Notes on Gary Becker's human capital and the economy

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Introduction

There's both obsolescence and depreciation in human capital. For example, you don't remember the math you were taught in high school. Also, computer knowledge today get's useless four years from now.

Human capital:

- Education is an important form, but not such a good measure of human capital. It's not the same to spend a year at U of C that at Hawaii State say.
- Training is also an important measure of human capital but we don't have good data on it.
- Health is a particularly important form of human capital.
- Labor market (wages).
- Inequality
- Intergenerational mobility
- Specialization (a specialist is worthless in an island)

Since modern economies are based on knowledge, it must be that human capital is important to understand them and to understand why is it that knowledge changes over time.

Claim 1 *An important feature of human capital is complementarity. E.g. an increase in life expectancy rises my incentive to invest in education.*

Proof. *Suppose $r = 0$. If I define W as wealth and E_s^y as earning in year y if schooling level s, then the change in wealth I get from increasing schooling is given by:*

$\Delta W_s = \Delta E_s^0 + \Delta E_s^1 + ... + \Delta E_s^N + ... + \Delta E_s^{N+L}$

Now, if before I was only going to live until N and now I get to live until $N + L \Rightarrow$ my income is larger and I get a higher investment in education. ■

The same is true the other way around. If I have more education, I will invest more in health since this way I can collect my returns longer. Think of the AIDS epidemic. It hits young people, so they won't invest in their education since their death rate increased and this slows economic growth.

What's the point of distinguishing between human (H) and physical (K) capital?

- Human capital is not good collateral (except in the armed forces)
- The production of H is different from that of K since most of it is produced inside the family and in schools. More generally, H is produced in not for profit organizations.

Investment in human capital at the family level

Suppose parents decide how much to invest in their children. Why do some families invest more in their children than others? Why do governments come into the provision of human capital?

Let's use overlapping generations.

Parents have 3 choices: how much to consume, how many kids to have and how much to invest in each kid.

For now, let us assume that parents have just one kid. Also, we'll assume that any individual lives for two periods such that when he is young his parents are old, and when he is old his kids are young. Parents invest in their kids when they are old, and the kid receives the payoff when he is old so his parents are dead. Why do parents invest in their kids then? If parents were completely selfish then they would not invest in their kid's human capital, or if the kid was their slave they could sell him. So, there must be some benefit for the parents.

We will assume parents are "altruistic" so their utility depends on that of their children In this way, even though they will be dead when the children are adults, the parents anticipate their kid will have a higher utility if they invest in him and have a higher utility themselves. Now, how much do parents invest depends on the degree of altruism, the productivity of the investment and the "productivity" of H in the kid's utility function.

0.1 A simple model of altruism

Notation 2 *Define:*

V_i - *Indirect utility function for parent or child* ($i.e.$ $i = \{p, c\}$)

$U()$ - *(concave) utility function of consumption*

C_i - *consumption*

Y_c - *investment per child*

H_i - *Human capital stock*

W_i - *wealth*

Assume the parents' utility function can be written as:

$$V_p = U(C_p) + aV_c \tag{1}$$

Where a is the degree of altruism of the parents. We will assume that $a \geq 0$ so that parents cannot disinvest in their kids. If $a = 0$ then parents are completely selfish and do not invest in their kids. The parents budget constraint is given by $Y_c + C_p = W_p$. The household production function of human capital depends only in the amount invested:

$$\begin{aligned} H_c &= f(Y_c) \\ \frac{\partial f(Y_c)}{\partial Y_c} &= f_y > 0,\ f_{yy} \leq 0 \end{aligned} \tag{2}$$

Assume that wealth is proportional to the level of human capital:

$$W_c = r_c H_c \tag{3}$$

where the factor of proportionality r may change between generations. We will also assume a one stock model of human capital, so there's one (perfectly) substitute kind of H. That is, $H^s = \sum_{i=1}^{N_c} H_c^j$.

Remark 3 *Notice that people take r_c as given by the market, but that it is not given for the economy as a whole. So, we have that:*
$\frac{\partial r}{\partial H^s} < 0$ *and* $\frac{\partial H^s}{\partial H_c^j} \approx 0$

All inequality in this model is determined by the different amounts of human capital each individual has. From equation 3 we have that $var(\log w) = var(\log H)$. Also notice that we do have bequests in this model but, since there is no other asset, they are given in form of human capital.

From all this, we get that the problem parents solve is:

$$\max_{C_p, Y_c} U(C_p) + aV_c + \lambda [W_p - C_p - Y_c]$$
$$\text{subject to equations 2 and 3}$$

with FOC's given by:

$$U'(C_p) = \lambda$$

$$a\frac{\partial V_c}{\partial Y_c} \leq \lambda \text{ with equality} \Leftrightarrow Y_c > 0$$

So, in the optimum, the marginal utility of spending on the kid (second condition) should be less than or equal the marginal utility the parent gets by spending on himself. Notice that if $a = 0$ (selfish parents), then $Y_c = 0$. On the other hand if $a > 0$, then the second condition can be written as:

$$a\frac{\partial V_c}{\partial Y_c} = a\frac{\partial V_c}{\partial W_c}\frac{\partial W_c}{\partial H_c}\frac{\partial H_c}{\partial Y_c} = aV_c' r_c f_y(Y_c) \leq \lambda$$

Notice that if $f_y = 0$ I still get $Y_c = 0$ so, we will impose an Inada condition: $\lim\limits_{Y_c \to 0} f_y(Y_c) = \infty$. This way, any altruistic parent invests something in his child $\Longrightarrow$ we can assume that it holds with equality: $a\frac{\partial V_c}{\partial Y_c} = \lambda$.

Definition 4 $R_y = r_c f_y(Y_c)$ *is the (gross) rate of return of investments in kids., i.e. what's the effect of investing now on the earnings of kids tomorrow.*

$\frac{\partial R_y}{\partial Y} \leq 0 \iff f_{yy} \leq 0$ *since* r_c *is fixed (for me) so it all depends on the technology*

We may have $f_{yy} = 0$ *but it's improbable since it takes time to produce H too*

We can rewrite the FOC as $R_y = \frac{U'}{aV_c'}$ i.e. the ratio of marginal utilities = interest rate. Now a plays the same role as an intergenerational discounting rate.

0.1.1 Comparative statics

We take the FOC's and the BC and differentiate them to get the SOC:

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & U'' & 0 \\ 1 & 0 & aV_c'\frac{\partial R_y}{\partial Y} + aR_y^2 V_c'' \end{vmatrix} = D = -U'' - aV_c'\frac{\partial R_y}{\partial Y} - aR_y^2 V_c'' > 0$$

Now, differentiate the FOC's with respect to W_p and form the linear system:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & U'' & 0 \\ 1 & 0 & aV_c'\frac{\partial R_y}{\partial Y} + aR_y^2 V_c'' \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial W_p} \\ \frac{\partial C_p}{\partial W_p} \\ \frac{\partial Y}{\partial W_p} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Then, apply Cramer's rule to get:

$$\frac{\partial Y_c}{\partial W_p} = \frac{\begin{vmatrix} 0 & 1 & 1 \\ 1 & U'' & 0 \\ 1 & 0 & 0 \end{vmatrix}}{D} = -\frac{U''}{D} > 0$$

Poorer families send kids to work earlier (invest less in their human capital). Notice that within limits poor parents could be more altruistic than rich ones and you would still see this. Now, from equation 2:

$$\frac{\partial H_c}{\partial W_p} = \frac{\partial f}{\partial Y_p}\frac{\partial Y_c}{\partial W_p} = -\frac{f_y U''}{D} > 0$$

Parents with higher human capital have kids with higher human capital since they invest more in them.

Capital market imperfections

First, assume that $f_{yy} = 0$ so we have constant returns to scale in the human capital production function. In this case the SOC are $D_1 = -U'' - aR_y^2 V'' > 0$. Here there is no obvious capital market imperfection since now $R_y = r_c f_y$ is constant. That is, everyone gets the same return of investing on their kids' human capital.

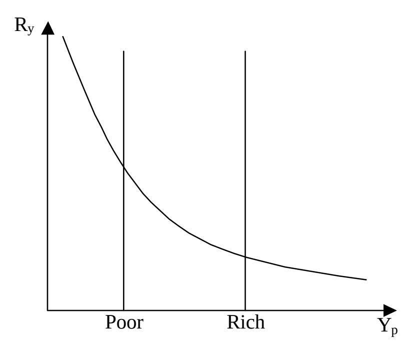
Now, assume that $f_{yy} \leq 0$ so we have decreasing returns. Now, we have the original SOC and $D > D_1 > 0$. If we look at the comparative statics:

$$\frac{\partial Y_c}{\partial W_p}|_{f_{yy}=0} = -\frac{U''}{D_1} > \frac{\partial Y_c}{\partial W_p}|_{f_{yy}<0} = -\frac{U''}{D}$$

$$\frac{\partial H_c}{\partial W_p}|_{f_{yy}=0} = -\frac{f_y U''}{D_1} > \frac{\partial H_c}{\partial W_p}|_{f_{yy}<0} = -\frac{f_y U''}{D}$$

So as rich people invest more and more their return decreases, but they still invest more than poor people.

- Why is this a capital market imperfection?



So $R_y^r < R_y^p$ and they face different returns. This doesn't limit how much the poor invest in their kids. It's only that this couldn't exist if capital markets were perfect. In this case, the poor could borrow from the rich, both getting a higher return until the rates were equal.

Would a situation of perfect capital markets equalize income? No, the poor kids would still have to pay the rich kids. It eliminates differences in earnings not differences in income.

If we ran $\log(W_c) = \alpha + \beta \log(W_p)$, would $\beta > 1$? Who knows? $\frac{\partial W_c}{\partial W_p} = r_c \frac{\partial H_c}{\partial W_p} > 0$ and $\beta = \frac{\partial H_c}{\partial W_p} \frac{W_p}{W_c} r_c = \frac{\partial H_c}{\partial W_p} \frac{W_p}{H_c} \gtrless 1$.

Changing altruism

Suppose that now we change a. Then, if we differentiate the FOC's and the BC with respect to a we get the linear system:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & U'' & 0 \\ 1 & 0 & aV_c'\frac{\partial R_y}{\partial Y} + aR_y^2 V_c'' \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial a} \\ \frac{\partial C_p}{\partial a} \\ \frac{\partial Y}{\partial a} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -V_c' R_y \end{pmatrix}$$

Again, using Cramer's rule and equation 3 to get:

$$\frac{\partial H_c}{\partial a} = \frac{f_y \frac{\partial Y_c}{\partial a}}{D} = \frac{f_y V_c' R_y}{D} > 0$$

$$\frac{\partial C_p}{\partial a} = -\frac{V_c' R_y}{D} < 0$$

Now, take someone with average income. If we increase a he would invest more than the average so their kids will have more than average income. **If** children with higher a parents are more altruistic than average themselves, over time rich people would seem to be more altruistic. In equilibrium we would observe $cov\,(a, W) > 0$.

0.2 A model of altruism with influence of parents' human capital

Now, we extend the technology so that parents with higher human capital directly increase the production of human capital of their kids.

$$\begin{aligned} H_c &= f\,(Y_c, H_p) \\ f_y &> 0,\ f_{yy} \leq 0,\ f_H \geq 0,\ f_{yH} > 0 \end{aligned} \tag{4}$$

This is known as *the recursive property of knowledge*. This is the argument used in favor of headstart programs. Taking a 20 year old who is lagging behind and getting him to average level is a lot harder because of this. Now, we have a different reason why H varies across the population. The fact that $f_{yH} > 0$ makes it not only an average effect, it's a marginal effect too!

Even if I have $f_{yy} < 0$ I can offset this by having more parent's human capital. One special case of this is $H_c = g\,(Y)\,H_p$ so we have constant returns to scale to parents' human capital. With $g' > 0$ and $g'' < 0$ then $R_y = r_c g' H_p$. It provides a link across generations:

$$\frac{H_c}{H_p} = \frac{H_{t+1}}{H_t} = g\,(Y)$$

This is the growth of human capital. The greater the parents' human capital the greater the kids' and maybe the greater the growth rate. This is the same trick we use at the economy level: I can grow without ever increasing Y! If we didn't have H_p then $\frac{H_c}{H_p} = \frac{f(Y)}{H_p}$. So, to have a constant growth you would need to increase Y continuously and that would decrease the return rate.

Let us go back to the household level. Suppose I increase H_p. How does this affect Y? I.e. how much (or less) would a parent with higher human capital spend on his kid. Take the compensated case (keep $\overline{W}_p$ constant):

1. Keep $C_p = \overline{C}_p$. Still parent's utility goes up. This is the wealth effect, since this increases H_c just by increasing H_p

2. Then, from the wealth effect: $\uparrow C_p$ and $\downarrow Y_c$

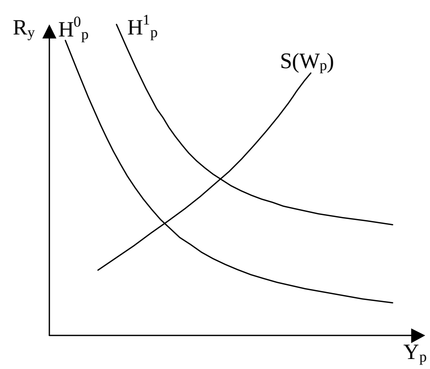
3. Now, $R_y = r_c f_y(Y, H_p)$ goes up! (substitution effect, I want to invest more)

So $\uparrow H_p \Rightarrow \left\langle \begin{array}{c} \uparrow H_c \rightarrow \uparrow V_p \rightarrow \uparrow C_p \rightarrow \downarrow Y_c \\ \uparrow R_y \rightarrow \uparrow Y_c \rightarrow \downarrow C_p \end{array} \right.$

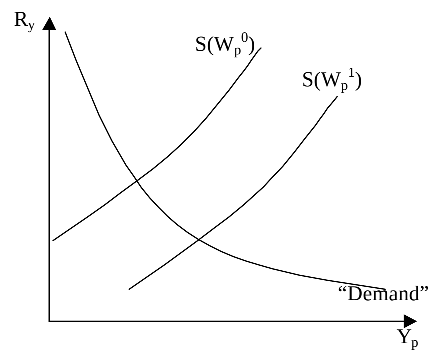
In the end we know that $\uparrow H_c, \uparrow W_p$, C_p? and Y_c?.

Before we had $H_c = f(Y_c), f_y > 0, f_{yy} < 0$ so that $R_y^r < R_y^p$. There's an opportunity to redistribute income from the rich to the poor, decrease inequality and increase efficiency: free lunch!

