INSTITUT DE MATHÉMATIQUES DE TOULOUSE

Master thesis

Quotients of the Cantor set for the action of a free group

Quentin Mareau

Supervised by Arnaud Hilion

Year 2023

Contents

I Introduction

The study conducted in this dissertation is part of the work of Coulbois, Hilion, and Lustig and consists primarily in studying quotients of the free group F_N by algebraic laminations. First, we turn our attention in the Section IV to the study of the particular case of the lamination $L([a,b])$. In this case, we can interpret ∂F_N in a suitable way to better understand the quotient $\partial F_N/L([a,b])$. This study involves a construction similar to the triadic Cantor set and leads to the following result:

$$\partial F_N/L([a,b]) \simeq \mathbb{S}^1$$

In the section V, we focus on a specific class of laminations: dual laminations to an $\mathbb{R}$-tree. To do this, we first aim to accurately describe these laminations using a key function in this work:

$$\mathcal{Q} : \partial F_N \longrightarrow \hat{T}\,.$$

Therefore, we can initially explore the connections between different definitions of dual laminations, such as

$$L^2_\Omega(T) = L^2_\mathcal{Q}(T).$$

To achieve both studies, we focus the Section III on particular spaces: $\mathbb{R}$-trees. This type of spaces can be viewed in multiple equivalent ways, each with its advantages. Therefore, we based from [1] for the study of this space. Then, since we study $\mathbb{R}$-trees endowed with an action by isometry, it was necessary to thoroughly understand the possible behaviours of such isometries. The main result, given in Theorem III.11, is that these isometries can be characterized on whether they have a fixed point or not, and their behaviours differ greatly accordingly. Finally, we need to consider a new topology on the $\mathbb{R}$-trees which gives important properties to the map $\mathcal{Q}$. Once this is accomplished, we use these properties to prove the following result given in the Theorem V.18:

$$\partial F_N/L^2(T) \simeq \hat{T}^{\text{obs}}$$

Another observation that can be made is that any action of the free group by homeomorphisms passes naturally to the quotient and yields an action by homeomorphisms on $\partial F_N/L^2$ where L^2 is an algebraic lamination. This is where the significance of Theorem VI.3 lies. It establishes the existence of a lift for any continuous action of a countable group on a compact metrizable Hausdorff space, resulting in a continuous action on the Cantor space. Since the set ∂F_N is homeomorphic to a Cantor space, a natural question arises from the above: Does every action on the quotient $\partial F_N/L$ by homeomorphisms stem from an action on ∂F_N by homeomorphisms? Otherwise, is it possible to characterize such actions? However, only the understanding of the objects involved and the various mentioned theorems could be addressed in this report, and this question remains open.

II Preliminaries

II.1 Hyperbolic spaces and Gromov boundary of free groups

For the good understanding all the notion of lamination and $\mathbb{R}$-trees, it is mandatory to first see what is an hyperbolic space. Then, it will allow us to define properly what's the Gromov boundary of an hyperbolic groups such F_N, a free group of rank N..

Definition II.1. Consider (X,d) a metric space. For $x,y,v \in X$ we define :

$$(x|y)_v = \frac{1}{2}(d(x,v)+d(y,v)-d(x,y))$$

Remark. The triangle inequality gives us that : $\forall x,y,v \in X, \quad (x|y)_v \geq 0$.

Definition II.2. A metric space (X,d) is δ-hyperbolic if for $x,y,z,v \in X$,

$$(x|y)_v \geq \min((x|z)_v,(y|z)_v) - \delta$$

Definition II.3. A triple $(a,b,c) \in \mathbb{R}^3$ is isosceles if :

$$a \geq \min(b,c),\ b \geq \min(a,c) \text{ and } c \geq \min(a,b)$$

In particular, this means that at least two of a, b, c are equal, and not greater than the third.

Remark. A metric space X is 0-hyperbolic if and only if for any points $x,y,z,v \in X$, the triple $((x|y)_v,(x|z)_v,(y|z)_v)$ is isosceles.

Definition II.4. Let (X,d) be a metric space which is geodesic, δ-hyperbolic and proper. A geodesic ray from a point $O \in X$ is an isometry $\varphi : \mathbb{R} \longrightarrow X$ such that $\varphi(0) = O$ and two geodesic rays from φ_1 and φ from O are said equivalent if it exists $M > 0$ such that :

$$\forall t \in \mathbb{R},\ d(\varphi_1(t),\varphi_2(t)) \leq M.$$

Let's now define $\partial X = \{[\varphi] \mid \varphi \text{ a geodesic ray from } O\}$ the Gromov boundary of X, where $[\varphi]$ denotes the equivalence class of φ.

Remark. The first remark is that this definition of the Gromov Boundary does not depend on the point O.

Definition II.5. For any $p,q \in \partial X$ and $O \in X$, we define $(p|q)_O = \inf_A \limsup_{s,t\to+\infty}(\varphi(s)|\psi(t))_O$ where $A = \{(\varphi,\psi) \mid [\varphi] = p \text{ and } [\psi] = q\}$. The topology over ∂X is generated by the sets

$$V(p,r) = \{q \in \partial X \mid (p|q)_O \geq r\}$$

where $a \in \partial X$ and $r > 0$.

We now present a sequential point of view of the Gromov boundary which we be more used later and related of the Gromov boundary of a free group. Fortunately, these two points of view are equivalents.

Definition II.6. Let (X,d) be a metric space which is geodesic, δ-hyperbolic and proper. One says that a sequence $(x_n)_{n\in\mathbb{N}}$ tends to a point at infinity if $(x_n|x_p)_w \xrightarrow[n,p\to+\infty]{} +\infty$ for some point $w\in X$. Two such sequences $(x_n),(y_n)$ are equivalents if $(x_n|y_p)_w \xrightarrow[n,p\to+\infty]{} +\infty$.

Definition II.7. For any $a,b\in\partial X$ and $w\in X$, we define $(a|b)_w = \inf_A \limsup_{n,p\to+\infty}(x_n|y_p)_w$ where $A=\{((x_n),(y_p)) \mid [(x_n)]=a \text{ and } [(y_p)]=b\}$. The topology over ∂X is generated by the sets :

$$V(a,r)=\{b\in\partial X \mid (a|b)_w\geq r\}$$

where $a\in\partial X$ and $r>0$.

Now, we will be focus on 0-hyperbolic spaces and on the boundary of free groups. Let F_N be a free group of rank N and $\mathcal{A}$ a basis of F_N.
Let's remind that we can define the word metric on F_N by : $\forall u,v\in F_N,\quad d_{\mathcal{A}}(u,v)=|u^{-1}v|_{\mathcal{A}}$. Moreover, for $u,v\in F_N$, let's define $u\wedge v$ the longest common prefix to u and v. In particular,

$$|u^{-1}v|_{\mathcal{A}}=|u|_{\mathcal{A}}+|v|_{\mathcal{A}}-2|u\wedge v|_{\mathcal{A}}$$

Proposition II.1. F_N endowed with the word metric (associated to $\mathcal{A}$) is 0-hyperbolic.
In particular, this property doesn't depend on the basis.

Proof. Let $u,v,w,g\in F(\mathcal{A})$, one has

$$\begin{aligned}(g^{-1}u|g^{-1}v)_{1_{F_N}} &= d(g,u)+d(g,v)-d(u,v)\\ &= |g^{-1}u|_{\mathcal{A}}+|g^{-1}v|_{\mathcal{A}}-|(g^{-1}u)^{-1}g^{-1}v|_{\mathcal{A}}\\ &= |g^{-1}u|_{\mathcal{A}}+|g^{-1}v|_{\mathcal{A}}-|u^{-1}v|_{\mathcal{A}}\\ &= (u|v)_g\end{aligned}$$

Thus it suffices to prove the proposition for $g=1_{F_N}$. Then, we have that $(u|v)_{1_{F_N}}=|u\wedge v|$ and it's clear that

$$|u\wedge v|\geq\min(|u\wedge w|,|w\wedge v|)$$

□

An important example of hyperbolic space which is a big part of our work are $\mathbb{R}$-trees. It will be shown that those spaces T are nothing but connected and 0-hyperbolic spaces. It's easy to see that the metric completion $\overline{T}$ of such a space is still an $\mathbb{R}$-tree. To understand the next definition, it suffices to have in mind that T is 0-hyperbolic :

Definition II.8. Let T be an $\mathbb{R}$-tree, for $p \in \partial T$ and $O, P \in \overline{T}$, we define

$$(p|P)_O = \inf_A \limsup_{t \to +\infty} (\varphi(t)|P)_O$$

where $A = \{(\varphi) \mid [\varphi] = p\}$.
Thus the topology over $\hat{T} = \overline{T} \sqcup \partial T$ is given by the natural topology over $\overline{T}$ and the neighborhoods

$$\tilde{V}(p,r) = V(p,r) \cup \{P \in T \mid (p|P)_O \geq r\}$$

where $p \in \partial T$ and $r > 0$.

Remark. This neighborhood basis of $[\varphi] \in \partial T$ correspond to the connected components C of $T \setminus \{P\}$ such that $\mathrm{Im}(\varphi) \cap C$ is not compact. In particular, this definition doesn't depend on φ.

Here we give a definition of the Gromov boundary ∂F_N of the free group F_N, which coincides with the definition of Gromov boundary of F_N endowed with the word metric (in particular, it doesn't depend on the basis).

Definition II.9. Let F_N be a free group of rank N and a basis $\mathcal{A}$. We define :

$$\partial F_N = \{(x_n)_{n \in \mathbb{N}^*} \mid \forall n \in \mathbb{N}^*,\ x_n \in \mathcal{A} \cup \mathcal{A}^{-1} \text{ and } x_{n+1} \neq x_n^{-1}\}$$

Moreover, for $X, Y \in \partial F_N$ and if $X \neq Y$, let's define $X \wedge Y$ the longest prefix in common to X and Y. Finally,

$$\forall X, Y \in \partial F_N, \qquad d(X,Y) = \begin{cases} 0 \text{ if } X = Y, \\ \exp(-|X \wedge Y|_{\mathcal{A}}) \text{ otherwise.} \end{cases}$$

Remark. We can notice that $(X|Y)_{1_{F_N}} = |X \wedge Y|_{\mathcal{A}}$.

Lemma II.2. ∂F_N is a compact space.

Proof. This space corresponds to a countable Cartesian product of finite spaces. Hence it's compact. □

Theorem II.3. ∂F_N is homeomorphic to the Cantor space K.

Proof. By the the theorem VI.1 and the latest lemma, it suffices to show that ∂F_N is totally disconnected and has no isolated point.
Let $X \in \partial F_N$, for all $n \in \mathbb{N}^*$, there exists $Y \in \partial F_N$ such that $Y \neq X$ and $|X \wedge Y|_{\mathcal{A}} \geq n$, ie $Y \in B(X, e^{-n})$. Therefore, ∂F_N has no isolated point.
Moreover, it's easy to see that for $\varepsilon > 0$ and $X \in \partial F_N$, we have $\overline{B(X,\varepsilon)} = B(X,\varepsilon)$. Thus, the topology of ∂F_N has a clopen basis so ∂F_N is totally disconnected. □

II.2 Automorphisms of free groups

For the sequel of this section, it's necessary to understand some properties of the automorphisms of F_N such as the Cooper's bounded cancellation property. To reach this result, we first need study the action of $\mathrm{Aut}(F_N)$ on $\widehat{F_N} = F_N \cup \partial F_N$, where F_N is a free group of rank $N \geq 2$ and $\mathcal{A}$ a basis of this group.

Definition II.10. Let $f \in \mathrm{Aut}(F_N)$, let's define the size of f by : $S(f) := \max_{x \in \mathcal{A}}(|f(x)|_{\mathcal{A}}, |f^{-1}(x)|_{\mathcal{A}})$.

Proposition II.4. Any $f \in \mathrm{Aut}(F_N)$ can be extended on an unique homeomorphism $\hat{f} : \widehat{F_N} \longrightarrow \widehat{F_N}$.

Proof. Let $X = (x_n)_{n\geq 1} \in \partial F_N$, denote $X_p = x_1 \dots x_p$ for $p \geq 1$. Let's write $r = |f(X_p)|_{\mathcal{A}}$ so that $f(X_p) = y_1 \dots y_r$ with $y_i \in \mathcal{A} \cup \mathcal{A}^{-1}$.
Thus, one has : $|X_p|_{\mathcal{A}} = |f^{-1}(f(X_p))|_{\mathcal{A}} = |f^{-1}(y_1)\dots f^{-1}(y_r)|_{\mathcal{A}} \leq S(f)r = S(f)|f(X_p)|_{\mathcal{A}}$.
Let $m(p) = \left\lfloor \dfrac{|X_p|_{\mathcal{A}}}{S(f)} \right\rfloor$ such that $|f(X_p)| \geq m(p)$.
For $k \leq p$, let's write $\mathrm{init}(X_p, k)$ the k-prefix of X_p. Since $|f(x_{p+1})|_{\mathcal{A}} \leq S(f)$, then in the worst case, after reduction, $|f(X_p)f(x_{p+1})|_{\mathcal{A}} = |f(X_p)|_{\mathcal{A}} - S(f)$.
Therefore, one has : $\mathrm{pre}(f(X_{p+1}), m - S(f)) = \mathrm{pre}(f(X_p), m - S(f))$.
Since $m(p) \xrightarrow[p \to +\infty]{} +\infty$, one has :

$$\forall n \geq 1,\ \exists K \geq 1\ \forall p, p' \geq K, \quad \mathrm{init}(f(X_p), n) = \mathrm{init}(f(X_{p'}), n).$$

It allows us to define $Y = \lim_{p \to +\infty} f(X_p)$ and $\hat{f}(X) = Y$. Since F_N is dense in $\hat{F}$, f must be unique. Finally, $\hat{f}$ is (uniformly) continuous by construction and $\widehat{f^{-1}} \circ \hat{f} = \hat{f} \circ \widehat{f^{-1}} = 1$. Thus, $\hat{f}$ is an homeomorphism. $\square$

Corollary II.5. There exist a natural left action of F_N on ∂F_N given by extension on ∂F_N of the left translation automorphism. $\square$

Proposition II.6. Let F be an F_N-invariant non-empty subset of ∂F_N, then F is dense in ∂F_N.

