Advanced Macroeconomics

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Welcome to Advanced Macroeconomics!

- Who am I?
- How has the course evolved?
- What are we going to be doing for the next 7 weeks?

Introduction to Overlapping Generations Models

Why OLG Models?

- Realism – people have finite lives
- Framework enables us to study issues related to fact that people grow old and die (e.g. public pension systems, age-consumption profiles)
- Mathematical simplicity

Overview of Course

Advanced Macroeconomics

- Goal
  - Provide rigorous approach with which you can analyze wide range of issues in dynamic or intertemporal macroeconomics
- Example topics
  - Why do most OECD countries have a public pension system?
  - How will population ageing affect the economy?
  - What and when should we tax?
  - Why is the return on equity higher than the return on bonds?
- First course in macro sequence

Overview of Course

Advanced Macroeconomics

- What do we need to tackle these questions?
- Develop a simple model
  - Time, resources, markets, population, preferences
- Extend the model
  - Government, infinitely lived assets, storage technology, production technology
- Read journal articles with applications involving richer model environments

General Information

The Team

- Instructor: Johanna Wallenius
- TA: Alexandre Mendonça

General Information

Office Hours

- Office hours on Zoom
  - Wednesdays 10-11 and by appointment

General Information

Course Website

- Handouts, assignments and solutions will be posted on Canvas
- Assignments will be submitted via Canvas

General Information

Reading

- Textbook: Introduction to Dynamic Macroeconomic Theory – An Overlapping Generations Approach by McCandless and Wallace
- Journal articles listed on syllabus
- Handouts (partly based on material from David Domeij)

General Information

Examination

- Your grade
  - Problem sets: 10%
  - Referee report and presentation: 10%
  - Final exam: 80%
  - Reflective essay: pass/fail

General Information
Problem Sets

- 5 problem sets (last one practice for exam)
  - Can work in groups of 2-3 (turn in one assignment per group)
    - Choose own group, register it on Canvas before 1st assignment!
  - Submit assignments via Canvas
  - Solutions discussed in seminars led by TA

General Information

Referee Report and Presentation

- Each group assigned a paper on which it prepares both a written report and a presentation with slides
  - Groups pre-assigned
- Each paper assigned to two groups
- On day of presentation, one group randomly selected to present paper, other group comments afterwards
- Presentations and discussions on Dec 5th and 7th
- Submit report and slides via Canvas by Dec 4th

- What if use generative AI tools (e.g., ChatGPT)?
  - Must document how tools were used
  - Weight in grading given to components ChatGPT and the like aren't good at
  - Here, mainly see ChatGPT and the like as helping improve language

General Information

Workload

- This course will require you to work hard
  - Putting in the effort will pay off
- First two weeks most intense (lots of notation and theory)
- Time to digest material

Time

- Time, t, is discrete and indexed by integers: $t \in \{-\infty, ...0, ..., +\infty\}$
- $t = 1$ is initial period
- $t \in \{-\infty, ...0\}$ part of economy's history and exogenously given

Introduction to Overlapping Generations Models

Environment

Population

- Overlapping generations models (alternative infinitely lived dynasty models)
- At each time, t, a new generation is born (called generation t)
- There are $N(t)$ members in this generation
- All generations live for two periods: generation t is young in period t and old in period $t+1$
- In period $t+1$, a new generation is born, generation $t+1$
- Generation t *overlaps* with generations $t-1$ and $t+1$

Resources

- In each period t there is only one good available: the t good
- For now, good is perishable (apples, oranges,...)
- In each period, economy endowed with $Y(t)$ units of the time t good
- Sequence $\{Y(t)\}_{t=1}^{\infty}$ denotes all current and future endowments