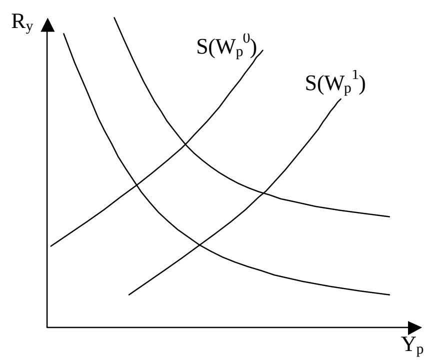
In the case we are discussing now $H_c = f(Y, H_p), f_y > 0, f_{yy} < 0, f_H \geq 0, f_{yH} > 0$. Now, if $f_H = 0 \Rightarrow cov(W_p, R_y) < 0$. However, if $f_H > 0$ it is possible to get $cov(W_p, R_y) > 0$. Suppose Y_c is independent of W_p then, for sure we have $cov(W_p, R_y) > 0$ because of $f_{Hy} > 0$. Then, I would want to give money to the rich to increase efficiency! That is, if $R^{r*} > R^{p*}$ then the rich would want to disinvest in the poor kids and invest in the rich kids. Graphically:



Here $S(W_p^i)$ tells me how much would be invested in a kid given a return and a real (compensated) income (W_p^i) of the parent. We draw the case of $W_p^1 > W_p^0$ so $cov(R, W_p) < 0$. That is, kids are equally productive it is just differences in income.



Now, if $cov\,(R, W_p) > 0$, rich parents invest more and get a higher rate of return since $f_{yH} > 0$. Even kids with same parents' income $\overline{W}_p$ have different returns since there is a true productivity difference. Now, the true case is between these two cases.



In this particular case I drew $cov\,(R, W_p) = 0$ and rich people would invest more. In any case rich people invest more but whether they have higher or lower returns is anyone's guess.

0.3 Altruism with differences in ability

Ability is a multidimensional variable, however we are going to deal with it as if it had a single dimension. As a result, we are going to get that, in average, rich guys are more able. Why do families tend to regress to the mean[1]? in our previous analysis we couldn't get this that easily. However, the data points to regression towards the mean. How do we explain this?

[1]Regression to the mean is attained when a regression of $W_c = \alpha + \beta W_p$ has a coefficient $0 < \beta < 1$. This means that kids of parents who have income above the mean of their generation will still have income above the mean of their own generation but this difference will not be as large as the one of their parents.

Suppose there's something we'll call ability (A_c) of children such that:

$$\begin{aligned} H_c &= f(Y_c, A_c, H_p) \\ f_A &> 0, f_{Ay} > 0 \end{aligned} \tag{5}$$

Notice that, since $W_c = r_c H_c = r_c f(Y_c, A_c, H_p)$ then, given $\overline{H}_c$ ability doesn't affect earnings. As before, the problem of the parent is to maximize equation 1 subject to 5 and 3. We will assume parents know A_c and ask the question of what happens when we increase it. The FOC's and SOC are the same as before. As before, we do the comparative statics by differentiating the FOC's and the BC with respect to A_c and forming the linear system:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & U'' & 0 \\ 1 & 0 & aV_c' \frac{\partial R_y}{\partial Y} + aR_y^2 V_c'' \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial A_c} \\ \frac{\partial C_p}{\partial A_c} \\ \frac{\partial Y_c}{\partial A_c} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ X \end{pmatrix}$$

The sign of $X = -aV_c' r_c f_{yA} - aR_y V_c'' r_c f_A$ is unknown. Now, using Cramer's rule:

$$\begin{aligned} \frac{\partial Y_c}{\partial A_c} &= \frac{ar_c \left(V_c' f_{yA} + R_y V_c'' f_A\right)}{D} ? \\ \frac{\partial C_p}{\partial A_c} &= -\frac{\partial Y_c}{\partial A_c} \\ \frac{\partial H_c}{\partial A_c} &= f_A + f_y \frac{\partial Y_c}{\partial A_c} = \frac{-U'' f_A - aV_c' \frac{\partial R_y}{\partial Y_c} f_A + aV_c' R_y f_{yA}}{D} > 0 \\ \frac{\partial V_c}{\partial A_c} &= V_c' r_c \frac{\partial H_c}{\partial A_c} > 0 \\ \frac{\partial V_p}{\partial A_c} &= \frac{-a^2 R_y^2 V_c'' r_c f_A V_c' - U'' f_A a V_c' r_c - a^2 V_c'^2 f_A r_c \frac{\partial R_y}{\partial Y_c}}{D} > 0 \end{aligned}$$

Where in the last equation we used the fact that $U' = aV_c' R_y$ from the FOC's. Can we give a more intuitive reason why this happens? I'm not sure but let's try:

$$\uparrow A_c \Rightarrow \left\langle \begin{array}{c} \uparrow H_c \rightarrow \uparrow V_c \rightarrow \uparrow V_p \rightarrow \uparrow C_p \rightarrow \downarrow Y_c \\ \uparrow R_y \rightarrow \uparrow Y_c \rightarrow \downarrow C_p \end{array} \right.$$

That is, the first effect of increasing ability is to increase H_c via equation 5, but this means that W_c increases and thus V_c and V_p, but this tends to increase C_p and thus decrease Y_c. On the other hand, since $f_{yA} > 0$ then an increase in ability rises R_y which tends to increase Y_c and thus decrease C_p.

0.3.1 The case of more than one kid

Suppose you have 2 kids, one smarter than the other ($A_1 > A_2$). Now, equation 1 would be $V_p = U(C_p) + a_1 V_{c1} + a_2 V_{c2}$ and the budget constraint is now

$Y_{c1}+Y_{c2}+C_p=W_p$. Assume you love them both the same so $a_1=a_2=a$. The efficient thing to do would be to invest until $R_{y1}=R_{y2}$ so you invest more on the smarter kid. But if you do this $V_{c1}>V_{c2}$. To see what you should do, just look at the FOC's:

$$\begin{aligned} U'(C_p) &= \lambda \\ aV'_{c1}R_{y1} &= \lambda \\ aV'_{c2}R_{y2} &= \lambda \end{aligned}$$

So you will make a compromise to equalize the last two equations. You will not equalize marginal utilities or returns . This is similar to a society's trade-off between equity and efficiency. The difference is that here I have **one** utility function while in a society it all depends on who is weighting everything.

0.3.2 The case of intergenerationally transmitted ability

Until now, we took ability as given. What if it's transmitted within the family?. Suppose there's some sort of inheritance equation:

$$A_c=(1-h)\,\overline{A}+hA_p+\varepsilon_c \tag{6}$$

We will assume that $0\leq h<1$ so that there is regression to the mean. Now, take expectations in both sides $E(A_c)=(1-h)\,E\left(\overline{A}\right)+hE(A_p)$ so that if richer parents are abler, then their kids are abler (in average). Also, since there are people with higher earnings, in average the abler children are going to get more human capital and more money. That is, in average abilities are higher for people with higher earnings (*sorting of families*).

Suppose there are perfect capital markets, that $H_c=f(Y_c,A_c)$ and that $\log(E_c)=\alpha+h\log(E_p)+\mu()$. So parents and children are related (in this case) only through h since there are perfect capital markets and no H_p in the production function. Here the only reason richer parents have richer kids is through ability:

$$\sigma^2(\log(E_c))=h^2\sigma^2(\log(E_p))+\sigma^2(\mu)$$

In equilibrium $\sigma^2(\log(E_c))=\sigma^2(\log(E_p))$ then:

$$\sigma^2(\log(E_c))=\frac{\sigma^2(\mu)}{1-h^2}$$

So, in equilibrium h magnifies shocks in μ and thus increases inequality across families.

Of course, if we don't have perfect capital markets not only h would matter but also the propensity to spend more on your kids when your income goes up. I.e. now $\log(E_c)=\alpha+(h+\beta)\log(E_p)+\mu()$ where β is the marginal propensity to spend (Y_c) when income rises. Now, we have a mixture in the coefficient so we can't extract h completely unless some assumptions hold.

0.4 Adding physical (and financial) capital

We usually have data on bequests made in physical capital. Introducing it enables us to get a better reading of the empirical relations that were ambiguous.

1. We said that if $H_c = f(Y_c)$ then $f_{yy} < 0$ implies richer families get lower rates of return (absent perfect capital markets).

2. If $H_c = f(Y_c, H_p)$ and $f_{yH} > 0$ then we are no longer sure about the sign of the return relations. It could be possible that $R_y^r > R_y^p$.

3. The same is true with ability.

It's been a common view that the poor are the ones getting higher rates of return so we should help them.

Condition 5 *There is a market in physical capital. So, if I increase the non-human capital, I do not affect the market. We didn't think this way of human capital. For our purposes, families take the rate of return as given in physical capital but they have a family specific return on human capital.*

Also, physical capital is separable so I can borrow on it.

With these conditions, I get that $R_k^i = R_k^j = R_k$. To keep it simple, we will assume an Inada condition. Namely, $\underset{Y_c \to 0}{lim} R_y = \infty$. So, if a family is not completely selfish there is going to be some investment in human capital.

We are going to use a portfolio view of the problem. We are going to assume parents want to maximize the income of the kid. This doesn't need to be true because of risk, or maybe human capital gives more utility or parents maybe worry (good samaritan). Also, kids know that if they spend too much today, it is in the parents' interest to bail them out (moral hazard). So, since human capital cannot be dissipated while physical capital can, may be parents prefer to in vest in human capital instead of giving physical capital to the kid.

Remark 6 *If the marginal rate of return on K is greater than the rate of return on H,i.e. if $R_k > R_y$, then you invest more on physical capital.*

Either we end up with $R_y = R_k$ or I end up in a corner solution with $R_y > R_k$ and no investment in K.

Also, by Inada it can't be that $R_k > R_y$ when $Y_c = 0$.

So, either you only invest in human capital or you invest on both.

Lets' look at the parents problem. As before, he wants to maximize equation 1 subject to his new budget constraint. Now, his expenditure is divided between consumption, investment in the kid's human capital and the physical capital bequest he'll leave the kid. On the other hand his earnings (E_p) are given by his previous earnings and the return he gets from the physical capital his parents left him. That is:

$$C_p + Y_c + K_c = W_p + R_k K_p = E_p$$

1. $H_c = f(Y_c), f_{yy} < 0$

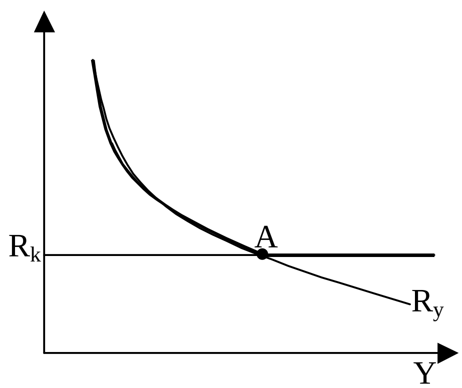
   FOC's:

   $U'(C_p) = \lambda$

   $aR_y V_c' = \lambda$

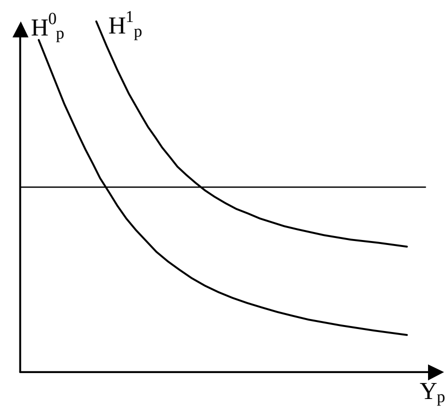
   $\frac{\partial V_c}{\partial K_c} = aR_k V_c' \leq \lambda$ with $= \iff K_c > 0$

   $\implies aR_y V_c' > aR_k V_c' \iff \boxed{R_y > R_k \implies K_c = 0}$



   We proved before that $\frac{\partial Y_c}{\partial W_p} > 0$. So, as wealth rises, they invest more on their kids until point A in the figure. From A on, I also add to non-human capital. If we look at bequests only as non-human capital, only rich parents give them (consistent with data). Now, the other families are also giving bequests but in human capital. So, bequests data is highly misleading, poor families **could** be giving a higher bequest (as percentage of their income) than rich families. Also, there's an incentive to bequest in human capital since it is not taxed and I can "depreciate" my investment in human capital immediately, while that is no the case with physical capital.

2. $H_c = f(Y_c, H_p)$



Richer families (with higher H_p) will have higher H_c. In principle, now we **could** have that poor families invest in both while rich families invest only on human capital. As before, we don't know what happens to Y_c as H_p rises. What have we solved then? Evidence says that only rich families give K_c so they must be the ones with lower rates of return at the margin. That is, in equilibrium (in average) $R_y > R_k$.

0.5 Human capital investment in adulthood

We invest in human capital since the very beginning since the return depends on how long I live. Also, it affects my productivity so I want to start early to have a big stock. Now, this investments continue over the lifetime. Consider a young adult who takes as given the stock of human capital that was invested in him (H_0). Think of a simple decision of the form: 2 categories A, B (e.g. go to college-don't go,occupation 1-occupation2, etc.)

There are $0, 1, ..., m$ periods of life remaining. Then at $t = 0$ you have two choices:

$A \rightarrow$ no investment: $\left\{w_A^1, w_A^2, ..., w_A^m\right\}$

$B \rightarrow$ investment of c: $\left\{w_B^1, w_B^2, ..., w_B^3\right\}$

Which one do you choose? It would be easy if you always earned more in one activity. Look at the "wealth" (PDV) created in each case:

$$V_y = \sum_{i=0}^{m} (1+r)^{-1} w_y^i t_y^i$$

Where t_y^i is hours worked in period i in activity y for $y = \{A, B\}$. Notice that the cost is implicitly included in $i = 0$ in the form of foregone earnings (net of tuition): $c = t_A^0 w_A^0 - t_B^0 w_B^0$. So, if we compare both options:

$$V_B - V_A = \sum_{i=1}^{m} \left(t_B^i w_B^i - t_A^i w_A^i\right) (1+r)^{-1} - c$$

So you would choose B if and only if the difference is positive. Now, assume that $t_y^i = t$ and that $w_B^i - w_A^i = \Delta w$ for all i. In this case, the $Benefits = \sum_{i=1}^{m} t\Delta w \left(1+r\right)^{-1} = t\Delta w R\frac{(1-R^m)}{1-R}$ where $R = (1+r)^{-1}$. Then as $m \to \infty, Benefits \to \frac{t\Delta w}{r}$. So, we have 5 determinants of whether to choose A or B:

1. c : the higher the cost the smaller the gain (or the bigger the loss) of investing (choosing B). Part of the cost is market (tuition) but most of it is idiosyncractic (foregone earnings). In the U.S. foregone earnings are like 75% of the cost (even more for graduate education). We can see that the more you advance the higher the cost of foregone earnings, so the benefit must rise accordingly.

