5303 - Advanced Macroeconomics
Assignment 2

Hugo - 42597@student.hhs.se
Marek - 42624@student.hhs.se
Marleena - 42620@student.hhs.se

Stockholm School of Economics

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Homework #2

(6.4a) Describe temp. equilibria for the economy:

$N(t)=100$, $u_t^h = c_t^h(t)\, c_t^h(t+1)$, $w_t^h = [2,1]$ $\forall h, t \geq 1$.
$A=100$, $d(t+1)=1$, $p^e(t+1)=1$ $\forall t \geq 1$.

Equilibrium cond #1: $p(t) = \dfrac{d(t+1)+p^e(t+1)}{r(t)}$

Eq. Cond #2 : $S_t(r(t)) = p(t)A$

The consumer maximizes:

$$\max_{\{c_t^h(t),\, c_t^h(t+1)\}} c_t^h(t)\, c_t^h(t+1)$$

$$\text{s.t. } c_t^h(t) + \frac{c_t^{h,q}(t+1)}{r(t)} = w_t^h(t) + \frac{w_t^h(t+1)}{r(t)} + a^h(t)\left[p(t) - \frac{p^e(t+1)+d(t+1)}{r(t)}\right]$$

Applying the first equilibrium condition, we can solve the budget constraint for $c_t^h(t+1)$:

$$c_t^{h,q}(t+1) = r(t)\left[-c_t^h(t) + w_t^h(t)\right] + w_t^h(t+1)$$

And plug into the utility ~~max~~ function to yield an unconstrained optimization problem:

$$\max_{c_t^h(t)} c_t^h(t)\left[r(t)\left(-c_t^h(t) + w_t^h(t)\right) + w_t^h(t+1)\right]$$

FOC:

$$\frac{\partial u}{\partial c(t)} \Rightarrow c_t^{h,q}(t+1) - r(t)\, c_t^h(t) = 0 \iff c_t^{h,q}(t+1) = r(t)\, c_t^h(t)$$

Plugging this into the budget constraint yields

$$2c_t^h(t) = w_t^h(t) + \frac{w_t^h(t+1)}{r(t)}$$

$$c_t^h(t) = \frac{w_t^h(t)}{2} + \frac{w_t^h(t+1)}{2r(t)} = 1 + \frac{1}{2r(t)}$$

$$s_t^h(t) = w_t^h(t) - c_t^h(t) = 2 - \left(1 + \frac{1}{2r(t)}\right) = 1 - \frac{1}{2r(t)}$$

By eq. cond #2,

$$S_t(r(t)) = 100\left(1 - \frac{1}{2r(t)}\right) = p(t)A = p(t)\cdot 100 \qquad (1)$$

By eq. cond #1,

$$p(t) = \frac{d(t+1)+p^e(t+1)}{r(t)} = \frac{2}{r(t)} \qquad (2)$$

Substituting expression ② into ① yields

$$100 - \frac{50}{r(t)} = \frac{2}{r(t)} \cdot 100$$

$$\Leftrightarrow$$

$$1 - \frac{1}{2r(t)} = \frac{2}{r(t)}$$

$$r(t) - \frac{1}{2} = 2$$

$$r(t) = \frac{5}{2}$$

Then by ②, $p(t) = \frac{2}{r(t)} = \frac{4}{5}$.

Thus we have:

$$s_t^h(t) = 1 - \frac{1}{2r(t)} = 1 - \frac{1}{5} = \frac{4}{5}$$

$$c_t^h(t) = 1 + \frac{1}{2r(t)} = 1 + \frac{1}{5} = \frac{6}{5}$$

$$c_t^{h,e}(t+1) = r(t)\, c_t^h(t) = \frac{5}{2} \cdot \frac{6}{5} = 3$$

Assuming(?) that consumers do not have perfect foresight, e.g. $p^e(t+1) \overset{?}{\neq} p(t+1)$, we need to see if $c_t^h(t+1) = c_t^{h,e}(t+1)$

Equation 6.7 in the textbook gives the market clearing condition: (applied to t+1)

$$\sum_{h=1}^{N(t)} c_t^h(t+1) + \sum_{h=1}^{N(t+1)} c_{t+1}^h(t+1) = \sum_{h=1}^{N(t)} \omega_t^h(t+1) + \sum_{h=1}^{N(t+1)} \omega_{t+1}^h(t+1) + D(t+1)$$

so since $D(t+1) = d(t+1)A = 1 \cdot 100$

$$100 \cdot c_t^h(t+1) + 100 \cdot \frac{6}{5} = 100 \cdot 1 + 100 \cdot 2 + 100 \cdot 1$$

$$c_t^h(t+1) = 4 - \frac{6}{5} = \frac{14}{5}$$

So $c_t^h(t+1) - c_t^{h,e}(t+1) = \frac{-1}{5}$.

