Handout 3: Indifference Curves, Budget Constraints, and Constrained Optimization

1 Introduction

We will focus in this handout on solving the standard problem of consumer choice. A consumer has a utility function over two goods, which relates quantities it consumes of each good to a level of well-being of each basket. Even though the consumers likes both goods, they cannot consume unlimited amounts. His consumption choice must be within his *budget constraint*, which is the set of all consumption baskets the consumer can afford given the prices of each good and his income. We will determine what is the optimal choice for the consumer that satisfies his budget constraint. We will separate this handout in four parts. First, we will explore the budget constraint and how price/income changes change the set of available baskets for a consumer. Second, we will explore the utility function and indifference curves. Third, we will put these two elements together and solve the consumer choice problem of utility maximization subject to the budget constraint. Finally, we will solve a consumer choice example that uses all of the theory developed and discusses briefly income and substitution effects. Through this handout, we will focus on a world with two goods. The principles developed here can be easily extended to a world with more than two goods (even infinite!).

The consumer choice problem is the first example of constrained choice, which is the basis of modern micro and macroeconomics. Consumers, workers, firms and even governments must make decisions under constraints and, as the class advances, we will always be referring back and applying the concepts seen in this handout to solve more advanced problems.

2 Budget Constraint

Consider a world with two goods, X and Y. Given prices for each good $\{P_X, P_Y\}$ and an income level I, the budget constraint shows the bundles of goods X, Y that can be afforded, that is:

$$p_X X + p_Y Y \leq I$$

The value $p_X X$ is the total amount spent on good X: price per unit times the number of units of good X. The total amount spent on both goods must be less or equal than the income I. In particular, one can always spend less than our total income. The set of points that spends exactly our income is the **budget constraint line**, given by:

$$p_X X + p_Y Y = I$$

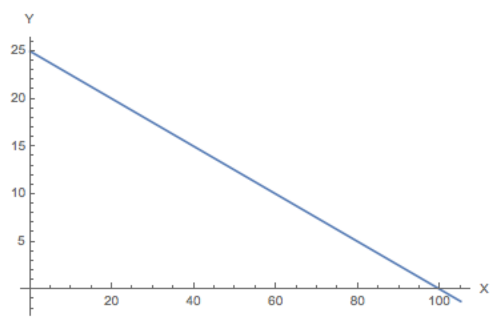
Example. Assume that $p_X = 1, p_Y = 4$ and $I = 100$. The budget line is given by

$$1X + 4Y = 100$$

or:

$$Y = \frac{100 - X}{4}$$
$$Y = 25 - \frac{X}{4}$$

Graphically:



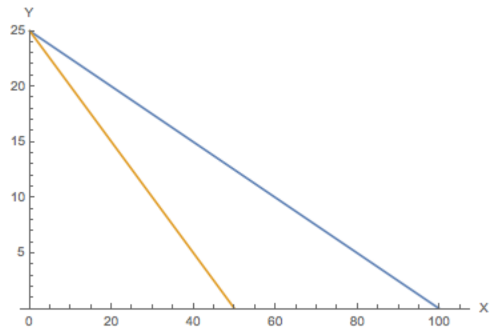
The consumer can purchase any bundle on or below the budget line. If he buys $X = 100$, he can afford 0 Y. If he buys $Y = 25$, he can afford 0 X. Plotting this line is usually easiest by finding the X- and Y-intercepts and then connecting the line. In general, the utility will be strictly increasing in all goods. This means that in general there is no loss of generality of using the budget constraint with an equality, i.e., $1X + 4Y = 100$. We focus now on what happens with the budget constraint if prices or income changes.

2.1 Price Changes

What happens when the price changes? Suppose P_X goes from \$1 to \$2: If we only buy X, the number we can afford goes from $X = 100$ to $X = 50$. If we only buy Y, the price is unchanged. The new budget constraint is given by:

$$2X + 4Y = 100 \Rightarrow Y = 25 - \frac{X}{2}$$

Therefore, we can see the budget constraint **tilts** when the price of 1 good changes.

