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In cylindrical and spherical geometries, the proper linear combination of standing waves results in a rotating wave field. Such a field looks like a traveling wave chasing its tail: a constant field rotating in space. These rotating wave fields can be considered to be a third major category of waves, similar to traveling and standing waves. While all three types of waves are interrelated, rotating waves offer the clearest insight into angular momentum and rotary motion in wave fields. This paper summarizes the field equations and physics for various rotating wave fields and goes on to discuss simple demonstrations of rotating waves on the surface of water.

I. INTRODUCTION

Most of us have a vivid picture of traveling waves (those long lines of ocean waves rolling into the shore) and of standing waves that we excited on jump ropes and chains when we were children. As physicists, we dwell on these images because traveling and standing waves are so central to modern physics and technology; and these vivid, mechanistic pictures help us to accept and understand the other, invisible traveling and standing waves. However, there is a third class of waves, equally important, which could also easily be planted in our minds' eyes: rotating waves. At least in simple cases, physicists usually tend to refer to these rotating waves as "wave fields having angular momentum," since indeed they have angular momentum, although the term "rotating waves" is also occasionally used.

Physicists typically first encounter wave fields having angular momentum in quantum mechanics and later in graduate electromagnetism, both in a rather mathematical context. It may surprise some, that mechanical waves, including those on the surface of water, can possess angular momentum in very close analogy to those in quantum mechanics, even to the point of being similarly proportional to the azimuthal index or quantum number m. When excited in a simple cylindrical container of water, such rotating waves will appear as traveling waves, traveling in circles, such that their rotary nature is self evident (and fascinating). In a beginning physics class, a demonstration of this would be a powerful introduction to angular momentum in wave fields. Use of the concise descriptive name "rotating waves," in parallel to "traveling waves" and "standing waves," would also help learning, conceptualizing, and conversing about these waves.

II. RELATIONSHIP BETWEEN ROTATING WAVES AND STANDING WAVES

To better understand rotating waves, consider waves on the surface of water in a cylindrical container. The vertical displacements $\zeta^{(1)}(r,\phi,t)$ and $\zeta^{(2)}(r,\phi,t)$ of the water surface of the two independent, degenerate *standing* wave modes of indices m,n are given by:[1]

$$\begin{aligned}\zeta^{(1)}(r,\phi,t)&=AJ_m(\kappa r)\cos m\phi\cos\omega t,\\ \zeta^{(2)}(r,\phi,t)&=AJ_m(\kappa r)\sin m\phi\cos(\omega t-\delta),\end{aligned}\tag{1}$$

where $m=0,1,2,3,4,\ldots,J_m(\kappa r)$ is an mth-order Bessel function, $\kappa=2\pi/\lambda$ is the wave number, $\omega=\kappa v_p$ is the angular frequency, v_p is the phase velocity for traveling waves of the same frequency on an open body of water, and δ is an arbitrary temporal phase shift between the two modes. The second index n is related to how many radial nodes occur from the center to the outer edge of the cylinder where $\partial\zeta/\partial r=0$ is the boundary condition on the cylindrical container wall, i.e., $\kappa a=u_{mn}$, where a is the container radius and u_{mn} is the nth root of $dJ_m(u)/du=0$. Thus the container's radius a and the indices m and n determine the radial wave number κ and in turn, the frequency ω. For $m=0$, the $\zeta^{(2)}$ solution equals zero everywhere, meaning only the $\zeta^{(1)}$ standing wave mode is nontrivial.

Each of the modes of Eq. (1), if taken by itself, possesses no angular momentum. However, linear combinations of $\zeta^{(1)}$ and $\zeta^{(2)}$ for $m\geqslant 1$ have angular momentum. For example, setting $\delta=\pi/2$ and adding $\zeta^{(1)}$ and $\zeta^{(2)}$ yields

$$\zeta(r,\phi,t)=AJ_m(\kappa r)\cos(m\phi-\omega t),\tag{2}$$

where $m=0,1,2,3,\ldots$. Equation (2) represents a pure rotating wave. Note that the $\cos(m\phi-\omega t)$ term is very similar to a $\cos(\kappa x-\omega t)$ term in a traveling wave: It allows the wave to maintain its shape as it propagates, in this case in the positive ϕ direction. Thus rotating waves appear quite different to the eye than do the standing wave modes of Eq. (1) which appear to vibrate between fixed nodal lines and continually change their shape. Figures 1 and 2 show computer drawings of two such rotating modes, for $m=3$, $n=2$, and $m=7$, $n=1$, respectively, although in reality these could be snapshots of either standing wave modes or rotating wave modes. It is the dynamics, the rotating wave crests, as indicated by arrows on the figures, that distinguish these as rotating modes. A rotating wave thus has the dynamics and appearance similar to that of a traveling wave, but has the modal structure or discrete line spectrum similar to that of a standing wave.