Introduction to Overlapping Generations Models

Environment

Consumption Allocations

- A consumption allocation describes who consumes what
- Let $c_t^h(s)$ stand for consumption of the time s good by individual h of generation t, where $h \in \{1, .., N(t)\}$
- Let c_t^h denote the ordered pair of consumptions over a lifetime:

$$c_t^h = [\overset{\text{young}}{c_t^h(t)}, \overset{\text{old}}{c_t^h(t+1)}]$$

- A *time t consumption allocation* is given by the set of consumptions of the young $\{c_t^h(t)\}_{h=1}^{N(t)}$ and of the old $\{c_{t-1}^h(t)\}_{h=1}^{N(t-1)}$

Introduction to Overlapping Generations Models

Environment

- A *consumption allocation* is the sequence of consumption allocations for all $t \in \{1, 2, ..., +\infty\}$
- A *feasible consumption allocation* is a consumption allocation that can be achieved given the total resources in the economy
- The *total consumption at time t* is denoted $C(t)$ and is given by

$$C(t) = \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t)$$

For an economy without storage capabilities we have:

Definition

A consumption allocation is feasible if the consumption path satisfies $C(t) \leq Y(t)$ for all $t \geq 1$.

Definition

A feasible consumption allocation is efficient if there is no alternative feasible allocation with more total consumption of some good and no less of any other good.

Don't confuse a feasible allocation with an affordable consumption profile!

Don't confuse efficiency with Pareto optimality! Need to introduce preferences before can talk about Pareto optimality.

- *Symmetric consumption allocations* are those in which all members of all generations consume the same consumption pair:

$$c_t^h(t) = c_s^j(s)$$

need not be

$$c_t^h(t) = c_t^h(t+1)$$

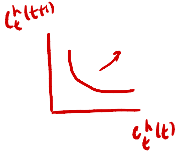
and

$$c_t^h(t+1) = c_s^j(s+1)$$

for any j and h belonging to generation t and $s : t, s \geq 1$

Introduction to Overlapping Generations Models

Preferences

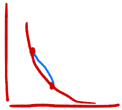


- Utility function for individual h of generation t written as

$$u_t^h = u_t^h\left(c_t^h(t), c_t^h(t+1)\right)$$

For now

- Only arguments are consumption when young and consumption when old (no leisure or consumption of children or other individuals)
- Strictly increasing (prefer more consumption to less)
- Differentiable



- Convex preferences (convex combination of two consumption bundles preferred to at least one of the consumption bundles)

Definition

Consumption allocation A is Pareto Superior to consumption allocation B if (i) no-one strictly prefers B to A, and (ii) at least one person strictly prefers A to B.

- If A is not Pareto superior to B and B is not Pareto superior to A, the two consumption allocations are said to be *noncomparable*

Definition

A consumption allocation is Pareto optimal if it is feasible and if there does not exist a feasible consumption allocation that is Pareto superior to it

- Pareto optimality is a quite weak concept and includes a large number of allocations, only a few of which may be socially desirable

$c_t(t)$ $c_t(t+1)$

Example

Let population be given by $N(t) = 1$ for all t. Suppose resources are given by $Y(t) = 4$ for all t. There is no technology for transforming one good into another. Preferences are common to all and represented by $u_t^h(x, y) = u(x, y) = x \cdot y$. Then the following allocation is feasible: $c_t^h(t) = 3$ 1 and $c_{t-1}^h(t) = 1$ 3 for all t. Notice this involves specifying how much members of generation -1 get to consume as old. Is this allocation Pareto optimal?

