DIRECTED POLYMERS, CRITICAL TEMPERATURE AND UNIFORM INTEGRABILITY

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Abstract. In this paper, we consider directed polymers in random environment. More accurately, we consider the critical temperature β_c between weak and strong disorder. We find a necessary condition that ensures that this critical temperature is generally different from its lower bound obtained with the second moment method introduced by E. Bolthausen. To do this, we study a condition of uniform integrability originally due to B. Derrida and M.R. Evans. We also illustrate our result by examples of environments that satisfy this condition.

Introduction

Directed polymers in random environment have been introduced by J.Z. Imbrie et T. Spencer in [IS88].
Let us denote Ω_n the paths of $\mathbb{Z}^d$ starting from 0, of length n and with jumps of length 1, i.e.

$$\Omega_n = \left\{\omega \in (\mathbb{Z}^d)^{n+1};\ \omega_0 = 0,\ \forall\ i = 1\dots n,\ \|\omega_i - \omega_{i-1}\|_1 = 1\right\}.$$

In this context, a directed polymer is a path $\omega \in \Omega_n$.
The polymer lives in a random environment denoted by $(g(i,x))_{i\in\mathbb{N},x\in\mathbb{Z}^d}$. We suppose these variables are independent with common distribution $\mathbf{Q}$. We denote, for all $\beta \in \mathbb{R}$ the *cumulant*

$$\lambda(\beta) = \ln \mathbf{Q}\left[e^{\beta g}\right],$$

and suppose this quantity is finite for all β positive. We restrict our study to positive values of β (Indeed, for β negative we substitute $-g$ for g).
For every path $\omega \in \Omega_n$, we define its *Hamiltonian* :

$$H_n(\omega) = \sum_{i=1}^{n} g(i,\omega_i).$$

Denoting by $\mathbf{P}$ the uniform distribution on Ω_n, we call the *partition function* of the system :

$$Z_n(\beta) = \mathbf{P}\left[e^{\beta H_n(X)}\right],$$

where β states for the inverse of the temperature. We sometimes forget the dependance on β to lighten the notations.
We will also talk about directed polymers on a tree, i.e. we substitute a regular tree with $2d$ branches for $\mathbb{Z}^d$ (cf. [BPP93]).

E. Bolthausen (cf. [Bol89]) showed that $W_n = Z_n e^{-n\lambda(\beta)}$ is a positive martingale, so converges to a positive and finite random variable W_∞. Moreover, a $0-1$ law shows that for all $\beta \in \mathbb{R}_+$,

$$\mathbf{Q}(W_\infty = 0) \in \{0, 1\}.$$

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Using the FKG inequality, F. Comets et N. Yoshida (cf. [CY06]) showed that $\beta \mapsto \mathbf{Q}\left[\sqrt{W_\infty(\beta)}\right]$ is decreasing. One can thus define the critical temperature β_c such that

$$W_\infty(\beta)\begin{cases} =0 & , \, if \ \beta<\beta_c \\ >0 & , \, if \ \beta>\beta_c \end{cases}.$$

The area $\beta<\beta_c$ is called *weak disorder* phase, whereas the area $\beta>\beta_c$ is called *strong disorder* phase.
We remark that when $\beta=0$, the polymer behaves like a simple random walk and then $W_\infty=1$: all the paths are equally distributed and there's no disorder.
When the dimension d equals 1 or 2, Ph. Carmona and Y. Hu (cf. [CH02]) in a Gaussian environment (and then F. Comets and N. Yoshida (cf. [CSY03]) in any environment) showed that $\beta_c=0$. In the following we focus on dimensions d greater than 3.

To characterize the weak disorder phase, Ph. Carmona and Y. Hu (cf. [CH02]) showed that this phase is equivalent to the assertion (W_n) is uniformly integrable. To begin with, we present a condition for uniform integrability introduced by B. Derrida et M.R. Evans (cf. [DE92]).
Let ω^1, ω^2 be two random walks in $\mathbb{Z}^d$ of length n starting from zero. For $(t,x)\in\mathbb{N}\times\mathbb{Z}^d$, we will denote $p(t,x)$ the probability that these two walks meet for the first time at time t in x, i.e.

$$p(t,x)=\mathbf{P}^{\otimes 2}\left(\omega_j^1\neq\omega_j^2,\ 1\le j<t,\ \omega_t^1=\omega_t^2=x\right). \tag{0.1}$$

Define next for $\alpha\in(1,2]$,

$$\rho(\alpha)=\sum_{t,x}p(t,x)^{\alpha/2}. \tag{0.2}$$

Theorem 1 ([DE92])**.** *Let $\beta\in\mathbb{R}_+$. If there exists $\alpha\in(1,2]$ such that*

$$\lambda(\alpha\beta)-\alpha\lambda(\beta)<-\ln\rho(\alpha),$$

then $(W_n(\beta))_n$ is uniformly integrable.

Using a second moment method, E. Bolthausen (cf. [Bol89]) found a lower bound β_2 for β_c. Using the previous theorem, we state a sufficient condition that ensures that β_2 is not the best lower bound.
Let us introduce the entropies

$$h_\nu(\alpha) \quad = \quad -\sum_{t,x}\frac{\ln p(t,x)^{\alpha/2}}{\rho(\alpha)}\ \frac{p(t,x)^{\alpha/2}}{\rho(\alpha)}, \tag{0.3}$$

$$h_{\mathbf{Q}}(\alpha) \quad = \quad \mathbf{Q}\left[\ln\frac{e^{\alpha\beta_\alpha g}}{\mathbf{Q}[e^{\alpha\beta_\alpha g}]}\ \frac{e^{\alpha\beta_\alpha g}}{\mathbf{Q}[e^{\alpha\beta_\alpha g}]}\right]. \tag{0.4}$$

We will show the following theorem.

