project n°3
Rotating & resonant waves in a cylindrical cavity…

1 Introduction

The objective of the projects proposed here is to evaluate your ability to state and to solve problems by yourself, on a topic related to the lecture Acoustics 1. The expected work is theoretical in nature, on a somewhat framed but also open subject. You are expected to work independently for a limited period of 5 weeks (note however that you can ask me for help or advice all along the project).

What prevails in the work proposed here is the ”obligation of means”, not the ”obligation of results” (as for a research work where it is not always easy to anticipate difficulties). The proposed evaluation format is a report and a 30-minutes individual interview (mainly for you to have a feedback on your work…). Beyond your knowledge of acoustics in general, you will be evaluated on skills that are those expected for a (young) researcher, e.g. :

- ⋆ ability to present concisely and clearly the approaches adopted and the results obtained,
- ⋆ ability to have a critical look at your work,
- ⋆ ability to suggest interesting future prospects,
- ⋆ etc …

The proposed scheduled is the following :

- ⋆ **Tuesday November** 10^{th}. Presentation of each subjects. Then you'll have a few days to chose one of the proposed subjects, and inform me about your choice by E-mail.
- ⋆ **November** 10^{th} → **December 18**th. All along the duration of the project, you can contact me for advice or guidelines. Please contact me by E-mail beforehand so that we take an appointement for a skype/discord/zoom meeting.
- ⋆ **Friday December 18**th. Deadline for sending me your report. The report part can take the form that suits you, as long as it clearly and effectively summarizes your work. You should send me this report as a pdf file, and you are strongly encouraged to write it with latex (ideally, it would be good if it could have the form of a research article, e.g. using the AIP template).

2 Description of the project, guidelines

2.1 Geometry of the problem and first elements of theory

The geometry of the problem considered in this project is the one described in Fig. 1. A cylindrical cavity having a radius R and a thichness $h \ll R$ is considered. This cavity will be submitted to external forcing at angular frequency ω, and as $h \ll R$, we'll consider in the following (or at least at the beginning) that the acoustic field does not depend on z but only on the azimuthal and radial coordinates, θ and r.

So the acoustic field should write as :

$$\tilde{p}(r,\theta) = \sum_{m,n} \tilde{A}_{mn} J_m\left(k_{mn} r\right) \left[\tilde{A}_{\theta_m} \cos\left(m\theta\right) + \tilde{B}_{\theta_m} \sin\left(m\theta\right)\right] \tag{1}$$

where m is an integer number, and where the possible values of k_{mn} are those imposed by the boundary condition stating that the lateral walls are rigid, i.e. $J_m'(k_{mn}r)|_{r=R} = 0$, which leads

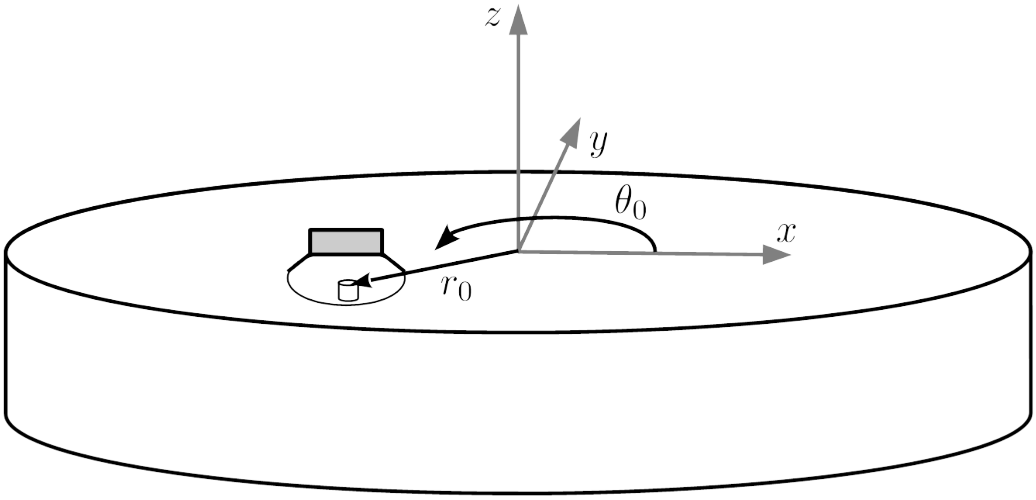

FIGURE 1 – Geometry of the problem considered

to the values which are given in my lecture notes (lecture n°3, slide 56). The amplitudes $\tilde{A}_{mn}$, $\tilde{A}_{\theta_m}$ and $\tilde{B}_{\theta_m}$ should depend on the excitation...

