Advanced macroeconomics problem set 1

Hugo - 42597@student.hhs.se,
Marek - 42624@student.hhs.se,
Marleena - 42620@student.hhs.se

November 2, 2023

(1.3) Prove that the allocation is feasible:

A cons. allocation is feasible if the consumption path satisfies
$C(t) \le Y(t) \quad \forall\, t \ge 1$

$$C(t) \le Y(t) \Rightarrow \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) \le \sum_{h=1}^{N(t)} \omega_t^h(t) + \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t) \quad \forall\, t \ge 1$$

We have $Y(t) = 2 \quad \forall\, t \ge 1$ and $N(t-1) = 2 \quad \forall\, t \ge 1$.

In period $t=1$:

$$C(t) = c_0^1(1) + c_0^2(1) + c_1^1(1) + c_1^2(1) = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + 3/4 = 2 \le Y(1) = 2.$$

So the allocation is feasible for $t=1$.

For $t=2$, we have

$$C(2) = c_1^1(2) + c_1^2(2) + c_2^1(2) + c_2^2(2) = \frac{3}{4} + \frac{1}{4} + \frac{1}{2} + \frac{1}{2} = 2 \le Y(2) = 2$$

Feasible!

~~For $t \ge 2$~~

For $t > 2$,

$$C(t) = \sum_{h=1}^{2} c_t^h(t) + \sum_{h=1}^{2} c_{t-1}^h(t)$$

Since $t-1 \ge 2$, we know:

$$C(t) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2 \le Y(t) = 2.$$

So the allocation is feasible for all $t \ge 1$.

Definition 1. An allocation is efficient if there is no alternative feasible allocation with more total consumption of some good and no less of any other good.

Definition 2. An allocation is Pareto optimal if there exists no feasible allocation which is Pareto superior to it.

Exercise 1.7 Show that if an allocation is Pareto optimal then it is efficient.

Solution. Proof by contraposition. Let $A = (\{c_t(t)_{h=1}^{N(t)}\}, \{c_{t-1}(t)_{h=1}^{N(t-1)}\})$ be a non-efficient allocation. By Exercise 1.5, there exists a period t^* such that $C(t^*) < Y(t^*)$. Construct a new allocation A^* by setting period t^* consumption for some individual g in generation t^* equal to

$$c_t^{g*}(t^*) = c_t^g(t^*) + Y(t^*) - C(t^*),$$

all else equal. Then, for all people h in all generations t we have $u_t^h(c_t^h(t), c_t^h(t+1)) = u_t^h(c^{h*}(t), c^{h*}(t+1))$, except for individual g in period t^* where a strictly increasing utility function u^g guarantees that,

$$u_{t^*}^g(c_{t^*}^{g*}(t), c_{t^*}^{g*}(t+1)) > u_{t^*}^g(c_{t^*}^g(t), c_{t^*}^g(t+1)).$$

The new allocation A^* is therefore Pareto superior and hence A is not Pareto optimal.

2.3 Find $S(r(t))$

ⓐ $N(t)=100$, $u_t^h = c_t^h(t)[c_t^h(t+1)]^\beta$, $\beta=1$, $[\omega_t^h(t), \omega_t^h(t+1)] = [2,1]$.

Each consumer maximizes:

$$\max \; c_t^h(t)(c_t^h(t+1))^\beta \quad \text{s.t.} \quad c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}.$$

or equivalently:

$$\max \; c_t^h(t)\left(\omega_t^h(t)r(t) + \omega_t^h(t+1) - c_t^h(t)r(t)\right)^\beta$$

FOC:

$$\partial u/\partial c_t^h(t) = (c_t^h(t+1))^\beta + c_t^h(t)\beta(c_t^h(t-1))^{\beta-1}(-r(t)) = 0$$

$$\Rightarrow \cancel{\frac{(c_t^h(t+1))^\beta}{c_t^h(}}$$

At this point, it is useful to apply $\beta=1$:

$$c_t^h(t+1) = c_t^h(t)r(t)$$

Plugging into the budget constraint:

$$2c_t^h(t) = \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}$$

$$c_t^h(t) = \frac{\omega_t^h(t)}{2} + \frac{\omega_t^h(t+1)}{2r(t)}$$

Savings = endowment − consumption (note since we have identical consumers, there will be no lending)

$$s_t^h = \omega_t^h(t) - c_t^h(t) = \frac{\omega_t^h(t)}{2} - \frac{\omega_t^h(t+1)}{2r(t)} = \frac{2}{2} - \frac{1}{2r(t)} = 1 - \frac{1}{2r(t)}$$