Subtracting $\zeta^{(1)}$ and $\zeta^{(2)}$ in Eq. (1) with $\delta=\pi/2$ also results in rotating waves and can be expressed with Eq. (2) if we interpret the m in Eq. (2) to be negative in this case, i.e., minus that in Eq. (1) when we subtract. Thus Eq. (2) represents modes for $m=\ldots-3,-2,-1,0,1,2,3,\ldots$. The modes of negative m's rotate in the $-\phi$ direction. Notice that this use of negative m's has eliminated the need to have pairs of modes, $\zeta^{(1)}$ and $\zeta^{(2)}$, share the same mode indices as in Eq. (1): Equation (2) assigns a unique index to every independent mode. It also more easily explains the difference between the two degenerate modes: A mode with a positive m value rotates counterclockwise, while its degenerate counterpart with equal magnitude but negative m

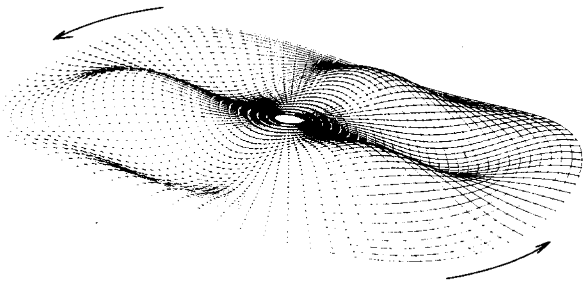

Fig. 1. A rotating wave mode on the surface of water composed of the two standing wave modes $m=3,n=2$ with a $\pi/2$ temporal phase shift between them. The arrows indicate the direction of wave propagation.

value rotates counterclockwise. The $m=0$ mode is cylindrically symmetric and does not rotate at all. We might also note that approximate rotating wave fields can be created in square, octagonal, etc., containers of water by combining the pairs of degenerate, independent modes in a similar way as we did above.

Equation (2) can also be expressed in complex form as

$$\zeta(r,\phi,t)=AJ_m(\kappa r)e^{i(m\phi-\omega t)}. \tag{3}$$

III. ANGULAR MOMENTUM IN ROTATING WAVE FIELDS

Next, we consider the question of angular momentum. At an intuitive level, since rotating waves look like traveling waves moving in circles, a simple rotating wave demonstration can be very convincing in itself that rotating waves carry angular momentum, provided an audience is aware that traveling waves carry the linear momentum.[2] (What child in the surf isn't?) To mathematically calculate the angular momentum of these rotating wave fields, we first consider the velocity potential[1] Φ consistent with the waves of Eq. (2), where the fluid velocity $\mathbf{v}=-\nabla\Phi$ and the fluid displacement $\zeta=\int \mathbf{v}dt$ is now a three dimensional vector:

$$\Phi=-\frac{\omega A}{\kappa\sinh\kappa h}J_m(\kappa r)\cosh\kappa z\sin(m\phi-\omega t), \tag{4}$$

where $z=0$ at the bottom of the container and $z=h$ at the water's surface. For all locations in the fluid reasonably below the surface, the fluid motion is purely sinusoidal and contains no net angular momentum when averaged over a wave cycle. However, points in the space that are submerged during only part of the wave cycle do contain net momentum. The net momentum per surface area (of the free water surface) is given by $\mathbf{p}=\rho\langle\zeta\mathbf{v}|_{z=h}\rangle$, where ζ is the z directed displacement of the water surface and $\langle...\rangle$ means the time average. The net angular momentum in the z direction per surface area is given by $l_z=\rho r\langle\zeta v_\phi\rangle$ making the total z-directed angular momentum,

$$L_z=\int \rho r\Big\langle AJ_m(\kappa r)\cos(m\phi-\omega t)(A\omega m/\kappa r)J_m(\kappa r)\times\coth\kappa h\cos(m\phi-\omega t)\Big\rangle\,dS. \tag{5}$$