(6.4e) Same as a, but for all h: $w_t^h = \begin{cases} [2,1] & t \text{ odd} \\ [1,1] & t \text{ even} \end{cases}$

From 6.4a, we know the consumption function for all h is

$$c_t^h(t) = \frac{w_t^h(t)}{2} + \frac{w_t^h(t+1)}{2r(t)}$$

Thus

$$c_t^h(t) = \begin{cases} 1 + \frac{1}{2r(t)} & t \text{ odd} \\ \frac{1}{2} + \frac{1}{2r(t)} & t \text{ is even.} \end{cases}, \text{ and}$$

$$s_t^h(t) = \begin{cases} 1 - \frac{1}{2r(t)} & t \text{ is odd} \\ \frac{1}{2} - \frac{1}{2r(t)} & t \text{ is even.} \end{cases}$$

By equilibrium condition #2, $S_t(r(t)) = p(t) \cdot A = 100p(t).\ \forall t \geq 1.$

$$S_t(r(t)) = \begin{cases} 100\left(1 - \frac{1}{2r(t)}\right) & t \text{ is odd} \\ 100\left(\frac{1}{2} - \frac{1}{2r(t)}\right) & t \text{ is even} \end{cases} = 100 \cdot p(t).$$

So we know $p(t) = \begin{cases} 1 - \frac{1}{2r(t)} & t \text{ is odd} \\ \frac{1}{2} - \frac{1}{2r(t)} & t \text{ is even} \end{cases}$

By eq. cond #1,

$$p(t) = \frac{d(t+1) + p^e(t+1)}{r(t)} = \frac{2}{r(t)}$$

Thus:

for odd t, $\frac{2}{r(t)} = 1 - \frac{1}{2r(t)} \Leftrightarrow 2 = r(t) - \frac{1}{2} \Leftrightarrow r(t) = \frac{5}{2}$

for even t, $\frac{2}{r(t)} = \frac{1}{2} - \frac{1}{2r(t)} \Leftrightarrow 2 = \frac{1}{2}r(t) - \frac{1}{2} \Leftrightarrow r(t) = 5$

Then we have

$$p(t) = \frac{2}{r(t)} = \begin{cases} \frac{4}{5} & t \text{ odd} \\ \frac{2}{5} & t \text{ even} \end{cases}$$

$$c_t^h(t) = \begin{cases} 1 + \frac{1}{2(5/2)} = \frac{6}{5} & t \text{ odd} \\ \frac{1}{2} + \frac{1}{2(5)} = \frac{3}{5} & t \text{ even} \end{cases}$$

$$s_t^h(t) = \begin{cases} 1 - \frac{1}{2(5/2)} = \frac{4}{5} & t \text{ odd} \\ \frac{1}{2} - \frac{1}{2(5)} = \frac{2}{5} & t \text{ even} \end{cases}$$

$$c_t^{h}(t+1) = r(t)c_t^h(t) = \begin{cases} \frac{5}{2} \cdot \frac{6}{5} = \frac{30}{10} = 3 & t \text{ odd} \\ 5 \cdot \frac{3}{5} = 3 & t \text{ even} \end{cases}$$

~~From 6.4a we have $c_t^h(t+1) = \frac{14}{5} < c_t^{n,e}(t+1)$ for odd t~~

From 6.4a, we have $c_{t-1}^h(t) = \frac{14}{5} < c_{t-1}^{n,e}(t)$ for odd t

For even t,

Again using the market clearing condition and $D(t) = 100, = d(t)A$:

$$\sum_{n=1}^{N(t-1)} c_{t-1}^{n}(t) + \sum_{n=1}^{N(t)} c_{t}^{n}(t) = \sum_{n=1}^{N(t-1)} w_{t-1}^{n}(t) + \sum_{n=1}^{N(t)} w_{t}^{n}(t) + D(t)$$

$$100 \cdot c_{t-1}^{n}(t) + 100 \cdot \frac{3}{5} = 100 \cdot 1 + 100 \cdot 1 + 100$$

$$c_{t-1}^{n}(t) = 3 - \frac{3}{5} = \frac{12}{5} < c_{t-1}^{n,e}(t) \text{ for even } t.$$