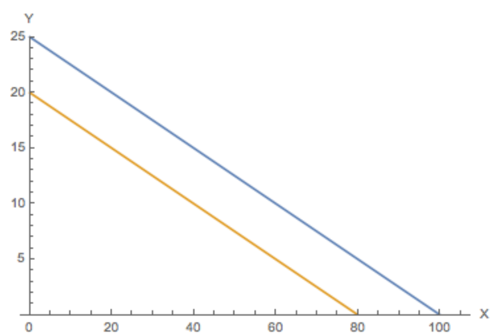


2.2 Income Changes

What if instead of having \$100 , the consumer has \$80 ? The new budget constraint is given by:

$$X + 4Y = 80 \Rightarrow Y = 20 - \frac{X}{4}$$

Our budget line moves inward, at the same slope $(-1/4)$. Mathematically, this makes sense because the intercept of the budget line is I/P_Y, which has decreased, but the slope given by the price ratio is unchanged.



2.3 General Case

More generally, we can determine the slope and the intercepts of the budget constraint symbolically In a $X - Y$ diagram, the budget constraint is a downward sloping straight line

$$P_X X + P_Y Y = I \Rightarrow Y = \frac{I}{P_Y} - \frac{P_X}{P_Y} X$$

The intercept of the line in the x-axis - when $Y = 0$ - is given by

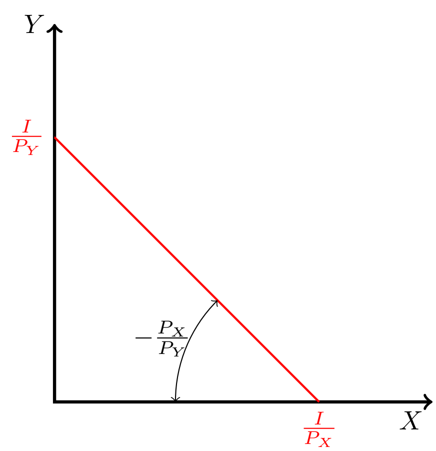
$$X = \frac{I}{P_X}$$

This represents the units of good X that the consumer can buy if she spends all of her income on good X. For instance, for an income of \$10 and $P_X = 2$, the intercept will be 5. The analogous holds for Y. The intercept on the y-axis, when $X = 0$ is given by $Y = \frac{I}{P_Y}$. The slope is given by $-P_X/P_Y$. The slope represents the fact that if the consumer reduces his consumption of good X by Δ units, it can increase the consumption of good Y by $\Delta P_X/P_Y$ units.

$$P_X X + P_Y Y = I \Rightarrow P_X (X - \Delta) + P_Y \left(Y + \frac{P_X}{P_Y}\Delta \right) = I$$

The *relative* price, P_X/P_Y captures this trade-off between X and Y in the budget constraint. We plot the budget constraint in Figure 1.

Figure 1: Budget Constraint



This concludes our discussion on the budget constraint. We now move on to the discussion of the utility function and indifference curves.

3 Indifference Curves and MRS

3.1 Indifference Curves

The utility function captures how much the agent likes baskets of goods X, Y. It is a function $U(X, Y)$ that relates the quantities consumed of each good to a number. The larger the number, the happier the agent is. An **indifference curve** is a set of points that give a consumer the same amount of utility. The idea of the consumer choice problem will be to find the "largest" indifference curve that is still within the budget constraint we explored in the last section. Formally, indifference curves are **level sets** of the utility function. For instance, if the utility of the consumer is given by:

$$U(X, Y) = X \times Y$$

the consumer is indifferent between $X = 1$ and $Y = 4$ and $X = 4$ and $Y = 1$, since $XY = 4$ in either case. The consumer is also indifferent between $X = 1$ and $Y = 4$, and $X = .5$ and $Y = 8$. In fact, the consumer is indifferent between $X = 1$ and $Y = 4$ and any combination of X, Y such that