Aggregate savings:

$$S_t(r(t)) = \sum_{h=1}^{100} s_t^h(r(t)) = 100\left(1 - \frac{1}{2r(t)}\right)$$

(2.3)

(b) $N(t) = 100$, $\beta = 1$, $u_t^h = c_t^h(t)\left(c_t^h(t+1)\right)^{\beta}$, $1/2$ cons have $[2, 1]$
$1/2$ have $[1, 1]$

From part (a), we know the individual savings function is:

$$s_t^h = \frac{w_t^h(t)}{2} - \frac{w_t^h(t+1)}{2r(t)} = \begin{cases} 1 - \frac{1}{2r(t)}, & h = 1, 2, \ldots, 50 \\ \frac{1}{2} - \frac{1}{2r(t)} & h = 51, \ldots, 100 \end{cases}$$

Then aggregate savings are:

$$\cancel{S_t(r(t)) = \sum_{h=1}^{100} s_t^h = \sum_{h=1}^{50}\left(1 - \frac{1}{2r(t)}\right) + \sum_{h=51}^{100}\left(\frac{1}{2} - \frac{1}{2r(t)}\right) = 50\left(1 - \frac{1}{2r(t)}\right) + 50\left(\frac{1}{2} - \frac{1}{2r(t)}\right)}$$

$\cancel{= \text{[illegible]}\, 50\, r(t)}$.

$$S_t(r(t)) = \sum_{h=1}^{100} s_t^h = \sum_{h=1}^{50}\left(1 - \frac{1}{2r(t)}\right) + \sum_{h=51}^{100}\left(\frac{1}{2} - \frac{1}{2r(t)}\right) = 50\left(1 - \frac{1}{2r(t)}\right) +$$

$$50\left(\frac{1}{2} - \frac{1}{2r(t)}\right) = 75 - \frac{50}{r(t)}$$

2.4 (a, e, f) Describe the competitive equilibria:

ⓐ $u_t^h = c_t^h(t)(c_t^h(t+1))^\beta$, $\beta = 1$, endowments for all $[2, 1]$, $N(t) = 100$.

In 2.3ⓐ, we determined that in this case, aggregate savings is given by:

$$S_t(r(t)) = \cancel{100(1 - \tfrac{1}{2}r(t))} \quad 100\left(1 - \frac{1}{2r(t)}\right)$$

In equilibrium, $S_t(r(t)) = 0$.

$$100\left(1 - \frac{1}{2r(t)}\right) = 0 \iff r(t) = \frac{1}{2}$$

Then, for all $t \geq 0$, again from 2.3 a, we have

$$c_t^h(t) = \frac{\omega_t^h(t)}{2} + \frac{\omega_t^h(t+1)}{2r(t)} = 1 + \frac{1}{2 \cdot \frac{1}{2}} = 2$$

$$c_t^h(t+1) = r(t)c_t^h(t) = \frac{1}{2} \cdot 2 = 1.$$

So in this economy, we have an equilibrium when

$s_t^h = 0$, $[c_t^h(t), c_t^h(t+1)] = [2, 1]$, $r(t) = \frac{1}{2}$ for all $t \geq 0$, $h = 1, \ldots, 100$

ⓔ Same as a, but endowments: $\begin{cases} [2, 1] & h = 1, \ldots, 60 \\ [1, 1] & h = 61, \ldots, 100 \end{cases}$

From 2.3a, we know individual savings are:

$$s_t^h = \frac{\omega_t^h(t)}{2} - \frac{\omega_t^h(t+1)}{2r(t)} = \begin{cases} 1 - \frac{1}{2r(t)} & h = 1, \ldots, 60 \\ \frac{1}{2} - \frac{1}{2r(t)} & h = 61, \ldots, 100. \end{cases}$$