2. R : a smaller R means that I am more reluctant to invest since now I discount the future more

3. m : as we said before, the longer I live the more I want to invest.

4. Δw

5. t : for example, women participated little in market activities. They didn't want to go to activities that required high t. This is why saw many female teachers but no so many engineers

Case 7 *In the 80's tuitions went up (big) but enrollment too! Is this a contradiction? No. there is no puzzle. Foregone earnings went down a lot in the 80's and 90's and the gains increased a lot too.*

0.5.1 Are tuition and foregone earnings perfect substitutes in the cost?

We said that the cost of investment was given by tuition plus foregone earnings. Now, if cost is the factor determining the decision, it would imply they are perfect substitutes. That is, same cost with different compositions would give the same results. However, with an imperfect capital market this is not true. Take two cases:

1. Suppose your earnings while in school went up and tuition increased by the same amount, so that cost remains constant. That is, if we let E_s be earnings while in school, E_w be earnings if not in school and T be tuition we have that:

   $E_s^1 - E_s^0 = d$ and $T^1 - T^0 = d$

   Are they perfect substitutes? Yes. Consider the cost in each case:

Earnings at school go up

	Total cost	Available resources
Before change	$E_w - E_s^0 + T^0 = c$	$E_s^0 - T^0$
After change	$E_w - E_s^1 + T^0 = c - d$	$E_s^1 - T^0 = E_s^0 - T^0 + d$

Tuition goes up

	Total cost	Available resources
Before change	$E_w - E_s^0 + T^0 = c$	$E_s^0 - T^0$
After change	$E_w - E_s^0 + T^1 = c + d$	$E_s^0 - T^1 = E_s^0 - T^0 - d$

That is, since the cahnge in available resources is enough to balance each other and the cost is the same, I can use my higher earnings to pay for my higher tuition and nothing has changed, independently of whether we have perfect or imperfect capital markets.

2. Now, take another case. Suppose that now my potential earnings out of school decrease by the same amount that tuition increases. That is $E_w^1 - E_w^0 = -d$ and $T^1 - T^0 = d$ so that:

Earnings out of school go down

	Total cost	Available resources
Before change	$E_w^0 - E_s + T^0 = c$	$E_s - T^0$
After change	$E_w^1 - E_s + T^0 = c - d$	$E_s - T^0$

Tuition goes up

	Total cost	Available resources
Before change	$E_w^0 - E_s + T^0 = c$	$E_s - T^0$
After change	$E_w^0 - E_s + T^1 = c + d$	$E_s - T^1 = E_s - T^0 + d$

In this case, it is not clear whether they are perfect substitutes when capital markets are imperfect. As we can see, there are less available resources when tuition rises. Then, if there is no borrowing, consumption must decrease if I am to remain studying.

0.5.2 What if cost is not given?

The cost might depend on your level of human capital at the beginning (and/or ability). Presumably, the higher H_0 the lower the cost, i.e.

$$C(H_0, A), C_H \leq 0$$

It helps explain why adult investments are different among race, gender, etc. without introducing ability or capital markets. Take family background, if I come from a broken home, I have less human capital invested in me and thus higher cost and lower schooling (ceteris paribus). The same is true for groups, say race; we don't have to think about differences in ability because of race but use human capital as a rationale.

Suppose you take 2 groups, same ability in the group, one with more human capital than the other and ask: if we observe the ability of those going on. would it be the same? The ability of the group with **less** human capital will be higher since only those with high ability would continue. That is, guys with higher human capital for the same ability level have lower cost of continuing thus lowering the ability level for that group.

0.6 Investments in health

You can invest varying amounts and the payoff is that you live longer (better) lives (no effect on wages assumed). Assume there is no uncertainty and that agents are wealth maximizers. That is, you live a certain length of life, and the more you invest the longer you will (certainly) live. The utility function of an agent who lives $m+1$ periods is given by:

$$V = \sum_{i=0}^{m} t_i w_i (1+r)^{-i} - C(M) \tag{7}$$

In the initial year you can invest M in health with convex cost $C(M)$, so that $C'(M) > 0$ and $C''(M) > 0$. Then the agent's problem consists on finding the optimal level M of investment in health. Well, now we have that m is a function of M. We'll assume that $\frac{\partial m(M)}{\partial M} > 0$ and $\frac{\partial^2 m(M)}{\partial M^2} < 0$. The FOC's are:

$$\frac{\partial V}{\partial M} = t_m w_m (1+r)^{-m} \frac{\partial m}{\partial M} - \frac{\partial C}{\partial M} = 0$$

What determines the length of life?

1. t_m if it was zero, I wouldn't want to live more (in this model retirement is worthless, we'll correct this limitation)
2. w_m the smaller the earnings the less valuable life is
3. C'

In any case, health is related to earnings, which are related to your investment in human capital, which depends on how long you will live to collect the returns on it. That is, as we said before, there are complementarities in human capital. You want to invest more in health if you invest more in education and the other way around. Actually, education is the most important factor in explaining investment in health in middle to rich societies. In societies like Africa it's obviously income.

We will not assume that young people don't want to invest in their health. For example, young people smoke because it will only affect them if they smoke for 20 years. However, they still have an incentive to invest since it also increases their health this year (even though we are not modelling quality of life).

Now, let's allow investment to be affected by individual choices. For example, it is not only whether there is a medicine that can cure me but whether I take it. To do this, we'll change equation 7 and assume that now M doesn't affect how long I live with certainty but it just affects my survival probability. Namely:

$$\begin{aligned} E(V) &= \sum_{i=0}^{\infty} t_i w_i (1+r)^{-i} S_i(M) - C(M) \qquad (8) \\ \text{Assume } S_i' &\geq 0, S_i'' < 0 \Rightarrow S_j \leq S_i \Longleftrightarrow j > i \end{aligned}$$

Where S_i is my survival probability from age 0 to age i. Now, we'll use a very simple model to determine probabilities. We will assume that $S_i = s^i(M)$, where $s(M)$ is the constant hazard rate such that $\lim\limits_{i\to\infty} S_i = 0$ and $\frac{\partial s(M)}{\partial M} > 0$. Also, $w_i = w$ and $t_i = t$. Then, we can rewrite equation 8 as:

$$E(V) = \sum_{i=0}^{\infty} twR^{i+1}s^i(M) - C(M) = \frac{twR}{1 - Rs(M)} - C(M) = \frac{wt}{R^{-1} - s} - C(M)$$

$$\begin{aligned} E(V) &= \frac{wt}{r+d} - C(M) \qquad (9) \\ \text{where } d &= 1 - s \text{ is the death rate} \end{aligned}$$

The discount rate is now the death rate plus the interest rate. They both affect my future income. The FOC's of this problem are given by:

$$C'(M) = -\frac{wt}{r+d}\frac{d'(M)}{d}\frac{d}{r+d}$$

This equation gives me the willingness to pay **at the margin.** That is, it is not whether I live or die but how much I am willing to pay for a marginal increase in the length of my life. Notice that I can implicitly for $M^* = M(w, t, r, d)$, which tells me that, any increase in my survival probability rises my investment in health.

Do people have control over the probability of living? Well, before we had rudimentary knowledge about it like that if we spent more we avoided starvation. However, this knowledge has exploded in the last 150 years, nutrition, exercise, controlling salt intake, drugs, etc. all affect my probability of living longer. Now, rewrite the FOC's:

$$-\frac{C'}{\frac{d'}{d}} \approx \frac{wt}{r+d} \qquad (10)$$

This equation is what we will call the value of life. As we can see, in this very simple model, the value of life is just the discounted value of future earnings. Suppose the probability of dying falls by 1/1000, then the value of life is what you gain because of this decrease in the probability of dying. If we look at earnings, a typical person earns $\approx \$1,000,000$ over his lifetime. However, value of life estimates are between 3 and 5 millions. What are we missing? We have to consider **leisure.** If you are not working, you still value the utility you get from leisure by **at least** the wage rate. If twice as much time is spent in the house vs the market, I value life at least at 3 times my earnings.

The difference in this is dictated by the fact that you pay marginally but gain average. If I spend one more dollar in decreasing my probability of dying, this costs me the marginal utility I loose from reducing consumption by \$1 but I multiply my gain by the level of utility I get over my life.

Now, we modify the value function of the consumer to account for the value of leisure. Namely:

$$V = \sum_{i=0}^{\infty} U\left(X_{i,}l_i\right)\beta^i S_i \tag{11}$$

We will assume there's a market of (fair) annuities so that the $E\left(Income\right) = E\left(Consumption\right)$. Of course, actual consumption might be smaller (or bigger) if you die young (old). The budget constraint is given by:

$$\begin{aligned} \sum_{i=0}^{\infty} X_i R^i S_i &= A + \sum_{i=0}^{\infty} w_i t_i S_i R^i - C\left(M\right) \\ where\ \frac{\partial S_i}{\partial M} &\geq 0 \end{aligned}$$

Now, if we assume that $\beta = R$ the FOC's are:

$$\begin{aligned} \frac{\partial U}{\partial X_i} R^i S_i &= \lambda R^i S_i \Rightarrow \frac{\partial U_i}{\partial X_i} = \lambda \\ \frac{\partial U}{\partial l_i} &\geq \lambda w_i \text{ with} = \text{if } t_i > 0 \end{aligned}$$

By the fair annuities market assumption the probability of dying does not enter since we don't need to worry about it. On the other hand, if we look at the second condition, we can see that people who voluntarily retire ($t_i = 0$) are getting more than the wage rate in value for their leisure, that's why they retire. Now, take the value function and take its derivative with respect to investments in health (i.e. $\frac{\partial V}{\partial M}$):

$$\sum_{i=0}^{\infty} U\left(X_{i,}l_i\right) R^i \frac{\partial S_i}{\partial M} = \lambda \left[C'\left(M\right) + \sum_{i=0}^{\infty}\left(X_i - w_i t_i\right) R^i \frac{\partial S_i}{\partial M}\right]$$

$$\Rightarrow C'\left(M\right) = \sum_{i=0}^{\infty}\left[\frac{U\left(X_{i,}l_i\right)}{\lambda} - X_i + w_i t_i\right] R^i \frac{\partial S_i}{\partial M} \rightarrow \text{Basic health equation} \tag{12}$$

- $\frac{U(X_{i,}l_i)}{\lambda}$ is the monetary value of my level of utility
- $w_i t_i - X_i$ is the difference between earnings and spending, so that when I improve my life at ages of positive savings it is better.

How much would I pay for an event that, marginally, rose my probability of surviving by a constant at every age? i.e. a $\frac{\frac{\partial S_i}{\partial M}}{S_i} = \frac{S'}{S} = \overline{s}$. Then from equation 12

$$\frac{C'(M)}{\overline{s}} = \sum_{i=0}^{\infty} \left[\frac{U(X_i, l_i)}{\lambda} - X_i + w_i t_i \right] R^i S_i \rightarrow \textbf{Value of life} \tag{13}$$

Now, assume that the utility function is homogeneous of degree d in X and l so that, by Euler's theorem we can write it as:

$$\begin{aligned} U &= \frac{1}{d}(U_X X + U_l l) \\ &\rightarrow \frac{U}{U_X} = \frac{U}{\lambda} = \frac{1}{d}\left(X + \frac{U_l}{U_X} l\right) \geq \frac{1}{d}(X + wl) \end{aligned}$$

Notice that the right hand side is your $\frac{1}{d}$*"full expenditure' if you are working, so it holds with equality. The lhs is even larger than that if you are not working. Now it all depends on the value of leisure, and $\frac{1}{d}$ gives the difference between marginal and average utility. That is, if $d < 1$ you get more than what you give up (i.e. $\frac{U}{\lambda} > x + wl$). So, **the value of life is discounted full income**. Discounted with the interest rate and the probability of surviving and adjusted by the difference between marginal and average utility. That is, if we plug this back in equation 13:

$$\frac{C'(M)}{\overline{s}} = \sum_{i=0}^{\infty} \left[\frac{1}{d}(X_i + w_i l_i) - X_i + w_i t_i \right] R^i S_i$$

Assuming homogeneity gives us that the value of life is proportional to full income. This is why value of life estimates using foregone earnings are too small. Then, a country with twice the income has $\approx$ twice the value of life. So a poor guy will not pay $\$8,000$ for AIDS medicine (like in Africa).

Why is it that with education we didn't look at everything and here we do? Does education just affect earnings? No.

Everything is discounted by R^i so any effect that takes 20 years will not be worth much. So, if you are young and can take an action that increases your probability of surviving when old (say quit smoking), you value it very little (but people generally quit when they get older). Young people are self-destructive. Now, it might be true that the utility is worth less when you are old so the gains would be smaller. In any case, once you reach 80 you are not discounting much and might spend a lot. Also, notice that the level of S affects it. So, if you are in an environment where you are very likely to die you will not invest much. If I am HIV positive, why would I eat rich foods to avoid cholesterol.

As a country increases it's life expectancy, the more they want to improve it (like an addiction).

What if education also makes you more productive in the household? More educated people "appear" to have better health, vote more, commit less crime, raise better kids. We should estimate the return to non-market activities too. Non-market time is twice the time spent on market activities. Then $\frac{t}{T} = \frac{1}{3}$, if $\Delta wt = 7\%$ then $\Delta wT \approx 20\%$. Why don't people have more education then? Maybe we don't have such a big return to non-market activities, or we have capital market imperfections or self-selection because of disutility of going to school.

0.7 Fertility

In our analysis of bequests we assumed each family had just one child. Now, we are going to model the decision where families can decide how many kids to have and how much to invest in each. This is called the **quality-quantity** interaction. We can think of population as the extensive margin and education and health as the intensive margins.

0.7.1 Malthusian economy

Assume we have parents choosing consumption and how many kids they have, but kids are twins, triplets, etc. That is, they are all the same and come at the same time. We'll assume parents spend a certain amount on their kids (food, etc.) and can have them working at any age.

Notation 8 *Let C be the parent's consumption, f be the amount (food) spent on each kid, W_p be the parent's income, W_y be each kid's income, n be the number of kids. Define the "price" of each kid as his net cost: $P_n = f - W_y$*

Then, the budget constraint is given by:

$$C + nf = W_p + W_y n \Leftrightarrow$$

$$\boxed{C + nP_n = W_p}$$

Then, if children are profitable ($P_n < 0$) I want to have as many kids as I can. In many societies kids are sent to work early because it's important to them: it may not be socially optimal, but privately it is. We will derive why, as countries get richer they want to get away from it.

