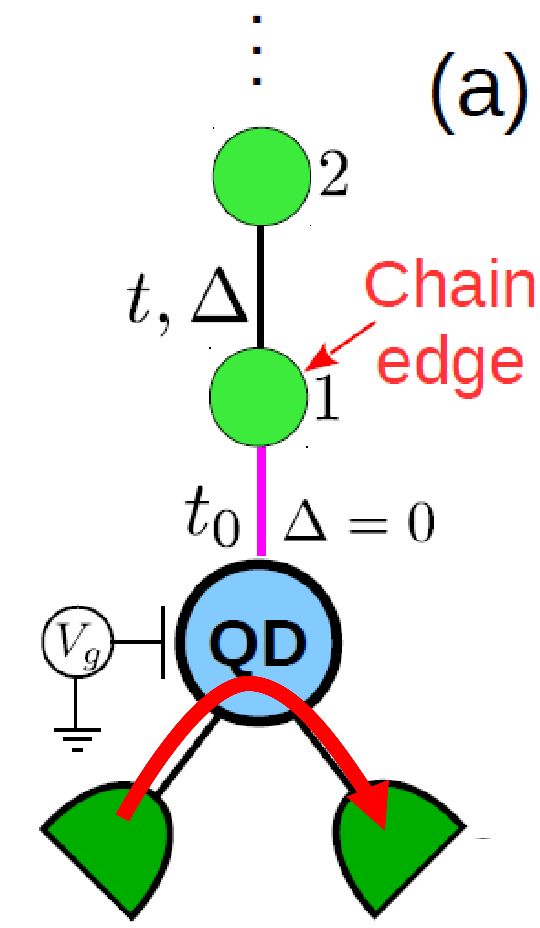
Gapless excitation at the interfaces



Detecting Majorana Fermions using Quantum dots

QD coupled to a Kitaev chain

Subtle leakage of a Majorana mode into a quantum dot,
Vernek, *et al.*, Phys. Rev. B **89**, 165314 (2014)



$$\varepsilon_{\mathtt{dot}}(V_g) = -eV_g$$

The Hamiltonian

$$H_{\text{dot-chain}} + H_{\text{leads}} + H_{\text{dot-leads}}$$

p-wave pairing

$$H_{\text{chain}} = -\mu \sum_{j=1}^{N} c_j^\dagger c_j - \frac{1}{2} \sum_{j=1}^{N-1} \left[t c_j^\dagger c_{j+1} + \Delta e^{i\phi} c_j c_{j+1} + H.c. \right]$$

Majorana Green's function

$$M_{\alpha i,\beta j}(\varepsilon) = -i \int_{-\infty}^{\infty} \Theta(\tau) \langle [\gamma_{\alpha i}(\tau), \gamma_{\beta j}(0)]_+ \rangle e^{i\varepsilon(\tau)} d\tau,$$

Electron Green's function

$$G_{ij}(\varepsilon) = \frac{1}{4} \left[M_{Ai,Aj} + M_{Bi,Bj}(\varepsilon) + i \left(M_{Ai,Bj} - M_{Bi,Aj} \right) \right]$$

