The Bogolubov-de Gennes Equations

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based on the joint work with Li Chen

previous work with V. Bach, S. Breteaux, Th. Chen and J. Fröhlich

Discussions with Rupert Frank and Christian Hainzl

Quantissima II, August 2017

Hartree and Hartree-Fock Equations

Starting with the many-body Schrödinger equation

$$i\partial_t \psi = H_n \psi,$$

for a system of n identical bosons or fermions and restricting it to the Hartree and Hartree-Fock states

$$\otimes_1^n \psi \qquad \text{and} \qquad \wedge_1^n \psi_i,$$

we obtain the Hartree and the Hartree-Fock equations.

There is a considerable literature on

- the derivation of the Hartree and Hartree-Fock equations
- the existence theory
- the ground state theory
- the excitation spectrum

Describing quantum fluids, like superfluids and superconductors, requires another conceptual step.

Non-Abelian random Gaussian fields

We think of Hartree-Fock states as non-Abelian generalization of random Gaussian fields. These fields (centralized) are uniquely characterized by the expectations of the 2nd order:

$$\langle \psi^*(y)\,\psi(x)\rangle. \tag{1}$$

We generalize this to (centralized) quantum fields, $\hat{\psi}(x)$, by assuming that the latter are uniquely characterized by the expectations of the 2nd order:

$$\langle \hat{\psi}^*(y)\,\hat{\psi}(x)\rangle. \tag{2}$$

These are exactly the Hartree-Fock states.

However, the above states are not the most general 'quadratic' states. The most general ones are defined by

$$\langle \hat{\psi}^*(y)\,\hat{\psi}(x)\rangle \quad \text{and} \quad \langle \hat{\psi}(x)\,\hat{\psi}(y)\rangle. \tag{3}$$

Quantum fluids

To sum up, the most general 'quantum Gaussian' states are the states defined by their quadratic expectations

$$\gamma(x,y) := \langle \hat{\psi}^*(y)\, \hat{\psi}(x)\rangle, \tag{4}$$

$$\alpha(x,y) := \langle \hat{\psi}(x)\, \hat{\psi}(y)\rangle. \tag{5}$$

α describes the (macroscopic) pair coherence (correlation amplitude, long-range order).

This type of states were introduced by Bardeen-Cooper-Schrieffer and further elaborated by Bogolubov.

In math language, these are the quasifree states. They give the most general one-body approximation to the $n-$body dynamics.

Let γ and α denote the operators with the integral kernels $\gamma(x,y)$ and $\alpha(x,y)$. Then, after stripping off the spin components,

$$\gamma = \gamma^* \geq 0 \text{ and } \alpha^* = \bar{\alpha} \text{ and a technical property,} \tag{6}$$

where $\bar{\sigma} = C\sigma C$ with C the complex conjugation.

Quasifree reduction

Following V. Bach, S. Breteaux, Th. Chen and J. Fröhlich and IMS, we map the solution ω_t of the Schrödinger equation

$$i\partial_t\omega_t(A) = \omega_t([A, H])\,,\ \forall A \in \mathfrak{W}. \tag{7}$$

to the family φ_t of quasifree states satisfying

$$i\partial_t\varphi_t(A) = \varphi_t([A, H]) \quad \forall \ \ \textit{quadratic} \ \ A. \tag{8}$$

We call this map the *quasifree reduction*.

Evaluating (8) on $\hat{\psi}^*(x, t)\hat{\psi}(y, t),\ \hat{\psi}(x, t)\hat{\psi}(y, t)$, yields a system of coupled nonlinear PDE's for (γ_t, α_t).

For the standard any-body hamiltonian, these give the (time-dependent) *Bogolubov-de Gennes (fermions) or Hartree-Fock-Bogolubov (bosons) equations.*

(In the latter case, one has also $\phi_t(x) = \langle\hat{\psi}(y, t)\rangle$.)

The BdG eqs give an equivalent formulation of the BCS theory.

Dynamics (Bosons)

Derivation (formal) and analysis of the dynamics for the generalized Gaussian states for bosons:

V. Bach, S. Breteaux, Th. Chen and J. Fröhlich and IMS. (See Grillakis and Machedon for some rigorous results on the deriv.)

