Appendices

Contents

A Implémentation de SAFE pour des modes de Lamb

A.1. Formalism

1. Variational form for elasticity equations

$$-\rho\omega^2 \int_\Omega u\delta u \, d\Omega + \int_\Omega \boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(\delta u) \, d\Omega = \int_{\partial\Omega} \boldsymbol{\sigma}(\delta u) \cdot \boldsymbol{n} \, d\Gamma, \quad \forall \, \delta u \tag{1}$$

2. Displacement fields $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}(x_2)e^{ik_1x_1}e^{-i\omega t}$

3. Double dot product $\boldsymbol{\sigma}(u) : \boldsymbol{\epsilon}(\delta u^\star) = \sigma_{ij}(u)\varepsilon_{ij}(\delta u^\star)$

$$\begin{aligned} \sigma_{11}\varepsilon_{11} &= k_1^2(\lambda + 2\mu)u_1\delta u_1 + -ik_1\lambda\nabla u_2\delta u_1 \\ \sigma_{12}\varepsilon_{12} &= \frac{\mu}{2}\left(-ik_1\nabla u_1\delta u_2 + \nabla u_1\delta\nabla u_1 + k_1^2 u_2\delta u_2 + ik_1u_2\nabla\delta u_1)\right) \\ \sigma_{22}\varepsilon_{22} &= ik_1\lambda u_1\nabla\delta u_2 + (\lambda + 2\mu)\nabla u_2\nabla\delta u_2 \end{aligned} \tag{2}$$

4. Explicit form for the weak form of elasticity equations,

$$\begin{aligned} \int_\Omega -\rho\omega^2 \, (u_1\delta u_1 + u_2\delta u_2) + k_1^2(\lambda + 2\mu)u_1\delta u_1 - ik_1\lambda\nabla u_2\delta u_1 - ik_1\mu\nabla u_1\delta u_2 + \mu\nabla u_1\delta\nabla u_1 \\ + \mu k_1^2 u_2\delta u_2 + ik_1\mu u_2\nabla\delta u_1) + ik_1\lambda u_1\nabla\delta u_2 + (\lambda + 2\mu)\nabla u_2\nabla\delta u_2 \, d\Omega = 0 \quad \forall \, \delta u \end{aligned} \tag{3}$$

where ∇u_i denotes differentiation of the i-th component of the displacement along x_2. Note that the surface term initally expressed vanishes from the no-traction boundary condition.

5. Discretized fields

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{N}(\xi)\boldsymbol{U}e^{ik_1x_1}e^{-i\omega t}, \tag{4}$$

with $\boldsymbol{N}$ containing the second-order Lagrange shape functions, and $\boldsymbol{U}$ is the vector of displacement amplitudes.

6. Elementary matrices

$$\begin{aligned} \boldsymbol{M}^{(e)} &= \frac{h}{2}\int_{-1}^{1} \boldsymbol{N}^\top \boldsymbol{N}\, d\xi, \quad \boldsymbol{K}^{(e)} = \frac{2}{h}\int_{-1}^{1} \nabla \boldsymbol{N}^\top \nabla \boldsymbol{N}\, d\xi, \\ \boldsymbol{C}_1^{(e)} &= \int_{-1}^{1} \boldsymbol{N}^\top \nabla \boldsymbol{N}\, d\xi, \quad \boldsymbol{C}_2^{(e)} = \int_{-1}^{1} \nabla^\top \boldsymbol{N}^\top \boldsymbol{N}\boldsymbol{U}\, d\xi, \end{aligned} \tag{5}$$

