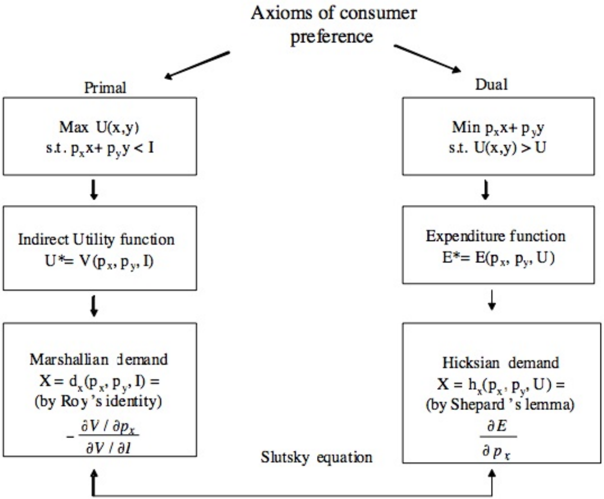
14.03 Microeconomic Theory & Public Policy Fall 2022

Lecture 6. Individual Demand: The Expenditure Function, Demand Curves, Income & Substitution Effects

David Autor (Prof), MIT Economics and NBER
Jonathan Cohen (TA), MIT Economics

Theory roadmap



Today's topics

1. **Expenditure function**
   - Minimum possible expenditure given prices and a minimum utility level
2. **Marshallian (uncompensated) demand curve**
   - Personal demand curve, holding **income** constant, allowing prices to vary
   - Derives from the indirect utility function, which takes prices and **income** as arguments
   - Exhibits **both** income and substitution effects
3. **Hicksian (compensated) demand curve**
   - Personal demand curve, holding **utility** constant, allowing prices to vary
   - Derives from the expenditure function, which takes price and **utility** as arguments
   - Exhibits **only** substitution effects

Expenditure Function

What is the expenditure function?

- Utility function (familiar)
  - Utility level for each bundle of goods
- Indirect utility function (familiar)
  - Maximum possible utility given prices and a maximum budget/income
- **Expenditure function**
  - Minimum possible expenditure given prices and a minimum utility level

Indirect utility function vs. expenditure function

- **Indirect utility function**

$$\max U(x,y)$$
$$s.t.\ p_x x + p_y y \quad \leq \quad I$$

Solve for x^*, $y^* \Rightarrow u^* = U(x^*,\, y^*)$ given $p_x, p_y, I \implies V\,(p_x, p_y, I)$

- **Expenditure function**

$$\min p_x x + p_y y$$
$$s.t.\ U(x,y) \quad \geq \quad V^*$$

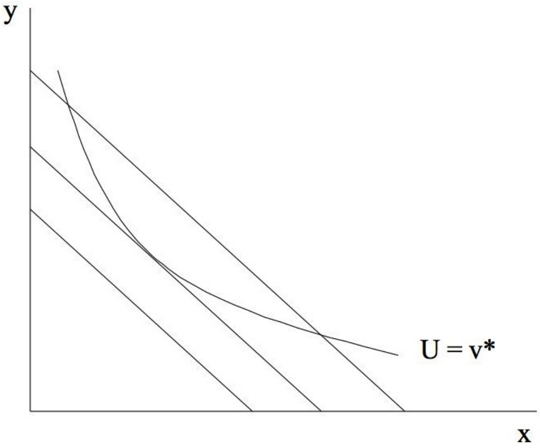
which gives $E = p_x x^* + p_y y^*$ for $U(x^*,\, y^*) = V^* \implies E\,(p_x, p_y, V^*)$

Expenditure function

Graphical interpretation

Expenditure function

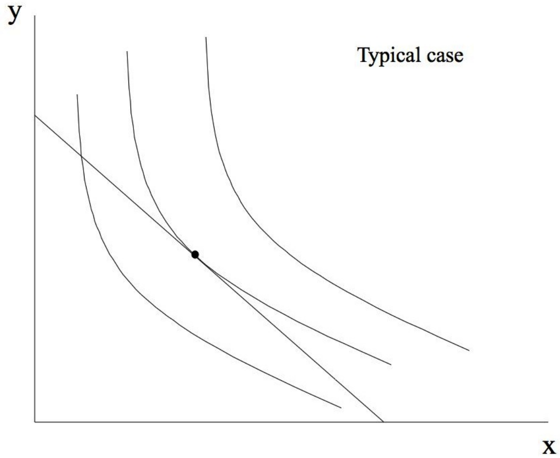
Graphical interpretation



Indirect utility function

Graphical interpretation

Indirect utility function
Graphical interpretation



Expenditure function
Example

$$\begin{aligned} \min E &= p_x x + p_y y \\ s.t.\ x^{.5} y^{.5} &\geq U_p \end{aligned}$$