Theorem 2. *If*

$$h_\nu(2)<h_{\mathbf{Q}}(2),$$

there exists β_* such that $\beta_2<\beta_*<\beta_c$.

We will check that for some centered poisson environments this condition is always satisfied even if the dimension equals 3. For Bernoulli environments it is satisfied only when the dimension equals to 4 or 5. Finally, for Gaussian environments, this condition is a sufficient condition to obtain the uniform integrability criterion of Theorem 1 and we improve the second moment bound if the dimension equals 5.

We want to stress that this result shows that, in general, the bound β_2 is different from the critical temperature β_c contrary to what some physicists expect (cf. [MG06]). On the other hand, a lower bound on β_c better than β_2 has been found by M. Birkner. However this result uses a conditional Sanov whose proof is not yet published (cf.[Bir04] and references herein).

In this article, we will first calculate in section 1 the moments of order α of the partition function to obtain a condition for uniform integrability. In a second time, we will study precisely the function ρ introduced in Theorem 1. In section 3 we study more precisely the condition of uniform integrability introduced by B. Derrida and M.R. Evans and then prove the Theorem 2. Finally we will illustrate this theorem with in section 4 some examples of environments.

1. Moments and Uniform Integrability condition

For seek of completness we present here a detailed proof of Theorem 1.

To show uniform integrability, B. Derrida and M.R. Evans used the following necessary condition.

Theorem (for example [Wil91])**.** *If there exists $\alpha > 1$ such that*

$$\sup_n \mathbf{Q}(W_n^\alpha) < +\infty,$$

then $(W_n)_n$ is uniformly integrable.

We restrict ourselves to $\alpha \in (1,2]$ in the following, so that one can make computations, as regards to what happens in the second moment method.
Recall that we wrote

$$p(t,x) = \mathbf{P}\left(\omega_j^1 \neq \omega_j^2,\ 1 \leq j < t,\ \omega_t^1 = \omega_t^2 = x\right).$$

In the following, when two random walks only meet at times $t_1 < \ldots < t_m$ and are then in $x_1, \ldots, x_m$, i.e.

$$\left\{\omega_0^1 = \omega_0^2 = 0,\ \omega_{t_i}^1 = \omega_{t_i}^2 = x_i,\ i = 1\ldots m,\ \omega_j^1 \neq \omega_j^2,\ j \notin \{t_1, \ldots, t_m\}\right\},$$

we write $r = ((t_1, x_1), \ldots, (t_m, x_m)) \in \left(\mathbb{N} \times \mathbb{Z}^d\right)^m$ and

$$\omega^1 \stackrel{r}{=} \omega^2. \tag{1.5}$$

Proof of Theorem 1. We first compute the second moment order of Z_n. To do this we introduce two independent random walks and then split the expectations according to their meeting times :

$$\begin{aligned}
Z_n^2 &= \mathbf{P}\left[e^{\beta(H_n(\omega^1)+H_n(\omega^2))}\right] \\
&= \sum_{m=0}^{n} \sum_{1\leq t_1 \leq \cdots \leq t_m \leq n} \sum_{(x_1,\ldots,x_m)} \mathbf{P}\left[e^{\beta(H_n(\omega^1)+H_n(\omega^2))} \mathbb{1}_{\omega^1 \stackrel{r}{=} \omega^2}\right] \\
&=: \sum_{m=0}^{n} \sum_{r \in (\mathbb{N}_n \times \mathbb{Z}_n^d)^m} Y(r).
\end{aligned}$$

We compute next the α order moments of Z_n when $\alpha \in (1,2]$, using the inequality

$$\left(\sum x_i\right)^\gamma \leq \sum x_i^\gamma, \qquad \gamma \in [0,1],\ x_i \geq 0. \tag{1.6}$$

We have,

$$\begin{aligned}
\mathbf{Q}\left[Z_n^\alpha\right] &= \mathbf{Q}\left[(Z_n^2)^{\alpha/2}\right] \\
&= \mathbf{Q}\left[\left\{\sum_{m=0}^n \sum_r Y(r)\right\}^{\alpha/2}\right] \\
&\leq \sum_{m=0}^n \sum_r \mathbf{Q}\left[Y(r)^{\alpha/2}\right].
\end{aligned}$$

Let's concentrate now on the quantity $Y(r)$. We define the partial hamiltonian :

$$H_{j_1}^{j_2}(\omega) = \sum_{i=j_1+1}^{j_2} g(i,\omega_i).$$

We can thus decompose, noting $\omega_{i,j} = (\omega_k)_{k\in\{t_i,\dots,t_j\}}$, $t_0 = 0$ and $t_{m+1} = n$,

$$\begin{aligned}
Y(r) &= \mathbf{P}\left[\prod_{i=1}^m e^{\beta\left(H_{t_{i-1}}^{t_i}(\omega^1)+H_{t_{i-1}}^{t_i}(\omega^2)\right)} \mathbb{1}_{\{\omega^1_{i-1,i} \overset{(t_i,x_i)}{=} \omega^2_{i-1,i}\}} \right. \\
&\qquad\qquad \left. \times e^{\beta\left(H_{t_m}^n(\omega^1)+H_{t_m}^n(\omega^2)\right)} \mathbb{1}_{\{\omega^1_{m,m+1}\neq\omega^2_{m,m+1}\}}\right]. \\
&= \prod_{i=1}^m Y_{i-1,i} \times \widetilde{Y}_{m,n},
\end{aligned}$$