No $2 \times 2 = 4$

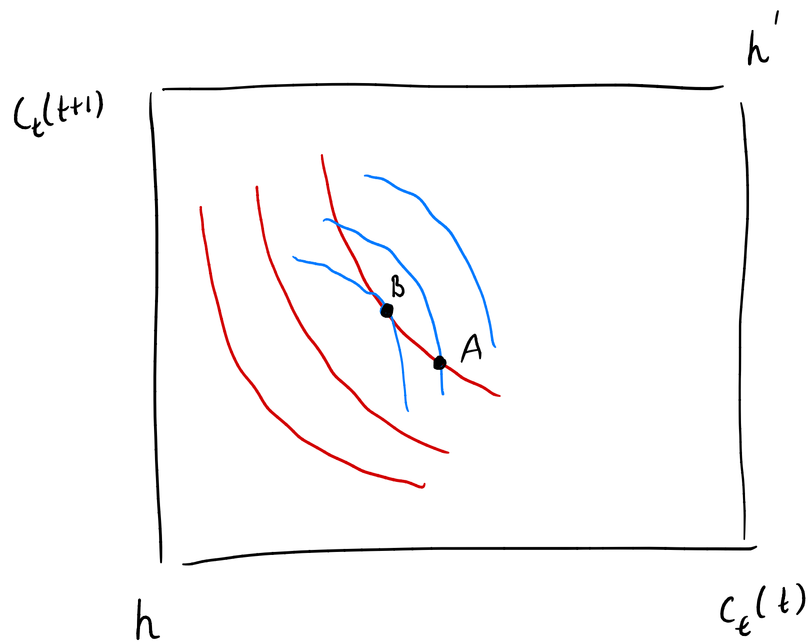
Necessary conditions for Pareto optimality

- Efficiency
- Equality of marginal rate of substitution for all members of a given generation
- MRS defined as

$$MRS = \frac{\frac{\partial u_t^h}{\partial c_t^h(t)}}{\frac{\partial u_t^h}{\partial c_t^h(t+1)}}$$

  The MRS at a given consumption allocation equals minus the slope of the indifference curve at that specific allocation

- MRS needs to be *sufficiently high*



h'
$c_t(t+1)$
B
A
h
$c_t(t)$

Competitive Equilibrium

Introduction

- Macroeconomics predicts outcomes of competitive market economy
- Competitive equilibrium concept designed for economies with private ownership, enforceable contracts and large number of people so that no single one can affect prices
- Competitive equilibrium is allocation mechanism for our model economy

Competitive Equilibrium

Private Ownership

- Private ownership = full right of disposal
- Real world limitations to private ownership
  1. Government taxes
  2. Theft
  3. Social and legal restrictions on disposal of wealth
  4. Limited control (stock owners vs. management)
  5. Common resources (e.g., air, public capital)

Competitive Equilibrium

Endowments

For now, ignore production and assume endowment economy

- In our model economy, each individual has a claim to some quantities of the goods available during his/her lifetime
- Denote endowments of person h of generation t by ordered pair

omega

$$\omega_t^h = \left[\omega_t^h(t), \omega_t^h(t+1)\right]$$

where $\omega_t^h(s) \geq 0$

- Stream of endowments known at birth
- All endowments privately owned (save taxes) and total endowment given by

$$Y(t) = \sum_{h=1}^{N(t)} \omega_t^h(t) + \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t)$$

Competitive Equilibrium

Trade

- Trade is only possible if there is private ownership
- Trade is exchange of goods
- With only one good in each period trade is equivalent to borrowing and lending
  - Lending and borrowing will only occur within generations (why?)
  - Contracts are assumed enforceable by law

Competitive Equilibrium

Lending

- Markets are competitive
  - When individuals interact they view prices as given and unaffected by their actions
- Only price in this economy is gross real interest rate

Competitive Equilibrium

Lending

negative lending is borrowing

- Let lending of individual h in generation t be denoted by $l^h(t)$, and measured in terms of time t goods
- In period $t+1$, individual h receives repayment of $r(t)l^h(t)$ of the time $t+1$ good

  $1+\tilde{r}(t) = r(t)$

- The gross real interest rate is denoted by $r(t)$
  - It is defined in terms of time $t+1$ goods per unit of time t good and known at time t

Competitive Equilibrium

Budget Constraints

- Individuals face the following budget constraints

  when young

$$c_t^h(t) = \omega_t^h(t) - l^h(t) \tag{1}$$

  and when old

$$c_t^h(t+1) = \omega_t^h(t+1) + r(t)l^h(t) \tag{2}$$

$$l^h(t) = \frac{c_t^h(t+1)}{r(t)} - \frac{\omega_t^h(t+1)}{r(t)}$$

- Combining the two budget constraints into a single life-time budget constraint we get

