Task 1 Deterministic model w. CRRA utility.

Our model :

$$\max_{\{c_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

Subject to :

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = W_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$ (the sum of consumption = sum of endowments).

- the consumer allocates consumption to T + 1 'goods' in time.
- relative price of c_t and c_0 is $1/(1+r)^t$
- total wealth is initial wealth + discounted wealth over the horizon.

1. Find Euler Equation when $\beta(1+r)$ is not assumed $=1$, and when

$$u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$$

Form a Lagrangian (discuss λ and not λ_t, general constraints / Lagrangian understanding).

$$\mathcal{L}(c_t, \lambda) = \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\gamma}-1}{1-\gamma} + \lambda \left(W_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t} - \sum_{t=0}^{T} \frac{c_t}{(1+r)^t} \right)$$

Take First-Order Conditions

$$c_t : \quad \beta^t c_t^{-\gamma} - \frac{\lambda}{(1+r)^t} = 0 \qquad (1)$$

$$\lambda : \quad W_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t} = \sum_{t=0}^{T} \frac{c_t}{(1+r)^t} \qquad (2)$$

From (1) for $t+1 / t$

$$\frac{\beta^{t+1}}{\beta^t} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} = \frac{\lambda}{\lambda} \frac{(1+r)^t}{(1+r)^{t+1}}$$

$$\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} = \frac{1}{(1+r)}$$

$$\left(\frac{c_{t+1}}{c_t} \right)^{\gamma} = \beta(1+r)$$

Task 1

2. Express c_t as a function of c_0 for any t

$$c_{t+1} = [\beta(1+r)]^{\frac{1}{\gamma}} c_t$$

$$\left(\frac{c_1}{c_0}\right)^{\gamma} = \beta(1+r) \qquad \text{then} \quad \left(\frac{c_2}{c_0}\right)^{\gamma} = \left(\frac{c_2}{c_1}\right)^{\gamma} \cdot \left(\frac{c_1}{c_0}\right)^{\gamma} = \beta(1+r) \cdot \beta(1+r)$$

$$\left(\frac{c_2}{c_1}\right)^{\gamma} = \beta(1+r)$$

$$(\dots)$$

$$c_t = [\beta(1+r)]^{\frac{t}{\gamma}} c_0$$

3. Substitute into the consolidated budget constraint

$$\underbrace{W_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}} - \sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = 0$$

call this $\bar{Y}$, our permanent income.

$$\bar{Y} = \sum_{t=0}^{T} \frac{[\beta(1+r)]^{t/\gamma}}{(1+r)^t} \cdot c_0$$

4. Express c_0 as a function of model parameters.

$$\bar{Y} = c_0 \sum_{t=0}^{T} \left[(\beta(1+r)^{1-\gamma})^{\frac{1}{\gamma}}\right]^t$$

This is a sum of an infinite geometric sequence

$$\sum_{t=0}^{T} r^t \quad \text{when} \quad r < 1 \quad \text{is} \quad \frac{1}{1-r}$$

$$\bar{Y} = \frac{c_0}{1 - [(\beta(1+r)^{1-\gamma})^{\frac{1}{\gamma}}]}$$

Task 1

5. In class,

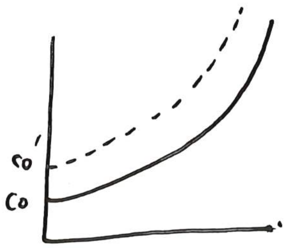
- consumption is constant over time
- consumption in time 0 is affected by income and wealth only through NPV.

Do these results survive our assumptions?

Our Euler says:

$$\frac{c_{t+1}}{c_t} = (\beta(1+r))^{\frac{1}{\gamma}}$$ so if $\beta(1+r)$ is not necessarily 1, then $\frac{c_{t+1}}{c_t}$ may not equal 1.

This means it will not be constant, it may grow or decay from c_0.



- $(\beta(1+r))^{\frac{1}{\gamma}}$ determines our path
- Our Budget constraint determines initial level c_0.
- The BC only looks at NPV of all income, so shifting this around makes no difference.

Economic intuition

- Consumption won't be equal unless relative patience is 1. If this is not the case then consumption will become relatively more expensive / less expensive, and the consumer will shift consumption accordingly.
- In this setting, we have perfect capital markets. The consumer can trade in any period costlessly, so they can borrow everything at $t=1$ and wait for income if it is well into the future, or if they receive all their income initially they can save and consume out of it. The market cares for the rest.

Task 1

6. Compute elasticity of consumption growth (c_{t+1}/c_t) w.r.t gross return $(1+r)$.

n.b. elasticity $\dfrac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \dfrac{\Delta y}{\Delta x} \cdot \dfrac{x}{y}$ as we let the changes get smaller $\dfrac{\Delta y}{\Delta x} \rightarrow \dfrac{dy}{dx}$

$$\frac{dy}{dx} \cdot \frac{x}{y} = \frac{dy}{y} \cdot \frac{x}{dx} = \frac{d\log(y)}{d\log(x)}.$$

So,

$$\log\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \log \beta(1+r)$$

$$\gamma \log\left(\frac{c_{t+1}}{c_t}\right) = \log\beta + \log(1+r)$$

$$\log\left(\frac{c_{t+1}}{c_t}\right) = \frac{\log\beta}{\gamma} + \frac{\log(1+r)}{\gamma}$$

$$\frac{d\log\left(\frac{c_{t+1}}{c_t}\right)}{d\log(1+r)} = \frac{1}{\gamma}$$

If agents are not sensitive to interest rate changes, they want a smooth consumption path (they may sacrifice extra income for this). This happens when γ is higher

$\gamma \rightarrow \infty$, then consumption is constant

$\gamma \rightarrow 0$, then consumption is all taken in the cheapest period.