For the pair interaction potential $v = \lambda\delta$ (where $\lambda \in \mathbb{R}$ and δ is the delta distribution), they are of the form,[1]

$$i\partial_t\phi_t = h\phi_t + \lambda|\phi_t|^2\phi_t + 2\lambda\rho_{\gamma_t}\phi_t + \lambda\bar{\phi}_t\rho_{\alpha_t} \tag{9}$$

$$i\partial_t\gamma_t = [h_{\gamma_t,\alpha_t}, \gamma_t]_- + \lambda[w_t, \alpha_t]_-, \tag{10}$$

$$i\partial_t\alpha_t = [h_{\gamma_t,\alpha_t}, \alpha_t]_+ + \lambda[w_t, \gamma_t]_+ + \lambda w_t, \tag{11}$$

where h is a one-particle Schrödinger operator, $\rho_\mu(x) := \mu(x;x)$,

$$\begin{aligned} w_t(x) &:= \rho_{\alpha_t}(x) + \phi_t^2(x), \\ h_{\gamma_t,\alpha_t} &:= h + 2\lambda(|\phi_t|^2 + \rho_{\gamma_t}). \end{aligned} \tag{12}$$

[1] $[A, B]_- = AB^* - BA^*$ and $[A, B]_+ = AB^T + BA^T$, with $A^T = \bar{A}^*$.

Dynamics (Fermions)

From now on, we concentrate on fermions.

It is convenient to organize the operators γ and α into the matrix-operator

$$\eta := \begin{pmatrix} \gamma & \alpha \\ \bar{\alpha} & \mathbf{1} \pm \bar{\gamma} \end{pmatrix} \tag{13}$$

Then

$$0 \leq \gamma = \gamma^* \ \leq 1 \ \text{ and } \ \alpha^* = \bar{\alpha} \text{ and a technical property} \tag{14}$$
$$\iff \ 0 \leq \eta = \eta^* \ \leq 1$$

As the generalized Gaussian states for fermions describe superconductors we have to couple the order parameter η to the electromagnetic field.

We describe the latter by the magnetic and electric potentials, a and ϕ.

Then states of the fermionic system are now described by the triple (η, a, ϕ), where $\eta \sim (\gamma, \alpha)$.

Bogolubov-de Gennes Equations

The many-body Schödinger equation implies the equations

$$i(\partial_t + i\phi)\eta = [H(\eta, a), \eta], \tag{15}$$
$$\partial_t(\partial_t a + \nabla\phi) = -\operatorname{curl}^* \operatorname{curl} a + j(\gamma, a), \tag{16}$$

where $j(\gamma, a)(x) := [-i\nabla_a, \gamma]_+(x, x)$, the superconducting current,

$$H(\eta, a) = \begin{pmatrix} h_{\gamma a} & v^\sharp \alpha \\ v^\sharp \bar{\alpha} & -\overline{h_{\gamma a}} \end{pmatrix},$$

where $v^\sharp : \alpha(x, y) \to v(x, y)\alpha(x; y)$, $v(x, y)$ is a pair potential, and

$$h_{\gamma a} = -\Delta_a + v^*\gamma, \tag{17}$$

with $\Delta_a := (\nabla + ia)^2$ and $v^*\gamma := v * \rho_\gamma,\ \rho_\gamma(x) := \gamma(x, x)$, the direct self-interaction energy. (We dropped the exchange energy.)

These are the celebrated Bogolubov-de Gennes equations (BdG eqs). They give an equivalent description of the BCS theory.

Conservation laws

BdG eqs conserve the energy $E(\eta, a, e) := E(\eta, a) + \frac{1}{2}\int |e|^2$, where

$$E(\eta, a) = \operatorname{Tr}\left((-\Delta_a)\gamma\right) + \frac{1}{2}\operatorname{Tr}\left((v * \rho_\gamma)\gamma\right) + \frac{1}{2}\operatorname{Tr}\left(\alpha^* v^\sharp \alpha\right) + \frac{1}{2}\int dx |\operatorname{curl} a(x)|^2 \tag{18}$$

and e is the electric field, and the particle number,

$$N := \operatorname{Tr}\gamma.$$

Theorem. The physically interesting stationary BdG solutions are critical points of the free energy

$$F_T(\eta, a) := E(\eta, a) - TS(\eta) - \mu N(\eta), \tag{19}$$

where $S(\eta) = -\operatorname{Tr}(\eta \ln \eta)$, the entropy, $N(\eta) := \operatorname{Tr}\gamma$.