Proof. Let $X \in \partial F_N$ and $Z = z_1 z_2 \dots \in F$ where $z_i \in \mathcal{A}^{\pm 1}$. For any $k \geq 1$, we denote X_k the k-prefix of X. If there exists $a \in \mathcal{A}^{\pm 1}$ such that $Z = aaa\dots$, and if $X = a^{-1}a^{-1}a^{-1}\dots$ then for $b \in \mathcal{A} \setminus \{a\}$ (which exists since $N \geq 2$), we have $X_k b Z \xrightarrow[k \to +\infty]{} X$. Otherwise $X_k Z \xrightarrow[k \to +\infty]{} X$. $\square$

Corollary II.7. **(Cooper's bounded cancellation)** Given an automorphism $f : F_N \longrightarrow F_N$ there is a positive integer C such that if $u, v \in F_N$ and $|uv|_{\mathcal{A}} = |u|_{\mathcal{A}} + |v|_{\mathcal{A}}$, then :

$$|f(u)|_{\mathcal{A}} + |f(v)|_{\mathcal{A}} - |f(uv)|_{\mathcal{A}} \leq C$$

.

Proof. $|f(u)|_{\mathcal{A}} + |f(v)|_{\mathcal{A}} - |f(uv)|_{\mathcal{A}} = 2|f(u^{-1}) \wedge f(v)|_{\mathcal{A}}$. Since f^{-1} is uniformly continuous on $\hat{F}$, it exists an $N \geq 1$ such that :

$$|f(u^{-1}) \wedge f(v)|_{\mathcal{A}} > N \Rightarrow |u^{-1} \wedge v|_{\mathcal{A}} \geq 1 \Rightarrow |uv|_{\mathcal{A}} < |u|_{\mathcal{A}} + |v|_{\mathcal{A}}$$

Considering the contrapositive, we have the result. $\square$

Given an automorphism $f : F_N \longrightarrow F_N$, and a basis $\mathcal{A}$ of F_N, we now write $\mathrm{BBT}(\mathcal{A}, f)$ the smallest such constant C. Moreover, if we consider another basis $\mathcal{B}$, it's well known that $\mathcal{B}$ is the image of $\mathcal{A}$ under some $f \in \mathrm{Aut}(F_N)$ and then :

Corollary II.8. Given two bases $\mathcal{A}$ and $\mathcal{B}$ of F_N there is a positive integer C such that if $u, v \in F_N$ and $|uv|_{\mathcal{A}} = |u|_{\mathcal{A}} + |v|_{\mathcal{A}}$, then :

$$|u|_{\mathcal{B}} + |v|_{\mathcal{B}} - |uv|_{\mathcal{B}} \leq C.$$

We write $\mathrm{BBT}(\mathcal{A}, \mathcal{B})$ the smallest such constant C. $\square$

Another point of view of the Cooper's bounded cancellation appears by considering the Cayley graphs of F_N with respect to $\mathcal{A}$ and $\mathcal{B}$. Indeed, if we write $T_{\mathcal{A}}$ and $T_{\mathcal{B}}$ these trees and $i : T_{\mathcal{A}} \longrightarrow T_{\mathcal{B}}$ the equivariant map related, this result can be rephrased as follow :

Proposition II.9. For any (possibly infinite) geodesic $[P, Q]$ in T_A, the image $i([P, Q])$ lies in the $\mathrm{BBT}(\mathcal{A}, \mathcal{B})$-neighbourhood in $T_{\mathcal{B}}$ of $[i(P), i(Q)]$. $\square$

II.3 Laminations

We now introduce laminations which is an object seen in different fields of mathematics. Indeed, this notion may appear in a more geometric context, for example, when we consider the foliation of differential manifold. Here, in the context of free groups, the notion of geodesic lamination on an hyperbolic surface, as studied by Thurston, led to a version which is more algebraic. This is the one presented in this part.

Let F_N be the free group of rank $N \geq 2$ endowed with the usual F_N-action. Let $\mathcal{A}$ be a basis of F_N which allows us to see ∂F_N as usual.

Definition II.11. We define $\partial^2 F_N = \partial F_N \times \partial F_n \setminus \Delta$ where Δ denotes the diagonal in $\partial F_N \times \partial F_N$. It follows that $\partial^2 F_N$ inherits a natural topology and F_N-action given by $w(X,Y) = (wX, wY)$ for $(X,Y) \in \partial^2 F_N$.
Moreover, we define a on $\partial^2 F_N$ the flip involution $(X,Y) \mapsto (Y,X)$ which is an F_N-equivariant homeomorphism.

Definition II.12. An algebraic lamination is a non-empty subset L^2 of $\partial^2 F_N$ which is closed, symmetric (ie. flip invariant), and F_N-invariant. Let's denote Λ^2 the set of algebraic lamination of F_N.

Definition II.13. For any element $w \neq 1$ in F_N, we denote $w^{+\infty}$ (respectively $w^{-\infty}$) the limit in ∂F_N of $(w^n)_{n \geq 1}$ (respectively $(w^{-n})_{n \geq 1}$). In particular, if $w = x_1 \ldots x_p y_1 \ldots y_q x_p^{-1} \ldots x_1^{-1}$ is a reduced word in $\mathcal{A}^{\pm 1}$, then :

$$w^{+\infty} = x_1 \dots x_p y_1 \dots y_q y_1 \dots y_q \dots \in \partial F_N$$

Definition II.14. Let $L(w)$ denote the algebraic lamination generated by $w \in F_N$ which is defined by :

$$L(w) = \{(vw^{-\infty}, vw^{+\infty}) \mid v \in F_N\} \cup \{(vw^{+\infty}, vw^{-\infty}) \mid v \in F_N\}$$

Example. Consider $F_2 = F(a,b)$ and $w = [a,b] = aba^{-1}b^{-1}$. We give here some examples of elements of $L(w)$. First note that

$$w^{+\infty} = aba^{-1}b^{-1}aba^{-1}b^{-1}\dots \quad \text{and} \quad w^{-\infty} = bab^{-1}a^{-1}bab^{-1}a^{-1}\dots$$

Thus,

$$(aw^{-\infty}, aw^{+\infty}) = (aaba^{-1}b^{-1}aba^{-1}b^{-1}\dots\ ,\ abab^{-1}a^{-1}bab^{-1}a^{-1}\dots)$$

and

$$(b^{-1}w^{-\infty}, b^{-1}w^{+\infty}) = (b^{-1}aba^{-1}b^{-1}aba^{-1}b^{-1}\dots\ ,\ ab^{-1}a^{-1}bab^{-1}a^{-1}\dots)$$

Here we give a picture of the above elements in the Cayley graph of F_2. This might help to have a geometric view of those objects which is very natural. In particular, this will be useful in the Section IV of this paper which focus on the study of $L(w)$ for $w = [a,b]$ (see Figure 9).

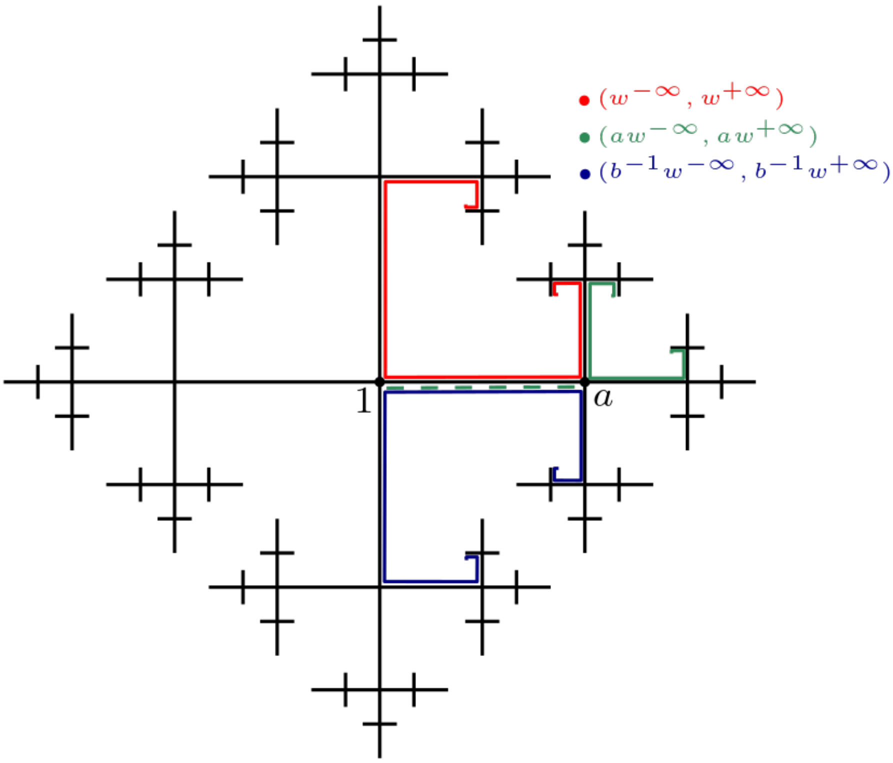

Figure 1: Picture of some elements of $L(w)$

Definition II.15. Let's define $\Sigma_{\mathcal{A}} = \{Z = (z_n)_{n\in\mathbb{Z}} \in (\mathcal{A}^{\pm 1})^{\mathbb{Z}},\ \ \forall n \in \mathbb{Z},\ z_{n+1} \neq z_n^{-1}\}$ the set of biinfinite words of $\mathcal{A}^{\pm1}$ endowed with the usual topology.

Definition II.16. Let's define the shift in $\Sigma_{\mathcal{A}}$ by $\sigma\ :\ (z_n)_{n\in\mathbb{Z}} \mapsto (z'_n)_{n\in\mathbb{Z}}$ where $z'_n = z_{n+1}$ for $n \in \mathbb{Z}$.
If $Z = (z_n)_{n\in\mathbb{Z}} \in \Sigma_{\mathcal{A}}$, one defines $Z^{-1} = (\tilde{z_n})_{n\in\mathbb{Z}}$ where $\tilde{z_n} = (z_{1-n})^{-1}$ for $n \in \mathbb{Z}$. One says that L is symmetric, if $L \subset \Sigma_{\mathcal{A}}$ verify that $L^{-1} = L$.

Notations. Let $Z = (z_n)_{n\in\mathbb{Z}} \in \Sigma_{\mathcal{A}}$, this notation $Z = \dots z_{-1}z_0 \cdot z_1z_2 \dots$ might be used.
Thus, $\sigma(Z) = \dots z_0z_1 \cdot z_2z_3 \dots$ and $Z^{-1} = \dots z_2^{-1}z_1^{-1} \cdot z_0^{-1}z_{-1}^{-1} \dots$

Proposition II.10. Shift and inversion are homeomorphisms of $\Sigma_{\mathcal{A}}$. □

Definition II.17. A symbolic lamination in $\mathcal{A}^{\pm1}$ is a closed non-empty subset of $L \subset \Sigma_{\mathcal{A}}$ which is symmetric and σ-invariant. Let's define $\Lambda_{\mathcal{A}}$ the set of symbolic laminations.

This new definition leads to a natural question : what's the link between algebraic and symbolic laminations ? As seen before, two biinfinite words might only differ from indexes. Thus, it's necessary to chose it arbitrarily.

For $(X,Y) \in \partial^2 F_N$ such that $|X \wedge Y|_{\mathcal{A}} = k$, let's define

$$X^{-1}Y = \dots x_{k+2}^{-1}x_{k+1}^{-1} \cdot y_{k+1}y_{k+2} \dots$$

Proposition II.11. The map $\rho_{\mathcal{A}} : \partial^2 F_N \longrightarrow \Sigma_{\mathcal{A}},\ (X,Y) \mapsto X^1Y$ is continuous and surjective. □

Proposition II.12. Given $L^2 \in \Lambda^2$, the subset $L_{\mathcal{A}} = \rho_{\mathcal{A}}(L^2)$ is a symbolic lamination.

Proof. First, we note that the flip on $\partial^2 F_N$ corresponds to the inversion in $\Sigma_{\mathcal{A}}$. Consider now $(X,Y) \in \partial^2 F_N$ and $w \in F_N$. We can remark that the biinfinite word $\rho_{\mathcal{A}}(w(X,Y))$ of $\Sigma_{\mathcal{A}}$ can differ from $X^{-1}Y$ only by an index shift. Conversely, let $X \wedge Y = X_k = Y_k$ with X_k, Y_k the k-prefixes of X and Y respectively. Then, for any $m \geq 0$ one has that $\rho_{\mathcal{A}}(Y_{k+m}^{-1}(X,Y)) = \sigma^m(X^{-1}Y)$ and for any $m \geq 0$, one has that $\rho_{\mathcal{A}}(X_{k-m}^{-1}(X,Y)) = \sigma^m(X^{-1}Y)$. Thus $\rho_{\mathcal{A}}$ induced a map from F_N-orbits of $\partial^2 F_N$ to σ-orbits of $\Sigma_{\mathcal{A}}$ which is a bijection.
Moreover, it's clear that for $L^2 \in \Lambda^2$, the subset $L_{\mathcal{A}} = \rho_{\mathcal{A}}(L^2)$ of $\Sigma_{\mathcal{A}}$ is closed. Thus $L_{\mathcal{A}}$ is a lamination. □

The last proof gives particularly that :

Corollary II.13. For any basis $\mathcal{A}$ of the free group F_N, the map

$$\rho_{\mathcal{A}}^2 : \Lambda^2(F_N) \longrightarrow \Lambda_{\mathcal{A}},\ L^2 \mapsto \rho_{\mathcal{A}}(L^2)$$

defines a bijection such that $(\rho_{\mathcal{A}}^2)^{-1}(L_{\mathcal{A}}) = \rho_{\mathcal{A}}^{-1}(L_{\mathcal{A}})$. □

III $\mathbb{R}$-trees

III.1 Definitions and properties

Definition III.1. A segment of a metric space X is the image of an isometry $\alpha : [a,b] \longrightarrow X$. $\alpha(a)$ and $\alpha(b)$ are called endpoints of the segment.

Definition III.2. One says a metric space X is geodesic if for $x,y \in X$, there exists a segment in X which has x,y as endpoints.

There exists some definitions of $\mathbb{R}$-tree which might look different. Actually, these definitions are all equivalents and give various points of view about the notion of $\mathbb{R}$-tree. The goal of the next results is to show this equivalence.

Definition III.3. An $\mathbb{R}$-tree is a geodesic metric space T with the following property : if two segments intersect in a single point, which is an endpoint of both, then their union is a segment.

Example. 1. In particular, a simplicial metric tree is an $\mathbb{R}$-tree.
2. Consider $(\mathbb{R}^2, d)$ endowed with the SNCF metric d with respect to 0 : if $x,y \in \mathbb{R}^2$ are on the same line, then $d(x,y)$ coincides with the euclidean metric and otherwise $d(x,y) = \|x\| + \|y\|$. Then $(\mathbb{R}^2, d)$ is an $\mathbb{R}$-tree which can be seen as an union of line linked in 0.
3. Consider $X = \{(x,y) \in \mathbb{R}^2,\ y \geq 0\}$ endowed with the metric d defined as :

$$\forall (x_1,y_1),(x_2,y_2) \in X,\ d((x_1,y_1),(x_2,y_2)) = \begin{cases} |y_1 - y_2| \text{ if } x_1 = x_2 \\ y_1 + y_2 + |x_1 - x_2| \text{ otherwise.} \end{cases}$$

Then X is an $\mathbb{R}$-tree which looks like a comb.