And in equilibrium, $S_t(r(t)) = 0$:

$$S_t(r(t)) = \sum_{h=1}^{100} s_t^h = \sum_{h=1}^{60}\left(1 - \frac{1}{2r(t)}\right) + \sum_{h=61}^{100}\left(\frac{1}{2} - \frac{1}{2r(t)}\right) = 60\left(1 - \frac{1}{2r(t)}\right) + 40\left(\frac{1}{2} - \frac{1}{2r(t)}\right)$$

$$= 80 - \frac{50}{r(t)} = 0 \quad \text{iff } r(t) = \frac{5}{8}$$

Again from 2.3a, consumption is

$$c_t^h(t) = \frac{\omega_t^h(t)}{2} + \frac{\omega_t^h(t+1)}{2\left(\frac{5}{8}\right)} = \begin{cases} 1 + \frac{4}{5} = \frac{9}{5} & h = 1, \ldots, 60 \\ \frac{1}{2} + \frac{4}{5} = \frac{13}{10} & h = 61, \ldots, 100 \end{cases}$$

$$c_t^h(t+1) = \begin{cases} \frac{9}{5} \cdot \frac{5}{8} = \frac{9}{8} & h = 1, \ldots, 60 \\ \frac{13}{10} \cdot \frac{5}{8} = \frac{13}{16} & h = 61, \ldots, 100 \end{cases}$$

Returning to individual savings, we have:

$$s_t^h = \begin{cases} 1 - \frac{1}{2(5/8)} = \frac{1}{5} \\ \frac{1}{2} - \frac{1}{2(5/8)} = -\frac{3}{10} \end{cases}$$

So we have an equilibrium where $h = 1, \ldots, 60$ lend $\frac{1}{5}$ each at $r(t) = \frac{5}{8}$ and consume $\left(\frac{9}{5}, \frac{9}{8}\right)$ and $h = 61, \ldots, 100$ borrow $\frac{3}{10}$ each and consume $\left(\frac{13}{10}, \frac{13}{16}\right)$.

2.4 f) $[1,1]$ ~~ta~~ t odd
$[2,1]$ t even

$$s_t^h = \frac{\omega_t^h(t)}{2} - \frac{\omega_t^h(t+1)}{2r(t)} = \begin{cases} \frac{1}{2} - \frac{1}{2r(t)} & t \text{ odd} \\ 1 - \frac{1}{2r(t)} & t \text{ even} \end{cases}$$

Aggregate savings are 0 in equilibrium:

For ~~even~~ odd t: $100\left(\frac{1}{2} - \frac{1}{2r(t)}\right) = 0 \iff r(t) = 1$

for even t: $100\left(1 - \frac{1}{2r(t)}\right) = 0 \iff r(t) = \frac{1}{2}$

For odd t, this means

$$c_t^h(t) = \frac{\omega_t^h(t)}{2} + \frac{\omega_t^h(t+1)}{2r(t)} = \frac{1}{2} + \frac{1}{2} = 1 \;;\; c_t^h(t+1) = c_t^h(t)r(t) = 1 \cdot 1 = 1$$

And for even t,

$$c_t^h(t) = \frac{\omega_t^h(t)}{2} + \frac{\omega_t^h(t+1)}{2r(t)} = 1 + \frac{1}{2\cdot\frac{1}{2}} = 2 \;;\; c_t^h(t+1) = c_t^h(t)r(t) = 2\cdot\frac{1}{2} = 1.$$

In this equilibrium, we have

$$c_t^h = \begin{cases} [1,1] & t \text{ odd} \\ [2,1] & t \text{ even} \end{cases}$$

with real gross interest rate 1 for odd generations and $\frac{1}{2}$ for even generations. Since consumers are ~~old~~ young (and old) at the same time as ~~only~~ identical consumers only, there is no borrowing or lending in this economy.

2.5 a, e, f

(a) Consider the allocation $[1.5, 1.5]$.
For all $t \geq 0$, and $h = 1, \ldots, 100$ $u_t^h([1.5, 1.5]) = (1.5)(1.5) = 2.25 > u_t^h([2,1]) = 2 \cdot 1 = 2$.

