Helicopter Rotor Model

Author: Marek M. Cel
Revision: 1
Date: 2020-01-12

iii. publicity and privacy rights pertaining to a person's image or likeness depicted in a Work;
iv. rights protecting against unfair competition in regards to a Work, subject to the limitations in paragraph 4(a), below;
v. rights protecting the extraction, dissemination, use and reuse of data in a Work;
vi. database rights (such as those arising under Directive 96/9/EC of the European Parliament and of the Council of 11 March 1996 on the legal protection of databases, and under any national implementation thereof, including any amended or successor version of such directive); and
vii. other similar, equivalent or corresponding rights throughout the world based on applicable law or treaty, and any national implementations thereof.

2. Waiver. To the greatest extent permitted by, but not in contravention of, applicable law, Affirmer hereby overtly, fully, permanently, irrevocably and unconditionally waives, abandons, and surrenders all of Affirmer's Copyright and Related Rights and associated claims and causes of action, whether now known or unknown (including existing as well as future claims and causes of action), in the Work (i) in all territories worldwide, (ii) for the maximum duration provided by applicable law or treaty (including future time extensions), (iii) in any current or future medium and for any number of copies, and (iv) for any purpose whatsoever, including without limitation commercial, advertising or promotional purposes (the "Waiver"). Affirmer makes the Waiver for the benefit of each member of the public at large and to the detriment of Affirmer's heirs and successors, fully intending that such Waiver shall not be subject to revocation, rescission, cancellation, termination, or any other legal or equitable action to disrupt the quiet enjoyment of the Work by the public as contemplated by Affirmer's express Statement of Purpose.

3. Public License Fallback. Should any part of the Waiver for any reason be judged legally invalid or ineffective under applicable law, then the Waiver shall be preserved to the maximum extent permitted taking into account Affirmer's express Statement of Purpose. In addition, to the extent the Waiver is so judged Affirmer hereby grants to each affected person a royalty-free, non transferable, non sublicensable, non exclusive, irrevocable and unconditional license to exercise Affirmer's Copyright and Related Rights in the Work (i) in all territories worldwide, (ii) for the maximum duration provided by applicable law or treaty (including future time extensions), (iii) in any current or future medium and for any number of copies, and (iv) for any purpose whatsoever, including without limitation commercial, advertising or promotional purposes (the "License"). The License shall be deemed effective as of the date CC0 was applied by Affirmer to the Work. Should any part of the License for any reason be judged legally invalid or ineffective under applicable law, such partial invalidity or ineffectiveness shall not invalidate the remainder of the License, and in such case Affirmer hereby affirms that he or she will not (i) exercise any of his or her remaining Copyright and Related Rights in the Work or (ii) assert any associated claims and causes of action with respect to the Work, in either case contrary to Affirmer's express Statement of Purpose.

4. Limitations and Disclaimers.

a. No trademark or patent rights held by Affirmer are waived, abandoned, surrendered, licensed or otherwise affected by this document.
b. Affirmer offers the Work as-is and makes no representations or warranties of any kind concerning the Work, express, implied, statutory or otherwise, including without limitation warranties of title, merchantability, fitness for a particular purpose, non infringement, or the absence of latent or other defects, accuracy, or the present or absence of errors, whether or not discoverable, all to the greatest extent permissible under applicable law.
c. Affirmer disclaims responsibility for clearing rights of other persons that may apply to the Work or any use thereof, including without limitation any person's Copyright and Related Rights in the Work. Further, Affirmer disclaims responsibility for obtaining any necessary consents, permissions or other rights required for any use of the Work.
d. Affirmer understands and acknowledges that Creative Commons is not a party to this document and has no duty or obligation with respect to this CC0 or use of the Work.

Contents

Notation

$a = \dfrac{dC_L}{d\alpha}$	[1/rad] lift curve slope
$A_R = \pi R^2$	[m^2] rotor disc area
B	[-] blade tip loss factor
c_b	[m] rotor blade chord
C_D	[-] drag coefficient
C_L	[-] lift coefficient
$C_T = \dfrac{T}{\rho A_R \Omega_R^2 R^2}$	[-] thrust coefficient
D	[N] drag
e	[m] flapping hinge offset
g	[m/s^2] gravity acceleration
J_b	[kg·m^2] rotor blade moment of inertia
L	[N] lift
m	[kg] mass
n_b	number of rotor blades
Q	[N·m] torque
r	[m] coordinate along blade span
$\vec{r} = [x, y, z]^T$	[m] coordinates vector
R	[m] rotor radius
$s = \dfrac{N_b c_b}{\pi R}$	[-] rotor solidity
S_b	[kg·m] blade first moment of mass
T	[N] thrust
V	[m/s] velocity
V_i	[m/s] induced velocity
V_{ih}	[m/s] induced velocity for hovering
$\vec{V} = [u, v, w]^T$	[m/s] velocity vector
α	[rad] angle of attack
β	[rad] rotor blade flapping angle

β_0	[rad] rotor coning angle
$\beta_1 c$, $\beta_1 s$	[rad] longitudinal and lateral tip-path angles
$\gamma = \dfrac{\rho a c_b R^2}{J_b}$	[-] blade Lock number
ε	[rad] rotor shaft inclination angle
θ_0	[rad] collective pitch angle
$\theta_1 s$, $\theta_1 c$	[rad] longitudinal and lateral cyclic pitch angle
$\lambda = \dfrac{w_{rw} - V_I}{\Omega_R R}$	[-] rotor inflow ratio
$\lambda_i = \dfrac{V_i}{\Omega_R R}$	[-] induced inflow ratio
$\lambda_{ih} = \dfrac{V_{ih}}{\Omega_R R}$	[-] induced inflow ratio for hovering
$\mu = \dfrac{V}{\Omega_R R}$	[-] rotor advance ratio
$\mu_C = \dfrac{V_C}{\Omega_R R}$	[-] normalized climb velocity
$\mu_D = \dfrac{V_D}{\Omega_R R}$	[-] normalized descent velocity
ρ	[kg/m^3] air density
χ	[rad] rotor wake angle
Ψ	[rad] rotor blade azimuth
$\vec{\omega} = [p, q, r]^T$	[rad/s] angular velocity vector
Ω_R	[rad/s] rotor revolution speed

Indices:

b	Body-Axis-System
b	rotor blade
ba	Blade Axis System
BE	blade element
c	Control Axis System
C	climb
cw	Control-Wind Axis System
D	descent
i	induced
ih	induced in hovering
r	Rotor Axis System
R	rotor
RH	rotor hub
rw	Rotor-Wind Axis System

X	x-axis component
Y	y-axis component
Z	z-axis component

Derivatives:

$\dot{u} = \frac{du}{dt}$	time derivative
$\bar{\beta} = \frac{d\beta}{d\Psi}$	azimuth derivative

Chapter 1

Introduction

1.1 Coordinate Systems Used for Rotor Calculations

1.1.1 Rotor Axis System

Origin of the Rotor Axis System is coincident with the rotor hub center, the x-axis is positive forwards, the y-axis is positive right and z-axis is positive downwards and coincident with the rotor shaft axis.

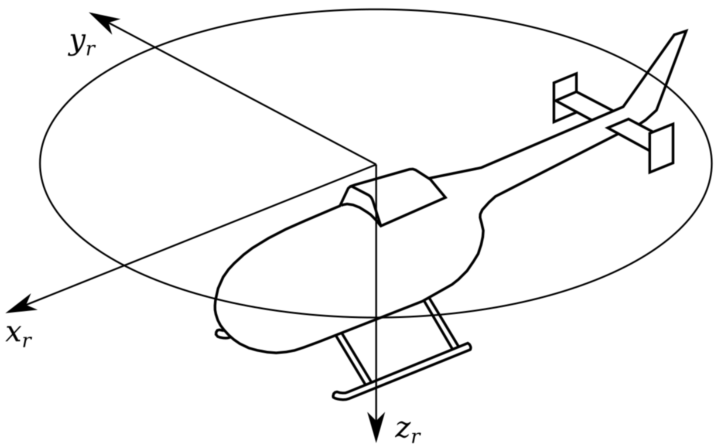

Figure 1.1: Rotor Axis System

1.1.2 Rotor-Wind Axis System

Rotor-Wind Axis System is very much like Rotor Axis System, the only difference is that it is rotated about z-axis in such a manner that x-axis points directly into relative wind, so there is no lateral airspeed component.