Since the BDG eqs are translation inv., the ground state energy and the number of particles are expected to be either 0 or ∞.

Gauge (magnetic) translational invariance

The BdG eqs equations are invariant under the *gauge* transforms

$$T_\chi^{\mathrm{gauge}} : (\gamma, \alpha, a, \phi) \to (e^{i\chi}\gamma e^{-i\chi}, e^{i\chi}\alpha e^{i\chi}, a + \nabla\chi, \phi + \partial_t \phi) \quad (20)$$

$\implies$ states related by a gauge transform are physically equiv.

For $a \neq 0$, the simplest class of states are the gauge translationally invariant ones. (Translationally invariant states $\iff a = 0$.)

Gauge (magnetically) transl. invariant states are invariant under

$$T_{bs} : (\eta, a) \to (T_{\chi_s}^{\mathrm{gauge}})^{-1} T_s^{\mathrm{trans}}(\eta, a), \quad (21)$$

for any $s \in \mathbb{R}^2$, where $\chi_s(x) := \frac{b}{2}(s \wedge x)$ (modulo ∇f).

The next result shows that, unlike the $b = 0$ translation invariant case, there are no non-normal magnetically translationally (MT)) invariant states.

Ground State

Recall $\eta \sim (\gamma, \alpha)$. The BdG eqs have the following classes of stationary solutions which are the candidates for the ground state:

1. Normal states: (γ, α, a), with $\alpha = 0$ ($\iff$ η is diagonal).
2. Superconducting states: (γ, α, a), with $\alpha \neq 0$ and $a = 0$.
3. Mixed states: (γ, α, a), with $\alpha \neq 0$ and $a \neq 0$.

For $a = 0$, the existence of superconducting and normal, translationally invariant solutions is proven by Hainzl, Hamza, Seiringer, Solovej.

Theorem. MT-invariance $\Longrightarrow$ normality ($\alpha = 0$).

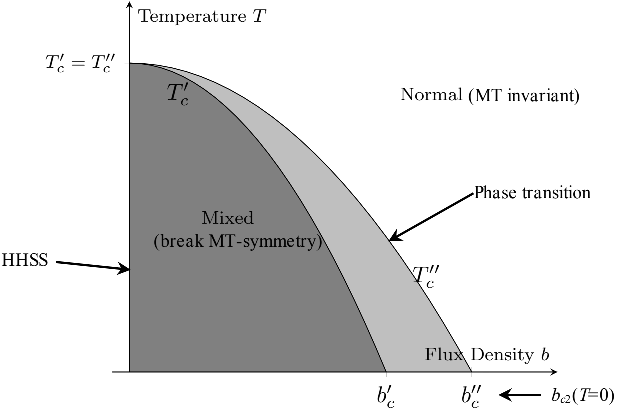
Corollary. Mixed states break the magnetic translational symmetry.

From now on, $d = 2$, i.e. we consider the cylinder geometry.

Results at a glance

Theorem [Li Chen-IMS] Let $b > 0$. Then $\exists\ 0 \leq T'_c(b) \leq T''_c(b)$ s.t.

- the energy minimizing states with $T > T''_c(b)$ are normal;
- the energy minimizing states with $T < T'_c(b)$ are mixed.



Normal states and symmetry breaking

Theorem. Drop the exchange term $v^\sharp\gamma$ and let $|\int v|$ be small.
Then $\forall T, b > 0$
(i) the BdG equations have a unique mt-invariant solution.

(ii) mt-invariance $\Longrightarrow$ normality $(\alpha = 0)$ $\Longrightarrow (\gamma_{T,b},\ 0,\ a_b)$, where

$$\gamma_{Ta} \qquad \text{solves} \qquad \gamma = f(\frac{1}{T}h_{\gamma,a}), \tag{22}$$

with $f(h) = (1+e^h)^{-1}$ the Fermi-Dirac distribution, and
$a_b(x) =$ magnetic potential with a constant magn. field b.