We assume preferences are separable in consumption and number of kids:

$$\begin{aligned} V &= u(C) + v(n) \\ u', v' &> 0 \text{ and } u'', v'' < 0 \end{aligned} \tag{14}$$

The FOC's of the parent are given by:

$$\begin{aligned} u'(C) &= \lambda \\ v'(n) &= \lambda P_n \end{aligned}$$

The SOC's are:

$$\begin{vmatrix} 0 & 1 & P_n \\ 1 & u" & 0 \\ P_n & 0 & v'' \end{vmatrix} = D = P_n^2 u'' + v'' < 0$$

and, the comparative statics of this model would give us:

$$\begin{pmatrix} 0 & 1 & P_n \\ 1 & u" & 0 \\ P_n & 0 & v'' \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial W_p} \\ \frac{\partial C}{\partial W_p} \\ \frac{\partial n}{\partial W_p} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \frac{\partial n}{\partial W_p} = \frac{P_n u''}{D} > 0$$

so, as income raises you want more kids (purely Malthusian result)

$$\begin{pmatrix} 0 & 1 & P_n \\ 1 & u" & 0 \\ P_n & 0 & v'' \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda}{\partial P_n} \\ \frac{\partial C}{\partial P_n} \\ \frac{\partial n}{\partial P_n} \end{pmatrix} = \begin{pmatrix} n \\ 0 \\ u' \end{pmatrix} \Rightarrow \frac{\partial n}{\partial P_n} = \frac{-nP_n u'' + u'}{D} < 0$$

So, "**even love responds to cost"** $_{\text{Gary Becker}}$

- Take the typical case of urban vs rural.

  Even though rural sector is poorer that urban sector, they have more kids. Why? Because the cost of raising a kid is lower (food is cheaper, they can work younger and have a higher productivity) so they have more kids.

What are the biggest forces we have neglected?

- Price component
- People have a quality dimension to kids, not just quantity

0.7.2 A Malthusian model with investment in human capital

Now, we will change the preferences of the parent. They no longer just decide how many kids to have, but also how much to invest in each. We keep the assumption that they are all equal so you invest the same in each kid. So, the new preferences are given by:

$$\begin{aligned} V &= u(C) + v(n) + \zeta(H) \\ u', v', \zeta, &> 0 \text{ and } u'', v'', \zeta'' < 0 \end{aligned} \tag{15}$$

We assume that parents can "buy" H at a fixed price P_H, so the budget constraint is given by:

$$C + nP_n + P_H nH = W$$

Notice that this budget constraint is not linear, we have nH in it. The FOC's are given by:

$$\begin{aligned} u'(C) &= \lambda \\ v'(n) &= \lambda(P_n + P_H H) = \lambda\pi_n \\ \zeta'(H) &= \lambda(P_H n) = \lambda\pi_H \end{aligned}$$

Here, π_n and π_H are the shadow prices of kids and human capital respectively. Notice that here the prices depend on quantities so that they are affected by your choices. In particular we can see that the cost of children rises in H and the cost of H rises in n. Why? If I have an additional child, I need to invest H in him as I am on the others. Also, if I decide to invest δH more in one, I need to invest δH more in the others too.

The SOC for this problem are harder, convex indifference curves are not enough because the budget constraint is concave too. To see this, look at the budget constraint keeping C constant (i.e. on a isoexpenditure line). We take differentials of the budget constraint:

$$P_n dn + P_H n dH + P_H H dn = 0 \Leftrightarrow \pi_n dn + \pi_H dH = 0$$

Then the slope of this line is given by:

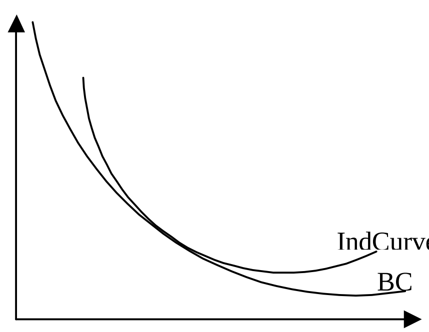
$$\frac{dn}{dH} = -\frac{\pi_H}{\pi_n} < 0$$

Now, if we take $n = n(H)$, we can get the second derivative of this curve:

$$\frac{d^2n}{dH^2}P_n + P_H H\frac{d^2n}{dH^2} + P_H\frac{dn}{dH} + P_H\frac{dn}{dH} = 0 \Leftrightarrow$$

$$\boxed{\frac{d^2n}{dH^2} = -\frac{2P_H\frac{dn}{dH}}{\pi_n} > 0}$$

Graphically what we have is:



Now, to get the second order conditions the math is a lot easier if we rewrite the model. I plug the budget constraint into the utility function, so the parent maximizes:

$$V = u\left(W - nP_n - P_H nH\right) + v\left(n\right) + \zeta\left(H\right)$$

FOC's:

$$\begin{aligned} -u'\left[P_n + P_H H\right] + v' &= -u'\pi_n + v' = 0 \\ -u'\left[P_H n\right] + \zeta' &= -u'\pi_H + \zeta' = 0 \end{aligned}$$

and we impose the SOC (i.e. the negative definiteness of the Hessian does not follow directly):

$$\left| \begin{array}{ll} u''\pi_n^2 + v'' & -u'P_H + u''\pi_H\pi_n \\ -u'P_H + u''\pi_H\pi_n & u''\pi_H^2 + \zeta'' \end{array} \right| = D$$

$$D = \left(u''\pi_n^2 + v''\right)\left(u''\pi_H^2 + \zeta''\right) - \left(-u'P_H + u''\pi_H\pi_n\right)^2 > 0$$

This weird interaction between quantity and quality fundamentally increases price elasticities (i.e. makes everything more responsive) and it can make children look like inferior goods. For example, suppose there is an increase in P_n so children are more costly. Then:

- First order$\rightarrow\downarrow n \rightarrow\uparrow H$
- Second order$\rightarrow\uparrow H \rightarrow\uparrow P_n$ even more! so it becomes more responsive

Formally:

$$\left(\begin{array}{ll} u''\pi_n^2 + v'' & -u'P_H + u''\pi_H\pi_n \\ -u'P_H + u''\pi_H\pi_n & u''\pi_H^2 + \zeta'' \end{array} \right) \left(\begin{array}{c} \frac{\partial n}{\partial P_n} \\ \frac{\partial H}{\partial P_n} \end{array} \right) = \left(\begin{array}{c} u' \\ 0 \end{array} \right)$$

$$\Rightarrow \begin{array}{r} \frac{\partial n}{\partial P_n} = \frac{u'\left[u''\pi_H^2 + \zeta''\right]}{D} < 0 \\ \frac{\partial H}{\partial P_n} = -\frac{u'\left[-u'P_H + u''\pi_H\pi_n\right]}{D} > 0 \end{array}$$

$\rightarrow$In both cases, the first term is what would happen if no interaction existed and the second one is result of this interaction.

Suppose now that income increases:

- First order$\rightarrow\uparrow n \rightarrow\uparrow H \rightarrow\uparrow C$
- Second order (hypothetical)$\rightarrow\uparrow H$ a lot$\rightarrow$ very high $\uparrow P_n \rightarrow\downarrow n$!

Formally:

$$\left(\begin{array}{ll} u''\pi_n^2 + v'' & -u'P_H + u''\pi_H\pi_n \\ -u'P_H + u''\pi_H\pi_n & u''\pi_H^2 + \zeta'' \end{array} \right) \left(\begin{array}{c} \frac{\partial n}{\partial W} \\ \frac{\partial H}{\partial W} \end{array} \right) = \left(\begin{array}{c} u''\pi_n \\ u''\pi_H \end{array} \right)$$

$$\Rightarrow \begin{array}{l} \frac{\partial n}{\partial W} = \frac{u''\pi_n\left[u''\pi_H^2+\zeta''\right]-u''\pi_H\left[u''\pi_H\pi_n-u'P_H\right]}{D} \gtreqless 0 \\ \frac{\partial H}{\partial W} = \frac{u''\pi_H\left[u''\pi_n^2+v''\right]-u''\pi_n\left[u''\pi_H\pi_n-u'P_H\right]}{D} \lesseqgtr 0 \end{array}$$

Both cannot be inferior at the same time. Why? The only way I got n to go down when W increased is to have H to increase a lot so both cannot be ambiguous. i.e. the inferiority of one depends on the superiority of the other.

Now, during the process of economic development, there is an increase in H, and we could have that at some point this causes a decrease in n, thus increasing H even more and decreasing n even more...That is, once we get this phenomenon it feeds in itself, which is exactly what happens when countries develop. This (higher $W \rightarrow$ higher $H \rightarrow$ lower n) is not only true across countries but also across groups within a country.

This has a couple of Macro implications:

1. Degree of regression to the mean: low income kids will regress to the mean slower (since they have less H and they have a lot of siblings) i.e. high fertility at the low end tends to keep them down because they get very little invested in them. So, I have a lot of forces working against regression to the mean.

2. Think of wage inequality in skilled vs unskilled terms. What are the determinants of this? Think of no regression to the mean (if you are poor you get no investment and are unskilled and vice versa). Suppose $n = d(W_p), d' > 0$ and that the level of skill is determined as $S_c = f(W_p), f' > 0$. Then, if we have two classes, skilled and unskilled we have that: $W_{st} > W_{ut} \Rightarrow n_{s,t+1} < n_{u,t+1}$. Then, the relative demand is given by: $\frac{W_{s,t+1}}{W_{u,t+1}} = \psi\left(\frac{n_{s,t+1}}{n_{u,t+1}}\right), \psi' < 0$.

   So, if we start with 100 and 100 of each class and say $n_s = 2$ and $n_u = 4$, next period the ratio is no longer 1:1 but 2:1. Then $\frac{W_{s,t+1}}{W_{u,t+1}} > \frac{W_{s,t}}{W_{u,t}}$ so that demography contributes to wage inequality.

0.7.3 Building preferences

We have been assuming that preferences are separable in quality and quantity of children. Is there some way to see how this utility function looks like?

Assume:

1. Utility function of parents is still separable but now they invest whatever they want in each kid:

$$V = u(C) + \sum_{i=1}^{n} v_i(H_i) \tag{16}$$

Now, take a given n and solve the subproblem of choosing how much to invest in each kid given a level of expenditure on kids $E^0 = W + C^*$

$$\max \sum_{i=1}^{n} v_i (H_i) \, s.t. \sum_{i=1}^{n} P_{H_i} H_i = E^0$$

FOC:

$$v_i' (H_i) \leq \mu P_{H_i} with = \Leftrightarrow H_i^* > 0$$

2. Assume $v_i = v$ for all i

3. $P_{H_i} = P_H$

$$\Rightarrow v' (H_i) = \mu P_H \rightarrow H_i^* = H^*$$

That is, we get the same preference results as before but now as a theorem. In order to get that people invest the same in each kid, we need to assume 2 and 3. Now, plug this back into equation 16:

$$V = u (C) + \sum_{i=1}^{n} v (H^*) = u (C) + nv (H^*) \tag{17}$$

So, we see it's not additively separable in n and H. That is we derived the form of the preferences. Notice that with these new preferences children and human capital are complements since $\frac{\partial^2 V}{\partial H \partial n} = v' > 0$. Here, the marginal utility of having another child is the level of utility of each child! That is, the more I invest in each child the higher the utility of having another one (ceteris paribus).

Case 9 *We could make other assumptions, for example we could assume that* $v = v_i$ *but* $P_{H_i} \neq P_H$. *Then* $v' (H_i) \leq \mu P_{H_i}$ *and:*

$H_i^* > H_j^* \Leftrightarrow P_{H_i} < P_{H_j}$

Take the case of boys vs girls. If boys provide more income $\Rightarrow P_{Hboys} < P_{Hgirls}$ *and* $H_{boys} > H_{girls}$ *and it's not that parents like boys more than girls.*

0.8 A general model of investment in human capital

Until now, we had no support at old age from kids to parents since we had only two periods. Adults died when their kids earned money so there was no possibility of getting supported by them. Now, we will consider a 3 period model: young, middle aged (adult) and old. As before, a person only has earnings when he is adult. A kid is supported by his dad and an old individual consumes out of his earnings and/or support he receives. There is no uncertainty (or full insurance); so, an individual's problem consists of choosing consumption at each level of life, bequests, investments in kids and how many kids to have.

Notation 10 *Let β be the one period discount rate, $i = \{c, m, o\}$ index whether a person is a child, middle aged or old, K_i be the amount of savings , E_i be earnings, S_i be transfers and let the rest of the variables be defined as before.*

0.8.1 One kid model with no support at old age

The utility function for a parent would be given by:

$$V_p = U(C_m) + \beta U(C_o) + \beta a V_c \tag{18}$$

Now, to get the budget constraint for a parent notice that:

- At middle age: $C_m + Y_c + K_p + f = E_p$
- At old age: $C_o = R_k K_p - B_c$

Putting this two together we get:

$$C_m + \frac{C_o}{R_k} + Y_c + \frac{B_c}{R_k} + f = E_p \tag{19}$$

So, the problem of a parent is to maximize equation 18 subject to the budget constraint given by equation 19. The FOC's of this problem are given by:

$$\left.\begin{array}{l} U'(C_m) = \lambda \\ \beta U'(C_o) = \frac{\lambda}{R_k} \end{array}\right\} \frac{U'(C_m)}{\beta U'(C_o)} = R_k$$

$$\beta a \frac{\partial V_c}{\partial E_c} \frac{\partial E_c}{\partial Y_c} = \lambda \text{ if } Y_c > 0$$

$$\beta a \frac{\partial V_c}{\partial E_c} 1 \leq \frac{\lambda}{R_k} \text{ with } = \Leftrightarrow B_c > 0$$

- Suppose $B_c > 0 \Rightarrow \frac{\partial E_c}{\partial Y_c} \equiv R_y = R_k$ as before, if you are giving bequests you must be getting the same return in both assets.
- Suppose $B_c = 0 \Rightarrow \frac{\partial E_c}{\partial Y_c} \equiv R_y > R_k$ so you give no bequests since the return to investing in the kid's human capital is larger. Now, in this case $\beta a \frac{\partial V_c}{\partial E_c} < \frac{\lambda}{R_k} = \beta U'(C_o)$ so that $a \frac{\partial V_c}{\partial E_c} < U'(C_o)$ the marginal utility you get from giving your kid an additional dollar is smaller than what you get from consuming it yourself. That is, $B_c = 0$ implies that you would want them to help you out. You get more utility from getting a dollar from them than what you lose from not having it.