Time averaging and using the relation[1] $\omega^2=g\kappa\tanh\kappa h$, we have

$$L_z=2\frac{m}{\omega}\int\frac{1}{4}\rho gA^2J_m^2(\kappa r)dS. \tag{6}$$

Using the time average of Eq. (2) squared, we can put Eq. (6) in the form,

$$L_z=2\frac{m}{\omega}\int\frac{1}{2}\rho g\langle\zeta^2\rangle\,dS=2\frac{m}{\omega}U_p, \tag{7}$$

where the potential energy density[3] equals $(1/2)\rho g\zeta^2$, and U_p is the total potential energy of the wave field. Using the fact that the total potential energy is one half the total energy U of the resonance,[4] we have

$$L_z=m(U/\omega). \tag{8}$$

Equation (8) generally holds in other wave systems as well. For example, Jackson[5] derives this relationship for classical electromagnetic fields. In quantum mechanics also, one might recall that $L_z=m\hbar=m(U/\omega)$. Therefore, both the classical and quantum mechanical waves have angular momentum proportional to an integer, the azimuthal mode number. There is however one important distinction: in quantum mechanics U/ω is equal to the constant $\hbar$ which means L_z only takes on values which are integer multiples of $\hbar$. However, in the classical systems, U/ω can take on a continuum of positive values making the total angular momentum, which is proportional to this ratio, also able to take on a continuum of values.

IV. ROTATING WAVES FIELDS THAT RADIATE

Rotating waves are not only limited to closed, resonant systems, but also can exist in open, radiating systems. The two mathematically independent, zero angular momentum radiation modes for the indices m,n from a localized source are given by:[6]

$$\zeta^{(1)}(r,\phi,t)=A\cos m\phi[J_m(\kappa r)\cos\omega t+N_m(\kappa r)\sin\omega t]$$
$$\zeta^{(2)}(r,\phi,t)=A\sin m\phi[J_m(\kappa r)\cos(\omega t-\delta)+N_m(\kappa r)\sin(\omega t-\delta)], \tag{9}$$

where $m=0,1,2,3,...,$ $N_m(\kappa r)$ is the mth-order Neumann function, and δ is the arbitrary temporal phase shift between modes. A picture of these would show expanding circular wave fronts, all concentric on the origin or source. By adding and subtracting these with $\delta=\pi/2$, we create the rotating mode:

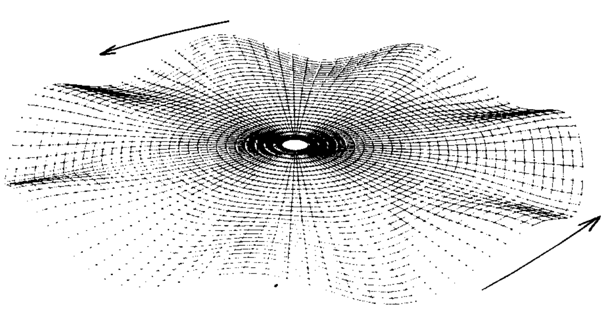

Fig. 2. The $m=7,n=1$ rotating wave mode. Most of the water motion in such a wall mode (a $n=1$ mode) occurs next to the cylindrical wall.

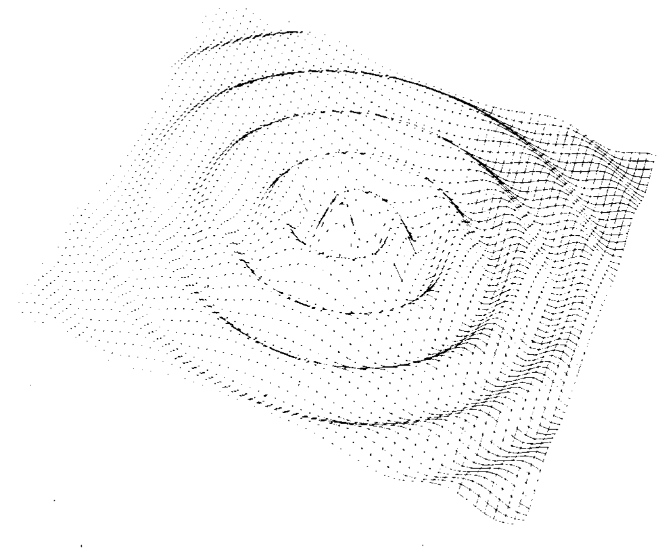

Fig. 3. A radiating, rotating wave field, i.e., spiral wave field.