Thus all $h = 1, \ldots, 100$ and $t \geq 0$ strictly prefer $[1.5, 1.5]$ to $[2, 1]$.
We finally check that generation $t = -1$ prefer this allocation: in time $t = 0$, they now consume 1.5; greater than the previous allocation of 1. So generation -1 strictly prefers $[1.5, 1.5]$. Thus $[1.5, 1.5]$ is Pareto Superior to $[2, 1]$, so it must be the case that $[2, 1]$ is not Pareto Optimal.

(e) In this case we had the equilibrium consumption allocation

$$c_t^h = \begin{cases} \left[\frac{9}{5}, \frac{9}{8}\right] & h = 1, \ldots, 60 \\ \left[\frac{13}{10}, \frac{13}{16}\right] & h = 61, \ldots, 100 \end{cases}$$

Suppose we ~~transfer~~ take $\frac{1}{5}$ from each young person at time t and give $\frac{1}{5}$ to each old person at time t. The new allocations are:

$$c_t^h = \begin{cases} \left[\frac{8}{5}, \frac{53}{40}\right] & h = 1, \ldots, 60 \\ \left[\frac{11}{10}, \frac{81}{80}\right] & h = 61, \ldots, 100 \end{cases}$$

This new allocation is a pareto improvement over the equilibrium allocation in 2.4e. For generations $t \geq 0$,

$h = 1, \ldots, 60$ prefer the new allocation since

$$u([8/5, 53/40]) = 2.12 > u([9/5, 9/8]) = 2.025$$

$h = 61, \ldots, 100$ similarly prefer the new allocation:

$$u([11/10, 81/80]) \approx 1.11 > u\left(\left[\frac{13}{10}, \frac{13}{16}\right]\right) \approx 1.06.$$

Generation $t = -1$ consumes a strictly higher amount as well at $t = 0$.

Thus this allocation is pareto superior to the equilibrium allocation, so the equilibrium allocation cannot be Pareto Optimal.

2.5 (f)

In 2.4 f, we found the equilibrium consumption allocation

$$E = \begin{cases} [1, 1] & t \text{ is odd} \\ [2, 1] & t \text{ is even} \end{cases}$$

~~Consider the consumption allocation~~ $\cancel{A = \begin{cases} [1, 1] & t \text{ is odd} \\ [2.5, 1.5] & t \text{ is even} \end{cases}}$

Suppose in each even time period t, we ~~trans~~ took .2 units of good t, from ~~the~~ each young (~~who~~ who have $[2,1]$ in the equilibrium allocation) and gave .2 to each old person (who have allocation $[1,1]$ in the equilibrium), and in each odd time period t, we transferred .15 ~~good~~ of the t good from each ~~old person~~ young person to each old person.

Then we would achieve the new consumption allocation A:

$$c_t^h = A = \begin{cases} [.85, 1.2] & t \text{ is odd} \\ [1.8, 1.15] & t \text{ is even} \end{cases} \qquad \forall\, t \geq 0.$$

Then all individuals h and all generations $t \geq 0$ strictly prefer A to E, since for odd generations,

$$u([.85, 1.2]) = 1.02 > u([1, 1]) = 1$$

and for even generations

$$u([1.8, 1.15]) = 2.07 > u([2, 1]) = 2$$

The old in $t=1$, i.e. the generation 0, also prefers the allocation A since they each receive .15 units more of good $t=1$ to consume.

Thus A is Pareto Superior to E, so it must be that E is not Pareto Optimal.

3.1 a, c, d

(a) The consumer maximizes:

$$\max\ c_t^h(t)\, c_t^h(t+1) \quad \text{s.t.} \quad c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = \omega_t^h(t) - \tau_t^h(t) + \frac{\omega_t^h(t+1) - \tau_t^h(t+1)}{r(t)}$$

or equivalently,

$$\max\ c_t^h(t)\left[r(t)\left(\omega_t^h(t) - \tau_t^h(t)\right) + \omega_t^h(t+1) - \tau_t^h(t+1) - r(t)\, c_t^h(t)\right]$$

