Introducing a Government

Why?

- Our economy so far suffers from an important problem
- Competitive equilibria are not necessarily Pareto optimal
- Can a government improve things?
- Turns out the answer is yes (of course it can make things worse too)

Introducing a Government

What does the government do?

Government

- Levies taxes
- Redistributes transfers
- Issues debt

Introducing a Government

What is special about a government?

Government can be viewed as an infinitely lived institution

Introducing a Government

Time Consistency

- Impose one strong assumption, namely *time consistency*:
  A policy at time $t + k$, that seemed optimal at time t, must be optimal to carry out when period $t + k$ arrives
- Time consistency puts strong restrictions on future plans of the government

Introducing a Government

How do we model taxes?

- A simple tax structure: lump-sum tax on endowments (transfer = negative tax)
- Tax structure commonly known at all dates
- Individual h of generation t will face the following taxes over his/her lifetime

$$t_t^h = \left[t_t^h(t), t_t^h(t+1) \right]$$

- With no government consumption, the budget constraint of the government equals

$$\sum_{h=1}^{N(t)} t_t^h(t) + \sum_{h=1}^{N(t-1)} t_{t-1}^h(t) = 0 \qquad (1)$$

Introducing a Government

Budget Constraints with Taxes

If pre-tax endowments for individual h of generation t are

$$\omega_t^h = \left[\omega_t^h(t), \omega_t^h(t+1)\right]$$

then post-tax endowments are

$$\omega_t^h = \left[\omega_t^h(t) - t_t^h(t), \omega_t^h(t+1) - t_t^h(t+1)\right]$$

The budget constraint when young is

$$c_t^h(t) \overset{=}{\leq} \omega_t^h(t) - t_t^h(t) - l^h(t) \tag{2}$$

and when old is

$$c_t^h(t+1) \overset{=}{\leq} \omega_t^h(t+1) - t_t^h(t+1) + r(t) l^h(t) \tag{3}$$

$\ell^h(t) = \ldots$

which implies a life-time budget constraint

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} \overset{=}{\leq} \omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)} \tag{4}$$

Introducing a Government

Consumption Decisions with Taxes

The competitive choice problem is to chose an affordable consumption basket that maximizes utility:

$$\max_{c_t^h(t)} u_t^h \left(c_t^h(t), r(t) \left[\omega_t^h(t) - t_t^h(t) - c_t^h(t) \right] + \omega_t^h(t+1) - t_t^h(t+1) \right)$$

The first order condition is still

$$\frac{du_t^h}{dc_t^h} = \frac{\partial u_t^h}{\partial c_t^h(t)} - r(t) \frac{\partial u_t^h}{\partial c_t^h(t+1)} = 0 \tag{5}$$

so the utility maximizing choice for $c_t^h(t)$ still satisfies

$$r(t) = \frac{\frac{\partial u_t^h}{\partial c_t^h(t)}}{\frac{\partial u_t^h}{\partial c_t^h(t+1)}} = MRS$$

Introducing a Government

Consumption Decisions with Taxes

Demand function for consumption when young can be written as

$$c_t^h(t) = \chi_t^h \left(r(t), \omega_t^h(t) - t_t^h(t), \omega_t^h(t+1) - t_t^h(t+1) \right) \quad (6)$$

and the savings function can then be written as

$$\begin{aligned} s_t^h\left(r(t)\right) &= \omega_t^h(t) - t_t^h(t) - \\ &\quad \chi_t^h \left(r(t), \omega_t^h(t) - t_t^h(t), \omega_t^h(t+1) - t_t^h(t+1) \right) \end{aligned} \quad (7)$$

Aggregate savings will thus equal

$$S_t\left(r(t)\right) = \sum_{h=1}^{N(t)} s_t^h \left(r(t), \omega_t^h(t) - t_t^h(t), \omega_t^h(t+1) - t_t^h(t+1) \right)$$

Introducing a Government

Aggregate Savings with Taxes

Since there is no intergenerational borrowing and lending, aggregate savings of the young must still equal zero in equilibrium

Introducing a Government

Pareto Optimality

In general, one can show that if a competitive equilibrium without a government is not Pareto optimal, then there exist a government tax-transfer policy that produces a Pareto superior allocation

Introducing a Government

Example with Taxes

Take the example economy from Handout 1:

Example

Let $u_t^h = \log c_t^h(t) + \beta \log c_t^h(t+1)$ *for all* h, t.
Also, let $\omega_t^h = [100, 2]$ *for all* h, t *and let* $N(t) = 100$ *for all* t.
Let $\beta = 1$.