Theorem. Suppose that $v \le 0,\ v \not\equiv 0$. Then,

- for $T > 0$ and b large, the normal solution is stable,
- for b and T small, the normal solution is unstable.

Open problem. Are minimizers among normal states MT invariant?

Mixed states

Let $\mathcal{L} = r(\mathbb{Z} + \tau\mathbb{Z})$, where $\tau \in \mathbb{C}, \operatorname{Im}\tau > 0$. We define

- Vortex lattice: $T_s^{\mathrm{trans}}(\eta, a) = T_{\chi_s}^{\mathrm{gauge}}(\eta, a)$, for every $s \in \mathcal{L}$ and a co-cycle $\chi_s : \mathcal{L} \times \mathbb{R}^2 \to \mathbb{R}$, and $\alpha \neq 0$.

(The condition $\alpha \neq 0$ rules out that (η, a) is magnetically translationally invariant and therefore a normal state.)

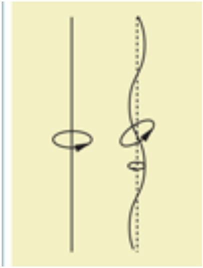
The magnetic flux is quantized ($\Omega_{\mathcal{L}}$ is a fundamental cell of $\mathcal{L}$):

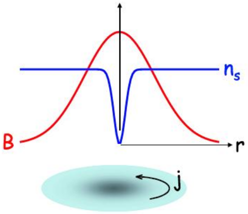
$$\frac{1}{2\pi} \int_{\Omega_{\mathcal{L}}} \operatorname{curl} a = c_1(\chi) \in \mathbb{Z}.$$

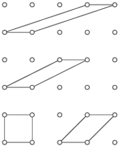
A vortex lattice solution is formed by magnetic vortices, arranged in a (mesoscopic) lattice $\mathcal{L}$.

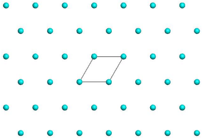
Magnetic vortices are localized finite energy solutions of a fixed degree, they are excitations of the homogeneous ground state.

Magnetic vortices and vortex lattices









Existence of vortex lattices

Theorem

(i) $\forall n$ *and* $\mathcal{L}$, $\exists$ *a solution* (η, a) *of the BdG eqs satisfying*

$$T_s^{\mathrm{trans}}(\eta, a) = \hat{T}_{\chi_s}^{\mathrm{gauge}}(\eta, a), \forall s \in \mathcal{L},$$

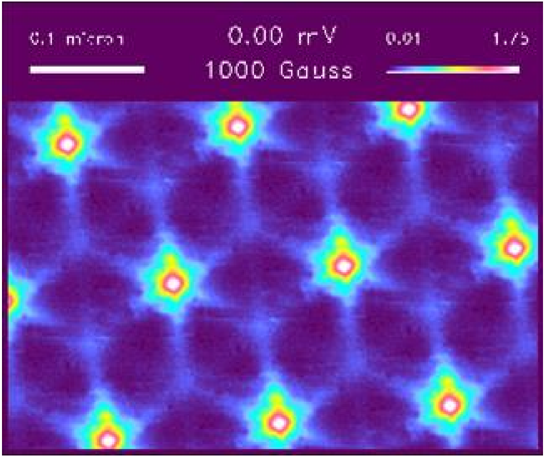
$$\int_{\mathcal{L}} \operatorname{curl} a = 2\pi n;$$

(ii) This solution minimizes the free energy F_T *on* $\Omega_{\mathcal{L}}$ *for* $c_1 = n$*;*

(iii) For $v \leq 0, v \not\equiv 0$ *and* T *and* b *sufficiently small, this solution is a vortex lattice (i.e.* $\alpha \neq 0$*);*

(iv) For $n > 1$*, there is a finer lattice, for which this solution is equivariant and* $c_1 = 1$*.*

Vortex Lattice. Experiment



Summary

- considered the Bogolubov-de Gennes equations, which are equivalent to the BCS theory of superconductivity
- introduced the key stationary solutions of BdG eqs, the competitors for the ground state: normal, superconducting and mixed (or intermediate) states
- described a rough phase diagram in the temperature - magnetic field plane
- discussed the magnetic translation symmetry and its spontaneous breaking
- presented an important class of the mixed states - the vortex lattices - demonstrating the symmetry breaking