1.1.3 Control Axis System

For most purposes, using the Rotor Axis System causes unnecessary complications. It is convenient to use no cyclic feathering axes system. [1] Origin of the Control Axis System is coincident with the origin of the Rotor Axis System, but it is rotated by angles of the swashplate roll and pitch so there is no cyclic feathering in this coordinate system.

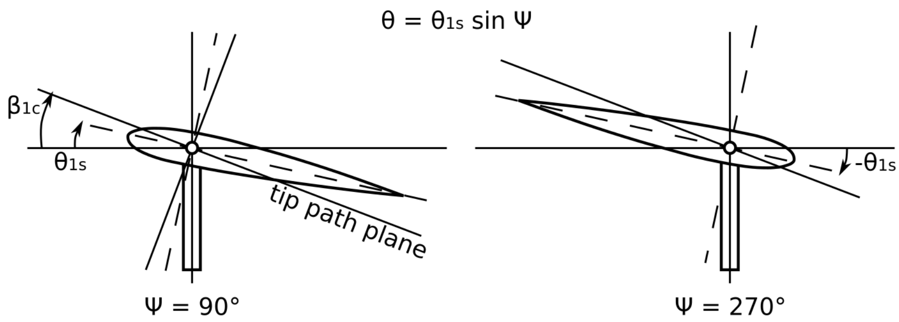

Figure 1.2: Rotor reference planes

1.1.4 Disc Axis System

Origin of the Disc Axis System is coincident with the origin of the Rotor Axis System, but it is rotated by angles of the rotor cone roll and pitch in such a manner that z-axis is perpendicular to the tip path plane so there is no cyclic flapping in this coordinate system.

1.1.5 Control-Wind Axis System

Control-Wind Axis System is very much like Control Axis System, the only difference is that it is rotated about z-axis in such a manner that x-axis points directly into relative wind, so there is no lateral airspeed component.

1.1.6 Blade Axis System

Blade Axis System is a coordinate system fixed to the rotor blade, its origin is coincident with the intersection point of blade feathering and flapping hinges axes. The x-axis is coincident with blade feathering axis and pointing outwards, the y-axis lies on XY plane of the Control Axis System and points towards blade leading edge, while the z-axis completes a right-handed coordinate system.

Chapter 2

Actuator Disc Method

2.1 Assumptions

Following assumptions are made for the purpose of modeling helicopter rotor aerodynamics:

— forces and moments generated by the rotor are considered to be quasi-steady,

— rotor lift force is a linear function of blade incidence angle and drag force is a quadratic function of lift, [2]

— rotor blades have 3 degrees of freedom movement,

— inflow is uniformly distributed over rotor disc, [2]

— reversed flow effects are ignored,

— airflow is considered to be quasi-steady and incompressible,

— thrust is considered to be parallel to the z-axis of the Control Axis System and magnitude of the thrust is considered to be magnitude of the resulting rotor force. [1]

2.2 Momentum Theory

2.2.1 Momentum Theory for Axial Flight

Mass flow through the rotor disc, momentum change and change in kinetic energy are given by the following formulas. [2]

$$\dot{m} = \rho A_1 V_C = \rho A_R \left(V_C + V_i\right) = \rho A_{i\infty} \left(V_C + V_{i\infty}\right) \tag{2.1}$$

$$T = \dot{m} \left(V_C + V_{i\infty}\right) - \dot{m} V_C = \dot{m} V_{i\infty} \tag{2.2}$$

$$T \left(V_C + V_{i\infty}\right) = \frac{1}{2}\dot{m} \left(V_C + V_{i\infty}\right)^2 - \frac{1}{2}\dot{m} V_C^2 = \frac{1}{2}\dot{m} \left(2V_C V_{i\infty} + V_{i\infty}^2\right) \tag{2.3}$$

Where:

$A_1 = \pi R_1^2$ [m^2] control volume section area

$A_R = \pi R^2$ [m^2] rotor disc area

$A_\infty = \pi R_\infty^2$ [m^2] far wake slipstream section area

$\dot{m}$ [kg/s] mass flow

V_C [m/s] climb velocity

V_i [m/s] induced velocity

$V_{i\infty}$ [m/s] far wake induced velocity

T [N] rotor thrust

From these relationships it can be deduced that induced velocity in the far wake is twice the rotor inflow. [2]

$$V_{i\infty} = 2V_i \tag{2.4}$$

Substituting equations (2.1) and (2.4) into (2.2) rotor thrust is given as follows:

$$T = 2\rho A_R \left(V_C + V_i\right) V_i \tag{2.5}$$

In hover flight, this equation can be expressed as:

$$T = 2\rho A_R V_{ih}^2 \tag{2.6}$$

Transforming equations (2.5) and (2.6) gives:

$$V_i = \frac{T}{2\rho A_R \left(V_C + V_i\right)} \tag{2.7}$$

$$V_{ih} = \sqrt{\frac{T}{2\rho A_R}} \tag{2.8}$$

Writing velocities in normalized form:

$$\lambda_i = \frac{V_i}{\Omega_R R} \tag{2.9}$$

$$\lambda_{ih} = \frac{V_{ih}}{\Omega_R R} \tag{2.10}$$

$$\mu_C = \frac{V_C}{\Omega_R R} \tag{2.11}$$

Rotor thrust coefficient is:

$$C_T = \frac{T}{\rho A_R \Omega_R^2 R^2} \tag{2.12}$$

Then equations (2.9) and (2.10) can be expressed as:

$$\lambda_i = \frac{C_T}{2\left(\mu_C + \lambda_i\right)} \tag{2.13}$$

$$\lambda_{ih} = \sqrt{\frac{C_T}{2}} \tag{2.14}$$

Combining these equations gives:

$$\lambda_{ih}^2 = \lambda_i \left(\mu_C + \lambda_i\right) \tag{2.15}$$

This equation can be transformed into following form:

$$\lambda_i = -\frac{\mu_C}{2} + \sqrt{\left(\frac{\mu_C}{2}\right)^2 + \lambda_{ih}^2} \tag{2.16}$$

For descent velocity $V_D = -V_C$ formula (2.16) can be written as:

$$\lambda_i = \frac{\mu_D}{2} - \sqrt{\left(\frac{\mu_D}{2}\right)^2 - \lambda_{ih}^2} \tag{2.17}$$

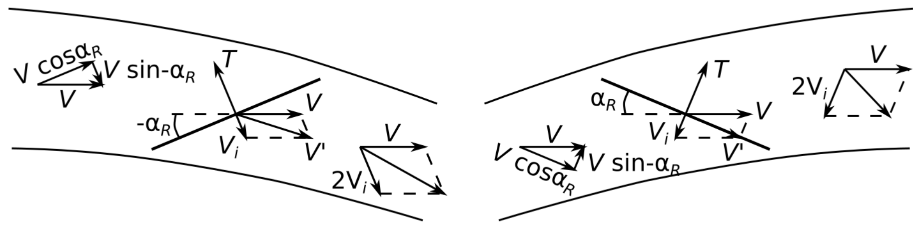

Figure 2.1: Flow trough a rotor in forward flight

This relationship is valid only in windmill brake state where the wake is fully established and the flow is upwards. [2] It can be assumed that such a condition occurs when descent velocity is two times greater than induced velocity in hover. [3]

Young's approximation can be used to determine induced velocity outside range of momentum theory application. [2]

$$\lambda_i = \lambda_{ih}\left(1 + \frac{\mu_D}{\lambda_{ih}}\right) \text{ for } 0 \leq \mu_D \leq -1.5\lambda_{ih} \tag{2.18}$$

$$\lambda_i = \lambda_{ih}\left(7 - 3\frac{\mu_D}{\lambda_{ih}}\right) \text{ for } -1.5\lambda_{ih} < \mu_D \leq -2\lambda_{ih} \tag{2.19}$$