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)} \tag{3}$$

Competitive Equilibrium

Consumption Decisions

- The competitive choice problem is to chose an affordable consumption basket that maximizes utility

$$\max_{[c_t^h(t), c_t^h(t+1)]} u_t^h \left(c_t^h(t), c_t^h(t+1) \right) \tag{4}$$

subject to

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}$$

$$c_t^h(t+1) = r(t)\omega_t^h(t) + \omega_t^h(t+1) - r(t)c_t^h(t)$$

or alternatively, to chose the consumption when young that maximizes

$$\max_{c_t^h(t)} u_t^h \left(c_t^h(t), r(t) \left[\omega_t^h(t) - c_t^h(t) \right] + \omega_t^h(t+1) \right)$$

Competitive Equilibrium

Consumption Decisions

- The first order condition (also called "Intertemporal Euler condition") is

$$\frac{du_t^h}{dc_t^h} = \frac{\partial u_t^h}{\partial c_t^h(t)} - r(t)\frac{\partial u_t^h}{\partial c_t^h(t+1)} = 0 \tag{5}$$

- The utility maximizing choice for $c_t^h(t)$ then satisfies

$$r(t) = \frac{\frac{\partial u_t^h}{\partial c_t^h(t)}}{\frac{\partial u_t^h}{\partial c_t^h(t+1)}} = MRS$$

- Eq. (5) implicitly defines a demand function for consumption when young

$$c_t^h(t) = \chi_t^h\left(r(t), \omega_t^h(t), \omega_t^h(t+1)\right) \tag{6}$$

Competitive Equilibrium

Savings Function

- Define savings of individual h in generation t as

$$\omega_t^h(t) - c_t^h(t)$$

- The savings function can then be written as

$$s_t^h\left(r(t), \omega_t^h(t), \omega_t^h(t+1)\right) = \omega_t^h(t) - \chi_t^h\left(r(t), \omega_t^h(t), \omega_t^h(t+1)\right) \quad (7)$$

natural log

ln

Example

Assume preferences given by $\log c_t^h(t) + \beta \log c_t^h(t+1)$*.*
Derive the savings function.

$$\max \log c_t^h(t) + \beta \log c_t^h(t+1)$$

$$\text{s.t. } c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w_t^h(t) + \frac{w_t^h(t+1)}{r(t)}$$

$$\rightarrow \log c_t^h(t) + \beta \log \left[r(t) w_t^h(t) + w_t^h(t+1) - r(t) c_t^h(t) \right]$$

<u>FOC</u>

$$c_t^h(t): \quad \frac{1}{c_t^h(t)} - \frac{r(t)\beta}{c_t^h(t+1)} = 0 \quad \rightarrow \quad c_t^h(t+1) = \beta r(t) c_t^h(t)$$

Budget constraint: $c_t^h(t) + \beta c_t^h(t) = w_t^h(t) + \dfrac{w_t^h(t+1)}{r(t)}$

$$(1+\beta) c_t^h(t) = w_t^h(t) + \frac{w_t^h(t+1)}{r(t)}$$

$$\rightarrow c_t^h(t) = \frac{w_t^h(t)}{1+\beta} + \frac{w_t^h(t+1)}{r(t)(1+\beta)}$$

$$s_t^h = w_t^h(t) - c_t^h(t)$$
$$= \frac{\beta}{1+\beta} w_t^h(t) - \frac{w_t^h(t+1)}{r(t)(1+\beta)}$$

A competitive equilibrium is a set of prices and quantities that satisfies the following two conditions:

(i) The quantities relevant for each individual maximize that individual's utility in the set of all affordable quantities, given prices and the individual's endowments