Suppose the government introduces a tax and transfer scheme that taxes the young 49 units and gives it to the old. Show that this is welfare improving.

$$\log c_t^h(t) + \beta \log c_t^h(t+1)$$

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w_t^h(t) - t_t^h(t) + \frac{w_t^h(t+1) - t_t^h(t+1)}{r(t)}$$

$$\rightarrow c_t^h(t+1) = r(t) w_t^h(t) - r(t)\, t_t^h(t) + w_t^h(t+1) - t_t^h(t+1) - r(t)\, c_t^h(t)$$

$$c_t^h(t): \quad \frac{1}{c_t^h(t)} - r(t)\frac{\beta}{c_t^h(t+1)} = 0 \rightarrow c_t^h(t+1) = \beta r(t)\, c_t^h(t)$$

Budget constraint: $c_t^h(t)\cancel{(1+\beta)} = \frac{w_t^h(t) - t_t^h(t)}{1+\beta} + \frac{w_t^h(t+1) - t_t^h(t+1)}{r(t)(1+\beta)}$

$$s_t^h = w_t^h(t) - t_t^h(t) - c_t^h(t)$$

$$= \frac{\beta}{1+\beta}\left(w_t^h(t) - t_t^h(t)\right) - \frac{\left(w_t^h(t+1) - t_t^h(t+1)\right)}{(1+\beta)\, r(t)}$$

$$\rightarrow \frac{(100-49)}{2} - \frac{(2+49)}{2r(t)} = 0 \rightarrow r(t) = 1$$

individual and aggregate savings zero here

$$c_t^h(t) = c_t^h(t+1) = 51$$

$$\log 51 + \log 51 > \log 100 + \log 2$$

Introducing a Government

Example with Optimal Transfer

Suppose government wants to implement transfer that results in Pareto optimal, symmetric allocations (i.e., look for smallest, symmetric tax/transfer).
Assume identical individuals and constant population.

Example

Let $u_t^h = c_t^h(t)c_t^h(t+1)$ *for all* h, t.
Also, let $\omega_t^h = [3, 1]$ *for all* h, t.

What if $\omega_t^h = [1, 3]$ instead?

Everyone is identical → no borrowing/lending

Let t^* denote optimal tax/transfer policy

$$c_t^h(t) = \omega_t^h(t) - t^*$$

$$c_t^h(t+1) = \omega_t^h(t+1) + t^*$$

(tax young ⟷ t^* positive

tax old ⟷ t^* negative)

$$u_t^h = (\omega_t^h(t) - t^*)(\omega_t^h(t+1) + t^*)$$

When endowments are [3, 1]

$$\rightarrow 3 + 2t^* - t^{*2}$$

$$t^*: \quad 2 - 2t^* = 0 \quad \rightarrow t^* = 1$$

What if endowments are [1, 3] ?

$\rightarrow t^* = -1$ But harm initial old !

Introducing a Government

Government Bonds

- Assume government issues *one period bonds*
  - A certain claim to *one* unit of the consumption good next period
- Assume government always honors its debt
- As before, only young generation is interested in lending, i.e., purchasing bonds

Introducing a Government

Government Bonds

- Suppose government issues $B(t)$ units of bonds in period t
- There are four ways government can finance repayment of debt in period $t+1$
    1. Tax the young of generation $t+1$ a total of $B(t)$ units of the good
    2. Tax the old of generation t a total of $B(t)$ units of the good
    3. Issue $B(t+1)$ units of new bonds that raise a total of $B(t)$ units of time $t+1$ good
    4. Some mix of 1-3 that adds up to $B(t)$ units of the good

Introducing a Government

Government Budget

(no government purchases/ consumption)

- Time t budget constraint of the government is

$$\sum_{h=1}^{N(t)} t_t^h(t) + \sum_{h=1}^{N(t-1)} t_{t-1}^h(t) + p(t)B(t) - B(t-1) = 0 \quad (8)$$

  where $p(t)$ is price of one government bond at time t

- Government's budget is said to be balanced when

$$\sum_{h=1}^{N(t)} t_t^h(t) + \sum_{h=1}^{N(t-1)} t_{t-1}^h(t) = 0$$

  Government runs a deficit (surplus) when the left hand side is negative (positive)