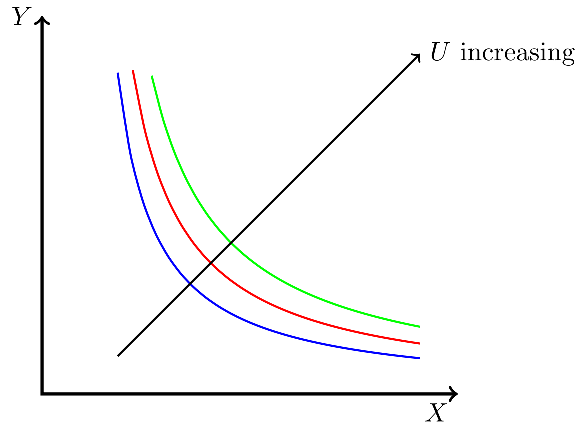
$$XY = 4$$

The set of all points where $XY = 4$ defines the indifference curve of at level 4 of utility. More generally, for any level v of utility, the indifference curve at this level is given by $XY = v$. Or, isolating Y:

$$Y = \frac{v}{X} \tag{1}$$

For larger values of v, the indifference curves shift northeast, as in Figure 2.

Figure 2: Indifference Curves

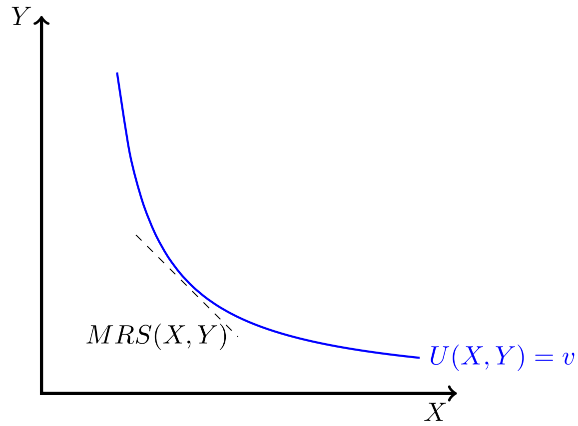


The indifference curve is this case decreasing in the $X - Y$ diagram. This means that we must compensate the consumer by *increasing* X if we are *decreasing* Y. The indifference curve is this case is also *convex*. This means that the agent has a taste for variety. For large values of X, a decrease of one unit of Y must be compensated by increasing X by large amounts (while the opposite is true for X small). Overall, there are many types of indifference curves, because there are many types of utility functions! You can always find indifference curves given a utility function by picking points that return the same value. The hard part can be going the other way: starting with an idea about your indifference curve/tradeoffs, and finding the right utility function.

3.2 Marginal Rate of Substitution

The *Marginal Rate of Subsitution*(MRS) represents the *slope* of the indifference curve. As this slope will be generally negative, we define the MRS as the absolute value of the slope of an indifference curve. Intuitively, it represents the rate at which the consumer is willing to trade off one good for another. This is represented in Figure 3.

Figure 3: MRS



Since we know the equation for the indifference curve at any level v for $U(X,Y) = XY$ - Eq. (1) - we can compute the MRS simply by taking the derivative of Y w.r.t X:

$$MRS = \frac{dY}{dX} = -\frac{v}{X^2} \tag{2}$$

Note that the MRS can always be written in terms of X and Y, since v can be written in terms of X and Y. In our example: $v = XY$:

$$MRS = -\frac{v}{X^2} = -\frac{Y}{X} \tag{3}$$

the MRS changes depending on the point X, Y we evaluate it. We denote the MRS then as a function of $MRS(X,Y)$ in the rest of this handout.

General Case. We derive now the MRS equation for a general utility function. This requires the knowledge of differentiation. If you are not comfortable with taking derivatives, you can skip this section and intuitively understand the formula for the MRS.

