An Introduction to Regularized Regression

Machine Learning and Causal Inference

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Thanks to Sendhil Mullainathan for sharing slides; see Mullainathan and Speiss (2017) JEP which covers much of this material

What we do in Econometrics: The Case of Regression

- Specify a model:

$$Y_i = f(X_i) + \epsilon_i = X_i\beta + \epsilon_i$$

- Data set has observations i=1,..,n
- Use OLS regression on the entire dataset to construct an estimate $\hat{\beta}$
- Discuss assumptions under which some components of $\hat{\beta}$ have a causal interpretation
- Consider that S_n (set of observed units, i=1,..,n) is a random sample from a much larger population.
- Construct confidence intervals and test the hypothesis that some components are equal to zero.
- Theorem: OLS is BLUE (Best Linear Unbiased Estimator)
  - Best = lowest-variance

Goals of Prediction and Estimation

- Goal of estimation: unbiasedness

$$E[\hat{f}] = f$$

- Goal of prediction: loss minimization

$$L(f) = E_{(x,y)}\ell(f(x), y)$$

$$\hat{f} \approx \min_{f \in \mathcal{F}} L(f)$$

  - E.g. $\ell(f(x), y) = (f(x) - y)^2$
  - Use the data to pick a function that does well on a new data point

Key assumptions in both cases

- Stationary data generating process

$$\begin{array}{l}\text{Data}\\ S_n = (y_i, x_i)\ iid\end{array}$$

- Estimation:
  - Interested in a parameter of that process
- Prediction:
  - Interested in predicting *y*

High v. Low Dimensional Analysis

- We have discussed prediction as a high dimensional construct

- Practically that is where it is useful

- But to understand how high dimensional prediction works we must unpack an implicit presumption
  - Presumption: Our known estimation strategies would be great predictors *if they were feasible*

A Simple OLS example

- Suppose we truly live in a linear world

$$y = \beta_0 + \beta_1 x_1 + \varepsilon$$

$$\varepsilon \sim \mathcal{N}(0, \sigma_\epsilon^2) \quad x_1 \sim N(0, 1)$$

- Write x = (1,x)

$$y = \beta x + \varepsilon.$$

OLS seems like a good predictor

$$L(\hat{f}^{\mathrm{OLS}}) = \mathrm{E}_{(y,x)}(\hat{\beta}'x - y)^2 = (\hat{\beta}_0 - \beta_0)^2 + (\hat{\beta}_1 - \beta_1)^2 + \sigma_\varepsilon^2$$

So wouldn't we want the $\hat{\beta}$ with $\mathrm{E}_{S_n}(\hat{\beta}) = \beta$?

Especially since it is known to be efficient

An Even Simpler Set-up

- Let’s get even lower dimensional

- No variables at all

- Suppose you get the data of the type:

$$y_i = \mu + \epsilon_i$$

- You would like to estimate the mean

Forming an estimator of the mean

- Minimize bias: $\alpha = 1$
- The sample mean is an unbiased estimator
  - Also what you would get from OLS regression on a constant

A prediction problem

- In the same setup, you are given *n* data points

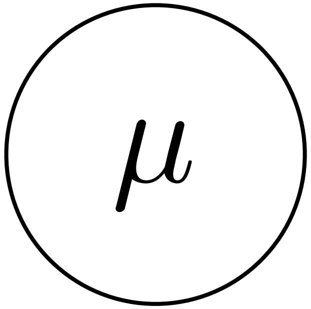
- You would like to guess the value of a new data point from the same distribution

- Goal: minimize quadratic loss of prediction

Best Predictor

$\ell(\hat{\mu}, y)]$

The higher alpha the lower the bias



The higher alpha the more variable across samples it is

Key problem

- The unbiased estimator has a nice property:

$$E[\hat{\mu}|\mu] = \mu$$

- But getting that property means large sample to sample variation of estimator

- This sample to sample variation means that in any particular finite sample I'm paying the cost of being off on all my predictions

Intuition

- I see your first test score. What should my prediction of your next test be?
  - Your first test score is an unbiased estimator
  - But it is very variable
- Note: “Bayesian” intuition
  - Even simpler: what was my guess before I saw any information
  - Shrink to that
  - In this example I’m shrinking to zero

But in a way you know this

- As empiricists you already have this intuition

Back to Simple OLS example

- Suppose we truly live in a linear world

$$y = \beta_0 + \beta_1 x_1 + \varepsilon$$

$$\varepsilon \sim \mathcal{N}(0, \sigma_\epsilon^2) \quad x_1 \sim N(0, 1)$$

- Write x = (1,x)

$$y = \beta x + \varepsilon.$$

A Simple Example

- You run a one variable regression and get

$$
\begin{aligned}
\hat{\beta}_0^{\mathrm{OLS}} &= 0 \pm .2 \\
\hat{\beta}_1^{\mathrm{OLS}} &= 2 \pm 10
\end{aligned}
$$

- Would you use the OLS coefficients to predict

- Or drop the first variable and use this:

$$\hat{\beta}_0^{OLS} = \arg\min_{\beta_0} \widehat{\mathbb{E}}_{S^n}(\beta_0 - y)^2 = \widehat{\mathbb{E}}_{S^n} y$$

Deciding whether to drop

- Suppose in the (impossible) case we got the true world right.
  - (0,2) are the right coefficients

- Of course OLS does perfectly (by assumption).

- But how would OLS do on new samples…where (0,2) being the generating coefficients?
  - We're giving OLS a huge leg up here.

OLS Performance

$$
\begin{aligned}
&\mathcal{L}_n(\mathrm{OLS}) - \sigma_\varepsilon^2 = \\
= \quad & \mathrm{E}_{(y,x)}\,\mathrm{E}_{S_n}[\beta' x - (\hat{\beta}^{\mathrm{OLS}})' x]^2 \\
= \quad & \mathrm{E}_{(y,x)}[\,\cancelto{0}{\underbrace{(\beta' x - (\mathrm{E}_{S_n}\hat{\beta}^{\mathrm{OLS}})' x)}_{\text{unbiased}}{}^2} + \mathrm{Var}_{S_n}((\hat{\beta}^{\mathrm{OLS}})' x)] \\
= \quad & \mathrm{Var}_{S_n}(\hat{\beta}_0^{\mathrm{OLS}}) + \mathrm{Var}_{S_n}(\hat{\beta}_1^{\mathrm{OLS}})
\end{aligned}
$$

What if we dropped the variable

$$\mathcal{L}_n(\mathrm{OLS}_0) - \sigma_\varepsilon^2 =$$

$$\begin{aligned}
&\mathcal{L}_n(\text{OLS}) - \sigma_\varepsilon^2 = \\
&= \mathrm{E}_{(y,x)}\,\mathrm{E}_{S_n}[\beta' x - (\hat\beta^{\text{OLS}})' x]^2 \\
&= \mathrm{E}_{(y,x)}[\underbrace{\cancelto{0}{(\beta' x - (\mathrm{E}_{S_n}\hat\beta^{\text{OLS}})' x)^2}}_{\text{unbiased}} + \mathrm{Var}_{S_n}((\hat\beta^{\text{OLS}})' x)] \\
&= \mathrm{Var}_{S_n}(\hat\beta_0^{\text{OLS}}) + \mathrm{Var}_{S_n}(\hat\beta_1^{\text{OLS}})
\end{aligned}$$

$$\begin{aligned}
&\mathcal{L}_n(\text{OLS}_0) - \sigma_\varepsilon^2 = \\
&= \mathrm{E}_{(y,x)}\,\mathrm{E}_{S_n}[\beta' x - (\hat\beta^{\text{OLS}_0})' x]^2 \\
&= \mathrm{E}_{(y,x)}[\underbrace{(\beta' x - (\mathrm{E}_{S_n}\hat\beta^{\text{OLS}_0})' x)^2}_{\text{bias}} + \mathrm{Var}_{S_n}((\hat\beta^{\text{OLS}_0})' x)] \\
&= (0-2)^2 + \mathrm{Var}_{S_n}(\hat\beta_0^{\text{OLS}_0}) + \cancelto{0}{\mathrm{Var}_{S_n}(\hat\beta_1^{\text{OLS}_0})}
\end{aligned}$$

$$\mathcal{L}_n(\text{OLS}) - \mathcal{L}_n(\text{OLS}_0) = \underbrace{\mathrm{Var}(\hat\beta_1^{OLS})}_{\text{variance}} - \underbrace{(0-2)^2}_{\text{bias}}$$

↑

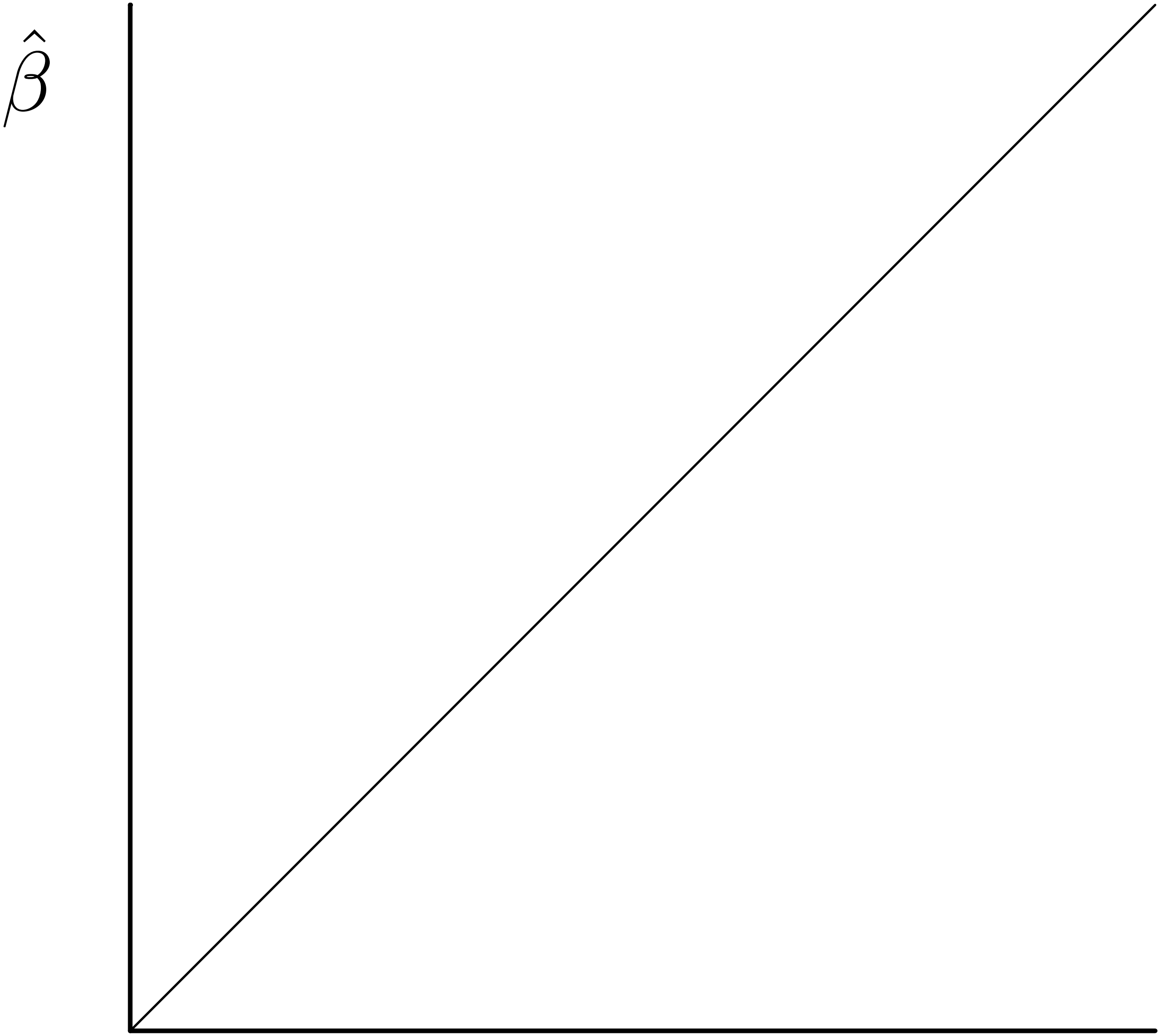
Your standard error worry!

