WEAK CONTROLLABILITY AND INVARIANT MEASURES

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ABSTRACT. This paper is devoted to the study of differential systems driven by degenerate noise. More particularly, we are interested in the notions of weak controllability on the one side and uniqueness of invariant measures on the other side. We provide conditions to prove each of these properties when the semigroup describing the flow is smooth. The notion of weak controlability introduced by M. Hairer ([7]) is used to show uniqueness of invariant measures for degenerate diffusions. We propose different applications to illustrate how this method works. We will also present a more dynamical proof of uniqueness of invariant measures for Hamiltonian systems using LaSalle's principle.

In this article, we present some results on degenerate differential equation systems. These results are of both kinds : controllability on the one side, uniqueness of invariant probability measures on the other. We are going to show how these notions are linked. More precisely, we are going to show how the knowledge of an invariant measure entails controllability in a weak sense (theorem 1 and corollary 2 below). On the other side, we will show how weak controllability entails the uniqueness of the invariant measure, assuming its existence, in theorem 3.

This article is divided in five parts. We begin with an introductory section, devoted to describe our main results and end by a discussion about what was previously known in control and invariant measure theories. The two following sections will be devoted to prove the two main statements. Thanks to these new theorems, we will develop some applications. We end this paper with another proof of uniqueness of invariant measures based on the dynamics of the process.

DEFINITIONS AND MAIN RESULTS

All diffusions considered in the following will be degenerate in the sense that control or Brownian motion only acts on some components of the dynamical system. Thus, we will decompose the state space $\mathbb{R}^n$ as follows :

$$\mathbb{R}^n = \mathbb{R}^{n-m} \oplus \mathbb{R}^m =: E \oplus E^\perp,$$

Henceforth, we will write for any $z \in \mathbb{R}^n$, $z = (x, y) \in E \times E^\perp$. To simplify the notations, for $f : \mathbb{R}^n \to E$ and $g : \mathbb{R}^n \to E^\perp$, we will write $f + g = \begin{bmatrix} f \\ g \end{bmatrix}$.

For dynamical systems, the space $E^\perp$ denotes the coordinates on which either the control or the Brownian motion acts.

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Weak controllability. First, we will study weak controllability. Let us consider the following deterministic system :

$$\text{(S)} \qquad \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x_t, y_t) \\ \dot{y}(t) & = & u(t), \end{array} \right.$$

where f is a smooth function and u is a continuous function called the *control.*

Definition 1 (Weak Controllability)**.** If for any starting point z_0 and any finishing open set A, there exists a control u and a time $T = T_{z,A}$ such that the solution (x_t, y_t) of system (S) satisfies

$$(x_0, y_0) = z_0, \qquad (x_T, y_T) \in A,$$

we will say that this system is *weakly controllable* (see figure 1).

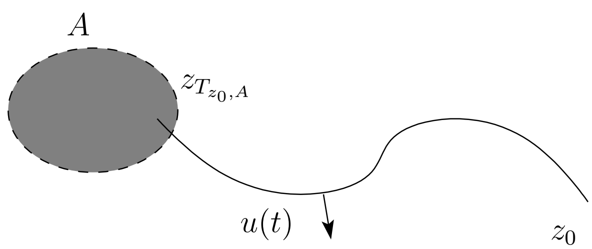

FIGURE 1. Illustration of the Weak Controllability property. We emphasized the dependence of the control time on both starting point and finishing open set.

The controllability theorem is the following. We refer the reader looking for more precise definitions to section 1.

Theorem 1 (Weak controllability)**.** *Assume there exists a smooth function* $g : \mathbb{R}^n \to E^\perp$ *such that* $div(f + g) = 0$*. Assume moreover that*

(1) *there exists a smooth function* H *with negative exponential moments such that*
$$\langle f + g, \nabla H \rangle = 0,$$
(2) *there exists a Lyapunov function* W *and a constant* $c > 0$ *such that*
$$\mathcal{L}W \leq cW,$$
where $\mathcal{L} = f \cdot \nabla_x + g \cdot \nabla_y - \nabla_y H \cdot \nabla_y + \Delta_y$
(3) *the Lie algebra* $\mathfrak{L}$ *generated by the generator* $\mathcal{L}$ *has full rank.*

Then the deterministic system (S) *is weakly controllable.*

Formally, to prove this weak controllability theorem, we are going to use an associated stochastic diffusion. Assumption (2) ensures the non-explosion of the diffusion, the property of function H implies the existence of a unique invariant measure and hypoellipticity ensures its regularity.

For example, one can deduce the following useful corollary.

Corollary 2. *For any confining, non-degenerate smooth function U having negative exponential moments and satisfying $\Delta_y U - (\partial_y U)^2 \leq CU$ for a constant $C > 0$, the system*

$$\begin{cases} \dot{x}_t &= \partial_y U(x,y) \\ \dot{y}_t &= u(t) \end{cases}$$

is weakly controllable.

The proof of this corollary is given in section 2.3. For example, this corollary includes odd degree polynomial control systems introduced by [11], but also a more wide class of systems where function f can be decomposed as an odd degree polynomial times a perturbation as $f = P(1 + \gamma \sin y)$ or $f = P(1 + e^{-\gamma y})$, for $\gamma \in \mathbb{R}_+$. We will study these examples in section 2.3.

Uniqueness of invariant measures. Second, we will use this controllability theorem to prove a result on the uniqueness of invariant measures for stochastic differential equations. Let us consider the stochastic differential equation

$$(\mathcal{S}) \qquad \begin{cases} dX_t &= f(X_t, Y_t)\ dt \\ dY_t &= g(X_t, Y_t)\ dt + \sqrt{2}\sigma(X_t, Y_t) dB_t, \end{cases}$$

where σ is an invertible matrix and $(B_t)_{t \geq 0}$ is a multidimensional Brownian motion. We will denote $Z_t = (X_t, Y_t)$ the solution of this stochastic differential equation.

Theorem 3 (Uniqueness of invariant measures)**.** *Assume the diffusion associated to* $(\mathcal{S})$ *does not explode, the Lie algebra associated to its generator has full rank and the deterministic system* (S) *is weakly controllable. Then, if there exists an invariant measure for the diffusion, it is unique.*

The idea of the proof is to reduce the stochastic differential equation $(\mathcal{S})$ to another stochastic differential equation $(\mathcal{S}^1)$ which has an invariant probability measure. Then we show that weak controllability for the control problem associated (S^1) implies weak controllability for (S). Finally, weak controllability for (S) implies uniqueness of the invariant measure for $(\mathcal{S})$.

Discussion.

- First notice that the controllability theorem 1 shows that for a large class of differential systems hypoellipticity entails weak controllability. Usually, hypoellipticity only entails reachability. Indeed, let us recall some background in control theory.

Definition 2 (see [19] p.511-516)**.** We consider the deterministic differential system (S).

- (S) is said *controllable* if for any two points $z_0,\ z_1 \in \mathbb{R}^n$, there exist a time T and a control such that
$$(x_0, y_0) = z_0,\ (x_T, y_T) = z_1.$$
- (S) is said *strongly controllable* if there exists a time T such that for any two points $z_0,\ z_1 \in \mathbb{R}^n$, there exists a control such that
$$(x_0, y_0) = z_0,\ (x_T, y_T) = z_1.$$

- The *reachable set* from z_0 in time T is defined by

$$R(z_0, \leq T) = \{z \in \mathbb{R}^n;\ \exists s \leq T \text{ and a control such that} \\ (x_0, y_0) = z_0,\ (x_s, y_s) = z\}$$

The controllability theorem (see [19] Theorem 11.4) says that if the Lie algebra has full rank in a neighborhood of z_0, then for any $T > 0$ the reachable set $R(z_0, \leq T)$ has nonempty interior. On the other side, if the deterministic system is strongly controllable (cf. [17]), then the diffusion is irreducible. The goal in this article is to use probability theory to show that, with our assumptions on the deterministic system, one can obtain *weak controllability*.
The difficulty is that we try to control a system with a drift that steers the system in a fixed subspace E. We show in this article that, if the Lie algebra has full rank and the system is approximately Hamiltonian, the noise allows one to visit every state position. We can then go in every direction. However, we cannot control the time spent to go from one point to the other.
We emphasize that in this article, the diffusions we consider have not to be strongly controllable or irreducible (we refer the reader interested in this case to [17]). Here, we just have a weak notion of irreducibility. The example given by M. Hairer in [7], shows that one may have weak controllability without strong controllability.

- We also notice that when the system is linear, there exits a necessary and sufficient condition to prove controllability in terms of Lie algebra. Ph. Carmona made use of it in [2] (see also references herein) in the Heat Conduction Networks setting.
- We will see that this weak controllability method provides more information on the behavior of the systems. For example, when we consider the stochastic damped Hamiltonian system in section 4.2 (see also [21]), we can prove its topological transitivity (see definition 11).

- The condition of stochastic differential equations on non-explosion $\mathcal{L}W \leq cW$ for a suitable Lyapunov function W is quite standard. We will see that it is quite easy to verify in the applications.
This article doesn't deal with existence of invariant measures. The reader interested in such results may read for the applications we consider [21], [18], [17], [2]. The condition usually needed is to find a Lyapunov function with $\sup_t P_t W < +\infty$. However, the method used is based on generator estimates such as $\mathcal{L}W \leq cW + b\mathbb{1}_K$ for a compact K, that we couldn't satisfy in our settings.
- To prove uniqueness we propose two ways, both using regularity. Indeed, regularity implies that ergodic invariant measures have disjoint supports. We first develop a method, inspired by M. Hairer [7], based on the existence of an invariant measure with full support. Second, using LaSalle's principle, we find an attractive point that belongs to the support of every invariant ergodic measure.
- All this article is written using the Strong Feller assumption (Definition 4). However, to prove our systems satisfy this assumption, we use Hörmander's condition (Theorem 5). For our applications this means we have to assume our potentials are non-degenerate (for example, they cannot be locally constant). Thus, we cannot reach the level of generality obtained in [1]. The following Asymptotically Strong Feller property, introduced by M. Hairer and J. Mattingly to study Stochastic Navier-Stokes equations, is more related to the path behavior of the dynamical

systems. We think that it is more appropriate to study potential degenerated diffusions. We will emphasize at the end of each section which results can be generalized using the Asymptotically Strong Feller property.

Definition 3 (Asymptotically Strong Feller)**.** (see [8] Definition 3.8) A Markov transition semigroup $(P_t)_t$ is called *Asymptotically Strong Feller* at x if there exists a totally separating system of pseudo-metrics $(d_n)_n$ and a sequence $t_n > 0$ such that

$$\inf_{\gamma\to 0}\limsup_{n\to\infty}\sup_{y\in\mathcal{B}(x,\gamma)}\|P_{t_n}(x,\cdot)-P_{t_n}(y,\cdot)\|_{d_n}=0,$$

where a totally separating system means that for all $x \neq y$, $d_n(x,y)\uparrow 1$.

We recall here a way to prove that a semigroup is Asymptotically Strong Feller.

Proposition 1. *(see [8] Proposition 3.11) Assume there exist two positive sequences $(t_n)_n$ and $(\delta_n)_n$ such that $(t_n)_n$ is nondecreasing and $\delta_n \to 0$. If for all $\phi : \mathbb{R}^n \to \mathbb{R}$ with $\|\phi\|_\infty$ and $\|\nabla\phi\|_\infty$ finite,*

$$|\nabla \mathbf{P}_{t_n}\phi(z)| \le C(\|z\|)\left(\|\phi\|_\infty + \delta_n\|\nabla\phi\|_\infty\right),$$

for a nondecreasing function C, then the semigroup is Asymptotically Strong Feller at z, with $d_n(x,y) = 1 \wedge |x-y|/\delta_n$.