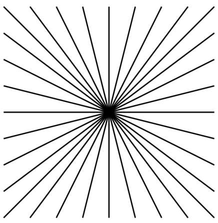

Figure 2: Picture of the example 2

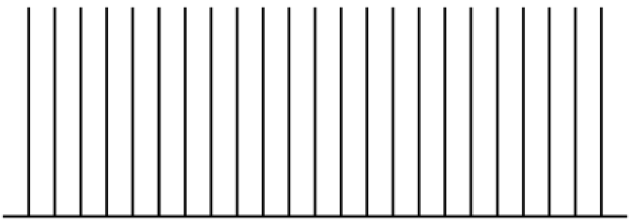

Figure 3: Picture of the example 3

Lemma III.1. Consider (T,d) an $\mathbb{R}$-tree. Then there exists an unique segment $[x,y]$ with endpoints $x,y \in X$. Moreover, for $x,y,z \in T$, there exists an unique $w \in T$ such that $[x,y] \cap [x,z] = [x,w]$ which is called the center of (x,y,z).

Proof. Let σ, τ be two segment with endpoints $x,y \in T$ (where $x \neq y$). We may assume without loss of generality that $\sigma \cap \tau = \{x,y\}$. Let $d = d(x,y)$ and $\varphi : [0,d] \longrightarrow T$ the isometry associated to σ, with $\varphi(0) = x$ and $\varphi(d) = y$.
We define now $\tau' = \varphi([0,d/2]) \cup \tau$ which is a segment since $\varphi([0,d/2]) \cap \tau = \{x\}$.
Let $z = \varphi(d/2)$ an endpoint of τ' with y. Thus, $d(z,y) = d(z,x) + d(x,y) > d$, but $d(z,y) = \dfrac{d}{2}$ which gives a contradiction. This property gives in particular that if σ and τ are two non disjoints segment, then $\sigma \cap \tau$ is a segment. This conclude the second statement. $\square$

Lemma III.2. Let x,y,z be points of a $\mathbb{R}$-tree (T,d), and write w the center of (x,y,z).

1. We have $[y,w] \cap [w,z] = \{w\}, [y,z] = [y,w] \cup [w,z]$ and $[x,y] \cap [w,z] = \{w\}$,
2. The center w depends only on the set $\{x,y,z\}$, not on the order in which the elements are written.

Proof. 1. Since $y,w \in [x,y]$, we have $[y,w] \subseteq [x,y]$. Similarly, $[w,z] \subseteq [x,z]$. So, if $u \in [y,w] \cap [w,z]$, then $u \in [x,y] \cap [x,z] = [x,w]$. Hence $u \in [x,w] \cap [y,w] = \{w\}$ (because $w \in [x,y]$). Thus, $[y,w] \cap [w,z] = \{w\}$, and necessarily $[y,z] = [y,w] \cup [w,z]$.

Now, since $w \in [x,y]$, we have $[x,y] = [x,w] \cup [w,y]$, so $[x,y] \cap [w,z] = ([x,w] \cap [w,z]) \cup ([y,w] \cap [w,z])$, and both intersections are equal to $\{w\}$ ($w \in [x,z]$).

2. We have by point 1. that :

$$
\begin{aligned}
[y,x] \cap [y,z] &= [y,x] \cap ([y,w] \cup [w,z]) \\
&= [y,w] \cup ([y,x] \cap [w,z]) \\
&= [y,w] \cup ([y,w] \cap [w,z]) \cup ([w,x] \cap [w,z])
\end{aligned}
$$

Now $[y,w] \cap [w,z] = \{w\}$ by 1. and $[w,x] \cap [w,z] = \{w\}$ since $w \in [x,z]$. Hence, $[y,x] \cap [y,z] = [y,w]$. Similarly, $[z,x] \cap [z,y] = [z,w]$. This concludes. $\square$

Theorem III.3. Let T be an $\mathbb{R}$-tree, then T is 0-hyperbolic and connected.

Proof. T is obviously connected since it is geodesic. Let's show that T is 0-hyperbolic. Let $x,y,z,v \in X$ and q,r,s the centers of $(x,v,y),(y,v,z)$ and (x,v,z) respectively. We may assume without loss of generality that :

$$d(v,q) \leq d(v,r) \leq d(v,s).$$

Since $(x|y)_v = d(v,q)$, $(y|z)_v = d(v,r)$ and $(x|z)_v = d(v,s)$, the goal is to show that $d(v,q) = d(v,r)$. Since $r,s \in [v,z]$ and $d(v,r) \leq d(v,s)$, then $r \in [v,s] \subset [v,x]$.
Thus $r \in [v,x] \cap [v,y] = [v,q]$. But $d(v,q) \leq d(v,r)$, so $q = r$ and T is 0-hyperbolic. $\square$

Lemma III.4. Consider (X,d) a 0-hyperbolic space. Let $x,y,v \in X$ such that there exists σ and τ two segments with respectively x,v and v,y as endpoints. Then :

1. If $x' \in \sigma$, then $x' \in \tau \iff d(v,x') \leq (x|y)_v$
2. If w is the point of σ such that $d(v,w) = (x|y)_v$, then $\sigma \cap \tau$ is a segment with w,v as endpoints.

Proof. If $d(x',v) > d(y,v)$, then $x' \notin \tau$ and $d(x',v) > (x|y)_v$. Hence, let's assume that $d(x',v) \leq d(y,v)$. Let $y' \in \tau$ such that $d(x',v) = d(y'v)$. Consider $\alpha = (x|y)_v, \beta = (x'|y)_v, \gamma = (x|x')_v, \alpha' = (x'|y')_v$.
We see that $\gamma = \dfrac{1}{2}(d(x,v) + d(x',v) - d(x,x')) = d(v,x') = d(v,y') = (y|y')_v$.

By hypothesis : $d(y',v)-d(x',y) \leq d(x',v)-d(x',y) \leq d(x',v)$. Hence :

$$\beta = \frac{1}{2}(d(x',v)+d(y,v)-d(x',y) \leq d(x,v) = \gamma.$$

. Moreover, $\alpha' = \frac{1}{2}(d(x',v)+d(v,y')-d(x',y')) = d(v,x')-\frac{1}{2}d(x',y') = \gamma - \frac{1}{2}d(x',y') \leq \gamma$.
Thus : $x' \in \tau \iff x' = y' \iff d(x',y') = 0 \iff \alpha' = \gamma$.
Then, (α',β,γ) is an isosceles triple since X is 0-hyperbolic, which implies $\alpha' = \beta$. Therefore $\alpha' = \gamma$ if and only if $\beta = \gamma$.
Similarly, (α,β,γ) is an isosceles triple so : $\beta = \gamma \iff \alpha \geq \gamma$.
At the end, we have $x' \in \tau$ if and only if $\alpha \geq \gamma$.
Then we can deduce the point 2. □

Theorem III.5. Consider (X,d) a geodesic space and 0-hyperbolic, then X is an $\mathbb{R}$-tree.

Proof. Let $x,y,v \in X$, σ,τ two segments with respectively x,v and y,v as endpoints. By the latest lemma, $\sigma \cap \tau$ is a segment with w,v as endpoints, where w is the point of σ such that $d(v,w) = (x|y)_v$. Let's now assume that $\sigma \cap \tau = \{v\}$, hence $d(v,w) = 0$ and $(x|y)_v = 0$. Therefore, we have $d(x,y) = d(x,v)+d(v,y)$.
To show that $\sigma \cup \tau$ is a segment, it suffices to show that $d(x,z) = d(x,v)+d(v,z)$ for $z \in \tau$ since σ and τ are two segment with endpoints x,v and v,y respectively. Since X is 0-hyperbolic, $(x|y)_v \geq \min((x|z)_v,(z|y)_v)$. Note that we showed that $(x|y)_v = 0$. so we have to show that $(x|z)_v \leq (z|y)_v$. Moreover, $(z|y)_v = d(z,v)$ and then :

$$(x|z)_v \leq (z|y)_v \iff \frac{1}{2}(d(x,v)+d(z,v)-d(x,z)) \leq d(z,v) \iff d(x,v) \leq d(x,z)+d(z,v).$$

The latest assertion is true by triangle inequality, then $(x|z)_v = 0$. Thus, $\sigma \cup \tau$ is a segment and then X is an $\mathbb{R}$-tree. □

Theorem III.6. Consider (X,d) a 0-hyperbolic metric space. There exists an $\mathbb{R}$-tree (T,d') and an isometry $\phi : X \longrightarrow T$.

Proof. See theorem 3.38 in [1]. □

Lemma III.7. Consider (T,d) an $\mathbb{R}$-tree and $v \in T$.

1. For $x,y \in X \setminus \{v\}$, $[x,v] \cap [v,y] \neq \{v\}$ if and only if x and y are in the same connected component of $T \setminus \{v\}$.

2. $T \setminus \{v\}$ is locally path connected, the components of $T \setminus \{v\}$ coincide with its path components, and they are open sets in T.

Proof. See Lemma 3.39 in [1]. □

Theorem III.8. Consider (X,d) a 0-hyperbolic and connected metric space, then X is geodesic.

Proof. Using the lemma III.6, there exists an $\mathbb{R}$-tree (T,d') and an isometry $\phi : X \longrightarrow T$. Then, it suffices to show that $Y = \phi(X)$ is geodesic. Let $x,y \in Y$, it exists a segment $\sigma \subset T$ with x,y

as endpoints, let's show that $\sigma \subset Y$.
Let's assume there is an point $v \in \sigma$ such that $v \notin Y$, if one writes τ, τ' the segments of T with endpoints x, v and v, y respectively, then $\tau \cap \tau' = \{v\}$.
Thus, by the latest lemma, x and y are not in the same component of $T \setminus \{v\}$. Let $C(x)$ be the component of x which is a clopen of T by the same lemma. Since Y is closed in T, then $C = Y \cap C(x)$ is a clopen of $Y \setminus \{v\} = Y$. However, $x \in C$ so $C \neq \emptyset$ and $y \notin C$ so $C \neq Y$, which leads to a contradiction since Y is connected. $\square$

Last theorem leads us to the next picture that show the equivalence between the different definitions :

$$\begin{tikzcd}
 & \boxed{\text{0-hyperbolic + connected}} \arrow[dl, bend right] & \\
\boxed{\text{0-hyperbolic + geodesic}} \arrow[rr] & & \boxed{\text{Geodesic + intersection}} \arrow[ul, bend right]
\end{tikzcd}$$

Here we give another characterization of $\mathbb{R}$-trees, which corresponds to the Theorem II.1.13 in [2] :

Theorem III.1. A metric space T is an $\mathbb{R}$-tree if and only if for $x, y \in T$, there exists an unique arc with endpoints x, y, and this arc is a segment. $\square$

III.2 Isometries of $\mathbb{R}$-trees

We now need to understand some good properties about isometries of $\mathbb{R}$-trees. Actually, we will see it's possible characterise two types of such isometries. Either an isometry fixes a point and is called elliptic, or it hasn't such point and acts by (non trivial) translation on an axis and is called hyperbolic.

Definition III.4. Consider T an $\mathbb{R}$-tree and γ an isometry of T. One says that γ is hyperbolic if it has no fixed point. If γ is not hyperbolic, then it is elliptic.

Definition III.5. Consider T an $\mathbb{R}$-tree, an axis A of T is the image of an isometry from $\mathbb{R}$ in T. An axis A is a translation axis for an isometry γ (or an axis for γ for short) of T if γ acts by (non trivial) translation on A.

Lemma III.9. Consider (T,d) an $\mathbb{R}$-tree and A an axis of T. Let $x \in T \setminus A$, there exists a unique $y \in A$ such that $[x,y] \cap A = \{y\}$. In particular : $d(x,y) = d(x,A)$.

Proof. Let $z \in A$, there exists a segment σ with x, z as endpoints. Consider the segment $\tau = \overline{\sigma \setminus A}$, there exists $y \in A$ such that $\tau = [x,y]$ and $\tau \cap A = \{y\}$.
Let y' be another point of A which verify this property. Then, we have that $[y,y'] \subset A$ and $[x,y] \cup [y,y'] = [x,y']$. However, $[x,y'] \cap A = \{y'\}$ so $[y,y'] = \{y\} = \{y'\}$.
In particular, that last part show that if $z \in A \setminus \{y\}$, then $d(x,z) = d(x,y) + d(y,z) > d(x,y)$. $\square$

Corollary III.10. Consider (T,d) an $\mathbb{R}$-tree and A, B two axes of T. Then there exists an unique $(x,y) \in A \times B$ such that $[x,y] \cap A = \{x\}$ and $[x,y] \cap B = \{y\}$. In particular : $d(x,y) = d(A,B)$.

Theorem III.11. Consider γ an isometry of a non-empty $\mathbb{R}$-tree T. Then there exists an axis for γ if and only if γ is hyperbolic. Moreover, if γ is hyperbolic, then its axis A is unique and : $l(\gamma) := \min_{x\in T} d(x,\gamma x) > 0$ et $A = \{x \in T,\ d(x,\gamma x) = l(\gamma)\}$

Proof. Assume that γ has an axis A and a fixed point $x \in T$. Necessarily $x \notin A$ and then, it exists a point $y \in A$ like in the lemma III.9. Moreover $\gamma([x,y]) = [\gamma(x),\gamma(y)] = [x,\gamma(y)]$ and $\gamma(y) \in A \setminus \{y\}$, hence $y \in [x,\gamma(y)]$. In particular $d(x,\gamma(y)) = d(x,y) + d(y,\gamma(y) > d(x,y)$ which leads to a contradiction since γ is an isometry.

Reciprocally, let's assume that γ has no fixed point in T, and let's find a construction of an axis for γ. Let $x \in T$, one has $x \in [x,\gamma x] \cap \gamma^{-1}([x,\gamma x])$. Then, it exists $x_1 \in [x,\gamma x] \cap \gamma^{-1}([x,\gamma x])$ such that $[x,\gamma x] \cap \gamma^{-1}([x,\gamma x]) = [x,x_1]$. In particular, $x_1 \in \gamma^{-1}([x,\gamma x])$, hence $\gamma x_1 \in [x,\gamma x]$ and by hypothesis, $\gamma x_1 \neq x_1$. Thus, either $\gamma x_1 \in [x,x_1]$ or $\gamma x_1 \in [x_1,\gamma x]$.