The FOC yields:

$$\frac{\partial u}{\partial c_t^h(t)} = c_t^h(t+1) - r(t)\, c_t^h(t) = 0 \iff c_t^h(t+1) = r(t)\, c_t^h(t)$$

Plugging into the budget constraint:

$$c_t^h(t) = \frac{\omega_t^h(t) - \tau_t^h(t)}{2} + \frac{\omega_t^h(t+1) - \tau_t^h(t+1)}{2r(t)}$$

Savings = post-tax endowment − consumption

$$s_t^h = \omega_t^h(t) - \tau_t^h(t) - \left(\frac{\omega_t^h(t) - \tau_t^h(t)}{2} + \frac{\omega_t^h(t+1) - \tau_t^h(t+1)}{2r(t)}\right)$$

$$s_t^h = +\frac{\omega_t^h(t) - \tau_t^h(t)}{2} - \frac{\omega_t^h(t+1) - \tau_t^h(t+1)}{2r(t)} = \frac{2-1}{2} - \frac{1+1}{2r(t)} = \frac{1}{2} - \frac{1}{r(t)}$$

In equilibrium, ~~[illegible]~~ aggregate savings are 0:

$$100\left(\frac{1}{2} - \frac{1}{r(t)}\right) = 0 \iff r(t) = 2$$

$$\text{Then } c_t^h(t) = \frac{2-1}{2} + \frac{1+1}{2\cdot 2} = \frac{1}{2} + \frac{1}{2} = 1$$

$$c_t^h(t+1) = c_t^h(t)\, r(t) = 2$$

The competitive equilibrium is characterized by consumption $[c_t^h(t), c_t^h(t+1)] = [1, 2]$ for all generations $t \geq 0$.

For all generations $t \geq 0$, every person $h = 1, \dots, 100$ is indifferent between the new equilibrium and the equilibrium without the scheme as described in 2.4 (a) since $u([2,1]) = 2 = u([1,2])$.

However, the old at time $t=0$ get to consume 1 additional unit of the $t=0$ good, so they strictly prefer the allocation with the scheme.

Since some consumers strictly prefer the allocation with the scheme, and no consumers prefer the allocation without the scheme, we conclude the equilibrium allocation with the social security scheme is Pareto superior to the equilibrium allocation without the scheme.

3.1 ⓒ $N(0) = 100 \qquad N(t) = 2N(t-1) \quad \forall\, t \geq 1$

~~From 3.1 a, we have~~

Tax revenue at time t is given by $t_t^h(t) \cdot N(t)$

$= 1 \cdot N(t) = 2N(t-1)$

Then the amount given to each old person is

$$\frac{1 \cdot N(t)}{N(t-1)} = 2 \qquad \text{So } t_t^h(t+1) = -2 \;\; \forall\; t \geq 0$$

E.g., in $t=1$, we have $N(1) = 200$ and $N(0) = 100$. Each young person is taxed $t_1^h(1) = 1$ and the old thus have $t_0^h(1) = -2$ since the government must balance its budget.

From 3.1ⓐ we have the savings function

$$s_t^h = \frac{w_t^h(t) - t_t^h(t)}{2} - \frac{w_t^h(t+1) - t_t^h(t+1)}{2r(t)}$$

$$s_t^h = \frac{2-1}{2} - \frac{1+2}{2r(t)} = \frac{1}{2} - \frac{3}{2r(t)}$$

In equilibrium, aggregate savings are 0:

$$N(t)\left(\frac{1}{2} - \frac{3}{2r(t)}\right) = 0 \iff r(t) = 3 \text{ since } N(t) > 0 \;\; \forall\, t \geq 0.$$

Again from 3.1ⓐ, we have consumption

$$c_t^h(t) = \frac{w_t^h(t) - t_t^h(t)}{2} + \frac{w_t^h(t+1) - t_t^h(t+1)}{2r(t)}$$

$$c_t^h(t) = \frac{2-1}{2} + \frac{1+2}{2 \cdot 3} = \frac{1}{2} + \frac{1}{2} = 1$$

$$c_t^h(t+1) = c_t^h(t)\, r(t) = 1 \cdot 3 = 3$$

For all people in generations $t \geq 1$, the new equilibrium consumption allocation $[1,3]$ is strictly preferred to the equilibrium allocation without the scheme, $[2,1]$ from 2.4a since $u([1,3]) = 3 > u([2,1]) = 2$.
Moreover, the old in ~~genera~~ time $t=1$ receive 2 units more of the $t=1$ good so they also strictly prefer the equilibrium allocation with the social security scheme.