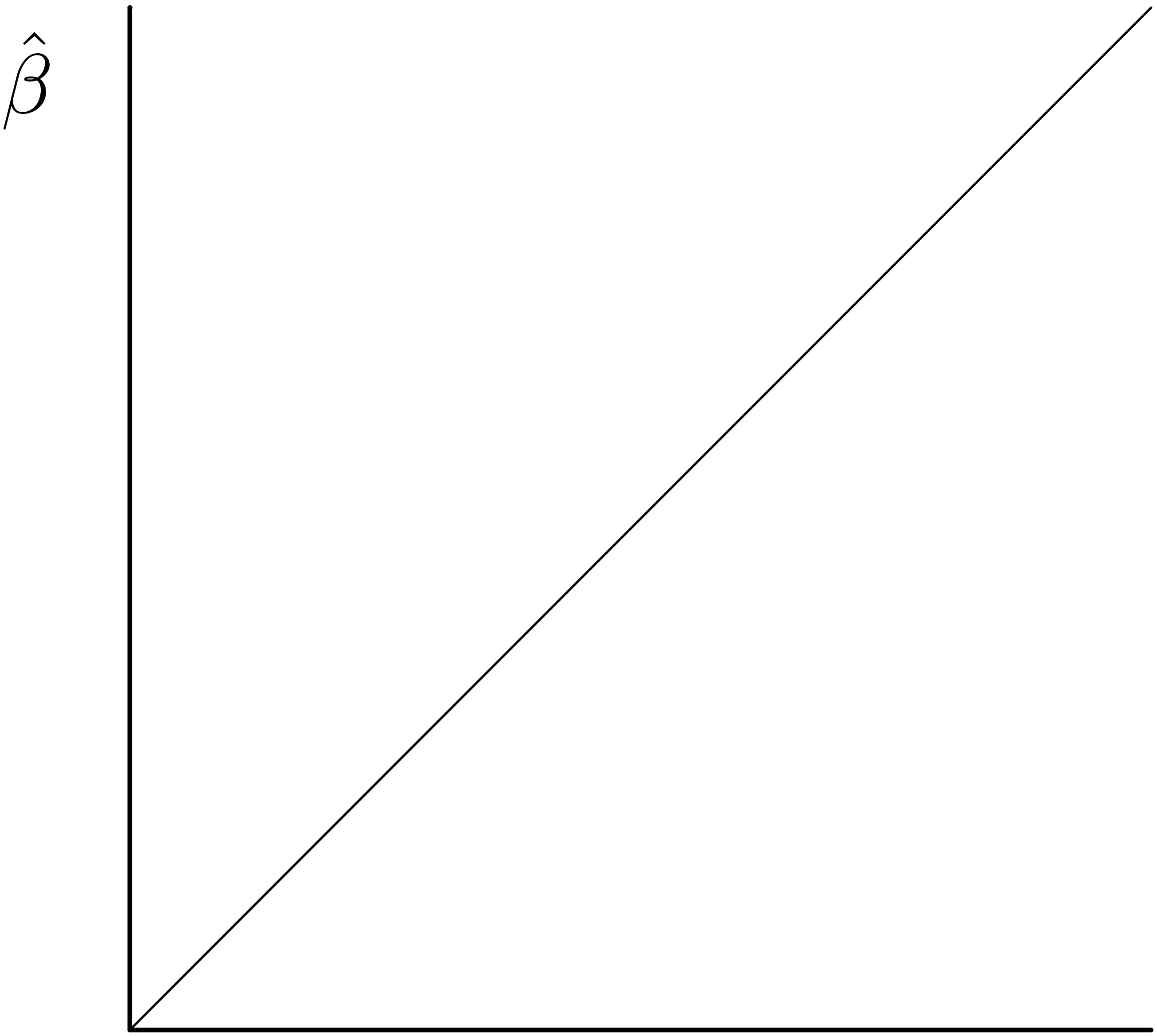
Where does your standard error intuition come from?

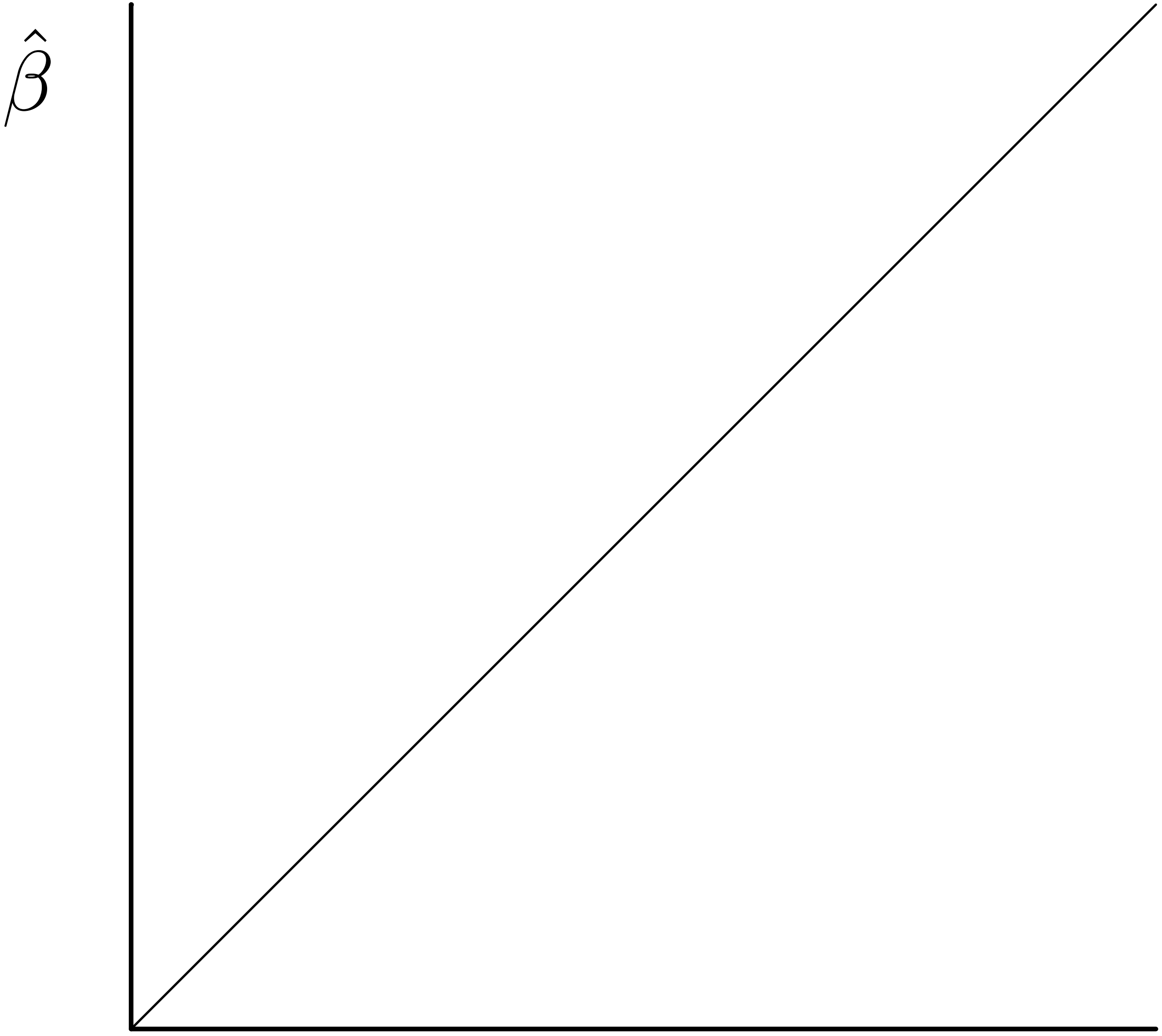
- You see a standard error

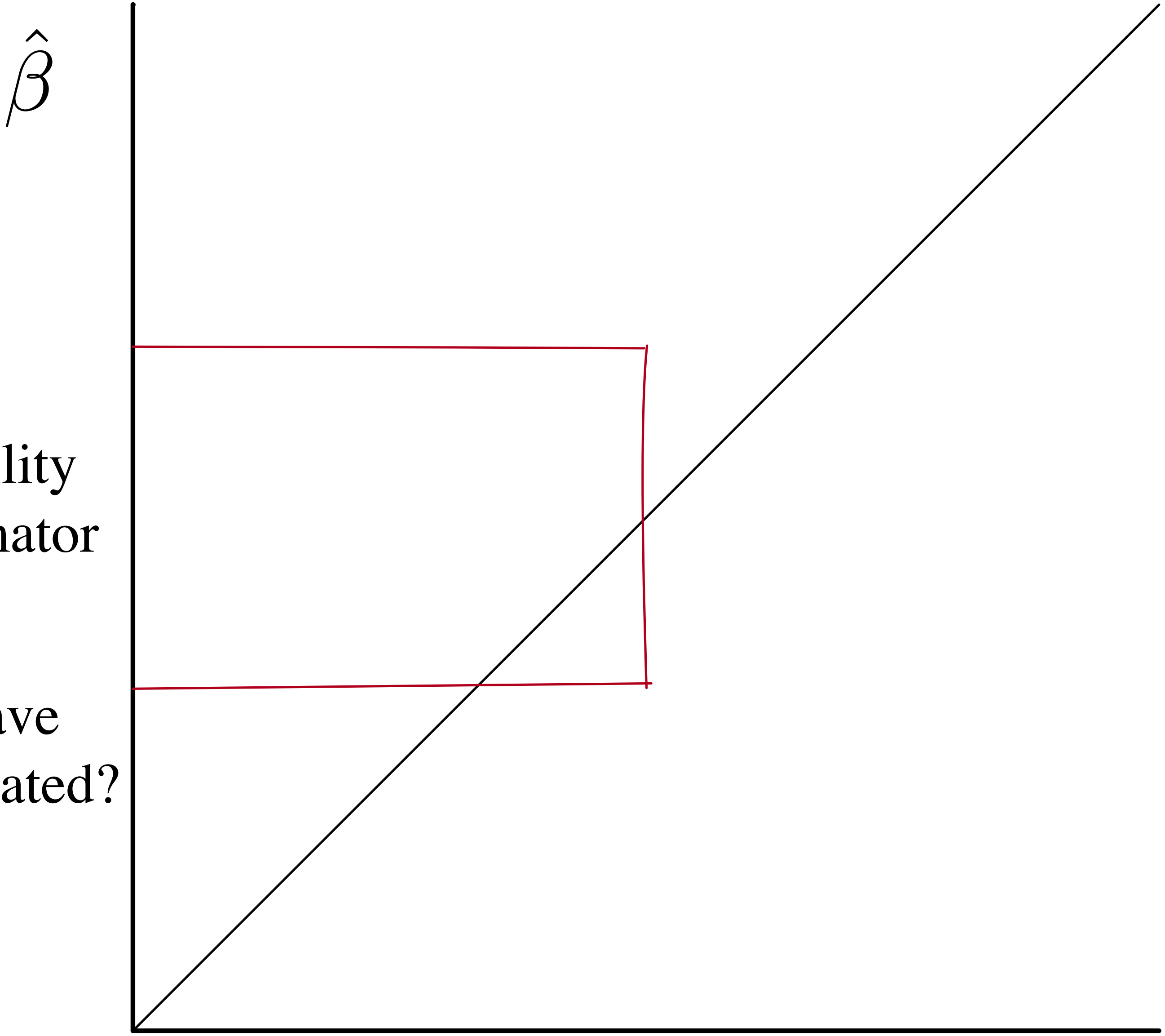
- You think “that variable is not ‘significant’” so you might not want to include it.

- But this is misleading


$\hat{\beta}$


$\hat{\beta}$


$\hat{\beta}$



Variability
of Estimator

ld have
been generated?

Your Standard Error Worry

- For hypothesis testing se tells you whether the coefficient is significant are not

- For prediction it’s telling you how variable an estimator using it really is

Dual purposes of the standard eror

- The standard error also tells you that even if you’re right on average:
  - Your estimator will produce a lot of variance
  - And then in those cases you make systematic prediction mistakes.
- Bias variance tradeoff
  - Being right on average on the coefficient is not equal to the best predictor.

The Problem Here

- Prediction quality suffers from:
  - Biased coefficients
  - Variability in *estimated* coefficients
    - Even if the true coefficient is 2, in any sample, we will estimate something else
- OLS is lexicographic
  - First ensure unbiased
  - Amongst unbiased estimators: seek efficiency
- Good predictions must trade these off

Two Variable Example

- Belaboring the point here…
- Assume now that we have two variables
  - As before, both normally distributed unit variance
- Your estimator produces

$$\begin{aligned}\hat{\beta}_0^{OLS} &= 0 \pm .2\\ \hat{\beta}_1^{OLS} &= 2 \pm 10\end{aligned}$$

What would you do now?