For a general utility function, take a *total differential*:

$$U(X,Y) = U(X,Y) \Rightarrow dU = MU_X(X,Y)dX + MU_Y(X,Y)dY$$

where $MU_X(X,Y)$ is the marginal utility with respect to X, that is:

$$MU_X(X,Y) \equiv \frac{\partial U(X,Y)}{\partial X}$$

In an indifference curve, it is the case that $dU = 0$, since the utility of all bundles is the same. In this case:

$$MU_X(X,Y)dX + MU_Y(X,Y)dY = 0 \Rightarrow \frac{dY}{dX} = -\frac{MU_X(X,Y)}{MU_Y(X,Y)}$$

If you are familiar with the *Implicit function Theorem* , note that this is an application of it. Therefore, for a general utility function

$$MRS(X,Y) = -\frac{MU_X(X,Y)}{MU_Y(X,Y)} \tag{4}$$

For $U(X,Y) = XY$, $MU_X = Y$ and $MU_Y = X$. Therefore

$$MRS(X,Y) = -\frac{Y}{X}$$

which is the same that we had before by using the equation for the indifference curve.

Interpretation. At a point (X,Y), the slope of the indifference curve is $-MRS(X,Y)$, which is the rate at which an agent is willing to trade goods X and Y. For instance, if $MRS = 2$, the agent is willing to trade 1 unit of X for MRS extra units of Y.

4 Solution of the Constrained Optimization Problem

We will now put together the budget constraint and indifference curves to derive the solution of the constrained optimization problem of a consumer. The problem of the consumer will be to maximize his utility subject to the budget constraint, that is

$$\max_{X,Y} U(X,Y) \quad \text{subject to} \quad P_X X + P_Y Y = I$$

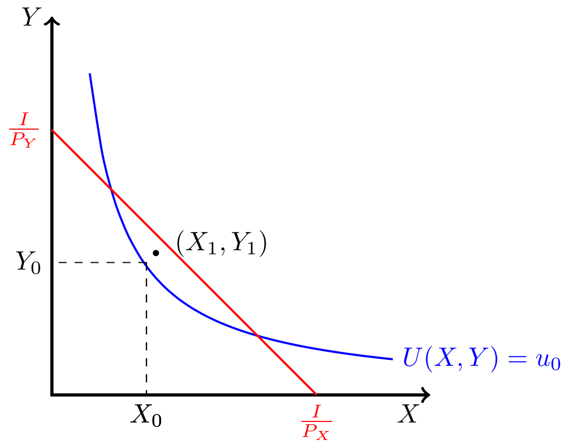
where the notation $\max_{X,Y}$ represents that the consumer is maximizing the function U choosing by choosing the values of X, Y, P_X is the price of good X, P_Y is the price of good Y and I is the income level.

We will solve the consumer problem using three methods: (i) a graphic solution and (ii) the

Lagrangean. The graphic solution is the more intuitive, while the Lagrangean is the more general, but requires some knowledge of multivariate calculus.

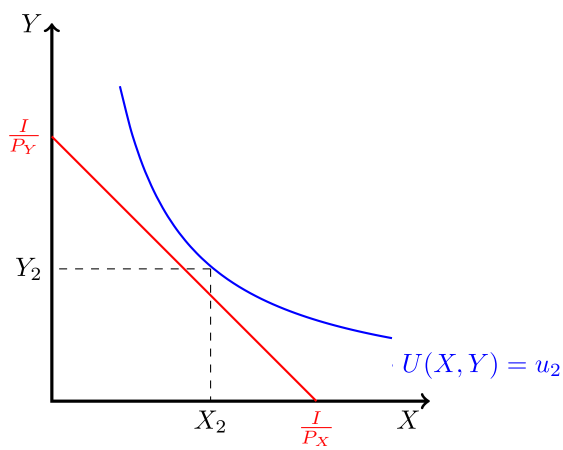
4.1 Graphic Solution

Consider any bundle (X_0, Y_0) at utility level u_0. Suppose that the indifference curve of the bundle is such that:

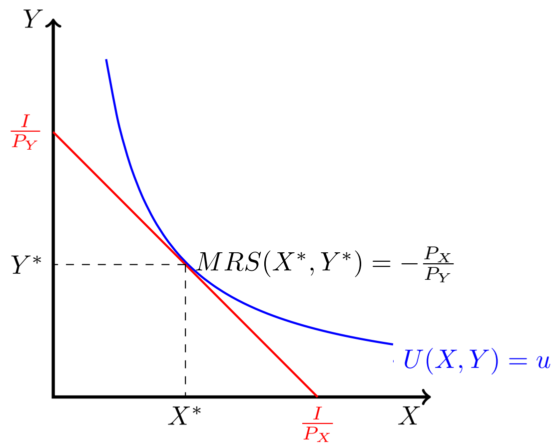


This *cannot* be optimal, that is, the point (X^*, Y^*) cannot be a utility maximizer. Since the utility of the agent is increasing northeast, we could pick (X_1, Y_1) that would give a higher utility (the indifference curve would be northeast of the one for u_0) and is still affordable (since it is on the budget line).

Consider now that (X_2, Y_2) at utility level u_2 is such that



This also *cannot* be optimal, because this level of consumption is not *feasible*, because the point lies outside of the budget set. For an optimum (X^*, Y^*), we must have then that: (i) the MRS, which is the absolute value of the slope of the indifference curve is equal to the absolute value of the slope of the budget set and (ii) the bundle is feasible.



The absolute value of the slope of the budget set is given P_X/P_Y. Therefore, our two optimality conditions are

$$MRS(X^*, Y^*) = -\frac{P_X}{P_Y} \tag{5}$$

$$P_X X^* + P_Y Y^* = I \tag{6}$$

4.2 Lagrangian

The Lagrangian of the constrained problem can be written as:

$$\mathcal{L}(X, Y, \lambda) = U(X, Y) + \lambda (I - P_X X - P_Y Y)$$

where λ is the Lagrange multiplier and we use the budget constraint in the above form to have that $\lambda \geq 0$. The FOC (First Order Condition) from the Lagrangian implies

$$\frac{\partial \mathcal{L}(X, Y, \lambda)}{\partial X} = 0 \Rightarrow MU_X(X, Y) = \lambda P_X \tag{7}$$

$$\frac{\partial \mathcal{L}(X, Y, \lambda)}{\partial Y} = 0 \Rightarrow MU_Y(X, Y) = \lambda P_Y \tag{8}$$

Dividing (7) by (8) jointly with the budget constraint:

$$MRS(X^*, Y^*) = -\frac{P_X}{P_Y} \tag{9}$$

$$P_X X^* + P_Y Y^* = I \tag{10}$$

which is the same system as we had from the graphical solution. Some places use the following *mechanical* device to represent that the budget constraint must be satisfied:

$$\frac{\partial \mathcal{L}(X, Y, \lambda)}{\partial \lambda} = 0 \Rightarrow I = P_X X + P_Y Y \tag{11}$$

This does not mean we are optimizing with respect to λ, but the derivative with respect to λ equals to zero implies the budget constraint.

4.3 Example

Suppose that $U(X, Y) = XY$, $P_X = 1, P_Y = 4$ and $I = 100$, which is the example we have been using throughout this handout. We have that:

$$MU_X = Y$$

and

$$MU_Y = X$$

Therefore, $MRS(X, Y) = -\frac{Y}{X}$. Therefore, we have that the system we derived from the graphical solution

$$MRS(X^*, Y^*) = -\frac{P_X}{P_Y} \tag{12}$$

$$P_X X^* + P_Y Y^* = I \tag{13}$$

can be written in this case as:

$$\frac{Y^*}{X^*} = \frac{1}{4}$$

$$X^* + 4Y^* = 100$$

In the first equation: $Y^* = \frac{X^*}{4}$. Replacing in the budget constraint:

$$X^* + 4\frac{X^*}{4} = 100 \Rightarrow X^* = 50$$

Replacing $X^* = 50$ in either equation:

$$Y^* = 12.5$$

We can also solve the optimization problem using the Lagrangian. The Lagrangian of the constrained problem can be written as:

$$\mathcal{L}(X, Y, \lambda) = XY + \lambda\,(100 - X - 4Y)$$

where λ is the Lagrange multiplier and we use the budget constraint in the above form to have that $\lambda \geq 0$. The FOC from the Lagrangian implies

$$\frac{\partial \mathcal{L}(X, Y, \lambda)}{\partial X} = 0 \Rightarrow Y^* = \lambda \tag{14}$$

$$\frac{\partial \mathcal{L}(X, Y, \lambda)}{\partial Y} = 0 \Rightarrow X^* = \lambda 4 \tag{15}$$

$$\tag{16}$$

Dividing (15) by (14), joint with the budget constraint:

$$X^* = 4Y^* \tag{17}$$

$$X^* + 4Y^* = 100 \tag{18}$$

which is the same system we had before.