Introducing a Government

No Arbitrage Condition

Claim:
One period bonds have the same risk-characteristics as private lending within generations, and they must therefore yield the same return, $r(t)$

Introducing a Government

No Arbitrage Condition

Proof:
Let $b^h(t)$ denote individual bond holdings
Then the budget constraints of the individuals are

$$c_t^h(t) = \omega_t^h(t) - t_t^h(t) - l^h(t) - p(t)b^h(t)$$

$$c_t^h(t+1) = \omega_t^h(t+1) - t_t^h(t+1) + r(t)l^h(t) + b^h(t)$$

Consequently the life-time budget constraint is $l^h(t) = \ldots$

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = \omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)} - b^h(t)\left[p(t) - \frac{1}{r(t)}\right] \quad (9)$$

Introducing a Government

No Arbitrage Condition

This implies that the individual's demand for bonds equals

$$b^h(t) = \begin{cases} 0 & \text{if } p(t) > 1/r(t) \\ \infty & \text{if } p(t) < 1/r(t) \\ ? & \text{if } p(t) = 1/r(t) \end{cases}$$

Since neither $b^h(t) = 0$ or $b^h(t) = \infty$ can be an equilibrium with positive bond holdings

$$p(t) = 1/r(t) \text{ or } r(t) = 1/p(t) \tag{10}$$

must be an equilibrium condition

Introducing a Government

Indeterminacy

Can't write out individual demand functions for particular assets, only demand function for net asset positions

- Same return $\Longrightarrow$ individual indifferent between two types of savings, only *net position* matters (*indeterminacy*)

Competitive Equilibrium with Government

Equilibrium Condition

Sum budget constraints of the young

$$\sum_{h=1}^{N(t)} c_t^h(t) = \sum_{h=1}^{N(t)} \omega_t^h(t) - \sum_{h=1}^{N(t)} t_t^h(t) - \sum_{h=1}^{N(t)} l^h(t) - p(t) \sum_{h=1}^{N(t)} b^h(t) \tag{11}$$

Given $\sum_{h=1}^{N(t)} l^h(t) = 0$ and $\sum_{h=1}^{N(t)} b^h(t) = B(t)$, we have aggregate savings equal

$$S_t\left(r(t)\right) = \sum_{h=1}^{N(t)} s_t^h(r(t)) = \sum_{h=1}^{N(t)} \left(\omega_t^h(t) - t_t^h(t) - c_t^h(t)\right) = p(t)B(t) \tag{12}$$

Competitive Equilibrium with Government

Equilibrium Condition

Summing the budget constraints of the old (keeping in mind that all debt in period $t-1$ was bought by generation $t-1$)

$$\sum_{h=1}^{N(t-1)} c_{t-1}^h(t) = \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t) - \sum_{h=1}^{N(t-1)} t_{t-1}^h(t)$$
$$+r(t-1)\sum_{h=1}^{N(t-1)} l^h(t-1) + \sum_{h=1}^{N(t-1)} b^h(t-1) \qquad (13)$$
$$= \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t) - \sum_{h=1}^{N(t-1)} t_{t-1}^h(t) + B(t-1)$$

where $\sum_{h=1}^{N(t-1)} l^h(t-1) = 0$ and $\sum_{h=1}^{N(t-1)} b^h(t-1) = B(t-1)$

Summing consumption of the young and of the old (eq. (11) and eq. (13)) we have

$$
\begin{aligned}
\sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) &= \sum_{h=1}^{N(t)} \omega_t^h(t) - \sum_{h=1}^{N(t)} t_t^h(t) \\
&+ \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t) - \sum_{h=1}^{N(t-1)} t_{t-1}^h(t) \\
&- p(t)B(t) + B(t-1)
\end{aligned}
$$

Competitive Equilibrium with Government

Equilibrium Condition

Imposing the government budget constraint gives

$$\sum_{h=1}^{N(t)} t_t^h(t) + \sum_{h=1}^{N(t-1)} t_{t-1}^h(t) + p(t)B(t) - B(t-1) = 0$$

which guarantees markets clearing in the goods market

$$\sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) = \sum_{h=1}^{N(t)} \omega_t^h(t) + \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t)$$

Competitive Equilibrium with Government

Equilibrium Condition

- To sum up: the equilibrium condition on savings is simply

$$S_t\left(r(t)\right) = \frac{B(t)}{r(t)} \tag{14}$$

where we have exploited the no-arbitrage condition on the price of bonds, i.e., $p(t) = 1/r(t)$ (eq. (10))

Example

Let $u_t^h = \log c_t^h(t) + \beta \log c_t^h(t+1)$ *for all* h, t.
Also, let $\omega_t^h = [2, 1]$ *for* h *odd and* $\omega_t^h = [1, 1]$ *for* h *even for all* t*, and let* $N(t) = 100$ *for all* t.
Let $\beta = 1$.