where we wrote

$$\begin{aligned}
\mathbf{Q}\left[Y_{i-1,i}^{\alpha/2}\right] &= e^{\beta\left(H_{t_{i-1}}^{t_i}(\omega^1)+H_{t_{i-1}}^{t_i}(\omega^2)\right)} \mathbb{1}_{\{\omega^1_{i-1,i} \overset{(t_i,x_i)}{=} \omega^2_{i-1,i}\}}, \\
\mathbf{Q}\left[\widetilde{Y}_{m,n}^{\alpha/2}\right] &= e^{\beta\left(H_{t_m}^n(\omega^1)+H_{t_m}^n(\omega^2)\right)} \mathbb{1}_{\{\omega^1_{m,m+1}\neq\omega^2_{m,m+1}\}}.
\end{aligned}$$

Hence, using the independance of the environment with respect to the temporal evolution :

$$\mathbf{Q}\left[Z_n^\alpha\right] \leq \sum_{m=0}^n \sum_r \prod_{i=1}^m \mathbf{Q}\left[Y_{i-1,i}^{\alpha/2}\right] \mathbf{Q}\left[\widetilde{Y}_{m,n}^{\alpha/2}\right].$$

Therefore, using Fatou's lemma,

$$\limsup_n \left[W_n^\alpha\right] \leq \sum_{m=0}^\infty \sum_{n\in(\mathbb{N}\times\mathbb{Z}^d)^n} \prod_{i=1}^m \mathbf{Q}\left[Y_{i-1,i}^{\alpha/2}\right] \limsup_n \mathbf{Q}\left[\frac{\widetilde{Y}_{m,n}^{\alpha/2}}{e^{n\alpha\lambda(\beta)}}\right].$$

On the one side, as the random walks never meet after time t_m and using Jensen's inequality,

$$\begin{aligned}
\mathbf{Q}\left[\widetilde{Y}_{m,n}^{\alpha/2}\right] &\leq \mathbf{P}\left(\omega_j^1 \neq \omega_j^2,\ t_m < j \leq n\right)^{\alpha/2} e^{\alpha(n-t_m)\lambda(\beta)} \\
\limsup_n \mathbf{Q}\left[\frac{\widetilde{Y}_{m,n}^{\alpha/2}}{e^{\alpha n\lambda(\beta)}}\right] &\leq P_m e^{-\alpha t_m\lambda(\beta)},
\end{aligned}$$

with

$$\begin{aligned}
P_m &= \mathbf{P}\left(\omega_{t_m}^1 = \omega_{t_m}^2 = x_m,\ \omega_j^1 \neq \omega_j^2,\ j > t_m\right) \\
&= \mathbf{P}\left(\omega_j^1 \neq \omega_j^2,\ j > 0\right) \\
&=: q_d,
\end{aligned}$$

thanks to Markov's property and since $d \geq 3$.

The environment is equally distributed, so we can write our upper bound

$$\begin{aligned}\limsup_n \mathbf{Q}\left[W_n^\alpha\right] &\leq q_d^{\alpha/2}\sum_{m=0}^\infty \sum_{r\in(\mathbb{N}\times\mathbb{Z}^d)^m}\prod_{i=1}^m \mathbf{Q}\left[e^{-\alpha(t_i-t_{i-1})\lambda(\beta)}Y_{i-1,i}^{\alpha/2}\right]\\ &\leq q_d^{\alpha/2}\sum_{m=0}^\infty\left\{\sum_{t_1\in\mathbb{N},x_1\in\mathbb{Z}^d}\mathbf{Q}\left[e^{-\alpha t_1\lambda(\beta)}Y_{0,1}^{\alpha/2}\right]\right\}^m.\end{aligned}$$

On the other side, using the independance of the environment, denoting $(t,x)=(t_1,x_1)$ and using Jensen's inequality,

$$\begin{aligned}\mathbf{Q}\left[\frac{Y_{0,1}^{\alpha/2}}{e^{\alpha t\lambda(\beta)}}\right] &= e^{-\alpha t\lambda(\beta)}\mathbf{Q}\left[\mathbf{P}\left[e^{\beta\sum_{i=1}^{t-1}g(i,x_i^1)+g(i,x_i^2)}e^{2\beta g(t,x)}\mathbb{1}_{\omega^1\overset{(t,x)}{=}\omega^2}\right]^{\alpha/2}\right]\\ &= e^{-\alpha t\lambda(\beta)}e^{\lambda(\alpha\beta)}\mathbf{Q}\left[\mathbf{P}\left[e^{\beta\sum_{i=1}^{t-1}g(i,x_i^1)+g(i,x_i^2)}\mathbb{1}_{\omega^1\overset{(t,x)}{=}\omega^2}\right]^{\alpha/2}\right]\\ &\leq e^{\lambda(\alpha\beta)-\alpha t\lambda(\beta)}\mathbf{QP}\left[e^{\beta\sum_{i=1}^{t-1}g(i,x_i^1)+g(i,x_i^2)}\mathbb{1}_{\omega^1\overset{(t,x)}{=}\omega^2}\right]^{\alpha/2}\\ &= e^{\lambda(\alpha\beta)-\alpha t\lambda(\beta)}e^{2\frac{\alpha}{2}\lambda(\beta)(t-1)}\mathbf{P}\left(\omega^1\overset{(t,x)}{=}\omega^2\right)^{\alpha/2}\\ &= e^{\lambda(\alpha\beta)-\alpha\lambda(\beta)}p(t,x)^{\alpha/2}.\end{aligned}$$

Finally, we find the following upper bound (recall definition (0.2) of ρ) :

$$\limsup_n \mathbf{Q}\left[W_n^{\alpha/2}\right]\leq q_d^{\alpha/2}\sum_{m=0}^\infty\left\{e^{\lambda(\alpha\beta)-\alpha\lambda(\beta)}\rho(\alpha)\right\}^m.$$

Therefore, if there exists $\alpha\in(1,2]$ such that

$$\lambda(\alpha\beta)-\alpha\lambda(\beta)<-\ln\rho(\alpha),$$

the martingale $(W_n(\beta))_n$ is uniformly integrable. □

Remark.

