Computational Bootcamp

Evaluating Expectations: Gauss Hermite Quadrature

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Evaluating expectations

- in economics, we very very often work with uncertainty and hence with expectations:

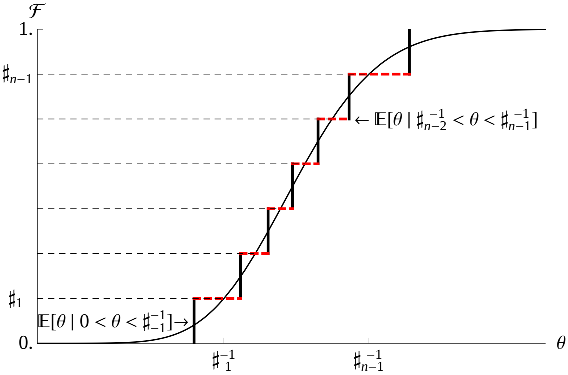
$$E\left[H(x)\right] = \int_{-\infty}^{\infty} H(x)f(x)dx$$

- but how can we do this numerically?

Possibilities to evaluate expectations

- by simulation:
  - draw shocks from distribution, compute outcome for each, take mean
  - VERY SLOW, IMPRECISE $\Rightarrow$ not advisable
- equiprobable grid points (Carroll, 2012):
  - discretize the support of the random variable such that each bin is equally likely; take midpoints of bins as node
  - compute outcome at each node, take average
- Gauss-Hermite Quadrature (see e.g. Judd (1998), Ch. 7):
  - choose nodes and weights efficiently

Equiprobable Grid Points



source: Carroll (2012)

$$\int_a^b H(x)f(x)dx \approx \frac{1}{N}\sum_{i=1}^{N} H(x_i)$$

advantage: same probability for each node $\Rightarrow$ easy to handle

Gauss-Hermite Quadrature

- Quadrature: choose nodes and weights efficiently

$$\int_a^b H(x)w(x)dx \approx \sum_{i=1}^{N} \omega_i H(\zeta_i)$$

$\Rightarrow$ only n nodes needed to perfectly approximate polynomial of degree $2n+1$

- Gauss-Hermite Quadrature:
  - $w(x) = e^{-x^2}, a = -\infty, b = \infty$:

$$\int_{-\infty}^{\infty} H(x)e^{-x^2}dx \approx \sum_{i=1}^{N} \omega_i H(\zeta_i)$$

  - useful for normally distributed random variables!

Gauss-Hermite Quadrature

How do we approximate an expectation that depends on a normally distributed random variable?

- assume ε is normally distributed:
  - $\varepsilon \sim N(\mu, \sigma^2)$
- for our optimization we need the expected value of a function of this variable, $H(\epsilon)$:

$$
\begin{aligned}
E\big[H(\varepsilon)\big] &= \int_{-\infty}^{\infty} H(\varepsilon)\; f(\epsilon)\; d\epsilon \\
&= \int_{-\infty}^{\infty} H(\varepsilon)\; \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\varepsilon-\mu)^2}{2\sigma^2}}\; d\epsilon
\end{aligned}
$$

$\rightarrow$ we are close to the form we need, but not quite there yet!

Gauss-Hermite Quadrature

- change of variables: $z := \frac{\varepsilon - \mu}{\sqrt{2}\sigma}$:

$$
\begin{aligned}
E\big[H(\varepsilon)\big] &= \int_{-\infty}^{\infty} H(\varepsilon)\, \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\varepsilon-\mu)^2}{2\sigma^2}}\, d\varepsilon \\
&= \int_{-\infty}^{\infty} H(\sqrt{2}\sigma z + \mu)\, \frac{1}{\sqrt{2\pi}\sigma} e^{-z^2}\, \sqrt{2}\sigma\, dz \\
&= \int_{-\infty}^{\infty} H(\sqrt{2}\sigma z + \mu)\, \frac{1}{\sqrt{\pi}} e^{-z^2}\, dz
\end{aligned}
$$

$\rightarrow$ this is the form we need!

Gauss-Hermite Quadrature

- approximate by

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} H(\sqrt{2}\sigma z + \mu)\ e^{-z^2}\ dz \approx \sum_{i=1}^{N} \frac{1}{\sqrt{\pi}} \omega_i H(\sqrt{2}\sigma\zeta_i + \mu)$$

  - ζ_i and ω_i are Gauss-Hermite nodes & weights
    $\rightarrow$ we get those from tables
  - don't forget to divide weighted sum by $\sqrt{\pi}$!
  - $H(\cdot)$ can be anything, including $\exp(\cdot)$
    - lognormal variable $y \sim \log N(\mu, \sigma^2)$ implies $y = \exp(\varepsilon)$
    - $E[y] \approx \sum_{i=1}^{N} \frac{1}{\sqrt{\pi}} \omega_i \exp(\sqrt{2}\sigma\zeta_i + \mu)$

Literature I

Carroll, C. D. (2012). Solution methods for microeconomic dynamic stochastic optimization problems.

Judd, K. L. (1998). Numerical Methods in Economics. MIT Press.