Assume that $\gamma x_1 \in [x,x_1]$ then let's define $a := [x_1,\gamma x_1] \subset [x,x_1]$. We can remark that a and γa have the same length and that a and γa have γx_1 as a common endpoint. Therefore $\gamma a = a$, ie $[x_1,\gamma x_1] = [\gamma x_1,\gamma^2 x_1]$. Since $\gamma x_1 \neq x_1$, it implies that $\gamma^2 x_1 = x_1$. Let's write $d = d(x_1,\gamma x_1)$, it exists an isometry $\varphi : [0,d] \longrightarrow T$ such that $\varphi([0,d]) = [x_1,\gamma x_1]$ with $\varphi(0) = x_1$ and $\varphi(d) = \gamma x_1$. Let's define $\psi = \gamma \circ \varphi$ which has the same property with $\psi(0) = \gamma x_1$ and $\psi(d) = x_1$. Necessarily, for $t \in [0,d]$, $\psi(t) = \varphi(d-t)$. Thus $\psi(d/2) = \varphi(d/2)$, then $\gamma(\varphi(d/2)) = \varphi(d/2)$, which is absurd. Hence, we have that $\gamma x_1 \in [x_1,\gamma x]$ which is the configuration given in the figure 4. Finally, one has :

$$\gamma([x_1,\gamma x_1]) \cap [x_1,\gamma x_1] \subset \gamma([x_1,\gamma x]) \cap [x,\gamma x] = \gamma([x_1,\gamma x] \cap \gamma^{-1}([x,\gamma x]) = \gamma(\{x_1\}) = \{\gamma x_1\}$$

Thus for any $N \in \mathbb{N}$, $\bigcup_{n=-N}^{N} [\gamma^n x_1, \gamma^{n+1} x_1]$ is a segment and then $A = \bigcup_{n\in\mathbb{Z}} [\gamma^n x_1,\gamma^{n+1} x_1]$ is an axis for γ. Moreover, we can remark that for any $y,z \in A$, we have $d(y,\gamma y) = d(z,\gamma z)$ and then $d(y,\gamma y) = \min_{z\in A} d(z,\gamma z)$.

Let's now prove the uniqueness of this axis. Assume there exists two axes A and B for γ. If $A \cap B \neq \emptyset$, then $A = B = \bigcup_{n\in\mathbb{Z}} [\gamma^n x,\gamma^{n+1} x]$ for $x \in A \cap B$.

Conversely, if $A \cap B = \emptyset$, then by the lemma III.9, it exists two unique points $x \in A$ and $y \in B$ such that $[x,y] \cap A = \{x\}$ and $[x,y] \cap B = \{y\}$. Thus $d(\gamma x,\gamma y) = d(\gamma x,x) + d(x,y) + d(y,\gamma y) > d(x,y)$, which leads to a contradiction.

In particular, this proves that the construction above doesn't depend on $x \in T$. But we the previous notations, one has $d(x,\gamma x) > d(x_1,\gamma x_1) = \min_{z\in A} d(z,\gamma z)$ if $x \notin A$. Therefore $l(\gamma)$ exists, verify that $l(\gamma) > 0$ and $A = \{x \in T,\ \ d(x,\gamma x) = l(\gamma)\}$. $\square$

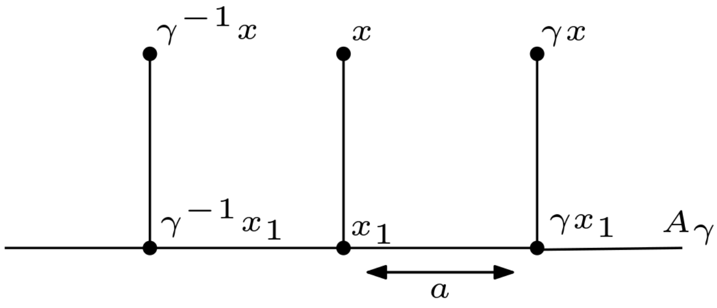

Figure 4

Corollary III.12. Let γ be an isometry of an $\mathbb{R}$-tree T. Then $l(\gamma)$ is well defined and called translation length of γ. Moreover, $l(\gamma) = 0$ if and only if γ is elliptic. $\square$

Proposition III.13. Let γ be an hyperbolic isometry of an non-empty $\mathbb{R}$-tree T. For any isometry δ of T, the isometry $\delta\gamma\delta^{-1}$ is hyperbolic and :

$$A_{\delta\gamma\delta^{-1}} = \delta A_\gamma \quad \text{et} \quad l(\delta\gamma\delta^{-1}) = l(\gamma)$$

Proof. Firstly, $\delta\gamma\delta^{-1}$ has no fixed point so is hyperbolic. Then :

$$l(\delta\gamma\delta^{-1}) = \min_{y\in T} d(y, \delta\gamma\delta^{-1}y) = \min_{x\in T} d(\delta x, \delta\gamma\delta^{-1}\delta x) = \min_{x\in T} d(x, \gamma x) = l(\gamma)$$

If $x \in A_\gamma$, then : $d(\delta x, \delta\gamma\delta^{-1}\delta x) = l(\gamma) = l(\delta\gamma\delta^{-1})$ so $\delta x \in A_{\delta\gamma\delta^{-1}}$ and $\delta A_\gamma \subset A_{\delta\gamma\delta^{-1}}$
If $y \in A_{\delta\gamma\delta^{-1}}$, then : $d(\delta^{-1}y, \gamma\delta^{-1}y) = d(\delta^{-1}y, \delta^{-1}\delta\gamma\delta^{-1}y) = d(y, \delta\gamma\delta^{-1}y) = l(\delta\gamma\delta^{-1}) = l(\gamma)$ so $\delta^{-1}y \in A_\gamma$. Thus, $y \in \delta A_\gamma$ and then $A_{\delta\gamma\delta^{-1}} \subset \delta A_\gamma$. $\square$

Proposition III.14. Let γ, δ be two hyperbolic isometries of an $\mathbb{R}$-tree T and their respective axes A_γ and A_δ. If $A_\gamma \cap A_\delta = \emptyset$, consider S the unique segment between A_γ and A_δ of length D. Then $S \subset A_{\gamma\delta}$ and $l(\gamma\delta) = l(\gamma) + l(\delta) + 2D$.

Proof. A first statement which is useful in this proof is that, if S is a segment meeting A_γ in a single point x, then γS is a segment disjoint from S and meeting A_γ only in γx.
Let's now write $S = [x, y]$ the segment of the proposition such that

$$S \cap A_\gamma = \{x\} \quad \text{and} \quad S \cap A_\delta = \{y\}.$$

According the above remark, one has that γS is a segment disjoint from S and meeting A_γ only in γx. Since $[x, \delta y] = [x, y] \cup [y, \delta y]$, then $\gamma[x, \delta y] = [\gamma x, \gamma y] \cup [\gamma y, \gamma\delta y]$. Thus

$$d(y, \gamma\delta y) = d(y, x) + d(x, \gamma x) + d(\gamma x, \gamma y) + d(\gamma y, \gamma\delta y) = l(\gamma) + l(\delta) + 2D.$$

Since $S \subset [y, \gamma\delta y]$, it suffices to show that $y \in A_{\gamma\delta}$. By the construction of the Theorem III.11, we must prove that

$$[y, \gamma\delta y] \cap (\gamma\delta)^{-1}[y, \gamma\delta y] = \{y\}.$$

Consider $z \in [y, \gamma\delta y]$ such that $z \neq y$. Hence, by the remark again, we have that $\delta[z,y]$ is a segment disjoint from $[z,y]$ and meeting A_δ only in δy.
Therefore, $\delta z \notin [\gamma^{-1}y, \delta y]$ and hence $\gamma\delta z \notin [y, \gamma\delta y]$.
All those arguments are resumed in the next figure : $\square$

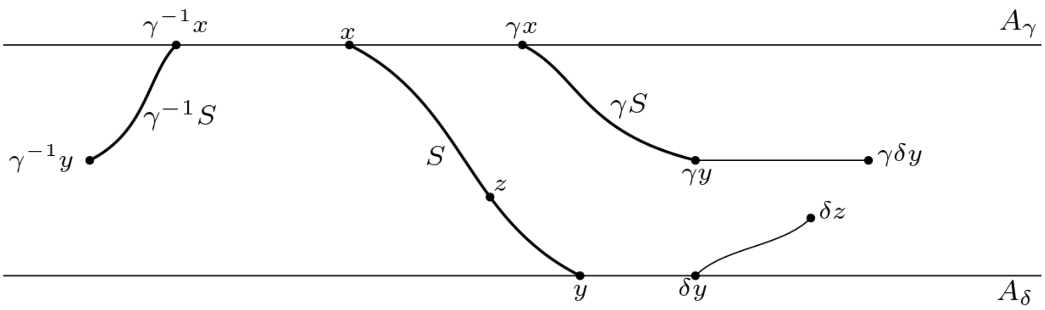

Figure 5

III.3 Observer's topology

The usual metric topology over $\hat{T} = \overline{T} \sqcup \partial T$ is first induced by the topology of $\overline{T}$ and then we add the open neighborhood of $[\varphi] \in \partial T$ given by $\tilde{V}([\varphi], r)$. The goal of this part is to introduce then observers' topology which is a weaker topology on this space than the metric one.
Then, some important properties of this topology will be shown. However, the purpose of this topology, which concerns the map $\mathcal{Q}$, won't be explained right now but in the Section V.

Definition III.6. A point $P \in \hat{T}$ is called extremal if $T \setminus \{P\}$ is connected, or equivalently, if P does not belong in the interior of any geodesic segment of $\hat{T}$

Proposition III.15. If $P \in \hat{T} \setminus T$, then P is extremal. $\square$

Definition III.7. Consider two distinct points $P, Q \in \hat{T}$, we define the direction $\mathrm{dir}_P(Q)$ of Q at P as the connected component of $\hat{T} \setminus \{P\}$ which contains Q.

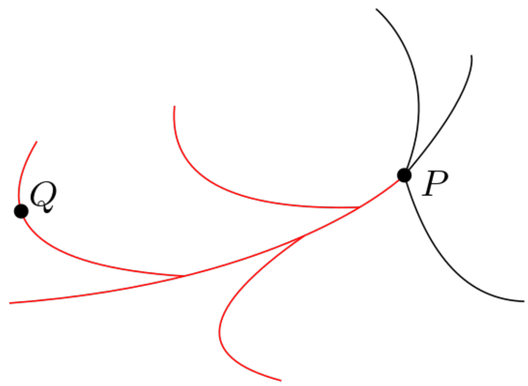

Figure 6: Picture to illustrate the direction $\mathrm{dir}_P(Q)$

Remark. An important thing to remark is that any direction is an open set of $\hat{T}$. This leads to the next definition :

Definition III.8. We define on $\hat{T}$ the observers' topology as the topology generated by the directions in $\hat{T}$.
From now, we denote $\hat{T}^{\text{obs}}$ the space $\hat{T}$ endowed with this topology. We still write $\hat{T}$ for this space endowed with the metric topology.

By definition, we have that the observers' topology is weaker than the metric topology. In particular, the identity map : $\hat{T} \longrightarrow \hat{T}^{\text{obs}}$ is continuous.

Definition III.9. Consider $(x_n)_{n\in\mathbb{N}}$ a sequence of point of $\hat{T}$. One says that this sequence is turning around a point $P \in \hat{T}$ if it stays in every direction at P only for a finite time.

Remark. By definition, a sequence can't turn around an extremal point. In particular, if there exists a sequence that turns around a point $P \in \hat{T}$, then $P \in \overline{T}$.

Proposition III.16. The observers' topology verify the next properties :

1. Every closed ball of $\overline{T}$ is closed is $\hat{T}^{\text{obs}}$.
2. Sequences that are turning around a point $P \in \overline{T}$ converges in $\hat{T}^{\text{obs}}$ to P.

Proof. 1. Consider $P \in \overline{T}$, $r > 0$ and $B = B(P, r) \subset \hat{T}$. For $x \in \partial B$, let $C(x)$ be the connected component of $\hat{T} \setminus \{x\}$ of P. Since B is connected, then $B \subset C(x)$. Then, it exists some points $(Q_{x,i})_{i\in I_x}$ such that $\bigcup_{i\in I_x} \mathrm{dir}_x(Q_{x,i}) = \hat{T} \setminus (C(x) \cup \{x\})$.
Thus, one has $\hat{T} \setminus \overline{B} = \bigcup_{x\in\partial B} \bigcup_{i\in I_x} \mathrm{dir}_x(Q_{x,i})$ is an open in $\hat{T}^{\text{obs}}$. The the closed ball $\overline{B}$ is closed in $\hat{T}^{\text{obs}}$.

2. Consider $(x_n)_{n\in\mathbb{N}}$ a sequence that are turning around a point $P \in \overline{T}$. Let's write $Q \in \hat{Q}$ and $V = \mathrm{dir}_Q(P)$. We can remark that $V^c \subset \mathrm{dir}_P(Q)$, hence $\{n \in \mathbb{N} \mid x_n \in V^c\}$ is finite since $\{n \in \mathbb{N} \mid x_n \in \mathrm{dir}_P(Q)\}$ is finite. □

Remark. There is no similar property for open balls, indeed an open ball in $\overline{T}$ is in general not open in $\hat{T}^{\text{obs}}$.

Proposition III.17. $\hat{T}$ and $\hat{T}^{\text{obs}}$ have exactly the same connected subsets and all of them are arcwise connected for both topologies.

Proof. A connected subset of $\hat{T}$ is connected for $\hat{T}^{\text{obs}}$. Moreover, since those subsets are arcwise for $\hat{T}$, they must be for $\hat{T}^{\text{obs}}$. Now consider C a connected subset for $\hat{T}^{\text{obs}}$ and assume that subset is not connected for the metric topology. Hence, it exists $P, Q \in C$ and $R \in]P, Q[$ such that $R \notin C$.
Let's write

$$\hat{T}^{\text{obs}} \setminus \{R\} = \bigsqcup_{i\in I} D_i$$

where the D_i are some direction from R. Thus, it exists i_0 such that $D_{i_0} = \mathrm{dir}_R(P)$. One defines

$$U = D_{i_0} \cap C \quad \text{and} \quad V = \bigsqcup_{i\neq i_0} D_i \cap C$$

Since $R \notin C$, one has $U \cup V = C$ and by definition $U \cap V = \emptyset$. Finally, those sets are not empty since $P \in U$ and $Q \in V$. This leads to a contradiction and therefore C is connected. $\square$

Consider a sequence $(P_n)_{n\in\mathbb{N}}$ of points in $\hat{T}$ and a base point $Q \in \hat{T}$. For $m \in \mathbb{N}$, let's define the set

$$I_m = \bigcap_{n\geq m} [Q, P_n] \subset \hat{T}$$

Since I_m is a segment, it exists $R_m \in \hat{T}$ such that $I_m = [Q, R_m]$. Moreover, $I_m \subset I_{m+1}$ and hence the set $\bigcup_{m\in\mathbb{N}} I_m$ is a segment. Thus, there exists a point $P \in \hat{T}$ such that

$$\bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} [Q, P_n] = [Q, P].$$

In particular, $R_m \xrightarrow[m\to+\infty]{} P$ for the metric topology (and hence as well for the observers' topology). This point P is called inferior limit from Q of the sequence $(P_n)_{n\in\mathbb{N}}$ and is denoted by $P = \liminf\limits_{n\to+\infty} {}_Q P_n$.