- When $\alpha=2$, $\rho(2)=1-q_d$ and the previous condition is the same as the one obtained by E. Bolthausen in [Bol89] with the so-called second moment method

$$\lambda(2\beta)-2\lambda(\beta)<-\ln(1-q_d).$$

- One can find in Griffin [Gri90] numerical values for q_d. To illustrate our talk we use the following approximations :

d	$1-q_d$
3	0.340
4	0.193
5	0.135

2. More on the function ρ

We write the uniform integrability condition previously obtained :

$$\exists\ \alpha\in(1,2] \text{ such that } \frac{\lambda(\alpha\beta)}{\alpha\beta}-\frac{\lambda(\beta)}{\beta}<\frac{-\ln\rho(\alpha)}{\alpha\beta}. \tag{2.7}$$

First we shall study the function ρ to improve our understanding of this condition.

2.1. **Characteristic values.** We have already remarked that $\rho(2) = 1 - q_d < +\infty$. Thus, the function ρ is defined on an interval $(\alpha_0, 2]$ where we wrote

$$\alpha_0 = \inf \{\alpha \in (1,2]; \rho(\alpha) < +\infty\} . \tag{2.8}$$

We prove in annexe A that $4/d < \alpha_0 \leq 1 + 2/d$ (recall that d is greater than 3).

Proposition 3. *The function $\alpha \mapsto \rho(\alpha)$ is non increasing on $(\alpha_0, 2]$.*

Proof. Indeed, ρ is differentiable on its domain of definition and

$$\rho'(\alpha) = \frac{1}{2} \frac{\sum_{t,x} \ln p(t,x) p(t,x)^{\alpha/2}}{\sum_{t,x} p(t,x)^{\alpha/2}}$$

is non positive since $p(t,x) < 1$, for all $(t,x) \in \mathbb{N} \times \mathbb{Z}^d$. □

We need in the next section the quantity α_1 defined below.

Proposition 4. *There exists $\alpha_1 \in (1,2)$ such that*

$$\rho(\alpha_1) = 1.$$

Proof. On the one side, when $\alpha = 2$,

$$\begin{aligned} \rho(2) &= \mathbf{P}\left(\exists\, t \in \mathbb{N},\ x \in \mathbb{Z}^d;\ \omega_t^1 = \omega_t^2 = x\right) \\ &= 1 - \mathbf{P}\left(\forall\, t \in \mathbb{N},\ \omega_t^1 \neq \omega_t^2\right) \\ &= 1 - q_d \\ &< 1. \end{aligned}$$

When d is greater than 3, we show in the appendix A that

$$\lim_{\alpha \downarrow \alpha_0} \rho(\alpha) = +\infty.$$

Using the preceding lemma, ρ is decreasing and continuous, so we are done. □

Remark. When the polymer lives on a regular tree with $2d$ branches, the random walks either meet on the first step or never meet. We have thus,

$$\ln \rho(\alpha) = (\alpha - 1) \ln(2d),$$

and $\alpha_0 = \alpha_1 = 1$.

2.2. **Graph of the function ρ.** In the following, to simulate $(p(t,x))_{t,x}$ we will look at the first point of intersection of N couples of independent random walks of length n. We refer the reader interested in the way we simulate $p(t,x)$ to the Annex C.

We conclude this section by presenting the graph of ρ in dimension 3. We emphasize on the value of α_1 thus obtained. The step definition of the coordinate α is 0.005.

3. The lower bound on β_c

In some cases, the bound given by the main theorem (see Theorem 2) yields a better lower bound for β_c than the one obtained with the second moment method. In this section, we try to find a nice condition that ensures β_2 is not the best bound.

Recall that we use the notations (0.3), (0.4) :

$$\begin{aligned} h_\nu(\alpha) &= -\sum_{t,x} \frac{\ln p(t,x)^{\alpha/2}}{\rho(\alpha)} \frac{p(t,x)^{\alpha/2}}{\rho(\alpha)}, \\ h_{\mathbf{Q}}(\alpha) &= \mathbf{Q}\left[\ln \frac{e^{\alpha\beta_\alpha g}}{\mathbf{Q}[e^{\alpha\beta_\alpha g}]} \frac{e^{\alpha\beta_\alpha g}}{\mathbf{Q}[e^{\alpha\beta_\alpha g}]}\right]. \end{aligned}$$

Remark. Since $p(t,x)^{\alpha/2}/\rho(\alpha) \leq 1$ and $x \mapsto x \ln x$ is concave, the quantities $h_\nu(\alpha)$ et $h_{\mathbf{Q}}(\alpha)$ are positive.