EXERCISE 8.1 Suppose $N(t) = 1, Y(t) = 1$ for all $t \geq 0, \lambda = 2$, $K(1) = 0$, and the utility function is $u_t^h = c_t^h(t)c_t^h(t+1)$, for all h and $t \geq 1$. Show that the following consumption allocation is feasible but not Pareto optimal:

$$c_t^h = [1/2, 1/2] \quad \text{for all } h \text{ and } t \geq 1, \quad \text{and} \quad c_0^h(1) = 1/2.$$

Definition *A path of total consumption, $C(1), C(2), C(3), \ldots, C(t)$, is feasible for an economy with storage if, given $K(1)$, there exists a nonnegative $K(t)$ sequence for $t > 1$ that satisfies* $K(t)$ time $t-1$ good.

$$C(t) + K(t+1) \leq Y(t) + \lambda K(t), \qquad \text{for all } t \geq 1. \qquad (8.1)$$

FEASABILITY:

$t = 1$:

$$C(1) + K(2) \leq Y(1) + \lambda K(1)$$
$$1/2 + 1/2 + K(2) \leq 1 + 2 \times 0$$
$$\Leftrightarrow K(2) = 0$$

$t = 2$:

$$C(2) + K(3) \leq Y(2) + \lambda K(2)$$
$$1/2 + 1/2 + K(3) \leq 1 + 2 \times 0$$
$$\Leftrightarrow K(3) = 0$$

Given consumption feasible as long as $K(t) = 0 \ \forall t$.

PARETO OPTIMALITY:

Let $x \in (0, 1/2)$ denote the investment in storage of individual h when young at all times, so that the new allocation is

$$\gamma_t^h = [1/2 - x, 1/2 + \lambda x]$$

with

$$u_t^h(\gamma_t^h) = (1/2 - x)(1/2 + 2x) = 1/4 + 1/2 x - 2x^2$$

which is higher than $u_t^h(c_t^h) = 1/4$ for $x \in (0, 1/4)$.

Further
$$(1/2 - x + 1/2 + \lambda x) + x \leq 1 + \lambda x$$
$$1 - x + \lambda x + x \leq 1 + \lambda x$$
$$0 \leq 0$$

shows the feasability of the alternative allocation.

$\Rightarrow \quad c_t^h = [1/2, 1/2]$ not Pareto optimal.

EXERCISE 8.4 Describe an economy without land with a stationary equilibrium that satisfies $r > \lambda$.

EXERCISE 8.5 Describe an economy without land with a stationary equilibrium that satisfies $r = \lambda$.

Definition *A perfect foresight competitive equilibrium for an economy with storage and land is a nonnegative sequence of land prices and interest rates and of total storage amounts, $K(t+1)$, that for all $t \geq 1$ satisfy*

(i) $S_t(r(t)) = p(t)\mathbf{A} + K(t+1)$,

(ii) $p(t) = (p(t+1) + d(t+1))/r(t)$,

(iii) $r(t) \geq \lambda$, *and*

(iv) $K(t+1)[1 - \lambda/r(t)] = 0$.

$r > \lambda$ ... return on loans greater than return on storage

$\Rightarrow$ agents won't invest in storage : $K = 0$

consequently no aggregate savings : $S(r) = K = 0$

the stationary equilibrium is therefore

$$\{r(t), K(t+1)\} = \{r, 0\} \quad \forall t \geq 1$$

with r determined from $S(r) = 0$

$r = \lambda$

$\Rightarrow$ agents indifferent between loans and storage, allowing for a stationary value of K

equilibrium given by $\{r(t), K(t+1)\} = \{\lambda, K\} \quad \forall t \geq 1$

with arbitrary individual division of loans / storage since only net position matters

#2 $N(t)=1,\ A=1,\ d(t)=1,\ \omega_t^h=[7,10],\ u_t^h = c_t^h(t)\cdot c_t^h(t+1) \quad \forall h,t$

(a) What allocations are feasible in this environment?