- Logic above suggests you would drop both variables?

- Or keep both variables?

- It really depends on how you feel about the variance (10)?

Calculation

$$\begin{aligned}\mathcal{L}_n(\mathrm{OLS}) - \mathcal{L}_n(\mathrm{OLS}_0) = {} & \overbrace{\mathrm{Var}(\hat{\beta}_1^{\mathrm{OLS}}) + \mathrm{Var}(\hat{\beta}_2^{\mathrm{OLS}})}^{\text{variance}} - \overbrace{((0-2)^2 + (0-2)^2)}^{\text{bias}} \\ & + \underbrace{2\rho_{12}\,\mathrm{Cov}(\hat{\beta}_1^{\mathrm{OLS}}, \hat{\beta}_2^{\mathrm{OLS}})}_{\text{covariance variance}} - \underbrace{2\rho_{12}(0-2)^2}_{\text{covariance bias}}\end{aligned}$$

$$\mathcal{L}_n(\mathrm{OLS}) - \mathcal{L}_n(\mathrm{OLS}_0) = \underbrace{\mathrm{Var}(\hat{\beta}_1^{OLS})}_{\text{variance}} - \underbrace{(0-2)^2}_{\text{bias}}$$

Hidden in Bias-Variance Tradeoff

- Covariance is central

- The standard error on several variables can be large, even though together their effect is highly consistent

- For prediction covariance between x matters

In a way this problem is not important

$$\mathcal{L}_n(\mathrm{OLS}) - \mathcal{L}_n(\mathrm{OLS}_0) = \underbrace{\mathrm{Var}(\hat{\beta}_1^{OLS})}_{\text{variance}} - \underbrace{(0-2)^2}_{\text{bias}}$$

- The variance term diminishes with sample size
  - Prediction-estimation wedge falls off as $\frac{1}{n}$
- But variance term increases with “variables”
  - Prediction-estimation rises with k
- So this is a problem when…
  - Function class high dimensional relative to data $\frac{k}{n}$

What this means practically

- In some cases what you already know (estimation) is perfectly fine for prediction
  - This is why ML textbooks teach OLS, etc.
  - They are perfectly useful for the kinds of prediction problems ML tries to solve *in low dimensional settings*
- But in high dimensional settings…
  - Note: high dimensional does not ONLY mean lots of variables! It can mean rich interactions.

So far…

- All this gives you a flavor of how the prediction task is not mechanically a consequence of the estimation task

- But it doesn’t really tell you **how** to predict
  - Bias variance tradeoff is entirely unactionable
  - What’s the bias?
  - What’s the variance?
  - This is not really a tradeoff you can make

- A different look at the same problem produces a practical insight though

Back to OLS

$$\hat{\beta}^{\mathrm{OLS}} = \arg\min_{\beta} \widehat{\mathbb{E}}_{S_n} (\beta' x - y)^2$$

AVERAGES NOTATION: $\widehat{\mathbb{E}}_{S_n}$ for sample ave. for sample S_n

$$\beta^*_{\text{prediction}} = \arg\min_{\beta} \mathrm{E}_{(y,x)} (\beta' x - y)^2$$

- The real problem here is minimizing the “wrong” thing: In-sample fit vs out-of-sample fit

Overfit problem

- OLS looks good with the sample you have
  - It's the best you can do *on this sample*

- Bias-variance improving predictive power is about improving *out of sample* predictive power

- Problem is OLS by construction overfits
  - We overfit in estimation

This problem is exactly why wide data is troubling

- Similarly think of the wide data case

- Why are we worried about having so many variables?

- We’ll fit very well (perfectly if $k > n$) in sample

- But arbitrarily badly out of sample

Understanding overfit

- Let’s consider a general class of algorithms

A General Class of Algorithms

- Let $L(f) = \int_{x,y} \ell(f(x), y) dP(x, y)$ for some loss function ℓ (e.g. squared error)
  - Note: *L* is an unknown function: we don't know *P*
- Consider algorithms of the form

$$\hat{f}_{A,S_n} = \arg\min_{f \in \mathcal{F}_{\mathcal{A}}} \hat{L}_{S_n}(f)$$

  - $\hat{L}_{S_n}$ is used here as shorthand for sample mean observations in sample S_n of size *n*
  - OLS is an *empirical loss minimizer*: it minimizes the sample average over observed data of the loss function

- So empirical loss minimization algorithms are defined by the function class they choose from
- For estimation what we typically do…
  - Show that empirical loss minimizers generate unbiasedness

Empirical Loss minimization

- Leads to unbiasedness/consistency
  - Fit the data you have…
  - In a frequentist world "on average" (across all S_n) this will produce the right thing
  - This is usually how we prove consistency/unbiasedness

- Other variants:
  - MLE

Some Notation

- Define

$$f^* = \arg\min_{f \in \mathcal{F}} L(f)$$

The best we can do

$$f_A^* = \arg\min_{f \in \mathcal{F}_A} L(f)$$

The best in the subset of functions that the algorithm looks at

  - Recall: *L* is infeasible b/c we don't know true data-generating process

- Contrast the latter with:

$$\hat{f}_{A,S_n} = \arg\min\nolimits_{f \in \mathcal{F}_{\mathcal{A}}} \hat{L}_{S_n}(f)$$

What the in-sample loss minimizer actually produces given a sample

Performance of Algorithm

- Performance of a predictor

$$L(\hat{f}_{A,S_n})$$

- Performance of an Algorithm

$$\mathcal{L}_n(A) := E_{S_n} L(\hat{f}_{A,S_n})$$

  - Algorithm’s expected loss
  - (Suppress S_n in some of the notation for estimator)

The performance of *A*

$$\mathcal{L}_n(A) = \underbrace{L(f^*)}_{\text{irreducible error}} + \overbrace{L(f_A^*) - L(f^*)}^{\text{approximation error}} + \boxed{\underbrace{\mathrm{E}_{S_n}(L(\hat{f}_A) - L(f_A^*)),}_{\text{estimation error}}}$$

Understanding estimation error:

$$\mathrm{E}_{S_n}\left(L(\hat{f}_A) - L(f_A^*)\right) = \mathrm{E}_{S_n}\left(\hat{L}(\hat{f}_A) - \hat{L}(f_A^*)\right) + \boxed{\mathrm{E}_{S_n}(L(\hat{f}_A) - \hat{L}(\hat{f}_A))}$$

“Wrong” function
looks good in-sample

Algorithm does
not see this

Basic Tradeoff

- These two terms go hand in hand:

$$\mathcal{L}_n(A) = \underbrace{L(f^*)}_{\text{irreducible error}} + \boxed{\overbrace{L(f^*_A) - L(f^*)}^{\text{approximation error}}} + \underbrace{\mathrm{E}_{S_n}(L(\hat{f}_A) - L(f^*_A))}_{\text{estimation error}},$$

$$\mathrm{E}_{S_n}(\overbrace{L(\hat{f}_A) - L(f^*_A)}^{\text{out-of-sample, }\geq 0}) = \mathrm{E}_{S_n}(\overbrace{\hat{L}(\hat{f}_A) - \hat{L}(f^*_A)}^{\text{in-sample, }\leq 0}) + \boxed{\underbrace{\mathrm{E}_{S_n}(L(\hat{f}_A) - \hat{L}(\hat{f}_A))}_{\text{unseen overfit}}}$$

Approximation – Overfit Tradeoff

- If we reduce set of *f* to reduce possible over-fit:

$$\underbrace{\mathrm{E}_{S_n}(L(\hat{f}_A) - \hat{L}(\hat{f}_A))}_{\text{unseen overfit}}$$

- Then we fit fewer "true functions" and drive up

$$\overbrace{L(f_A^*) - L(f^*)}^{\text{approximation error}}$$

- Only way to avoid this is if we knew information about *f** so we could shrink the set

Unobserved overfit

- So the problem of prediction really is managing unobserved overfit

$$\underbrace{L(\hat{f}_A)}_{\text{unobserved out-of-sample}} = \overbrace{\hat{L}(\hat{f}_A)}^{\text{observed in-sample}} + \underbrace{(L(\hat{f}_A) - \hat{L}(\hat{f}_A))}_{\text{unobserved overfit}}$$

- We do well in-sample. But some of that “fit” is overfit.

Return to the original example

OLS

Greater Chance To Overfit

OLS_0

Less Chance To Overfit

- We drove down overfit by doing a constrained optimization

Basic Tradeoff at the Heart of Machine Learning

- Bigger function classes…
  - The more likely we are to get to the truth (less approximation)
  - The more likely we are to overfit
- So we want to not just minimize in-sample error given a class of functions
- We also want to decide on the class of functions
  - More expressive means less approximation error
  - More expressive means more overfit

Let’s do the same thing here

Unconstrained

$$\hat{f}_{A,S_n} = \arg\min_{f \in \mathcal{F}_{\mathcal{A}}} \hat{L}_{S_n}(f)$$

But we are worried about $\underbrace{\mathrm{E}_{S_n}(L(\hat{f}_A) - \hat{L}(\hat{f}_A))}_{\text{unseen overfit}}$

So why not do this instead?

$\arg\min_{f \in \mathcal{F}_{\mathcal{A}}} \hat{L}_{S_n}(f)$

s.t. $R(f) \leq c$ Complexity measure: tendency to overfit

Return to the original example

OLS	OLS_0
Greater Overfit	Less Overfit
Better approximation	Worse approximation
More **Expressive** *R(f) higher*	Less **Expressive** *R(f) lower*

- Reduce overfit by approximating worse
- Choose less expressive function class

Constrained minimization

- We could do a constrained minimization

- But notice that this is equivalent to:

$$\hat{f}_{A_\lambda, S_n} = \arg\min_{f \in \mathcal{F}_{\mathcal{A}}} \hat{L}_{S_n}(f) + \underbrace{\lambda R(f)}_{\text{want: } \approx L(f) - \hat{L}(f)}$$

- Complexity measure should capture tendency to overfit

Basic insight

- Data has signal and noise

- More expressive function classes-
  - Allow us to pick up more of the signal
  - But also pick up more of the noise

- So the problem of prediction becomes the problem of *choosing expressiveness*

Overall Structure

- Create a regularizer that:
  - Measures expressiveness
- Penalize algorithm for choosing more expressive functions
  - Tuning parameter lambda
- Let it weigh this penalty against in-sample fit

Linear Example

- Linear function class $x \mapsto \beta' x \; (\beta \in \mathbb{R}^{k+1})$

- Regularized linear regression

$$\hat{\beta}_\lambda^R = \arg\min_{\beta \in \mathbb{R}^{k+1}} \widehat{\mathbb{E}}_{S_n} (\beta' x - y)^2 - \lambda R(\beta)$$

Regularizers for Linear Functions

- Linear functions more expressive if use more variables

$$R(\beta) = \sum_{j=1}^{k} 1_{\beta_j \neq 0}$$

- Can transform coefficients

$$R(\beta) = \sum_{j=1}^{k} |\beta_j|^p$$

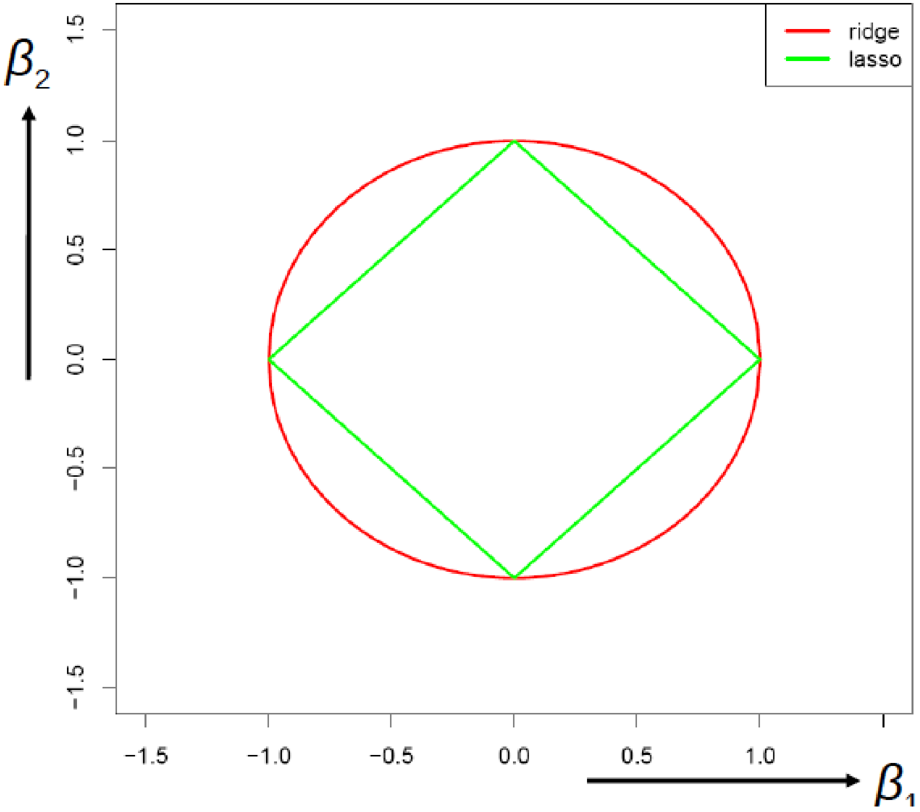
Computationally More Tractable

- Lasso

$$\mathcal{F}_{1,c} = \{f_\gamma; \sum_{j=1}^{k} |\gamma_j| \le c\}$$

- Ridge

$$\mathcal{F}_{2,c} = \{f_\gamma; \sum_{j=1}^{k} \gamma_j^2 \le c\}$$



What makes a good regularizer?