2.2.2 Momentum Theory in Forward Flight

In forward flight induced velocity in the far wake is twice the flow at the rotor. [2] Expression for thrust is given as follows:

$$T = \dot{m}2V_i = (\rho A_R V')\, 2V_i \tag{2.20}$$

Transforming this equation for induced velocity gives:

$$V_i = \frac{T}{2\rho A_R V'} \tag{2.21}$$

Where V' is the resultant velocity at the rotor.

$$V' = \sqrt{V^2\cos^2\alpha_R + (V\sin\alpha_R - V_i)^2} \tag{2.22}$$

Writing velocities in normalized form:

$$\lambda_i = \frac{V_i}{\Omega_R R} \tag{2.23}$$

$$\mu_X = \frac{u_{rw}}{\Omega_R R} \tag{2.24}$$

$$\mu_Z = \frac{w_{rw}}{\Omega_R R} \tag{2.25}$$

Where:

$$u_{rw} = V \cos \alpha_R \tag{2.26}$$

$$w_{rw} = V \sin \alpha_R \tag{2.27}$$

Substituting equations (2.23), (2.24), (2.25) and (2.22) into (2.21) formula for the normalized induced velocity can be expressed as follows.

$$\lambda_i = \frac{C_T}{2\sqrt{\mu_X^2 + (\mu_Z - \lambda_i)^2}} \tag{2.28}$$

In high speed flight summing helicopter translational velocity and velocity due to rotor shaft rotation causes strong non-uniformities of rotor induced velocity. Glauert's model is used to describe this phenomena. [2, 4]

$$\lambda_i (r, \Psi) = \lambda_{i0} + \frac{r}{R} \lambda_{1c} \cos \Psi \tag{2.29}$$

Where:

$$\lambda_{1c} = \lambda_{i0} \tan \left(\frac{\chi}{2} \right) \quad \text{for } \chi < \frac{\pi}{2} \tag{2.30}$$

$$\lambda_{1c} = \lambda_{i0} \cot \left(\frac{\chi}{2} \right) \quad \text{for } \chi > \frac{\pi}{2} \tag{2.31}$$

The wake angle is given by the following formula:

$$\chi = \arctan \left(\frac{\mu}{\lambda_{i0} - \mu_Z} \right) \tag{2.32}$$

Where λ_{i0} is given by formula (2.28).

2.3 Forces Acting on the Blade Segment

Determining forces and moments acting on segment of the blade is made, assuming that blade is composed of aerodynamically independent, narrow strips of elements. [3] High aspect ratio of the blade justifies usage of two-dimensional flow, while lift loss at the blade tip and root can be accounted by using tip-loss factor. [2, 3, 4]

Control-Wind Axis System is used to determine forces and moments generated by the rotor, such computed forces and moments are the transformed to the Body Axis System.

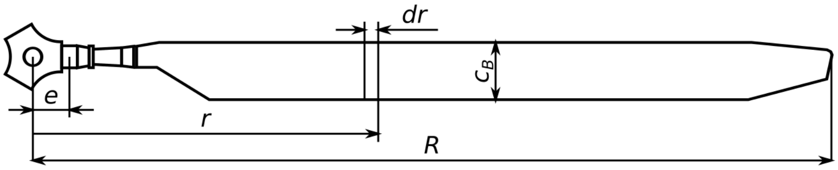

Figure 2.2: Rotor blade element

Lift and drag acting on a blade section is given by the following expressions:

$$dL = \frac{1}{2}\rho U^2 (r, \Psi) C_L c_b dr \tag{2.33}$$

$$dD = \frac{1}{2}\rho U^2 (r, \Psi) C_D c_b dr \tag{2.34}$$

Lift and drag coefficients are given as follows: [2]

$$C_L = a\alpha_{BE} (r, \Psi) \tag{2.35}$$

$$C_D = \delta_0 + \delta_2 C_T^2 \tag{2.36}$$

Blade section angle of attack is given by the following formula.

$$\alpha_{BE} (r, \Psi) = \theta + \phi (r, \Psi) \tag{2.37}$$

Where:

$$\phi (r, \Psi) = \arctan \frac{U_P (r, \Psi)}{U_T (r, \Psi)} \tag{2.38}$$

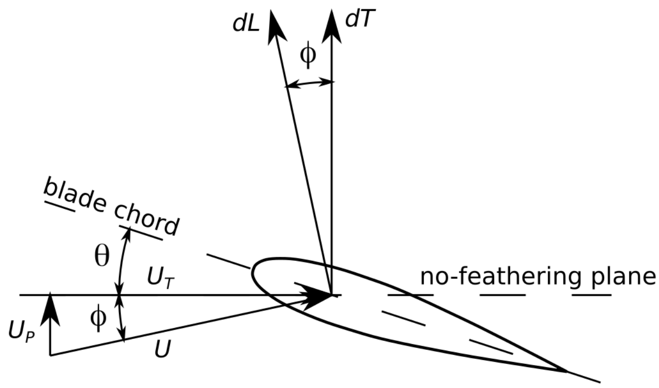

Figure 2.3: Velocity components at the blade element

Linear velocity of the rotor hub and angular velocity expressed Rotor Axis System are given by the following formulas: [3, 4]

$$\vec{V}_{RH,r} = \boldsymbol{T}(\varepsilon)\left(\vec{V}_{O,b} + \vec{\omega}_b \times \vec{r}_{RH,b}\right) \tag{2.39}$$

$$\vec{\omega}_r = \boldsymbol{T}(\varepsilon)\,\vec{\omega}_b \tag{2.40}$$

Rotattion matrix $\boldsymbol{T}(\varepsilon)$ is given as:

$$\boldsymbol{T}(\varepsilon) = \begin{bmatrix} \cos\varepsilon & 0 & \sin\varepsilon \\ 0 & 1 & 0 \\ -\sin\varepsilon & 0 & \cos\varepsilon \end{bmatrix} \tag{2.41}$$

Following formulas can be used to transform this values to Control Axis System:

$$\vec{V}_{RH,c} = \boldsymbol{T}(\theta_{1c}, \theta_{1s})\,\vec{V}_{RH,r} \tag{2.42}$$

$$\vec{\omega}_c = \boldsymbol{T}(\theta_{1c}, \theta_{1s})\,\vec{\omega}_r \tag{2.43}$$

Where rotation matrices are:

— for counterclockwise direction of rotor:

$$\boldsymbol{T}(\theta_{1c}, \theta_{1s}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{1c} & -\sin\theta_{1c} \\ 0 & \sin\theta_{1c} & \cos\theta_{1c} \end{bmatrix} \begin{bmatrix} \cos\theta_{1s} & 0 & -\sin\theta_{1s} \\ 0 & 1 & 0 \\ \sin\theta_{1s} & 0 & \cos\theta_{1s} \end{bmatrix} \tag{2.44}$$

— for clockwise direction of rotor:

$$\boldsymbol{T}(\theta_{1c}, \theta_{1s}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_{1c} & \sin\theta_{1c} \\ 0 & -\sin\theta_{1c} & \cos\theta_{1c} \end{bmatrix} \begin{bmatrix} \cos\theta_{1s} & 0 & -\sin\theta_{1s} \\ 0 & 1 & 0 \\ \sin\theta_{1s} & 0 & \cos\theta_{1s} \end{bmatrix} \tag{2.45}$$

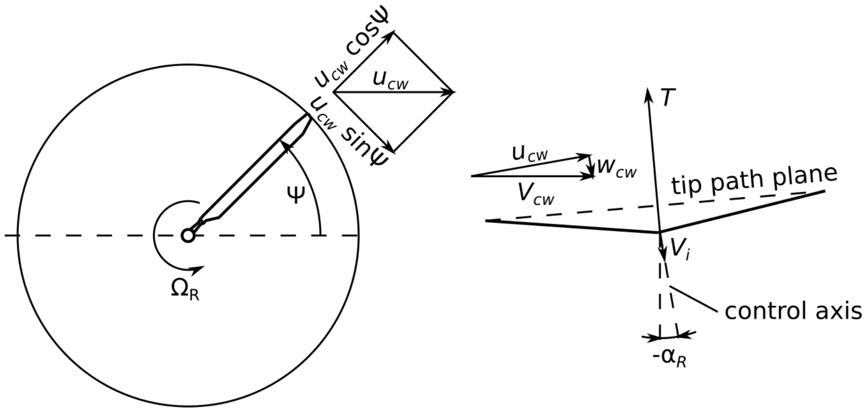

Figure 2.4: Air velocities at the blade element

Following formulas can be used to transform linear and angular velocity vector to Control-Wind Axis System:

$$\vec{V}_{RH,cw} = \boldsymbol{T}\left(\beta\right)\vec{V}_{RH,c} \tag{2.46}$$

$$\vec{\omega}_{cw} = \boldsymbol{T}\left(\beta\right)\vec{\omega}_{c} \tag{2.47}$$

Rotation matrix $\boldsymbol{T}\left(\beta\right)$ is given by formula (**??**).