Equation 6.7 in the textbook gives us the feasibility requirement for economies with infinitely lived assets:

$$\sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + \sum_{h=1}^{N(t)} c_t^h(t) \leq \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t) + \sum_{h=1}^{N(t)} \omega_t^h(t) + d(t)A \quad \forall t \geq 0$$

So $$\sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + \sum_{h=1}^{N(t)} c_t^h(t) \leq 10+7+1\cdot 1 = 18$$

or equivalently $c_{t-1}^h(t) + c_t^h(t) \leq 18$.

(b) Solving for the stationary equilibrium:

From 6.4a, we have

$$c_t^h(t) = \frac{\omega_t^h(t)}{2} + \frac{\omega_t^h(t+1)}{2r(t)} = \frac{7}{2} + \frac{5}{r(t)}$$

So savings are: $s_t^h(t) = \omega_t^h(t) - c_t^h(t) = 7/2 - 5/r(t) = S_t(r(t))$ since $N(t)=1$.

We know in equilibrium, $S_t(r(t)) = p(t)\cdot A$, so

$$\frac{7}{2} - \frac{5}{r(t)} = p(t)\cdot 1. \qquad (1)$$

& in equilibrium, $p(t) = \dfrac{d(t+1) + p^e(t+1)}{r(t)}$

so $$p(t) = \frac{1+p^e(t+1)}{r(t)} \qquad (2)$$

To find our stationary equilibrium, we let $p(t) = p^e(t+1) = p \quad \forall t$ and $r(t) = r \quad \forall t$.

Then rearranging eq. 2 yields

$$pr = 1 + p \iff p = \frac{1}{r-1}$$

Substituting into eq. 1, we have

$$\frac{1}{r-1} = \frac{7}{2} - \frac{5}{r} \iff r = \frac{7}{2}r(r-1) - 5(r-1)$$

$$\iff \frac{7}{2}r^2 - \frac{7}{2}r - r - 5r + 5 = 0 \iff \frac{7}{2}r^2 - \frac{19}{2}r + 5 = 0$$

$\iff 7r^2 - 19r + 10 = 0$ by the quadratic formula, $r = 2$ or $r = \frac{5}{7}$.

If $r = 2$, $p = \frac{1}{2-1} = 1$.

If $r = \frac{5}{7}$, $p = \frac{1}{\frac{5}{7}-1} < 0 \rightarrow$ prices are nonneg in equilibrium, so $r = 2$.

Thus $\forall t \geq 1$

$$c_t^h(t) = \frac{7}{2} + \frac{5}{r(t)} = \frac{7}{2} + \frac{5}{2} = 6$$

$$s_t^h(t) = w_t^h(t) - c_t^h(t) = 7 - 6 = 1$$

$c_t^h(t+1) = r(t)\, c_t^h(t)$ by 6.4a, so $c_t^h(t+1) = 2 \cdot 6 = 12$

Feasibility cond. holds with equality, markets clear

The initial old consume their endowment + crop yield + sale price of land, so

$c_t^h(t+1) = 10 + 1 + 1 = 12$ when $t = 0$

(c) Suppose gov takes 2/5 of crop as a tax to dist. among old.

We first derive the savings function: by maximizing utility:

$$\max_{\{c_t^h(t),\, c_t^h(t+1)\}} c_t^h(t) \cdot c_t^h(t+1) \text{ s.t. } w_t^h(t) - c_t^h(t) - p(t)a^h(t) = \frac{-w_t^h(t+1) + c_t^h(t+1)}{r(t)}$$

$$- \frac{a^h(t)(1-\frac{2}{5})d(t+1) - a^h(t)p^e(t+1) + x}{r(t)}$$ where x is the transfer to the old.

Rearranging the lifetime BC yields

$$c_t^h(t+1) = r(t)[w_t^h(t) - c_t^h(t) - p(t)a^h(t)] + w_t^h(t+1) + a^h(t)(\tfrac{3}{5})d(t+1) + a^h(t)p^e(t+1) + x$$

Substituting the above into $c_t^h(t) \cdot c_t^h(t+1)$ yields an unconstrained optimization problem with FOC

(1) $\partial u / \partial c_t^h(t) = c_t^h(t+1) - r(t)c_t^h(t) = 0 \iff c_t^h(t+1) = r(t)c_t^h(t)$.