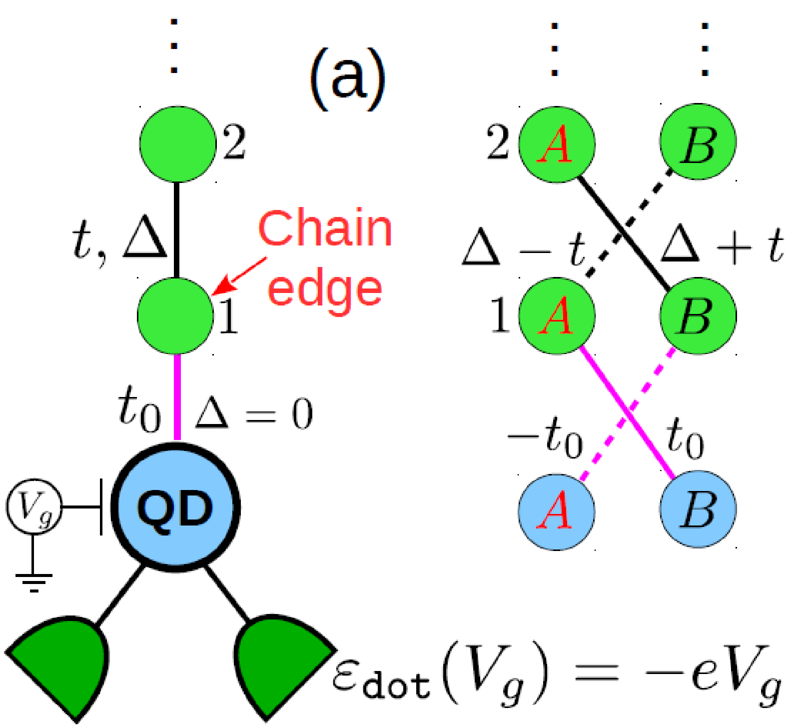
Conductance: $G/G_0 \propto \mathrm{Im} G_{00}(\varepsilon = 0)$

Majorana edge modes

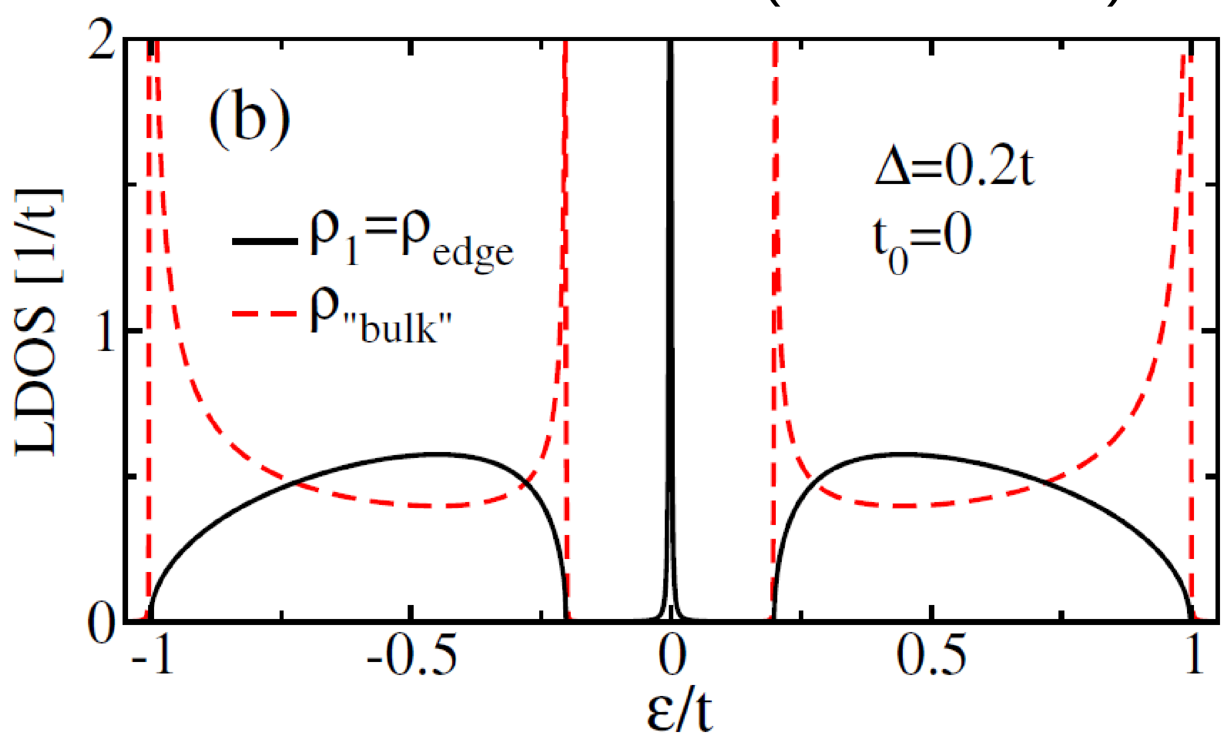
$$c_j = \frac{e^{-i\phi/2}}{2}\left(\gamma_{Bj} + i\gamma_{Aj}\right)$$
$$c_j^\dagger = \frac{e^{i\phi/2}}{2}\left(\gamma_{Bj} - i\gamma_{Aj}\right)$$