$$\zeta(r,\phi,t) = A[J_m(\kappa r)\cos(m\phi-\omega t) - N_m(\kappa r)\sin(m\phi-\omega t)], \tag{10}$$

where $m = \ldots -3,-2,-1,0,1,2,3,\ldots$. This wave rotates and moves outward with a pinwheel or spiral galaxy appearance as shown in Fig. 3. It might also be called a spiral wave. These wave fields can conveniently be expressed in complex notation using Hankel[7] functions $H_m^{(1)}(x) \equiv J_m(x) + iN_m(x)$, $H_m^{(2)}(x) \equiv J_m(x) - iN_m(x)$ as

$$\zeta(r,\phi,t) = AH_m^{(1)}(\kappa r)e^{i(m\phi-\omega t)}, \tag{11}$$

where the $H_m^{(2)}$ function should be used for converging spiral wave fields.

In the far field, we can approximate the Bessel and Neumann functions[7] in Eq. (10) with $J_m(x) \cong (2/\pi x)^{1/2}\cos(x - m\pi/2 - \pi/4)$ and $N_m(x) \cong (2/\pi x)^{1/2}\sin(x - m\pi/2 - \pi/4)$ to get

$$\zeta(r,\phi,t) \cong \sqrt{\frac{2}{\pi\kappa r}}\, A \cos\left(\kappa r + m\phi - \omega t - \frac{m\pi}{2} - \frac{\pi}{4}\right). \tag{12}$$

Having the r, ϕ, and t dependence all inside the same argument of a cosine function most clearly shows the interrelationship between the r, ϕ, and t coordinates and that this is a wave field that travels in the r and ϕ directions. (Compare this to the traveling wave: $\cos(\boldsymbol{\kappa}\cdot\mathbf{r} - \omega t)$ moving in the $\boldsymbol{\kappa}$ direction.)

V. THREE-DIMENSIONAL ROTATING WAVES

Three-dimensional *cylindrical* resonators are not much more complex than their two dimensional counterparts. For example, starting with the normal *acoustical* standing wave modes,[8] shifting phases, and adding as we did before, we get the rotating pressure p wave field in a cylindrical acoustic resonator:

$$p(r,\phi,z,t) = AJ_m(\kappa r)\sin \kappa_z z \sin(m\phi - \omega t), \tag{13}$$

where $m = \ldots -2,-1,0,1,2,3\ldots$ and κ_z is the z-directed wave number $[\omega = v_p(\kappa^2 + \kappa_z^2)^{1/2}]$. If we instead have z-directed, rotating traveling waves propagating as in a long pipe or circular wave guide, then Eq. (13) changes to

$$p(r,\phi,z,t) = AJ_m(\kappa r)\sin(\kappa_z z + m\phi - \omega t). \tag{14}$$

A somewhat similar rotating wave field is seen in circularly polarized electromagnetic radiation[9] which can be expressed in cylindrical, complex form as

$$E_r(r,\phi,z,t) = E_0 e^{i(\kappa z + \phi - \omega t)},$$
$$E_\phi(r,\phi,z,t) = E_0 e^{i(kz + \phi - wt + \pi/2)}. \tag{15}$$

Equation (15) is the cylindrical coordinate equivalent of the more common Cartesian coordinate $E_x = E_0 \exp i(\kappa z - \omega t)$, $E_y = E_0 \exp i(\kappa z - \omega t + \pi/2)$. We used the cylindrical coordinate form because it most clearly shows the rotating wave form having a $(\kappa z + m\phi - \omega t)$ argument, $m = 1$ in this case, as in Eq. (14). Also related to Eqs. (13) and (14), as well as to Eq. (10), are the rotating modes of propagation in a circular, electromagnetic wave guide, as well as the rotating and spiraling modes of electromagnetic radiation from a long array of identical radiators arranged along the z axis.