- You might think…
  - Bayesian assumptions
  - Example: Ridge
- A good regularizer can build in beliefs
- Those are great and useful when available
- But central force is tendency to overfit
- Example:
  - Even if true world were not sparse or priors were not normal you'd still do this

Summary

- Regularization is one half of the secret sauce

- Gives a single-dimensional way of deciding of capturing expressiveness

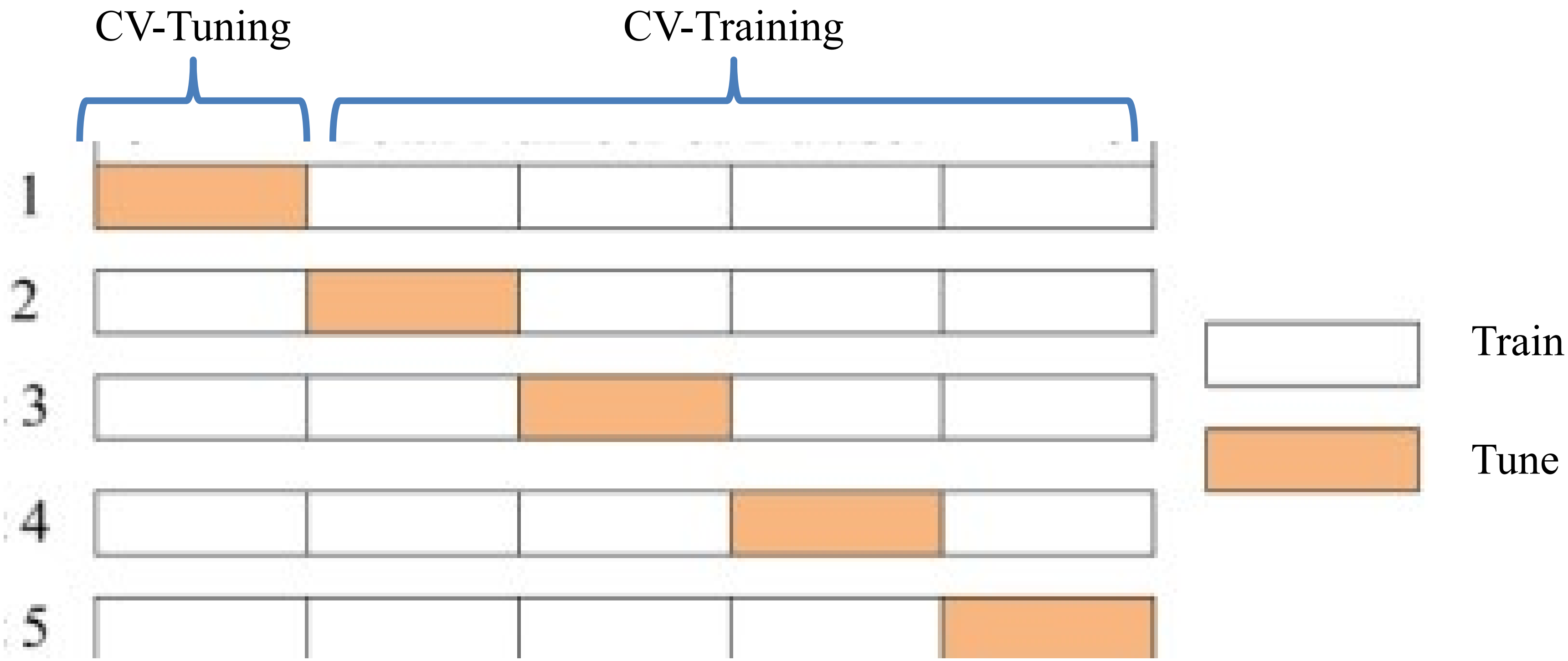
$$\hat{f}_{A_\lambda,S_n} = \arg\min_{f\in\mathcal{F}_\mathcal{A}} \hat{L}_{S_n}(f) + \boxed{\lambda} R(f)$$

- Still missing ingredient is lambda

Choosing lambda

- How much should we penalize expressiveness?
- How do you make the over-fit approximation tradeoff?
- The **tuning** problem.
- Use cross-validation

How Does Cross Validation Work?



Tuning Set = 1/5 of Training Set

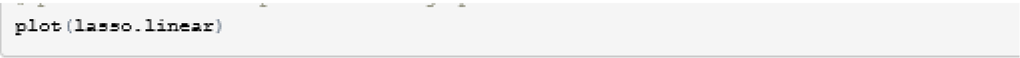
Cross-Validation Mechanics

- Loop over cross-validation samples
  - Train a deep tree on CV-training subset

- Loop over penalty parameters λ
  - Loop over cross-validation samples
    - Prune the tree according to penalty
    - Calculate new MSE of tree
  - Average (over c-v samples) the MSE for this penalty