Majorana-electron transformation
(performed even for the QD site)

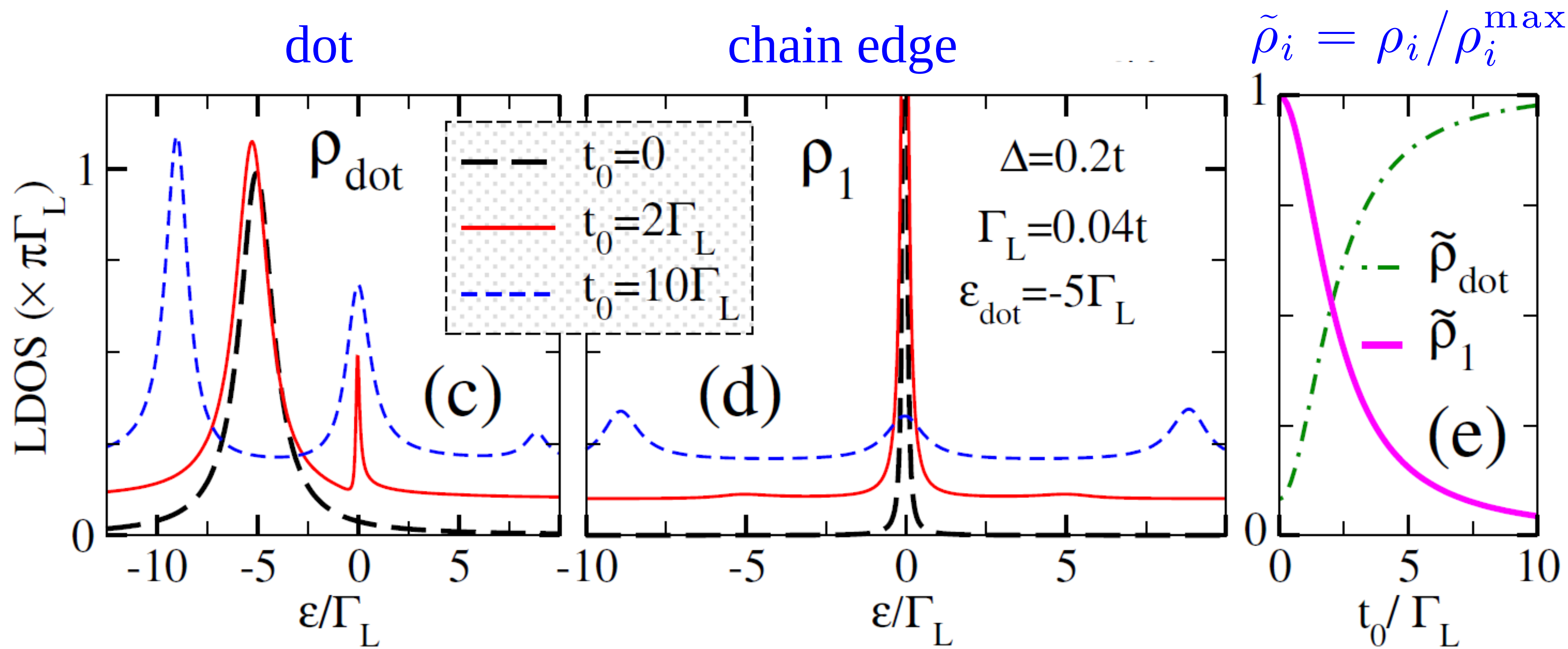
Clifford algebra: $[\gamma_\alpha, \gamma_\beta]_+ = 2\delta_{\alpha\beta}$