1. Notations and first result

For sake of completeness, we recall some general properties of diffusion processes (see [16] for example).
In this section, we begin by giving some classical notations and results about diffusions and hypoellipticity. Then, we recall a previous result obtained by M. Hairer in [7] : an invariant measure with full support is unique. We end this section with an application to Hamiltonian systems that illustrates the previous results.

1.1. Some results on semigroups.

In this section, we recall some general properties of diffusions.

The semigroup. Let us denote $(P_t)_{t\ge 0}$ the semigroup of the diffusion $(Z_t)_{t\ge 0}$, i.e. for any measurable function ϕ,

$$P_t\phi(z) = \mathbf{E}_z\left[\phi(Z_t)\right].$$

We will usually consider the semigroup as a distribution acting on the functions via the formula

$$P_t\phi(z) = \int \phi(y)\ P_t(z,dy).$$

When the function is an indicator function, we will denote for A Borelian,

$$P_t(z,A) = P_t\mathbb{1}_A(z).$$

The results we will present are based on regularity properties.

Definition 4 (Strong Feller)**.** A semigroup $(P_t)_{t\ge 0}$ is said to be *strong Feller* if for any bounded measurable function f and $t > 0$, the function $z \mapsto P_t f(z)$ is continuous.

The formal adjoint of the semigroup $(P_t)_{t\geq 0}$ acts on probability measures via the following formula. Let μ be a probability measure and A a Borelian set,

$$P_t^\star \mu(A) = \int P_t(z, A)\ \mu(dz).$$

Definition 5 (Invariant measure)**.** μ is said to be *invariant* for the diffusion if

$$P_t^\star \mu = \mu,\ \forall\ t \geq 0.$$

The generator. We will denote by $\mathcal{L}$ the generator of the diffusion $(Z_t)_{t\geq 0}$ and $\mathcal{D}_\mathcal{L}$ its domain. Let us recall that stochastic diffusion system $(\mathcal{S})$ is associated to the second order differential operator

$$\mathcal{L} = f \cdot \nabla_x + g \cdot \nabla_y + \sigma\sigma^\star \Delta_y, \tag{2}$$

where $\sigma^\star$ denotes the conjugate of σ, ∇_x (resp. ∇_y) stands for the gradient restricted to the vectorial subspace E (resp. $E^\perp$).
To the generator $\mathcal{L}$ is associated the formal adjoint $\mathcal{L}^\star$ defined by

$$\mathcal{L}^\star = -\nabla_x f - f \cdot \nabla_x - \nabla_y g - g \cdot \nabla_y + \sigma\sigma^\star \Delta_y. \tag{3}$$

We assumed here that σ is constant. However, every further result could be generalized to smooth functions σ.

The Fokker-Planck equation. Semi-group and generator are linked via the Fokker-Planck equation. For any function $f \in \mathcal{D}_\mathcal{L}$,

$$\partial_t P_t f = \mathcal{L} P_t f = P_t \mathcal{L} f. \tag{FP}$$

1.2. **Non explosion of SDE's.** To show well-posedness of the diffusion, we will use the following characterisation.

Definition 6 (Lyapunov function)**.** A function W on $\mathbb{R}^n$ is called a *Lyapunov function* if

(1) for any $z \in \mathbb{R}^n$, $W(z) \geq 1$,
(2) W is confining, i.e.

$$\lim_{|z|\to\infty} W(z) = +\infty.$$

Theorem 4 (Non-explosion, cf. [17] Theorem 5.9 or [14] Theorem 2.1)**.** *If there exists a smooth Lyapunov function W and a constant $C > 0$ such that*

$$\mathcal{L}W \leq CW,$$

then the diffusion is defined for all times and satisfies

$$P_t W(x) \leq W(x) e^{ct}.$$

Moreover, the domain $\mathcal{D}_\mathcal{L}$ contains all $\mathcal{C}^2$ functions dominated by W.

In the mechanical systems we consider, we will usually use a function of the Hamiltonian as a Lyapunov function.

Remark 1. Since a Lyapunov function is greater than 1, it is enough to check that

$$\mathcal{L}W \leq c.$$

We will use this fact in section 4.1

1.3. **Existence of invariant measure.** Using the previous definitions, we have the following proposition.

Proposition 2. *A measure μ is invariant if and only if*

$$\mathcal{L}^\star \mu = 0.$$

We will use this property to show the existence of explicit invariant measures for Hamiltonian diffusions. We will look for invariant measures of Gibbs type $\mu_H(z) = e^{-H(z)}/Z\ dz$, where $H : \mathbb{R}^n \to \mathbb{R}_+$ is a smooth function standing for the energy of the system. We will check that function $z \mapsto e^{-H(z)}$ is integrable and denote Z the normalizing constant.

Remark 2. The fact that μ_H is a probability measure is quite important in the following since we will use ergodic properties of probability measures.

The following example will be used to carry out the proofs of weak controllability and uniqueness results.

Exemple 1 (Fundamental Example). For H a smooth function with negative exponential moments, we consider the following Hamiltonian stochastic system,

$$(\mathcal{H}) \qquad \left\{ \begin{array}{rcl} dX_t & = & \nabla_y H\ dt \\ dY_t & = & -\nabla_x H\ dt - \nabla_y H\ dt + \sqrt{2} dB_t. \end{array} \right.$$

We have

$$\mathcal{L}^\star = -\nabla_y H \cdot \nabla_x + \nabla_x H \cdot \nabla_y + \nabla_y H \cdot \nabla_y + \Delta_y H + \Delta_y.$$

Thus, if we consider the *probability measure*

$$\mu(dx, dy) = \frac{e^{-H}}{Z}\ dx\ dy,$$

where Z is the normalizing constant, we obtain

$$\begin{aligned} \frac{\mathcal{L}^\star \mu}{\mu} &= \nabla_y H \cdot \nabla_x H - \nabla_x H \cdot \nabla_y H - |\nabla_y H|^2 + \Delta_y H - \Delta_y H + |\nabla_y H|^2 \\ &= 0, \end{aligned}$$

and μ is an invariant probability measure.

1.4. **Hypoellipticity and regularity.** To show hypoellipticity of the operators, we will use Lie algebras. Let us decompose any second order operator P :

$$\text{(P)} \qquad P = c\ Id + X_0 + \sum_{i=1}^{r} X_i^2,$$

where $(X_i, 0 \le i \le r)$ are smooth vector fields and c is a smooth function.

Definition 7 (Lie algebra)**.** The Lie algebra $\mathfrak{L}$ generated by the vector fields $(X_i, 0 \le i \le r)$ is the algebra containing $(X_i, 0 \le i \le r)$ stable by Lie bracket, i.e.

$$\mathfrak{L} = \left\{ X_i,\ 0 \le i \le r,\ [Y_1, Y_2],\ Y_1,\ Y_2 \in \mathfrak{L} \right\}.$$

We will use the following theorem.

Theorem 5 (Hörmander's condition, Theorem 22.2.1. [10] p. 353)**.** *If for any point $z \in \mathbb{R}^n$, the Lie algebra $\mathfrak{L}(z)$ has full rank n, then the operator P is hypoelliptic. We will call this condition the* Hörmander's condition.

Remark 3. The way $\mathcal{L}$ and $\mathcal{L}^\star$, defined in (2) and (3), are linked allows us to say that $\mathcal{L}$ satisfies Hörmander's condition if and only if $\mathcal{L}^\star$ does.

The following remark explains how hypoellipticity and regularity are linked.

Remark 4 (Hypoellipticity and regularity). Let us recall some definitions about hypoellipticity.
A distribution v is said to be $\mathcal{C}^\infty$ if there exists a smooth function φ such that for any smooth function ϕ,

$$v(\phi) = \int \varphi(x)\phi(x)\ dx.$$

An operator P is said to be hypoelliptic (cf. [9] p. 110) if for any distribution v,

$$sing\ supp\ v = sing\ supp\ Pv,$$

where the singular support (cf. [9] p. 42) of v is the set of points $z \in \mathbb{R}^n$ such that for any open set $A \ni z$, v restricted to A is smooth.

In particular, we note that when P is hypoelliptic, the solution of

$$Pv = 0$$

is smooth.

Proposition 3. *If the generator $\partial_t + \mathcal{L}$ is hypoelliptic, then the semigroup $(P_t)_{t\geq 0}$ has a smooth density with respect to Lebesgue's measure.*
If the formal adjoint $\mathcal{L}^\star$ of the generator is hypoelliptic, then any invariant measure μ has a smooth density with respect to Lebesgue's measure.

Proof. We just stress that the Fokker-Planck equations (FP) ensure that

$$\begin{aligned} (\partial_t - \mathcal{L})\, P_t(x, dy) &= 0 \\ \mathcal{L}^\star \mu &= 0 \end{aligned}$$

The conclusion follows directly from the definition of hypoellipticity. □

Corollary 6. *If the generator is hypoelliptic, then the semigroup $(P_t)_{t\geq 0}$ is* strong Feller*.*

1.5. **Support of ergodic measures.** In this section, we show under regularity conditions that if there exists an invariant measure of full support, then this invariant measure is unique and ergodic (see [7]).

Theorem 7 (Full support theorem)**.** *If there exists a full support measure μ_H invariant under an operator $\mathcal{L}$ with full rank Lie algebra, then μ_H is the* unique *invariant measure. Moreover, this measure is* ergodic*.*

Remark 5. Hypoellipticity will be used to have strong Feller semigroups *and* to use the density of invariant measures with respect to Lebesgue's measure.

We recall here the results presented by G. Da Prato in [4].

Proposition 4 (Convex hull, see [3] Proposition 3.2.7 p.29). *The set of invariant measures with respect to a stochastically continuous semigroup is the convex hull of the set of invariant ergodic measures.*

For any $z \in \mathbb{R}^n$, $\epsilon > 0$, $\mathcal{B}(z, \epsilon)$ denotes the open ball centred in z of radius ϵ.

Definition 8 (Support of measures). For any measure μ, the *support* of μ is the set

$$Supp\ \mu = \left\{ z \in \mathbb{R}^n;\ \forall \epsilon > 0,\ \mu\left(\mathcal{B}(z, \epsilon)\right) > 0 \right\}.$$

Remark 6. The support of a measure is a closed set.

To prove the theorem, we are going to split $\mathbb{R}^n$ into supports of ergodic measures. We first notice that these supports are non-empty and disjoints.

Proposition 5 (cf. [4], Proposition 7.8 p.97). *Let μ, ν be two distinct invariant ergodic measures under the same* strong Feller *semigroup. Then their supports are disjoint, i.e.*

$$Supp\ \mu \bigcap Supp\ \nu = \emptyset.$$

Proposition 6. *Assume that the adjoint diffusion generator is hypoelliptic. Then, for any invariant measure μ,*

$$\overset{\circ}{Supp}\ \mu \neq \emptyset.$$

Proof. Let μ be an invariant measure. Using the definition 4 of hypoellipticity and the link between $\mathcal{L}^\star$ and μ (see Proposition 2), μ has a smooth density ρ_μ with respect to Lebesgue's measure. Then, we can write

$$Supp\ \mu = \overline{\{\rho_\mu > 0\}}.$$

Thus, as soon as $\mu \neq 0$, there exists $x_0 \in Supp\ \mu$ such that $\rho(x_0) > 0$. Since ρ is continuous, ρ is strictly positive on a neighbourhood of x_0 and the proposition is proved. □

We are now in a position to prove the main theorem of this section.