- Choose the penalty λ^* that gives the best average MSE

LASSO c-v Example



```
plot(lasso.linear)
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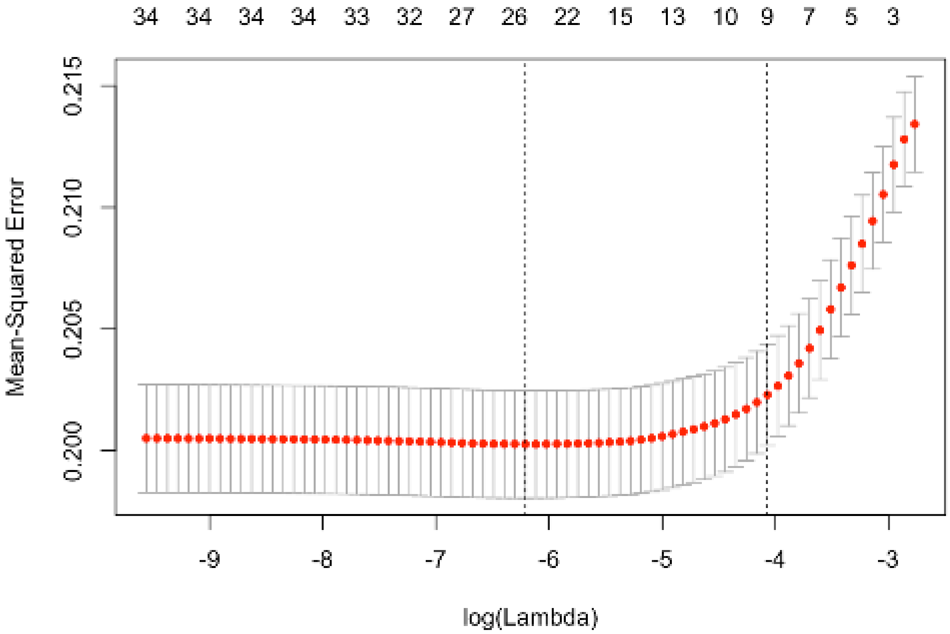


Creating Out-of-Sample **In Sample**

- Major point:
  - Not many assumptions
  - Don’t need to know true model.
  - Don’t need to know much about algorithm
- Minor but important point
  - To get asymptotics right we need to make some regularity assumptions
- Side point (to which we return)
  - We’d like to choose best algorithm for sample size *n*
  - But this will not do that. **Why?**

Why does this work?

1. Not just because we can split a sample and call it out of sample

   – It’s because the thing we are optimizing is observable (easily estimable)

This is more than a trick

- It illustrates what separates prediction from estimation:
  - I can’t ‘observe’ my prior.
    - Whether the world is truly drawn from a linear model
  - But prediction quality is observable
- Put simply:
  - Validity of predictions are measurable
  - Validity of coefficient estimators require structural knowledge

This is the essential ingredient to prediction: Prediction quality is an empirical quantity not a theoretical guarantee

Why does this work?

1. It’s because the thing we are optimizing is observable

2. By focusing on prediction quality we have reduced dimensionality

To understand this…

- Suppose you tried to use this to choose coefficients
  - Ask which set of coefficients worked well out-of sample.
- Does this work?
- Problem 1: Estimation quality is unobservable
  - Need the same assumptions as algorithm to know whether you “work” out of sample
    - If you just go by fit you are ceding to say you want best predicting model
- Problem 2: No dimensionality reduction.
  - You’ve got as many coefficients as before to search over

$$\hat{\beta}_{\lambda}^{R} = \arg\min_{\beta \in \mathbb{R}^{k+1}} \mathbb{E}_{S_n} (\beta' x - y)^2 + \lambda R(\beta)$$

Method	$R(\beta)$
OLS	0