Thus, we conclude the equilibrium with the social security scheme is Pareto superior to the equilibrium without.

Exercise 3.1 d)

In comparison to the equilibrium achieved in 3.1 (c), since population growth eventually ceases in this case, it is not feasible to maintain transfers of 2 to the old indefinitely.

In this case, the old of generations 0-9 would receive transfers of 2, but as population growth stops at t=10, generations t $\geq$ 10 will receive transfers of only 1 in the period in which they're old.

Thus, we would have consumption of [1,2] for generations t $\geq$ 10 and [1,3] for t = 1,2,...,9.

In the United States now, as population growth slows, the current Social Security system is no longer able to collect sufficient money from the working population to maintain the size of social security transfers received by the old. As this trend continues, social security transfers will by necessity become smaller and smaller.

Exercise 3.2 Show that tax schemes $(z_c, z_y) \in \mathbb{R}^2_+$ such that $(1 - z_y)/(1 + z_c) = g$ for some $g \in (0, 1)$ are equivalent.

Solution. Two tax schemes are equivalent if they induce the same consumption and savings decision. With the taxes, each individual in each period face constraints

$$c_t^h(t) \leq (1 - z_y)\omega_t^h(t) - l^h(t) - z_c c_t^h(t),$$

$$c_t^h(t+1) \leq (1 - z_y)(\omega_t^h(t+1) + r(t)l^h(t)) - z_c c_t^h(t+1).$$

Divide the second equation by $r(t)(1 - z_y)$, add it to the first and factorize. This yields lifetime budget constraint

$$c_t^h(t) + \frac{c_t^h(t+1)}{(1 - z_y)R(t)} \leq \frac{1 - z_y}{1 + z_c}(\omega_t^h(t) + \frac{\omega_t^h(t+1)}{(1 - z_y)R(t)}),$$

$$c_t^h(t) + \frac{c_t^h(t+1)}{(1 - z_y)R(t)} \leq g(\omega_t^h(t) + \frac{\omega_t^h(t+1)}{(1 - z_y)R(t)}).$$

By assumption g is constant between all tax schemes. The only channel through which the taxes might affect savings decisions is via the after-tax interest rate $R(t)(1 - z_y)$. However, market-clearing in a competitive equilibrium implies aggregate savings equal to

$$S_t((1 - z_y)R(t)) = 0.$$

The savings function S_t is strictly increasing, therefore the after-tax interest rate needs to be equal for all pairs (z_y, z_c). We conclude that the taxes are equivalent.

3.3

(a) According to the setup, taxes are:

$$t_1^h = \begin{cases} -1/4, & t=0; \\ (1/4, 0), & t=1; \\ (0,0), & \forall t>1. \end{cases}$$

Corresponding consumption and savings are (using results derived in Ex. 3.1a)

$$c_1^h(1) = 1/2 \left(2 - 1/4 + \frac{1-0}{r(1)}\right) = 7/8 + \frac{1}{2r(1)},$$
$$s_1^h(1) = 2 - 1/4 - \left(7/8 + \frac{1}{2r(1)}\right) = 7/8 - \frac{1}{2r(1)}.$$
$$c_1^h(t) = 1/2 \left(2 - 0 + \frac{1-0}{r(t)}\right) = 1 + \frac{1}{2r(t)},$$
$$s_1^h(t) = 2 - \left(1 + \frac{1}{2r(t)}\right) = 1 - \frac{1}{2r(t)}, \forall t > 1.$$

Aggregate Savings are therefore:

$$S_1(r(1)) = 100\left(7/8 - \frac{1}{2r(1)}\right),$$
$$S_1(r(t)) = 100\left(1 - \frac{1}{2r(t)}, \forall t>1.\right.$$

Competitive equilibrium implies:

$$S_1(r(1)) = 0 \iff r(1) = 4/7,$$
$$S_1(r(t)) = 0 \iff r(t) = 1/2, \forall t>1.$$