Proof of the uniqueness Theorem 7. First let us consider ergodic invariant measures. Observe we can link any ball included in $\mathbb{R}^n$ to a rational number. The preceding proposition 6 says there is an open ball in each support of any invariant measure. Moreover, proposition 5 ensures that these supports are disjoint. Thus, the set of invariant ergodic measures is countable. We will denote it $(\mu_i)_{i\in D}$, where D is a subset of $\mathbb{N}$.

Since μ_H is an invariant measure, using Choquet's theorem (see [15] p.19) and proposition 4, it can be written as a convex combination of ergodic measures. There exists a family of non-negative real numbers $(p_i)_{i\in D}$ with $\sum_i p_i = 1$ and

$$\mu_H = \sum_{i \in D} p_i \mu_i.$$

We can thus write the support of μ_H as the disjoint union

$$Supp\ \mu_H = \bigsqcup_{i;\ p_i \neq 0} Supp\ \mu_i.$$

If in the preceding decomposition we have at least two distinct ergodic measures, then the sets *Supp* μ_i are disjoints closed sets and *Supp* μ_H is not connected. But we have supposed that μ_H has full support, so *Supp* $\mu_H = \mathbb{R}^n$ and

$$\mu_H = \mu_{i_0} \text{ is ergodic.}$$

Since μ_H has full support $\mathbb{R}^n$ and all ergodic measures have disjoints supports, μ_H is the *unique* ergodic measure. □

Remark. First notice that, according to [8] Theorem 3.16, the disjoint support property 5 is still true in the Asymptotically Strong Feller setting. So, to check that we can weaken the hypothesis in the previous theorem 3, it remains to show that ergodic measures are countable. The strategy in the Asymptotically Strong Feller setting is quite different. We are going to show that for every $z \in \mathbb{R}^n$, there exists $\delta_z > 0$ such that at most one ergodic measure μ satisfies

$$Supp\ \mu \cap \mathcal{B}(z, \delta_z) \neq \emptyset.$$

Proposition 7. *Let μ be an invariant ergodic measures with respect to an ASF semigroup, $z \in Supp\ \mu$. There exists $\delta_z > 0$ such that, for any invariant ergodic measure $\nu \neq \mu$,*

$$\mathcal{B}(z, \delta_z) \cap Supp\ \nu = \emptyset.$$

Proof. Let μ be an invariant ergodic measure, z a point in the support of μ. Using the Asymptotically Strong Feller property, there exists $\gamma_0 > 0,\ n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\sup_{y \in \mathcal{B}(z,\gamma_0)} \|P^\star_{t_n}\delta_z - P^\star_{t_n}\delta_y\|_{\delta_n} \leq \frac{1}{4}.$$

Let us suppose that there exists another distinct invariant ergodic measure ν such that $\nu\left(\mathcal{B}(z,\gamma_0)\right) > 0$. Then, using Lemma 10.7 in [13],

$$\|P^\star_{t_n}\mu - P^\star_{t_n}\nu\|_{\delta_n} \leq 1 - \frac{\alpha}{2},$$

where $\alpha = min\left(\mu\left(\mathcal{B}(z,\gamma_0)\right), \nu\left(\mathcal{B}(z,\gamma_0)\right)\right)$. Since measures $\nu,\ \mu$ are invariant under (P_t), we get

$$\|\mu - \nu\|_{\delta_n} \leq 1 - \frac{\alpha}{2}.$$

Finally, let $n \to \infty$ to obtain

$$\|\mu - \nu\|_{TV} \leq 1 - \frac{\alpha}{2}.$$

Thus, measures μ and ν are not mutually singular. Since they are ergodic, they are equal, which is impossible.
Thus, for all ergodic invariant measure ν,

$$Supp\ \nu \cap \mathcal{B}(z,\gamma_0) = \emptyset.$$

□

1.6. **Langevin processes with degenerate diffusion in momentums.** We end this section with an application that summarizes the theorems we've just introduced. We study a mechanical system described in [1]. We show that the diffusion is well defined and has an explicit full support invariant measure. Then, using Hörmander's condition, we show that its generator is hypoelliptic. Finally, the uniqueness theorem 7 implies the uniqueness of this invariant measure.

Let us consider the Langevin equation described in [1].

$$\text{(L)} \qquad \left\{ \begin{array}{ccl} dx & = & v\ dt \\ \begin{bmatrix} dv \\ d\omega \end{bmatrix} & = & \begin{bmatrix} -\partial_x U \\ 0 \end{bmatrix} dt - cC \begin{bmatrix} v \\ \sigma\omega \end{bmatrix} dt + \alpha C^{1/2} \begin{bmatrix} dB_v \\ dB_\omega \end{bmatrix}, \end{array} \right.$$

where $U : \mathbb{R} \to \mathbb{R}$ is a smooth confining function with negative exponential moments, $c,\ \alpha,\ \sigma$ are non-negative constants and C is the following matrix :

$$C = \begin{bmatrix} 1 & 1/\sigma \\ 1/\sigma & 1/\sigma^2 \end{bmatrix}.$$

The result we obtain is less general than the one of [1], since we need a non-degeneracy assumption on the potential.

Assumption 1. Let us suppose that for any $x \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that

$$\partial_x^{m+2} U(x) \neq 0.$$

Theorem 8. *Under the non-degeneracy assumption 1, the Langevin processes with degenerate diffusion in momentums* (L) *has a unique invariant measure.*

Even if the previous system cannot be seen as an immediate application of the fundamental example $(\mathcal{H})$ of stochastic systems, we will be able to use the same tools.
First notice that

$$\begin{aligned} \mathcal{L} &= v\ \partial_x - (\partial_x U + cv + c\omega)\,\partial_v - \frac{c}{\sigma}(v+\omega)\partial_\omega + \frac{\alpha^2}{2}\left(\partial_v + \frac{1}{\sigma}\partial_\omega\right)^2 \\ &= X_0 + X_1^2, \end{aligned}$$

thus we can easily show that

$$\begin{aligned} \mathcal{L}^\star = -v\ \partial_x - \omega\partial_\theta + &(\partial_x U + cv + c\omega)\ \partial_v + C\ Id \\ &+ \frac{c}{\sigma}(v+\omega)\ \partial_\omega + \frac{c}{\sigma}\ Id \\ &+ \frac{\alpha^2}{2}\left(\partial_v + \frac{1}{\sigma}\partial_\omega\right)^2. \end{aligned}$$

Let's take for the Hamiltonian function,

$$H(x,v,\omega) = U(x) + \frac{1}{2}v^2 + \frac{\sigma}{2}\omega^2.$$

Non-explosion. First of all, we notice that

$$\begin{aligned}\frac{\mathcal{L}e^{\theta H}}{e^{\theta H}} &= \theta\left\{vU'(x) - \left(U'(x)+cv+c\omega\right)v - \frac{c}{\sigma}(v+\omega)\sigma\omega + \frac{\alpha^2}{2} + \frac{\alpha^2}{2\sigma}\right\}\\ &\qquad + \frac{\theta^2\alpha^2}{2}\left(v+\omega\right)^2\\ &= -\theta c\left(v+\omega\right)^2 + \theta\alpha^2\left(1+\frac{1}{\sigma}\right) + \frac{\theta^2\alpha^2}{2}\left(v+\omega\right)^2\end{aligned}$$

Take $\theta = 2c/\alpha^2$ to show that the previous quantity is then lower than a non-negative constant. So, the non-explosion theorem 4 used with Lyapunov function $e^{2c(H+\inf H)/\alpha^2}$ insures that the diffusion is defined for all $t \in \mathbb{R}_+$.

Invariant measure. Recall that U has exponential negative moments and so we have for any real constant β,

$$\frac{\mathcal{L}^\star e^{-\beta H}}{e^{-\beta H}} = \left(-c + \beta\frac{\alpha^2}{2}\right)\beta\left(v^2 + v\omega + \omega^2\right).$$

Take $\beta = 2c/\alpha^2$ to see that

$$\mu_H = \frac{e^{-\beta H}}{Z}\, dz,$$

is an invariant measure with full support (see Proposition 2).

Hypoellipticity. Let us describe the Lie algebra $\mathfrak{L}$ associated to the generator $\mathcal{L}$. First of all, with our previous notations,

$$\{X_0, X_1\} \subset \mathfrak{L}.$$

We recursively compute the following brackets.

- $[X_0, X_1] = \partial_x \underbrace{-c\partial_v - \frac{c}{\sigma}\partial_\omega}_{\in\mathfrak{L}} \underbrace{-\frac{c}{v}\partial_v - \frac{c}{\sigma^2}\partial_\omega}_{\in\mathfrak{L}}.$

  Thus, the first bracket iteration gives
  $$\partial_x \in \mathfrak{L}.$$
- $[\partial_x, X_0] = -\partial_x^2 U\partial_v$, and then, by iteration, for all $m \in \mathbb{N}$,
  $$\partial_x^{m+2}U\partial_v \in \mathcal{L}.$$
  Thus, if we suppose that for any $x \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that
  $$\partial_x^{m+2}U(x) \neq 0,$$
  we get $\partial_v \in \mathfrak{L}$.
- Finally, for any $z \in \mathbb{R}^3$, we get
  $$\{\partial_x,\ \partial_v,\ \partial_\omega\} \in \mathfrak{L}(z),$$
  which we can write as follows
  $$dim\ \mathfrak{L}(z) = 3.$$

So, the Lie algebra $\mathfrak{L}$ has full rank and satisfies Hörmander's condition. Thus, using theorem 5, the generator $\mathcal{L}$ is hypoelliptic.

We conclude this section with the proof of the announced theorem.

Proof of theorem 8. The degenerate Langevin diffusion (L) is hypoelliptic using the non-degeneracy assumption 1 and has a full support invariant measure. Thus, using the uniqueness theorem 7, this invariant measure is unique and ergodic. □

2. The weak controllability problem

This section is devoted to the proof of the weak controllability theorem 1 and its corollary 2.

2.1. **Equivalence of controllability problems.** We start this section by noticing a simple fact about controllable systems.

Definition 9 (Equivalence)**.** Two deterministic systems are said *equivalent* if the weak controllability of any of them implies the weak controllability of the other.

Define the deterministic system $(S(g))$ as

$$(S(g)) \qquad \left\{ \begin{array}{lcl} \dot{x}_t & = & f(x_t, y_t) \\ \dot{y}_t & = & g(x_t, y_t) + u(t). \end{array} \right.$$

Theorem 9 (Equivalence between dynamical systems)**.** *For any smooth function g, deterministic systems* (S) *and* $(S(g))$ *are equivalent.*

Proof. Let $z_0 \in \mathbb{R}^n$ and A an open subset of $\mathbb{R}^n$. If the initial system is weakly controllable, there exists a time $T > 0$ and a control function u such that

$$(x_0, y_0) = z_0, \qquad (x_T, y_T) \in A.$$

Let $\nu(t) = u(t) - g(x_t, y_t)$. Then, (x, y) is a solution of the differential equation

$$\left\{ \begin{array}{lcl} \dot{x}_t & = & f(x_y, y_t) \\ \dot{y}_t & = & g(x_t, y_t) + \nu(t). \end{array} \right.$$

The system $(S(g))$ is thus weakly controllable. □

2.2. **Existence of controls : proof of Theorem 1.** We are now going to use the previous equivalence theorem 9 to prove the weak controllability theorem 1.