Now, if we consider that external forcing is provided by a loudspeaker through a small hole on the upper wall at position (r_0, θ_0), and that at the end a given volume velocity per unit volume q_0 is imposed by this source, then the acoustic pressure field should satisfy the following inhomogeneous wave equation below :

$$\frac{\partial^2 \tilde{p}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{p}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{p}}{\partial \theta^2} + k^2 \tilde{p} = -i\omega q_0 \delta\left(r - r_0\right)\delta\left(\theta - \theta_0\right) \tag{2}$$

with $k = \omega / c_0$, and where δ represents the dirac delta distribution function. Reporting the form of solution (1) in the inhomogeneous wave equation leads to :

$$\sum_{m,n} \left[k^2 - k_{mn}^2\right] \tilde{A}_{mn} J_m\left(k_{mn} r\right) \left[\tilde{A}_{\theta_m} \cos\left(m\theta\right) + \tilde{B}_{\theta_m} \sin\left(m\theta\right)\right] = -i\omega q_0 \delta\left(r - r_0\right)\delta\left(\theta - \theta_0\right) \tag{3}$$

The amplitudes of the modes should be obtained by making use of orthogonality. For instance, if one applies the operation $\frac{1}{2\pi}\int_0^{2\pi}(\dots)\cos(m'\theta).d\theta$ to the equation above, then one gets :

$$\sum_{n} \left[k^2 - k_{m'n}^2\right] \tilde{A}_{m'n} J_{m'}\left(k_{m'n} r\right) \frac{A_{\theta_{m'}}}{2} = -i\omega q_0 \delta\left(r - r_0\right)\cos\left(m'\theta_0\right) \tag{4}$$

if $m' \neq 0$ and

$$\sum_{n} \left[k^2 - k_{0n}^2\right] \tilde{A}_{0n} J_0\left(k_{0n} r\right) A_{\theta_0} = -i\omega q_0 \delta\left(r - r_0\right) \tag{5}$$

if $m' = 0$...In the same way, applying the operation $\frac{1}{2\pi}\int_0^{2\pi}(\dots)\sin(m'\theta).d\theta$, one gets

$$\sum_{n} \left[k^2 - k_{m'n}^2\right] \tilde{A}_{m'n} J_{m'}\left(k_{m'n} r\right) \frac{A_{\theta_{m'}}}{2} = -i\omega q_0 \delta\left(r - r_0\right)\sin\left(m'\theta_0\right) \tag{6}$$

if $m' \neq 0$ and

$$\sum_{n} \left[k^2 - k_{0n}^2\right] \tilde{A}_{0n} J_0\left(k_{0n} r\right) A_{\theta_0} = 0 \tag{7}$$

if $m' = 0$...As the original angle $\theta = 0$ can be chosen more or less arbitrarily, then if we chose $\theta_0 = 0$ the problem simply writes as :

$$\sum_n \left[k^2 - k_{m'n}^2\right] \tilde{A}_{m'n} J_{m'}\left(k_{m'n} r\right) \frac{A_{\theta_{m'}}}{2} = -i\omega q_0 \delta\left(r - r_0\right) \tag{8}$$

if $m' \neq 0$ and

$$\sum_n \left[k^2 - k_{0n}^2\right] \tilde{A}_{0n} J_0\left(k_{0n} r\right) A_{\theta_0} = -i\omega q_0 \delta\left(r - r_0\right) \tag{9}$$

if $m' = 0$...and finally a unified results gives :

$$\sum_n \left[k^2 - k_{mn}^2\right] \frac{\tilde{A}_{mn}}{2 - \delta_m^0} J_m\left(k_{mn} r\right) = -i\omega q_0 \delta\left(r - r_0\right) \tag{10}$$

where m' was replaced with m and where the amplitude $\tilde{A}_{\theta_m}$ was included in the (new) amplitude $\tilde{A}_{mn}$. Next [1] the process can be continued so as to calculate the amplitude $\tilde{A}_{mn}$ as functions of the frequency and the position of the source term (to that purpose, you will need to use the definition of a scalar product for Bessel functions proposed in the last slides of lecture 4...) and the final solution should writes as

$$\tilde{p}(r,\theta) = \sum_{m,n} \tilde{A}_{mn} J_m\left(k_{mn} r\right) \cos\left(m\theta\right) \tag{11}$$

where the expressions of A_{mn} are known.