Assuming that flapping angle is positive upwards and writing velocity components as:

$$\vec{V}_{RH,cw} = \left[u_{cw}, 0, w_{cw}\right]^T \tag{2.48}$$

$$\vec{\omega}_{cw} = \left[p_{cw}, q_{cw}, 0\right]^T \tag{2.49}$$

Air velocity at the blade segment is:

— for counterclockwise direction of rotor:

$$U_T = \Omega_R r \cos\beta + u_{cw}\sin\Psi \tag{2.50}$$

$$\begin{split} U_P = w_{cw}\cos\beta - V_i\cos\beta - \dot{\beta}r - u_{cw}\sin\beta\cos\Psi \\ + p_{cw}r\sin\Psi + q_{cw}r\cos\Psi \end{split} \tag{2.51}$$

— for clockwise direction of rotor:

$$U_T = \Omega_R r \cos\beta + u_{cw} \sin\Psi \tag{2.52}$$

$$\begin{split} U_P = w_{cw}\cos\beta - V_i\cos\beta - \dot{\beta}r - u_{cw}\sin\beta\cos\Psi \\ - p_{cw}r\sin\Psi + q_{cw}r\cos\Psi \end{split} \tag{2.53}$$

Assuming that for small angles:

$$\sin\beta \approx \beta \tag{2.54}$$

$$\cos\beta \approx 1 \tag{2.55}$$

This expressions can be simplified to:

— for counterclockwise direction of rotor:

$$U_T = \Omega_R r + u_{cw}\sin\Psi \tag{2.56}$$

$$U_P = w_{cw} - V_i - \dot{\beta}r - u_{cw}\beta\cos\Psi + p_{cw}r\sin\Psi + q_{cw}r\cos\Psi \tag{2.57}$$

— for clockwise direction of rotor:

$$U_T = \Omega_R r + u_{cw}\sin\Psi \tag{2.58}$$

$$U_P = w_{cw} - V_i - \dot{\beta}r - u_{cw}\beta\cos\Psi - p_{cw}r\sin\Psi + q_{cw}r\cos\Psi \tag{2.59}$$

Using normalized velocities:

$$\mu = \frac{u_{cw}}{\Omega_R R} \tag{2.60}$$

$$\lambda = \frac{w_{cw} - V_i}{\Omega_R R} \tag{2.61}$$

Expressions for air velocity at the blade segment can be written in the following form:

— for counterclockwise direction of rotor:

$$U_T = \Omega_R r + \mu\Omega_R R\sin\Psi \tag{2.62}$$

$$U_P = \lambda\Omega_R R - \dot{\beta}r - \mu\Omega_R R\beta\cos\Psi + p_{cw}r\sin\Psi + q_{cw}r\cos\Psi \tag{2.63}$$

— for clockwise direction of rotor:

$$U_T = \Omega_R r + \mu\Omega_R R \sin\Psi \tag{2.64}$$

$$U_P = \lambda\Omega_R R - \dot{\beta} r - \mu\Omega_R R\beta\cos\Psi - p_{cw} r \sin\Psi + q_{cw} r \cos\Psi \tag{2.65}$$

2.4 Rotor Thrust

Assuming that for small angles:

$$\phi = \arctan\frac{U_P}{U_T} \approx \frac{U_P}{U_T} \tag{2.66}$$

$$U \approx U_T \tag{2.67}$$

$$dT \approx dL \tag{2.68}$$

Expression for the blade segment angle of attack is given as follows:

$$\alpha_{BE} = \theta + \frac{U_P}{U_T} \tag{2.69}$$

Then expression (2.35) can be written as. [1]

$$C_L = a\left(\theta + \frac{U_P}{U_T}\right) \tag{2.70}$$

Substituting expression (2.33) and taking into account simplifications (2.66), (2.67) and (2.68) then rotor thrust is given by the following formula.

$$dT \approx \frac{1}{2}\rho a c_b U_T^2 \left(\theta + \frac{U_P}{U_T}\right) dr \tag{2.71}$$

Transforming this relationship gives:

$$dT \approx \frac{1}{2}\rho a c_b \left(\theta U_T^2 + U_P U_T\right) dr \tag{2.72}$$

Total thrust generated by the rotor of n_b blades can be determined by integrating differential equation (2.72) first with respect to the azimuth then along the blade span. [1] Total thrust is given as follows:

$$T = \frac{n_b}{2\pi}\int_0^{2\pi}\int_0^{BR}\frac{dT}{dr} dr d\Psi \tag{2.73}$$

Where B is a tip loss factor.

Substituting (2.72) into (2.73) gives:

$$T = \frac{1}{2}\rho a c_b n_b \left(\theta \frac{1}{2\pi}\int_0^{2\pi}\int_0^{BR} \frac{U_T^2}{dr} dr d\Psi + \frac{1}{2\pi}\int_0^{2\pi}\int_0^{BR} U_P U_T dr d\Psi\right) \tag{2.74}$$

Neglecting helicopter angular velocity expressions for U_T^2 and $U_P U_T$ can be written as:

$$U_T^2 = r^2\Omega_R^2 + 2\Omega_R^2 R r\mu \sin\Psi + \mu^2 R^2 \Omega_R^2 \sin^2\Psi \tag{2.75}$$

$$\begin{aligned} U_P U_T = {} & \lambda\Omega_R^2 R r - \dot{\beta}\Omega_R r^2 - \beta\mu\Omega_R^2 R r \cos\Psi \\ & + \lambda\mu\Omega_R^2 R^2 \sin\Psi - \dot{\beta}\mu\Omega_R R r \sin\Psi - \beta\mu^2\Omega_R^2 R^2 \sin\Psi\cos\Psi \end{aligned} \tag{2.76}$$

Expression for the blade flapping angle can be written as follows: [2, 1, 5]

$$\beta(\Psi) = \beta_0 + \beta_{1c}\cos\Psi + \beta_{1s}\sin\Psi \tag{2.77}$$

Assuming constant rotor revolution speed, blade flapping angle derivatives with respect to time can be written as derivatives with respect to the azimuth. [1]

$$\dot{\beta} = \frac{d\beta}{dt} = \frac{d\beta}{d\Psi}\frac{d\Psi}{dt} = \bar{\beta}\Omega_R = \Omega_R\left(\beta_{1s}\cos\Psi - \beta_{1c}\sin\Psi\right) \tag{2.78}$$

$$\ddot{\beta} = \frac{d^2\beta}{dt^2} = \frac{d^2\beta}{d\Psi^2}\left(\frac{d\Psi}{dt}\right)^2 = \bar{\bar{\beta}}\Omega_R^2 = -\Omega_R^2\left(\beta_{1c}\cos\Psi + \beta_{1s}\sin\Psi\right) \tag{2.79}$$

Then expressions for U_T^2 and $U_P U_T$ can be written as:

$$U_T^2 = r^2\Omega_R^2 + 2\Omega_R^2 R r\mu \sin\Psi + \mu^2 R^2 \Omega_R^2 \sin^2\Psi \tag{2.80}$$

$$\begin{aligned} U_P U_T = {} & \lambda\Omega_R^2 R r - \beta_{1s}\Omega_R^2 r^2 \cos\Psi + \beta_{1c}\Omega_R^2 r^2 \sin\Psi \\ & - \beta\mu\Omega_R^2 R r\cos\Psi + \lambda\mu\Omega_R^2 R^2 \sin\Psi - \beta_{1s}\mu\Omega_R^2 R r \sin\Psi\cos\Psi \\ & + \beta_{1c}\mu\Omega_R^2 R r \sin^2\Psi - \beta\mu^2\Omega_R^2 R^2 \sin\Psi\cos\Psi \end{aligned} \tag{2.81}$$

Knowing that: [1]

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\Psi d\Psi = \frac{1}{2\pi}\int_0^{2\pi} \cos\Psi d\Psi = 0 \tag{2.82}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin^2\Psi d\Psi = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\Psi d\Psi = \frac{1}{2} \tag{2.83}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \sin\Psi\cos\Psi d\Psi = 0 \tag{2.84}$$

Rotor thrust can be written as:

$$T = \frac{1}{2}\rho a c_b n_b \Omega_R^2 R^3 B \left[\frac{\lambda B}{2} + \frac{\theta}{3}\left(B^2 + \frac{3}{2}\mu^2\right) + \frac{\beta_{1c} B}{4}\mu\right] \tag{2.85}$$

Using expression (2.12) rotor thrust coefficient is given by the following formula:

$$C_T = \frac{1}{2} a s B \left[\frac{\lambda B}{2} + \frac{\theta}{3}\left(B^2 + \frac{3}{2}\mu^2\right) + \frac{\beta_{1c} B}{4}\mu\right] \tag{2.86}$$

Where s is rotor solidity.