First step : equivalent problem. Using the assumption on the function f and the previous equivalence theorem 9, there exist functions g, H such that the initial dynamical system (S) is equivalent to

$$(\widetilde{S}) \qquad \left\{ \begin{array}{lcl} \dot{x}(t) & = & f(x_t, y_t) \\ \dot{y}(t) & = & g(x_t, y_t) - \nabla_y H + u(t), \end{array} \right.$$

with $div\,(f + g) = 0$.

Second step : stochastic system and invariant measure. We now consider the stochastic system *associated* to the previous deterministic system $(\widetilde{S})$:

$$(\widetilde{\mathcal{S}}) \qquad \left\{ \begin{array}{lcl} dx(t) & = & f(x_t, y_t)\ dt \\ dy(t) & = & g(x_t, y_t)\ dt - \nabla_y H\ dt + \sqrt{2} dB_t, \end{array} \right.$$

where (B_t) is a m-dimensional Brownian motion.

Non-explosion Using the Lyapunov function W given in the assumption, we obtain by theorem 4 that the stochastic system is well defined.

Invariant measure We notice that, since $div(f+g)=0$, the formal adjoint of the generator can be written

$$\mathcal{L}^\star = -f\cdot\nabla_x - g\cdot\nabla_y + \Delta_y - \nabla_y H\cdot\nabla_y - \Delta_y.$$

Therefore,

$$\begin{aligned}\frac{\mathcal{L}^\star e^{-H}}{e^{-H}} &= \langle f+g, \nabla H\rangle \\ &= 0.\end{aligned}$$

Uniqueness By assumption, e^{-H} is integrable and the corresponding probability measure $e^{-H}/Z\ dx\ dy$ has full support.

Hence, using the uniqueness theorem 7 and the hypoellipticity of $\mathcal{L}$, we get that μ_H is the unique ergodic invariant measure.
In the following step, we are going to use this ergodicity to deduce weak controllability.

Third step : from ergodicity to recurrence. To solve the weak controllability problem, we are going to use a realization of the corresponding stochastic system for which we establish a recurrence property.

Proposition 8 (Recurrence)**.** *If the semigroup is strong Feller and there exists an invariant ergodic measure μ_H, every measurable subset A of $\mathbb{R}^n$ with $\mu_H(A)>0$ is recurrent.*

Let us recall that the semigroup can be extended to a probability measure on continuous paths (see [17] p. 10) and introduce some basic notations linked to the dynamics of the stochastic system $(\widetilde{\mathcal{S}})$. We are going to use an approach similar to the one used in D. Revuz and M. Yor [16] p. 424, but with slightly different definitions. Let θ_t be the time shift operator on the trajectories, i.e.

$$\theta_t \circ Z_\cdot = Z_{t+\cdot}.$$

We introduce two main quantities : function h measures the recurrence of the diffusion, σ-field $\mathcal{J}$ is usually called the invariant σ-field :

$$\begin{aligned} h(z) &= \mathbf{P}_z\left\{\limsup_t\left(\mathbb{1}_{Z_t\in A}\right)=1\right\}, \\ \mathcal{J} &= \sigma\left\{\Gamma\subset\mathcal{C}([0,\infty],\mathbb{R}^n),\ \theta_t^{-1}(\Gamma)=\Gamma\right\}.\end{aligned}$$

Lemma 1. *When the semigroup is strong Feller and the invariant measure is ergodic, the function h is constant.*

Proof. The Markov property insures that h is invariant because for any $z\in\mathbb{R}^n$,

$$\begin{aligned} P_t h(z) &= \mathbf{E}_z[h(Z_t)] \\ &= \mathbf{E}_z\left[\mathbf{P}_{Z_t}\left\{\limsup_s\left(\mathbb{1}_{Z_s\in A}\right)=1\right\}\right] \\ &= \mathbf{P}_z\left[\limsup_s\left(\mathbb{1}_{Z_{t+s}\in A}\right)=1\right] \\ &= h(z).\end{aligned}$$

On the other side, $h(Z_t)$ is a martingale,

$$\begin{aligned}\mathbf{E}_z[h(Z_{t+s})|\mathcal{F}_s] &= \mathbf{E}_{Z_s}[h(Z_t)]\\ &= P_t h(Z_s)\\ &= h(Z_s).\end{aligned}$$

But $h(Z_t)$ is bounded, so it is convergent, i.e. there exists a random variable Z $\mathcal{J}$-measurable such that

$$Z = \lim_{t\to\infty} h(Z_t),\ \mathbf{P}_{\mu_H} - a.s.$$

Since μ_H is ergodic (due to the full support theorem 3), the sigma-field $\mathcal{J}$ is trivial and

$$Z = constant,\ \mathbf{P}_{\mu_H} - a.s.$$

But, $h(z) = \mathbf{E}_z[Z]$. Thus,

$$h = constant,\ \mu_H - a.s.$$

Finally, the semigroup is strong Feller and h is bounded, so $P_t h = h$ is continuous. Thus,

$$h \text{ is constant.}$$

□

To prepare the proof of proposition 8, we recall the following well-known recurrence theorem.

Theorem 10 (Poincaré's Recurrence Theorem)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and θ a measurable application that conserves the measure μ. Then, for every $A \subset \Omega$ such that $\mu(A) > 0$, for almost every $z \in A$, there exists an increasing sequence k_n such that*

$$\theta^{k_n} z \in A,\ \forall\ n \in \mathbb{N}.$$

Proof of recurrence proposition 8. We are going to apply the Poincaré's recurrence theorem to the measure P_{μ_H} induced by μ_H on the continuous functions on $\mathbb{R}^+$, invariant by the shift θ_t.
Let A be a nonempty open set, $E_A = \{Z; Z_0 \in A\}$. Since μ_H has full support $\mathbb{R}^n$,

$$\mathbf{P}_{\mu_H}(E_A) = \mu_H(A) > 0.$$

Thus there exists $F \subset E_A$ such that $\mathbf{P}_{\mu_H}(F) = \mathbf{P}_{\mu_H}(E_A)$ and for any $Z \in F$, there exists a sequence t_n such that

$$\theta^{t_n} Z \in E_A,\ \forall\ n \in \mathbb{N},$$

i.e. $Z_{t_n} \in A,\ \forall\ n \in \mathbb{N}$. Finally, for almost every $z = Z_0 \in A$, we have

$$h(z) = 1.$$

But, using the preceding lemma, h is constant, so

$$h \equiv 1.$$

Every open subset of $\mathbb{R}^n$ is recurrent. □

Fourth step : conclusion.

Proof of the weak controllability theorem 1. For all $z \in \mathbb{R}^n$, let τ_A be the hitting time of the open set A. From the previous lemma, we get

$$\mathbf{P}_z\left(\tau_A < +\infty\right) = 1.$$

Let Γ_z be the set of Brownian trajectories such that $\tau_A < +\infty$ and $\mathcal{W}$ be the Wiener measure. Then,

$$\begin{array}{rcl} \mathcal{W}\left(B \in \Gamma_z\right) & = & \mathbf{P}_z\left(\tau_A < +\infty\right) \\ & = & 1. \end{array}$$

Hence, $\Gamma_z \neq \emptyset$ and we can choose a trajectory $B(\omega)$ in Γ_z such that $\tau_A(\omega) < +\infty$,

$$Z_0(\omega) = z,\ Z_{\tau_A(\omega)}(\omega) \in A.$$

Thus, using the path of this Brownian motion as a control, if we denote $T = \tau_A(\omega)$, we obtain

$$z_0 = z,\ z_T \in A.$$

We conclude that $(\widetilde{S})$ is weakly controllable and by the equivalence theorem 9 that (S) is weakly controllable. □

Remark.

- In this section, we couldn't avoid the Strong Feller assumption. The trick is to find a realization of a Brownian motion that steers the system from its initial state z to any open set A. To do so, we proved the recurrence of the process and we used the continuity of the recurrence function h. Indeed, the ergodic argument entails only the almost sure recurrence. Even if Asymptotically Strong Feller property means formally that two processes starting from neighbours points tend to move closer and closer, we couldn't prove mathematically that this property entails recurrence from any initial point.
- Recurrence property ensures that one can choose T as large as we want.
- As soon as the diffusion is strong Feller, for every measurable $A \subset \mathbb{R}^n$, $t \mapsto P_t(z, A)$ is continuous and thus $\int_0^\infty \mathbb{1}_{Z_s^z \in A}\ ds > 0$ if $\mu_H(A) > 0$. Therefore the diffusion is μ_H-irreducible (see [14] p.520).

2.3. **A class of weakly controllable systems : Corollary 2.** To end this section, we present a class of non-trivial weakly controllable systems. These deterministic systems are inspired from Hamiltonian systems, especially from the fundamental example $\mathcal{H}$.

Proof of corollary 2. We just check that the corresponding differential system

$$\left\{ \begin{array}{rcl} \dot{x}_t & = & \partial_y U(x,y) \\ \dot{y}_t & = & -\partial_x U(x,y) - \partial_y U(x,y) + u(t) \end{array} \right.$$

satisfies the assumptions of Theorem 1.
First, the generator is

$$\mathcal{L} = \partial_y U\ \partial_x - \partial_x U(x,y)\ \partial_y - \partial_y U(x,y)\ \partial_y + \Delta_y.$$

We thus have,

$$\left\{\partial_y,\ \partial_y^{m+1} U\ \partial_x\right\} \in \mathfrak{L}.$$

But, since the potential U is non-degenerate, for any $(x, y) \in \mathbb{R}^2$, there exists an integer m such that

$$\partial_y^{m+1} U(x, y) \neq 0.$$

Finally, for any $z \in \mathbb{R}^2$,

$$dim\ \mathfrak{L}(z) = 2.$$

Second, we notice that U (more precisely $U - min\ U + 1$) is a Lyapunov function and

$$\begin{aligned} \mathcal{L}U &= \Delta_y U - (\partial_y U)^2 \\ &\leq CU. \end{aligned}$$

Thus, the stochastic diffusion associated to this deterministic differential system does not explode.

Finally, we have

$$\langle \partial_y U(x, y) - \partial_x U(x, y), \nabla U \rangle = 0,$$

so we can take $H = U$ for the Hamiltonian in the previous theorems. □

Remark 7.

- In section 4, we will sometime replace condition $\Delta_y U - (\partial_y U)^2 \leq CU$ by condition $\Delta_y U \leq C$. Indeed, we then have

$$\begin{aligned} \frac{\mathcal{L}e^U}{e^U} &= \Delta_y U \\ &\leq C \end{aligned}$$

  and the stochastic diffusion is well-defined.
- We notice that the examples in [11] of uncontrollable polynomial systems are not in contradiction with the previous theorems. Indeed, the primitive of the polynomial $y \mapsto U(x, y) = y - x^3(1 + y^2 + x^2)$ is not confining!

We end this section with some particular examples.