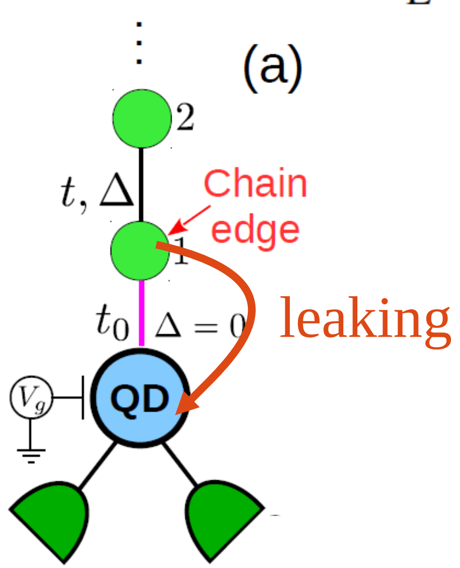


$t \approx 5\Delta \approx 10\mathrm{meV}$ (Mourik-2012).



Majorana leaking



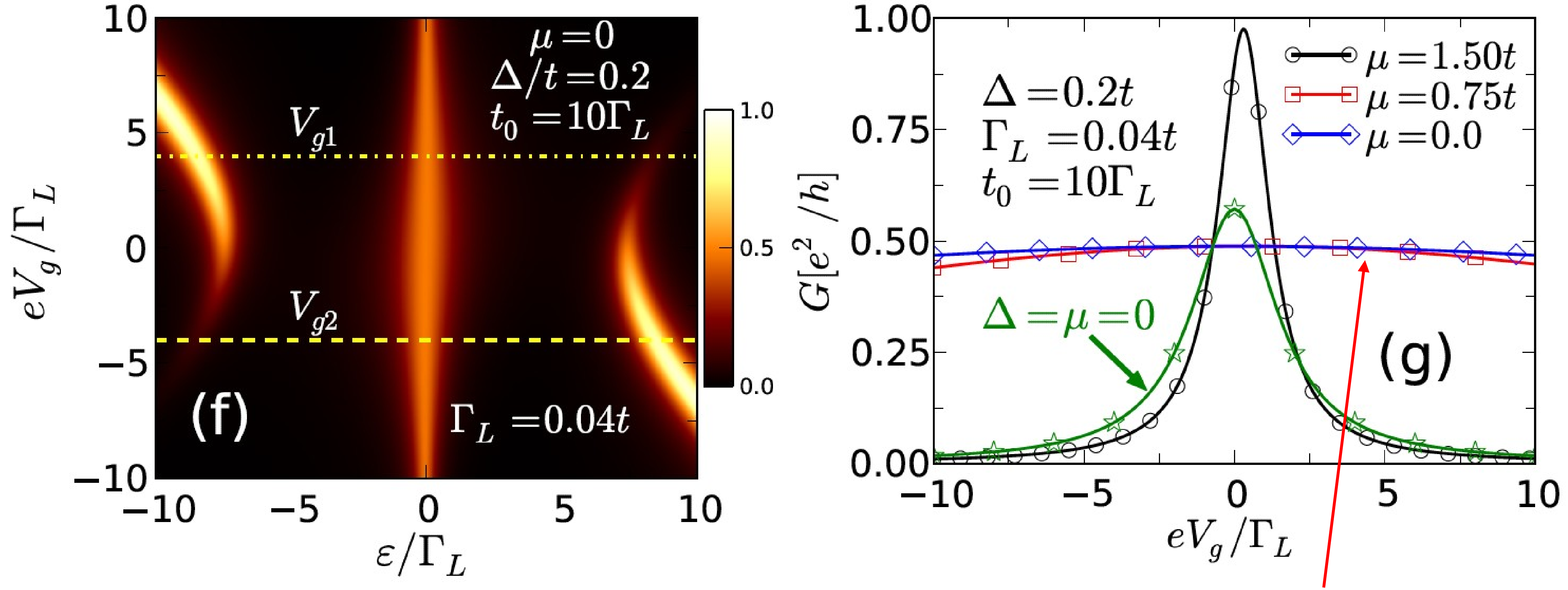


→ Majorana zero mode disappears from the edge of the chain and shows up at the dot.