7. Discretization $\forall \delta \boldsymbol{U}$, weak form for one element and rearranging the terms

$$\begin{aligned} &\int_{\Omega_e} k_1^2\left(\mu\delta\boldsymbol{U}_2^\top\boldsymbol{M}^{(e)}\boldsymbol{U}_2 + (\lambda+2\mu)\delta\boldsymbol{U}_1^\top\boldsymbol{M}^{(e)}\boldsymbol{U}_1\right) + ik_1\left(\mu\delta\boldsymbol{U}_2^\top\boldsymbol{C}_1^{(e)}\boldsymbol{U} - \mu\delta\boldsymbol{U}_1^\top\boldsymbol{C}_2^{(e)}\boldsymbol{U}_2\right. \\ &\quad \left. + \lambda\delta\boldsymbol{U}_1^\top\boldsymbol{C}_1^{(e)}\boldsymbol{U}_2 - \lambda\delta\boldsymbol{U}_2^\top\boldsymbol{C}_2^{(e)}\boldsymbol{U}_1\right) + \mu\delta\boldsymbol{U}_1^\top\boldsymbol{K}^{(e)}\boldsymbol{U}_1 + (\lambda+2\mu)\delta\boldsymbol{U}_2^\top\boldsymbol{K}^{(e)}\boldsymbol{U}_2 \\ &\quad - \rho\omega^2\delta\boldsymbol{U}_1^\top\boldsymbol{M}^{(e)}\boldsymbol{U}_1 - \rho\omega^2\delta\boldsymbol{U}_2^\top\boldsymbol{M}^{(e)}\boldsymbol{U}_2\, d\Omega_e = 0 \end{aligned} \tag{6}$$

8. General form for the problem

$$\left(k_1^2\boldsymbol{A}_2 + ik_1\boldsymbol{A}_1 + \boldsymbol{A}_0 - \omega^2\boldsymbol{M}\right)\boldsymbol{U} = 0. \tag{7}$$

Rewrites as a generalized eigenvalue problem,

$$\left[\begin{pmatrix} -\boldsymbol{A}_1 & \boldsymbol{A}_0 - \omega^2\boldsymbol{M} \\ \boldsymbol{I} & \boldsymbol{0} \end{pmatrix} - k_1\begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix}\right]\begin{pmatrix} k_1\boldsymbol{U} \\ \boldsymbol{U} \end{pmatrix} = 0 \tag{8}$$

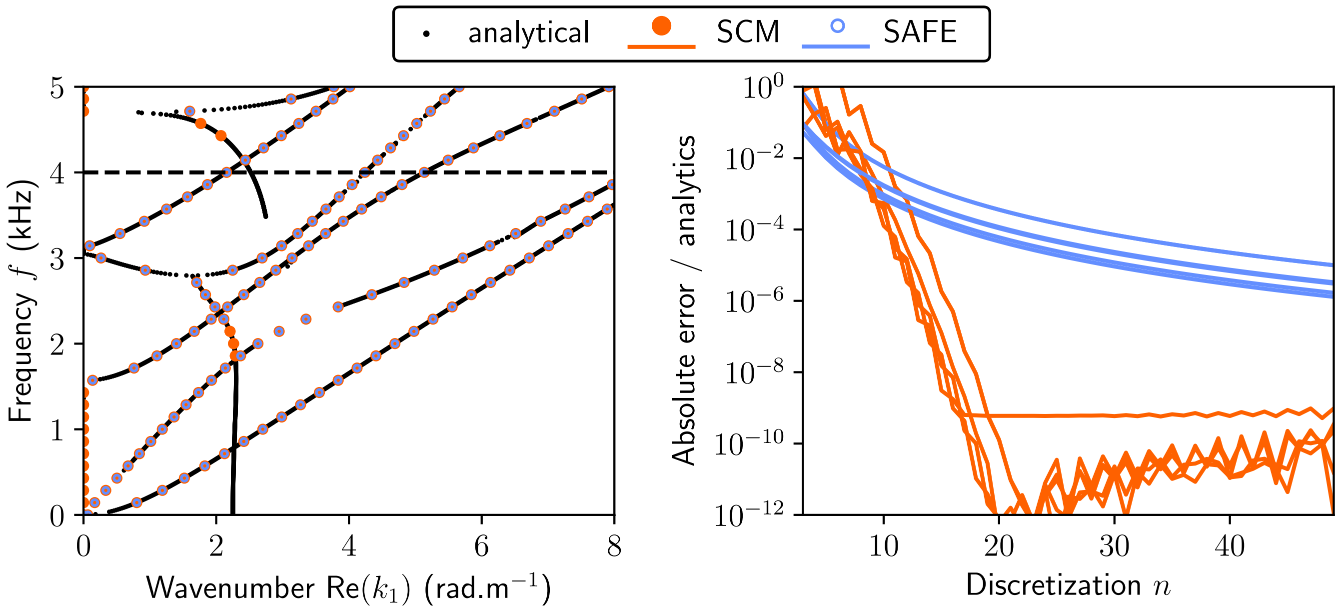

Figure 1: a) Dispersion relation for the Lamb modes with the root-finding results, alongside the spectral collocation and semi-analytical finite elements solutions. b) Convergence curve of the 2 methods at a given frequency $f = 4$ kHz. The multiple lines depicts the convergence of each wavenumber.