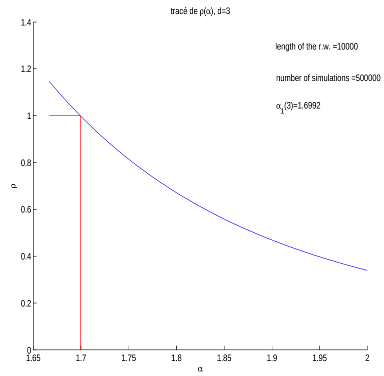

FIGURE 2.1. Graph of ρ in dimension $d = 3$

3.1. **The domain of definition of β_α.** We can define for all $\alpha \in [\alpha_1, 2]$ the critical temperature β_α (possibly infinite) such that

$$\beta_\alpha = \sup \left\{ \beta \in \mathbb{R}_+,\ \lambda(\alpha\beta) - \alpha\lambda(\beta) < -\ln \rho(\alpha) \right\}.$$

To begin with, as λ is strictly convex and $\lambda(0) = 0$,

$$\beta \mapsto \frac{\lambda(\beta)}{\beta} \text{ is non decreasing.}$$

So, a necessary condition to have uniform integrability criterion (2.7) is $\ln \rho(\alpha) > 0$, which we can write

$$\rho(\alpha) < 1.$$

Proposition 5. *The function $\alpha \mapsto \beta_\alpha$ is defined on $[\alpha_1, 2]$.*
Moreover, when $\alpha = \alpha_1$,

$$\beta_{\alpha_1} = 0.$$

Proof. Going back to the definition,

$$\frac{\lambda(\alpha_1\beta_{\alpha_1})}{\alpha_1\beta_{\alpha_1}} - \frac{\lambda(\beta_{\alpha_1})}{\beta_{\alpha_1}} = 0.$$

Since $\beta \mapsto \frac{\lambda(\beta)}{\beta}$ is non decreasing and $\alpha_1 > 1$,

$$\begin{aligned} \alpha_1\beta_{\alpha_1} &= \beta_1 \\ \beta_1 &= 0. \end{aligned}$$

$\square$

Remark. When $\beta_c < +\infty$ (for example in a gaussian environment cf. [CY06]), $\beta_\alpha < +\infty$ and it satisfies :

$$\lambda(\alpha\beta_\alpha) - \alpha\lambda(\beta_\alpha) = -\ln \rho(\alpha). \tag{3.9}$$

We suppose in the sequence that $\beta_c < +\infty$ and use the previous equation to define β_α.

3.2. **Proof of theorem 2.** We study in this section the behavior of $\alpha \mapsto \beta_\alpha$ when this quantity is finite on $(\alpha_1, 2]$.
Writing

$$\psi(\alpha, \beta) = \lambda(\alpha\beta) - \alpha\lambda(\beta) + \ln \rho(\alpha),$$

the chain rule and the definition (3.9) of β_α give

$$\frac{\partial \beta_\alpha}{\partial \alpha} = -\frac{\partial_\alpha \psi}{\partial_\beta \psi}(\alpha, \beta_\alpha).$$

Observe that since λ is strictly convex, λ' is increasing and for all $\alpha \geq \alpha_1 > 1$,

$$\frac{\partial \psi}{\partial \beta} = \alpha(\lambda'(\alpha\beta) - \lambda'(\beta)) > 0.$$

Thus, the variations of $\alpha \mapsto \beta_\alpha$ depend on the sign of

$$\frac{\partial \psi}{\partial \alpha}(\alpha, \beta_\alpha) = \beta_\alpha \lambda'(\alpha\beta_\alpha) - \lambda(\beta_\alpha) + \frac{1}{2}\sum_{t,x} \frac{\ln p(t,x) p(t,x)^{\alpha/2}}{\sum_{t,x} p(t,x)^{\alpha/2}}.$$

The function $\alpha \mapsto \beta_\alpha$ has a maximum on $(\alpha_1, 2)$ if

$$\left.\frac{\partial \beta_\alpha}{\partial \alpha}\right|_{\alpha=2} < 0,$$

i.e.

$$\beta_2 \lambda'(2\beta_2) - \lambda(\beta_2) + \frac{\rho'(2)}{2\rho(2)} > 0.$$

Thus, using the definition (3.9) of β_2,

$$\underbrace{2\beta_2\lambda'(2\beta_2) - \lambda(2\beta_2)}_{h_{\mathbf{Q}}(2)} \underbrace{- \ln \rho(2) + \frac{\rho'(2)}{\rho(2)}}_{-h_\nu(2)} > 0,$$

as regards to the definitions of the entropies (0.3) and (0.4). We thus obtain the necessary condition

$$h_\nu(2) > h_{\mathbf{Q}}(2).$$

Proof of theorem 2 is thus completed.

Remark. When the environment is Gaussian, using Annex B, this sufficient condition is in fact necessary. We expect this result is still true for the environments introduced in section 4, but we could'nt carry out the computations.

We can define, when it exists, an optimal couple denoted $(\alpha_*, \beta_{\alpha_*})$. It satisfies the relation $\frac{\partial \psi}{\partial x}(\alpha_*, \beta_{\alpha_*}) = 0$, which we can write :

$$\beta_{\alpha_*}\lambda'(\alpha_*\beta_{\alpha_*}) - \lambda(\beta_{\alpha_*}) = -\frac{1}{2}\sum_{t,x} \ln p(t,x) \frac{p(t,x)^{\alpha/2}}{\rho(\alpha)}.$$

Finally, using the definition (3.9) of β_α, we have

$$\alpha_*\beta_{\alpha_*}\lambda'(\alpha_*\beta_{\alpha_*}) - \lambda(\alpha_*\beta_{\alpha_*}) = h_\nu(\alpha_*).$$

Remark. When the polymer lives on a regular tree with $2d$ branches, the function ρ has a maximum in $\alpha_* = \alpha_1 = 1$, and the critical temperature β_c^t satisfies the following relation :

$${\beta_c}^t \lambda'({\beta_c}^t) - \lambda({\beta_c}^t) = \ln(2d).$$

We end this section with a lemma concerning the behavior of $h_{\mathbf{Q}}(2)$ with respect to the dimension d.