4.4 Pitfalls

Our analysis so far is based on the tangency condition of MRS and relative prices. For some preferences this condition is not necessary or sufficient to guarantee that the point we determine is the optimal one.

Example 1. Suppose now that we maximize:

$$\max_{X,Y} \; -X \cdot Y \text{ s.t. } X + 4Y \leq 100 \tag{19}$$

which is our lead example with a minus in the utility function. In this case

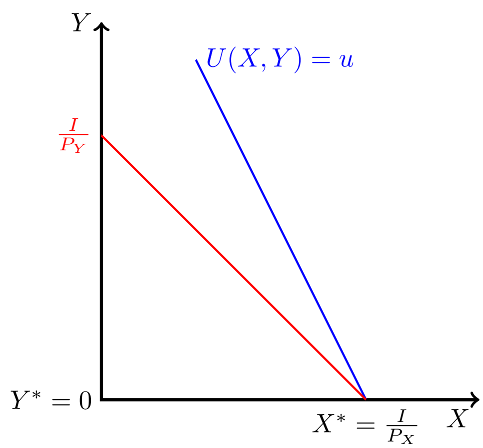
$$MRS(X,Y) = -\frac{-Y}{-X}$$

so we will be solving the same system as before. The difference is that in this case we are solving for the *minimum utility*. The maximum utility in this case is achieved by $X^* = Y^* = 0$. This pitfall can be avoided by computing the second order condition.

Example 2. Suppose now that we maximize:

$$\max_{X,Y} \; X + Y \text{ s.t. } X + 4Y \leq 100 \tag{20}$$

In this case $MRS(X,Y) = -1$ is not a function of (X,Y), so we cannot equalize $MRS(X,Y) = -\frac{P_X}{P_Y}$. This is the case where X, Y are perfect substitutes. In this case, the consumer will buy all of his income in good X (since it is the cheaper one in this case.). Graphically:



5 Applied Example

Objective. *In this example we will apply the concepts of budget constraints and indifference curves to find the optimal consumption bundle of an agent. The budget constraint is the mathematical representation of the combination of goods that are affordable at a given level of income and prices for each good. The indifference curves represent points where a consumer has the same level of utility. We will use the what is commonly called as the tangency condition between the budget constraint and indifference curves. This example is the basis of most of what is done in economics, even in frontier* *research.*

After we solve the initial problem, we will change the prices and income of the agent to understand how his/her decisions are affected by it. In economics, we call this type of exercise where we vary parameters of a problem to understand how the solutions change with it of comparative statics.

John lives in a word of two goods, A and B, and has a utility function - which denotes how much john likes each bundle with combinations of good A and B - given by [1]

$$U_J = A^{1/2}B^{1/2}$$

John earns an income of 2. The price of good A is \$ 1 and of good B is \$ 2.

1. Write down John's budget constraint and maximization problem.
2. Calculate John's MRS between goods A and B.
3. Solve for John's optimal choice of goods A and B as a function of his income and prices. What is his utility with this consumption?
4. Compute the share of his income John spends on each good.
5. Suppose that the price of good B now is \$4 instead of \$2. What is John's optimal choice of goods? Interpret your results through income and substitution effects (without computing them).
6. Suppose John's income and prices of both goods all double simultaneously. What is the percentage change in his consumption of goods A and B from your answer in item 3?

Solution.

