JEB064 2022/2023 Sample solution to Homework 4

IDE in Tullock lottery (3 points)

Find all iterated-dominated equilibria (IDE) in a Tullock lottery of two symmetric players. Remember, IDE are predictions when the players are payoff-maximizers but it is not sure that their beliefs about the opponent are formed correctly.

Sample solution Any (x_1, x_2) such that $x_1 \in (0, \frac{R}{4}]$ and $x_2 \in (0, \frac{R}{4}]$ is IDE:

- Step 1: We know $x_i > R$ is strictly dominated by $x_i = 0$; in the former case, $\pi_i < 0$, whereas in the latter case, $\pi_i \geq 0$. Therefore, we eliminate $x_1 > R$ and $x_2 > R$.
- Step 2: Notice that for any $x_{-i} \in (0, R]$, the best response is $X_i(x_{-i}) = \sqrt{x_{-i}R} - x_{-i} \leq \frac{R}{4}$. This means that profits are decreasing when $x_i > \frac{R}{4}$. Therefore, in the reduced game, we eliminate $x_1 > \frac{R}{4}$ and $x_2 > \frac{R}{4}$.
- Step 3: Now use that profits x_i are single-peaked at $X_i(x_{-i})$ and $X_i(x_{-i})$ is an *increasing* function for $x_{-i} \in (0, \frac{R}{4}]$. This implies that no additional strict dominance that can be applied. Prove by contraction. Take any $x_1 \in [0, \frac{R}{4}]$ and any $x'_1 \in [0, \frac{R}{4}]$ where $x_1 \neq x'_1$. There exists $x_2 \in [0, \frac{R}{4}]$ such that $x_1 = X_1(x_2)$. Therefore, by single-peakedness of profits, $\pi_1(x_1, x_2) > \pi_1(x'_1, x_2)$. Similarly, there exists $x'_2 \in [0, \frac{R}{4}]$ such that $x'_1 = X_1(x'_2)$. Therefore, by single-peakedness of profits, $\pi_1(x_1, x'_2) > \pi_1(x'_1, x'_2)$.

  The two strict inequalities implies that x_1 does not strictly dominate x'_1 and x'_1 does not strictly dominate x_1. Intuitively, the reason is that *any* $x_i \in (0, \frac{R}{4}]$ is a best response to some $x_{-i} \in (0, \frac{R}{4}]$.

Finally, consider $x_1 = 0$. Even if $\lim_{x_2 \to 0^+} X_1(x_2) = 0$, we know that by discontinuity of payoffs, the best response is also discontinuous (positive), $X_1(0) > 0$. Therefore, we can say that $x_1 = 0$ is strictly dominated by an infinitesimally larger $x'_1 = \epsilon$. (But I admitted a solution where IDEs include also $x_1 = 0$ and $x_2 = 0$ because this is just a small technical issue associated with a continuous/infinite strategy space.)

Budgeting (4 points)

Assume three parties, A, B and C, and two types of public expenditures, $x \geq 0$ and $y \geq 0$. Parties have the following (Euclidean) preferences over budgets of total size $B = x + y$:

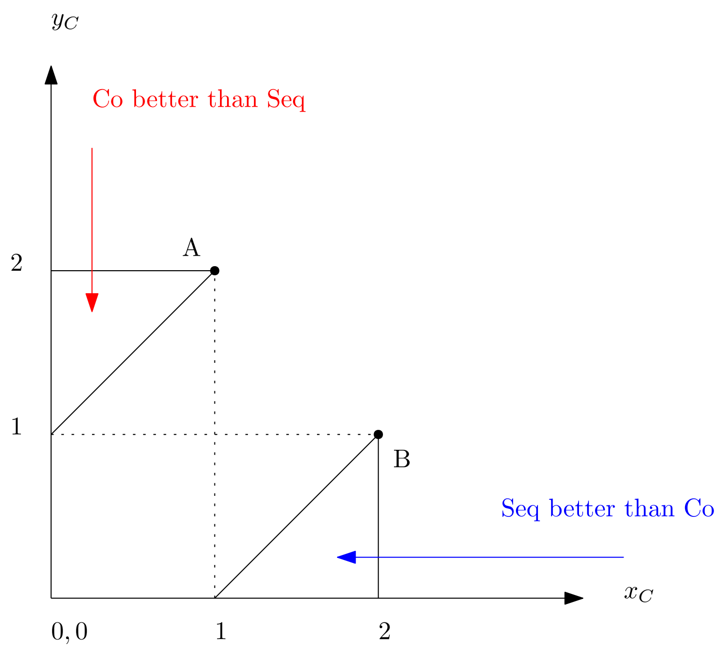
$$
\begin{aligned}
u_A &= -(1 - x)^2 - (2 - y)^2 \\
u_B &= -(2 - x)^2 - (1 - y)^2 \\
u_C &= -(x_C - x)^2 - (y_C - y)^2
\end{aligned}
$$