Exemple 2. The following three systems are weakly controllable,

$$(S.1) \qquad \left\{ \begin{array}{rcl} \dot{x} &=& 3y^2 + 4y^3 + \gamma \sin y \\ \dot{y} &=& u(t) \end{array} \right.$$

$$(S.2) \qquad \left\{ \begin{array}{rcl} \dot{x} &=& (3y^2 + 4y^3)(1 + \gamma \sin y) \\ \dot{y} &=& u(t) \end{array} \right.$$

$$(S.3) \qquad \left\{ \begin{array}{rcl} \dot{x} &=& (3y^2 + 4y^3)(1 + e^{-y}) \\ \dot{y} &=& u(t) \end{array} \right.$$

Indeed, we can consider respectively,

$$\begin{aligned} U_1(x, y) &= (y^4 + y^3 - \gamma \cos y) + \frac{x^2}{2} \\ U_2(x, y) &= y^4 + y^3(1 - 4\gamma \cos y) + 3\gamma y^2(4 \sin y - \cos y) - \cdots \\ & \quad \cdots + 6\gamma y(\sin y + 4 \cos y) + 6\gamma(\cos y - 4 \sin y) + \frac{x^2}{2} \\ U_3(x, y) &= y^4 + y^3(1 - 4e^{-y}) - 15y^2 e^{-y} - 30ye^{-y} - 30e^{-y} + \frac{x^2}{2}. \end{aligned}$$

Finally, for each of the previous systems, when y is large, $\Delta_y U_i$ behaves like y^2, $(\partial_y U_i)^2$ like y^6 and U_i like y^4. Thus, $\frac{\Delta_y U_i - (\partial_y U_i)^2}{U_i}$ goes to $-\infty$ and thus is bounded above.

3. Uniqueness of invariant measures

The aim of this section is to provide a proof of the uniqueness of invariant measure theorem 3. First, we recall the Stroock-Varadhan's support theorem. We consider a stochastic system

$$dZ_t = F(Z_t)\ dt + \sigma(Z_t) \circ\ dB_t,$$

where $F,\ \sigma$ are smooth functions and $\circ$ denotes the Stratonovitch integral. We denote

$$\mathcal{S}_{t_0,z_0} = \left\{ z_{t_0};\ \exists \psi \in \mathcal{C}^-,\ z_t = z_0 + \int_0^t F(z_s)\ ds + \int_0^t \sigma(z_s)\psi(s)\ ds \right\},$$

where $\mathcal{C}^-$ stands for the set of piecewise continuous functions from $[0,\infty)$ to $\mathbb{R}^n$.

Theorem 11 (Support theorem, cf. [20] section 5)**.** *Using the previous notations,*

$$Supp\ P_{t_0,z_0} = \overline{\mathcal{S}_{t_0,z_0}}.$$

3.1. **From Itô to Stratonovitch.** The system $(\mathcal{S})$ we consider in theorem 3 is written with Itô integrals. To apply the support theorem, we have to translate it in the Stratonovitch sense. Thus, this system can be written

$$\left\{ \begin{array}{rcl} dX_t &=& f(X_t,Y_t)\ dt \\ dY_t &=& g(X_t,Y_t)\ dt - \frac{1}{2}\widetilde{\sigma}(X_t,Y_t)\ dt + \sigma(X_t,Y_t) \circ\ dB_t, \end{array} \right.$$

where $\widetilde{\sigma}_i = \sum_{j=1}^n \sum_{k=1}^n \frac{\partial \sigma_{ij}}{\partial z_k}\sigma_{kj}$, $i \in \{1,\dots,n\}$. We are interested in the deterministic control problem

$$\left\{ \begin{array}{rcl} \dot{x}_t &=& f(x_t,y_t) \\ \dot{y}_t &=& g(x_t,y_t) - \frac{1}{2}\widetilde{\sigma}(x_t,y_t) + \sigma(x_t,y_t)u(t). \end{array} \right.$$

However, using the equivalence theorem 9 and the inversibility of σ, we are brought to the deterministic control problem

$$\left\{ \begin{array}{rcl} \dot{x}_t &=& f(x_t,y_t) \\ \dot{y}_t &=& u(t). \end{array} \right.$$

3.2. **Weak controllability and uniqueness.** We consider the stochastic system $(\mathcal{S})$ and the associated deterministic system (S). We show that if the deterministic system is weakly controllable and the stochastic one behaves regularly, then the invariant measure is unique if it exists.

Lemma 2. *Assume the deterministic system* (S) *is weakly controllable. Let A be an open subset of $\mathbb{R}^n$. For all $z \in \mathbb{R}^n$, there exists $T > 0$ such that*

$$P_T(z,A) > 0.$$

Proof. Let $z \in \mathbb{R}^n$. By assumption, there exists $z_0 \in A$ and a continuous control such that

$$(x_0,y_0) = z,\ (x_T,y_T) = z_0.$$

Thus, using the support theorem, $z_0 \in Supp\ P_T(z,\cdot)$ and

$$P_T(z,A) > 0.$$

□

Remark 8. We do not need any regularity assumptions on the semigroup in the previous lemma.

Lemma 3. *Assume the deterministic system* (S) *is weakly controllable and the semigroup* (P_t) *is strong Feller. Then, for any measure* μ *invariant for the semigroup* (P_t)*,*

$$Supp\ \mu = \mathbb{R}^n.$$

Proof. Indeed, let $z \in Supp\ \mu$, A an open set in $\mathbb{R}^n$ and T defined as in the previous lemma. Since $u \mapsto P_T(u, A)$ is continuous, there exists a neighbourhood $\mathcal{B}(z, \epsilon)$ of z and a constant $c > 0$ such that for any $u \in \mathcal{B}(z, \epsilon)$, $P_T(u, A) \geq c$:

$$\begin{aligned}
\mu(A) &= \int_{\mathbb{R}^n} P_T(u, A)\ \mu(du) \\
&\geq \int_{\mathcal{B}(z,\epsilon)} P_T(u, A)\ \mu(du) \\
&\geq c\mu(\mathcal{B}(z, \epsilon)) \\
&> 0,
\end{aligned}$$

as $z \in Supp\ \mu$. □

Proof of the uniqueness theorem 3. Recall that non-explosion and full rank algebra properties are assumed. Using the previous lemma, $Supp\ \mu = \mathbb{R}^n$. Thus theorem 7 entails the uniqueness of the invariant measure μ. Moreover, the invariant measure is ergodic. □

Remark. Although regularity is not needed to prove lemma 2, strong Feller property is essential to prove the invariant measure has full support. In the Stochastic Navier-Stokes setting, contraction of the flow is used to prove uniqueness of invariant measures. In LaSalle's principle argument below, we will use this idea. However, if we suppose only weak controllability and Asymptotically Strong Feller properties, we couldn't obtain such a result.

4. Applications

Let us resume the strategy we are going to use in the following examples. We consider a stochastic differential equation. First, we consider the associated weak controllability problem. We translate this problem into an equivalent problem. For this latter, we guess an explicit invariant measure of full space support to the stochastic equation associated which induces its weak controllability. Finally, we conclude using the previous section.

4.1. **Heat conduction networks in contact with heat baths.** Let us consider the heat conduction network model of [12]. Let $G = (\mathcal{V}, \sim)$ be a connected graph with vertex set $\mathcal{V}$. Two vertices $i \neq j$ are nearest neighbours if there is an edge between them : $i \sim j$. Every node $i \in \mathcal{V}$ holds an atom of momentum p_i and position q_i. The total energy inside the system is given by the Hamiltonian

$$H(q, p) = \sum_{i \in \mathcal{V}} \left(V(q_i) + \frac{1}{2} \sum_{j \sim i} U(q_i - q_j) + \frac{1}{2} p_i^2 \right).$$

The *pinning* potential V and the *interaction* potential U are both assumed to be smooth functions that are confining, i.e.

$$\lim_{x\to\pm\infty} U = +\infty, \quad \lim_{x\to\pm\infty} V = +\infty.$$

For sake of simplicity in the computations of the Hörmander's condition, we shall assume that U is polynomial. We shall therefore not reach the level of generality of [5] but are more general than the harmonic case treated in [2]. We notice that with this model, we are not able to prove the existence of the invariant measure for the following system.

Among the particles, there is a non empty subset $\partial\mathcal{V} \subset \mathcal{V}$, called the boundary set. Each atom $i \in \partial\mathcal{V}$ at the boundary set is connected to a heat bath at temperature T_i. This interaction is modeled by an Ornstein-Uhlenbeck process. Thus, the dynamic of the oscillators is described by the system of stochastic differential equations

$$\text{(HCN)} \qquad \left\{ \begin{array}{rcl} dq_i(t) & = & p_i(t)\ dt \\ dp_i(t) & = & -\partial_{q_i} H\ dt + \left(-\frac{1}{2}p_i\ dt + \sqrt{T_i}\ dB_i(t)\right) \mathbb{1}_{i\in\partial\mathcal{V}}. \end{array} \right.$$

We now introduce the assumption that turns out to be sufficient in the general potential case and necessary and sufficient in the harmonic case (see section 4.1), to ensure uniqueness of an invariant probability measure.
Let $N = |\mathcal{V}|$ be the number of oscillators, $\mathcal{V} = \{1, \ldots, N\}$. Let us define Λ the operator on the canonical base of $(e_i)_{i\in\mathcal{V}}$ of $\mathbb{R}^N$ by

$$(\mathcal{C}) \qquad \Lambda e_i = \sum_{j\sim i} e_j.$$

Definition 10 (AET)**.** The couple $(G, \partial\mathcal{V})$ is said to be Asymmetrical Energy Transmitting (AET) if the smallest vector space containing $(e_i,\ i \in \partial\mathcal{V})$, stable under Λ has full rank, i.e.

$$span\left\{\Lambda^k e_i,\ k \geq 0,\ i \in \partial\mathcal{V}\right\} = \mathbb{R}^N.$$

Remark 9. For these dynamical systems, the uniqueness of an invariant measure is an issue by itself. If condition (AET) is not satisfied and the heat baths have the same temperature, we can show that the invariant measure is not unique (see lemma 4 and proposition 9) in the harmonic setting (see J.-P. Eckmann and E. Zabey [6]).

We are now able to state the following theorem.

Theorem 12. *If the graph $(G, \partial\mathcal{V})$ is AET, then the diffusion defined by the system* (HCN) *has a unique invariant measure (if it exists).*

Let us first notice that

$$\mathcal{L} = \sum_{i\in\mathcal{V}} \partial_{p_i} H\ \partial_{q_i} - \partial_{q_i} H\ \partial_{p_i} + \frac{1}{2} \sum_{i\in\partial\mathcal{V}} \left(-p_i\partial_{p_i} + T_i\partial^2_{p_i}\right),$$

and similarly,

$$\mathcal{L}^\star = -\sum_{i\in\mathcal{V}} \partial_{p_i} H\ \partial_{q_i} - \partial_{q_i} H\ \partial_{p_i} + \frac{1}{2} \sum_{i\in\partial\mathcal{V}} \left(Id + p_i\ \partial_{p_i} + T_i\partial^2_{p_i}\right).$$

Non-explosion. Observe that

$$\begin{aligned}\mathcal{L}H &= \frac{1}{2}\sum_{i\in\partial\mathcal{V}}(-p_i^2+T_i)\\ &\leq \sum_{i\in\partial\mathcal{V}}T_i\end{aligned}$$

Thus, if we consider the Lyapunov function $W = H - \inf H + 1$, we have

$$\mathcal{L}W \leq CW,$$

and the diffusion is well defined (cf. Theorem 4).

Hypoellipticity. We show that AET condition ensures that the Lie algebra $\mathfrak{L}$ generated by the generator $\mathcal{L}$ has full rank. To do so, we compute some brackets.