2.5 Rotor Torque

The torque on a blade element is given by the following formula. [1, 4]

$$dQ = r\left(dD\cos\phi - dL\sin\phi\right) dr \tag{2.87}$$

Assuming that for small angles:

$$\sin\phi \approx \phi \tag{2.88}$$

$$\cos\phi \approx 1 \tag{2.89}$$

And assuming that drag coefficient is constant along blade span, expression (2.87) can be written as: [1, 4]

$$dQ = \frac{1}{2}\rho U_T^2 C_D c_b r dr - \frac{1}{2}\rho U_T^2 C_L c_b r \phi dr \tag{2.90}$$

Torque due to the profile drag can be expressed as: [4]

$$Q_p = \frac{n_b}{2\pi}\int_0^R \int_0^{2\pi} \frac{1}{2}\rho U_T^2 C_D c_b r d\Psi dr \tag{2.91}$$

Substituting (2.75) and integrating this equation first with respect to the azimuth then along the blade span gives:

$$Q_p = \frac{1}{2}\rho n_b c_b \Omega_R^2 R^4 C_D \left(\frac{1}{4} + \frac{1}{4}\mu^2\right) \tag{2.92}$$

Induced torque is given by the following formula: [4]

$$Q_i = \frac{n_b}{2\pi}\int_0^R\int_0^{2\pi}\frac{1}{2}\rho U_T^2 C_L c_b r\phi d\Psi dr \tag{2.93}$$

Substituting (2.66) and (2.70) gives:

$$Q_i = \frac{n_b}{2\pi}\frac{1}{2}\rho a c_b\int_0^R\int_0^{2\pi}\left(\theta U_P U_T r + U_P^2 r\right) d\Psi dr \tag{2.94}$$

Neglecting helicopter angular velocity expressions for U_P^2 can be written as:

$$\begin{aligned} U_P^2 = \dot\beta^2 r^2 + 2\beta\dot\beta\mu\Omega_R R r\cos\Psi - 2\dot\beta\lambda\Omega_R R r \\ + \beta^2\mu^2\Omega_R^2 R^2\cos^2\Psi - 2\beta\lambda\mu\Omega_R^2 R^2\cos\Psi + \lambda^2\Omega_R^2 R^2 \end{aligned} \tag{2.95}$$

Substituting (2.76) and (2.95) into (2.94) and integrating equation (2.94) first with respect to the azimuth then along the blade span gives:

$$\begin{aligned} Q_i = \frac{1}{2}\rho a c_b n_b\Omega_R^2 R^4 \Bigg[\frac{1}{3}\lambda\theta + \frac{1}{2}\lambda^2 - \frac{1}{8}\left(\beta_{1c}^2 + \beta_{1s}^2\right) \\ + \frac{1}{2}\mu^2\left(\frac{\beta_0^2}{2} - \frac{3}{8}\beta_{1c}^2 - \frac{1}{8}\beta_{1s}^2\right) - \frac{1}{2}\mu\lambda\beta_{1c} + \frac{1}{3}\mu\beta_0\beta_{1s}\Bigg] \end{aligned} \tag{2.96}$$

The total rotor torque is given as follows: [4, 5]

$$Q = Q_p - Q_i \tag{2.97}$$

$$\begin{aligned} Q = \frac{1}{2}\rho a c_b n_b\Omega_R^2 R^4 \Bigg[\frac{C_D}{4a}\left(1+\mu^2\right) - \frac{1}{3}\lambda\theta - \frac{1}{2}\lambda^2 + \frac{1}{8}\left(\beta_{1c}^2 + \beta_{1s}^2\right) \\ - \frac{1}{2}\mu^2\left(\frac{\beta_0^2}{2} - \frac{3}{8}\beta_{1c}^2 - \frac{1}{8}\beta_{1s}^2\right) + \frac{1}{2}\mu\lambda\beta_{1c} - \frac{1}{3}\mu\beta_0\beta_{1s}\Bigg] \end{aligned} \tag{2.98}$$

Rotor torque coefficient can be written as:

$$\begin{aligned} C_Q = \frac{1}{2}as \Bigg[\frac{C_D}{4a}\left(1+\mu^2\right) - \frac{1}{3}\lambda\theta - \frac{1}{2}\lambda^2 + \frac{1}{8}\left(\beta_{1c}^2 + \beta_{1s}^2\right) \\ - \frac{1}{2}\mu^2\left(\frac{\beta_0^2}{2} - \frac{3}{8}\beta_{1c}^2 - \frac{1}{8}\beta_{1s}^2\right) + \frac{1}{2}\mu\lambda\beta_{1c} - \frac{1}{3}\mu\beta_0\beta_{1s}\Bigg] \end{aligned} \tag{2.99}$$

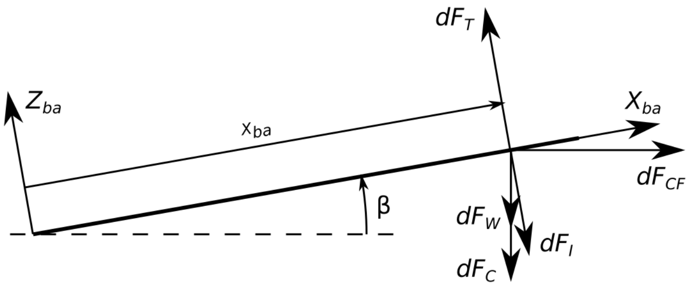

Figure 2.5: Forces acting on the blade element

2.6 Flapping Coefficients

Expressions for blades flapping coefficients can be derived from the equation of moments equilibrium about flapping hinge using method described in [5].

Moments equilibrium about flapping hinge can be written as follows: [1]

$$M_I + M_{CF} + M_C + M_T + M_W = 0 \tag{2.100}$$

Where:

M_I [N·m] moment due to inertia forces of flapping

M_{CF} [N·m] moment due to centrifugal force

M_C [N·m] moment due to Coriolis force

M_T [N·m] moment due to thrust

M_W [N·m] moment due to weight

2.6.1 Moments of Inertia Forces

Assuming that rotor revolution speed is constant, helicopter angular velocities are constant and rotor blades are able to move about flapping hinge axis while neglecting helicopter yaw motion and blades pitching and lagging motion then centrifugal force can be written as:

$$dF_{CF} = m_b \Omega_R^2 r dr \tag{2.101}$$

Component of the Coriolis force laying on the flapping plane is given:

— for counterclockwise direction of rotor:

$$dF_C = 2m_b p_{cw} \Omega_R r \cos \Psi dr - 2m_b q_{cw} \Omega_R r \sin \Psi dr \tag{2.102}$$

— for clockwise direction of rotor:

$$dF_C = -2m_b p_{cw} \Omega_R r \cos \Psi dr - 2m_b q_{cw} \Omega_R r \sin \Psi dr \tag{2.103}$$