Expenditure function

Example

$$\begin{aligned} \min E &= p_x x + p_y y \\ s.t.\ x^{.5}y^{.5} &\geq U_p \end{aligned}$$

$$L = p_x x + p_y y + \lambda\left(U_p - x^{.5}y^{.5}\right)$$

$$\begin{aligned} \frac{\partial L}{\partial x} &= p_x - \lambda .5x^{-.5}y^{.5} = 0 \\ \frac{\partial L}{\partial y} &= p_y - \lambda .5x^{.5}y^{-.5} = 0 \\ \frac{\partial L}{\partial \lambda} &= U_p - x^{.5}y^{.5} = 0 \end{aligned}$$

Expenditure function

Example continued

The first two of these equations simplify to:

$$x = \frac{p_y y}{p_x}$$

We substitute into the constraint $U_p = x^{.5} y^{.5}$ to get

$$\begin{aligned} U_p &= \left(\frac{p_y y}{p_x}\right)^{.5} y^{.5} \\ x^* &= \left(\frac{p_y}{p_x}\right)^{.5} U_p, \; y^* = \left(\frac{p_x}{p_y}\right)^{.5} U_p \end{aligned}$$

Expenditure function
Example continued

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$$\begin{aligned} E^* &= p_x \left(\frac{p_y}{p_x}\right)^{.5} U_p + p_y \left(\frac{p_x}{p_y}\right)^{.5} U_p \\ &= 2 p_x^{.5} p_y^{.5} U_p \end{aligned}$$

Expenditure Function
What is it good for?

- The expenditure function is an essential tool for public policy analysis. Allows 'monetizing' otherwise incommensurate trade-offs to evaluate costs and benefits. Facilitates cost-benefit analysis.
- We don't know what 'utils' are, but "how much money" is well-defined.
- The expenditure function permits us to calculate a 'money metric.'

Expenditure function $\leftrightarrow$ indirect Utility Function

Expenditure function $\leftrightarrow$ Indirect Utility Function

$$\begin{aligned} V(p_x, p_y, I_0) &= U_0 \\ E(p_x, p_y, U_0) &= I_0 \\ V(p_x, p_y, E(p_x, p_y, U_0)) &= U_0 \\ E(p_x, p_y, V(p_x, p_y, I_0)) &= I_0 \end{aligned}$$

From indirect utility to expenditure Function:
Back to Cobb-Douglas example

The dual problem gave us expenditures (budget requirement) as a function of utility and prices.

$$x_p^* = \frac{I}{2p_x},\ y_p^* = \frac{I}{2p_y},\ U^* = \left(\frac{I}{2p_x}\right)^{.5}\left(\frac{I}{2p_y}\right)^{.5}$$

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Now plug these into expenditure function:

$$E^* = 2U_p p_x^{.5} p_y^{.5} = 2\left(\frac{I}{2p_x}\right)^{.5}\left(\frac{I}{2p_y}\right)^{.5} p_x^{.5} p_y^{.5} = I$$

Individual (i.e., Personal) Demand Curves

Individual demand

Now, let's use the indirect utility function and the expenditure function to get demand functions. Up to now, we have been solving for:

- Utility as a function of prices and budget
- Expenditure as a function of prices and utility

Implicitly we have already found demand schedules—a demand schedule is immediately implied by an individual utility function. For any utility function, we can solve for the quantity demanded of each good as a function of its price, holding the price of all other goods constant *and* holding *either* income *or* utility constant.