(ii) The quantities clear all markets at all dates

Competitive Equilibrium

Finding an Equilibrium

Steps in finding an equilibrium:

1. Impose conditions (i) and (ii) and see what conditions these imply for the sequence of prices and interest rates
2. Search for a sequence of prices and interest rates that fulfil these conditions
3. Check if the resulting quantities generated by the sequence of prices and interest rates indeed fulfil conditions (i) and (ii)

Competitive Equilibrium

Finding an Equilibrium

Step 1a): Condition (ii) $\Rightarrow$ goods market clearing condition

$$\sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) = Y(t) = \sum_{h=1}^{N(t)} \omega_t^h(t) + \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t)$$

No intergenerational trade occurs $\Rightarrow$ aggregate borrowing and lending within a generation must sum to zero:

$$\sum_{h=1}^{N(t)} l^h(t) = 0$$

$\Rightarrow$ Thus aggregate consumption of the young must equal aggregate endowments of the young:

$$\sum_{h=1}^{N(t)} c_t^h(t) = \sum_{h=1}^{N(t)} \omega_t^h(t) - \sum_{h=1}^{N(t)} l^h(t) = \sum_{h=1}^{N(t)} \omega_t^h(t) \tag{8}$$

(and the same for the old)

Competitive Equilibrium

Finding an Equilibrium

Step 1b): Condition (i) $\Rightarrow$ individual consumption given by eq. (6)

Combined with eq. (8), we have the following equilibrium condition:

$$\sum_{h=1}^{N(t)} \chi_t^h \left(r(t), \omega_t^h(t), \omega_t^h(t+1) \right) = \sum_{h=1}^{N(t)} \omega_t^h(t) \tag{9}$$

for all dates $t \geq 1$

One equation with one endogenous variable, $r(t)$

Alternatively, the equilibrium condition can, using the savings function in eq. (1), be defined in terms of aggregate savings

$$S_t\left(r(t)\right) = \sum_{h=1}^{N(t)} \omega_t^h(t) - \sum_{h=1}^{N(t)} \chi_t^h\left(r(t), \omega_t^h(t), \omega_t^h(t+1)\right)$$

or

$$S_t\left(r(t)\right) = 0 \tag{10}$$

for all $t \geq 1$

In an economy with the only assets being private borrowing and lending, the following proposition must hold:

Proposition

If the quantities and the sequence $r(t)$ are a competitive equilibrium, then $\{r(t)\}$ satisfies

$$S_t\left(r(t)\right) = 0$$

for every t.

Competitive Equilibrium

Proposition

Proposition

If $\{r(t)\}$ satisfies Eq. (10) for every t, then there exist quantities such that they and the $r(t)$ sequence are a competitive equilibrium.

Competitive Equilibrium

Pareto Optimality

- In general, a competitive equilibrium does not result in Pareto optimal allocations

Example

Let $u_t^h = \log c_t^h(t) + \beta \log c_t^h(t+1)$ *for all* h, t.
Also, let $\omega_t^h = [100, 2]$ *for all* h, t *and let* $N(t) = 100$ *for all* t.
Let $\beta = 1$.

Solve for the competitive equilibrium.

Competitive Equilibrium
Example

Then, the equilibrium condition reads

$$S_t(r(t)) = \sum_{h=1}^{N(t)} s_t^h(t) = 0$$

or

$$\sum_{h=1}^{N(t)} \frac{\beta \omega_t^h(t)}{1+\beta} - \frac{\omega_t^h(t+1)}{(1+\beta)\, r(t)} = 0$$

or

$$100 \left(\frac{100}{2} - \frac{2}{2r(t)} \right) = 0$$

thus

$$r(t) = \frac{1}{50}$$

Competitive Equilibrium

Example

- Why is the rate of return less than one?
- All individuals would like to smooth consumption, thus save a lot, but no one is willing to borrow
- Thus the rate of return falls until individuals are indifferent between lending and borrowing
- Moreover,

$$s_t^h(t) = 0$$

  and

$$c_t^h = \omega_t^h$$

- Clearly this allocation is not Pareto optimal
- There seems to be scope for improvements – Is there a role for a government?