Subset selection	$\|\beta\|_0 = \sum_{j=1}^{k} \mathbb{1}_{\beta_j \neq 0}$
Lasso	$\|\beta\|_1 = \sum_{j=1}^{k} \lvert\beta_j\rvert$
Ridge	$\|\beta\|_2^2 = \sum_{j=1}^{k} \beta_j^2$
Elastic Net	$\alpha \|\beta\|_1 + (1-\alpha)\|\beta\|_2^2$

Bayesian Interpretation of Ridge

Consider the regression

$$Y_i = \sum_{k=1}^{K} \beta_k \cdot X_{ik} + \varepsilon_i$$

with

$$\varepsilon_i | X_{i1}, \ldots, X_{iK} \sim \mathcal{N}(0, \sigma^2)$$

Suppose we put a prior on the β_k:

$$\beta_k \sim \mathcal{N}(0, \tau^2)$$

and all the β_k independent. Assume σ^2 is known.

Bayesian Interpretation of Ridge

Then the posterior distribution is proportional to

$$p(\beta|\text{data}) \propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left(Y_i - \sum_{k=1}^{K}\beta_k \cdot X_{ik}\right)^2\right)\prod_{k=1}^{K}\exp\left(-\frac{\beta_k^2}{2\tau^2}\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left(Y_i - \sum_{k=1}^{K}\beta_k \cdot X_{ik}\right)^2 - \sum_{k=1}^{K}\frac{\beta_k^2}{2\tau^2}\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\left(Y_i - \beta' X_i\right)^2 - \frac{\beta'\beta}{2\tau^2}\right)$$

Bayesian Interpretation of Ridge

So, the posterior is normal, and the posterior mean minimizes

$$\sum_{i=1}^{N}\left(Y_i - \beta' X_i\right)^2 + \beta'\beta \cdot \frac{\sigma^2}{\tau^2}$$

$$= \sum_{i=1}^{N}\left(Y_i - \beta' X_i\right)^2 + \frac{\sigma^2}{\tau^2} \cdot \|\beta\|^2$$

This leads to the posterior mean

$$\left(\mathbf{X}'\mathbf{X} + I_K \cdot \sigma^2/\tau^2\right)^{-1}\mathbf{X}'\mathbf{Y}.$$

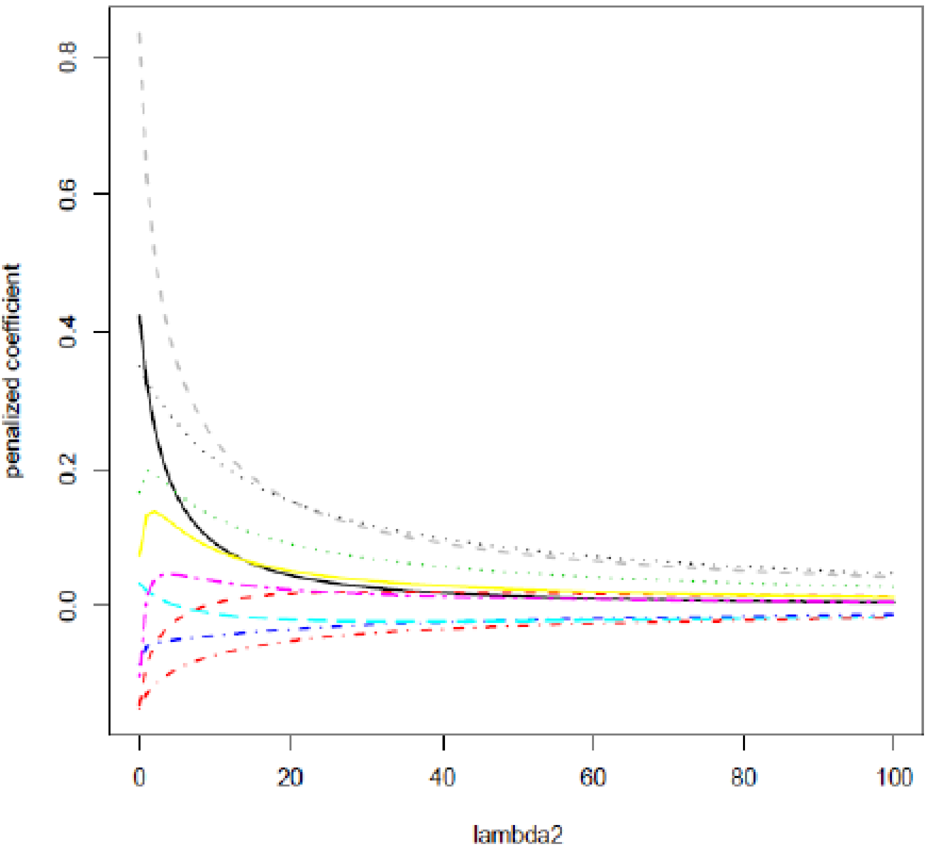
If the $\mathbf{X}'\mathbf{X}$ matrix is diagonal, all elements of β would be shrunk towards zero by the same fraction. With a non-diagonal matrix the degree of shrinkage varies.

POST-Lasso

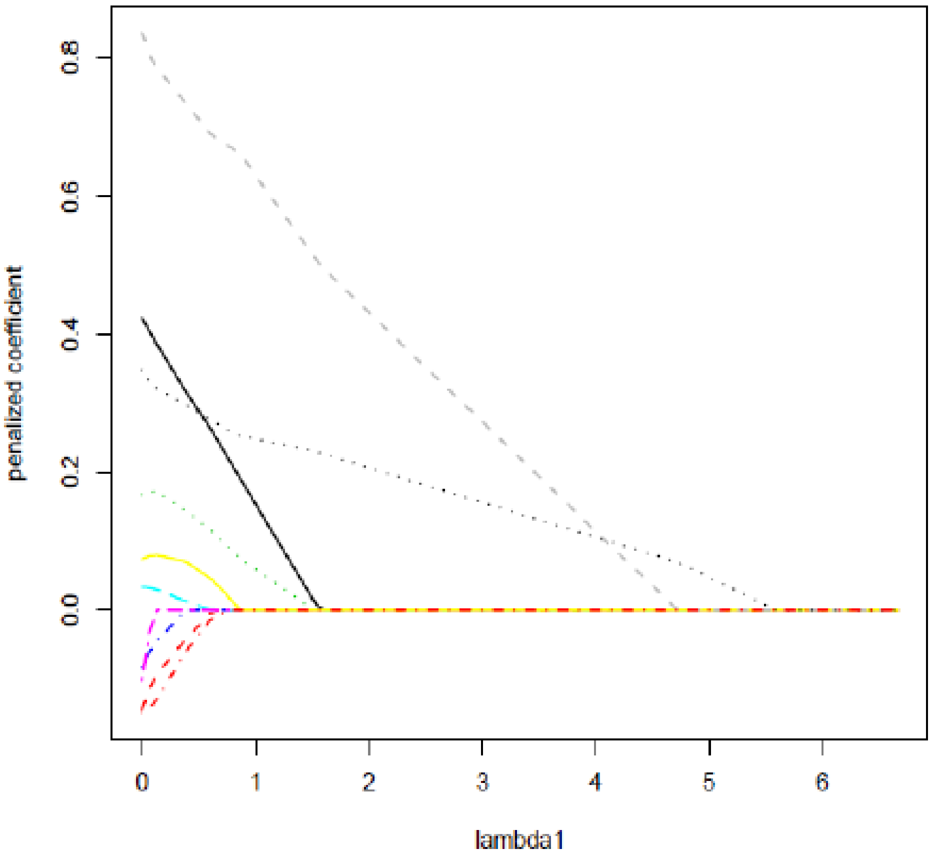
- Important distinction:
  - Use LASSO to choose variables
  - Use OLS on these variables

- How should we think about these?

Ridge regularization path



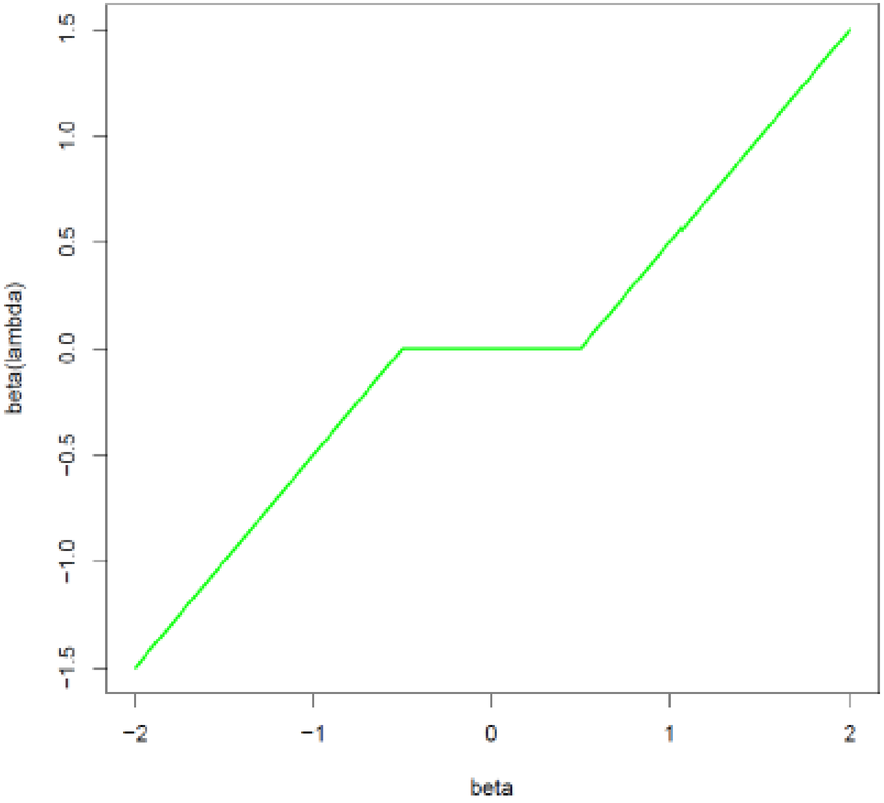