Suppose the government in period 1 wishes to borrow 5 units of the time t good and transfer it to the old from generation 0. It will pay off the debt by taxing the young of generation 2. Solve for the equilibrium.

$$S_t^h = \frac{W_t^h(t)}{1+\beta} - \frac{W_t^h(t+1)}{(1+\beta)\,r(t)} \qquad \text{(same as before)}$$

$$s_t^h = \begin{cases} 1 - \dfrac{1}{2r(t)}, & h \text{ odd} \\[2ex] \dfrac{1}{2} - \dfrac{1}{2r(t)}, & h \text{ even} \end{cases}$$

$$S_t(t) = 50\left[1 - \frac{1}{2r(t)}\right] + 50\left[\frac{1}{2} - \frac{1}{2r(t)}\right]$$

$$= 75 - \frac{50}{r(t)} = 5 \quad \text{in } t=1$$

$$\rightarrow r(1) = \frac{5}{7}$$

$$s_1^h(1) = \begin{cases} 1 - \dfrac{1}{2 \cdot 5/7} = 0.3 & , \text{odd} \\ \dfrac{1}{2} - \dfrac{1}{2 \cdot 5/7} = -0.2 & , \text{even} \end{cases}$$

odd have additional savings of 0.1 units each

→ 5 in total, which corresponds to government bonds

payoff scheme doesn't affect decisions of generation 1

tax young of generation 2

Total taxes

$\frac{B(1)}{r(1)} = 5$ and $r(1) = 5/7 \rightarrow B(1) = 3\ 4/7$

Tax young of generation 2 $3\ 4/7$ to payoff bonds

Competitive Equilibrium with Government

Ricardian Equivalencet

Proposition – Ricardian Equivalence

Fix an "initial" government (tax and debt) policy and consider an associated "initial" competitive equilibrium. Now consider a new tax policy that preserves the initial policy's present value (at the initial equilibrium interest rates) of each individual's tax liability. This policy is equivalent to the initial one in the following sense. There exists a debt policy such that the initial allocation and interest rates, together with the new tax and debt policy are an equilibrium.

Proof – Ricardian Equivalence

For proof see McCandless and Wallace

Government wishes to make some purchases.

Can tax now or or borrow now and tax later.

Won't change prices or consumption pattern.

→ True as long as change from tax now to borrow now and tax later doesn't change present value of anyone's income.

Competitive Equilibrium with Government

Example of Ricardian Equivalence

Example

Consider two government policies.

(1) Government imposes taxes equal to 0.1 unit of time 1 good on everyone in generation 1.

(2) Government borrows 0.1 unit of time 1 good from every member of generation 1 and taxes each member of generation 1 at time 2 whatever is needed to pay off bonds.

Assume all members of generation 1 identical.

Competitive Equilibrium with Government

Example of Ricardian Equivalence

$$c_1^h(1) = \omega_1^h(1) - t_1^h(1) - b^h(1)$$
$$c_1^h(2) = \omega_1^h(2) - t_1^h(2) + r(1)b^h(1)$$

When tax of 0.1 unit of time 1 good is collected:
$[c_1^h(1), c_1^h(2)] = [\omega_1^h(1) - 0.1, \omega_1^h(2)]$

When government borrows of 0.1 unit of time 1 good and taxes at time 2:
$t_1^h(2) = r(1)b^h(1) = r(1)0.1$

Present value of tax liability for person h the same

Competitive Equilibrium with Government

Example of Ricardian Equivalence

Consumption at time 1 reduced by 0.1 because buy bond.
Consumption at time 2 unchanged because taxes returned in form of payment on bond,

Consumption pattern with government bond:
$[c_1^h(1), c_1^h(2)] = [\omega_1^h(1) - 0.1, \omega_1^h(2)]$