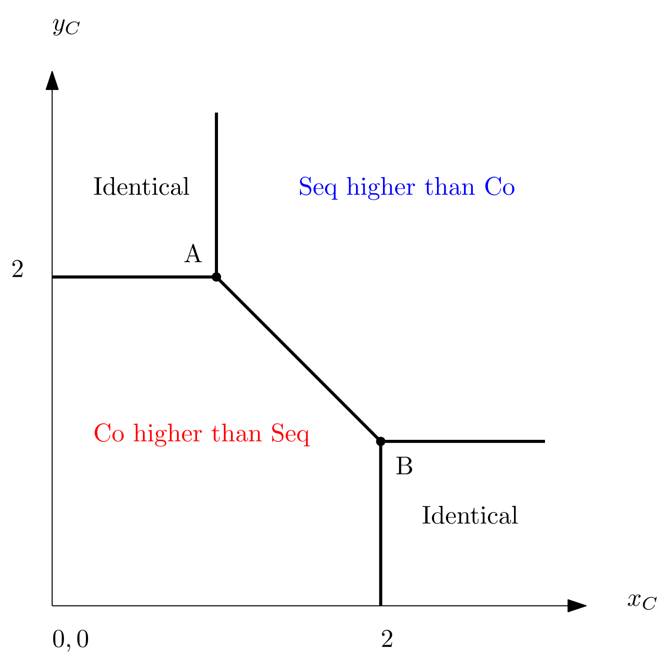
In other words, utility of a party is decreasing in Euclidean distance from the first-best optimal budget.

1. For which (x_C, y_C) does sequential budgeting lead to a budget that is first-best optimal for A?
2. For which (x_C, y_C) does coordinated budgeting lead to a budget that is first-best optimal for A?
3. Is it true that coordinated budgeting is always better for A than sequential budgeting? If yes, prove. If not, prove.
4. For which (x_C, y_C) does sequential budgeting lead to a budget that is first-best optimal for C?
5. For which (x_C, y_C) does coordinated budgeting lead to a budget that is first-best optimal for C?
6. Prepare a figure that illustrates for which (x_C, y_C) sequential budgeting leads to a lower/identical/higher budget than coordinated budgeting.

Sample solution In sequential budgeting, the budget depends on medians in dimensions x and y. In coordinated budgeting, the budget depends on medians in dimensions $x - y$ and $x + y$. In this specific example, (i) sequential budgeting implements budgets in the square, $x \in [1, 2]$ and $y \in [1, 2]$, whereas (ii) coordinated budgeting implements budgets on a line, $x \in [1, 2]$ and $y \in [1, 2]$ such that $x + y = 3$.

1. The sequential budget is $(x, y) = (1, 2)$ for any $x_C \leq 1$ and $y_C \geq 2$.
2. The coordinated budget is $(x, y) = (1, 2)$ for any $x_C - y_C \leq -1$.
3. No. Take Figure 1. In the red triangle, coordinated budgeting leads to $(x, y) = (1, 2)$ which is first-best optimal for A; sequential budgeting leads to a worse budget $(x, y) = (1, y_C,)$. But in the blue triangle, coordinated budgeting leads to $(x, y) = (2, 1)$ which is worse than $(x, y) = (x_C, 1)$ that is adopted under sequential budgeting.
4. The sequential budget is $(x, y) = (x_C, y_C)$ for any $x_C \in [1, 2]$ and $y_C \in [1, 2]$.
5. The coordinated budget is $(x, y) = (x_C, y_C)$ if $x_C - y_C \in [-1, 1]$ and $x_C + y_C = 3$.
6. We use that the coordinated budget always gives $x + y = 3$. The two procedures give an identical budget when (x_C, y_C) is in NW and SE squares and when (x_C, y_C) is on the diagonal. Below the diagonal, the sequential budget is lower than the coordinated budget. Above the diagonal, the sequential budget is higher than the coordinated budget. See also Figure 2.





Price leadership and sequential (ir)rationality (6 points)

Consider a market where a mass of consumers of size 1 have valuation of the good $\alpha \in [0, 1]$ that is uniformly distributed. Two firms, $i = 1, 2$, are able to produce the good. Each firm has a production technology with a fixed cost and zero variable costs. Their fixed costs are $0 < f_2 < f_1 < \frac{1}{4}$. In other words, Firm 1 is a *less efficient* company and Firm 2 is a *more efficient* company. Both firms are price-setters.