Lasso regularization path



In the orthonormal case, i.e. $\mathbf{X}^T\mathbf{X} = \mathbf{I} = (\mathbf{X}^T\mathbf{X})^{-1}$:

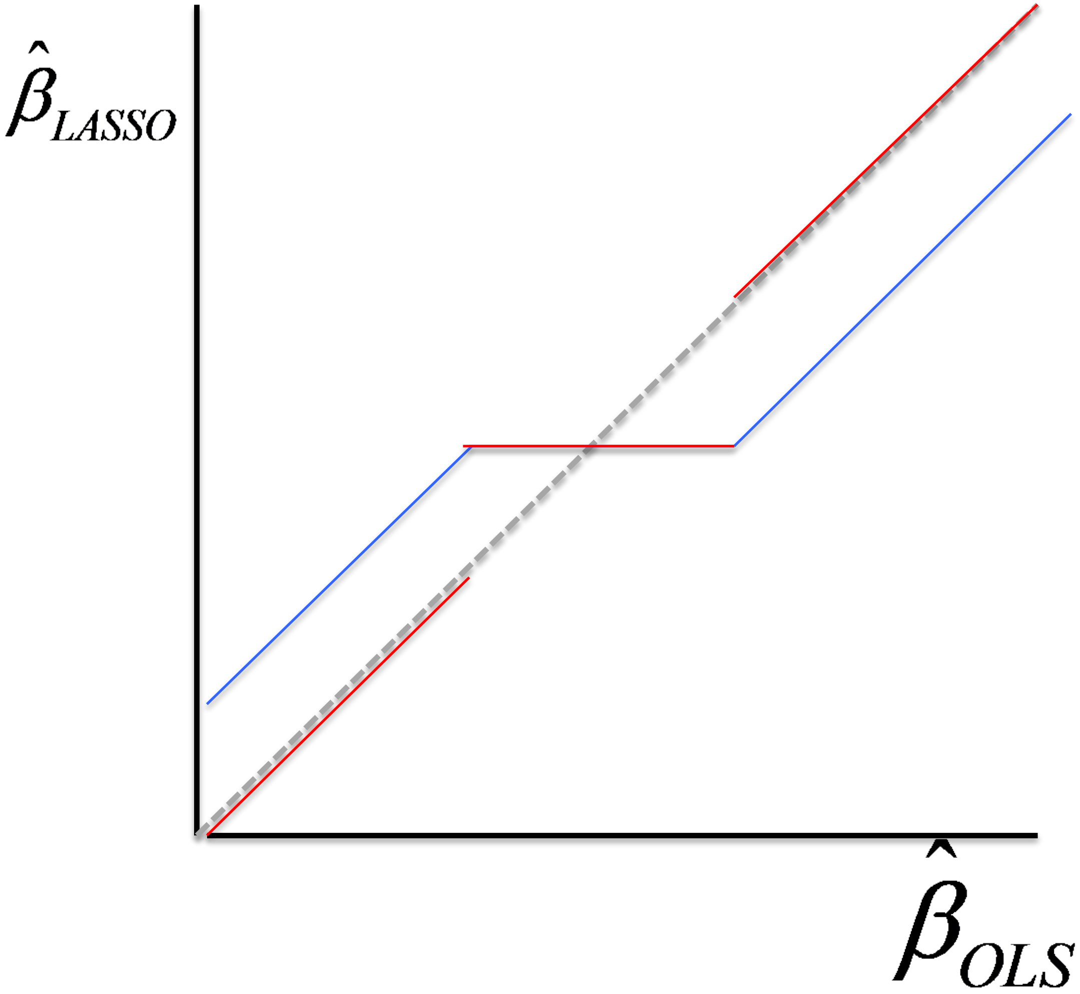
$$\hat{\beta}_j(\lambda_1) \quad = \quad \mathrm{sgn}(\hat{\beta}_j)\,(|\hat{\beta}_j| - \lambda_1/2)_+$$

That is, the lasso estimate is related to the OLS estimate via the so-called *soft threshold function* (depicted here for λ=1).

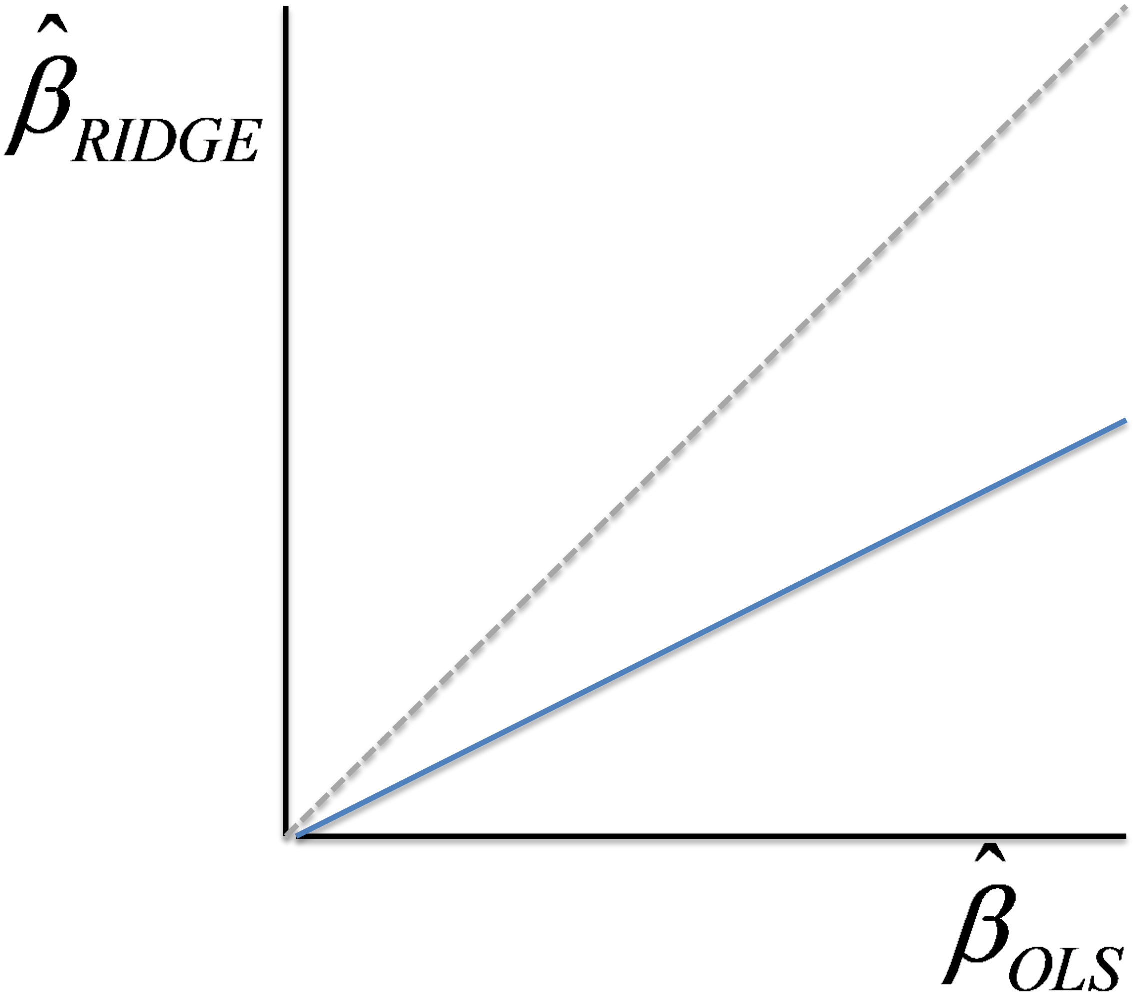


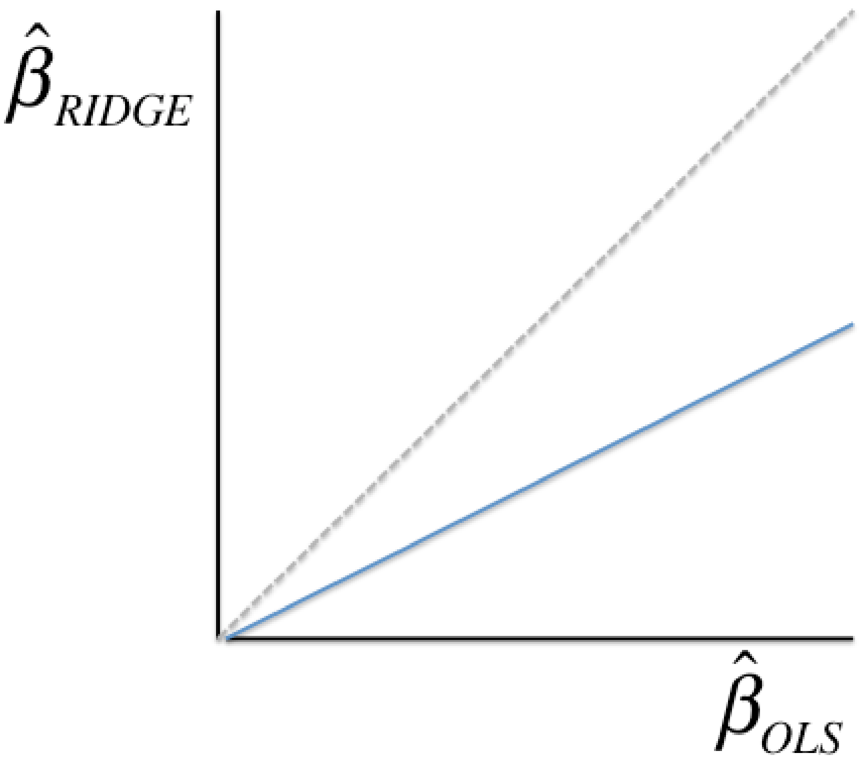
Why not Hard Thresholding?

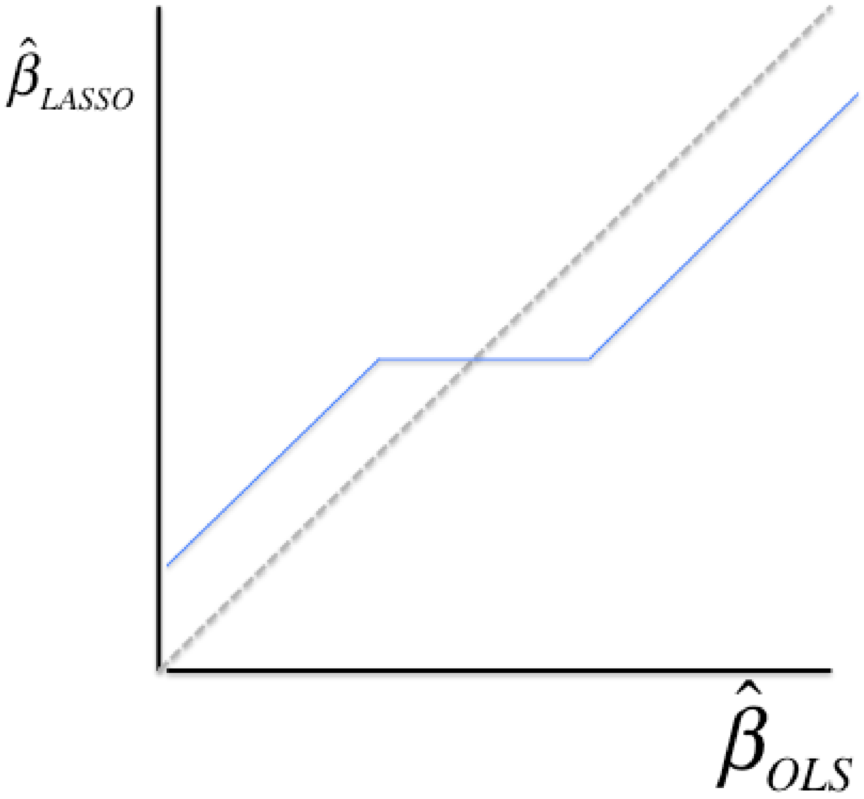
Soft Thresholding



Orthonormal: $\hat{\beta}_{RIDGE} = \dfrac{\hat{\beta}_{OLS}}{1+\lambda}$





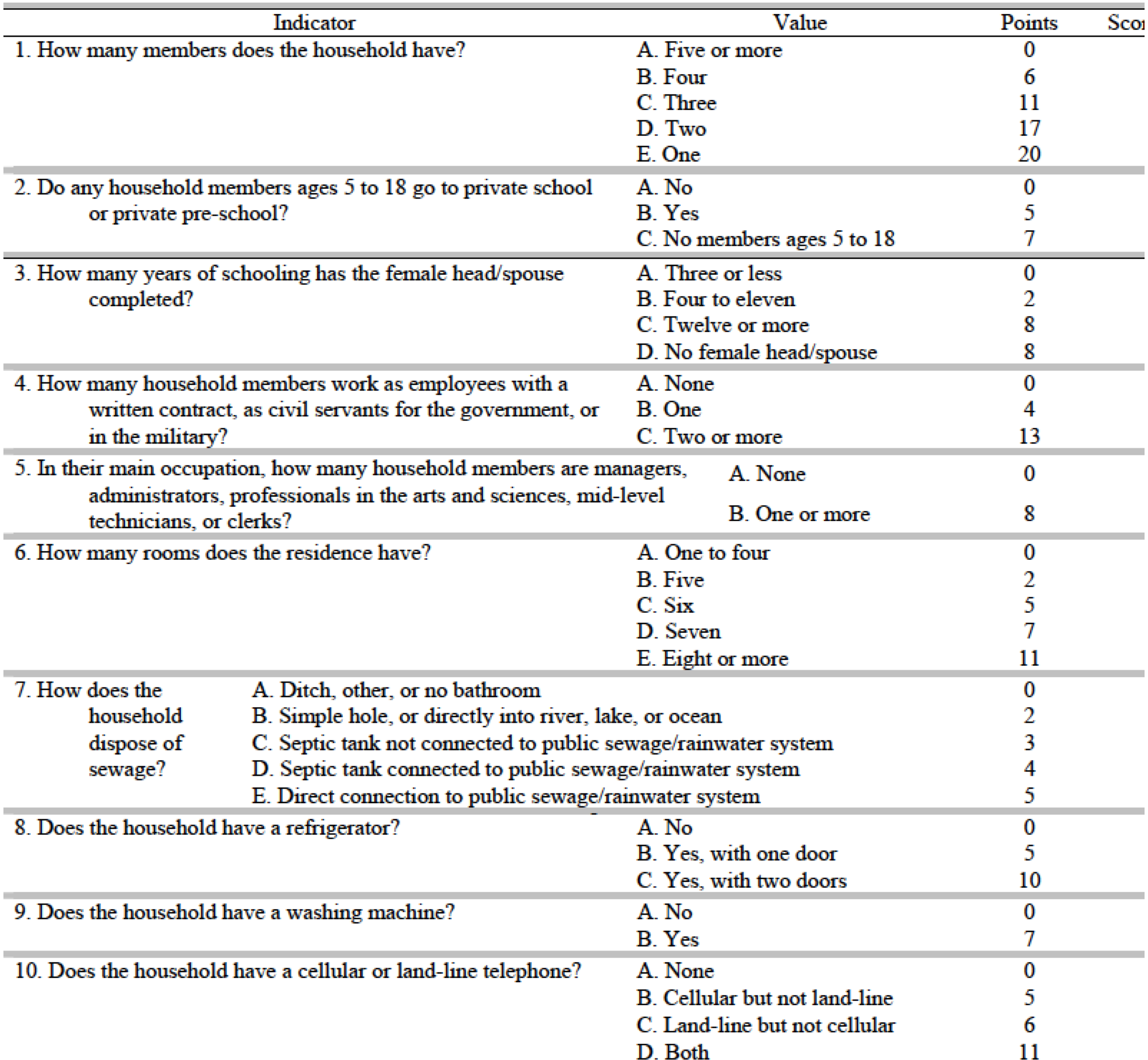


Can be very misleading

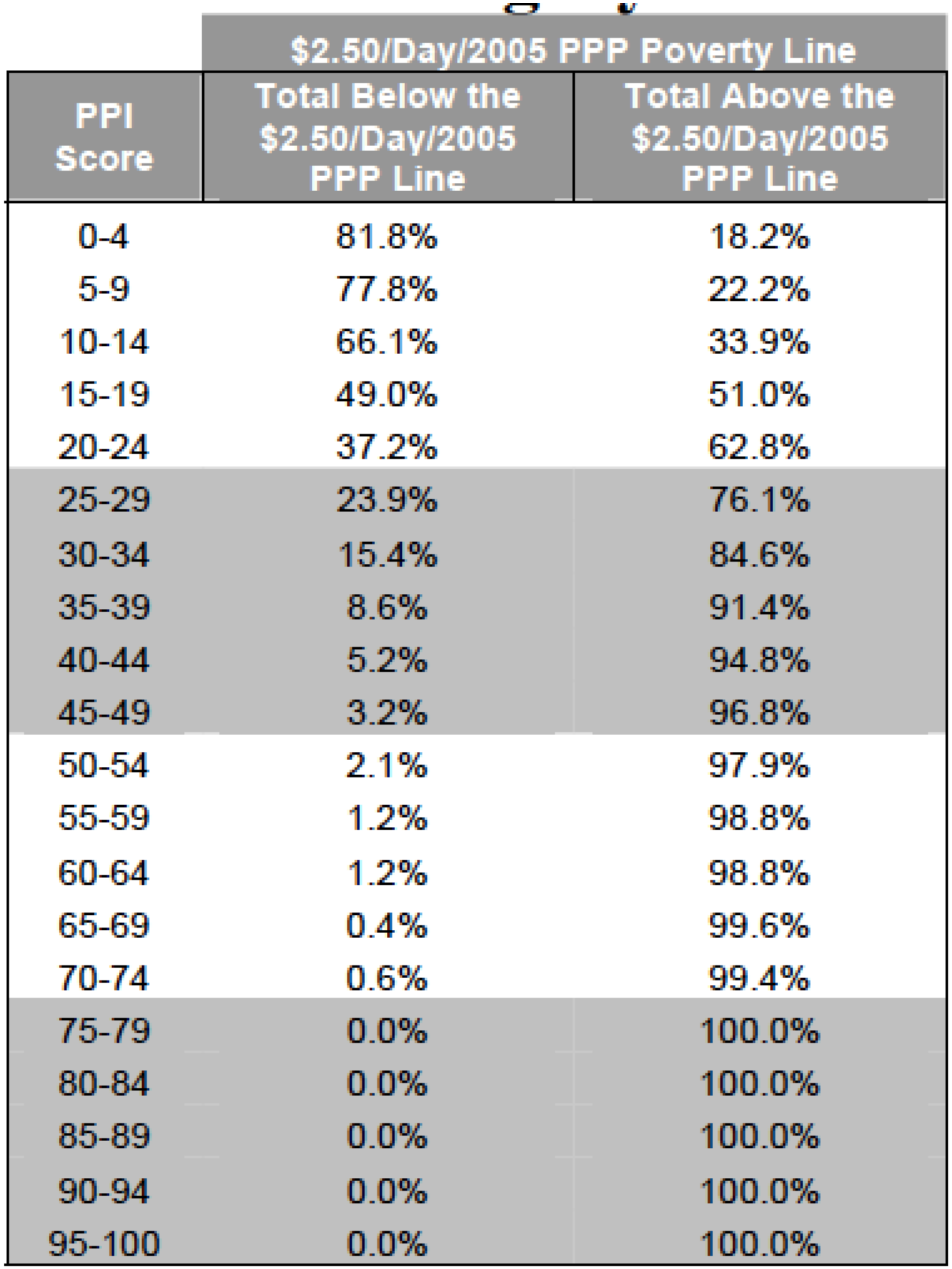
A Basic Policy Problem

- Every transfer program in the world must determine…
  - Who is eligible for the transfer
- Typical goal of redistributive programs
  - Transfer to neediest
- But identifying the neediest is easier said than done

Typical Poverty Scorecard



Indicator	Value	Points	Sco
1. How many members does the household have?	A. Five or more	0	
	B. Four	6	
	C. Three	11	
	D. Two	17	
	E. One	20	
2. Do any household members ages 5 to 18 go to private school or private pre-school?	A. No	0	
	B. Yes	5	
	C. No members ages 5 to 18	7	
3. How many years of schooling has the female head/spouse completed?	A. Three or less	0	
	B. Four to eleven	2	
	C. Twelve or more	8	
	D. No female head/spouse	8	
4. How many household members work as employees with a written contract, as civil servants for the government, or in the military?	A. None	0	
	B. One	4	
	C. Two or more	13	
5. In their main occupation, how many household members are managers, administrators, professionals in the arts and sciences, mid-level technicians, or clerks?	A. None	0	
	B. One or more	8	
6. How many rooms does the residence have?	A. One to four	0	
	B. Five	2	
	C. Six	5	
	D. Seven	7	
	E. Eight or more	11	