In the case of a *spherical* resonator, one often sees standing wave solutions in terms of spherical harmonics $Y_{lm}(\theta,\phi)$. However, rotating wave fields are best written with the θ and ϕ dependence separated. For this we use the equivalent form containing the associated Legendre[10] function $P_l^m(\cos\theta)$. Adding two of the normal standing wave solutions of a spherical acoustic resonator with the appropriate phase shift yields the rotating wave:

$$p(R,\theta,\phi,t) = Aj_l(\kappa R)P_l^m(\cos\theta)\sin(m\phi - \omega t), \tag{16}$$

where $j_l(\kappa R)$ is the spherical Bessel function (lower case j and n are used to denote *spherical* Bessel functions, whereas upper case J and N are used for the common cylindrical Bessel functions). Both the cylindrical and spherical rotating modes look, in a snap shot, just the same as the standing wave modes they are created from, however dynamically they are very different. Instead of the oscillatory appearance with stationary nodal surfaces that the standing wave modes have, a rotating mode appears as a frozen field, that keeps its shape in time, and rotates around the z axis. These modes have angular momentum equal to mU/ω as derived earlier.

Spherical radiation patterns, or multipole radiation,[5,11] are very important in advanced electromagnetism. Here, for simplicity, we will examine rotating *acoustic* multipole fields. Rotating electromagnetic multipole fields can be written in an analogous, but more complicated, form. By adding appropriately phase shifted acoustical modes, we get the rotating acoustic multipole radiation terms as:

$$p(R,\theta,\phi,t) = AP_l^m(\cos\theta)[\,j_l(\kappa R)\cos(m\phi - \omega t) - n_l(\kappa R)\sin(m\phi - \omega t)], \tag{17}$$

where $n_l(\kappa R)$ is the spherical Neumann function. This can be expressed in complex notation conveniently by using the spherical Hankel functions $h_l^{(1)} \equiv j_l + in_l$:

$$p(R,\theta,\phi) = AP_l^m(\cos\theta)h_l^{(1)}(\kappa R)e^{i(m\phi - \omega t)}. \tag{18}$$

By using the far-field approximation, $h_l^{(1)}(x) \cong x^{-1}\exp(x - \frac{1}{2}l\pi - \frac{1}{2}\pi)$, we get

$$p(R,\theta,\phi,t) \cong (A/\kappa R)P_l^m(\cos\theta)e^{i(\kappa R + m\phi - \omega t - \frac{1}{2}l\pi - \frac{1}{2}\pi)}, \tag{19}$$

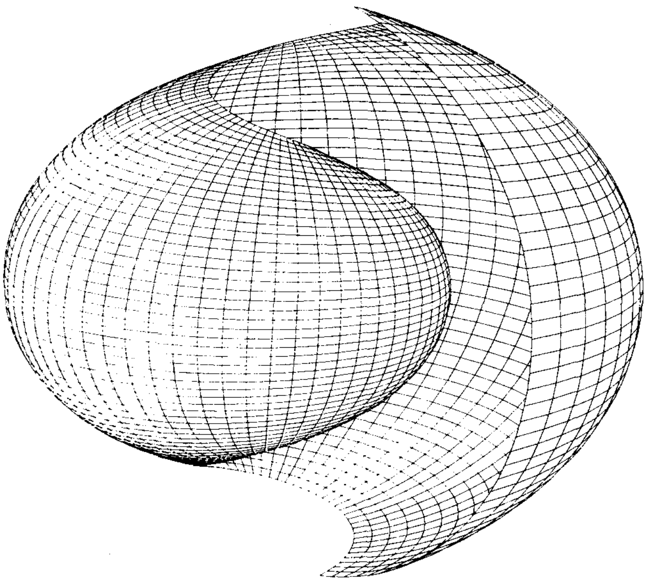

Fig. 4. The wave crest or maxima of a three-dimensional spiraling acoustic wave field emitted by a rotating dipole ($l=2, m=1$).

which clearly shows the interaction of the R, ϕ, and t coordinates. Figure 4 is a plot of the wave crest of this three-dimensional spiraling wave, which resembles the spiraling shell of some sea creature.

It is also interesting that the properties of both spherical harmonics and of associated Legendre functions allow m for these classical acoustical wave fields to take on integer values between $-l$ and $+l$, i.e., $m=-l,-l+1,...,1,0,1,...,l$, analogous to quantum mechanics.