Marshallian (uncompensated) demand

In our previous example where:

$$U(x, y) = x^{.5} y^{.5}$$

we derived:

$$\begin{aligned} x(p_x, p_y, I) &= 0.5 \frac{I}{p_x} \\ y(p_x, p_y, I) &= 0.5 \frac{I}{p_y} \end{aligned}$$

In general we will write these demand functions (for individuals) as:

$$\begin{aligned} x_1^* &= d_1(p_1, p_2, ..., p_n, I) \\ x_2^* &= d_2(p_1, p_2, ..., p_n, I) \\ &... \\ x_n^* &= d_n(p_1, p_2, ..., p_n, I) \end{aligned}$$

Marshallian (uncompensated) demand

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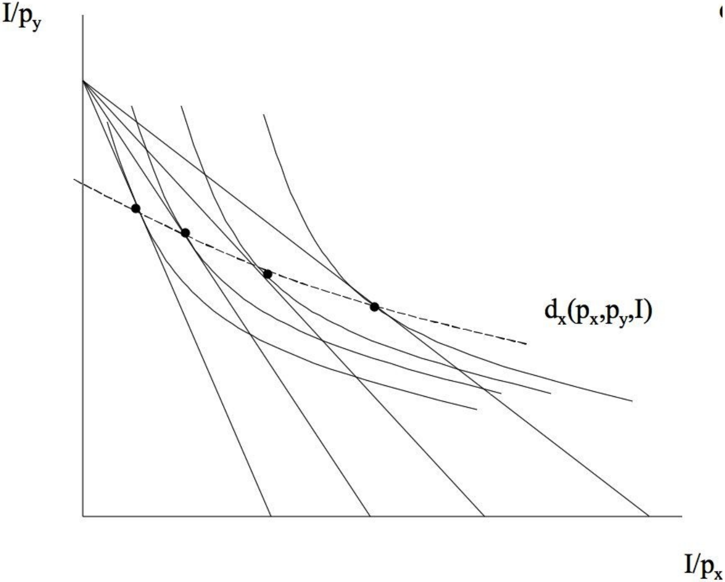


Hicksian (compensated) Demand

Similarly we derived that:

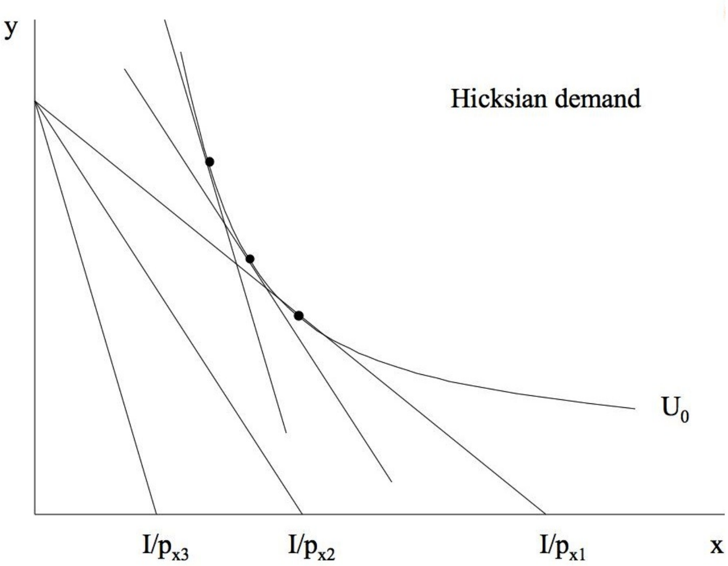
$$\begin{aligned} x(p_x, p_y, U) &= \left(\frac{p_y}{p_x}\right)^{.5} U_p \\ y(p_x, p_y, U) &= \left(\frac{p_x}{p_y}\right)^{.5} U_p \end{aligned}$$

In general we will write these demand functions (for individual) as:

$$\begin{aligned} x^*_{1,c} &= h_1(p_1, p_2, ..., p_n, U) \\ x^*_{2,c} &= h_2(p_1, p_2, ..., p_n, U) \\ &\dots \\ x^*_{n,c} &= h_n(p_1, p_2, ..., p_n, U) \end{aligned}$$

Hicksian (compensated) demand

Hicksian (compensated) Demand



Income and Substitution Effects
(Normal, Inferior, and Giffen Goods)

Main question

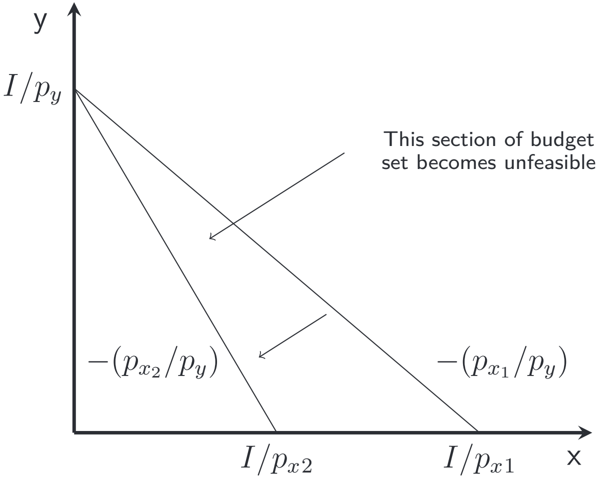
What happens to demand for a given good when its price increases and income is kept constant?