It might be hard to see what this limit really is. Actually an example that can help to understand this limit is given in the next figure :

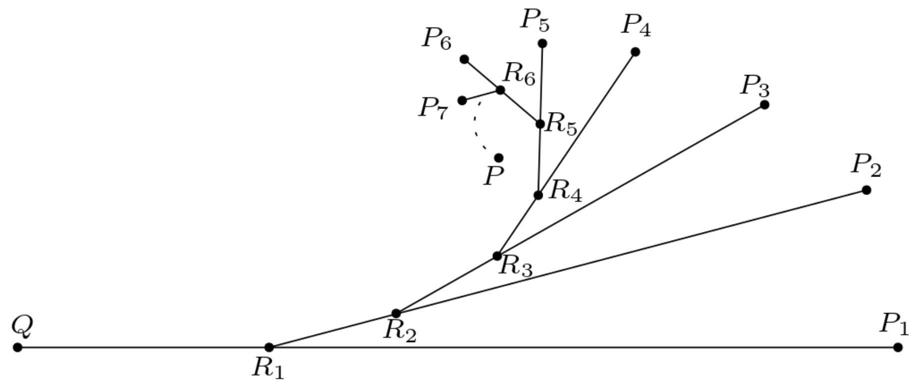

Figure 7: One interpretation of $P = \liminf\limits_{n\to+\infty} {}_Q P_n$

Lemma III.18. Consider $(P_n)_{n\in\mathbb{N}}$ a sequence of points pf $\hat{T}^{\text{obs}}$, and D be any direction of $\hat{T}$. Then one has :

1. If all P_n are contained in D, then for any $Q \in \hat{T}^{\text{obs}}$ the inferior limit $\liminf\limits_{n\to+\infty} {}_Q P_n$ lies to $\overline{D}$.

2. If for some $Q \in \hat{T} \setminus D$, the inferior limit $\liminf\limits_{n\to+\infty} {}_Q P_n$ belongs in D, then all of the P_n are eventually contained in D as well.

3. If for some $Q \in \hat{T}^{\text{obs}}$, the inferior limit $\liminf\limits_{n\to+\infty} {}_Q P_n$ belongs in D, then infinitely many of the P_n are contained in D as well.

Proof. 1. If all the P_n are contained in D, then all the $R_m \in \overline{D}$. Hence $\liminf\limits_{n\to+\infty} {}_Q P_n = \lim\limits_{m\to+\infty} R_m \in \overline{D}$.
2. Since $\liminf\limits_{n\to+\infty} {}_Q P_n \in D$ and D is open, then the R_m are eventually contained in D. Consider

$R_m \in D$ and write $D = \mathrm{dir}_U(R_m)$. Since Q is not contained in D, then

$$\bigcap_{n \geq m} [Q, P_n] = [Q, R_m] = [Q, U] \cup [U, R_m].$$

Hence, $U \in [Q, P_n[$ for any $n \geq m$ and then $P_n \in D$.
3. According to the point 2, it suffices to work on the case where $Q \in D$. Assume now that the P_n are eventually outside D. Let's write $D = \mathrm{dir}_U(Q)$. An argument similar to the one of the point 2 gives that for n large enough, one has $U \in [Q, P_n]$. Hence $\liminf_{n \to +\infty} {}_Q P_n \notin D$, which leads to a contradiction. $\square$

Remark. The point 3 looks like very close to the point 2 in the latest lemma. Nevertheless, the next drawing shows a case when $\liminf_{n \to +\infty} {}_Q P_n$ belongs in D but infinitely many of P_n lies outside of $D = \mathrm{dir}_U(Q)$.

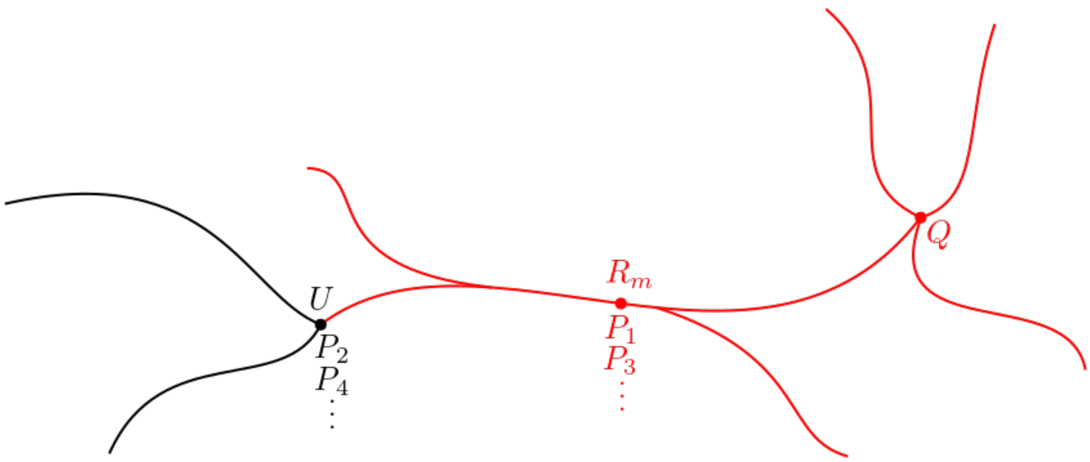

Figure 8

Lemma III.19. If a sequence $(P_n)_{n \in \mathbb{N}}$ converges in $\hat{T}^{\mathrm{obs}}$ to a point $P \in \hat{T}^{\mathrm{obs}}$, then for any $Q \in \hat{T}$ one has

$$P = \liminf_{n \to +\infty} {}_Q P_n.$$

Proof. Let D be a direction that contain P, then it contains all the P_n for n large enough by virtue of the observers' topology. By the point 1. of the latest lemma, one has that $R = \liminf_{n \to +\infty} {}_Q P_n \in \overline{D}$. Now, let's assume that $R \neq P$, and consider $\tilde{R} \in]P, R[$ and $D = \mathrm{dir}_{\tilde{R}} P$. We have that $R \notin \overline{D}$ which gives a contradiction. $\square$

As seen just before, the inferior limit verify some good properties useful to work with. However, it is mainly used since this one always exist. Moreover, we will see that it is useful for the definition of the map $\mathcal{Q}$ which is mandatory for our work. Here is a use of these properties to understand some link between T and $\hat{T}^{\mathrm{obs}}$.

Proposition III.20. $\hat{T}^{\mathrm{obs}}$ is Hausdorff. Moreover, if T is separable, then $\hat{T}^{\mathrm{obs}}$ is separable and compact.

Proof. Consider two distinct point $P,Q \in \hat{T}^{\mathrm{obs}}$ and let R be a point of $]P,Q[$. Then

$$\mathrm{dir}_R(P) \cap \mathrm{dir}_R(Q) = \emptyset.$$

Hence $\hat{T}^{\mathrm{obs}}$ is Hausdorff.
Assume now that T is separable and note χ_0 a countable subset and dense in T. For any P,Q distinct points of χ_0, we choose $R_{P,Q} \in]P,Q[$ and define

$$\chi = \{R_{P,Q} \in \hat{T} \mid (P,Q) \in \chi_0 \times \chi_0,\ P \neq Q\}$$

Thus, χ is a dense countable subset of $\hat{T}$ which intersects all non trivial geodesics of $\hat{T}$ and the family of open sets $(\mathrm{dir}_P(Q))_{P,Q \in \chi}$. These open sets and their finite intersections gives a countable basis for the observers' topology. Therefore $\hat{T}^{\mathrm{obs}}$ is metrizable and separable.
Consider now a sequence $(P_n)_{n \in \mathbb{N}}$ of points in $\hat{T}^{\mathrm{obs}}$ and denote $(D_i)_{i \in \mathbb{N}}$ a countable family of directions that generates the open sets of $\hat{T}^{\mathrm{obs}}$. By using a diagonal extraction, we can assume that for each direction D_i, the sequence is eventually inside or outside of D_i. Let $Q \in \hat{T}^{\mathrm{obs}}$ and consider $P = \liminf_{n \to +\infty}{}_Q P$. By virtue of the point 2 of the Lemma III.18, any direction D_i that contains P must contain infinitely main of the P_n. Hence, we can adapt our previous extraction to ensure that $P_n \xrightarrow[n \to +\infty]{} P$. □

Remark. Let T be an $\mathbb{R}$-tree which is minimal or has dense orbits. If T is endowed with an action of a finitely generated group (like a free group) by isometries, then T is separable.

IV Example for $L([a,b])$

We are here studying the space $\partial F_2/L(w)$ where $w=[a,b]=aba^{-1}b^{-1}$. We already looked at this example of lamination earlier to draw some biinfinite words over the Cayley graph in the Figure 1. The way we will study this example make us see the Cayley graph and those biinfinite rays as follow :

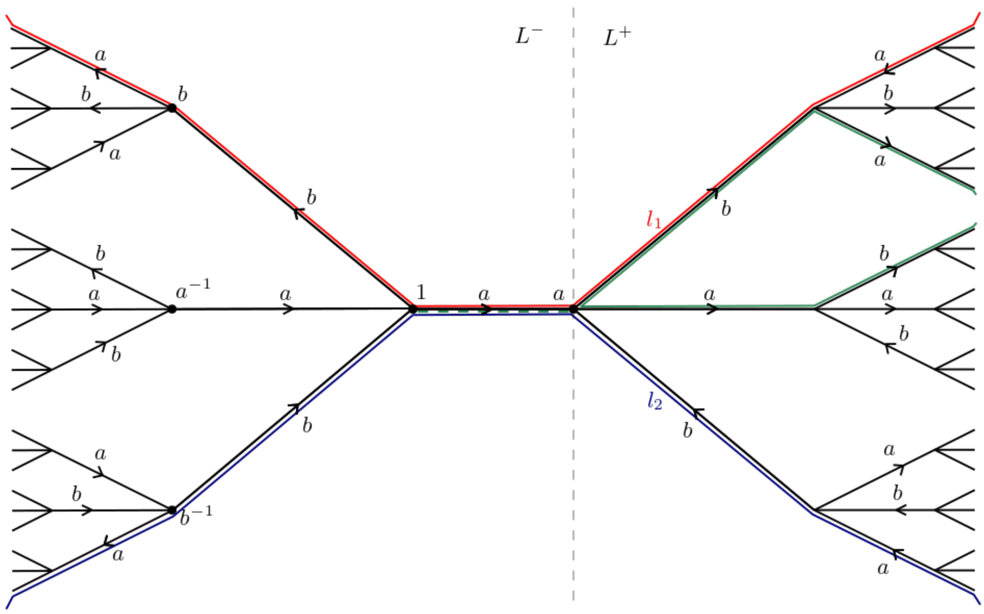

Figure 9

The reader would be convinced that these two picture both represent the Cayley graph of F_2 and the same biinfinite words of $\mathcal{A}=\{a,b,a^{-1},b^{-1}\}$. Thus, it would be here clearer that the left and right sides of this drawing can be related to the Cantor space. Indeed, the classical triadic Cantor space is kind of similar since at each step, you have to decide between two intervals instead of three.

Following to this point of view, we define $L^+=\{(x_1x_2\ldots,y_1y_2\ldots)\in\partial F_2 \mid x_1=y_1=a\}$, $l_1=(w^{-\infty},w^{+\infty})$ and $l_2=(b^{-1}w^{-\infty},b^{-1}w^{+\infty})$. According to these definitions, we have

$$L(w)=L^+\sqcup L^-\sqcup l_1^{\pm 1}\sqcup l_2^{\pm 1}$$

where $L^-=L(w)\setminus(L^+\sqcup l_1^{\pm 1}\sqcup l_2^{\pm 1})$.

Thus we consider the cylinder $C_a=\{(x_n)_{n\geq 1}\in\partial F_N \mid x_1=a\}$. For $X,Y\in C_a$, we write $X\sim Y$ if $(X,Y)\in L^+$. Then, the first step is to show that $C_a/L^+=C_a/\sim$ is homeomorphic to $[0,1]$.