0.8.2 The general model

We have a 3 period model and we will assume that kids earn no money (to ease notation)and let us use the utility function we derived before (see equations 16 and 17) so that the utility of a parent is now given by:

$$V_p = U(C_m) + \beta U(C_o) + \beta a n V_c \tag{20}$$

The budget constraint would be derived from:

- At middle age: $C_m + nY_c + K_p + nf = W_p - S_p$
- At old age: $C_o = R_k K_p - nB_c + nS_c$

That is, now the investment is made per kid so investment and fixed cost are multiplied by n. The earnings of the parent now subtract the transfer he makes to his parents (S_p) and the earnings of an old aged individual now include the transfers he receives from each kid (S_c). So, the budget constraint is given by:

$$C_m + \frac{C_o}{R_k} + \frac{nB_c}{R_k} + n(f + Y_c) = W_p + \frac{nS_c}{R_k} - S_p \tag{21}$$

We are going to take S_p and S_c as exogenous[2]. In this case, the FOC's are:

$$\left.\begin{array}{l} U'(C_m) = \lambda \\ \beta U'(C_o) = \frac{\lambda}{R_k} \end{array}\right\} \frac{U'(C_m)}{\beta U'(C_o)} = R_k$$

$$\beta a n \frac{\partial V_c}{\partial E_c} \frac{\partial E_c}{\partial Y_c} = \lambda n \text{ if } Y_c > 0$$

$$\beta a n \frac{\partial V_c}{\partial E_c} \leq \frac{\lambda n}{R_k} \text{ with } = \Leftrightarrow B_c > 0$$

$$\beta a V_c = \lambda \left(f + \frac{B_c}{R_k} + Y_c - \frac{S_c}{R_k} \right) = \lambda \pi_n$$

- Notice that the last FOC (the one we take with respect to n) depends only on the **level** of utility, the higher the level of utility of my kids, the more I want to have them.
- As before, if $B_c = 0$ then $R_y > R_k$
- There is a problem of time consistency, why would your kids want to take care of you?

[2]Actually, parents influence the upbringing of their kids to make them feel more altruistic (or guilty) towards them. See Becker's Nobel lecture

- Would an increase in S_c increase fertility? $\frac{B_c - S_c}{R_k}$ is what is relevant. Take everything as given and assume that $B_c = 0$ and there's an increase in S_c. Then, parents are happy with this and tend to have more kids. As we explained before, they obtain a higher utility from getting a dollar themselves than to giving it to their kids.

  However, since they have diminishing marginal utility, as we keep increasing S_c at some point the marginal utility of their kids becomes more important and $B_c > 0$. From that point on, any increase in S_c will be matched by an increase in B_c. That is, they would give it back to them. Why? If I have $B_c > 0$ I could have gotten more from my kids by cutting B_c. So, if I see an increase in S_c, if I didn't want it before why would I want it now.

  The fact that I have $\Delta S_c = \Delta B_c$ is know as **Ricardian equivalence**. Notice that we can't simply test this since we have people with zero bequests who would take the increase themselves and people with positive bequests who would show Ricardian equivalence. So, only if $B_c = 0$ would an increase in S_c increase n and investment per kid **could** decrease.

The next step would be to analyze social security from this. As societies get richer, parents with $B_c > 0$ would not need child support. To incorporate social security, we can think of a government providing income by taxing the workers. Suppose you put a head tax in each worker so that you collect depending on number of people. However, the yield will depend on the number of retired people you have. The yield is given by $R_{t+1} = \frac{N_t T}{N_{t+1}}$, but $\frac{N_t}{N_{t+1}}$ is determined by fertility (and death rates) so that the higher the fertility of everyone the better I am. That is, population growth plays an important role.

Over half the world population has below replacement fertility rates. In those countries, pay-go social security is hard to operate without high tax rates. A pay-go systems tends to discourage fertility, even though high fertility is what finances the system. Suppose they impose a tax but everyone is identical and altruistic. Basically they take money from your kids and give it to you, but you offset that by increasing B_c thus increasing the cost of each kid. Why isn't if offset by what you get? Because you take as given the average fertility rate so what you get doesn't depend on you fertility rate but on the economy's.

Human capital and the economy

Until now, we have talked about micro foundations. Let's use this to explain economy wide events. Basically, the variables we have stressed are important to understand specialization in the economy and the growth process.

0.9 Specialization and the division of labor

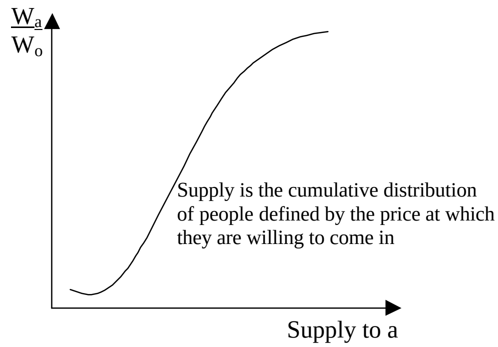
Adam Smith has a chapter in the wealth of nations called "the division of labor is limited by the extent of the market". Let's look at the division of labor. It has to do with heterogeneity, it is the main factor behind exchange and trade. In the 15^{th} century, Italy had like 150 occupations in a city. Now, we have tens of thousands of them. Why do we get so much specialization?

- People have different talents
- Investment

We'll concentrate in thinking that everyone is identical and what differentiates them is their investment. In the Roy model, people have different talents with different prices so that you enter the sector where you have a comparative advantage. I.e. I might be a better than average programmer but an even better lawyer, so I'll be a lawyer. The problem with this view (according to Becker) is that skills are not given they are produced. Ability depends on the investment you make but if they are endogenous, the Roy problem becomes very simple. If I invest in a smaller set of skills instead of spreading it, the return is higher. I'll do better investing in one occupation than if I try to be both an economist and a lawyer.

So, we need an analysis of what determines this investment. If we had a case where if you are good at something you are lousy at the rest, then it would be very easy. Well, this is what happens with investment. We'll assume that when you specialize you go into it.

Remark 11 *The elasticity of supply is largely determined by heterogeneity in the population. If we have equal people, supply will be very elastic. Say, people* *with high abilities enter at low prices and people with low ability need a higher price to enter.*



One of the important characteristics of investment in human capital is that the rate of return tends to increase the more time you put into it. So, activities that take a lot of investment usually are not part time (like a lawyer). But this means that there are **increasing returns** to investment in human capital if we **specialize**. Why do we incur the cost of having to organize specialized people? Because we have increasing returns.

We will assume there are a bunch of tasks needed to produce one product. We will treat the general problem of m tasks (where m can be very large). All people are the same (equally able and with the same level of initial human capital) and there is equal financing. Each task takes some investment: you spend some time investing, so that the more time you invest the more skilled you become. So, if we let t be the amount of hours available an individual must divide his time between investing in a task (h) and working (l), such that $t = l + h$. There might be complementarities, but we will assume them away.

Define a production function of human capital given by:

$$\begin{aligned} H &= g(h) \\ g' &> 0, g'' \leq 0 \end{aligned} \tag{22}$$

So we have no increasing returns from here (the economics of the problem will give us the increasing returns). Now, the task output you get is given by the amount of time you spend working times the effectiveness of that time:

$$Y = lH \tag{23}$$

Now, an individual does not care about H or l but only cares for Y. That is, his goal is to maximize equation 23 subject to equation 22 and $t = l + h$. The FOC's of this problem are given by:

$$\left.\begin{array}{l} H = \lambda \\ g'(h)\, l = \lambda \end{array}\right\} l = \frac{g(h)}{g'(h)}$$

Now, if we assume that $g = ch^{\theta}$ for $0 < \theta \leq 1$. Then the solution would be given by:

$$l^* = \frac{1}{1+\theta}t \text{ and } h^* = \frac{\theta}{1+\theta}t$$

If $\theta = 1$ then $l^* = h^* = \frac{1}{2}$. As θ falls, h^* falls relative to l^*. Now, look at Y :

$$\begin{array}{ll} \text{General case} & Y = \frac{[g(h)]^2}{g'(h)} \\ g = ch^{\theta} & Y = \frac{c}{\theta}\left(\frac{\theta}{1+\theta}\right)^{1+\theta} t^{1+\theta} \end{array}$$

In the first case, we get $[g(h)]^2$, i.e. the square means we get increasing returns. In the second case, we get $t^{1+\theta}$ so Y is not growing proportionally to time, it's growing more than proportionally. If $\theta = 1$ then $Y = \frac{ct^2}{4}$. The point is that, if you double t you more than double Y, so I'll go to a corner (I'll specialize).

Now, suppose we have more than one task, say two. As before, we have that $t = l_1 + l_2 + h_1 + h_2$, $Y_i = l_i H_i$ and $H_i = g_i(h_i)$ for $i = \{1, 2\}$. What do we maximize now? Assume that we need both tasks to produce output, in particular assume that:

$$Q = f(Y_1, Y_2) = \min[Y_1, Y_2] \tag{24}$$

Assume that I have to produce both tasks so $t = t_1 + t_2 = (l_1 + h_1) + (l_2 + h_2)$. Then, given t_i I solve the i^{th} problem and get (as before):

$$\begin{array}{ll} \text{General case} & l_i^* = \frac{g(h_i^*)}{g'(h_i^*)} \text{ and } h_i^* = t_i - l_i^* \\ g_i = c_1 h_i^{\theta_i} & l_i^* = \frac{1}{1+\theta_i}t_i \text{ and } h_i^* = \frac{\theta_i}{1+\theta_i}t_i \end{array}$$

How do we find the optimal t_i? It must be that $Y_1^* = Y_2^*$, so that in the particular case we are using:

$$Y_1^* = \frac{c_1}{\theta_1}\left(\frac{\theta_1}{1+\theta_1}\right)^{1+\theta_1} t_1^{1+\theta_1} = \frac{c_2}{\theta_2}\left(\frac{\theta_2}{1+\theta_2}\right)^{1+\theta_2} t_2^{1+\theta_2} = Y_2^*$$

If, we specialize to the case where $\theta_1 = \theta_2 = \theta$ then:

$$c_1 t_1^{1+\theta} = c_2 t_2^{1+\theta}$$

If $c_1 = c_2$, then we would divide the $t's$ equally; otherwise, the higher the c the lower the t. Now, further specialize to the case where $c_1 = c_2 = c$ and $\theta = 1$. Then, we have that:

$$t_i = \frac{t}{2}; Y_i = \frac{c}{4}\left(\frac{t}{2}\right)^2 = \frac{ct^2}{16}$$

$$\boxed{Q = \frac{ct^2}{16}}$$

It looks the same as before with one task. What's the difference? What does the formula show? It doubles the t but it **also** doubles the denominator, so I am less productive than before. Look at the case of m tasks, assuming the same as before:

$$\begin{aligned} Y_i^* &= \frac{[g_i(h_i^*)]^2}{g_i'(h_i^*)} = \lambda \text{ for all } i \\ \sum t_i^* &= t \text{ and } l_i^* + h_i^* = t_i \end{aligned}$$

If we look at the particular case of $\theta = 1, c_i = c \Rightarrow g_i = ch_i$ we get that:

$$Q^* = \frac{ct^2}{4m^2} \text{ since } t_i^* = \frac{t}{m}$$

Even if $\theta = 1$ as m gets large, you cut very much into your productivity. We still have increasing returns to scale, but by having to cut into time to dedicate to each task you cut into your productivity. **Solution?: exchange and specialization**.

So, we want to form a team, a group of specialized individuals working together to produce output. If we form a team, we get rid of the m^2 disadvantage. Suppose we have m tasks and n team members (all identical so they could use the market to produce). Each person has t time available so each puts all his time into one task:

$$Y_i^* = \frac{c}{\theta}\left(\frac{\theta}{1+\theta}\right)^{1+\theta} t^{1+\theta} = \frac{ct^2}{4}$$

Where the last equality assumed that $\theta = 1$ (best case scenario). In this case we get that total production is given by:

$$\boxed{Q^* = \frac{ct^2}{4}}$$

Just as before right? Wrong, this is total output, before I got this just for me. Now I have to share it among the members of the team. Suppose we divide it equally, then I would get:

$$Q_j^* = \frac{Q^*}{m} = \frac{ct^2}{4m} > \frac{ct^2}{4m^2}$$

In any case, it is better than what I get if I have to do all the tasks myself but not as good as the case of just one task. One was falling at a rate m^2 in this case it just falls at a rate m. That is, there is a cost to having a lot of tasks but the cost is linear while the return is quadratic. If $m = 15$ you get 15 times more per capita income than by each one producing it themselves.

Case 12 *Division of labor in the household*

Suppose you have a couple of equally productive people. Will there be a gain in the division of labor? Yes, if there are tasks with some complementarity and there are gains to investment. Think of household and market with similar production functions to the ones we have. Now, $m = 2$ and $n = 2$.It is more efficient to have one member work in the market and other in the household.

Why have women historically been the ones in the household? More challenging, it's not implied by the model of investment in human capital per se. 2 possible explanations:

1. *Discrimination against women in the market*
2. *Intrinsic difference, women may have absolute advantage, but relatively more productivity in the household.*

You don't need a lot of either to explain this. Suppose it's true but people are the same. A little bit of discrimination can cause this and the other way around. Segregation occurs very easily (to avoid discrimination or to gain by biological differences).

0.10 The division of labor and the extent of the market

Why could Smith be right? Is he?

Suppose there are m skills where m is very large and they are independent so there are huge gains from specialization. But, the market may be limited and prevent specialization. One interpretation is, given we have just one product, that the extent of the market is given by the number of available workers. Let N be the number of identical workers, then:

1. If $N > m$ and $Q = \min\{Y_1, ..., Y_m\}$

   How many people would produce each task? $\frac{N}{m} = \frac{1}{s}$. So, if we have $m = 1000$ and $N = 2000$ we have two teams with m members each. If $\frac{1}{s}$ people compete and everyone end up earning the same, here division of labor is not limited by the extent of the market.

2. If $N < m$ and $Q = \min\{Y_1, ..., Y_m\}$

   Now, I don't have enough people to go around so I'll have $\frac{m}{N} = s$ tasks per person. In this case, the division of labor is limited by the extent of the market. In a very small town you have a surgeon, while in a city like Chicago you have a heart surgeon, a brain surgeon, etc.

   Suppose $g = ch^{\theta}$, and that a person puts time t_i time into the i^{th} task. Then, $Y_i^* = kt_i^{1+\theta}$ where $t_i = \frac{t}{s}$, and total production would be:

   $$Q^* = Y_i^* = k\left(\frac{t}{m}\right)^{1+\theta} N^{1+\theta}$$

and per capita income would be:

$$q^* = \frac{Y_i^*}{N} = k\left(\frac{t}{m}\right)^{1+\theta} N^{\theta}$$

They don't want to do a little bit of each task because of increasing returns. So, as $m > N$ per capita income rises with N. When $N = m$ per capita income doesn't grow anymore. Notice that getting more teams actually buys me competition even when I have increasing returns.