Means we care about $\partial d_x(p_x, p_y, I)/\partial p_x$.

Two effects:

1. It shifts the budget set inward toward the origin for the good whose price has risen. This component is the 'income effect.'
2. It changes the slope of the budget set so that the consumer faces a different set of market trade-offs. This component is the 'substitution effect.'

Effect of a price increase on the budget set



Substitution effect

What happens to consumption of X if

$$\frac{p_x}{p_y} \uparrow$$

while utility is held constant?

Provided that the axiom of diminishing MRS applies, we'll have

$$\frac{\partial h_x(p_x, p_y, U)}{\partial p_x} < 0$$

Holding utility constant, the substitution effect is *always* negative.

Income effect

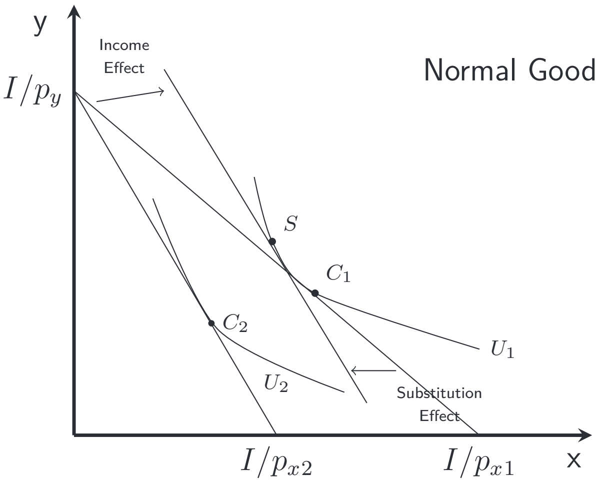
Defined as

$$\partial d_x(p_x, p_y, I)/\partial I$$

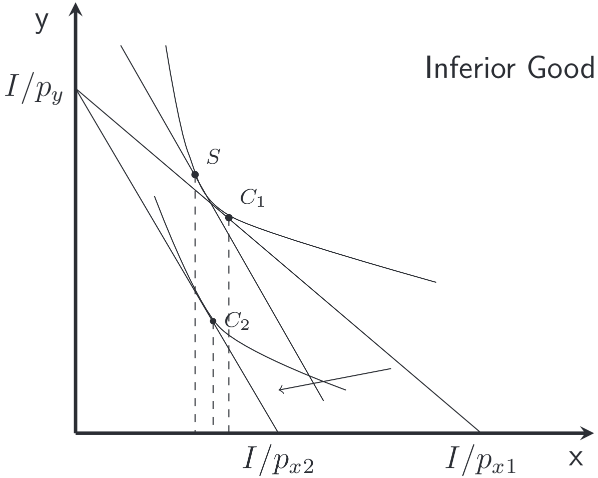
Can be either negative or positive.

- If positive, good X is said to be a “normal” good.
- If negative, good X is said to be an “inferior” good.
- Inferior goods can be further subdivided in “weakly” and “strongly” inferior goods:
  - “Weakly inferior” goods:
    $\partial d_x(p_x, p_y, I)/\partial I < 0$ and $\partial d_x(p_x, p_y, I)/\partial p_x < 0$.
  - “Strongly inferior” (Giffen) goods:
    $\partial d_x(p_x, p_y, I)/\partial I < 0$ and $\partial d_x(p_x, p_y, I)/\partial p_x > 0$.

Income and substitution effects: Normal good


Normal Good
y
x
Income
Effect
Substitution
Effect
I/p_y
I/p_{x2}
I/p_{x1}
S
C_1
C_2
U_1
U_2

Income and substitution effects: Inferior good


y
Inferior Good
I/p_y
S
C_1
C_2
I/p_{x2}
I/p_{x1}
x

Normal, Inferior, and Giffen Goods

Summary

For a normal good ($\frac{\partial d_x}{\partial I} > 0$), the income and substitution effects are complementary.

For an inferior good ($\frac{\partial d_x}{\partial I} < 0$), the income and substitution effects are countervailing.

For a Giffen good, the income effect dominates: $\left|\frac{\partial d_x}{\partial I} \cdot X\right| > \left|\frac{\partial h_x}{\partial p_x}\right|$ note that they are both negative.