(2) $\partial u / \partial a^h(t) = c_t^h(t) \cdot [-p(t)r(t) + \frac{3}{5}d(t+1) + p^e(t+1)] = 0$

$$\iff p(t) = \frac{\frac{3}{5}d(t+1) + p^e(t+1)}{r(t)}$$

Plugging FOCs 1 and 2 into the lifetime BC yields

$$w_t^n(t) - c_t^n(t) - p(t)a^n(t) = \frac{-w_t^n(t+1)}{r(t)} + c_t^n(t) - a^n(t)p(t) + \frac{x}{r(t)}$$

$$\Leftrightarrow w_t^n(t) + \frac{w_t^n(t+1)}{r(t)} - \frac{x}{r(t)} = 2c_t^n(t) \Leftrightarrow c_t^n(t) = \frac{1}{2}\left(w_t^n(t) + \frac{w_t^n(t+1) - x}{r(t)}\right)$$

$$c_t^n(t) = \frac{1}{2}\left(7 + \frac{10-x}{r(t)}\right)$$

~~Thus~~ And $s_t^n(t) = w_t^n(t) - c_t^n(t) = \frac{7}{2} - \frac{10-x}{2r(t)}$

The government must balance its budget: total transfers = total taxes
$\Rightarrow -D(t)\cdot\frac{2}{5} = N(t)\cdot x \Leftrightarrow -d(t)A\frac{2}{5} = N(t)x \Leftrightarrow -\frac{2}{5} = x$

Thus $c_t^n(t) = \frac{1}{2}\left(7 + \frac{10 - \frac{2}{5}}{r(t)}\right) = \frac{7}{2} + \frac{5 + \frac{1}{5}}{r(t)} = \frac{7}{2} + \frac{26}{5r(t)}$

$s_t^n(t) = 7 - c_t^n(t) = \frac{7}{2} - \frac{26}{5r(t)} = S_t(r(t))$ since $N(t)=1$

We know that in equilibrium, $S_t(r(t)) = p(t)A = p(t)\cdot 1$, so

$$\frac{7}{2} - \frac{26}{5r(t)} = p(t)$$

And by FOC 2, $p(t) = \frac{\frac{3}{5}d(t+1) + p^e(t+1)}{r(t)} = \frac{\frac{3}{5} + p^e(t+1)}{r(t)}$

Letting $p^e(t+1) = p(t) = p$ and $r(t) = r$ we can solve for a stationary eq.:

$$\frac{7}{2} - \frac{26}{5r} = p \quad \& \quad p = \frac{\frac{3}{5} + p}{r} \Leftrightarrow p = \frac{3/5}{r-1}$$

$$\Rightarrow \frac{7}{2} - \frac{26}{5r} = \frac{3/5}{r-1} \Leftrightarrow \frac{7}{2}r(r-1) - \frac{26}{5}(r-1) = \frac{3}{5}r$$

$$\Rightarrow \frac{7}{2}r^2 - \frac{7}{2}r - \frac{26r}{5} + \frac{26}{5} - \frac{3}{5}r = 0$$

$$\Rightarrow \frac{7}{2}r^2 - \frac{93}{10}r + \frac{26}{5} = 0 \text{ has roots } \frac{13}{7}, \frac{4}{5}.$$

If $r = \frac{13}{7}$, then $p = \frac{\frac{3}{5}}{\frac{13}{7} - 1} = \frac{7}{10}$

If $r = \frac{4}{5}$, $p = \frac{3}{5}/((4/5)-1) < 0$ not an equilibrium land price. so $r = \frac{13}{7}, p = \frac{7}{10}$

Thus in the stationary equilibrium we have; $\forall t \geq 1$,

$$c_t^y(t) = \frac{7}{2} + \frac{26}{5 \cdot (13/7)} = 6.3 \quad ; \quad s_t^y(t) = 7 - 6.3 = 0.7 ;$$

$$c_t^o(t+1) = c_t^y(t) r(t) \text{ by FOC ①} \Rightarrow c_t^o(t+1) = 6.3 \cdot \frac{13}{7} = 11.7$$

For $t=0$,

$$c_t^o(t+1) = w_t^o(t+1) + (1 - \tfrac{2}{5}) d(t+1) A + p(t+1) A - X$$
$$= 10 + \tfrac{3}{5} \cdot 1 \cdot 1 + \tfrac{7}{10} \cdot 1 + \tfrac{2}{5}$$
$$= 11.7$$

So the initial old are worse off than in part b without the scheme, where they consumed 12 units in $t=1$.