Why Might Competitive Equilibria NOT Be Pareto Optimal?

- Used to thinking of competitive equilibria as Pareto optimal
- 'Double infinity' key to understanding why they need not be in OLG model
  - Infinite number of agents
  - Infinite number of dated goods

Competitive Equilibrium

Why Might Competitive Equilibria NOT Be Pareto Optimal?

- To gain intuition, consider extreme example
  - Agents receive endowments in both periods of life, but only care about consumption when old
  - Agents would like to trade endowment when young for goods when old, but no one to trade with
  - Need low interest rate, $r = 0$
  - Confiscating, in perpetuity, a lump-sum amount τ from young and giving it to old improves every generation's situation

Competitive Equilibrium

Why Might Competitive Equilibria NOT Be Pareto Optimal?

- More generally, in economy with population growth, as long as interest rate is below population growth this type of sequence of transfers from young to old is Pareto improving
- Note that welfare improving transfer must run from young to old (otherwise hurt initial old) and be perpetual

Competitive Equilibrium

Why Might Competitive Equilibria NOT Be Pareto Optimal?

- Oddity of OLG model?
  - Lots of theoretical work around strange welfare properties and other special features
  - But can we really argue against realism of a model that rests on principle that live in a world where new generations always coming along?
  - Makes models where first welfare theorem always holds look like special case?

$$u_t^h = c_t^h(t) \cdot c_t^h(t+1)$$

Rich are endowed with 3 units when young and 1 unit when old.

Poor are endowed with 2 units when young and 1 unit when old.

Half are rich, half are poor.

$$\max \quad c_t^h(t)\cdot c_t^h(t+1)$$

$$\text{s.t.} \quad c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w_t^h(t) + \frac{w_t^h(t+1)}{r(t)}$$

$$\rightarrow c_t^h(t+1) = r(t)w_t^h(t) + w_t^h(t+1) - r(t)c_t^h(t)$$

FOC

$$c_t^h(t): \quad c_t^h(t+1) - r(t)c_t^h(t) = 0 \rightarrow c_t^h(t+1) = r(t)c_t^h(t)$$

$$\rightarrow \cancel{2}\,c_t^h(t) = \frac{w_t^h(t)}{2} + \frac{w_t^h(t+1)}{2r(t)}$$

$$s_t^h = w_t^h(t) - c_t^h(t)$$

$$= \frac{w_t^h(t)}{2} - \frac{w_t^h(t+1)}{2r(t)}$$

$$\left.\begin{aligned} \text{poor}: &\ 1 - \frac{1}{2r(t)} \\ \text{rich}: &\ \frac{3}{2} - \frac{1}{2r(t)} \end{aligned}\right\} \quad \frac{1}{2}\left(1 - \frac{1}{2r(t)}\right) + \frac{1}{2}\left(\frac{3}{2} - \frac{1}{2r(t)}\right) = 0$$

$$\rightarrow r(t) = 2/5$$

$$\rightarrow \text{poor}: \ 1 - \frac{1}{2 \cdot \frac{2}{5}} = -\frac{1}{4}$$

poor borrow $\frac{1}{4}$, pay back

$$\frac{1}{4} \cdot \frac{2}{5} = \frac{1}{10}$$

$$\text{rich}: \ \frac{3}{2} - \frac{1}{2 \cdot \frac{2}{5}} = \frac{1}{4}$$

$$\left[\frac{9}{4}, \frac{9}{10}\right]$$

$$\left[\frac{11}{4}, \frac{11}{10}\right]$$ smoother consumption profile

$$\frac{9}{4} \cdot \frac{9}{10} > 2 \cdot 1$$

$$\frac{11}{4} \cdot \frac{11}{10} > 3 \cdot 1$$

utility comparison relative to consuming endowments