Lemma 6. *Let d the dimension where the polymer lives. The function $d \mapsto h_{\mathbf{Q}}(2)$ is non decreasing.*

Proof. To begin with, let's remark (cf. [OS96]) that the function $d \mapsto 1 - q_d$ is non increasing. Thus, $d \mapsto -\ln(1 - q_d)$ is non decreasing.
On the other side, as λ is strictly convex, $\beta \mapsto \lambda(2\beta) - 2\lambda(\beta)$ is non decreasing. Thus, $d \mapsto \beta_2$ is non decreasing.
Finaly, one can check that $\beta \mapsto 2\beta\lambda'(2\beta) - \lambda(2\beta)$ is non decreasing, that allows us to conclude that $d \mapsto h_{\mathbf{Q}}(2)$ is non decreasing. $\square$

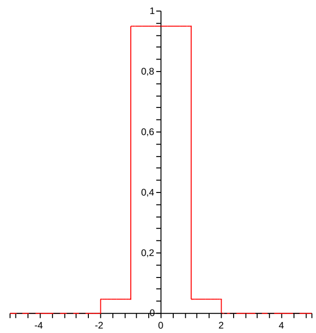

FIGURE 4.2. Repartition function of binomial(1/100,5) law

3.3. **Simulations.** Numerical simulations give us a value for $h_\nu(2)$. To obtain them, we simulate N random walks in dimension d, of length n and look at their first meeting time. We write $\tilde{q}_d$ the numerical approximation of q_d thus obtained.

d	$\tilde{q}_d - q_d$	$h_\nu(2)$	n	N
3	-0.005	4.808	1000	100000
4	-0.003	3.855	1000	50000
5	-0.001	3.608	1000	500000

4. Examples of environments

To illustrate our theoretical results, we present here some environments. Numerical simulations show that the behavior of $\alpha \mapsto \beta_\alpha$ is closely related to the law of the environment. Moreover, when d is greater than 3, there exists an environment which satisfies $\beta_c > \beta_2$.
Binomial environment described below doesn't have a cumulant defined on the whole real line. However, we can do the same reasonning on their definition interval, that's what we do in the following.
We can also verify, using the upper bond

$$\beta_c \leq \inf \left\{ \beta \in \mathbb{R}_+,\ \ln(2d) < \beta\lambda'(\beta) - \lambda(\beta) \right\},$$

that in the following examples, $\beta_c < +\infty$.

4.1. **A *binomial* environment.** We consider in this section a symmetric environment built from a binomial law $\mathcal{B}(n,p)$. Thus, we suppose

$$\mathbf{Q} = (1-p)^n \delta_0 + \frac{1}{2} \sum_{j \in \mathbb{Z}_n^\star} C_n^{|j|} p^{|j|} (1-p)^{n-|j|} \delta_j,$$

where δ is the Dirac-measure and $\mathbb{Z}_n = \{-n, \dots, n\}$.
We thus have :

$$\lambda(\beta) = \ln\left[(pe^\beta + 1 - p)^n + (pe^{-\beta} + 1 - p)^n \right] - \ln(2).$$

As we cannot inverse easily the equations established in the previous section, we solve them numerically.
Let $p = 1/100$, $n = 5$. The law of the environment is drawn in figure 4.1.

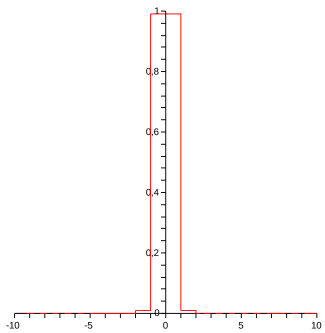

FIGURE 4.3. Repartition function of the Poisson (0.0001) law

We can finally establish the following tabular :

d	$h_{\mathbf{Q}}(2)$	$h_\nu(2)$	$\beta_{\alpha_*} > \beta_2$
3	4.140	4.8	*no*
4	6.228	3.8	*yes*
5	7.421	3.6	*yes*

Remark.
- Even if this environment is bounded, the behavior of the polymer is different from the behavior of a polymer in a Bernoulli environment for which $\beta_2 = +\infty$.
- We can also notice that the behavior of the polymer is related to parameter p. Indeed, if $p = 1/2$ and $d = 5$, then $\beta_2 = +\infty$.

4.2. A *Poisson* environment.

Let

$$\mathbf{Q} = e^{-k}\delta_0 + \frac{1}{2}\sum_{j\in\mathbb{Z}^\star} e^{-k}\frac{k^{|j|}}{|j|!}\delta_j .$$

We have,

$$\lambda(\beta) \quad = \quad \ln\frac{e^{-k}}{2} + \ln\left\{e^{ke^{\beta}} + e^{ke^{-\beta}}\right\}.$$

We consider $k = 0.0001$ (cf. figure 4.2).