Thank-you for your attention

Ginzburg-Landau Equations

Discovery of the vortex lattices are a crown achievement of theory of superconductivity. They were predicted by *A. A. Abrikosov* on the basis of the Ginzburg-Landau equations:

$$-\Delta_a \psi = \kappa^2(1 - |\psi|^2)\psi$$
$$\mathrm{curl}^* \,\mathrm{curl}\, a = \mathrm{Im}(\bar{\psi}\nabla_a \psi)$$

where $(\psi, a) : \mathbb{R}^d \to \mathbb{C} \times \mathbb{R}^d,\ d = 2, 3$, $\nabla_a = \nabla - ia$, $\Delta_a = \nabla_a^2$, the covariant derivative and covariant Laplacian, respectively, and κ is the Ginzburg-Landau (material) constant.

These equations describe equilibrium states of superconductors (mesoscopically) and of the $U(1)$ Yang-Mills-Higgs model of particle physics.

Formally, they approximate the stationary BdG in the mesoscopic regime.

GLE: Interpretation and dynamics

Superconductivity: $\psi : \mathbb{R}^d \to \mathbb{C}$ is called the *order parameter*; $|\psi|^2$ gives the density of (Cooper pairs of) superconducting electrons. $a : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic potential. $\mathrm{Im}(\bar{\psi}\nabla_a\psi)$ is the superconducting current.

Particle physics: ψ and a are the Higgs and $U(1)$ gauge (electro-magnetic) fields, respectively. (Part of Weinberg - Salam model of electro-weak interactions/a standard model.)

Time-dependent equations: The corresponding time-dependent equations are complex nonlinear Schrödinger and nonlinear (relativistic) wave equations coupled to a Maxwell equation.

Key problem: Dynamical stability of Abrikosov lattices.

GLE on Riemann surfaces

Abrikosov vortex lattices $\iff$ $\mathcal{L}-$equivariant functions and vector fields (one forms) $\iff$ *sections* and *connections* of the line bundle over a complex torus, $\mathbb{T} = \mathbb{C}/\mathcal{L}$.

$\implies$ reformulate the Ginzburg-Landau equations as equations on $\mathbb{T}$:

$$\Delta_a \psi = \kappa^2(|\psi|^2 - 1)\psi, \tag{23a}$$

$$d^* da = \mathrm{Im}(\bar{\psi}\nabla_a \psi). \tag{23b}$$

Here ψ is a section and a, a connection one-form on a $U(1)$ line bundle $L \to \mathbb{T}$, $\Delta_a = \nabla_a^* \nabla_a$, ∇_a and ∇_a^* are the covariant derivative and its adjoint, and d and d^* are the exterior derivative and its adjoint, which replace curl and curl*.

The complex torus, $\mathbb{T}$ is one of the simplest Riemann surfaces, but we can consider (23) on an arbitrary Riemann surface X.

Returning to the covering space

By the key uniformization theorem for Riemann surfaces, a Riemann surface X of genus ≥ 2 is of the form

$$X = \mathbb{H}/\Gamma,$$

for some discrete subgroup $\Gamma \subset PSL(2,\mathbb{R})$ (Fuchsian group) acting freely (i.e. without fixed points) on the Poincaré half-plane

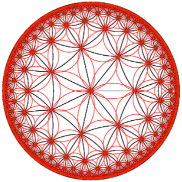
$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

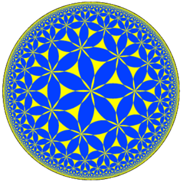
(η acts on $\mathbb{H}$ as $\gamma\ z = \frac{az+b}{cz+d},\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \eta$.)

Lifting the GLEs to $\mathbb{H}$, it becomes analogous to the original GLEs but with $\mathbb{C}$ replaced by $\mathbb{H}$ and the lattice $\mathcal{L}$ by a *Fuchsian* group, Γ.

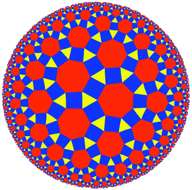
E.g. the $\mathcal{L}-$gauge invariance is replaced by $\Gamma-$gauge invariance.