A firm pays the fixed cost only if its sales are positive. In other words, by setting a sufficiently large price $p_i > 1$, Firm i effectively decides to not operate on the market. We also assume that firms have ϵ-spite. It means that a firm maximizes profits, but when faced with multiple strategies that deliver identical profits, the firm picks up a strategy that minimizes profits of the other firm.

1. Suppose Firm 1 (the less efficient company) is the price leader. In Stage 1, Firm 1 announces p_1. In Stage 2, Firm 2 announces p_2. Find a subgame-perfect Nash equilibrium. Write equilibrium prices and profits.

2. Find a Nash equilibrium that is not subgame-perfect. Write equilibrium prices and profits.

3. Suppose Firm 2 (the more efficient company) is price leader. That is, in Stage 1, Firm 2 announces p_2. In Stage 2, Firm 1 announces p_1. Find a subgame-perfect Nash equilibrium. Write equilibrium prices and profits.

4. Find a Nash equilibrium that is not subgame-perfect. Write equilibrium prices and profits.

Without loss of generality, suppose that if $p_1 = p_2$, all customers buy from the more efficient company.

Sample solution From the class, we know that the monopolist price is $p^m = \frac{1}{2}$. Now derive the price q_i below which Firm i that sells to all customers decides to leave the market. The price satisfies $q_i(1 - q_i) = f_i$. Precisely, $q_i = \frac{1}{2} - \sqrt{\frac{1}{4} - f}$. Notice that $q_2 < q_1 < p^m$.

- Firm 1 is price-leader and all players are sequentially rational.

  Consider follower's best response. For any $p_1 > q_2$, the follower sets $p_2 = p_1$ (follower's market with positive follower's profits). For any $p_1 \leq q_2$, the follower sets $p_2 > p_1$ (leader's market with zero follower's profits and negative leader's profits).

  Out of prices p_1 that implement the follower's market, the leader prefers the lowest price $q_2 + \epsilon$; the leader's profits are almost zero and follower's profits are zero. Out of prices p_1 that implement the leader's market, the leader prefers the highest price q_2; the leader's loss is minimized. Since $q_2 + \epsilon < q_1$, the leader prefers the price $q_2 + \epsilon$ because zero profits are better than a loss.

  In equilibrium, $p_1^* = p_2^* = q_2 + \epsilon$, only the follower (the more efficient Firm 2) sells and both the leader and follower earn (almost) **zero** profits. (Notice: Since $\epsilon > 0$ is allowed to be arbitrarily small, the 'lowest price' $q_2 + \epsilon$ strictly speaking doesn't exist unless we impose a positive lower bound on ϵ. Therefore, I accept a solution that a NE doesn't exist given that the strategy set is not finite. But normally we think that in any price competition, ϵ cannot be arbitrarily small; e.g., think of a lower bound by 1 cent.)

- Firm 1 is price-leader and the follower is not sequentially rational.

  Consider an equilibrium with $p_1 = p_2 \in (q_2, q_1)$ where only the follower sells but the follower now earns large **positive** profits. Follower doesn't deviate (by a higher price, she would destroy profits; by a lower price, she would decrease profits). Leader doesn't deviate to a lower price if she thinks that follower responds to a lower price by leaving the market; this belief is possible in Nash equilibrium but is not possible in a subgame-perfect Nash equilibrium.

- Firm 2 is price-leader and all players are sequentially rational.

  Follower's best response is constructed like above. Namely, for any $p_2 > q_1$, the follower sets $p_1 = p_2 - \epsilon$ (follower's market with positive follower's profits). For any $p_2 \leq q_1$, the follower sets $p_1 > p_2$ (leader's market with zero follower's profits).

  Out of prices p_1 that implement the follower's market, the leader prefers the lowest price $q_1 + \epsilon$; the leader's profits are identical (zero), but the follower's profits are minimized. Out of prices p_1 that implement the leader's market, the leader prefers the highest possible price q_1 (it is closest to the monopolist price $\frac{1}{2}$); the leader's profits are maximized. The leader prefers the price q_1 over the price $q_1 + \epsilon$ because positive profits are better than zero profits.

  In equilibrium, $p_1^* = p_2^* = q_1$, only the leader (again, the more efficient Firm 2) sells and the leader earns **positive** profits.

- Firm 2 is price-leader and the follower is not sequentially rational.

  Consider an equilibrium with $p_1 = p_2 \in (q_2, q_1)$ where only the leader sells but earns **lower** profits. Follower doesn't deviate (a higher price has no effect; by a lower price, she would begin to sell but at negative profits). Leader doesn't deviate to a higher price if she thinks that the follower responds to a higher price by competing; this belief is possible in Nash equilibrium but is not possible in a subgame-perfect Nash equilibrium.