VI. PHYSICAL ROTATING WAVES

Do rotating waves exist in nature? On a macroscopic scale, waves driven by spinning motions, such as the water waves below a hurricane would be an example of these wave fields. On an even larger scale, spiral galaxies have rotating wave fields that modulate their densities and give them their spiral shapes.[12,13] Reference 14 shows a beautiful visualization of the spiraling gravity wave field around two orbiting black holes. On a microscopic scale, certainly circularly polarized light, i.e., spin 1 or spin -1 photons, represent rotating wave fields. So does the spiraling radiation field from a relativistic circling charged particle.[15] Another place to look is the quantum mechanical wave field in a hydrogen atom as derived in most quantum mechanics texts.[16] The first clue that they are rotating wave fields and not simple standing wave fields is that for a given l index, m takes on values from $-l$ to $+l$, instead of from 0 to l with double degeneracy as in the standing wave case. Further examination of the texts reveals that the wave functions are

$$\begin{aligned}\psi_{nlm}(R,\theta,\phi,t) &= R_{nl}(R)\,Y_{lm}(\theta,\phi)\,e^{-i\omega t}\\ &= R_{nl}(R)\left(\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right)^{1/2}\\ &\quad\times P_l^m(\cos\theta)\,e^{i(m\phi-\omega t)}.\end{aligned} \tag{20}$$

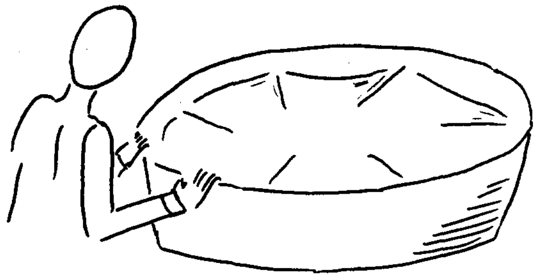

Fig. 5. Rotating wave field excited in a child's wading pool by grasping and cyclically pushing at two spaced points on the pool's rim with one's hands.

The $\exp[i(m\phi-\omega t)]$ factor shows this to clearly be a rotating mode. [Compare this with Eq. (18)]. Finally, the nonzero angular momentum which is proportional to m further confirms it to be a rotating mode.

There are certainly examples of rotating wave fields in technology. The rotating magnetic fields in many electric motors comprise such fields, as do circularly polarized microwave beams, fields in ferrite circulators, optical gyros, and the fields in free electron laser wigglers in the reference frame of the electrons. Also, acoustical emissions from spinning turbines, propellers, or rotors are further examples. Experimental motors have been made using surface rotating waves,[17] and the author's own work relates to thermoacoustic engines and refrigerators using rotating acoustic fields.[18–20] In a recent literature search of my own on the related topic of ring resonators (which can support standing wave modes as well as rotating wave modes), there were surprisingly few authors who mentioned rotating modes, even though ring resonators are ideal to be treated with this concept. If the concept of rotating waves were introduced to physicists and engineers in elementary physics classes, perhaps only with a demonstration, many more rotating wave devices would most likely be developed as a result.

Partially to verify the real existence of rotating waves to myself at a sensory level and partially to develop a demonstration, I have experimented with rotating wave fields on the surface of water. Figure 5 shows a child's wading pool in which I was able to easily excite rotating wave fields with my hands. Cyclically pushing on the side with one hand excited standing waves. Using two spaced hands, one pushing right after the other pushed to approximate a $\pi/2$ phase shift, I could consistently produce rotating waves in the range of $|m|\leqslant 6$ and $n\leqslant 2$. Small floating objects and slow moving ripples are not carried along by the waves and serve to show that it is the waves, and not the water that is rotating. Cyclically pushing on the wall with electromechanical devices of various sorts (motors, speakers, etc., in the 1- to 60-Hz range) also worked well. Because of perturbations (or deviations from a perfect cylinder of the walls) which can convert standing waves into rotating waves,[17,18] often only a single phase pusher was required to generate rotating waves. Cloudy or colored water in smooth walled containers works best and the speed control of electromechanical pushers has to be fairly good ($\pm 5\%$). Radiating rotating waves, i.e., spiral waves, are trivial to create by swirling the tip of a stick in a large container or body of water, as indicated in Fig. 6. In general, the people who observed these rotating waves of water