Periodicity w.r.to Γ

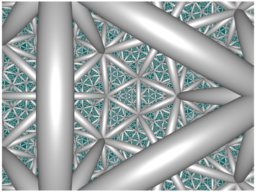




Tiling of the hyperbolic plane with equilateral triangles



Rhombitriheptagonal tiling



icosahedral honeycomb

Constant curvature connection on $\tilde{L}$

Theorem

Given the hyperbolic metric $h^{\mathrm{hyperb}} := |dz|^2/(\operatorname{Im} z)^2$ on $\mathbb{H}$, and $n \in \mathbb{Z}$, the unique constant curvature connection on $\tilde{L}$ of the degree n is given by

$$a^b = by^{-1}dx, \;\; b = \frac{\pi n}{g-1}.$$

It is equivariant with the automorphy factor

$$\rho(\gamma, z) = \left[\frac{c\bar{z} + d}{cz + d}\right]^{-\frac{\pi n}{g-1}}, \;\; \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{R}). \tag{24}$$

Summary

- I gave a thumbnail sketch of key PDEs of quantum physics concentrating on the Bogolubov-de Gennes equations. The latter describe the remarkable quantum phenomenon of superconductivity.
- There are many fundamental questions about these equations which are completely open.
- I introduced the key special solutions of BdGeqs: normal, superconducting and mixed (or intermediate) states.
- An important class of the mixed states are the vortex lattices.
- I discussed recent results on existence and stability of the normal and vortex lattice states.

Thank-you for your attention

Stationary Bogoliubov-de Gennes equations

We consider stationary solutions to BdG eqs of the form

$$\eta_t := T^{\mathrm{gauge}}_{\chi(t)} \eta_*, \tag{25}$$

with η_* and $\dot{\chi} \equiv \mu$ independent of t, χ independent of x, and a independent of t and $\phi = 0$. We have

Proposition

(25), *with* η_* *and* $\dot{\chi} \equiv -\mu$ *independent of* t, *is a solution to* (15) *iff* η_* *solves the equation*

$$[H_{\eta a}, \eta] = 0, \tag{26}$$

where

$$H_{\eta a} := \begin{pmatrix} h_{\gamma a \mu} & v^\sharp \alpha \\ v^\sharp \alpha^* & -\bar{h}_{\gamma a \mu} \end{pmatrix}, \quad h_{\gamma a \mu} := h_{\gamma a} - \mu. \tag{27}$$

Stationary Bogoliubov-de Gennes equations

For any reasonable function f, solutions of the equation

$$\eta = f\left(\frac{1}{T} H_{\eta a}\right), \tag{28}$$

solve $[H_{\eta a}, \eta] = 0 \Longrightarrow$ give stationary solutions of BdG eqs.
Physics:

$$f(h) = (1 + e^{h/T})^{-1} \quad \text{(the Fermi-Dirac distribution)} \tag{29}$$

Let $f^{-1} =: g'$. Then the stationary Bogoliubov-de Gennes equations can be written as

$$H_{\eta a} - Tg'(\eta) = 0, \tag{30}$$

$$\operatorname{curl}^* \operatorname{curl} a = j(\eta, a). \tag{31}$$

Free energy

Theorem

The stationary BdG eqs are the Euler-Lagrange equations for the free energy

$$F_T(\eta, a) := E(\eta, a) - TS(\eta) - \mu N(\eta), \qquad (32)$$

where $S(\eta) = -\operatorname{Tr}(\eta \ln \eta)$, *the entropy,* $N(\eta) := \operatorname{Tr} \gamma$, *the particle number.*

Lifting sections and connections to $\mathbb{H}$

Proposition. A connection $\nabla_A = d - iA$ and a section Ψ are in one-to-one correspondence with a one-form $\tilde{A}$ and function $\tilde{\Psi}$ on $\tilde{X} = \mathbb{H}$, which are *gauge $\Gamma-$invariant*, i.e. satisfy the relations

$$\gamma^* \tilde{\Psi} = \rho_\gamma \tilde{\Psi}, \ \gamma^* \tilde{A} = \tilde{A} + \rho_\gamma^{-1} d\rho_\gamma, \ \forall \gamma \in \eta, \tag{33}$$

where $\gamma^* \tilde{\Psi}(z) = \Psi(\gamma\ z)$, etc., for some automorphy factor, $\rho_\gamma(z) \equiv \rho(\gamma, z)$, i.e. a map $\rho : \Gamma \times \mathbb{H} \to U(1)$ satisfying the co-cycle relation

$$\rho(\gamma \cdot \delta, z) = \rho(\gamma, \delta\ z)\rho(\delta, z).$$

The existence of normal states

We give a key idea of the proof of existence of normal states with non-vanishing magnetic fields.