- For any $i \in \partial\mathcal{V}$,
$$\partial_{p_i} \in \mathfrak{L}.$$
- $[\partial_{p_i}, X_0] = \partial_{q_i} \underbrace{-\frac{1}{2}\mathbb{1}_{i\in\partial\mathcal{V}}\partial_{p_i}}_{\in\mathfrak{L}}.$
We shall deduce that for any $i \in \partial\mathcal{V}$,
$$\partial_{q_i} \in \mathfrak{L}.$$
- $[X_0, \partial_{q_i}] = \underbrace{\left(V''(q_i) + \sum_{j\sim i} U''(q_j - q_i)\right)\partial_{p_i}}_{\in\mathfrak{L},\ i\in\partial\mathcal{V}} + \sum_{j\sim i} U''(q_i - q_j)\partial_{p_j}.$
So, computing recursively these brackets, we obtain, for all $m \in \mathbb{N}$,
$$\sum_{j\sim i} U^{(m+2)}(q_i - q_j)\partial_{p_j} \in \mathfrak{L}.$$
Finally, since U is polynomial, let's say of highest monomial $a_m x^m$, $m!a_m \sum_{j\sim i}\partial_{p_j} \in \mathfrak{L}$. Hence, with obvious notations, for all $i \in \partial\mathcal{V}$,
$$\Lambda\left(\partial_{p_i}\right) \in \mathfrak{L}.$$
Computing the bracket with X_0 yields for all $i \in \partial\mathcal{V}$, $\Lambda\left(\partial_{q_i}\right) \in \mathfrak{L}$ and then, iterating, $\Lambda^2\partial_{p_i} \in \mathfrak{L}, \ldots$
- Since we supposed AET condition is satisfied, we then have for all $z \in \mathbb{R}^{2N}$,
$$dim\ \mathfrak{L}(z) = 2N.$$

Hörmander's condition is thus satisfied and the hypoellipticity assumptions are satisfied by this diffusion (see Theorem 5).

Remark 10. To apply Hörmander's condition in the general setting where U is not a polynomial, we could suppose that for all $q \in \mathbb{R}^N$, there exists an m such that the matrix $\left(U^{(m)}(q_i - q_j)\right)_{i,j}$ is invertible (see also Assumption ($H2$) in [18]). However, this condition seems to be far from AET assumption which, we recall, is optimal in the harmonic setting.

The equivalent system. The key observation to make in this settings is that the deterministic control system is

$$\text{(HCNd)} \qquad \left\{ \begin{array}{lcl} \dot{q}_i(t) & = & p_i(t) \\ \dot{p}_i(t) & = & -\partial_{q_i} H + \left(-\frac{1}{2}p_i + \sqrt{T_i}\ u_i(t)\right) \mathbb{1}_{i\in\partial\mathcal{V}}, \end{array} \right.$$

which is obviously equivalent to the system where all the temperatures are equal $(T_i = T = 1,\ \forall\ i \in \partial\mathcal{V})$:

$$(\widetilde{HCNd}) \qquad \left\{ \begin{array}{lcl} \dot{q}_i(t) & = & p_i(t) \\ \dot{p}_i(t) & = & -\partial_{q_i} H + \left(-\frac{1}{2}p_i + u_i(t)\right) \mathbb{1}_{i\in\partial\mathcal{V}}, \end{array} \right.$$

Then, the corresponding stochastic system is

$$\left\{ \begin{array}{lcl} dq_i(t) & = & p_i(t)\ dt \\ dp_i(t) & = & -\partial_{q_i} H\ dt + \left(-\frac{1}{2}p_i\ dt + dB_i(t)\right) \mathbb{1}_{i\in\partial\mathcal{V}}. \end{array} \right.$$

Hopefully, since in the fundamental example $(\mathcal{H})$, we can easily find a simple invariant measure for this system. Indeed, let us remark that

$$\begin{aligned} \frac{\mathcal{L}^\star e^{-\beta H}}{e^{-\beta H}} &= \frac{1}{2}\sum_{i\in\partial\mathcal{V}} 1 - \beta p_i^2 - \beta + \beta^2 p_i^2 \\ &= \frac{1}{2}\sum_{i\in\partial\mathcal{V}} (1-\beta)(1-\beta p_i^2). \end{aligned}$$

Let's take $\beta = 1$ to see that

$$\mathcal{L}^\star e^{-H} = 0.$$

Proof of theorem 12. The non-explosion and hypoellipticity assumptions are obviously satisfied for the simplified system where all the temperatures are equal. So, this system is weakly controllable (see theorem 1) and so is the first one. Finally using the ergodic theorem 3 we can conclude that the heat conduction network system has a unique invariant measure (if it exists). □

Non uniqueness of invariant measures. We end this section with a lemma showing that harmonic heat conduction networks that do not satisfy the AET condition have several invariant measures.

We start with the example of a diamond shaped system (see figure 2). Here the number of particles is 4, the edges and the boundary set are

$$1 \sim 2 \sim 3 \sim 4,\ \partial\mathcal{V} = \{1, 4\}.$$

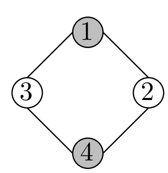

FIGURE 2. The diamond heat conduction network that does not satisfy AET.

We suppose that the potentials are harmonic $U(x) = V(x) = \frac{x^2}{2}$ and thus the Hamiltonian is

$$H(q,p) = \frac{1}{2}\sum_{1\le i\le 4}(p_i^2+q_i^2)+ \frac{1}{2}\left((q_1-q_2)^2+(q_1-q_3)^2+(q_2-q_4)^2+(q_3-q_4)^2\right).$$

We also suppose that the temperatures of the baths are all equal to T.

Lemma 4 (Diamond)**.** *When G is a diamond not satisfying condition AET, there exist an infinite number of invariant probability measures of Gibbs type*

$$\mu_{\beta,\gamma}(dz) = \frac{1}{Z_{\beta,\gamma}} e^{-\beta H(z)-\gamma K(z)}\ dz,$$

where $\beta = \frac{1}{T}$, $\gamma > 0$ and K is a quadratic polynomial.

Proof. We can write the generator

$$\mathcal{L} = \sum_i (p_i\partial_{q_i} - \partial_{q_i}H\partial_{p_i}) + \frac{1}{2}\sum_{i=1,4}\left(-p_i\partial_{p_i} + T\partial_{p_i}^2\right).$$

Let us take

$$K(q,p) = (p_3-p_2)^2 + 3(q_3-q_2)^2.$$

It is then a straightforward computation to show that

$$\mathcal{L}^\star e^{-\beta H - \gamma K} = 0,\ \forall\gamma>0.$$

□

The next proposition shows that this behavior is common to all heat conduction networks with equal temperatures that do not satisfy AET condition. To lighten the notations, we decompose the generator

$$\begin{aligned}\mathcal{L} &= \sum_i (p_i\partial_{q_i} - \partial_{q_i}H\partial_{p_i}) + \frac{1}{2}\sum_{i\in\partial\mathcal{V}}\left(-p_i\partial_{p_i} + \sqrt{T}\partial_{p_i}^2\right)\\ &= \mathcal{L}_H + \mathcal{L}_R.\end{aligned}$$

Proposition 9. *Let us consider a harmonic network with quadratic potentials and equal heat bath temperatures. If $(G,\partial\mathcal{V})$ is not AET, then there exists an infinite number of invariant probability measures of Gibbs type.*

Proof. First notice that we can decompose the formal adjoint of the generator

$$\frac{\mathcal{L}^\star e^f}{e^f} = -\mathcal{L}_H f + \frac{1}{2}\sum_{i\in\partial\mathcal{V}} 1 + p_i\partial_{p_i} f + T\left(\partial_{p_i}^2 f + (\partial_{p_i} f)^2\right).$$

Let us write $\beta = 1/T$ the inverse of the heat baths temperature. We want to find a function K such that for any scalar γ, $\mu_{\beta,\gamma}$ is an invariant probability measure, where

$$\mu_{\beta,\gamma}(dz) = \frac{e^{-\beta H(z)-\gamma K(z)}}{Z}\ dz.$$

Let us consider a quadratic polynomial

$$K(q,p) = \alpha\langle z,q\rangle + \langle z,p\rangle^2,$$

where $\langle z, q\rangle = \sum_{i=1}^{N} z_i q_i$ is the usual scalar product, α is a scalar and z a vector. We want to determine $\alpha,\ z$ such that K does not depend on $(p_i,\ i \in \partial\mathcal{V})$ to obtain $\mathcal{L}_R K = 0$ and such that

$$\mathcal{L}_H K = 0.$$

A simple computation provides

$$\begin{aligned}
\mathcal{L}_H K &= \sum_i \partial_{p_i} H \partial_{q_i} K - \partial_{q_i} H \partial_{p_i} K \\
&= \sum_i p_i \partial_{q_i} K - \partial_{p_i} K \left(q_i + \sum_{j\sim i} (q_i - q_j) \right) \\
&= \sum_i p_i 2 z_i \alpha \langle z, q\rangle - 2 z_i \langle z, p\rangle \left(\Gamma q\right)_i \\
&= 2\alpha \langle z, q\rangle\langle z, p\rangle - 2\langle z, p\rangle\langle z, \Gamma q\rangle,
\end{aligned}$$

where $\Gamma = D - \Lambda$ with D a diagonal operator $(Dq)_i = \left(1 + \sum_{j\sim i} 1\right) q_i$. Since $D,\ \Lambda$ are symmetric, Γ is symmetric and we have

$$\mathcal{L}_H K = 2\langle z, p\rangle\langle \alpha z - \Gamma z, q\rangle.$$

Since $(G, \partial\mathcal{V})$ is not AET, recall that E_Λ, the smallest vector space containing $\{e_i,\ i \in \partial\mathcal{V}\}$ and stable by Λ, satisfies

$$dim(E_\Lambda) < N.$$

Since Γ is symmetric and $\Gamma E_\Lambda = E_\Lambda$, Γ leaves stable its orthogonal $\Gamma E_\Lambda^\perp = E_\Lambda^\perp$. So, since $dim(E_\Lambda^\perp) \geq 1$, Γ has an eigenvector $z \in E_\Lambda^\perp$ with eigenvalue $\alpha \in \mathbb{R}$:

$$\Gamma z = \alpha z.$$

Then, $z \notin span\{e_i,\ i \in \partial\mathcal{V}\}$, and $K(q,p)$ does not depend on the variables $p_i,\ i \in \partial\mathcal{V}$.

That concludes the proof of this lemma. □

4.2. **Stochastic damped Hamiltonian systems.** In this section, we are interested in the dynamical system considered by L. Wu in [21]. Let (x, y) be a couple of vectors in $\mathbb{R}^{2N}$.

$$\text{(D)} \qquad \left\{ \begin{array}{lcl} dx_t &=& y_t\ dt \\ dy_t &=& -\nabla V(x_t)\ dt - c(x_t, y_t) y_t\ dt + \Sigma(x_t, y_t)\ dB_t, \end{array} \right.$$

where $V : \mathbb{R}^N \to \mathbb{R}^N$ is a smooth function, $c,\ \Sigma$ are matrices with smooth entries and Σ is symmetric. Henceforth, we will suppose that Σ is invertible.

Let us assume that $\Sigma \leq \sigma\ Id$ and $c^s = \frac{c+c^\star}{2} \geq \kappa\ Id$,

Theorem 13. *The previous stochastic damped process* (D) *has a unique invariant measure.*

Remark 11 (Existence of invariant measures). The degeneracy of this equation comes essentially form the degeneracy of the damping matrix c which is invertible only for big values of x in [21]. L. Wu uses this fact to show the existence of the invariant measure, using a modified Hamiltonian. The total degeneracy of the damping in the heat conduction network is an obstacle we couldn't overcome to prove existence.