Thus, optimal consumption allocations are:

$$c_0^h(1) = 1 + 1/4 = 5/4,$$
$$c_1^h(1) = 7/8 + \frac{1}{2}\frac{7}{4} = 7/4,$$
$$c_1^h(2) = 4/7 \; 7/4 = 1$$
$$c_1^h(t) = (2,1), \forall t>1,$$

giving rise to utilities:

$$u_1^h = 7/4 \; 1^1 = 7/4,$$
$$u_t^h = 2 \; 1^1 = 2, \forall t>1.$$

(b) The new scheme is:

$$t_t^h = \begin{cases} -1/4, & t=0; \\ (0,x), & t=1; \\ (0,0), & \forall t>1; \end{cases}$$

with $p(1)B(1) = 25$.

Analogously to (a):

$$c_1^h(1) = \frac{1}{2}\left(2 + \frac{1-x}{r(1)}\right) = 1 + \frac{1-x}{2r(1)},$$
$$s_1^h(1) = 2 - \left(1 + \frac{1-x}{2r(1)}\right) = 1 - \frac{1-x}{2r(1)},$$
$$\Rightarrow$$
$$S_1(r(1)) = 100\left(1 - \frac{1-x}{2r(1)}\right) = 100 - \frac{50(1-x)}{r(1)},$$

Equilibrium condition reads:
(assume $r(t) = 1/p(t)$)

$$S_1(r(1)) = p(1)B(1) \iff r(1) = \frac{2(1-x)}{3}.$$

Furthermore:

$$\sum_h t_1^h(1) + \sum_h t_0^h(1) - B(0) + p(1)B(1) = 0,$$
$$0 - 1/4 \; 100 - 0 + 25 = 0$$
$$0 = 0 \; \checkmark$$

and

$$\sum_h t_2^h(2) + \sum_h t_1^h(2) - B(1) + p(2)B(2) = 0$$
$$0 - 100x - 25r(1) + 0 = 0$$
$$\iff 100 - 25\frac{2(1-x)}{3} = 0$$
$$\iff x = 1/7 \Rightarrow r(1) = 4/7.$$

Thus

$$c_0^h(1) = 1 + 1/4 = 5/4,$$
$$c_1^h(1) = 1 + \frac{1-1/7}{2}\frac{7}{4} = 7/4,$$
$$c_1^h(2) = 4/7 \; 7/4 = 1,$$
$$c_t^h(t) = (2,1), \forall t>1,$$
$$u_1^h = 7/4 \; 1^1 = 7/4,$$
$$u_t^h = 2 \; 1^1 = 2 \; \forall t>1.$$

(c) Taxes change to:

$$t_t^h = \begin{cases} -1/4, & t=0; \\ (0,0), & t=1; \\ (x,0), & t=2; \\ (0,0), & \forall t>1; \end{cases}$$

and $p(1)B(1) = 25$.

$$\Rightarrow c_1^h(1) = 1 + \frac{1}{2r(1)},$$
$$s_1^h(1) = 2 - \left(1 + \frac{1}{2r(1)}\right) = 1 - \frac{1}{2r(1)},$$
$$S_1(r(1)) = 100 - 50/r(1).$$
$$S_1(r(1)) = p(1)B(1)$$
$$\iff r(1) = 2/3$$

$$\sum_h t_1(1) + \sum_h t_0^h(1) - B(0) + p(1)B(1) =$$
$$0 - 1/4 \; 100 - 0 + 25 = 0$$
$$0 = 0 \; \checkmark$$

Summarizing:

$$c_0^h(1) = 1 + 1/4 = 5/4,$$
$$c_1^h(1) = 1 + \frac{1}{2}\frac{3}{2} = 7/4,$$
$$c_1^h(2) = 2/3 \; 7/4 = 7/6,$$
$$c_2^h(2) = 11/12 + \frac{1}{2}\frac{11}{6} = 11/6$$
$$c_2^h(3) = 6/11 \; \frac{11}{6} = 1,$$
$$c_t^h(t) = (2,1), \forall t>2.$$