Competitive Equilibrium with Government

Ricardian Equivalence

- Is it possible to transfer taxes over generations of an OLG economy, say from "current old" to "future young", without affecting, in some sense, their choices?
- In general, answer is NO
- Need way of connecting generations so they behave as if they are infinitely lived $\rightarrow$ altruism

Competitive Equilibrium with Government

Rolling Over Government Debt

- Rolling over debt = financing payments of outstanding debt by issuing new bonds
- At t the equilibrium condition is

$$S_t(r(t)) = \frac{B(t)}{r(t)}$$

- At $t+1$ the equilibrium condition, if debt is rolled over, is

$$S_{t+1}(r(t+1)) = \frac{B(t+1)}{r(t+1)} = B(t)$$

At $t+2$, the equilibrium condition, if debt is rolled over again, is

$$S_{t+2}(r(t+2)) = \frac{B(t+2)}{r(t+2)} = B(t+1) = r(t+1)B(t)$$

If debt is rolled over forever, the amount of bonds issued in period $t+j$ equals

$$\begin{aligned} B(t+j) &= r(t+j)B(t+j-1) \\ &= r(t+j)r(t+j-1)B(t+j-2) \\ &= r(t+j)r(t+j-1)r(t+j-2)B(t+j-3) \\ &= r(t+j)r(t+j-1).....r(t+1)B(t) \end{aligned}$$

or

$$B(t+j) = B(t)\Pi_{k=1}^{j} r(t+k)$$

Competitive Equilibrium with Government

Rolling Over Government Debt

Consider three different cases:

1. **Case 1**: $r(t+k) = 1$ for all k

$$B(t+j) = B(t)$$

   Amount of issued debt constant (stationary equilibrium)

2. **Case 2:** $r(t+k) = \bar{r} < 1$ for all k

$$B(t+j) = B(t)\Pi_{k=1}^{j}\bar{r} = B(t)\bar{r}^{j}$$

   As time moves on we have

$$\lim_{j \longrightarrow \infty} B(t+j) = \lim_{j \longrightarrow \infty} B(t)\bar{r}^{j} = 0$$

   Debt goes to zero (another stationary equilibrium)

Competitive Equilibrium with Government

Rolling Over Government Debt

3. **Case 3:** $r(t+k) = \bar{r} > 1$ for all k

$$B(t+j) = B(t)\Pi_{k=1}^{j}\bar{r} = B(t)\bar{r}^{j}$$

As time moves on we have

$$\lim_{j \longrightarrow \infty} B(t+j) = \lim_{j \longrightarrow \infty} B(t)\bar{r}^{j} = \infty$$

Debt goes to infinity

Can't be an equilibrium, since eventually required financing will exceed aggregate endowment of the young

Example

Let $u_t^h = c_t^h(t)c_t^h(t+1)$ *for all* h, t.
Also, let $\omega_t^h = [2, 1]$ *for all* h, t *and* $N(t) = 100$ *for all* t.

Suppose government wants to raise 50 units of $t = 1$ good (and give to current old). What if government instead wants to raise 49 units? Or 51 units?

$$u_t^h = c_t^h(t) \cdot c_t^h(t+1)$$

$$\rightarrow S_t = \frac{\omega_t^h(t)}{2} - \frac{\omega_t^h(t+1)}{2r(t)}$$

endowment $[2, 1]$

$$\rightarrow S_t = 1 - \frac{1}{2r(t)}$$

$$S(t) = 100 - \frac{50}{r(t)}$$

Government wants to raise 50 units in $t=1$

$$\rightarrow S(1) = 100 - \frac{50}{r(1)} = 50 \rightarrow r(1) = 1$$

$$\rightarrow B(1) = 50$$

$$S(2) = \frac{B(2)}{r(2)} = 50 \rightarrow \text{Borrow } 50 \text{ every period}$$

every individual saving 0.5

Pareto superior to borrowing smaller amount

consumption $[1.5, 1.5]$

Suppose government instead wanted to raise 49 units in $t=1$

$$S(1) = 100 - \frac{50}{r(1)} = 49 \quad \rightarrow r(1) \sim 0.98$$

$$\rightarrow B(1) = 49 \cdot 0.98 \sim 48$$

$$S(2) = 100 - \frac{50}{r(2)} = 48 \quad \rightarrow \quad r(2) \sim 0.96 \quad \ldots$$

$\rightarrow$ amounts get smaller and smaller

If try to borrow more than 50 units, borrowing gets bigger every period $\rightarrow$ infeasible