We are going to use only the non-degeneracy of Σ to prove uniqueness of the invariant measure.

Let's take for Hamiltonian

$$H(x,y) = V(x) + \frac{1}{2}|y|^2.$$

First notice that we can write the generator

$$\mathcal{L} = \nabla_y H \cdot \nabla_x - \nabla_x H \cdot \nabla_y - c(x,y)y \cdot \nabla_y + \frac{1}{2}\sum_{i,j} \Sigma(x,y)^{(2)}_{i,j} \partial^2_{y_i y_j},$$

where $\Sigma(x,y)^{(2)}_{i,j} = \left(\Sigma^2\right)_{i,j}$; and for its formal adjoint

$$\begin{aligned}\mathcal{L}^* = &- \nabla_y H \cdot \nabla_x - \nabla_x H \cdot \nabla_y + cy \cdot \nabla_y + \sum_{i,j} \partial_{y_i}(c_{ij} y_j)\ Id + \nabla_y \\ &+ \frac{1}{2}\sum_{i,j} \partial^2_{y_i y_j} \Sigma(x,y)^{(2)}_{i,j} + \partial_{y_i}\Sigma(x,y)^{(2)}_{i,j}\partial_{y_i} + \partial_{y_j}\Sigma(x,y)^{(2)}_{i,j}\partial_{y_i} \\ &+ \frac{1}{2}\sum_{i,j} \Sigma(x,y)^{(2)}_{i,j}\partial^2_{y_i y_j}.\end{aligned}$$

Non-explosion. We recall (see [21]) that a simple computation yields

$$\mathcal{L}H = \frac{tr\Sigma^2}{2} - c(x,y)^s \cdot y.$$

But, since we supposed that $\Sigma \le \sigma\ Id$ and $c^s \ge \kappa\ Id$, we have

$$\mathcal{L}H \le C.$$

Thus, translating the Hamiltonian H, we can find a Lyapunov function such that

$$\mathcal{L}W \le CW.$$

The diffusion is well defined.

Hypoellipticity. Let us show how Hörmander's condition is obtained.
First notice that for all $k \in \{1,\dots,N\}$,

$$\sum_j \Sigma_{kj}\partial_{y_j} \in \mathfrak{L}.$$

Since Σ is invertible, using linear combinations we obtain that for all $j \in \{1,\dots,N\}$,

$$\partial_{y_j} \in \mathfrak{L}.$$

Thus, using first order brackets,

$$[X_0, y_i] = \partial_{x_i} \underbrace{- \sum_{j,k} \partial_{y_i}(c_{j,k} y_k)\partial_{y_k},}_{\in \mathfrak{L}}$$

Finally we have immediately

$$\forall\ i \in \{1,\dots,n\},\ \partial_{x_i} \in \mathfrak{L} \quad \text{ and } dim\ \mathfrak{L}(z) = 2N.$$

The hypoellipticity assumption (Theorem 5) is thus satisfied.

The equivalent system. Using the equivalence theorem, since the matrix Σ is non degenerate, we are going to consider the following equivalent deterministic system,

$$\left\{\begin{array}{lcl} \dot{x}_t & = & y_t \\ \dot{y}_t & = & -\nabla V(x_t) - cy_t + u(t), \end{array}\right.$$

where $c(x,y) \equiv \kappa\ Id$ is a constant matrix. We then notice that we can simply write

$$\mathcal{L}^* = -\nabla_y H \cdot \nabla_x - \nabla_x \cdot \nabla_y + \kappa y \cdot \nabla_y + n\kappa\ Id + \nabla_y.$$

And then for any real β,

$$\frac{\mathcal{L}^\star e^{-\beta H}}{e^{-\beta H}} = -\kappa\beta|y|^2 + \kappa n - \beta n + \beta^2|y|^2.$$

Let's take $\beta = \kappa > 0$, to see that the probability measure

$$\mu_H(dz) = \frac{e^{-\kappa H}}{Z}\ dz$$

is invariant under the dynamics of the system.
Thus, the system

$$\left\{\begin{array}{lcl} \dot{x}_t & = & y_t \\ \dot{y}_t & = & -\nabla V(x_t) - c(x,y)y_t + \ u(t) \end{array}\right.$$

is weakly controllable and the initial system (D) has a unique invariant measure (see Theorem 3). That concludes the proof of theorem 13.

Definition 11 (Topological transitivity)**.** A semigroup (P_t) is said *Topologically transitive* if for all nonempty open subset O of $\mathbb{R}^n$, for all $z \in \mathbb{R}^n$,

$$R(z,O) := \int_0^{+\infty} e^{-t} P_t(z,O)\ dt > 0.$$

Corollary 14. *The diffusion defined by the system* (D) *is topologically transitive.*

Proof. The weak controllability of the deterministic system ensures that the diffusion satisfies : for any $z \in \mathbb{R}^n$, $O \subset \mathbb{R}^n$ open set, there exists a time T such that

$$P_T(z,O) > 0.$$

Moreover, hypoellipticity assumptions ensures that $t \mapsto P_t(z,O)$ is continuous. Thus, the quantity

$$R(z,O) = \int e^{-t} P_t(z,O)\ dt > 0$$

that concludes our proposition. $\square$

Remark 12. The topological transitivity of the process is used to apply theorems on large deviations principles (cf. [21] Theorem 2.4). This method of weak controllability provides a proof of topological transitivity. This property was only an assumption in [21].

5. LaSalle's principle and invariant measures

The idea of this method is based on the contraction properties used in the lectures of J. Mattingly [13] to prove uniqueness of the invariant measure for Stochastic Navier-Stokes equations. Our aim is to prove that the contraction point is in the support of every invariant ergodic measure. Then, since ergodic measures have disjoint supports, we will prove that the invariant measure is unique. The main difficulty here is that we couldn't prove a Gronwall inequality for our diffusions. That's why we introduce the LaSalle's principle which provides a way to control the deterministic evolution.
In this section, we will consider the Hamiltonian system

$$(\widetilde{S}) \qquad \left\{ \begin{array}{lcl} dX_t & = & f(X_t, Y_t)\ dt \\ dY_t & = & g(X_t, Y_t)\ dt + \nabla_y H\ dt + \sigma(X_t, Y_t)\ dB(t). \end{array} \right.$$

As usual, we will denote Z_t the solution of this stochastic system and z_t the solution of the deterministic system where noise has been suppressed (i.e. when $\sigma \equiv 0$).

We would like to notice that in this section, we can suppose that the semigroup is Asymptotically Strong Feller.
We will show in appendix A that in the heat conduction networks with harmonic potential setting, the Asymmetric Energy Transmitting condition (see definition 10) implies Asymptotically Strong Feller condition.

5.1. **LaSalle's principle.** In this section, we recall LaSalle's principle. This principle is a generalization of Lyapunov's method. It is used to show (when the derivative of the Lyapunov function is not *definite* negative) that the solution of a differential system has an attractive point (cf. [19] p. 198).

Definition 12 (Invariant set)**.** $M \subset \mathbb{R}^n$ is said *invariant* if all the trajectories starting in M stay in M, i.e. for any $z_0 \in M$, for all $t \geq t_0$,

$$z_t^{z_0} \in M.$$

We will write $\dot{H}(x_t) = \partial_t H(x_t)$.

Theorem 15 (LaSalle's principle)**.** *Let us suppose there exists a function $H : \mathbb{R}^n \to \mathbb{R}_+$ of class $\mathcal{C}^1$ satisfying the following conditions : for all $c > 0$,*

(1) $\Omega_c = \{z;\ H(z) \leq c\}$ *is bounded,*
(2) $\dot{H}\big|_{\Omega_c} \leq 0$.

We will denote

$$S = \left\{ z \in \Omega_c;\ \dot{H}(z) = 0 \right\},$$

and we consider the biggest invariant subset M of S.
Then, for any $z_0 \in \Omega_c$,

$$z_t^{z_0} \xrightarrow[t\to\infty]{} M.$$

Proof. Let $z_0 \in \Omega_c$. according to the assumptions,

$$t \mapsto H\left(z_t^{z_0}\right)$$

is non-increasing and non-negative, thus convergent. Let us write

$$c_0 = \lim_{t\to\infty} H\left(z_t^{z_0}\right) \text{ and } L = \left\{ z;\ \exists\, t_n,\ z_{t_n}^{z_0} \to z \right\}.$$

L is invariant, hence
$$H(z) = c_0 \implies \dot{H}(z) = 0.$$
Thus, $L \subset M$ and the proof is done. □

Remark 13.

- When S is reduced to an equilibrium point and $\Omega_c \uparrow \mathbb{R}^n$, this principle implies that for any starting point, the trajectory converges to the equilibrium point.
- Since the function $\dot{H}$ is non-positive on Ω_c, every trajectory starting in Ω_c stays in Ω_c.
- In the Stochastic Navier-Stokes equations (cf. [13]), one can determine the speed of decrease to the equilibrium point of the deterministic system. In the example we consider, we couldn't find such a result. We are going to use the previous remark to control the dynamics of the system.

5.2. **Uniqueness of invariant measures.** Henceforth, we will suppose that the Hamiltonian function has a unique minimizer c such that $H(c) = 0$ (e.g. this assumption is satisfied as soon as the Hamiltonian is strictly convex) and that $|\nabla_y H(z)| = 0$ implies $y \equiv 0$ (it is usually true since $\nabla_y H(z) = y$).

We also suppose that the equation

$$(\widetilde{H}) \qquad \left\{ \begin{array}{rcl} \dot{x}_t & = & f(x,0) \\ 0 & = & g(x,0), \end{array} \right.$$

has only one solution given by $z_t = c$.

Remark 14. In the heat conduction network setting, this assumption means that when the particles linked to the reservoirs are pinned, the system can be only in the equilibrium position. One can show that when the potentials are harmonic, this assumption is implied by the AET condition (see appendix A).

We are going to show the following theorem.

Theorem 16. *If the previous assumptions are satisfied, then for every ergodic measure μ,*
$$c \in Supp\ \mu.$$

First notice that since H is continuous at c, for all $\epsilon > 0$, there exists $\eta > 0$ such that
$$K_\eta := \{x;\ H(x) < \eta\} \subset \mathcal{B}(c, \epsilon).$$
Let us sketch the method of proof of Theorem 16 : from LaSalle's principle, every solution of the deterministic system associated to $(\widetilde{H})$ goes to c. Hence, it hits K_η. As soon as the dynamical system enters K_η, it stays there, using again LaSalle's principle. If one consider all the solutions starting form the ball of radius R, we show that after a finite time all these solutions are in K_η. Finally, using the Stroock-Varadhan Support Theorem 11 we will prove uniqueness of the invariant measure.

For any $z \in \mathbb{R}^n$, let us denote T_z the hitting time of K_η starting from z, i.e.
$$T_z = \inf\left\{t \geq 0;\ z_t^{z_0} \in K_\eta\right\}.$$

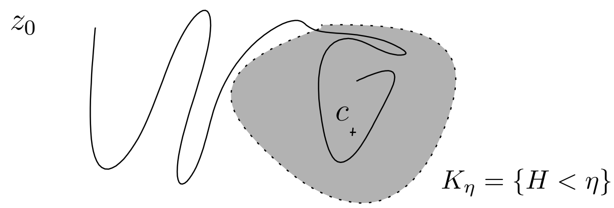

FIGURE 3. Dynamics of the Hamiltonian system in the absence of noise.