Moreover,

$$\sum t_2^h(2) + \sum_h t_1^h(2) - B(1) + p(2)B(2) = 0$$
$$100x - 0 - 25r(1) + 0 = 0$$
$$\iff 100x - 25 \; 2/3 = 0$$
$$\iff x = 1/6.$$

It follows that

$$c_2^h(2) = \frac{1}{2}\left(2 - 1/6 + \frac{1-0}{r(2)}\right) = \frac{11}{12} + \frac{1}{2r(2)},$$
$$s_1^h(2) = 2 - 1/6 - \left(11/12 + \frac{1}{2r(2)}\right) = \frac{11}{12} - \frac{1}{2r(1)},$$
$$S_2(r(2)) = 100\left(\frac{11}{12} - \frac{1}{2r(2)}\right) = 0$$
$$\iff r(2) = 6/11.$$

$$\Rightarrow u_1^h = 7/4 \left(7/6\right)^1 = \frac{49}{24} > 7/4$$
$$u_2^h = 11/6 \; 1^1 = 11/6 < 2$$
$$u_t^h = 2 \; 1, \forall t > 2.$$

Individuals $\begin{cases} \text{better off}, & t=1; \\ \text{worse off}, & t=2; \\ \text{indifferent}, & t=0, \forall t>2. \end{cases}$

3.4

(a) Following the derivation in Ex. 3.3, the tax scheme is now:

$$t_1^h = \begin{cases} -1/4 & t = 0; \\ (0,0)', & \forall t > 0. \end{cases}$$

with $p(1)B(1) = 25$.

Corresponding consumption and savings are (using results derived in Ex. 3.3)

$c_1^h(1) = \frac{1}{2}(2 + \frac{1}{r(1)}) = 1 + \frac{1}{2r(1)}$,

$s_1^h(1) = 2 - (1 + \frac{1}{2r(1)}) = 1 - \frac{1}{2r(1)}$,

$\Rightarrow$

$S_1(r(1)) = 100(1 - \frac{1}{2r(1)}) = 100 - \frac{50}{r(1)}$,

Equilibrium condition reads:
(assume $r(t) = 1/\pi(t)$)

$S_1(r(1)) = p(1)B(1) \Leftrightarrow r(1) = \frac{2}{3} \Rightarrow B(1) = 50/3$.

Thus $c_0^h(1) = 1 + 1/4 = 5/4$,
$c_1^h(1) = 1 + \frac{1}{2}\frac{3}{2} = 7/4$,
$c_1^h(2) = \frac{2}{3}\, 7/4 = 7/6$.

Further: $p(2)B(2) = B(1) = 50/3$

$\Rightarrow S_2(r(2)) = p(2)B(2)$

$\Leftrightarrow r(2) = 3/5 \Rightarrow B(2) = 50/3\, r(2) = 10$

$c_2^h(2) = 1 + \frac{1}{2}\frac{5}{3} = 11/6$

$c_2^h(3) = 3/5 \times \frac{11}{6} = 11/10$

Finally: $S_3(r(3)) = p(3)B(3)$

$\Leftrightarrow r(3) = 5/9 \Rightarrow B(3) = 50/9$

$c_3^h(3) = 1 + \frac{1}{2}\frac{9}{5} = 19/10$

$c_3^h(4) = 5/9\; 19/10 = 19/18$

It follows that for $t \to \infty$

$B(t) = 0, \quad c_t^h = (2,1), \quad r(t) = 1/2$

Thus the economy converges to the equilibrium without government.

But since

$c_0^h = 5/4 > 1$,
$u_1^h = 7/4\; 7/6 = 49/24 > 2$,
$u_2^h = 11/6\; 11/10 = 121/60 > 2$,
$u_3^h = 19/10\; 19/18 = 361/180 > 2$,
$\vdots$
$u_\infty^h = 2$,

it is Pareto superior to it.

However, policy

$t = (0.5, -0.5)$ with $c_t^h = (1.5, 1.5)$

makes everyone better off:

$c_0^h(1) = 3/2 > 5/4$

$u_t'^h = 9/4$

$> u_t^h \;;\; t = 1, \dots, +\infty$

Conclude: This equilibrium is not Pareto optimal.