Moment of inertia forces of flapping is given by the following formula:

$$M_I = -\ddot{\beta} \int_0^R m_b r^2 dr \tag{2.104}$$

Assuming that rotor blades are homogeneous rods blade first moment of mass and moment of inertia can be written as: [5]

$$J_b \approx \int_0^R m_b r^2 dr \tag{2.105}$$

$$S_b \approx \int_0^R m_b r dr \tag{2.106}$$

Hence:

$$M_I = -\ddot{\beta} J_b \tag{2.107}$$

Taking into account (2.54) and (2.55) moment of centrifugal forces is:

$$M_{CF} = -\int_0^R \beta m_b \Omega_R^2 r^2 dr = -\Omega_R^2 \beta \int_0^R m_b r^2 dr \tag{2.108}$$

Substituting (2.105) into (2.108) gives:

$$M_{CF} = -\Omega_R^2 \beta J_b \tag{2.109}$$

Moment of Coriolis forces can be writes as:

— for counterclockwise direction of rotor:

$$\begin{aligned} M_C = 2\int_0^R m_b p_{cw} \Omega_R r^2 \cos \Psi dr - 2\int_0^R m_b q_{cw} \Omega_R r^2 \sin \Psi dr = \\ = 2p_{cw} \Omega_R J_b \cos \Psi - 2q_{cw} \Omega_R J_b \sin \Psi \end{aligned} \tag{2.110}$$

— for clockwise direction of rotor:

$$M_C = -2\int_0^R m_b p_{cw}\Omega_R r^2 \cos\Psi dr - 2\int_0^R m_b q_{cw}\Omega_R r^2 \sin\Psi dr =$$
$$= -2p_{cw}\Omega_R J_b \cos\Psi - 2q_{cw}\Omega_R J_b \sin\Psi \quad (2.111)$$

Using approximation (2.106) moment due to weight can expressed as:

$$M_W = -g\int_0^R m_b r dr = -gS_b \quad (2.112)$$

2.6.2 Moment of Thrust

Using equation (2.72) expression for moment of thrust about flapping hinge can be written as follows:

$$M_T = \int_0^{BR} dTr = \frac{1}{2}\rho a c_b \int_0^{BR} \left(\theta U_T^2 + U_P U_T\right) dr \quad (2.113)$$

2.6.3 Equilibrium of Moments about Flapping Hinge

Substituting into (2.100) expressions for moments of thrust, weight, inertia, centrifugal and Coriolis forces moments equilibrium equation is given as:

— for counterclockwise direction of rotor:

$$\int_0^{BR} dTr - \ddot{\beta}J_b - \beta\Omega_R^2 J_b + 2p_{cw}\Omega_R J_b \cos\Psi - 2q_{cw}\Omega_R J_b \sin\Psi - gS_b = 0 \quad (2.114)$$

$$-J_b\ddot{\beta} - J_b\beta\Omega_R^2 + \int_0^{BR} dTr = 2J_b q_{cw}\Omega_R \sin\Psi - 2J_b p_{cw}\Omega_R \cos\Psi + gS_b \quad (2.115)$$

— for clockwise direction of rotor:

$$\int_0^{BR} dTr - \ddot{\beta}J_b - \beta\Omega_R^2 J_b - 2p_{cw}\Omega_R J_b \cos\Psi - 2q_{cw}\Omega_R J_b \sin\Psi - gS_b = 0 \tag{2.116}$$

$$-J_b\ddot{\beta} - J_b\beta\Omega_R^2 + \int_0^{BR} dTr = 2J_b q_{cw}\Omega_R \sin\Psi + 2J_b p_{cw}\Omega_R \cos\Psi + gS_b \tag{2.117}$$

Dividing both sides of equations (2.115) and (2.117) by $J_b\Omega_R^2$ gives:

$$\bar{\bar{\beta}} + \beta = \frac{1}{J_b\Omega_R^2}\int_0^{BR} dTr - 2\frac{q_{cw}}{\Omega_R}\sin\Psi + 2\frac{p_{cw}}{\Omega_R}\cos\Psi - \frac{gS_b}{J_b\Omega_R^2} \tag{2.118}$$

$$\bar{\bar{\beta}} + \beta = \frac{1}{J_b\Omega_R^2}\int_0^{BR} dTr - 2\frac{q_{cw}}{\Omega_R}\sin\Psi - 2\frac{p_{cw}}{\Omega_R}\cos\Psi - \frac{gS_b}{J_b\Omega_R^2} \tag{2.119}$$

Substituting expressions (2.75) and (2.76) into (2.118) and (2.119) gives:

$$\begin{aligned} M_T = \int_0^{BR} dTr &= \frac{1}{2}\rho a c_b \int_0^{BR} \left(\theta U_T^2 + U_P U_T\right) r dr = \\ &= \frac{1}{2}\rho a c_b R^4\Omega_R^2 \left(\frac{B^4}{4}\theta + \frac{2}{3}B^3\theta\mu\sin\Psi + \frac{B^2}{2}\theta\mu^2\sin^2\Psi \right. \\ &\quad + \frac{B^3}{3}\lambda - \frac{B^4}{4}\bar{\beta} - \frac{B^3}{3}\beta\mu\cos\Psi + \frac{B^4}{4}\frac{p_{cw}}{\Omega_R}\sin\Psi \\ &\quad + \frac{B^2}{2}\lambda\mu\sin\Psi - \frac{B^2}{2}\beta\mu^2\sin\Psi\cos\Psi - \frac{B^3}{3}\bar{\beta}\mu\sin\Psi \\ &\quad \left. + \frac{B^3}{3}\frac{p_{cw}}{\Omega_R}\mu\sin^2\Psi + \frac{B^4}{4}\frac{q_{cw}}{\Omega_R}\cos\Psi + \frac{B^3}{3}\frac{q_{cw}}{\Omega_R}\mu\sin\Psi\cos\Psi\right) \end{aligned} \tag{2.120}$$

$$\begin{aligned} M_T = \int_0^{BR} dTr &= \frac{1}{2}\rho a c_b \int_0^{BR} \left(\theta U_T^2 + U_P U_T\right) r dr = \\ &= \frac{1}{2}\rho a c_b R^4\Omega_R^2 \left(\frac{B^4}{4}\theta + \frac{2}{3}B^3\theta\mu\sin\Psi + \frac{B^2}{2}\theta\mu^2\sin^2\Psi \right. \\ &\quad + \frac{B^3}{3}\lambda - \frac{B^4}{4}\bar{\beta} - \frac{B^3}{3}\beta\mu\cos\Psi - \frac{B^4}{4}\frac{p_{cw}}{\Omega_R}\sin\Psi \\ &\quad + \frac{B^2}{2}\lambda\mu\sin\Psi - \frac{B^2}{2}\beta\mu^2\sin\Psi\cos\Psi - \frac{B^3}{3}\bar{\beta}\mu\sin\Psi \\ &\quad \left. - \frac{B^3}{3}\frac{p_{cw}}{\Omega_R}\mu\sin^2\Psi + \frac{B^4}{4}\frac{q_{cw}}{\Omega_R}\cos\Psi + \frac{B^3}{3}\frac{q_{cw}}{\Omega_R}\mu\sin\Psi\cos\Psi\right) \end{aligned} \tag{2.121}$$