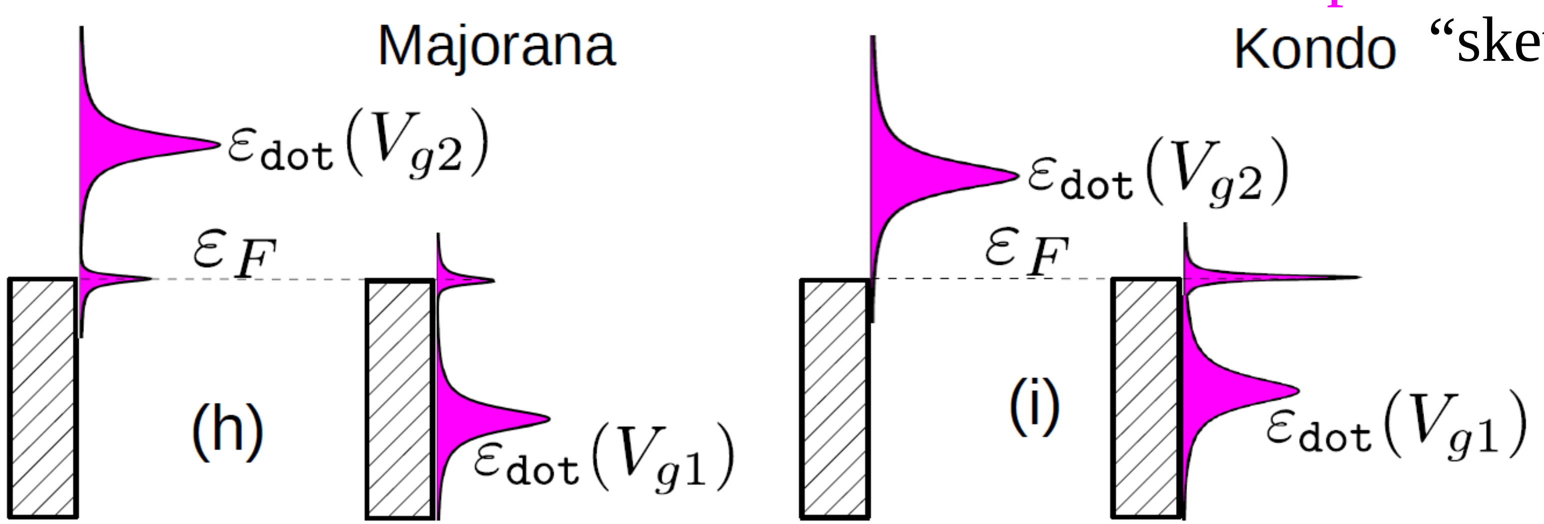
→ The QD can be thought as a new site of the chain, so the zero mode is always at the edge.

QD coupled-system seems NOT to be a non invasive way to probe Majorana bound states.