Numerical simulations show that the uniform integrability criterion (3.9) is better than the second moment method whatever the dimension is.

d	$h_{\mathbf{Q}}(2)$	$h_\nu(2)$	$\beta_{\alpha_*} > \beta_2$
3	6.418	4.8	*yes*
4	10.295	3.8	*yes*
5	12.726	3.6	*yes*

4.3. **The Gaussian environment.** When the environment is gaussian $\mathcal{N}(0,\sigma^2)$, all the previous equations are solvable. Indeed,

$$\lambda(\beta) = \frac{\beta^2}{2}.$$

We thus obtain

$$\beta_\alpha^2 = \frac{-2\ln\rho(\alpha)}{\alpha(\alpha-1)},$$

and we can then establish :

d	$h_{\mathbf{Q}}(2)$	$h_\nu(2)$	$\beta_{\alpha_*} > \beta_2$
3	2.158	4.8	*no*
4	3.290	3.8	*no*
5	4.004	3.6	*yes*

Remark. For the previous environments, changing the parameter of the law was influencing the behavior of β_{α_*}. In particular, the more the environment was concentrated around 0, the greater $h_{\mathbf{Q}}(2)$ was. On the contrary, for gaussian environments, the entropy $h_{\mathbf{Q}}(2)$ doesn't depend on the variance.

Appendix A. Domain of definition of ρ

We look for the domain of definition of ρ. More precisely, we recall the definitions (2.8) and (0.1) :

$$\alpha_0 = \inf\left\{\alpha\in(1,2];\rho(\alpha)<+\infty\right\},$$
$$p(t,x) = \mathbf{P}_0\left(\omega_j^1\neq\omega_j^2, j<t, \omega_t^1=\omega_t^2=x\right).$$

We write in the following $p_{t,x} := p(t,x)$ and show the following proposition.

Proposition 7. *For all d greater than 3,*

$$\frac{4}{d} \le \alpha_0^{(d)} \le 1+\frac{2}{d}.$$

Understanding how the quantities $p_{t,x}$ behave when t is large is not so easy. Indeed, it seems that both conditions "be in x at time t" and "$\omega_j^1\neq\omega_j^2,\ j<t$" are important. Probably, these quantities are related to the behavior of self-avoiding random walks. We present here the best approximation we were able to establish on α_0.

A.1. **The lower bound.** To establish the lower bound, we use the inequality (1.6). Indeed, recall that

$$\rho(\alpha) = \sum_{t\in\mathbb{N}}\sum_{x\in\mathbb{Z}^d} p_{t,x}^{\alpha/2}.$$

But $\alpha\in(1,2]$, so that

$$\begin{aligned}\rho(\alpha) &\ge \sum_{t\in\mathbb{N}}\left(\sum_{x\in\mathbb{Z}^d} p_{t,x}\right)^{\alpha/2}\\ &= \sum_{t\in\mathbb{N}}\mathbf{P}\left(\omega_t^1=\omega_t^2\right)^{\alpha/2}\\ &= \sum_{t\in\mathbb{N}}\mathbf{P}(\omega_{2t}=0)^{\alpha/2},\end{aligned}$$

where ω denotes a simple random walk starting from 0 in $\mathbb{Z}^d$.
Finally, recall that when $d\ge 3$,

$$\mathbf{P}(\omega_{2t}=0)\sim_{t\to\infty} t^{-d/2}.$$

So as soon as $\alpha \leq \frac{4}{d}$, we have $\rho(\alpha) = +\infty$ and

$$\alpha_0 \geq \frac{4}{d}.$$

A.2. **The upper bound.** To establish the upper bound, we suppress the avoiding condition in the definition of $p_{t,x}$. Indeed, let

$$r_{t,x} = \mathbf{P}\left(\omega_t^1 = x\right).$$

We have

$$p_{t,x} \leq r_{t,x}^2.$$

Proving the following lemma will complete the proof of the upper bound on α_0.

Lemma 8. *Let $\bar\rho(\alpha) = \sum_{t,x} r_{t,x}^\alpha$.*
$\bar\rho(\alpha)$ is finite if and only if $\alpha \leq 1 + 2/d$.

Proof. To obtain this result, we use the central limit theorem (cf. Lemma 1.2.1 in [Law96]) :

$$\left|\mathbf{P}(\omega_t = x) - \bar r_{t,x}\right| = O_{t\to\infty}\left(\frac{1}{t^{1+d/2}}\right),$$

where $\bar r_{t,x} := 2\left(\frac{d}{2\pi t}\right)^{d/2} e^{-\frac{d\|x\|^2}{2t}}$.
Thus, uniformly in x,

$$r_{t,x} \sim_{t\to\infty} \bar r_{t,x}. \tag{1.10}$$

Moreover, we remark that $\bar r_{t,x}$ depends only on the norm of x. We write

$$N(\sqrt{R}) = \sharp\left\{x \in \mathbb{Z}^d,\ R \leq x \leq R+1\right\}.$$

$\bar\rho$ converges if and only if

$$\sum_{t=0}^{\infty} \frac{1}{t^{\alpha d/2}} \sum_{R=0}^{t} N(\sqrt{R}) e^{-\frac{\alpha d R}{2t}} \ \textit{converges}.$$

Using lemma 9 below, the last serie has the same behavior as

$$\sum_{t=0}^{\infty} \frac{1}{t^{\alpha d/2}} \sum_{R=0}^{t} R^{d/2-1} e^{-\frac{\alpha d R}{2t}}.$$

We use the serie-integral comparison theorem to obtain that this serie behaves like

$$\sum_{t=0}^{\infty} \frac{1}{t^{\alpha d/2}} \int_0^t r^{d/2-1} e^{-\frac{\alpha d r}{2t}}\ dr,$$

that can be written

$$\sum_{t=0}^{\infty} \frac{t^{d/2}}{t^{\alpha d/2}} \int_0^1 u^{d/2-1} e^{-\alpha u/2}\ du.$$

Finally we have the equivalence

- $\bar\rho(\alpha) < +\infty$,
- $\alpha \geq 1 + \frac{2}{d}$.

and the lemma is shown. □

We conclude this section with the following lemma.