Knowing that:

$$\sin\Psi\cos\Psi = \frac{\sin(2\Psi)}{2} \tag{2.122}$$

$$\sin^2\Psi = \frac{1-\cos(2\Psi)}{2} \tag{2.123}$$

$$\cos^2\Psi = \frac{1+\cos(2\Psi)}{2} \tag{2.124}$$

$$\cos\Psi\sin(2\Psi) = \frac{\sin\Psi+\sin(3\Psi)}{2} \tag{2.125}$$

$$\sin\Psi\sin(2\Psi) = \frac{\cos\Psi-\cos(3\Psi)}{2} \tag{2.126}$$

Substituting (2.120) into (2.118) and (2.121) into (2.119) gives:

$$\begin{aligned}
&\bar{\bar{\beta}} + \bar{\beta}\frac{\gamma}{2}\left(\frac{B^4}{4} + \frac{B^3}{3}\mu\sin\Psi\right) + \beta\left(1 + \frac{B^3}{6}\gamma\mu\cos\Psi + \frac{B^2}{8}\gamma\mu^2\sin(2\Psi)\right) = \\
&= \frac{\gamma}{2}\left(\frac{B^3}{3}\lambda + \frac{B^4}{4}\frac{p_{cw}}{\Omega_R}\sin\Psi + \frac{B^2}{2}\lambda\mu\sin\Psi + \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu - \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu\cos(2\Psi)\right.\\
&\qquad + \frac{B^4}{4}\frac{q_{cw}}{\Omega_R}\cos\Psi + \frac{B^3}{6}\frac{q_{cw}}{\Omega_R}\mu\sin(2\Psi) + \frac{B^4}{4}\theta\\
&\qquad \left. + \frac{2}{3}B^3\theta\mu\sin\Psi + \frac{B^2}{4}\theta\mu^2 - \frac{B^2}{4}\theta\mu^2\cos(2\Psi)\right)\\
&\qquad - 2\frac{q_{cw}}{\Omega_R}\sin\Psi + 2\frac{p_{cw}}{\Omega_R}\cos\Psi - \frac{gS_b}{J_b\Omega_R^2}
\end{aligned} \tag{2.127}$$

$$\begin{aligned}
&\bar{\bar{\beta}} + \bar{\beta}\frac{\gamma}{2}\left(\frac{B^4}{4} + \frac{B^3}{3}\mu\sin\Psi\right) + \beta\left(1 + \frac{B^3}{6}\gamma\mu\cos\Psi + \frac{B^2}{8}\gamma\mu^2\sin(2\Psi)\right) = \\
&= \frac{\gamma}{2}\left(\frac{B^3}{3}\lambda - \frac{B^4}{4}\frac{p_{cw}}{\Omega_R}\sin\Psi + \frac{B^2}{2}\lambda\mu\sin\Psi - \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu + \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu\cos(2\Psi)\right.\\
&\qquad + \frac{B^4}{4}\frac{q_{cw}}{\Omega_R}\cos\Psi + \frac{B^3}{6}\frac{q_{cw}}{\Omega_R}\mu\sin(2\Psi) + \frac{B^4}{4}\theta\\
&\qquad \left. + \frac{2}{3}B^3\theta\mu\sin\Psi + \frac{B^2}{4}\theta\mu^2 - \frac{B^2}{4}\theta\mu^2\cos(2\Psi)\right)\\
&\qquad - 2\frac{q_{cw}}{\Omega_R}\sin\Psi - 2\frac{p_{cw}}{\Omega_R}\cos\Psi - \frac{gS_b}{J_b\Omega_R^2}
\end{aligned} \tag{2.128}$$

Where γ is a blade Lock number.

Transforming equations (2.127) and (2.128) gives:

$$
\begin{aligned}
\bar{\bar{\beta}} + \bar{\beta}\frac{\gamma}{2}\left(\frac{B^4}{4} + \frac{B^3}{3}\mu\sin\Psi\right) + \beta\left(1 + \frac{B^3}{6}\gamma\mu\cos\Psi + \frac{B^2}{8}\gamma\mu^2\sin(2\Psi)\right) = \\
= \left[\frac{\gamma}{2}\left(\frac{B^3}{3}\lambda + \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu + \frac{B^4}{4}\theta + \frac{B^2}{4}\theta\mu^2\right) - \frac{gS_b}{J_b\Omega_R^2}\right] \\
+ \left[\frac{\gamma}{2}\left(\frac{B^4}{4}\frac{q_{cw}}{\Omega_R}\right) + 2\frac{p_{cw}}{\Omega_R}\right]\cos\Psi \\
+ \left[\frac{\gamma}{2}\left(\frac{B^4}{4}\frac{p_{cw}}{\Omega_R} + \frac{B^2}{2}\lambda\mu + \frac{2}{3}B^3\theta\mu\right) - 2\frac{q_{cw}}{\Omega_R}\right]\sin\Psi \\
+ \left[\frac{\gamma}{2}\left(-\frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu - \frac{B^2}{4}\theta\mu^2\right)\right]\cos(2\Psi) \\
+ \left[\frac{\gamma}{2}\left(\frac{B^3}{6}\frac{q_{cw}}{\Omega_R}\mu\right)\right]\sin(2\Psi)
\end{aligned}
\tag{2.129}
$$

$$
\begin{aligned}
\bar{\bar{\beta}} + \bar{\beta}\frac{\gamma}{2}\left(\frac{B^4}{4} + \frac{B^3}{3}\mu\sin\Psi\right) + \beta\left(1 + \frac{B^3}{6}\gamma\mu\cos\Psi + \frac{B^2}{8}\gamma\mu^2\sin(2\Psi)\right) = \\
= \left[\frac{\gamma}{2}\left(\frac{B^3}{3}\lambda - \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu + \frac{B^4}{4}\theta + \frac{B^2}{4}\theta\mu^2\right) - \frac{gS_b}{J_b\Omega_R^2}\right] \\
+ \left[\frac{\gamma}{2}\left(\frac{B^4}{4}\frac{q_{cw}}{\Omega_R}\right) - 2\frac{p_{cw}}{\Omega_R}\right]\cos\Psi \\
+ \left[\frac{\gamma}{2}\left(-\frac{B^4}{4}\frac{p_{cw}}{\Omega_R} + \frac{B^2}{2}\lambda\mu + \frac{2}{3}B^3\theta\mu\right) - 2\frac{q_{cw}}{\Omega_R}\right]\sin\Psi \\
+ \left[\frac{\gamma}{2}\left(\frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu - \frac{B^2}{4}\theta\mu^2\right)\right]\cos(2\Psi) \\
+ \left[\frac{\gamma}{2}\left(\frac{B^3}{6}\frac{q_{cw}}{\Omega_R}\mu\right)\right]\sin(2\Psi)
\end{aligned}
\tag{2.130}
$$

Differentiating equation (2.77) gives:

$$\bar{\beta} = (\beta_{1s}\cos\Psi - \beta_{1c}\sin\Psi) \tag{2.131}$$

$$\bar{\bar{\beta}} = -(\beta_{1c}\cos\Psi + \beta_{1s}\sin\Psi) \tag{2.132}$$

Substituting expressions (2.131) and (2.132) into equations (2.129) and (2.130) as well as using trigonometric identities (2.122), (2.123), (2.124), (2.125) and (2.126) gives:

$$
\begin{aligned}
&\beta_0 + \beta_{1s}\gamma\frac{B^2}{8}\left(\frac{\mu^2}{2} + B^2\right)\cos\Psi + \beta_{1c}\gamma\frac{B^2}{8}\left(\frac{\mu^2}{2} - B^2\right)\sin\Psi \\
&\quad + \beta_{1c}\gamma\frac{B^3}{6}\mu\cos(2\Psi) + \left(\beta_{1s}\gamma\frac{B^3}{6}\mu + \beta_0\gamma\frac{B^2}{8}\mu^2\right)\sin(2\Psi) \\
&\quad + \beta_{1c}\gamma\frac{B^2}{16}\mu^2\sin(3\Psi) - \beta_{1s}\gamma\frac{B^2}{16}\mu^2\cos(3\Psi) = \\
&= \left[\frac{\gamma}{2}\left(\frac{B^3}{3}\lambda + \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu + \frac{B^4}{4}\theta + \frac{B^2}{4}\theta\mu^2\right) - \frac{gS_b}{J_b\Omega_R^2}\right] \\
&\quad + \left[\frac{\gamma}{2}\left(\frac{B^4}{4}\frac{q_{cw}}{\Omega_R} - \frac{B^3}{3}\beta_0\mu\right) + 2\frac{p_{cw}}{\Omega_R}\right]\cos\Psi \\
&\quad + \left[\frac{\gamma}{2}\left(\frac{B^4}{4}\frac{p_{cw}}{\Omega_R} + \frac{B^2}{2}\lambda\mu + \frac{2}{3}B^3\theta\mu\right) - 2\frac{q_{cw}}{\Omega_R}\right]\sin\Psi \\
&\quad + \left[\frac{\gamma}{2}\left(-\frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu - \frac{B^2}{4}\theta\mu^2\right)\right]\cos(2\Psi) \\
&\quad + \left[\frac{\gamma}{2}\left(\frac{B^3}{6}\frac{q_{cw}}{\Omega_R}\mu\right)\right]\sin(2\Psi)
\end{aligned}
\tag{2.133}
$$