Pinning of the Majorana zero mode

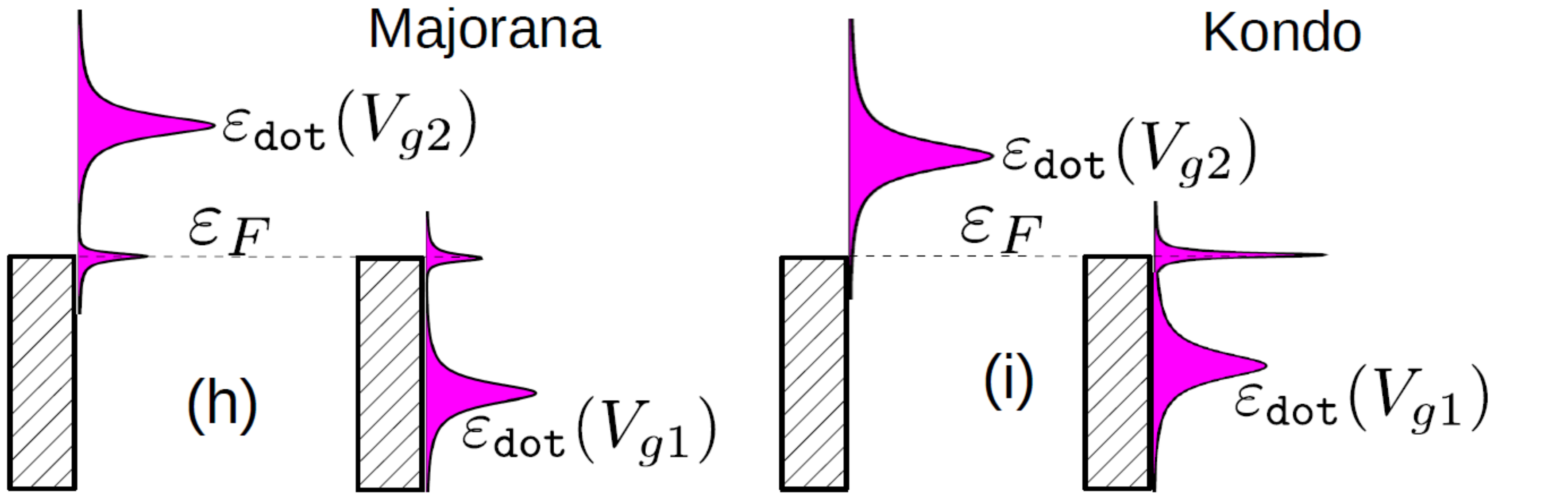


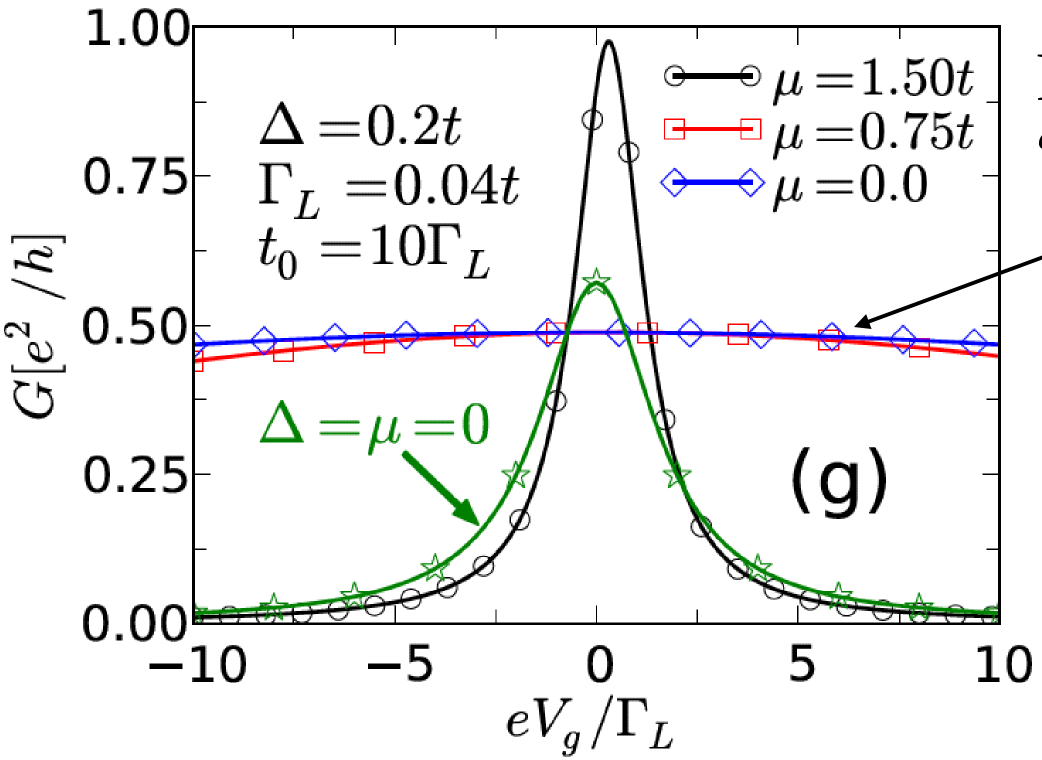
“Half” conductance plateau



“sketch”