We will denote

$$T = \sup_{z \in \mathcal{B}_R} T_z.$$

Lemma 5. *Using the previous notations, we have*

$$T < +\infty.$$

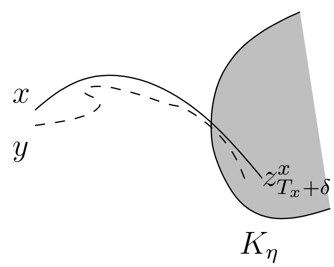

FIGURE 4. Description of the deterministic hitting times.

Proof. The idea of the proof is described in figure 4. We prove a property that looks like upper semi-continuity for $x \mapsto T_x$, i.e. there exists $\delta > 0$ such that for all y in a neighbourhood of x,

$$T_y \le T_x + \delta.$$

We are going to prove it ad absurdio. If there exists (x_n) such that $T_{x_n} \to +\infty$, using compactness, there exists a subsequence (always denoted (x_n)) such that

$$x_n \to x.$$

Yet, $T_x < +\infty$. Thus there exists $\delta > 0$ such that at time $T_x + \delta$, the particle is inside K_η, i.e.

$$\mathcal{B}\left(z^x_{T_x+\delta}, \epsilon\right) \subset K_\eta.$$

Using the continuity with respect to initial conditions, there exists $\widetilde{\epsilon} > 0$ such that for all $y \in \mathcal{B}(x, \widetilde{\epsilon})$,

$$\left|z^y_{T_x+\delta} - z^x_{T_x+\delta}\right| \le \frac{\epsilon}{2} \Rightarrow z^y_{T_x+\delta} \in K_\eta.$$

Thus, $T_y \le T_x + \delta$ for all $y \in \mathcal{B}(x, \widetilde{\epsilon})$. But that's impossible for x_n when n is large enough. $\square$

Proposition 10. *With T and K_η defined as before, for all $z \in \mathcal{B}(0,R)$,*

$$P_T(z, K_\eta) > 0.$$

Proof. Using the support theorem 11, for all $x \in \mathcal{B}_R$, denoting $\widetilde{K_\eta}$ a neighbourhood of K_η included in $\mathcal{B}_\epsilon$,

$$P\left(Z_T^z \in \widetilde{K_\eta}\right) = P_T\left(z, \widetilde{K_\eta}\right) > 0.$$

□

Proof of Theorem 16. First notice that

$$\begin{aligned}
\partial_t H(z_t) &= \nabla_x H(z_t) \cdot \partial_t x_t + \nabla_y H(z_t) \cdot \partial_t y_t \\
&= \nabla_x H(z_t) \cdot f(z_t) + \nabla_y H(z_t) \cdot g(z_t) - |\nabla_y H(z_t)|^2 \\
&= |\nabla_y H(z_t)|^2 .
\end{aligned}$$

So, using the assumptions, we will be able to apply LaSalle's principle.
Let μ be an ergodic invariant measure.

- Since μ is non equally null, there exists a ball $\mathcal{B}_R$ such that $\mu(\mathcal{B}_R) > 0$.
- Since H is continuous, there exists η such that $K_\eta \subset \mathcal{B}(c, \epsilon)$.
- Using the previous definitions, for all $\eta > 0$, there exists T such that for all $t \geq T$, for all $z \in \mathcal{B}_R$,
$$P_T(z, K_\eta) > 0.$$
- Since μ is an invariant measure,
$$\begin{aligned}
\mu(\mathcal{B}(c,\epsilon)) &\geq \mu\left(K_\eta\right) \\
&= P_T^* \mu\left(K_\eta\right) \\
&= \int_{\mathbb{R}^n} P_T\left(z, K_\eta\right)\ \mu(dx) \\
&\geq \int_{\mathcal{B}_R} P_T\left(z, K_\eta\right)\ \mu(dx) \\
&> 0.
\end{aligned}$$

Finally, we have shown that $c \in Supp(\mu)$. □

Corollary 17. *The invariant measure is unique, if it exists.*

Proof. Let us remark that all ergodic invariant measures with respect to an ASF semigroup have disjoint supports (see [8], Theorem 3.16). Thus, if there are two invariant measures, c will be in both supports. That's impossible! □

Appendix A. Heat Conduction Networks : the harmonic case

We present in this appendix some computations. We show that in the harmonic case, heat conduction networks satisfy both ASF and stability assumptions when they are AET. Even if we know that in this case, the stochastic process is strong Feller, the following properties ensure that this method should be applicable to more general settings.

ASF condition. In this section, we use the condition developed by M. Hairer and J. Mattingly to show ASF property (see [8] Equation (4.9)). For any function v in the Cameron-Martin space, let ρ_t be the solution of the differential system

$$\partial_t \rho_t = D(f+g)(Z_t^z)\rho_t - \sigma v_t.$$

Proposition 11. *(see [8]) If there exists a function v in the Cameron-Martin space such that*

$$\mathbf{E}_z \left| \int_0^t v(s)\ dB_s \right| < +\infty;\ \mathbf{E}_z \|\rho_t\| \to 0,$$

then the semigroup (P_t) is Asymptotically Strong Feller at x.

In the heat conduction network problem with harmonic potentials, ρ_t is the solution of the differential system

$$\left\{ \begin{array}{rcl} \partial_t \rho_i & = & \rho_{i+N} \\ \partial_t \rho_{i+N} & = & -\left(1 + \sum_{j\sim i} 1\right) \rho_i + \Lambda \rho_i - \rho_{i+N} \mathbb{1}_{i\in\partial\mathcal{V}} - \sigma_i v_i \mathbb{1}_{i\in\partial\mathcal{V}}. \end{array} \right.$$

Proposition 12. *When $v \equiv 0$, we have,*

$$\|\rho_t\|_\infty \to 0.$$

Proof. Let us denote D the diagonal matrix with entries $1+\sum_{j\sim i} 1$, Λ the connectivity matrice defined in $(\mathcal{C})$ and Id_∂ the diagonal matrix with entries $(Id_\partial)_{ii} = \mathbb{1}_{i\in\partial\mathcal{V}}$. We consider the differential system,

$$\partial_t \rho = \left(\begin{array}{cc} 0 & Id \\ -D+\Lambda & -Id_\partial \end{array} \right) \rho - \left(\begin{array}{cc} 0 & 0 \\ 0 & Id_\partial \end{array} \right) v.$$

We thus obtain the general solution,

$$\rho_t = e^{Mt} \left(Id - \int_0^t e^{-Ms} \left(\begin{array}{cc} 0 & 0 \\ 0 & Id_\partial \end{array} \right) v(s)\ ds \right) \rho_0,$$

where $M = \left(\begin{array}{cc} 0 & I \\ -D+\Lambda & -Id_\partial \end{array} \right)$.

We can show that M is of negative type. Indeed, let λ be an eigenvalue of M, $\rho = (q,p)$ an associated eigenvector.

$$\left\{ \begin{array}{rcl} p & = & \lambda q \\ (-D+\Lambda)\, q - Id_\partial p & = & \lambda p. \end{array} \right.$$

We thus have to study the following equation :

$$(-D+\Lambda)\, q - \lambda Id_\partial q = \lambda^2 q.$$

Thus for all $k \in \mathbb{N}$,

$$\Lambda^k (-D+\Lambda)\, q = \Lambda^k \lambda^2 q + \Lambda^k \lambda Id_\partial q.$$

So for all $i \in \{1,\dots,N\}$,

$$\left\langle (-D+\Lambda)\, q, \Lambda^k e_i \right\rangle = \lambda^2 \left\langle q, \Lambda^k e_i \right\rangle + \lambda \left\langle q, I_\partial \Lambda^k e_i \right\rangle.$$

Now choose $i \in \partial\mathcal{V}$ to see that

$$\left\langle (-D+\Lambda)\, q, \Lambda^k e_i \right\rangle = (\lambda^2 + \lambda) \left\langle q, (I + I_\partial)\Lambda^k e_i \right\rangle.$$

Using AET condition, we can rebuilt q in the second part of the scalar product,

$$\langle(-D+\Lambda)\,q,q\rangle = (\lambda^2+\lambda)\underbrace{\langle q,(I+I_\partial)q\rangle}_{\geq 0}.$$

Let us recall that we have with $d(i)=1+\sum_{j\sim i}1$,

$$\begin{aligned}\langle(-D+\Lambda)\,q,q\rangle &= \sum_i -d(i)x_i^2+\sum_{j\sim i}x_ix_j\\ &= -\sum_i\left(\frac{1}{2}\sum_{j\sim i}(x_i-x_j)^2+x_i^2\right)\end{aligned}$$

We thus obtain,

$$\lambda^2+\lambda\in\mathbb{R}_-.$$

We denote $\lambda=\rho e^{i\theta}$.

$$\left\{\begin{array}{rcl}\rho^2(\cos^2\theta-\sin^2\theta)+\rho\cos\theta &=& 0\\ \rho\sin\theta\,(2\cos\theta+1) &=& 0.\end{array}\right.$$

Either $\rho\sin\theta=0$, then $\rho\cos\theta<0$, using the study of the polynomial $\lambda^2+\lambda$; either $\rho\cos\theta=-\frac{1}{2}<0$.
Finally, $Re\lambda<0$ and matrix M is of finite type. □

Remark 15. We notice here that we used the trivial noise control $v\equiv 0$. This means that a small perturbation of the initial position tends to disappear.

Stability condition. In this section, we want to prove that if the graph $(G,\partial\mathcal{V})$ is AET, $z_t\equiv 0$ is the unique solution of the system

$$\left\{\begin{array}{rcl}\dot{q}_i(t) &=& p_i(t)\\ \dot{p}_i(t) &=& -q_i(t)-\sum_{j\sim i}(q_i-q_j)-p_i(t)\mathbb{1}_{i\in\partial\mathcal{V}}\\ p_i(t)\mathbb{1}_{i\in\partial\mathcal{V}} &=& 0.\end{array}\right.$$

Thus, we will be able to apply LaSalle's principle and the above results.

We will denote $(e_i)_{1\leq i\leq n}$ the canonical basis of $\mathbb{R}^n$.
Rewriting the equation associated to particle $i\in\partial V$,

$$\langle z,e_{i+N}\rangle\equiv 0.$$

But, if we derive the preceding equation with respect to the time parameter, since $p_i=0$ on ∂V,

$$\sum_{j\sim i}p_j\equiv 0.$$

Thus, we can rewrite this equation (recall that Λ is symmetric)

$$\langle x,\Lambda e_{i+N}\rangle\equiv 0.$$

By a trivial recurrence, we obtain for any $k\in\mathbb{N}$, $i\in\partial V$, $t\in\mathbb{R}_+$,

$$\langle x,\Lambda^k e_{i+N}\rangle = 0.$$

But AET condition ensures that

$$Vect\left(\Lambda^k e_{i+N},\ i\in\partial V,\ k\in\mathbb{N}\right)=\mathbb{R}^N,$$

thus,

$$\langle x, e_{i+N}\rangle \equiv 0,\ \forall i \in V.$$

Finally we obtain

$$q_i = constant,\ \forall i \in \{1, \ldots, N\}.$$

Since we supposed that $c = 0$ is the only solution of equation $H(c) = 0$, we obtain

$$x = 0$$

and the previous results can be applied to the heat conduction network with harmonic potentials.

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