A.2. Results

Analytical results are obtained by applying the root-finding approach using the Müller method presented in the paper. Solutions are sought from the classical Lamb modes equations, derived in many books such as Royer. Figure 1a) shows the results of the three methods for a given frequency range. They all match well. In order to compare the two numerical methods in play, a brief convergence study has been done using the analytical results from the root-finding method as a reference.

The convergence curve of the results given in Fig. 1b) is unsurprisingly in agreement with the literature. It is calculated at a frequency $f = 4$ kHz: the solutions along the dashed line in Fig 1a) are selected. The parameter n used for the discretization is shared between the two methods and corresponds to the number of elements taken for the discretization in each method. SAFE matrices have a size $2(2n + 1)$, with while SCM matrices have a size $2n$. The discretization is swept from $n = 3$ to $n = 50$.

The SCM convergence shows a steep slope and reaches a plateau around $n = 20$, while the SAFE curve follows the expected slope of a finite elements method that use second-order shape functions. When comparing at a given discretization order, the error from the SAFE is much lower that that of SAFE by several order of magnitudes. In order to see comparable convergence, it would require to use higher-order shape functions or use spectral elements instead.

A.3. Point sur la biblio

- ”Wave propagation along transversely periodic structures” cite [1]
- ”Finite element model for waves guided along solid systems of arbitrary section coupled to infinite solid media” [2]
- ”SAFE-PML approach for modal study of waveguides with arbitrary cross sections immersed in inviscid fluid” [3]
- ”A coupled SAFE-2.5D BEM approach for the dispersion analysis of damped leaky guided waves in embedded waveguides of arbitrary cross-section” [4] + ”Dispersion analysis of leaky guided waves in fluid-loaded waveguides of generic shape” and [5] ”Dispersion analysis of leaky guided waves in fluid-loaded waveguides of generic shape”
- Livre J. Rose [6]
- Semi-Analytical Finite Element (SAFE) method for plotting Lamb waves dispersion curves of an aluminum plate and comparison with Disperse software [7]

B Details on the Müller method

B.1. The root-finding algorithm

equations, iterative scheme etc.

[8]

$$
\begin{aligned}
f(x_0) &= a(x_0 - x_2)^2 + b(x_0 - x_2) \\
f(x_1) &= a(x_1 - x_2)^2 + b(x_1 - x_2) \\
f(x_2) &= a(x_2 - x_2)^2 + b(x_2 - x_2)
\end{aligned}
\tag{9}
$$

Note that we have dropped the subscript"2" from the function for conciseness. Because we have three equations, we can solve for the three unknown coefficients, a, b,and c. Because two of the terms in Eq. (7.20) are zero, it can be immediately solved for $c = f(x_2)$. Thus, the coefficient c is merely equal to the function value evaluated at the third guess, x_2. This result can then be substituted into Eqs. (7.18) and (7.19) to yield two equations with two unknowns:

$$
\begin{aligned}
f(x_0) - f(x_2) &= a(x_0 - x_2)^2 + b(x_0 - x_2) \\
f(x_1) - f(x_2) &= a(x_1 - x_2)^2 + b(x_1 - x_2)
\end{aligned}
$$

Algebraic manipulation can then be used to solve for the remaining coefficients, a and b. One way to do this involves defining a number of differences,

$$
\begin{aligned}
h_0 &= x_1 - x_0 & h_1 &= x_2 - x_1 \\
\delta_0 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} & \delta_1 &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\end{aligned}
$$

These can be substituted into Eqs.(7.21) and (7.22) to give

$$
\begin{aligned}
(h_0 + h_1)b - (h_0 + h_1)^2 a &= h_0\delta_0 + h_1\delta_1 \\
h_1 \quad b - \quad h_1^2 \quad a &= \quad h_1\delta_1
\end{aligned}
$$

which can be solved for a and b.The results can be summarized as (7.24)

$$
\begin{aligned}
a &= \frac{\delta_1 - \delta_0}{h_1 + h_0} \\
b &= ah_1 + \delta_1 \\
c &= f(x_2)
\end{aligned}
$$