Majorana vs Kondo





Pinning of the Majorana zero mode
“Half” conductance plateau

Spin-orbit interaction

Spin-orbit interaction

Dirac Equation

$$\left(c\alpha\cdot\mathbf{p}+\beta m_o c^2+V\right)\psi=E\psi$$

$$\alpha=\begin{pmatrix}0 & \sigma\\ \sigma & 0\end{pmatrix}\qquad \beta=\begin{pmatrix}\mathbf{I}_{2\times 2} & 0\\ 0 & -\mathbf{I}_{2\times 2}\end{pmatrix}$$

σ are Pauli matrices

$$\psi=\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$ denotes a four-component spinor, where ψ_A e ψ_B are two-component spinors

$$\begin{cases}\sigma\cdot\mathbf{p}\psi_B=\dfrac{1}{c}\left(\tilde{E}-V\right)\psi_A\\ \sigma\cdot\mathbf{p}\psi_A=\dfrac{1}{c}\left(\tilde{E}-V+2m_0c^2\right)\psi_A\end{cases}\qquad \tilde{E}=E-m_0c^2$$

$$\sigma\cdot\mathbf{p}\left[\frac{e^2}{\tilde{E}-V+2m_0c^2}\right]\sigma\cdot\mathbf{p}\psi_A=\left(\tilde{E}-V\right)\psi_A$$

Spin-orbit interaction

$$\sigma \cdot \mathbf{p} \left[\frac{e^2}{\tilde{E} - V + 2m_0c^2} \right] \sigma \cdot \mathbf{p}\psi_A = \left(\tilde{E} - V \right) \psi_A$$

Non-relativistic approximation: The Pauli Equation

$$\frac{e^2}{\tilde{E} - V + 2m_0c^2} \approx \frac{1}{2} \; 1 - \frac{\tilde{E} - V}{2m_0c^2} + \cdots \Bigg]$$

Minimal coupling

ψ_A Not normalized

$$\tilde{\psi}_A = \left(1 + \frac{p^2 + e^2\hbar\sigma \cdot \mathbf{B}}{8m_0^2c^2} \right) \psi_A$$

normalized

Up to order of $\dfrac{v^2}{c^2} \approx \dfrac{\tilde{E} - V}{2m_0c^2}$

Pauli Hamiltonian

$$\left[\frac{p^2}{2m_0} + V + \frac{e\hbar}{2m_0}\sigma \cdot \mathbf{B} \boxed{- \frac{e\hbar\sigma \cdot \mathbf{p} \times \mathcal{E}}{4m_0^2c^2}} - \frac{e\hbar^2}{8m_0^2c^2}\nabla \cdot \mathcal{E} \right.$$

$$\left. - \frac{p^4}{8m_0^3c^2} - \frac{e\hbar p^2}{4m_0^3c^2}\sigma \cdot \mathbf{B} - \frac{(e\hbar B)^2}{8m_0^3c^2} \right] \tilde{\psi}_A = \tilde{E}\tilde{\psi}_A$$

Spin couples to momentum

Spin-orbit in 1 and 2D systems

$$H_{\mathrm{P}} = \frac{e\hbar\sigma \cdot \mathbf{p} \times \mathcal{E}}{4m_0^2c^2}$$