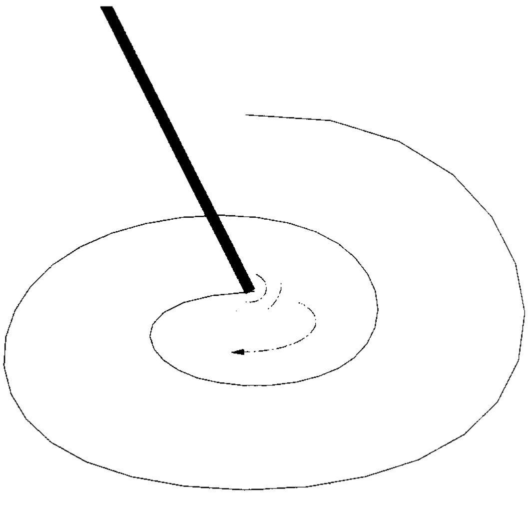

Fig. 6. A spiral wave field excited on a large water surface by swirling the tip of a stick in the water.

with me were fascinated by the moving form and symmetry of rotating waves, particularly of the wave fields for higher m and n values, having many peaks and valleys yet keeping a constant shape as they rotated. These higher modes are also hardest to create in a clear form because they are tightly spaced in frequency, making it hard not to excite more than one at one time. This is an excellent physics demonstration: simple, visually stimulating with connections to classical waves fields, rotary motion, angular momentum, and quantum mechanics.

Similar demonstrations could also be done by exciting the rotating waves on a stretched rubber membrane acoustically driven by a nearby loud speaker and perhaps with a strobe light to "slow them down." The difficulty of stretching the membrane uniformly in both the x and y directions might be a problem, but might also be turned into an advantage to allow excitation of rotating waves with a single phase driver as mentioned above with respect to water waves. I have yet to do it, but find intriguing the concept of exciting a rotating acoustical wave field in a suitable cylindrical or square room to check if my ears would tell me that the sound was everywhere circulating clockwise or counterclockwise, i.e., chasing its tail, with no audible beginning or end.

VII. CONCLUSION

Rotating waves have the appearance of traveling waves propagating in circles, but have the modal structure or discrete line spectra of standing waves and should be considered a third, important type of waves, similar to traveling waves and standing waves. While all three wave types are interrelated, rotating waves offer the clearest insight into angular momentum and rotary motion in wave fields. A good, visible example of rotating waves can be excited on the surface of water. Because their rotary nature is clearly visible and they have many of the same properties as the rotating waves in quantum mechanics and electromagnetic multipole radiation, rotating water waves make a good introduction to these more complex, invisible rotating wave fields.

ACKNOWLEDGEMENT

I gratefully acknowledge the support of the Office of Naval Research in my thermoacoustic research which involves the use of rotating wave fields and substantially contributed to my understanding of rotating waves.

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[3]W. C. Elmore and M. A. Heald, *Physics of Waves* (Ref. 1), p. 203.

[4]Because of the orthogonality of the standing wave modes of Eq. (1), the *potential* energy of the rotating wave of Eq. (2) equals the sum of the potential energies of the standing wave modes that comprise it from Eq. (1), i.e., $\int(1/2)\rho g\zeta^2 dS=\int(1/2)\rho g(\zeta^{(1)}+\zeta^{(2)})^2 dS=\int(1/2)\rho g[(\zeta^{(1)})^2+(\zeta^{(2)})^2]dS=U_p^{(1)}\cos^2\omega t+U_p^{(2)}\sin^2\omega t$ with the cross terms vanishing and where $U_p^{(1)}$ and $U_p^{(2)}$ are the maximum potential energies (in time) of the two standing wave modes. Since the amplitudes of the two modes of Eq. (1) are equal, $U_p^{(1)}$ equals $U_p^{(2)}$ and the potential energy of the rotating mode is therefore equal to $U_p^{(1)}(\cos^2\omega t+\sin^2\omega t)=U_p^{(1)}$ and thus is constant in time (as one might expect for a constant wave profile rotating in space). Similarly, the *kinetic* energy of the rotating mode equals the sum of the kinetic energies of the two constituent standing wave modes and is also constant in time with $U_k=U_k^{(1)}$. Since in standing wave modes the maximum potential energy (in time) equals the maximum kinetic energy ($U_p^{(1)}=U_k^{(1)}$), in the rotating wave mode, $U_p=U_p^{(1)}=U_k^{(1)}=U_k$, i.e., the potential energy equals the kinetic energy and each equals 1/2 the total wave energy.

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