Exercise: Numerical Integration by Gauss-Hermite Quadrature

Stockholm Doctoral Program in Economics
Computational Bootcamp

Kathrin Schlafmann

When we solve economic models numerically we almost always have to evaluate expectations. Expectations are integrals of the following form:

$$E[H(x)] = \int_a^b H(x) f(x) dx \tag{1}$$

where $f(x)$ is the density of our random variable x. Integrals cannot be directly evaluated numerically but have to be approximated. If $x \sim N(\mu, \sigma)$ and $a = -\infty, b = \infty$, we can use Gauss-Hermite Quadrature to approximate the integral (i.e. the expectation we are interested in) as follows:

$$E[H(x)] \;=\; \int_{-\infty}^{\infty} H(x) f(x) dx \;\approx\; \sum_{i=1}^{N} \frac{1}{\sqrt{\pi}} \omega_i H(\sqrt{2}\sigma\zeta_i + \mu) \tag{2}$$

where ζ_i and ω_i are the Gauss-Hermite nodes and weights, respectively. We have hence transformed the integral into a weighted sum - and sums are easy to numerically evaluate.

In this exercise we are going to go through some simple examples to see how Gauss-Hermite Quadrature works in practice. Two codes are provided for you:

- `GaussHermite.m`: function that returns Gauss-Hermite nodes and weights
- `ExerciseQuadrature.m`: code you need to complete to solve the exercises

Questions

1. Assume we have a random shock $\epsilon \sim N(0, 1)$. Calculate $E[\epsilon]$.
2. Now assume $\epsilon \sim N(\mu, \sigma^2)$ with $\mu = 2$ and $\sigma = 2$. Calculate $E[\epsilon]$.
3. Assume you want to get the expected value of firm profits $E[\Pi] = E[Y - wL - rK]$ where the wage rate $w = 1$, labor input $L = 10$, interest rate $r = 0.1$ and capital $K = 100$ is fixed but output is random with $Y \sim N(100, 5^2)$. Calculate $E[\Pi]$.

4. Assume again that you want to evaluate expected profits, but now labor input is random and output depends on labor: $L \sim N(10, 1)$ and $Y(L) = 5 \cdot K^{\alpha} L^{1-\alpha}$ with $\alpha = 0.3$. Calculate $E[\Pi]$.

5. Let's turn to income expectations. Assume income is lognormally distributed with $Y \sim \log N(-\frac{\sigma^2}{2}, \sigma^2)$ and $\sigma = 0.1$. Calculate expected income $E[Y]$.

6. Let's assume income is a random walk in logs, i.e. $Y_t = Y_{t-1} \cdot \varepsilon$ where $\varepsilon \sim \log N(-\frac{\sigma^2}{2}, \sigma^2)$ and $\sigma = 0.1$. Calculate expected income $E[Y_{t+1}|Y_t]$.

7. Assume again the simpler case where income is lognormally distributed: $Y \sim \log N(3, 0.3^2)$ and $\sigma = 0.1$. Further assume that the tax code is a step function:

$$T(Y) = \begin{cases} 0 & \text{if } Y < 10 \\ 0.2 \cdot Y & \text{if } 10 \leq Y < 20 \\ 0.3 \cdot Y & \text{if } 20 \leq Y \end{cases}$$

Calculate the expected after-tax income $E[Y - T(Y)]$.

8. Rerun all previous questions for different numbers of quadrature nodes. Are your results sensitive to the number of nodes? Why?