However, all other agents prefer this scheme: their utility now is $6.3 \times 11.7 = 73.71$, which is greater than their utility before the scheme, $6 \times 12 = 72$.

Exercise 1. Consider the following economy. Preferences are given by

$$u_t^h = \log c_t^h(t) + \beta \log c_t^h(t+1), \quad \text{where} \quad \beta = \frac{2}{3}, N = 100.$$

There is no financial assets but private borrowing and lending. Endowments for type-1 households are $\omega_t^h = [3, 1]$ for $h = 1, ..., 50$ and equal to $\omega_t^h = [1, 3]$ for $h = 51, ..., 100$.

1. Solve for $r(t)$ and s_t^h in the competitive equilibirium.
2. Now suppose that you are not allowed to borrow, so that $0 \leq s_t^h$. What happens to $r(t)$? Why?

Solution. The interest rate $r(t)$ and savings s_t^h is determined by the lifetime utility maximization problem

$$\begin{aligned} \max \quad & \log c_t^h(t) + \beta \log c_t^h(t+1) \\ \text{s.t} \quad & c_t^h(t) \leq \omega_t^h(t) - l^h(t) \\ & c_t^h(t+1) \leq \omega_t^h(t+1) + r(t) l^h(t). \end{aligned}$$

The problem lagrangian is

$$L = \log c_t^h(t) + \beta \log c_t^h(t+1) + \mu_t(c_t^h(t) - \omega_t^h(t) + l^h(t)) + \mu_{t+1}(c_t^h(t+1) - \omega_t^h(t+1) - r(t) l^h(t)).$$

The first order conditions for an optimal solution are

$$\frac{1}{c_t^h(t)} - \mu_t = 0,$$

$$\frac{\beta}{c_t^h(t+1)} - \mu_{t+1} = 0,$$

$$-\mu_t + \mu_{t+1} r(t) = 0.$$

Solve the first and second equation for the multipliers and substitute into the third, this yields

$$c_t^h(t+1) = \beta r(t) c_t^h(t).$$

Substitute into the lifetime budget-constraint and solve for $c_t^h(t)$,

$$c_t^h(t) = \frac{1}{1+\beta}(w_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}).$$

Plug in endowments and β for both types,

$$c_t^h(t) = \frac{9}{5} + \frac{3}{5r(t)}, \quad h = 1, ..., 50$$

$$c_t^h(t) = \frac{3}{5} + \frac{9}{5r(t)}, \quad h = 51, ..., 100.$$

Using optimal period t consumption we find individual savings equal to

$$s_t^h(t) = \frac{\beta}{1+\beta}\omega_t^h(t) - \frac{1}{1+\beta}\frac{\omega_t^h(t+1)}{r(t)}.$$

Again plugging in values,

$$s_t^h(t) = \frac{6}{5} - \frac{3}{5r(t)}, \quad h = 1, ..., 50$$

$$s_t^h(t) = \frac{2}{5} - \frac{9}{5r(t)}, \quad h = 51, ..., 100.$$

Summing savings over both types of individuals we get aggregate savings which must equal zero,

$$S_t(r(t)) = 80 - \frac{120}{r(t)} = 0.$$

Solve for the interest rate $r(t)$, it is

$$r(t) = \frac{3}{2}.$$

Plug it into the savings expressions, it gives us

$$s_t^h(t) = \frac{4}{5}, \quad h = 1, ..., 50$$

$$s_t^h(t) = \frac{-4}{5}, \quad h = 51, ..., 100.$$

b) Suppose borrowing is restricted so that $0 \leq s_t^h$. Since there is a counter-party saver for each borrower and there cant be any borrowers, it implies no one saves. The interest rate must cause the type one individuals not to save or borrow

$$\frac{6}{5} - \frac{3}{5r(t)} = 0,$$

giving us an interest rate equal to

$$r(t) = \frac{1}{2}.$$

Exercise 3 Consider the following OLG enviorment where everyone lives for two periods and $N(t) = 1$. There is no sotrage or capital. Half of the population is poor and the endowment profile is $\omega = [5, 3]$ and the other half is rich with endowment profile $\omega = [5, 4]$. Preferences are described by $u_t = \ln c_t(t) + \ln c_t(t+1)$. Wasteful government purschaes are given by $G(t) = \frac{3}{2}$. Suppose all te poor consume the same and likewise with the rich.