Consider the classical construction of the Cantor set but with each interval cut in 5 pieces instead of 3. As said before, in this case, one has to chose at each step between 3 intervals. Thus, we can associate to each interval of the n-th step a number σ of n digits of base 3. We right I_σ such an interval as in the Figure 10. We write K the limit set of this construction

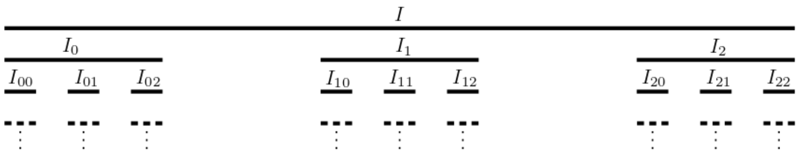

Figure 10: Construction of the set K

Moreover, it's necessarily to associate to each interval an element of $\mathcal{A}$ in such a way it's coherent with the Figure 9 (see Figure 11). For this step, we define to each an application τ as follow :

- $\tau(I_0) = b$, $\tau(I_1) = a$ and $\tau(I_2) = b^{-1}$.
- Recursively, if σ is number of n digits in base 3, we can define τ for $I_{\sigma 0}, I_{\sigma 1}$ and $I_{\sigma 2}$ as in the table 1:

If $\tau(I_\sigma) =$	a	b	a^{-1}	b^{-1}
$\tau(I_{\sigma 0}) =$	b	a^{-1}	b^{-1}	a
$\tau(I_{\sigma 1}) =$	a	b	a^{-1}	b^{-1}
$\tau(I_{\sigma 2}) =$	b^{-1}	a	b	a^{-1}

Table 1

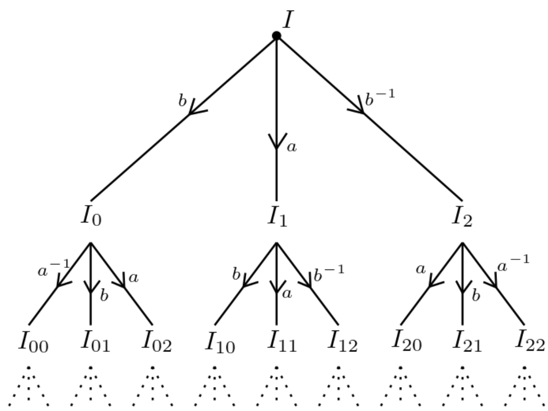

Figure 11

This map induces an homeomorphism $\psi : C_a \longrightarrow K$. Let $y \in K$, for any $n \geq 1$, there exists a unique number σ_n of n digits in base 3 such that $y \in I_{\sigma_n}$. Thus, there exists an unique $X = ax_2x_3 \cdots \in C_a$ such that

$$\forall n \geq 1,\ \tau(I_{\sigma_n}) = x_{n+1}$$

Therefore, we define $\psi(X) = y$ which is, by construction, an homeomorphism. In another hand, the sequence $(\sigma_n)_{n\geq 1}$ induce a sequence $\tilde{\sigma} = (a_k)_{k\geq 1}$ such that:

$$\forall n \geq 1, \quad \sigma_n = a_1 a_2 \ldots a_n.$$

Then, let's define $\theta(y) = \sum_{k=1}^{+\infty} \frac{a_k}{3^k} \in I$. For $y, z \in K$, let's write $y \sim_\theta z \iff \theta(y) = \theta(z)$. Then, it's a known fact that $K/\sim_\theta \simeq [0,1]$. Therefore, it suffices to show that:

$$\forall X, Y \in C_a, \quad X \sim Y \iff \psi(X) \sim_\theta \psi(Y)$$

Here, there appears the interest of such a complex construction for ψ. Let's see how it works on an example. Consider $(X,Y) = (aw^{-\infty}, aw^{+\infty}) \in L^+$, we write $\tilde{\sigma}_X$ and $\tilde{\sigma}_Y$ the sequences associated to $\psi(X)$ and $\psi(Y)$ respectively. Thus, one has

$$\tilde{\sigma}_X = 02222\dots \quad \text{and} \quad \tilde{\sigma}_Y = 1000\dots$$

Thus, it's clear that $\theta(\psi(X)) = \theta(\psi(Y))$. Let's now prove this statement.

It suffices to show that for $(X,Y) = (vw^{-\infty}, vw^{+\infty}) \in L^+$, one has $\theta(\psi(X)) = \theta(\psi(Y))$. As said in the Proposition II.12, the image of (X,Y) by $\rho_{\mathcal{A}}$ is nothing but a shift of the biinfinite word $w^{-\infty} \cdot w^{+\infty} = \dots aba^{-1}b^{-1} \cdot aba^{-1}b^{-1} \dots$.

This means that we have to only check the property in 4 cases depending on this shift which can be written as:

$$(X,Y) = (a\alpha_2 \dots \alpha_r bab^{-1}a^{-1}b\dots \;,\; a\alpha_2 \dots \alpha_r aba^{-1}b^{-1}a\dots)$$

$$(X,Y) = (a\alpha_2 \dots \alpha_r ab^{-1}a^{-1}ba\dots \;,\; a\alpha_2 \dots \alpha_r b^{-1}aba^{-1}b^{-1}\dots)$$

$$(X,Y) = (a\alpha_2 \dots \alpha_r b^{-1}a^{-1}bab^{-1}\dots \;,\; a\alpha_2 \dots \alpha_r a^{-1}b^{-1}aba^{-1}\dots)$$

$$(X,Y) = (a\alpha_2 \dots \alpha_r a^{-1}bab^{-1}a^{-1}\dots \;,\; a\alpha_2 \dots \alpha_r ba^{-1}b^{-1}ab\dots)$$

with $\alpha_2, \dots, \alpha_r \in \mathcal{A}^{\pm 1}$ and all infinite words reduced. Let's see what happens in the first case with $\alpha_r = a$ (all others cases are similar). Thus, there exists $x_1, \dots, x_{r-1} \in \{0,1,2\}$ such that:

$$\tilde{\sigma}_X = x_1 \dots x_{r-1} 0222\dots \quad \text{and} \quad \tilde{\sigma}_Y = x_1 \dots x_{r-1} 1000\dots$$

Therefore, we have $\theta(\psi(X)) = \theta(\psi(Y))$. In another hand, consider two distinct points $x, y \in K$, $X = \psi^{-1}(x)$ and $Y = \psi^{-1}(y)$. Then, one has that $\theta(x) = \theta(y)$ if and only if the sequences $\tilde{\sigma}_X$ and $\tilde{\sigma}_Y$ look like

$$\tilde{\sigma}_X = x_1 \dots x_{r-1} x_r 222\dots \quad \text{and} \quad \tilde{\sigma}_Y = x_1 \dots x_{r-1} \overline{x_r} 000\dots$$

with $x_r \in \{0,1\}$ and $\overline{x_r} = x_r + 1$ (or the same but with X and Y swapped). And in this case, it's easy to see that $(X,Y) \in L^+$.

As said before, this prove that $C_a/L^+ \simeq [0,1]$. Considering $C = \partial F_n \setminus C_a$, we have by symmetry that $C/L^- \simeq [0,1]$. Then we have to consider l_1 and l_2, which glue these intervals in some sense. At the end, this gives that

$$\partial F_N / L([a,b]) \simeq \mathbb{S}^1.$$

V Dual lamination and map $\mathcal{Q}$

V.1 Bounded backtracking

An important property for $\mathbb{R}$-trees is the bounded backtracking property (BBT) which is mandatory to define some necessary objects for the following. Fortunately, we will see that the $\mathbb{R}$-trees we use to work with all satisfy this property.

Consider T an $\mathbb{R}$-tree endowed with a F_N-action by isometries. We described earlier the behavior of isometries of $\mathbb{R}$-trees which allows us to define the translation length $\|w\|_T$ where $w \in F_N$, which correspond to the translation length of the isometry associated to w.

Proposition V.1. We say that such an $\mathbb{R}$-tree T verifies the BBT property if it satisfies one of the following equivalent assertions :

1. For any $P \in T$, there exists $C > 0$ such that if $v, w \in F_N$ satisfy that $|vw|_{\mathcal{A}} = |u|_{\mathcal{A}} + |v|_{\mathcal{A}}$, then
$$d(vP, [P, vwP]) \leq C$$
2. For $P \in T$, there exists $C > 0$ such that if $v_1, v_2 \in F(\mathcal{A}) = F_N$ are reduced words, then
$$d((v_1 \wedge v_2)P, [v_1P, v_2P]) \leq C$$

Proof. $(1 \Rightarrow 2)$ Let $v = v_1^{-1}(v_1 \wedge v_2)$ and $w = (v_1 \wedge v_2)^{-1}v_2$. One has
$$|v|_{\mathcal{A}} = |v_1|_{\mathcal{A}} - |v_1 \wedge v_2|_{\mathcal{A}} \quad \text{and} \quad |w| = |v_2| - |v_1 \wedge v_2|_{\mathcal{A}}.$$
Moreover $vw = v^{-1}w$ so $|vw|_{\mathcal{A}} = |v_1|_{\mathcal{A}} + |v_2|_{\mathcal{A}} - 2|v_1 \wedge v_2|_{\mathcal{A}}$. Thus $|vw|_{\mathcal{A}} = |v|_{\mathcal{A}} + |w|_{\mathcal{A}}$, hence for $P \in T$, it exists $C > 0$ (independent on v_1 and v_2) such that $d(v_1^{-1}P, [P, vwP]) \leq C$. Therefore $d(v^{-1}(v_1 \wedge v_2)P, [P, v_1^{-1}v_2P]) \leq C$ and then $d((v_1 \wedge v_2)P, [v_1P, v_2P]) \leq C$.
$(2 \Rightarrow 1)$ $|vw|_{\mathcal{A}} = |v|_{\mathcal{A}} + |w|_{\mathcal{A}}$ if and only if $v^{-1} \wedge w = 1$. Thus there exist $C > 0$ (independent on v and w) such that $d(P, [v^{-1}P, wP] \leq C$ and then $d(vP, [P, vwP]) \leq C$. □

Remark. Here we see that this constant C depends on the basis $\mathcal{A}$. Thus, if T satisfies BBT and $P \in T$, we will write $\mathrm{BBT}(\mathcal{A}, P)$ the smallest such $C > 0$.

Corollary V.2. Let T be an $\mathbb{R}$-tree that satisfies BBT. Then for $P \in T$, one has :

1. Let $u = x_1 \ldots x_n \in F(\mathcal{A}) = F_N$ and a prefix $v = x_1 \ldots x_m$ of u. Then
$$d(vP, [P, uP]) \leq \mathrm{BBT}(\mathcal{A}, P)$$
2. Any cyclically reduced word $u \in F(\mathcal{A}) = F_N$ satisfies
$$d(P, uP) \leq 2\mathrm{BBT}(\mathcal{A}, P) + \|u\|_T$$

Proof. 1. Let's write $w = w_{m+1} \dots w_n$ and use the first property BBT. Thus $d(vP, [P, vwP]) \leq C$ with $vw = u$.
2. Since u is cyclically reduced, we have that u and u^2 are subwords of u^3.Using the first point, we have that $d(uP, [P, u^3P]) \leq \mathrm{BBT}(\mathcal{A}, P)$ and $d(uP, [P, u^3]) \leq \mathrm{BBT}(\mathcal{A}, P)$.
Thus $d(P, [u^{-1}P, u^2P]) \leq \mathrm{BBT}(\mathcal{A}, P)$ and $d(uP, [u^{-1}P, u^2P]) \leq \mathrm{BBT}(\mathcal{A}, P)$.
Moreover, by construction of A_{u^2}, we have that $[u^{-1}P, uP] \cap [P, u^2P] = [P, u^2P] \cap u^{-1}([P, u^2P]) \subset A_{u^2}$. Finally $A_u = A_{u^2}$ and then we have

$$d(uP, P) = d(P, A_u) + \|u\|_T + d(uP, A_u) \leq 2\mathrm{BBT}(\mathcal{A}, P) + \|u\|_T$$

□

Some kind of $\mathbb{R}$-trees are very important for the following, we give here some definitions and properties.

Definition V.1.
- An $\mathbb{R}$-tree T is minimal if any F_N-invariant subtree is either empty or T.
- An $\mathbb{R}$-tree T is called free if the action of F_N on T is free.

Proposition V.3. Every free $\mathbb{R}$-tree satisfies BBT.

Proof. See Corollary 3.3 in [3] □

Lemma V.4. Let T be a free $\mathbb{R}$-tree with dense F_N-orbits. For any $P \in T$, there exists a basis $\mathcal{A}$ such that $\mathrm{BBT}(\mathcal{A}, P) > 0$ is arbitrarily small.

Proof. See the Lemma 3.2 in [4] □

V.2 Dual lamination

As before, let T be an $\mathbb{R}$-tree with a left action by isometries of F_N and $\mathcal{A}$ a fixed basis of F_N.

Definition V.2. For any $\varepsilon > 0$, we define

$$\Omega_\varepsilon(T) = \{w \in F_N \mid \|w\|_T < \varepsilon\}$$

which is invariant under conjugation (according to the Proposition III.13) and inversion. Thus we define

$$\Omega^2_\varepsilon(T) = \bigcup_{w \in \Omega_\varepsilon(T)} L^2(w) \subset \partial^2 F_N$$

which is F_N- and flip-invariant.

Definition V.3. Let T be an $\mathbb{R}$-tree with dense F_N-orbits, then $\Omega^2_\varepsilon(T)$ is non-empty and we define the algebraic lamination

$$L^2_\varepsilon(T) = \overline{\Omega^2_\varepsilon(T)} \subset \partial^2 F_N.$$

Then, we define the dual algebraic lamination associated to T as follows :

$$L^2_\Omega(T) = \bigcap_{\varepsilon>0} L^2_\varepsilon(T)$$

Definition V.4. Let $L^1_{\mathcal{A}}(T)$ be the set of infinite words $X \in \partial F(\mathcal{A})$ such that $(X_i P)_{i\geq 1}$ is bounded in T for some $P \in T$, with X_i the i-prefix of X.

Remark. It's clear that this definition does not depend on the point P. However, even if $\partial F_N = \partial F(\mathcal{A})$ is independent on $\mathcal{A}$, it's not that clear that we have a similar result for $L^1_{\mathcal{A}}$.

Proposition V.5. The canonical identification $\partial F(\mathcal{A}) = \partial F_N$ gives an identification $L^1_{\mathcal{A}}(T) \subset \partial F_N$ which does not depend on the basis $\mathcal{A}$. In particular, it justifies the notation $L^1(T)$ for the following.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two basis of F_N. Let $X \in \partial F_N$ and let $A \in \partial F(\mathcal{A})$ and $B \in \partial F(\mathcal{B})$ the infinite words corresponding to X. We only have to show that if $A \in L^1_{\mathcal{A}}(T)$, then $B \in L^1_{\mathcal{B}}(T)$. Suppose that $A \in L^1_{\mathcal{A}}(T)$ so $(A_i P)_{i\in\mathbb{N}}$ is bounded for some $P \in T$. By the Proposition II.9, if $C = \mathrm{BBT}(\mathcal{A},\mathcal{B})$, one has

$$\forall j \in \mathbb{N},\ \exists i \in \mathbb{N} \quad d_{\mathcal{A}}(A_i, B_j) \leq C$$

Thus, let's define $S_P = \max_{\alpha\in\mathcal{A}\cup\mathcal{A}^{-1}} \{d(P,\alpha P)\} < +\infty$. Then we write A_i and B_j in $F(\mathcal{A})$ as

$$A_i = x_1 x_2 \dots x_i \quad \text{and} \quad B_j = y_1 y_2 \dots y_r$$

According to the preceding property, we have that $m = |i-r| \leq C$.
Thus, if $w = A_i^{-1} B_j \in F_N$, then $w = z_1 z_2 \dots z_m \in F(\mathcal{A})$. Finally, one has

$$d(A_i P, B_j P) = d(P, wP) \leq d(P, z_m P) + d(z_m P, z_m z_{m-1} P) + \dots + d(z_2 \dots z_m P, wP) \leq m S_p \leq C S_p$$

Since $(A_i P)_{i\in\mathbb{N}}$ is bounded, then $(B_j P)_{i\in\mathbb{N}}$ is as well. □

Lemma V.6. Let T be an $\mathbb{R}$-tree which satisfies BBT and $X \in \partial F_N \setminus L^1(T)$. For any $P \in T$, we have that $d(P, X_i P) \xrightarrow[i\to+\infty]{} +\infty$.