(7.25) (7.26) To find the root, we apply the quadratic formula to Eq. (7.17). However, because of potential round-off error, rather than using the conventional form, we use the alternative formulation [Eq.(3.13)] to yield

$$
x_3 - x_2 = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}
$$

or isolating the unknown x_3 on the left side of the equal sign,

$$
x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}
$$

Note that the use of the quadratic formula means that both real and complex roots can be located. This is a major benefit of the method. In addition, Eq. (7.27a) provides a neat means to determine the approximate error. Because the left side represents the difference between the present (x_3) and the previous (x_2) root estimate, the error can be calculated as

$$
\varepsilon_a = \left| \frac{x_3 - x_2}{x_3} \right| 100\%
$$

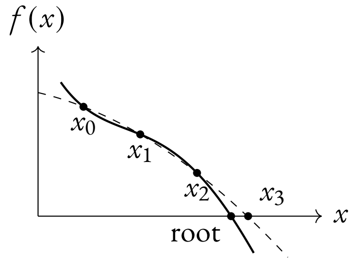

Figure 2: iteration muller

B.2. A complete routine for the automated solving of dispersion relations

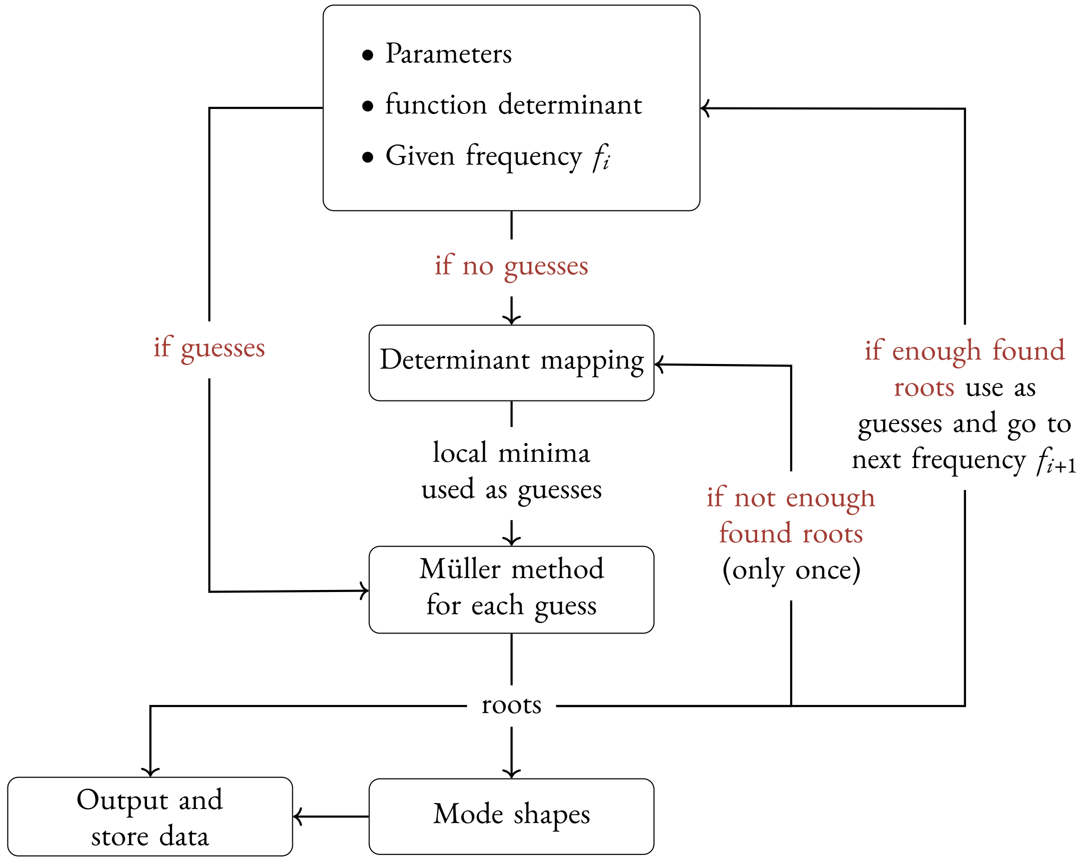

Figure 3: Description of the routine

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[1] M. V. Predoi et al. "Wave Propagation along Transversely Periodic Structures". *The Journal of the Acoustical Society of America* 121.4 (2007), 1935–1944.

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[3] P. Zuo and Z. Fan. "SAFE-PML Approach for Modal Study of Waveguides with Arbitrary Cross Sections Immersed in Inviscid Fluid". *Journal of Sound and Vibration* 406 (2017), 181–196.

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