7. How does the household dispose of sewage?	A. Ditch, other, or no bathroom	0	
	B. Simple hole, or directly into river, lake, or ocean	2	
	C. Septic tank not connected to public sewage/rainwater system	3	
	D. Septic tank connected to public sewage/rainwater system	4	
	E. Direct connection to public sewage/rainwater system	5	
8. Does the household have a refrigerator?	A. No	0	
	B. Yes, with one door	5	
	C. Yes, with two doors	10	
9. Does the household have a washing machine?	A. No	0	
	B. Yes	7	
10. Does the household have a cellular or land-line telephone?	A. None	0	
	B. Cellular but not land-line	5	
	C. Land-line but not cellular	6	
	D. Both	11	



	$2.50/Day/2005 PPP Poverty Line	
PPI Score	Total Below the $2.50/Day/2005 PPP Line	Total Above the $2.50/Day/2005 PPP Line
0-4	81.8%	18.2%
5-9	77.8%	22.2%
10-14	66.1%	33.9%
15-19	49.0%	51.0%
20-24	37.2%	62.8%
25-29	23.9%	76.1%
30-34	15.4%	84.6%
35-39	8.6%	91.4%
40-44	5.2%	94.8%
45-49	3.2%	96.8%
50-54	2.1%	97.9%
55-59	1.2%	98.8%
60-64	1.2%	98.8%
65-69	0.4%	99.6%
70-74	0.6%	99.4%
75-79	0.0%	100.0%
80-84	0.0%	100.0%
85-89	0.0%	100.0%
90-94	0.0%	100.0%
95-100	0.0%	100.0%

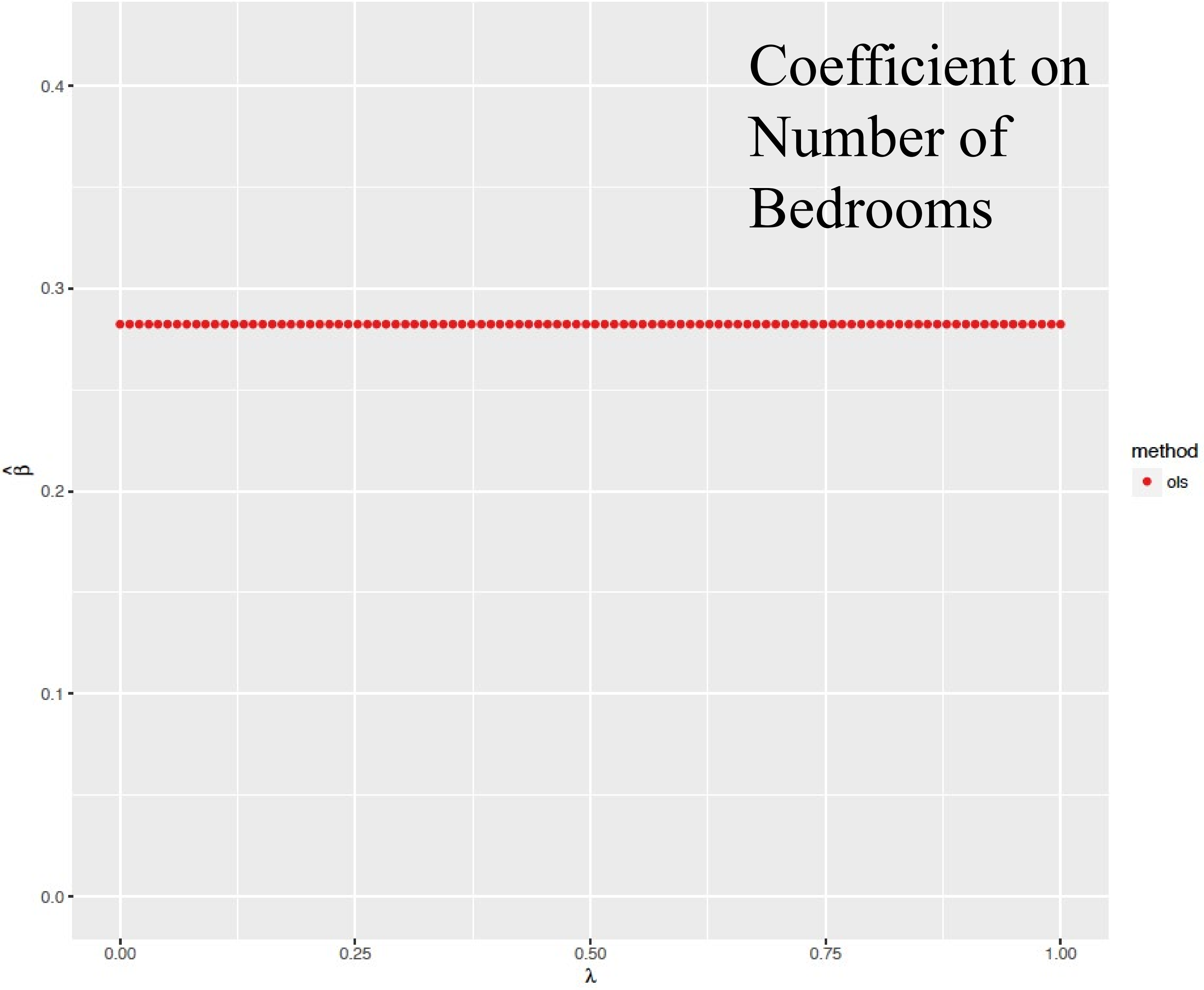
Can we do better?

- This component of targeting is a pure prediction problem
- We fundamentally care about getting best predictive accuracy
- Let’s use this example to illustrate the mechanics of prediction

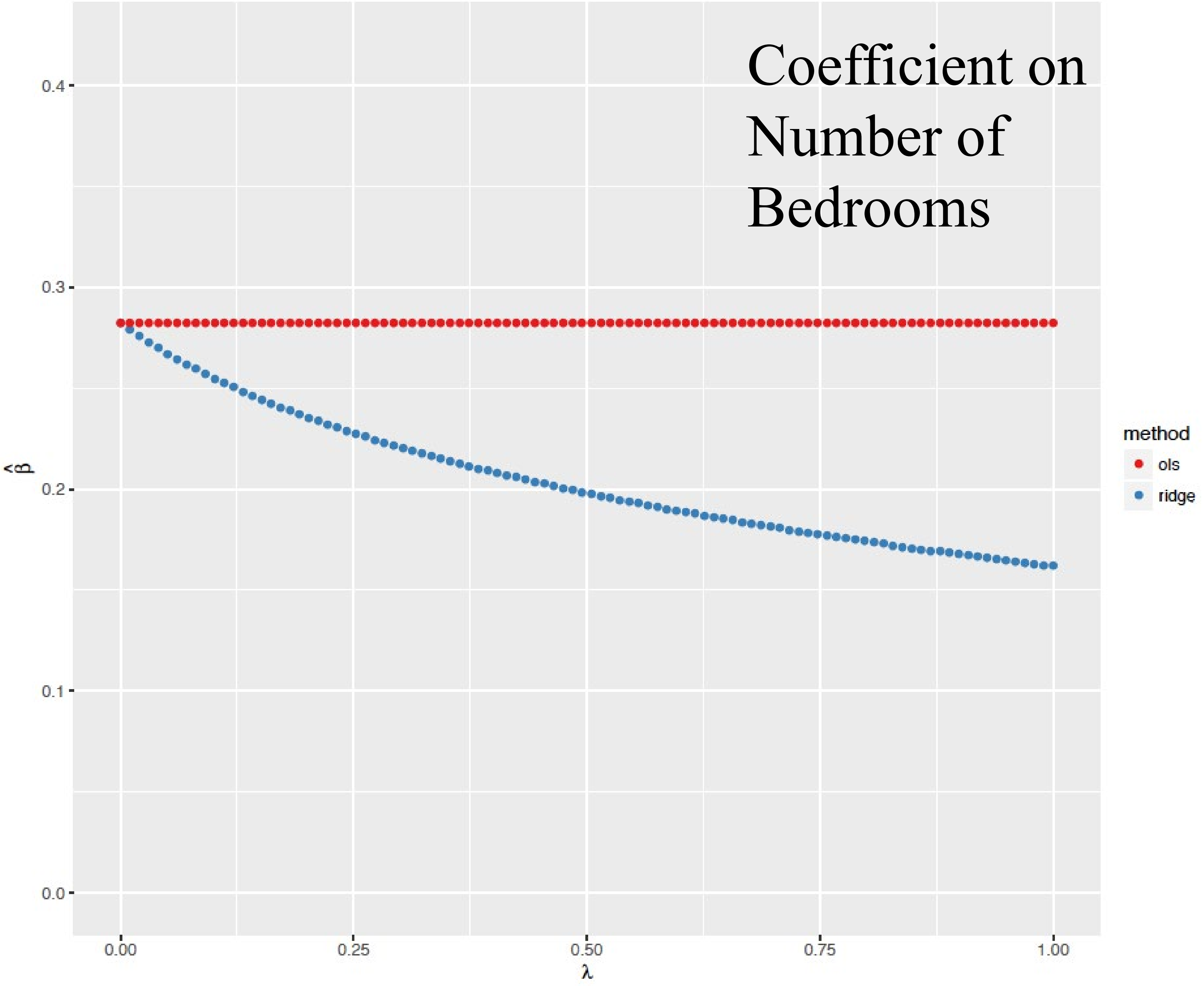
Brazilian Data

- The data:
  - 44,787 data points
  - 53 variables
  - Not very wide?
- Median
  - Annual consumption (in dollars): 3918
  - 348.85 monthly income
- 6 percent below 1.90 poverty line
- 14 percent below the 3.10 poverty line

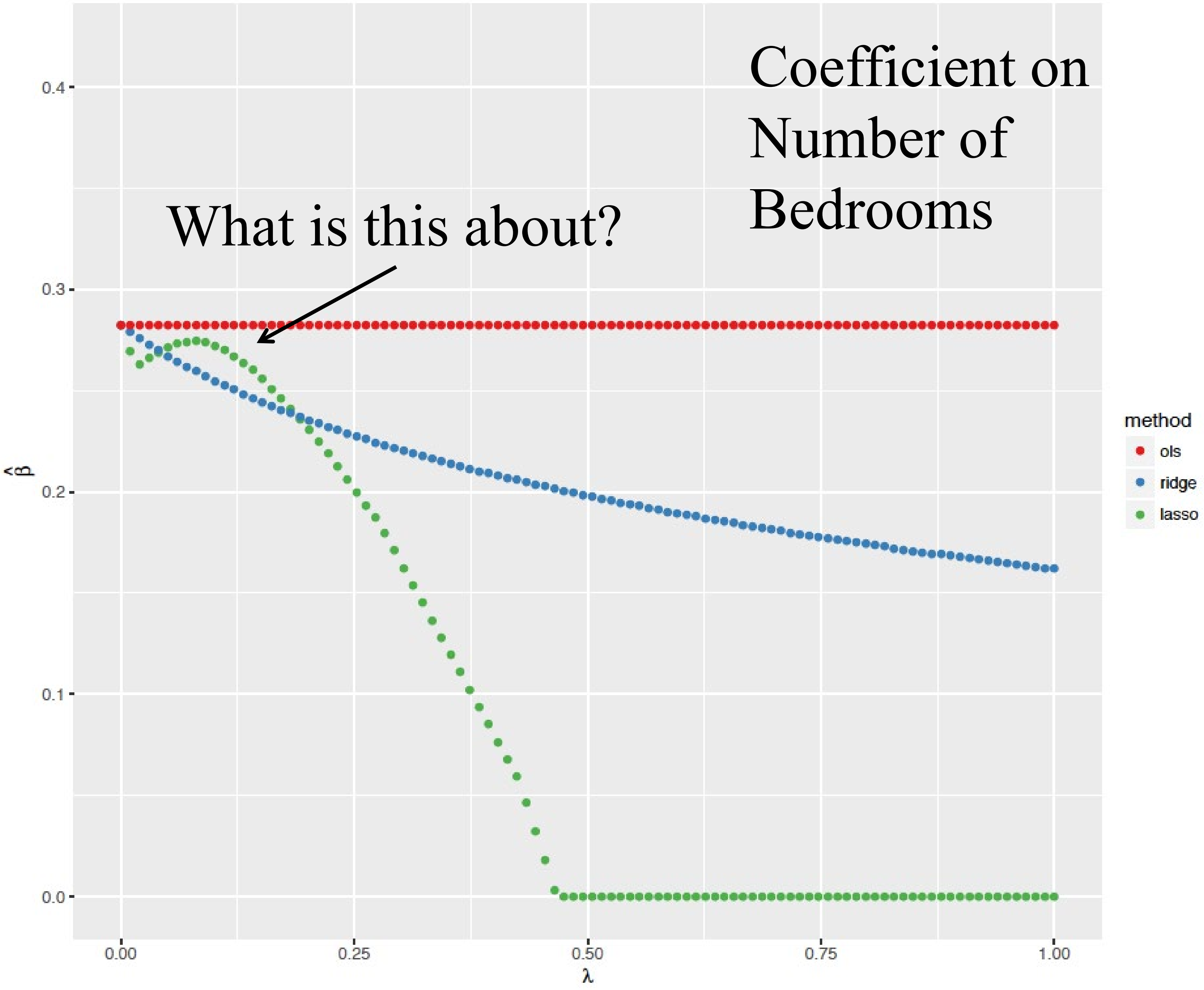
Coefficient on Number of Bedrooms



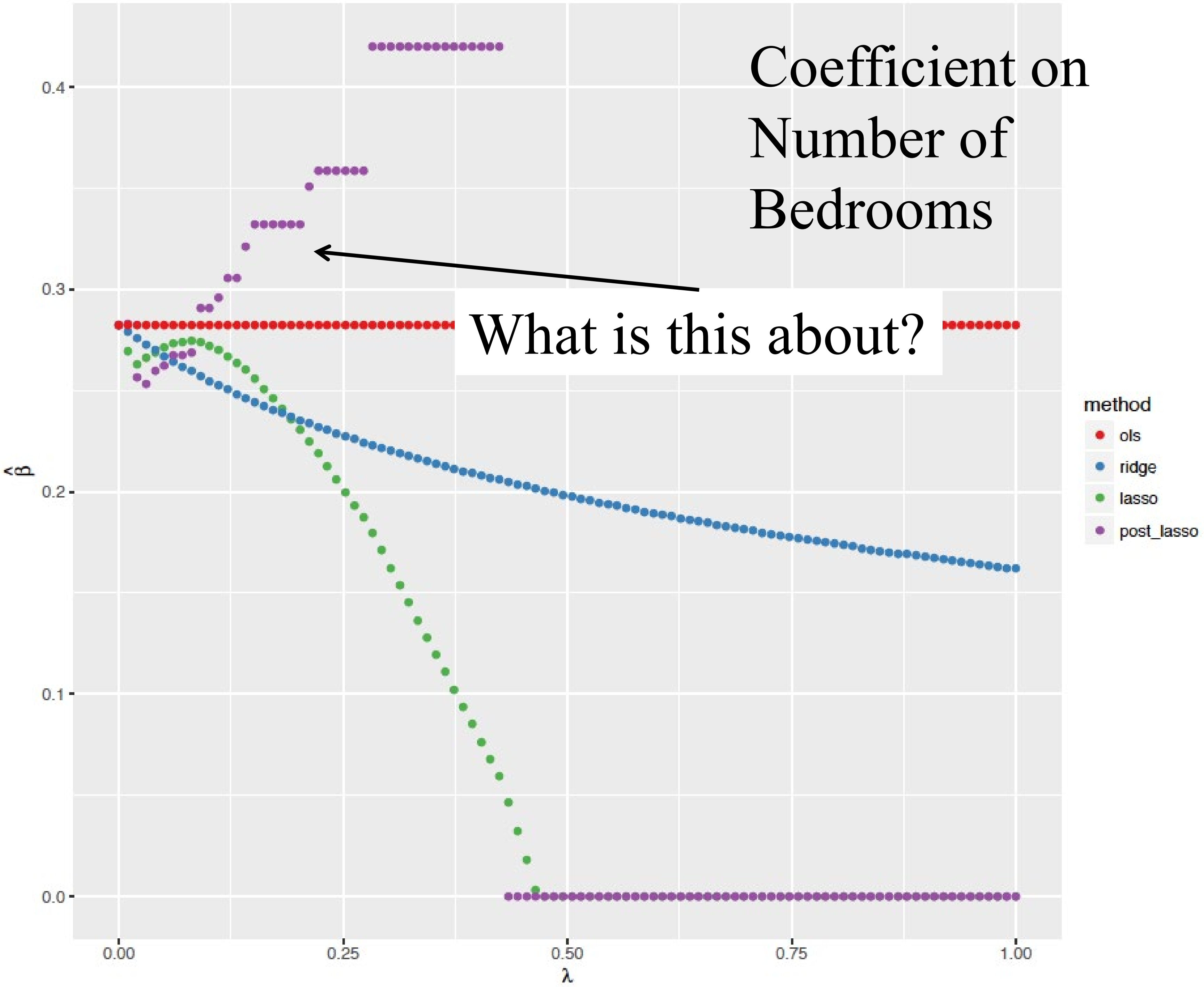
Coefficient on Number of Bedrooms



Coefficient on Number of Bedrooms

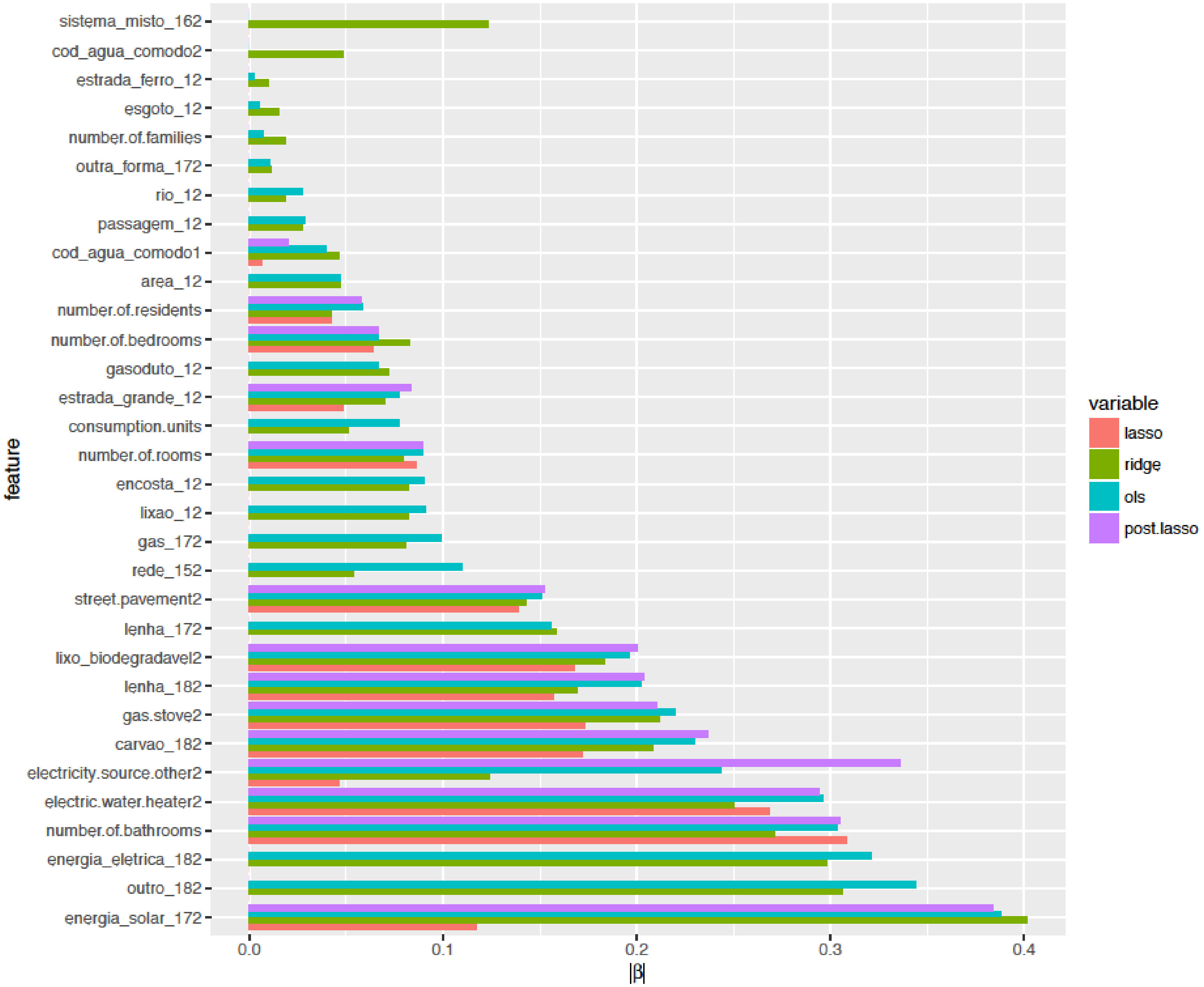


What is this about?



Coefficient on Number of Bedrooms

What is this about?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