Proof. Let $P \in T$ and let's write $C = \mathrm{BBT}(\mathcal{A}, P)$. Since $(X_i P)$ is not bounded, it exists a subsequence $(X_{\varphi(i)} P)$ such that $d(P, X_{\varphi(i)} P) \xrightarrow[i\to+\infty]{} +\infty$. Let $M > 0$, it exists $n \in \mathbb{N}$ such that for $i \geq n$, one has $d(P, X_{\varphi(i)} P) \geq M$.
Consider $j \geq \varphi(n)$, it exists $i \geq n$ such that $j \geq \varphi(i)$. Since T satisfies BBT and $X_{\varphi(i)}$ is a subword of X_j, we have that $d(X_{\varphi(i)} P, [P, X jP]) \leq C$. Let R be the point of $[P, X jP]$ such that $d(X_{\varphi(i)} P, [P, X_j P]) = d(X_{\varphi(i)}, R)$. Hence $d(X_{\varphi(i)} P, R) \leq C$ and $d(P, R) \geq M - C$. Since, $d(P,R) \leq d(P, X_j P)$ and C only depends on P and $\mathcal{A}$, this show that $d(P, X_j) \xrightarrow[j\to+\infty]{} +\infty$. □

Remark. Since the subset $L^1(T)$ of ∂F_N is F_N-invariant, then it must be dense in ∂F_N unless it is empty (see Proposition II.6).

Lemma V.7. If T is an $\mathbb{R}$-tree which satisfies BBT and $X \in L^1(T)$, then for any $P \in T$ and $\lambda > 2$, there exists $K \geq 1$ such that

$$\forall i, j \geq K, \quad d(X_iP, X_jP) \leq \lambda \mathrm{BBT}(\mathcal{A}, P).$$

Remark. In particular, for $P \in T$, there exists $K \geq 1$ such that

$$\forall i, j \geq K, \quad d(X_iP, X_jP) \leq 3\mathrm{BBT}(\mathcal{A}, P).$$

Proof. Let $\varepsilon > 0$ and $M = \sup_{k \geq 1} d(P, X_kP) < +\infty$. Consider $K \geq 0$ such that

$$M - \varepsilon \leq d(P, X_KP) \leq M.$$

Let $C = \mathrm{BBT}(\mathcal{A}, P)$ and $i, j \geq K$, the points X_iP and X_jP must be in one of the configuration shown in the Figure 12. Thus, if we consider $\varepsilon \leq C$, it's not hard to see that we have

$$d(X_iP, X_jP) \leq 2C + 2\varepsilon.$$

$\square$

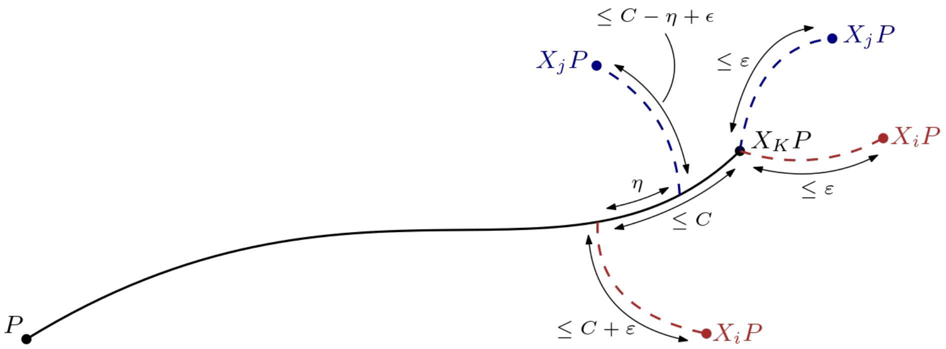

Figure 12

V.3 The map $\mathcal{Q}$ and the lamination $L^2_{\mathcal{Q}}(T)$

In this section, we only consider $\mathbb{R}$-trees that satisfies the BBT property. Using this property, the idea is to define a natural map as an extension of the action of F_N on $\hat{T}$ over ∂F_N. Firstly, if $X \in \partial F_n \setminus L^1(T)$, let's define

$$\mathcal{Q}(X) = \lim_{i \to +\infty} X_iP$$

where $(X_i)_{i \geq 1}$ is the sequence of prefixes of X and $P \in T$. The previous notation leads to a natural questions : does this limit exists and depends it on $(X_i)_{i \geq 1}$ and P ?

Proposition V.8. Let T be an $\mathbb{R}$-tree that satisfies the BBT property and $X \in \partial F_n \setminus L^1(T)$. For any $P \in T$ and any sequence $(X_i)_{i \geq 1}$ which converges to X, $\lim_{i \to +\infty} X_iP$ exists and is independent on P and $(X_i)_{i \geq 1}$.

Proof. Let's write $C = \mathrm{BBT}(\mathcal{A}, P)$.

Existence : As said in the Lemma V.6 , for any $M > 0$, it exists $K \geq 1$ such that for any $i \geq K$, we have $d(P, X_i P) \geq M$. Consider $j \geq i \geq K$ and $R \in [P, X_j P]$ such that $d(P,R) = (X_i P | X_j P)_P$. In particular, $d(X_i P, [P, X_j]) = d(X_i P, R)$. Since T satisfies BBT, so $d(X_i P, Q) \leq C$. Hence, $d(P,R) \geq M - C$. This shows that $(X_i P | X_j P)_P \xrightarrow[i,j \to +\infty]{} +\infty$.

Independence on $(X_i)_{i \geq 1}$: As said in the Lemma V.6 , for any $M > 0$, it exists $K \geq 1$ such that for any $i \geq K$, we have $d(P, X_i P) \geq M$. Consider now $(Y_i)_{i \geq 1}$ a sequence of $F_N = F(\mathcal{A})$ such that $Y_i \xrightarrow[i \to +\infty]{} X$ in ∂F_N. There exist $n \in \mathbb{N}$ such that

$$\forall i, j \geq n,\ \exists k \geq K,\ X_i \wedge Y_j = X_k$$

Consider such a k and let $R \in [X_i P, Y_j P]$ the point such that $d(P,R) = (X_i P | Y_j P)_P$. By using the second property of BBT, one has that $d(X_k P, R) \leq d(X_k P, [X_i P, Y_j P]) \leq C$. Hence $d(P,R) \geq M - C$, which shows that $(X_i P | Y_j P)_P \xrightarrow[i,j \to +\infty]{} +\infty$.

Independence on P : Let $P, Q \in T$, since F_N acts on T by isometry, it's clear that :

$$(X_n P | X_n Q)_P = d(P, X_n P) + d(P, X_n Q) - d(X_n P, X_n Q) = d(P, X_n P) + d(P, X_n Q) - d(P, Q) \xrightarrow[n \to +\infty]{} +\infty$$

Thus, consider $M > 0$, it exists $K \geq 1$ such that : $\forall n \geq K,\ (X_n P | X_n Q)_P \geq M$.
Consider $n, m \geq k$ and R the point of $[P, X_n P]$ such that $d(P,R) = (X_n P | X_n Q)_P$. Let now $\tilde{R}$ be the point of $[P, X_n P]$ such that $d(P, \tilde{R}) = (X_n P | X_m Q)_P$. By symmetry of the roles of P and Q, we can assume that $n \leq m$ without loss of generality. This leads to two (not incompatible) cases :

- If $\tilde{R} \in [R, X_n P]$, then $d(P, \tilde{R}) \geq d(P, R) \geq M$.
- If $\tilde{R} \in [P, R]$, then $R \in [\tilde{R}, X_n Q]$ and $d(\tilde{R}, R) \leq d(\tilde{R}, X_n Q)$. Also, since T satisfies BBT, one has that $d(X_n Q, \tilde{R}) = d(X_n Q, [P, X_m Q]) \leq C$.
  This gives that $d(R, \tilde{R}) \leq C$ and $d(P, \tilde{R}) \geq M - C$

Both cases allow us to give as conclusion that $(X_n P | X_m Q)_P \xrightarrow[n,m \to +\infty]{} +\infty$ ☐

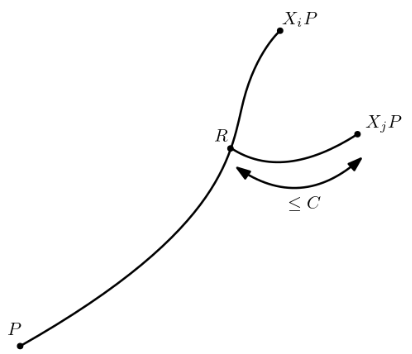

Figure 13: Drawing for the existence

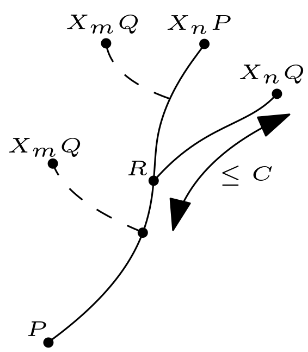

Figure 14: Drawing of the 2 cases in the third part

It would be natural to try to define an extension of $\mathcal{Q}$ over ∂F_N. The problem here is that when $(X_iP)_{i\geq 1}$ is not bounded, this sequence doesn't necessarily converge. Nevertheless, if we add some properties on T, it's always possible to locate points at infinity and to refine this region. All these questions behind this function consist of the Gilbert Levitt and Martin Lustig's work in [5]. For our work, we will use the next lemma :

Lemma V.9. Let T be a minimal $\mathbb{R}$-tree with dense F_N-orbits and trivial arc stabilizers and $X \in \partial F_N \setminus L^1(T)$. For any $P \in T$ and $Q \in \overline{T}$ there exists a sequence $(w_i)_{i\geq 1}$ of F_N such that $w_i \xrightarrow[i\to+\infty]{} X$ and $w_iP \xrightarrow[i\to+\infty]{} Q$. Conversely, if $\tilde{w}_i \xrightarrow[i\to+\infty]{} X$ and $(\tilde{w}_i)_{i\geq 1}$ converges to some point $R \in \overline{T}$, then $R = Q$.

Remark. According to the Lemma V.3, any minimal $\mathbb{R}$-tree with dense F_N-orbits and small satisfies the hypothesis of the previous lemma.

From now, we consider T a minimal $\mathbb{R}$-tree with dense F_N-orbits and free (which implies trivial arc stabilizers). In this context we can define the application $\mathcal{Q} : \partial F_N \longrightarrow \hat{T}$ by :
For any $X \in \partial F_N$, $\mathcal{Q}(X)$ is the only point in $\hat{T}$ such that it exists a sequence $(w_i)_{i\geq 1}$ of F_N that converges to X and such that $w_iP \xrightarrow[i\to+\infty]{} \mathcal{Q}(X)$ for some $P \in T$. The Lemma V.7 gives :

Proposition V.10. Let $P \in T$, and let $\mathcal{A}$ be a basis of F_N. Then for every $X = x_1x_2\ldots \in L^1_{\mathcal{A}}(T)$, there exists $K \geq 1$ such that for every $X_k = x_1\ldots x_k$ with $k \geq K$, one has

$$d(X_kP, \mathcal{Q}(X)) \leq 3\mathrm{BBT}(\mathcal{A}, P).$$

Proposition V.11. $\mathcal{Q}$ is F_N-invariant.

Proof. Consider a sequence $(w_i)_{i\geq 1}$ of F_N and $P \in T$ such that :

$$w_i \xrightarrow[i\to+\infty]{} X \in \partial F_N \quad \text{and} \quad w_iP \xrightarrow[i\to+\infty]{} \mathcal{Q}(X)$$

Then for $w \in F_N$, one has :

$$ww_i \xrightarrow[i\to+\infty]{} wX \quad \text{and} \quad ww_iP \xrightarrow[i\to+\infty]{} w\mathcal{Q}(X)$$

Hence $\mathcal{Q}(wX) = w\mathcal{Q}(X)$ by definition.

□

Proposition V.12. The restriction $\mathcal{Q} : \partial F_N \setminus L^1(T) \longrightarrow \partial T$ is injective.

Proof. Let $P \in T$ and $X, Y \in \partial F_N \setminus L^1(T)$, one has

$$\begin{aligned}
\mathcal{Q}(X) = \mathcal{Q}(Y) &\implies \lim_{i \to +\infty} X_i P = \lim_{i \to +\infty} Y_i P \\
&\implies (X_i P | Y_i P)_P \xrightarrow[i \to +\infty]{} +\infty \\
&\implies d((X_i \wedge Y_i)P, P) \xrightarrow[i \to +\infty]{} +\infty \\
&\implies |X_i \wedge Y_i|_{\mathcal{A}} \xrightarrow[i \to +\infty]{} +\infty \\
&\implies X = Y
\end{aligned}$$

Note that the third implication comes from the second BBT property. $\square$

Remark. Actually, this restriction is continuous too but this property will be studied in a next part for a weaker topology. Nevertheless, those results are false for $\mathcal{Q} : \partial F_N \longrightarrow \hat{T}$ in general. However, it will be shown later that $\mathcal{Q}$ is surjective (see Proposition V.17). For now, we admit this statement.

Definition V.5. We define the map $\mathcal{Q}^1 : L^1(T) \longrightarrow \overline{T}$ induced by $\mathcal{Q}$. According to preceding properties of $\mathcal{Q}$, we have that $\mathcal{Q}^1$ is F_N-equivariant and surjective.

Definition V.6. Let T be a free $\mathbb{R}$-tree with dense F_N-orbits. Let's define a F_N- and flip-invariant subset of $\partial^2 F_N$

$$L^2_{\mathcal{Q}}(T) = \{(X, X') \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(X')\}$$

Remark. Since the restriction $\mathcal{Q} : \partial F_N \setminus L^1(T) \longrightarrow \partial T$, if $(X, X') \in L^2_{\mathcal{Q}}(T)$, then $X, X' \in L^1(T)$.

Definition V.7. Let's define a map

$$\mathcal{Q}^2 : L^2_{\mathcal{Q}}(T) \longrightarrow \overline{T},\ (X, X') \mapsto \mathcal{Q}(X) = \mathcal{Q}(X')$$

which is F_N-equivariant and flip-invariant.

Proposition V.13. Let T be a free $\mathbb{R}$-tree with dense F_N-orbits. Then the subset $L^2_{\mathcal{Q}}(T)$ is closed, and the map $\mathcal{Q}^2$ is continuous for the metric topology.