1. Suppose the tax rate on the endowments of the young is 30% and that the old are not taxed. Find the competitve equilibrium.

2. Suppose the tax/debt policy is changed in period 0 from that described in (a). In period 0 no taxes are levied. Consider two alternative senarios.

   (a) From peridod 1 onwards, a proportional tax rate of 3/7 is levie on the old. Find the competitive equilibrium, including the quantity of governmtn debt issued in ach period. Does it feature the same consumption allocation as in (a)? Why or why not?

   (b) From period 1 onwrads, all the old people have to pay a lump-sum of 3/2 to the government. Fidn the competitive equilibrium. Does it feature the same consumption allocation as in (a)? Why or why not?

Solution. The budget constraint for each period with taxes is

$$c_t^h(t) = \omega_t^h(t) - l^h(t),$$

$$c_t^h(t) = \omega_t^h(t+1) + l^h(t)r(t).$$

The maximization problem then becomes

$$\begin{aligned}
\max \quad & \ln c_t^h(t) + \ln c_t^h(t+1) \\
\text{s.t} \quad & c_t^h(t) = \omega_t^h(t)(1-T) - l^h(t) \\
& c_t^h(t+1) = \omega_t^h(t+1) + r(t)l^h(t).
\end{aligned}$$

The Lagrangian for the problem is

$$L = \ln c_t^h(t) + \ln c_t^h(t+1) + \mu_t(\omega_t^h(t)(1-T) - l^h(t) - c_t^h(t)) + \mu_{t+1}(\omega_t^h(t+1) + r(t)l^h(t) - c_t^h(t+1)).$$

A first order condition for a maximum is that the derivatives of period t, $t+1$ consumption and lending is zero. They are

$$\frac{1}{c_t^h(t)} - \mu_t = 0,$$
$$\frac{1}{c_t^h(t+1)} - \mu_{t+1} = 0,$$
$$-\mu_t + \mu_{t+1} = 0.$$

Solving for the multipliers and substituting them into the third equation gives us

$$c_t^h(t+1) = c_t^h(t)r(t).$$

Substitution into the lifetime budget constraint and solving for $c_t^h(t)$ yields

$$c_t^h(t) = \frac{\omega_t^h(t)(1-T)}{2} + \frac{\omega_t^h(t+1)}{2r(t)}.$$

The savings equals the difference between endowment and consumption together with taxation,

$$s_t^h(t) = \omega_t^h(t) - c_t^h(t) - \frac{3}{10}\omega_t^h(t) = \omega_t^h(t) - \frac{\omega_t^h(1-T)}{2} - \frac{\omega_t^h(t+1)}{2r(t)} - \frac{3}{4}.$$

This means the poor will save

$$s_t^h(t) = \frac{7}{4} - \frac{3}{2r(t)},$$

and the rich will save

$$s_t^h(t) = \frac{7}{4} - \frac{2}{r(t)}.$$

Since aggregate savings equal zero the interest rate is

$$\frac{1}{2}(\frac{7}{4} - \frac{3}{2r(t)}) + \frac{1}{2}(\frac{7}{4} - \frac{2}{r(t)}) = 0,$$
$$r(t) = 1.$$

The interest rate equals $r(t) = 1$. Plugging this value into the consumption for each period and savings for the poor and rich we find

$$c_t^{poor}(t) = \frac{13}{4},$$
$$c_t^{poor}(t+1) = \frac{13}{4},$$
$$s_t^{poor}(t) = \frac{1}{4},$$
$$c_t^{rich}(t) = \frac{15}{4},$$
$$c_t^{rich}(t+1) = \frac{15}{4},$$
$$s_t^{rich}(t) = \frac{-1}{4}.$$