Name that good: Normal or Inferior

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Obtaining Demand Curves via the Envelope Theorem: Shephard's Lemma and Roy's Identity

Shephard's Lemma

Relates d_x to the expenditure function.

This helps to measure to compute the magnitude of the income effect following a price change.

How does it work?

Recall the expenditure minimization problem that yields $E(p_x, p_y, \overline{U})$.

$$\min_{X,Y} p_x X + p_y Y \ s.t. \ U(X,Y) \geq \overline{U}.$$

Shephard's Lemma continued: Langragian

The Lagrangian for this problem:

$$\ell = p_x X + p_y Y \; + \lambda(\overline{U} - U(X, Y)).$$

First order conditions:

$$\begin{aligned}
\frac{\partial \ell}{\partial X} &= p_x - \lambda U_x = 0, \\
\frac{\partial \ell}{\partial y} &= p_y - \lambda U_y = 0, \\
\frac{\partial \ell}{\partial \lambda} &= \overline{U} - U(X, Y).
\end{aligned}$$

$$\lambda = \frac{p_x}{U_x} = \frac{p_y}{U_y}.$$

Shephard's Lemma continued: Here's the magic

$$\frac{d\ell(X,Y,\lambda)}{dp_x} \quad = \quad X + \left(p_x \frac{\partial X}{\partial p_x} - \lambda U_x \frac{\partial X}{\partial p_x}\right) + \left(p_y \frac{\partial Y}{\partial p_x} - \lambda U_y \frac{\partial Y}{\partial p_x}\right)$$

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$$\frac{d\ell(X,Y,\lambda)}{dp_x} \;=\; X + \left(p_x\frac{\partial X}{\partial p_x} - \lambda U_x\frac{\partial X}{\partial p_x}\right) + \left(p_y\frac{\partial Y}{\partial p_x} - \lambda U_y\frac{\partial Y}{\partial p_x}\right)$$

Recall the following equations from above:

$$\begin{aligned} p_x &= \lambda U_x \\ p_y &= \lambda U_y. \end{aligned}$$

Shephard's Lemma continued: Here's the magic

$$\frac{d\ell(X,Y,\lambda)}{dp_x} = X + \left(p_x\frac{\partial X}{\partial p_x} - \lambda U_x\frac{\partial X}{\partial p_x}\right) + \left(p_y\frac{\partial Y}{\partial p_x} - \lambda U_y\frac{\partial Y}{\partial p_x}\right)$$

Recall the following equations from above:

$$\begin{aligned} p_x &= \lambda U_x \\ p_y &= \lambda U_y. \end{aligned}$$

Substituting in:

$$\begin{aligned} &= X + \left(p_x\frac{\partial X}{\partial p_x} - p_x\frac{\partial X}{\partial p_x}\right) + \left(p_y\frac{\partial Y}{\partial p_x} - p_y\frac{\partial Y}{\partial p_x}\right) \\ &= X + 0 + 0 \\ &= X. \end{aligned}$$

That's the envelope theorem at work

Shephard's Lemma: Intuition

Bottom line: at optimum: $E = \ell$ so

$$\frac{\partial E(p_x, p_y, U)}{\partial p_x} = h_x(p_x, p_y, U)$$

This equation is known as **Shephard's Lemma**

- In words: To hold utility constant given the price change, your expenditures must rise by the price change times the initial level of consumption.
- Concrete example. If you buy 2 cups of coffee a day and the price of coffee rises by 1 cent per cup, how much do we need to compensate you to hold utility constant? To a first approximation, 2 cents.
- Shephard's lemma holds only for small price changes. For a non-negligible price change, the consumer would re-optimize her bundle to re-equate the MRS with the new price ratio.

Closing the loop: Roy's identity

If we applied the same calculations as we did to get Shephard's lemma but on a budget-constrained utility maximization problem instead, we'd find that:

$$-\frac{\partial V(p_x, p_y, I)/\partial p_x}{\partial V(p_x, p_y, I)/\partial I} = d_x(p_x, p_y, I)$$

Called **Roy's identity**. Also an application of the *envelope theorem*

Summary

- Two effects of price increase: substitution effect, income effect.
- Three types of goods: normal, (weakly) inferior, Giffen (strongly inferior) goods.
- Shephard's lemma: recover compensated demand from expenditure function.
- Roy's identity: recover uncompensated demand from indirect utility function.