Proof. Let $(X_n, Y_n)_{n \in \mathbb{N}}$ be sequence of $L^2_{\mathcal{Q}}(T)$ that converges to $(X, Y) \in \partial^2 F_N$. Let $P \in T$, $\mathcal{A}$ be a basis of F_N and define $C = \mathrm{BBT}(\mathcal{A}, P)$. Since $X \neq Y$, we can define $h = X \wedge Y \in F(\mathcal{A})$.
The convergence $(X_n, Y_n) \xrightarrow[n \to +\infty]{} (X, Y)$ implies that $X_n \xrightarrow[n \to +\infty]{} X$ and $Y_n \xrightarrow[n \to +\infty]{} Y$. Then, one has that $X, Y \in L^1(T)$ and for $n \in \mathbb{N}$ large enough, that's clear that $X_n \wedge Y_n = h$. Hence, for $k \geq 1$ large enough, the k-prefixes $X_{n,k}, Y_{n,k}$ of X_n and Y_n respectively, verify that $X_{n,k} \wedge Y_{n,k} = h$.
By the Proposition V.10, for $k \geq 1$ large enough, one has

$$d(X_{n,k}P, \mathcal{Q}(X_n)) \leq 3C \quad \text{and} \quad d(Y_{n,k}P, \mathcal{Q}(Y_n)) \leq 3C.$$

By hypothesis, $Q(X_n) = Q(Y_n)$ and then one has that for any $k \geq 1$ large enough

$$d(X_{n,k}P, Y_{n,k}P) \leq 6C$$

Moreover, by the second property of BBT, one has : $d(hP, [X_{n,k}P, Y_{n,k}P]) \leq C$. Finally, since $X, Y \in L^1(T)$, if $n \in \mathbb{N}$ and $k \geq 0$ are large enough, one has

$$d(X_{n,k} \wedge X, Q(X)) \leq 3C \quad \text{and} \quad d(Y_{n,k} \wedge Y, Q(Y)) \leq 3C$$

By remarking that for $n \in \mathbb{N}$ and $k \geq 1$ large enough, one has: $(X_{n,k} \wedge X) \wedge (Y_{n,k} \wedge Y) = h$, we can assume that

$$d(hP, Q(X)) \leq 10C \quad \text{and} \quad d(hP, Q(Y)) \leq 10C$$

By the Lemma V.4, we can make C arbitrarily small. Thus $Q(X_n) \xrightarrow[n \to +\infty]{} Q(X)$, $Q(Y_n) \xrightarrow[n \to +\infty]{} Q(Y)$ and $Q(X) = Q(Y)$. □

Corollary V.14. The set $L^2_Q(T) \subset \partial^2 F_N$ is an algebraic lamination. □

Proposition V.15. Let T be a free $\mathbb{R}$-tree with dense F_N-orbits. Then

$$L^2_\Omega(T) = L^2_Q(T).$$

Proof. See Proposition 8.5 in [4] □

We gave previously two definition of the dual lamination in two different context. However, the latest result gives that these definitions are equivalents and justifies the notation of $L^2(T)$ for the dual algebraic lamination associated to T. Actually, there exists one last definition which is the algebraic lamination defined by the recurrent language associated to $L^1(T)$. We decided not to present this one since it is not needed in the purpose of the present report.

V.4 Continuity of $\mathcal{Q}$ over the observers' topology

In this part, we wants to show that the dual laminations of an $\mathbb{R}$-tree introduced before totally characterize the space $\hat{T}^{\text{obs}}$. To achieve this, we need to show that $\mathcal{Q}$ satisfies some properties.

Consider T be a free $\mathbb{R}$-tree with dense F_N-orbits. As seen in the latest part, the function $\mathcal{Q} : \partial F_N \longrightarrow \hat{T}$ is not necessarily continuous. However, it can be shown that for any point $P \in \hat{T}^{\text{obs}}$, any $X \in \partial F_N$ and any sequence $(w_i)_{i \in \mathbb{N}}$ such that $w_i \xrightarrow[i \to +\infty]{} X$, one has

$$\liminf_{i \to +\infty} {}_Q w_i P = \mathcal{Q}(X)$$

Also, this fact help us to see the link between the map $\mathcal{Q}$ and the weaker observers' topology on $\hat{T}$. In particular, it allows us to show that $\mathcal{Q}$ is continuous for this topology.

Proposition V.16. The map $\mathcal{Q} : \partial F_N \longrightarrow \hat{T}^{\text{obs}}$ is continuous.

__Proof.__ Consider a sequence $(X_k)_{k\in\mathbb{N}}$ a sequence of ∂F_N such that $X_k \xrightarrow[i\to+\infty]{} X \in \partial F_N$. Since $\hat{T}^{\mathrm{obs}}$ is compact, it suffices to show that $(Q(X_k))_{k\in\mathbb{N}}$ has an unique subsequential limit which is $\mathcal{Q}(X)$. Thus, we can assume that $\mathcal{Q}(X_k)$ converges to some point $Q \in \hat{T}^{\mathrm{obs}}$ and we must show that $Q = \mathcal{Q}(X)$.

We suppose that $Q \neq \mathcal{Q}(X)$ and consider $S \in]Q, \mathcal{Q}(X)[$. Then, we choose for each $k \in \mathbb{N}$ a sequence $(w_{k,i})_{i\in\mathbb{N}}$ of words in F_N such that $w_{k,i} \xrightarrow[i\to+\infty]{} X_k \in \partial F_N$.

Consider now $D = \mathrm{dir}_S(Q)$, one has that $X_k \in D$ for k large enough. Moreover, if $P \in \hat{T}^{\mathrm{obs}} \setminus D$, we have that for i large enough, $w_{k,i}P \in D$ by the point 3 of the Lemma III.18. Thus, we can extract diagonally a subsequence $(w_{k,i(k)})_{k\in\mathbb{N}}$ such that

$$\forall k \in \mathbb{N},\ w_{k,i(k)}P \in D \quad \text{and} \quad w_{k,i(k)} \xrightarrow[k\to+\infty]{} X$$

Therefore, one has $\liminf_{k\to+\infty} {}_P w_{k,i(k)}P = \mathcal{Q}(X)$ and by the point 1 of the Lemma III.18, it implies that $\mathcal{Q}(X) \in \overline{D}$ which gives a contradiction. □

Thanks to this property, we can finally give a proof of the surjectivity of $\mathcal{Q}$:

Proposition V.17. The map $\mathcal{Q}$ is surjective.

__Proof.__ Since $\mathcal{Q}$ is continuous, the image of $\mathcal{Q}$ is a F_N-invariant compact subset of $\hat{T}^{\mathrm{obs}}$. Furthermore, the F_N-orbits are dense in $\overline{T}$ for the metric topology and hence fore the observers' topology. Thus, the F_N orbit of any point in $\overline{T}$ is dense in $\hat{T}^{\mathrm{obs}}$. Then, it suffices to consider the orbit of a point $X \in \partial F_N$ such that $\mathcal{Q}(X) \in \overline{T}$, which exists according to the Lemma V.9. □

We introduced before the notion of dual algebraic lamination $L^2(T)$ and it has been proven that

$$L^2(T) = \{(X,Y) \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}$$

when F_N-orbits are dense in T.

For $X, Y \in \partial F_N$, consider the equivalence relation $\sim$ given by

$$X \sim Y \iff \mathcal{Q}(X) = \mathcal{Q}(Y)$$

and define $\partial F_N / L^2(T) = \partial F_N / \sim$. This space is endowed with the quotient topology and we write $\pi : \partial F_N \longrightarrow \partial F_N / L^2(T)$ the canonical projection.

Thus, the map $\mathcal{Q}$ naturally splits over π as a map $\varphi : \partial F_N / L^2(T) \longrightarrow \hat{T}^{\mathrm{obs}}$.

Theorem V.18. The map $\varphi : \partial F_N / L^2(T) \longrightarrow \hat{T}^{\mathrm{obs}}$ is a homeomorphism.

__Proof.__ The map φ is clearly injective and hence bijective. Due to the topology over $\partial F_N / L^2(T)$, φ is continous. Since π is continuous surjective, and ∂F_N compact Hausdorff, then $\partial F_N / L^2(T)$ is as well. Therefore, as any continous surjective map from a compact Hausdorff space, it is an homeomorphism. □

VI Cantor space and group action

In last parts, the goal was to well describe some quotient of ∂F_N by laminations. Furthermore, it's easy to notice that any action of the free group by homeomorphisms passes naturally to the quotient and yields an action by homeomorphisms on $\partial F_N/L^2$ where L^2 is an algebraic. Thus, a question would be : Does every action on the quotient $\partial F_N/L$ by homeomorphisms stem from an action on ∂F_N by homeomorphisms? Even if this question is not studied here, the study can be motivated by the following :

Definition VI.1. A topological space X is totally disconnected if any connected component of X is a singleton.

Remark. Any subset $A \subset X$ of at least 2 elements can't be connected.

Proposition VI.1. Let X be a Hausdorff topological space. If for any distinct points $x,y \in X$, there exists a clopen U such that $x \in U$ and $U \subset X \setminus \{y\}$, then X is totally disconnected.

Proof. Consider a subset $A \subset X$ which contains two distinct points $x,y \in X$. By hypothesis, there exists a clopen U such that $x \in U$ and $y \notin U$. Thus $U \cap A$ is an clopen of A and hence C can't be connected. Therefore, the connected components are singletons and X is totally disconnected. □

Proposition VI.2. Let X be a Hausdorff topological space. If there exists a clopen basis for this topology, then X is totally disconnected.

Proof. Let $(U_i)_{i\in I}$ be a clopen basis of X. Consider $x \neq y$ two elements of X, there exists U open such that $x \in U$ and $U \subset X \setminus \{y\}$. Since $(U_i)_{i\in I}$ is a basis, it exists $i \in I$ such that U_i verify the same property. The latest proposition concludes. □

Theorem VI.1. **(Cantor)** Let X be an Hausdorff metrizable space which is compact, totally disconnected and without isolated point, then $X \simeq K$. □

Theorem VI.2. **(Alexandroff-Urysohn)** Let X be an Hausdorff metrizable space which is compact, then there exists $F : K \longrightarrow X$ which is continuous and surjective. □

Theorem VI.3. Let Γ be a countable group and X and Hausdorff metrizable space which is compact. Let $\alpha : \Gamma \times X \longrightarrow X$ be a left action by homeomorphism on X. Then it exists a left action by homeomorphism $\widehat{\alpha} : \Gamma \times K \longrightarrow K$ and $\chi : K \longrightarrow X$ an equivariant surjective map.

Proof. Let $F : K \longrightarrow X$ be a continuous surjection given by the Alexandroff-Urysohn Theorem. Consider $T = \{\tau : \Gamma \longrightarrow K\}$ endowed with the pointwise convergence topology. Then, T is metrizable compact and without isolated point. Moreover, for any $\gamma \in \Gamma$ and any open subset $U \subset K$, we can define $V_{\gamma,U} = \{\tau \in T,\ \tau(\gamma) \in U\}$. Thus, $(V_{\gamma,U})$ is a clopen basis of T.
Consider $S = \{\tau \in T,\ \forall \gamma_1, \gamma_2 \in \Gamma,\ \gamma_1 \cdot F(\tau(\gamma_2)) = F(\tau(\gamma_1\gamma_2))\}$ a closed subset of T. Thus, S is metrizable compact and $(V_{\gamma,U} \cap S)$ is a clopen basis of S, then is totally disconnected. However, S might have some isolated points so we need to consider $Y := S \times K \simeq K$.
We define $\phi : S \longrightarrow X$ as : $\forall \tau \in S,\ \phi(\tau) = F(\tau(1_\Gamma))$. Let's show that ϕ is surjective : let $x \in X$, for any $\gamma \in \Gamma$, it exists $z_\gamma \in F^{-1}(\{\gamma \cdot x\}) \in K$. We can define $\tau_x\ :\ \gamma \mapsto z_\gamma$ in T. Let $\gamma_1, \gamma_2 \in \Gamma$,

one has : $\gamma_1 \cdot F(\tau_x(\gamma_2)) = \gamma_1 \cdot F(z_{\gamma_2}) = \gamma_1 \cdot \gamma_2 \cdot x = F(z_{\gamma_1\gamma_2}) = F(\tau_x(\gamma_1\gamma_2))$.
Moreover, we have $\phi(\tau_x) = F(\tau(1_\Gamma)) = F(z_{1_\Gamma}) = 1_\Gamma \cdot x = x$ so ϕ is surjective.
Let's now define an action of Γ onto T as follow : for any $\gamma, \gamma' \in \Gamma$ and $\tau \in T$, one has $(\gamma \cdot \tau)(\gamma') = \tau(\gamma'\gamma)$. Then, for any $\gamma \in \Gamma$ and $\tau \in S$, one has

$$\gamma \cdot \phi(\tau) = F(\tau(\gamma)) = F((\gamma \cdot \tau)(1_\Gamma)) = \phi(\gamma \cdot \tau).$$

Thus S is stable for this action and we define an action on Y as

$$\forall \gamma \in \Gamma,\ (\tau, z) \in Y,\ \gamma \cdot (\tau, z) = (\gamma \cdot \tau, z).$$

Finally, we define $\chi(\tau, z) = \phi(\tau)$ for $(\tau, z) \in Y$ which is equivariant for the previous action. $\square$

The latest theorem might motivate the study of the question seen before. Indeed, for any action on $\partial F_N / L^2$ by homeomorphism, it is possible to lift an action on ∂F_N which is homeomorphic to the Cantor space. But, the proof given before for this statement doesn't give any construction for such a lift. Moreover, it looks reasonable to think that the Cantor given in this same proof will differ a lot from the initial space ∂F_N. Therefore, for the following it would be interesting to, at least, see if it's possible to characterize actions on ∂F_N that come from action on ∂F_N.

Bibliography

[1] Steven Neil Evans. *Probability and real trees*. Springer, 2008.

[2] John W. Morgan and Peter B. Shalen. Valuations, trees, and degenerations of hyperbolic structures, 1984.

[3] Damien Gaboriau, Andre Jaeger, Gilbert Levitt, and Martin Lustig. An index for counting fixed points of automorphisms of free groups, 1998.

[4] Thierry Coulbois, Arnaud Hilion, and Martin Lustig. $\mathbb{R}$-trees and laminations for free groups. ii. the dual lamination of an $\mathbb{R}$-tree, 2008.

[5] Gilbert Levitt and Martin Lustig. Irreducible automorphisms of F_N have north-south dynamics on compactified outer space, 2008.

[6] Thierry Giordano and Pierre de la Harpe. Moyennabilité des groupes dénombrables et actions sur les espaces de cantor, 1997.

[7] Frédéric Paulin. The gromov topology on $\mathbb{R}$-trees, 1988.

[8] Daryl Cooper. Automorphisms of free groups have finitely generated fixed point sets, 1985.

[9] Thierry Coulbois, Arnaud Hilion, and Martin Lustig. Non-unique ergodicity, observers' topology and the dual algebraic lamination for $\mathbb{R}$-trees, 2007.

[10] Thierry Coulbois, Arnaud Hilion, and Martin Lustig. $\mathbb{R}$-trees and laminations for free groups. i. algebraic laminations, 2008.