Now, if it were really true that the division of labor is limited by the extent of the market we shouldn't see more than one person doing any one task in a small town. How can we explain that? We can't, so there must be something else limiting the division of labor other than the extent of the market. How do we explain it? **Coordination costs**. We need coordinators (entrepreneurs of various forms). We as economists usually think of it this way, this is Coase's transaction costs idea. So, when there's specialists you need to coordinate them since we have fixed proportions. If someone doesn't produce his share he screws it up for everyone. Notice that they are not given, it is a function of the degree of specialization.

Let n be the number of members of any one team. Then, the number of two way interactions is $n(n-1)$. Let $c(n)$ be the coordination costs per member of the team so that $c'(n) > 0$. Then, net per capita income:

$$I(n) = Y(n) - c(n)$$

We have a different maximization problem now: choose n to maximize per capita income. In this way, even when $N > m$ we will have that the size of the team is smaller than m. That is, the question is, why even when we have $m > N$ we might get more than one team? The FOC of the problem is given by:

$$\frac{\partial I(n)}{\partial n} = \frac{\partial Y(n)}{\partial n} \gtreqless \frac{\partial c(n)}{\partial n}$$

and the SOC would be $c_{nn}(n) > Y_{nn}(n)$.

1. Suppose that $Y_n(n) > c_n(n)$ for all $n \leq N \leq m$

   Then, $n = N \leq m$ so that everyone is in the same team and it's only the extent of the market limiting specialization.

2. $Y_n(n) < c_n(n)$ for all $n > 1$

   Then, we would see no specialization, e.g. peasant households.

3. For $1 < n^* < N$

   There will be some specialization ($n^* > 1$) but it's not to the extreme of market limiting. Suppose $m = 10000$ and $N = 5000$, it might appear that the extent of market is limiting but if $n^* = 1000$ we will see 5 teams, so it's not the extent of the market limiting. We could go up to two teams and it wouldn't be the extent of the market limiting specialization.

Take an example with:

$$\begin{aligned} I &= cn^{\theta} - \lambda n^{\beta} \\ \beta &> \theta \end{aligned}$$

In this case, $n^* = \left(\frac{c\theta}{\lambda\beta}\right)^{\frac{1}{\beta-\theta}}$ rising in c, the productivity of specialization; and falling in λ, the level effect of coordination costs. So, higher $\lambda' s$ means higher costs, less specialization and smaller teams. This is more or less Hayek's idea. In Russia, it was very unspecialized (high $\lambda' s$), a steel plant also raised hogs to feed its workers!

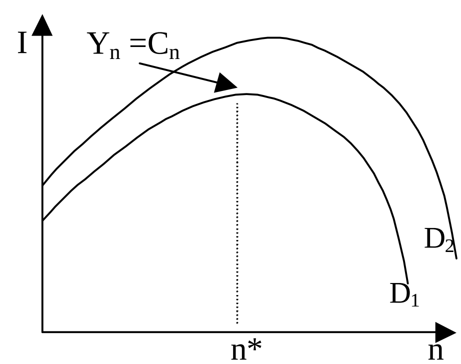
Now, coordination costs may depend on other factors like population density. They might depend on population density, e.g. $\lambda(D)$ with $\lambda' \leq 0$. So, cities would increase density and reduce these costs. That's why they exist (not because there is a bigger market).

So we have that $I^* = I^*(\lambda, \beta, \theta)$ with $\frac{\partial I^*}{\partial \lambda} < 0$. Also, we say that $\frac{\partial I^*}{\partial D} = \frac{\partial I^*}{\partial \lambda}\frac{\partial \lambda}{\partial D} > 0$. That is, both the division of labor and per capita income should be higher in cities. But as an economist that is a problem. What about equilibrium? Why don't everyone goes to the city? Because higher human capital people are in large cities. Why is there such a sizable city size effect? Population density is not only created by allocation but also by population growth.

If $D = \alpha P \Rightarrow \lambda(D) = \lambda(\alpha P)$ so that $\frac{\partial \lambda}{\partial P} \leq 0$. This is a force why per capita income might be rising in level of population. That is, even if density is not proportional to population, as long as $\frac{\partial D}{\partial P} > 0$, we have that:

$$\frac{\partial I^*}{\partial P} = \frac{\partial I^*}{\partial \lambda}\frac{\partial \lambda}{\partial D}\frac{\partial D}{\partial P} > 0$$

This is a source of increasing returns and it is consistent with competition too! As the amount of people grows, the number of teams might increase or decrease, so we **could** have competition.



So, if we increase density to D_2 we increase both n and I.

Now, we assumed that each person started with a certain amount of human capital (general human capital). Why do general human capital interact with specific human capital? It seems that the more general skills I have, the more productive I will be in specific human capital. So, now:

$$Y = Y(H, n) \text{ where } H \text{ is general human capital}$$

Suppose $Y = cn^{\theta}H^{\gamma}$, then all I need is $\frac{\partial Y}{\partial H} > 0$ (i.e. $\gamma > 0$) and $\frac{\partial^2 Y}{\partial H \partial n} > 0$. So, as I increase H for any degree of specialization I rise my productivity. What about coordination costs? Since we don't know, we will assume there is no effect. So, we have the FOC:

$$\frac{\partial I}{\partial n} = Y_n - c_n = 0$$

$$\Rightarrow \frac{\partial^2 I}{\partial H \partial n} = Y_{nH} + Y_{nn}\frac{\partial n}{\partial H} - c_{nn}\frac{\partial n}{\partial H} = 0$$

$$\Rightarrow \boxed{\frac{\partial n}{\partial H} = \frac{Y_{nH}}{C_{nn} - Y_{nn}} > 0}$$

That is, the optimal division of labor rises with general human capital. Now, if $I = cn^{\theta}H^{\gamma} - \lambda n^{\beta}$ we get that $n^* = \left(\frac{c\theta}{\lambda\beta}\right)^{\frac{1}{\beta-\theta}} H^{\frac{\gamma}{\beta-\theta}}$ which is what I had before times an interaction term with general human capital. Also, we get that:

$$I^* = K_1 \lambda^{-\frac{\theta}{\beta-\theta}} H^{\frac{\gamma\beta}{\beta-\theta}}$$

So, both per capita income and n rise in H. Then, if we increase H by 1% we increase I by more than γ since $\frac{\beta}{\beta-\theta} > 1$. That is, it is not only the direct effect of H (given by γ) but also that we are increasing the degree of specialization. So, an increase in H should lead to greater specialization and greater per capita income.

Remember we mentioned Hayek. Well, now that we include H it is even more important. Russia did fine when producing steel but couldn't produce good planes, etc. So, as economies become more specialized the coordination gets harder.

Can we look at this as a growth model? We said that as H increases both n and I increase. But, it is not only that specialization is induced by growth in H but also that growth in H might be induced by growth in specialization. The rate of return on H is given by:

$$\frac{\partial I}{\partial H} = \frac{K\gamma\beta}{\beta - \theta} H^{\frac{\gamma\beta}{\beta-\theta} - 1}$$

It tells me how this marginal product moves with H. Now, if $\frac{\gamma\beta}{\beta-\theta} = 1$ I have no increase or decrease of the marginal return with H. Notice , I can assume $\gamma < 1$ (diminishing returns) and still get growth. That is, we get endogenous non diminishing returns via the interaction with specialization.

Now, come back to cities. People in cities have larger H because they have invested more in their general human capital. So, now we have an answer to the question of why cities have higher income in equilibrium. People with higher human capital tend to go to cities. Now, there has been an important change in coordination costs with the advent of telecommunications. This makes physical proximity less important, so the advantage of large cities is going down and we see more specialization at a larger distance. However, you can't go to a good restaurant in internet so cities won't disappear.

0.11 Human capital and growth

Specialization is an example of why increasing returns might rise in a model. These models are the opposite to how we started looking at it. Think of Malthus, it was very important for his argument to have decreasing returns. We think Malthusian principles are more relevant to agricultural, traditional societies. Now, the Malthusian model is very interesting for various reasons.

- As progress of how income is determined at the economy level it was the first precise dynamic model.
- For much of the world up to recent time it was actually very relevant
- Some say he had a theory of population, we actually think he had a theory of wages (average income) that has repercussions to population.

Two building blocks:

1. Diminishing returns at the aggregate level to more workers
2. Family based model of fertility and mortality

The crossing of these two forces gave him equilibrium. Now, can we be neo-Malthusian? Food production doesn't seem to be a problem but we certainly have fixed resources so we can't have sustainable growth. Can we? So, understanding the Malthusian model is not only understanding the first dynamic consistent model.

0.11.1 The Malthusian model

We had a model of fertility that stated that $V(c,n) = u(c) + v(n)$. Now, we will use an overlapping generations model. Let p be the price of children, then the budget constraint is $pn + C = W$. If we solve this problem, the FOC's are:

$$\begin{aligned} u'(C) &= \lambda \\ v'(n) &= \lambda p \end{aligned}$$

We showed that $\frac{\partial n^*}{\partial W} > 0$. He talked about people marrying earlier or later as the adjusting variable. He also had mortality going down as income rose, so both forces rose population. The mortality force seems to be less important historically. We'll ignore it and stress the fertility.

Malthus tried to determine W, if we can do it then we have an equilibrium model. He looked at productivity and recognized that it is a function of weather, disease, machinery, etc. But in addition of shocks he said there is also the effect of density: the higher the labor supply the lower the productivity (decreasing returns). So, he had a link between factor inputs and income:

$$\begin{aligned} Q &= A(t) F(L, K) \\ \frac{\partial Q}{\partial L} &= AF_L > 0 \text{ and } \frac{\partial^2 Q}{\partial L^2} = AF_{LL} < 0 \end{aligned} \tag{25}$$

In his model he had $K =$ land, in the neoMalthusian K is some factor like fuel. Also, $\frac{\partial Q}{\partial A} = F(L, K) > 0$. Now, define $W = \frac{F(K,L)}{L} = \frac{Q}{L}$. His **fundamental** assumption is thus that $K = K^0$ fixed. But then, as population increases, we get decreasing returns. Now:

$$\frac{\partial W}{\partial L} = \frac{\frac{\partial Q}{\partial L}}{L} - \frac{Q}{L^2} = \frac{Q}{L^2}(\varepsilon - 1) < 0$$

So, $W = f\left(\frac{L}{K}\right)$ with $\frac{\partial f}{\partial L} < 0$.

We have two relations now:

1. Suppose we have a disease epidemic that cuts the population down. Then, the income of the survivors goes up, they tend to have more children thus increasing the rate of growth of population, thus **tending** to come back to where you started.

2. Suppose you have technological improvement in agriculture. Then, their income goes up and they tend to have more kids.

Will this bring me back to the original state in any of these cases?

Let $L(t) = N(t)$ be the number of adults (children do not work). Then, $L(t+1) = n(t) L(t)$. That is, every adult has n kids.

On the production side $W(t) = f(L(t))$ and we can say that $n(t) = D(W(t))$ with $D' > 0$. Then, $n(t) = D(f(L(t)))$ and:

$$\frac{\partial n(t)}{\partial L(t)} = \frac{\partial n}{\partial W}\frac{\partial W}{\partial L} < 0$$

So, if you increase population you lower fertility! and there is a force that **tends** to regulate this system[3]. Is there a steady state in W, n, L? Is it stable? Is there more than one?

If there is a steady state in population, then n must be constant at the replacement level $n = 1$ (i.e. each parent produces only one child). In modern

[3]Notice that we have then that the lower W the lower n. In modern economies this is the opposite of what we see.

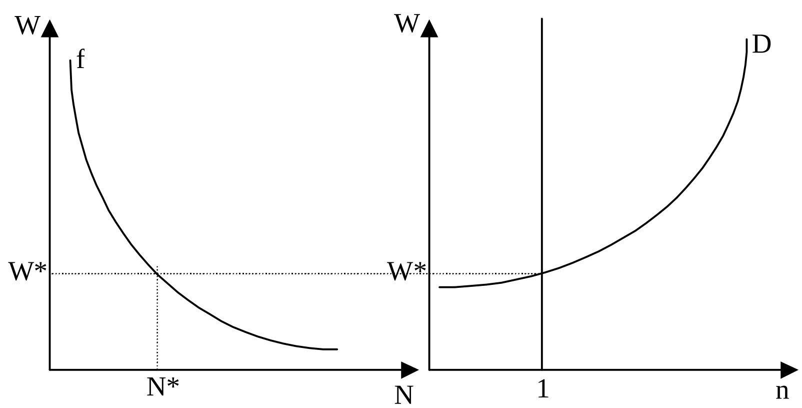
growth theory, what's the steady state fertility rate? It's not determined in the model (exogenous). Malthus on the other hand had a precise value. Does this Malthusian model have such a steady state outcome? Well, if $n = 1 = D(W^*)$ then:

$$W^* = D^{-1}(1)$$

This equation determines wages in equilibrium. Notice that diminishing returns do not matter, in equilibrium wages depend only on family behavior. Then, we can go to our other relation and get that:

$$L^* = f^{-1}(W^*)$$

Here is where we use diminishing returns, when solving for the steady state level of population. That is, we just found a (unique) steady state. Graphically:



So, Malthus did not need diminishing returns to solve for W, diminishing returns were crucial to guarantee L to be a steady state. Actually, Malthus assumed that "the passion between the sexes" was constant (stable preferences). Now, the equilibrium population level is less important. What determines welfare in steady state is W, which only depends on family behavior (n^*).

Now, take a linear approximation and define:

$$\begin{aligned} n(t) &= a + bW(t) \\ W(t) &= \alpha - \beta N(t) \end{aligned}$$

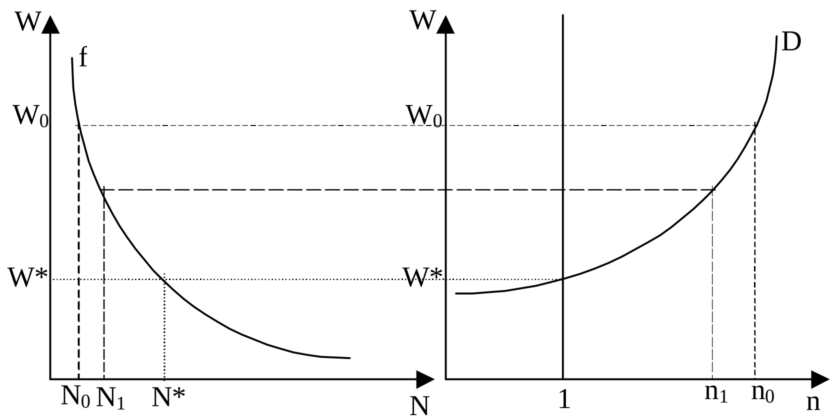
Then:

$$W^* = \frac{1-a}{b} \tag{26}$$

$$N^* = \frac{a-1}{b\beta} + \frac{\alpha}{\beta} \tag{27}$$

Notice that W^* only depends on the parameters of the demand function, while N^* depends on all parameters. Now, suppose you double A (technology). What happens to wages? **Nothing**. Here and increase in technology increases α, which doesn't enter the equilibrium wage given by equation 26. So, once and for all changes in technology do not affect wages. the conclusion that Malthus got from this was that we should not give money to the poor since they would just have more kids, increasing N^*, until W^* was the same as before. Is W^* the subsistence wage? No, there's nothing implied here about subsistence.