1. John's budget constraint is given by all of the combinations of goods A and B he can afford with his income. The amount he spends on good A is given by $P_A \times A$

$$P_A A + P_B B \leq I \Rightarrow A + 2B \leq 2$$

His optimization problem is to maximize his utility subject to his budget constraint. We write the budget constraint as an equality instead of an inequality due to fact that John likes both

[1] For simplicity, we will use A and B to denote both the name of the goods and the quantity of each good John consumes.

goods (the utility is strictly increasing in both), which means that John will spend all of his money.

$$\max_{A,B} \quad A^{1/2}B^{1/2} \qquad \text{subject to} \qquad A + 2B = 2$$

2. John's MRS represents the rate at which John is willing to trade the two goods. Visually, it is the slope of the indifference curve. We can compute the MRS as the ratio of marginal utilities, as in the following equation

$$MRS(A,B) = -\frac{MU_A}{MU_B}$$

For more mathematically inclined students, the theorem that provides this result is the *Implicit Function Theorem*. In our example:

$$MRS(A,B) = -\frac{A^{-.5}B^{.5}}{A^{.5}B^{-.5}} = -\frac{B}{A}$$

At a point (A, B), the slope of the indifference curve is $-\frac{B}{A}$, which is the rate at which John is willing to trade goods A and B. For instance, if $A = 1$ and $B = .25$, John is willing to trade X units of A for $.25X$ extra units of B.

3. To solve for John's optimal bundle, we must solve the following system of two equations (MRS = - price ratio + Budget-Constraint) and two unknowns. The first condition is known as the tangency condition, which says that the rate at which John is willing to trade between goods A and B, his MRS, is the same as the ratio that the market is willing to trade these goods, which is given by the price ratio of A and B, that is:

$$MRS(A,B) = -\frac{p_A}{p_B}$$

In our example:

$$-\frac{B}{A} = -\frac{1}{2} \Rightarrow B = \frac{1}{2}A$$

By solving the tangency condition, we can get what is the optimal consumption of one good in terms of the other. To solve for the consumption level of each good, we replace $B = \frac{1}{2}A$ in the budget constraint, since any basket John buys must be within his budget constraint:

$$1A + 2\frac{1}{2}A = 2 \Rightarrow A = 1$$

Replacing $A = 1$ equation back in the $B = \frac{1}{2}A$ condition:

$$B = \frac{1}{2}A = \frac{1}{2}$$

Therefore, the solution of John's optimization problem is given by

$$A = 1 \text{ and } B = \frac{1}{2} \tag{21}$$

His utility with this consumption is:

$$U_J(1, 1/2) = 1^{.5}(1/2)^{.5} = \frac{1}{\sqrt{2}}$$

4. Total amount spent with good A is given by $P_A A$. Therefore, the share of income I spent with good A, denoted by s_A, is given by

$$s_A = \frac{AP_A}{I}$$

$$s_A = \frac{A}{2} = \frac{1}{2} \text{ and } s_B = B = \frac{1}{2}$$

5. If P_B doubles, John's consumption of A does not change. The MRS equal to relative prices now delivers

$$-\frac{B}{A} = -\frac{1}{4} \Rightarrow B = \frac{1}{4}A$$

In the budget constraint:

$$1A + 4\frac{1}{4}A = 2 \Rightarrow A = 1$$

Replacing this equation back in the optimality condition:

$$B = \frac{1}{4}A = \frac{1}{4}$$

Therefore, the solution of John's optimization problem is given by

$$A = 1 \text{ and } B = \frac{1}{4} \tag{22}$$

Interpretation. The substitution effect will make John consume less of good B (since it is now more expensive) and more of good A (which is now relatively less expensive), while the income effect would make John consume less of both goods. For *these preferences* , the income and substitution effects cancel out for good A, such that there is no effect in consumption. For B, both effects act in the same way and reduce the demand.

6. If income and prices grow by the same proportion, this does not change the budget constraint and the solution of the problem is the same as the original one. Mathematically, if prices and income grow by 2:

$$2P_A A + 2P_B B = 2I$$

which is equivalent to

$$P_A A + P_B B = I$$

since we can divide both sides by two. The problem is the same as the original. Thus, $A = 1$, $B = 1/2$.

$\square$