2.2 Description of the project

The ultimate goal of this project would be to investigate how generating an acoustic vortex (say, a rotating wave) inside this resonator. Before the announcement of the lockdown, I was planning that an experimental part could be done, since I have the hardware available (only a few holes should be made) but unfortunately I am afraid that it might not be possible...we'll see.

Anyway, what you should do at first would be to read the provided article by Ceperley published in JASA a few decades ago, where much details are given about rotating waves. I also provided another paper which was published by Santillo et al. more recently, which also deals with generation of rotating waves (but in unbounded media), and presents a simple experiment with educational/demonstration purpose (the possibility of driving the rotation of a small object by acoustic waves is also treated in this paper).

The idea for generating rotating waves would be, first, to exploit modes with nodal lines (e.g. mode 10, 20 or 11) whose spatial distribution depends on θ, and second, to make use of *several* point sources at different angular position for which the control of their phasing should enable the control the standing/traveling wave nature of the θ-dependent component of the acoustic pressure field. For instance, intuition tells us that if one use for instance a point source at $(r_0, \theta_0 = 0)$ and two other ones at $(r_0, \theta_0 = 2\pi/3)$ and at $(r_0, \theta_0 = 4\pi/3)$, then if the angular frequency is that of the mode 10 and these two source are phase-shifted by $2\pi/3$ and $4\pi/3$, respectively (and relative to the first source), then we may expect having a rotating wave...Also, if one play with mode 11, then one may be able to observe two counter-rotating wave (because of the nodal circle...). So a proposed program would be :

1. but I stop here since this will be your job ;-)...

⋆ to investigate first the frequency response function of the cavity with a single source, and to retrieve the resonances associated to the different modes mn of the cylindrical cavity. To that purpose, you should employ a modal theory, and you may investigate the impact of the source position on frequency response functions

⋆ next, to focus on specific modes, and then to make use of the superposition principle so as to compute an acoustic field generated by several phase-shifted sources that would promote the generation of the "acoustic vortices" discussed above.

2.3 Should you have further time at disposal...

As I mentioned previously, I was thinking about experimental validations of this, but I think it might be complicated because of the lockdown...Also, I don't have a clear idea about how much time and efforts you would need for the works mentioned above. Now, *should you have further time at disposal*, I could suggest you to continue working on the use of modal analysis for cylindrical waves, since there is one of the tutorial labs (for other M1 students) for which such an approach could be advantageously performed. The device considered in this lab is the one schematically presented in Fig. 2.

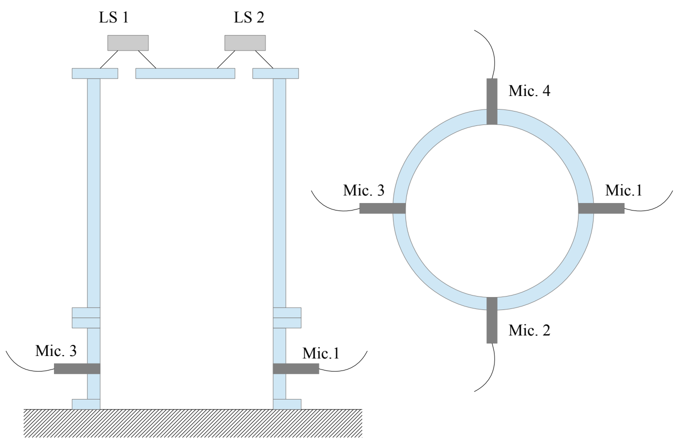

FIGURE 2 – Schematic drawing of the problem considered.

You could provide a critical comparison of experiments (I can give you some experimental data) and theory (the latter being to be developed by you...) in this system where non plane wave modes can become propagative, and therefore lead to complex frequency response functions (e.g. see the slide 32 of my lecture 4, where 2D ducts are treated).