$$
\begin{aligned}
&\beta_0 + \beta_{1s}\gamma\frac{B^2}{8}\left(\frac{\mu^2}{2} + B^2\right)\cos\Psi + \beta_{1c}\gamma\frac{B^2}{8}\left(\frac{\mu^2}{2} - B^2\right)\sin\Psi \\
&\quad + \beta_{1c}\gamma\frac{B^3}{6}\mu\cos(2\Psi) + \left(\beta_{1s}\gamma\frac{B^3}{6}\mu + \beta_0\gamma\frac{B^2}{8}\mu^2\right)\sin(2\Psi) \\
&\quad - \beta_{1s}\gamma\frac{B^2}{16}\mu^2\cos(3\Psi) + \beta_{1c}\gamma\frac{B^2}{16}\mu^2\sin(3\Psi) = \\
&= \left[\frac{\gamma}{2}\left(\frac{B^3}{3}\lambda - \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu + \frac{B^4}{4}\theta + \frac{B^2}{4}\theta\mu^2\right) - \frac{gS_b}{J_b\Omega_R^2}\right] \\
&\quad + \left[\frac{\gamma}{2}\left(\frac{B^4}{4}\frac{q_{cw}}{\Omega_R} - \frac{B^3}{3}\beta_0\mu\right) - 2\frac{p_{cw}}{\Omega_R}\right]\cos\Psi \\
&\quad + \left[\frac{\gamma}{2}\left(-\frac{B^4}{4}\frac{p_{cw}}{\Omega_R} + \frac{B^2}{2}\lambda\mu + \frac{2}{3}B^3\theta\mu\right) - 2\frac{q_{cw}}{\Omega_R}\right]\sin\Psi \\
&\quad + \left[\frac{\gamma}{2}\left(\frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu - \frac{B^2}{4}\theta\mu^2\right)\right]\cos(2\Psi) \\
&\quad + \left[\frac{\gamma}{2}\left(\frac{B^3}{6}\frac{q_{cw}}{\Omega_R}\mu\right)\right]\sin(2\Psi)
\end{aligned}
\tag{2.134}
$$

Neglecting all blade flapping harmonics above the first [1] equations (2.133) and (2.134) can be simplified as follows:

$$\begin{aligned}
&\beta_0 + \beta_{1s}\gamma\frac{B^2}{8}\left(\frac{\mu^2}{2} + B^2\right)\cos\Psi + \beta_{1c}\gamma\frac{B^2}{8}\left(\frac{\mu^2}{2} - B^2\right)\sin\Psi = \\
&\qquad = \left[\frac{\gamma}{2}\left(\frac{B^3}{3}\lambda + \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu + \frac{B^4}{4}\theta + \frac{B^2}{4}\theta\mu^2\right) - \frac{gS_b}{J_b\Omega_R^2}\right] \\
&\qquad\quad + \left[\frac{\gamma}{2}\left(\frac{B^4}{4}\frac{q_{cw}}{\Omega_R} - \frac{B^3}{3}\beta_0\mu\right) + 2\frac{p_{cw}}{\Omega_R}\right]\cos\Psi \\
&\qquad\quad + \left[\frac{\gamma}{2}\left(\frac{B^4}{4}\frac{p_{cw}}{\Omega_R} + \frac{B^2}{2}\lambda\mu + \frac{2}{3}B^3\theta\mu\right) - 2\frac{q_{cw}}{\Omega_R}\right]\sin\Psi
\end{aligned} \tag{2.135}$$

$$\begin{aligned}
&\beta_0 + \beta_{1s}\gamma\frac{B^2}{8}\left(\frac{\mu^2}{2} + B^2\right)\cos\Psi + \beta_{1c}\gamma\frac{B^2}{8}\left(\frac{\mu^2}{2} - B^2\right)\sin\Psi = \\
&\qquad = \left[\frac{\gamma}{2}\left(\frac{B^3}{3}\lambda - \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu + \frac{B^4}{4}\theta + \frac{B^2}{4}\theta\mu^2\right) - \frac{gS_b}{J_b\Omega_R^2}\right] \\
&\qquad\quad + \left[\frac{\gamma}{2}\left(\frac{B^4}{4}\frac{q_{cw}}{\Omega_R} - \frac{B^3}{3}\beta_0\mu\right) - 2\frac{p_{cw}}{\Omega_R}\right]\cos\Psi \\
&\qquad\quad + \left[\frac{\gamma}{2}\left(-\frac{B^4}{4}\frac{p_{cw}}{\Omega_R} + \frac{B^2}{2}\lambda\mu + \frac{2}{3}B^3\theta\mu\right) - 2\frac{q_{cw}}{\Omega_R}\right]\sin\Psi
\end{aligned} \tag{2.136}$$

2.6.4 Final Form

Equations (2.135) and (2.136) can be transformed to get blade flapping coefficients:

— for counterclockwise direction of rotor:

$$\beta_0 = \frac{\gamma}{2}\left(\frac{B^3}{3}\lambda + \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu + \frac{B^4}{4}\theta + \frac{B^2}{4}\theta\mu^2\right) - \frac{gS_b}{J_b\Omega_R^2} \tag{2.137}$$

$$\beta_{1c} = 2\mu\left(\lambda + \frac{4}{3}B\theta\right)\frac{1}{\dfrac{\mu^2}{2} - B^2} + \left(B^4\frac{p_{cw}}{\Omega_R} - 16\frac{q_{cw}}{\gamma\Omega_R}\right)\frac{1}{B^2\left(\dfrac{\mu^2}{2} - B^2\right)} \tag{2.138}$$

$$\beta_{1s} = -\frac{4}{3}\beta_0\mu\frac{B}{\dfrac{\mu^2}{2} + B^2} + \left(B^4\frac{q_{cw}}{\Omega_R} + 16\frac{p_{cw}}{\gamma\Omega_R}\right)\frac{1}{B^2\left(\dfrac{\mu^2}{2} + B^2\right)} \tag{2.139}$$

— for clockwise direction of rotor:

$$\beta_0 = \frac{\gamma}{2}\left(\frac{B^3}{3}\lambda - \frac{B^3}{6}\frac{p_{cw}}{\Omega_R}\mu + \frac{B^4}{4}\theta + \frac{B^2}{4}\theta\mu^2\right) - \frac{gS_b}{J_b\Omega_R^2} \tag{2.140}$$

$$\beta_{1c} = 2\mu\left(\lambda + \frac{4}{3}B\theta\right)\frac{1}{\dfrac{\mu^2}{2} - B^2} - \left(B^4\frac{p_{cw}}{\Omega_R} + 16\frac{q_{cw}}{\gamma\Omega_R}\right)\frac{1}{B^2\left(\dfrac{\mu^2}{2} - B^2\right)} \tag{2.141}$$

$$\beta_{1s} = -\frac{4}{3}\beta_0\mu\frac{B}{\dfrac{\mu^2}{2} + B^2} + \left(B^4\frac{q_{cw}}{\Omega_R} - 16\frac{p_{cw}}{\gamma\Omega_R}\right)\frac{1}{B^2\left(\dfrac{\mu^2}{2} + B^2\right)} \tag{2.142}$$

Chapter 3

Blade Element Method

Bibliography

[1] A. Gessow and G. C. Myers, *Aerodynamics of the Helicopter*. Frederick Ungar Publishing, 1985.

[2] G. D. Padfield, *Helicopter Flight Dynamics: The Theory and Application of Flying Qualities and Simulation Modelling*. Blackwell Publishing, 2007.

[3] W. Z. Stepniewski, *Rotary-Wing Aerodynamics. Volume I: Basic Theories of Rotor Aerodynamics*. Dover Publications, 1984.

[4] A. R. S. Bramwell, G. Done, and D. Balmford, *Bramwell's Helicopter Dynamics*. Butterworth-Heinemann, 2001.

[5] M. L. Mil, A. V. Nekrasov, A. S. Braverman, L. N. Grodko, and M. A. Leykand, *Helicopters: Calculation and Design. Volume 1: Aerodynamics*. National Aeronautics and Space Administration, TT-F-494, 1967.