Recall: (η, a) is a normal state $\iff \alpha = 0$ (η is diagonal)

When $\alpha = 0$, the BdG equations reduces to the equations for γ and a:

$$\gamma = g^\sharp(\frac{1}{T} h_{\gamma,a}), \quad \operatorname{curl}^* \operatorname{curl} a = j(\gamma, a) \tag{34}$$

where, recall, $j(\gamma, a)(x) := \frac{1}{2}[-i\nabla_a, \gamma](x, x)$.

We show that the second equation is automatically satisfied, i.e. the superconducting current vanishes, for $a = a_b$ and γ is magnetically translation invariant.

The existence of normal states

We define $t_s^{\mathrm{mt}} := t_{g_s}^{\mathrm{gauge}} t_s^{\mathrm{trans}}$, where $g_s(x) := \frac{b}{2} s \wedge x$,

$$t_\chi^{\mathrm{gauge}} : \gamma \mapsto e^{i\chi}\gamma e^{-i\chi}, \quad t_h^{\mathrm{trans}} : \gamma \mapsto U_h \gamma U_h^{-1},$$

for any sufficiently regular function $\chi : \mathbb{R}^d \to \mathbb{R}$, and any $h \in \mathbb{R}^d$. Let t^{refl} be a conjugation by reflections.

Proposition

If a trace class operator $\tilde{\gamma}$ satisfies $t_h^{\mathrm{mt}}\tilde{\gamma} = \tilde{\gamma}$, then $\tilde{\gamma}(x,x) = \tilde{\gamma}(0,0)$ for all x. If, in addition, $t^{\mathrm{refl}}\tilde{\gamma} = -\tilde{\gamma}$, then $\tilde{\gamma}(x,x) = 0$.

Proof.

Due to $t_h^{\mathrm{mt}}\tilde{\gamma} = \tilde{\gamma}$, the integral kernel of $\tilde{\gamma}$ obeys $e^{ig_h(x)}\tilde{\gamma}(x+h, y+h)e^{-ig_h(y)} = \tilde{\gamma}(x,y)$. Taking $y = x$ gives $\tilde{\gamma}(x+h, x+h) = \tilde{\gamma}(x,x)$, which implies $\tilde{\gamma}(x,x) = \tilde{\gamma}(0,0)$. $t^{\mathrm{refl}}\tilde{\gamma} = -\tilde{\gamma}$ implies $\tilde{\gamma}(-x,-y) = -\tilde{\gamma}(x,y)$, which gives $\tilde{\gamma}(-x,-x) = -\tilde{\gamma}(x,x)$, which implies $\tilde{\gamma}(x,x) = \tilde{\gamma}(0,0) = 0$. $\square$

The existence of normal states

Recall that $j(\gamma, a)(x) := \frac{1}{2}[-i\nabla_a, \gamma](x, x)$. Consider the operator $\tilde{\gamma} := \frac{1}{2}[-i\nabla_{a_b}, \gamma]$.

If γ is magnetically translation invariant, then so is $\tilde{\gamma}$. If γ is even under the reflections, then $\tilde{\gamma}$ is odd.

Applying the proposition above to $\tilde{\gamma} = \frac{1}{2}[-i\nabla_{a_b}, \gamma]$, where γ a magnetically translationally invariant and even trace class operator gives $j(\gamma, a_b) = 0$.

Since $\operatorname{curl}^* \operatorname{curl} a_b = \operatorname{curl}^* b = 0$, this proves $\operatorname{curl}^* \operatorname{curl} a_b = j(\gamma, a_b)$, which is the second equation in (34). $\square$