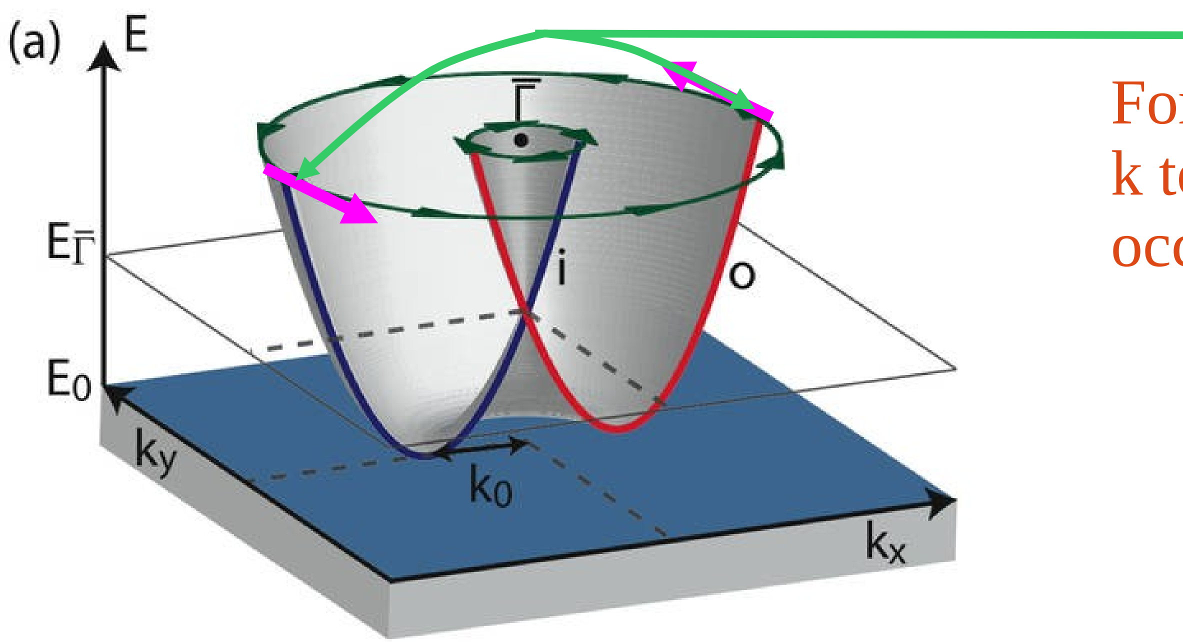
$$\mathcal{E} = \frac{1}{e}\nabla\mathbf{V}$$ is the electric field

Rashba Spin-orbit in 2D

$$H_{\mathrm{R}} = \int d^2k\psi_{\mathbf{k}}^{\dagger}\mathcal{H}_0(\mathbf{k})\psi_{\mathbf{k}}$$

$$\psi_{\mathbf{k}} = \begin{pmatrix} \psi_{\mathbf{k}\uparrow} \\ \psi_{\mathbf{k}\downarrow} \end{pmatrix}$$

$$\mathcal{H}_0(\mathbf{k}) = \left(\frac{k^2}{2m} - \mu\right)\mathbf{I} + \alpha(\sigma_x k_y - \sigma_y k_x)$$



Forbidden scatterings from k to -k, unless a spin flip occurs.

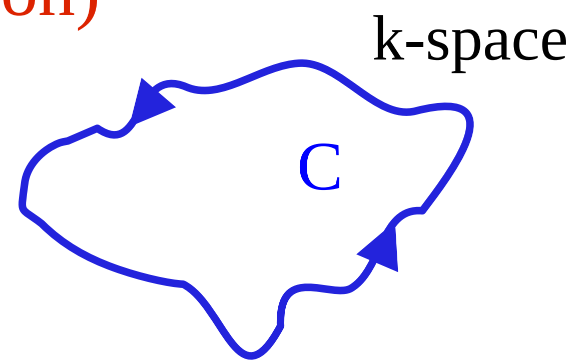
Berry phase and Berry connection

$$\mathcal{H}(\mathbf{k}) = \hat{\mathbf{h}}(\mathbf{k}) \cdot \sigma$$

Upon diagonalization we get $\Psi_n(\mathbf{k})\rangle$, the bands “n”.

$$\mathbf{A}_n(\mathbf{k}) = i\langle \Psi_n(\mathbf{k}) | \nabla_k \Psi_n(\mathbf{k}) \rangle$$ (Berry connection)

$$\gamma_n = \oint_C \mathbf{A}_n(\mathbf{k}) \cdot d\mathbf{k}$$ (Berry phase)



$$\mathbf{\Omega}_n(\mathbf{k}) = \nabla \times \mathbf{A}_n(\mathbf{k})$$ (Berry curvature)

Chern number

$$\mathcal{C}_n = \frac{1}{2\pi} \oint_S \mathbf{\Omega}_n(\mathbf{k}) \cdot d\mathbf{S} \Rightarrow C = \frac{1}{4\pi} \int_{occ} d^2k \hat{\mathbf{h}} \cdot \left(\frac{\partial \hat{\mathbf{h}}}{\partial k_x} \times \frac{\partial \hat{\mathbf{h}}}{\partial k_y} \right)$$