Lemma 9. *Using the previous notations,*

$$N(\sqrt{R}) \sim_{R\to\infty} C_d R^{d/2-1}.$$

Proof. We just use the equivalent

$$\sharp\left\{\mathbb{Z}^d \cap B(0,R)\right\} \sim_{R\to\infty} C_d R^d,$$

and a Taylor development of order 1. $\square$

Appendix B. Variations of $\alpha \mapsto \beta_\alpha$: the Gaussian case

Proposition 10. *When the environment is Gaussian, if there exists α_* such that $\alpha \mapsto \beta_\alpha$ is increasing on $[\alpha_1, \alpha_*)$ and decreasing on $(\alpha_*, 2]$, then it is unique.*

When the environment g is gaussian, we can inverse the uniform integrability condition (3.9). Indeed, the cumulant is

$$\lambda(\beta) = \frac{\beta^2}{2}.$$

Uniform integrability condition allows us to define :

$$\frac{\beta_\alpha^2}{2} = -\frac{\ln \rho(\alpha)}{\alpha(\alpha-1)}.$$

Proof. The variations of β_α are the same as the ones of $\beta_\alpha^2/2$ and we can notice that

$$\begin{aligned}
\partial_\alpha\left(\frac{\beta_\alpha^2}{2}\right) &= -\frac{\sum_{t,x} \ln p^{1/2}(t,x)\frac{p^{\alpha/2}(t,x)}{\rho(\alpha)}}{\alpha(\alpha-1)} + \frac{2\alpha-1}{\alpha^2(\alpha-1)^2}\ln\rho(\alpha) \\
&= \frac{1}{\alpha^2(\alpha-1)^2}\left\{(2\alpha-1)\ln\rho(\alpha) - \alpha(\alpha-1)\sum_{t,x}\ln p^{1/2}\frac{p^{\alpha/2}}{\rho(\alpha)}\right\} \\
&=: \frac{\Psi(\alpha)}{\alpha^2(\alpha-1)^2}.
\end{aligned}$$

Therefore we have to study the sign of Ψ, and to do so, we study its variations

$$\begin{aligned}
\partial_\alpha \Psi(\alpha) &= 2\ln\rho(\alpha) + (2\alpha-1)\sum_{t,x}\ln p^{1/2}\frac{p^{\alpha/2}}{\rho(\alpha)} - (2\alpha-1)\sum_{t,x}\ln p^{1/2}\frac{p^{\alpha/2}}{\rho(\alpha)} \\
&\qquad -\alpha(\alpha-1)\sum_{t,x}(\ln p^{1/2})^2\frac{p^{\alpha/2}}{\rho(\alpha)} \\
&\qquad +\alpha(\alpha-1)\left(\sum_{t,x}\ln p^{1/2}\frac{p^{\alpha/2}}{\rho(\alpha)}\right)^2 \\
&= 2\ln\rho(\alpha) - \alpha(\alpha-1)Var_{\nu_\alpha}\left(\ln p^{1/2}\right) \\
&\le 0,
\end{aligned}$$

where we wrote ν_α the measure on $\mathbb{N}\times\mathbb{Z}^d$ defined by $\nu_\alpha(t,x) = \frac{p(t,x)^{\alpha/2}}{\rho(\alpha)}$. Thus, as $\Psi(\alpha_1) > 0$, there exists $\alpha_* \in \overline{\mathbb{R}}$ such that $\alpha \mapsto \beta_\alpha$ is increasing on $[\alpha_1, \alpha_*)$ and decreasing on $(\alpha_*, +\infty)$. $\square$

Appendix C. Program

We give here the matlab program used to compute $p(t,x)$. The result is a row vector. This vector doesn't contain the information of which t and which x are considered.

```
%P(d,n,N) 1<i<n
%d : dimension
%n : maximal length of the r.w.
%N : number of experiments

%Initialising the counter
%colomn : [coord of the meeting point; moment; counter]
R=zeros(1,d+1);

for i=1:N
    %Constructing the two r.w.
    x=2*floor(2*rand(1,2*n))-1;   %direction of the jump
    t=floor(d*rand(1,2*n))+1;     %jumping coord.

    %number of the jumping coordinate increments
    t=t+[0:d:d*(2*n-1)];

    %Increments matrix
    xi=zeros(2*d,n);
    xi(t)=x;

    X=cumsum(xi,2);

    clear xi;clear x;clear t;

    %row vector : 1 if the r.w. are at the same point, 0 otherwise
    z=prod(double(X(1:d,:)==X(d+1:2*d,:)));
    X=X(1:d,:);
    %X : [coord time]
    X=[(1:n)' X'];

    if z==zeros(1,length(z))
        ;
    else
        I=find(z');        %meeting times
        k=I(1);
        X=X(k,:);           %meeting points
        R=[R;X];
    end
    clear X; clear z;clear k;
end
%suppress the first row
[r1 r2]=size(R);
R=R(2:r1,:);
r1=r1-1;

%order the rows in alphabetic order
R=sortrows(R);
%1 if 2 successive rows are equal in R, 0 otherwise
z=prod(double(R==[zeros(1,d+1);R(1:r1-1,:)]),2);
I=find(z==zeros(length(z),1));
clear z;
```

```
R=R(I,:);
i=length(I);

%number of r.w. meeting in each site
c=[I(2:i);r1+1]-I;

y=c/N;
```

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