How about the dynamics? Think of the case of a disease that wipes out part of the population. Graphically:



In this case, population jumps from $N^* \rightarrow N_0$, but as a result per capita income increases to W_0, which increases fertility to a point like n_0. Given that fertility is now above replacement, next period population is larger at say N_1 which decreases per capita income more thus reducing fertility but not to the original replacement rate, etc. This means that our steady state is stable.

What if there is constant technological improvement. Now take a log-linear approximation to the model so that:

$$\log n\left(t\right) = a + b \log W\left(t\right) \tag{28}$$

$$\log W\left(t\right) = \alpha\left(t\right) - \beta \log N\left(t\right) = At - \beta \log N\left(t\right) \tag{29}$$

$$\log n\left(t\right) = \log N\left(t+1\right) - \log N\left(t\right) \tag{30}$$

Is there a steady state where $\log W\left(t\right) = \log W^*, \log n\left(t\right) = \log n^*$ and $\log N\left(t\right) = \log N^*$? If we want it to exist, it must be true that:

$$\frac{d \log W^*}{dt} = A - \beta \frac{d \log N^*}{dt} = 0 \tag{31}$$

But this means that if wages are to be constant and I have positive technological change, population must be rising (and thus $n^* > 1$). Why? From 30 we know that $n(t) = \frac{N(t+1)}{N(t)}$, so using equation 31:

$$\frac{d \log N^*}{dt} = \frac{\frac{dN^*}{dt}}{N^*} = \frac{A}{\beta} = g_N$$

$$\Rightarrow n^* = 1 + g_N = 1 + \frac{A}{\beta} > 1$$

Now, we can substitute this result in to equation 28 to get an expression for equilibrium per capita income:

$$\log W^* = \frac{\log\left(1 + \frac{A}{\beta}\right) - a}{b} \tag{32}$$

Notice some interesting things. Before we had that wages only depended on supply while now they depend on everything, so the faster the diminishing returns (the higher the β) the slower the equilibrium n^* and the lower the wage. This is different from the neoclassical model. Here we can solve for the rate of growth of the population once we allow for exogenous technical change. Even with continued technological change we do not get increasing per capita income, since the neoclassical model has constant returns to scale and we have decreasing. If we allowed $\beta = 0$, this would no longer be true. Fertility is still constant but at a higher level than before.

This model looks good throughout history from 0 to 1800. We had very slow population growth and almost no income growth. However, it breaks down at middle 19 century. Why? We began to see persistent growth in per capita income. Also, we no longer see that as income increases fertility does too.

Notice that in the Malthusian model we had decreasing returns to scale because we assumed $K = K^0$. But it might be consistent with constant or increasing returns to scale once we allow K to move. In fact, cities are the embodiment of this increasing returns property we think modern economies have.

The other thing we can change is the assumption that fertility increases with income by introducing the quantity-quality interaction. We can think that since people with higher income tend to have higher value of time, they will not have that many kids since they take time to raise. Also, they will tend to substitute towards higher human capital kids. This is in fact related to cities. In cities the cost of children is higher since people are richer and thus have a higher value of time. Why? We said that cities have higher density and thus higher specialization. Remember we had an equation relating this to per capita income $I^* = C_1 \lambda^{-\frac{\theta}{\beta-\theta}} H^{\frac{\gamma\beta}{\beta-\theta}} \rightarrow \frac{\partial^2 I^*}{\partial H \partial \lambda} = -\frac{\theta}{\beta-\theta} \frac{\gamma\beta}{\beta-\theta} C_1 \lambda^{-\frac{\theta}{\beta-\theta}-1} H^{\frac{\gamma\beta}{\beta-\theta}-1} < 0$. So, human capital is more profitable in cities even aside of people having higher incomes.

Now, suppose that fertility is greater in rural areas, but demand for agricultural products is growing at smaller rates than for other products. Then, I will have migration towards cities. Now, if parents know this, they will invest more in the human capital of their kids since they know they have a higher probability of moving to urban areas[4]. So, cities have higher wages because they get more able people, lower fertility and higher human capital, and they grow in size with economic development.

Cities are also important in the creation of knowledge. Take Silicon Valley, why do we have such a place? "Ideas are in the air" said Marshall (i.e. there are externalities in these areas). Good and bad, they steal labor and ideas from you and vice versa. So, it has higher than average turnover and lower than average unemployment. Also, research universities are located in cities. Is it an accident or is there something special about cities?

0.11.2 Beyond Malthus, economic growth in the last 150 years

We are going to take what we have done and address the challenge to extend the model and explain what has happened in the last 150 years, still getting the Malthusian model as a special case. in this model, we want households that, under some conditions, want to invest in human capital and under some don't. The same for the number of kids and economic growth.

Preferences are given by:

$$V_t = U_t\left(C_t\right) + a_t\left(n_t\right)V_{t+1} \tag{33}$$

The case of one child

In this case, preferences would be given by:

$$V_t = U_t\left(C_t\right) + a_t V_{t+1}$$

Now, if we substitute recursively (forward):

$$V_t = U_t\left(C_t\right) + a_t U_{t+1}\left(C_{t+1}\right) + a_t a_{t+1} V_{t+2}$$

That is, the t^{th} generation parents' utility depends on their kid's and grand-child's consumption. If we kept substituting recursively[5]:

$$V_t = \sum_{j=0}^{\infty}\left[\prod_{i=0}^{j} a_{t+i}\right] U_{t+j}\left(C_{t+j}\right)$$

So, the t^{th} generation's utility is given by the utility they get from all their descendants. Now we have dynasties. Actually, if we allowed n to change we

[4]We can't completely explain international migration because we only have one product. We might have more products with different skill intensities.

[5]We make the notational assumption that $\prod_{i=0}^{0} x_i = 1$.

might get families dying out. Assume that $a_t = a$ and $U_t = U$ for all t. Then, the dynastic utility function is given by:

$$V_t = \sum_{j=0}^{\infty} a^j U\left(C_{t+j}\right)$$

To get finite utility we must impose the transversality condition: $0 \leq a < 1$. That is, it is not only that you discount the future, but that you love yourself more than you do your descendants.

Is the family behavior time consistent? Yes, since I have time contingent plans. There is conflict between children and parents. Children want more since they do not care about you, but this is not time inconsistency. If, for example, we had preferences with merit goods:

$$V_t = U\left(C_t\right) + af\left(C_{t+1}\right)$$

This is not time consistent. How would you like them to spend it? Only on their consumption, but they will divide it among their consumption and their kid's consumption.

Allowing for the choice of fertility

As we saw, we can say that either the head of the dynasty decides everything or that preferences are recursive. Now, we have that the head has to decide over two variables: how many kids to have and how much human capital to accumulate. Now, we go back to the preferences we derived in equation 17. In this case, they would be:

$$V_t = U\left(C_t\right) + an_t V_{t+1}$$

Since all kids are the same, they all behave the same. So, if we substitute recursively:

$$\begin{aligned} V_t &= \sum_{i=0}^{\infty} N_{t+i} a_i U\left(C_{t+i}\right) \\ N_{t+i} &= \prod_{j=0}^{i} n_{t+j} \end{aligned}$$

That is, N_i is the number of descendants in the i^{th} generation. For example, if $n = 2$ then in the second generation you have 4 descendants in the third you have 8, etc. Now, the head's utility indirectly depends on their grandchildren utility. We could simply add a term for grandchildren to allow a less indirect relation.

Until now, we have not assumed a changes with n. That is, that the marginal utility we get from having one more kid is simply given by aV_{t+1}. However, we might want to introduce concavity in the marginal utility we get from kids.

In this way, the more kids I have, the less altruistic I am towards each. In particular, we will assume that:

$$a = a(n_t)$$

with $a' < 0$. The particular functional form we will use is:

$$a(n_t) = \alpha n_t^{-\varepsilon}$$

where α is "pure" altruism and $\varepsilon < 1$. Then:

$$V_t = \sum_{i=0}^{\infty} \alpha^i N_{t+i}^{1-\varepsilon} U(C_{t+i})$$

Suppose we further assume that the instantaneous utility function is of the form:

$$\begin{aligned} U(C) &= \frac{C^\sigma}{\sigma} \\ \text{with } 0 &< \sigma < 1 \end{aligned}$$

So, preferences are:

$$V_t = \frac{1}{\sigma} \sum_{i=0}^{\infty} \alpha^i N_{t+i}^{1-\varepsilon-\sigma} (C_{t+i} N_{t+i})^\sigma$$

Take N_{t+i} and C_{t+i} constant. Is this value we start from a possible solution to the problem. Suppose we increase N_{t+i} and decrease C_{t+i} such that $N_{t+i}C_{t+i}$ remain constant. If $1 - \varepsilon - \sigma < 0$ then a decrease in N would increase utility and lower cost of having children so I would be better off in both ways. This can't be an equilibrium, so $\varepsilon + \sigma > 1$. We could still have $\varepsilon = 0$ as before, i.e. constant marginal utility of having kids.

Look at the problem of parents deciding how many kids to have and how much to invest in their kids. Before we assumed that parents allocated goods between them and their kids, now we will worry about allocating time between working, producing kids and investing in them. Take a Leontieff cost function so that there is a minimum of f goods and v units of time per child. So, if there is no investment in the human capital of the kid $(H = 0)$ vn is the total time spent on kids and fn is the total goods allocated to kids, where we assume that v and f are exogenous. However, this minimum investment creates an endowment (unskilled labor say) of human capital H^0. Now, in order to have better quality kids, parents need to invest more that the minimum time in them. Let h be investment in human capital of kids so hn is total investment in kids. That is, if we let l be time spent working, a parent's time budget constraint is given by:

$$l + (v + h)n = T \tag{34}$$

Consumption depends on l, but we will assume that each unit of time worked has a certain efficiency given by the human capital you have. That is, the production function of consumption goods is:

$$Q = \gamma l \left[bH^0 + H \right] \tag{35}$$

so the production function has constant returns to scale. Here b gives me the trade-off between unskilled human capital and investment in human capital. In other words, $\frac{1}{b}$ units of H^0 are equivalent to one unit of H in production. If we substitute from the time constraint and plug in the budget constraint we get:

$$c + nf = Q = \gamma \left[T - (h + v)\, n\right] \left[bH^0 + H\right]$$

$$\Rightarrow c + nf + \gamma (h + v)\, n \left[bH^0 + H\right] = \gamma T \left[bH^0 + H\right] = \textbf{Full Income} \tag{36}$$

We will assume that kids can't work. Before, we had that the parents human capital entered the production function of the kid's human capital. Now, we will have something similar, but we can justify it in a different way. There are some "teachers" that affect the kids H, where each family is a "classroom". So, the kid's human capital production function is:

$$H_{t+1} = f\left(h_t, H_t\right) \text{ with } f_h > 0, f_H > 0$$

or, if we take the same functional form as the production function of goods (equation 35):

$$H_{t+1} = Ah_t \left[dH_t^0 + H_t\right]^\beta \tag{37}$$

where $0 < \beta \leq 1$. Here d plays the same role b played in goods production. $\frac{1}{d}$ units of H^0 are equivalent to 1 unit of H in producing human capital. Now, assume that there is some underlying process of specialization (say teachers) affecting this reduced form, so that it wipes out the diminishing returns and $\beta = 1$. This is the same thing we had before, where specialization created constant returns out of a decreasing return technology.

We think that raw labor is less important in producing human capital than goods so that $d \leq b$. A particular case could be $d = 0$, i.e. there is no raw labor in producing human capital. We sill, however, assume that $d = b = 1$ to ease notation. So, once we impose all our assumptions our model consists of the following equations:

$$V_t = U\left(C_t\right) + a\left(n_t\right) n_t V_{t+1} = \frac{C_t^\sigma}{\sigma} + \alpha n_t^{1-\varepsilon} V_{t+1}, \varepsilon + \sigma < 1 \tag{38}$$

$$Q_t = C_t + n_t f = \gamma l_t \left[H_t^0 + H_t\right] = \gamma \left[T - \left(h_t + v_t\right) n_t\right] \left[H_t^0 + H_t\right] \tag{39}$$

$$H_{t+1} = Ah_t \left[H_t^0 + H_t\right] \tag{40}$$

The FOC's of this problem are given by:

$$U'\left(C_t\right) = \lambda_t \tag{41}$$

$$(1 - \varepsilon)\, \alpha n_t^{-\varepsilon} V_{t+1} = MUn = MCn = \lambda_t \left[\gamma \left(h_t + v_t\right)\left(H_t^0 + H_t\right) + f\right] \tag{42}$$

We see the quality-quantity interaction again. Now, we are still missing the condition on investments in human capital. We can get no investment and a kids still gets H^0 anyway, so he would still get earnings. The FOC here is:

$$\begin{aligned}\frac{\partial V_t}{\partial h_t} &= \frac{\partial U_t}{\partial C_t}\frac{\partial C_t}{\partial h_t} + \alpha n_t^{1-\varepsilon}\frac{\partial V_{t+1}}{\partial H_{t+1}}\frac{\partial H_{t+1}}{\partial h_t} \leq 0 \\ \frac{\partial V_t}{\partial h_t} &= -U'(C_t)\gamma n_t\left(H_t^0 + H_t\right) + \alpha n_t^{1-\varepsilon}V'_{t+1}A\left(H_t^0 + H_t\right) \leq 0\end{aligned}$$

$$U'_t \geq \frac{A}{\gamma}\alpha n_t^{-\varepsilon}V'_{t+1} \text{ with } =\Leftrightarrow h_t^* > 0 \tag{43}$$

Now, we can rewrite V'_{t+1} in terms of U'_{t+1}. How? Take the envelope condition:

$$V'_t = \alpha n_t^{1-\varepsilon}V'_{t+1}Ah_t + \lambda_t\left[\gamma\left(h_t + v_t\right)\right]$$

and substitute in equations 43 and 41 to get:

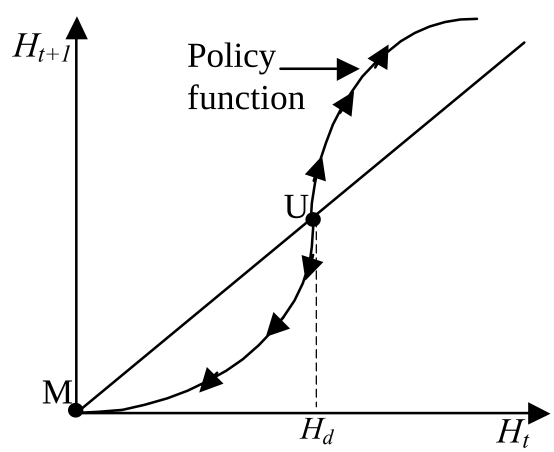
$$V'_{t+1} = U'_{t+1}\gamma\left[T - v_{t+1}n_{t+1}\right]$$

and substitute back in equation 43 to get:

$$\frac{U'_t}{\alpha n_t^{-\varepsilon}U'_{t+1}} \geq A\left[T - v_{t+1}n_{t+1}\right] \equiv \mathbf{R}_t \text{ with } =\Leftrightarrow h_t^* > 0$$

This last equation has the usual interpretation that the marginal rate of substitution between periods equals productivity of investing in human capital. Why is it T in there? Well, the only time that is not productive is the time my kids spend producing kids, otherwise they increase their productivity on goods and their kids that's why we subtract $v_{t+1}n_{t+1}$.