The tax finance all govnerment spending since the tax times the endowment is

$$\frac{3}{10} \cdot 5 = \frac{3}{2} = G(t).$$

Therefore, there is no need for government debt in this environment.

b) A tax $T = \frac{3}{7}$ is put on the old from period $t = 1$. Set up the same maximization problem but with the tax on the old and allowing for the government to issue debt

$$\begin{aligned} \max \quad & \ln c_t^h(t) + \ln c_t^h(t+1) \\ \text{s.t} \quad & c_t^h(t) = \omega_t^h(t) - l^h(t) - p(t)b^h(t) \\ & c_t^h(t+1) = \omega_t^h(t+1)(1-T) + r(t)l^h(t) + b^h(t). \end{aligned}$$

$$L = \ln c_t^h(t) + \ln c_t^h(t+1) + \mu_t(\omega_t^h(t) - l^h(t) - p(t)b^h(t) - c_t^h(t)) + \mu_{t+1}(\omega_t^h(t+1)(1-T) + r(t)l^h(t) + b^h(t) - c_t^h(t+1))$$

The first order condition will be the same as in a), resulting in equilibrium condition

$$c_t^h(t+1) = c_t^h(t)r(t).$$

However, the lifetime budget constraint is different, substituting values results in

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = \omega_t^h(t)\frac{\omega_t^h(t+1)(1-T)}{r(t)} - b^h(t)(p(t) - \frac{1}{r(t)}),$$

$$c_t^h(t) = \frac{\omega_t^h(t)}{2} + \frac{\omega_t^h(t+1)(1-T)}{2r(t)}.$$

The savings function then becomes

$$s_t^h(t) = \omega_t^h - c_t^h(t) = \frac{\omega_t^h(t)}{2} - \frac{\omega_t^h(t+1)(1-T)}{2r(t)}.$$

Plugging in values for the rich and the poor,

$$s_t^{rich}(t) = \frac{5}{2} - \frac{8}{7r(t)}.$$

$$s_t^{poor}(t) = \frac{5}{2} - \frac{6}{7r(t)}.$$

The economy equilibrium is characterized by no arbitrage $r(t) = \frac{1}{p(t)}$, that $S(t) = P(t)B(t)$ and government needing to raise $\frac{3}{2}$. This helps pin down the interest rate, it is

$$S(t) = P(t)B(t),$$

$$\frac{1}{2}(\frac{5}{2} - \frac{6}{7r(t)}) + \frac{1}{2}(\frac{5}{2} - \frac{8}{7r(t)}) = \frac{3}{2},$$

$$r(t) = 1.$$

The interest rate is again equal to $r(t) = 1$. Plugging in the interest rate, we find consumption and savings for poor and rich equal to

$$c_t^{poor}(t) = \frac{47}{14},$$

$$c_t^{poor}(t+1) = \frac{47}{14},$$

$$s_t^{poor}(t) = \frac{23}{14},$$

$$c_t^{rich}(t) = \frac{51}{14},$$

$$c_t^{rich}(t+1) = \frac{51}{14},$$

$$s_t^{rich}(t) = \frac{19}{14}.$$

This equilibrium is not the same as in part a). Due to the change in taxation and allowing for government borrowing, the consumer budget constraint changed and hence optimal behaviour too.

ii) Government issues bonds in $t = 0$ to finance $G = \frac{3}{2}$. From part a) we derived the optimal consumption and savings equations for both rich and poor. The equilibrium condition now becomes

$$\frac{1}{2}(\frac{1}{2} \cdot 5 - \frac{1}{2}\frac{\frac{3}{2}}{r(t)} + \frac{1}{2}(\frac{1}{2} \cdot 5 - \frac{1}{2}\frac{\frac{5}{2}}{r(t)}) = \frac{3}{2},$$

$$r(t) = 1.$$

This results in the exact same consumption and savings as in a)

$$c_t^{poor}(t) = \frac{13}{4},$$

$$c_t^{poor}(t+1) = \frac{13}{4},$$

$$c_t^{rich}(t) = \frac{15}{4},$$

$$c_t^{rich}(t+1) = \frac{15}{4},$$

In conclusion the present value of the budget constraint remains the same for both the rich and the poor so individuals chose the same bundls.