Suppose inequality held so $h_t^* = 0$. Children just have H^0, but this would be a steady state with no investment in kids in any generation. Graphically, this steady state is represented by point M:



Notice that if you start to the left of U ($H < H_d$) you would move back to M. Remember we have equation 40 and we can rewrite equation 43 as:

$$\frac{U_t'}{U_{t+1}'} \geq \alpha n_t^{-\varepsilon}\mathbf{R}_t \text{ with } \Longleftrightarrow h_t^* > 0 \tag{44}$$

Then, if we are at a point to the left of U, we know that $H_{t+1} < H_t$, but this means that eventually we return to the point $H_{t+1} = H_t = H^0$, where by definition $C_{t+1} = C_t$ and $h_t^* = 0$. So equation 44 is now:

$$1 \geq \alpha n_m^{-\varepsilon}\mathbf{R} = \alpha n_m^{-\varepsilon} A\left[T - v n_m\right]$$

This case of no human capital is what we call a Malthusian equilibrium. Notice that n_m needs to be large enough, so no investment means lots of kids. That is, since people have a lot of kids, the rate of return on human capital is low and incomes are low since everyone has H^0. From equations 41 and 42 we see that the marginal cost of having one more kid is:

$$MCn = U_t'\left[\gamma v_t H_t^0 + f\right]$$

Since kids only have H^0, foregone earnings are unimportant in the cost of kids. We can show that, if we take the power utility function we defined before:

$$\frac{\partial n}{\partial H} > 0 \Leftrightarrow s = \frac{f}{\gamma v\left[H^0 + H\right] + f} \geq 1 - \sigma \tag{45}$$

That is, around M it is very likely that $\frac{\partial n}{\partial H} > 0$. That is, the cost of having kids increases very little and the income effect is very large. In this case, if somehow you increase your human capital, you will have more kids, thus reducing the return to investments in human capital , thus investing less until you return to point M.

Now, there is a certain myopia in the Malthus equilibrium. If for some reason the economy got to the right of U, it would continue to grow. That is, as you accumulate H you tend to work against the Malthusian model. Notice that in this case, foregone earnings are very important since H_t is larger and the condition of equation 45 is less likely to be met. Actually at some point it wont' hold so $\frac{\partial n}{\partial H} < 0$ for large enough H. Foregone earnings become all the cost eventually.

If income increases because of higher H :

- I want to have more kids (income effect)
- Cost of time rises (substitution effect)

In malthusian equilibrium income effect dominates, while in Becker's equilibrium the substitution effect dominates. Notice, that as fertility begins to fall, $R_t = A\left[T - v n_m\right]$ rises. This stimulates more investment in H and less n, etc.

So, one of the neglects of Malthus was time costs. The question now is, is there a steady state with growth in this model, or is it explosive? That is, is there an equilibrium where

$$n_g^*, h_g^*, \frac{H_{t+1}}{H_t} = \frac{C_{t+1}}{C_t} = 1 + g^*$$

Since we want a steady state different from the Malthusian, we assume that $h_t^* > 0$. Remember that from equation 43 we know that:

$$U_t' = \frac{\alpha n_t^{-\varepsilon} A}{\gamma} V_{t+1}'$$

As, we said before, as the economy grows we can neglect costs not associated with foregone earnings (i.e. H^0 and f). If we in fact neglect them, from equations 41 and 42:

$$U_t' \left[\gamma \left(h^* + v\right) H_t\right] = \alpha \left(1 - \varepsilon\right) n_t^{-\varepsilon} V_{t+1}$$

putting together the last two equations we wrote, we get that:

$$\frac{(1-\varepsilon)}{A} \frac{V_{t+1}}{V_{t+1}'} \frac{1}{H_{t+1}} = \left(v + h^*\right) \frac{H_t}{H_{t+1}} = \frac{(v + h^*)}{1 + g^*} \tag{46}$$

Now, look at our budget constraint (equation 39) when we neglect f and H^0 and impose the steady state it says that:

$$C_{t+1} = \gamma l^* H_{t+1}$$

And, if we use the functional form for the utility function we defined in equation 38, impose the steady state and substitute iteratively:

$$V_{t+1} = \frac{C_{t+1}^{\sigma}}{\sigma} [1 + \alpha n_g^{1-\varepsilon} \left(1 + g^*\right)^{\sigma} + \left(\alpha n_g^{1-\varepsilon}\right)^2 \left(1 + g^*\right)^{2\sigma} ...]$$

and it's derivative would be:

$$V_{t+1}' = C_{t+1}^{\sigma - 1} \frac{\partial C_{t+1}}{\partial H_{t+1}} [1 + \alpha n_g^{1-\varepsilon} \left(1 + g^*\right)^{\sigma} + \left(\alpha n_g^{1-\varepsilon}\right)^2 \left(1 + g^*\right)^{2\sigma} ...]$$

Putting all these together, we get that equation 46 can be written as:

$$\frac{(1-\varepsilon)}{A\sigma} = \frac{(v + h^*)}{1 + g^*}$$

Furthermore, if we notice that given that we are ignoring f and H^0 then from equation 40:

$$\frac{H_{t+1}}{H_t} = 1 + g^* = A h^*$$

we get that:

$$\frac{(1-\varepsilon)}{\sigma} = \frac{Av}{1+g^*} + \frac{1+g^*}{(1+g^*)} = \frac{vA}{1+g^*} + 1$$

$$\boxed{1+g^* = \tfrac{v\sigma A}{1-\varepsilon-\sigma}} \tag{47}$$

This is our last expression for the growth rate. What it tells us is that if $\frac{v\sigma A}{1-\varepsilon-\sigma} > 1$ then we have a steady state with constant fertility, constant (positive) investment in human capital and consumption and human capital stock growing at a constant rate g^*. That is, we get a steady state with positive growth. Notice that there is no α in the growth rate, so altruism doesn't affect your growth rate. This doesn't mean it has no effect. It will affect the equilibrium level of fertility. We can actually show that:

$$\frac{\partial n^*}{\partial \alpha} > 0, \frac{\partial n^*}{\partial T} > 0, \frac{\partial n^*}{\partial A} > 0, \frac{\partial n^*}{\partial v} < 0$$

so that when you increase α you save more in the form of children.

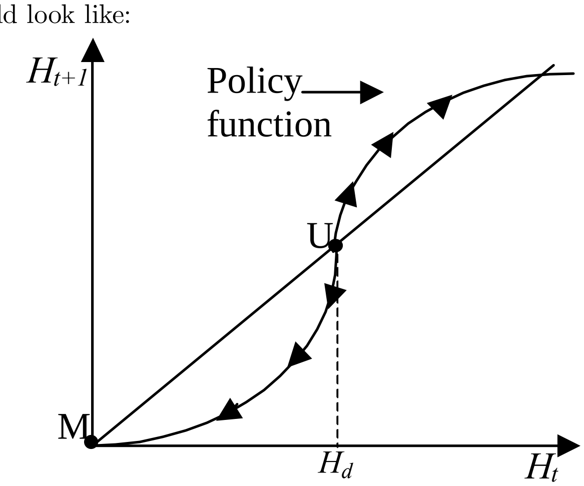
From this model, we can see why, in a modern economy, most earnings are due to human capital and fertility will be relatively low (it's lower than what it would be in the malthusian steady state). What we cannot say is how low this fertility rate is. we cannot say whether it is $n = 1$ in Malthus or $n < 1$ in an economy with growth.

Now, to get the steady state with growth it was essential to assume that there is no diminishing returns in human capital production. If we had something like:

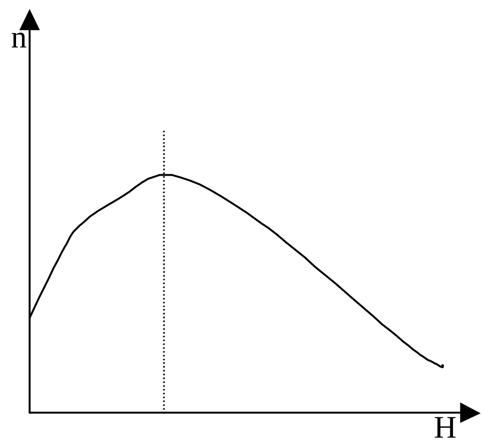
$$H_{t+1} = Ah_t H_t^{\beta} \text{ with } 0 < \beta \leq 1$$

$$\frac{H_{t+1}}{H_t} = Ah_t H_t^{\beta-1}$$

Here we cannot have perpetual growth since $\beta - 1 < 0$. The policy function in this case would look like:



We argued that we got constant returns from specialization counterbalancing the decreasing returns. We could explicitly say that $\beta = \beta\,(\text{specialization})$. So, Malthus was right. Around M the economy doesn't choose to go to an equilibrium with growth and no specialization. But Malthus was Myopic, the broader picture for fertility is:



So, to which equilibrium you go depends on two factors: the initial stock of human capital of the population and the policy function. H_d is determined by where the policy function is located. Suppose you start in a Malthusian equilibrium, do you ever get out of there. If nothing changes, no. But a lot of things may change:

1. Technology.

   Say it increases A enough, so that the condition $1 > \alpha n_m^{-\varepsilon} A\,[T - vn_m]$ no longer holds. Notice that γ doesn't play an important role in our analysis. In the Malthusian steady state it affects per capita income but not growth.

2. Declines in mortality

   It is a form of technological progress. The only way mortality enters our analysis is through T, by considering it to be available time in a life time. In this case, demographic transition can be the cause even though in steady state, T does not affect the growth rate.

What about the ideas of sustainable growth? That is, the idea that if we have growing population and diminishing returns we cannot grow forever. There may be decreasing returns in an economy, but also increasing returns to the population:

- We can have specialization if it was limited by the extent of the market.
- How much you want (not need) to invest in innovation depends on the market. Now, the cost of getting that innovation is independent, the only

thing that large population generates is a bigger incentive to invest in getting the innovation.

We can divide the economy in 2 sectors. A "rural" sector subject to diminishing returns and cities where you have specialization and division of labor. Assume that there is a natural resource fixed in supply **needed** to produce consumer goods, so it faces decreasing returns; but that this factor is not used in producing knowledge (the human capital sector). If we let $L_t = \frac{L^0}{P_t}$ be the per capita amount of fixed factor, the production function of (per capita) consumption goods is now given by:

$$\begin{aligned} Q &= \gamma L^{1-s} \left(l \left(H^0 + H\right)\right)^s \\ 0 &< s < 1 \end{aligned}$$

In this case, the malthusian equilibrium (no human capital) is given by:

$$Q = c + nf = \gamma L^{1-s} \left(lH^0\right)^s$$

Before we had that $s = 1$. Now, suppose that $n_m > 1$ so that population is growing (i.e. $P_{t+1} > P_t$).What happens in this case? L_r becomes smaller and smaller, and Q gets smaller and smaller. If it were true that $\frac{\partial n}{\partial Q} > 0$ around the malthusian equilibrium, then the population would tend to decrease! until:

$$P = P^*, \text{ so } n^* = n\left(Q\left(P^*\right)\right) = 1$$

So, through decreasing returns, we get back to $n = 1$. We go back to low income, stationary population and, endogenously, no investment in human capital. We could have a lot of mortality so the "net" n is the one that gives us stationarity (and fertility could still be very big).

Now, suppose we have an exogenous innovation that increases γ. This increases per capita income (Q) and thus increases n. This in turn increases population, thus decreasing Q and thus decreasing n again until $n = 1$. So, it seem that now we only have Malthus. But, that is not all, we can extend our analysis to allow an increase in population to change A (e.g. via innovations). This would make it more likely to get to the right of H_d. We can have the specialization argument too, so that my production function of human capital would be:

$$H_{t+1} = A\left(P_t\right) h_t \left[H^0 + H_t\right]^{\beta(\text{density})}$$

If there is diminishing returns to consumption goods then, will we see steady state growth? As before, forget about f and H^0. Then:

$$Q_t = C_t = \gamma L_t^{1-s} \left(l_t H_t\right)^s$$

$$H_{t+1} = A h_t H_t$$

We can show that in this case

$$\begin{aligned} 1 + g_H &= \frac{\widetilde{\sigma} v A}{1 - \varepsilon - \widetilde{\sigma}} \\ \text{where } \widetilde{\sigma} &= s\sigma \end{aligned}$$

So, if $s = 1$ we go back to what we had before. It does reduce the growth rate, and if important enough **could** eliminate the possibility of growth. What about consumption?

$$\frac{C_{t+1}}{C_t} = \left(\frac{L_{t+1}}{L_t}\right)^{1-s} \left(\frac{H_{t+1}}{H_t}\right)^{s} = (1 + g_L)^{1-s} (1 + g_H)^{s}$$

We have two things:

1. Assume $L = \overline{L}$. Then $1 + g_L = 1$, and you would still have growth in consumption (but smaller because of s)

2. Now, if $g_L \neq 0$, then $\left(\frac{L_{t+1}}{L_t}\right)^{1-s} = \left(\frac{P_t}{P_{t+1}}\right)^{1-s} = \left(\frac{P_{t+1}}{P_t}\right)^{s-1}$ So, if population is growing at a constant rate g_P, then:

   $$\frac{C_{t+1}}{C_t} = (1 + g_P)^{s-1} (1 + g_H)^{s}$$

   So we **could** have growing population and still have increasing consumption.