Seminar 4 - Exercises

1. A firm has production function $y = 0.2\ln(x_1) + 0.8\ln(x_2)$.
   - Construct Lagrange function, derive first order conditions and derive relationship between amount of inputs ($MRTS$) and prices of inputs (w_1 and w_2).
     A: $F.O.C$: $w_1 = \lambda\dfrac{0.2}{x_1}$, $w_2 = \lambda\dfrac{0.8}{x_2}$, $y = 0.2\ln(x_1) + 0.8\ln(x_2)$, $TRS = -\dfrac{x_2}{4x_1} = -\dfrac{w_1}{w_2}$
   - Calculate conditional demand functions $x_1(w, y)$ and $x_2(w, y)$.
     A: $x_1 = e^y\left(\dfrac{w_2}{4w_1}\right)^{\frac{4}{5}}$, $x_2 = e^y\left(\dfrac{4w_1}{w_2}\right)^{\frac{1}{5}}$

2. Consider Leontief production function in the form $y = \min\{ax_1, bx_2\}$
   - Prove that cost function equal $c(w, y) = y(\frac{w_1}{a} + \frac{w_2}{b})$ and derive conditional demand functions.
     A: Express conditional demands: $x_1 = \dfrac{y}{a}$, $x_2 = \dfrac{y}{b}$ and plug them into general form of cost function $c = w_1x_1 + w_2x_2$. *Note:* for perfect complements, the cost minimizing choice of inputs does not depend on the input prices.
   - What are minimal costs for x_1 and x_2 when $a = 3$, $b = 2$ $w_1 = 4$, $w_2 = 2$ and $y = 8$?
     A: $c = \dfrac{56}{3}$

3. Earl sells lemonade in a competitive market on a busy street corner. His production function is $f(l, h) = l^{\frac{1}{3}}h^{\frac{1}{3}}$, where output is measured in liters. l is the number of kilos of lemons he uses, and h is the number of hours he spends squeezing them.
   - Does Earl have constant/decreasing/increasing returns to scale?
     A: decreasing
   - If he is going to produce y units of lemonade in the cheapest way possible, what is the number of kilos of lemons and the number of hours he will use?
     A: $l(w_l, w_h, y) = \sqrt{y^3\dfrac{w_h}{w_l}}$; $h(w_l, w_h, y) = \sqrt{y^3\dfrac{w_l}{w_h}}$
   - What is the cost to Earl of producing y units at factor prices w_l and w_h?
     A: $c(w_l, w_h, y) = 2\sqrt{y^3w_lw_h}$

4. A university cafeteria produces square meals, using only one input and a rather remarkable production process. We are not allowed to say what that ingredient is, but an authoritative kitchen source says that "fungus is involved." The cafeteria production function is $f(x) = x^2$, where x is the amount of input and $f(x)$ is the number of square meals produced.
   - Does the cafeteria have constant/decreasing/increasing returns to scale?
     A: increasing
   - How many units of input does it take to produce 144 square meals? If the input costs w per unit, what does it cost to produce 144 square meals?
     A: 12; $c = 12w$
   - How many units of input does it take to produce y square meals?
     A: $\sqrt{y}$
   - If the input costs w per unit, what is the average cost of producing y square meals?
     A: $AC(w, y) = \dfrac{w}{\sqrt{y}}$

5. For the production function $f(x_1, x_2) = (9x_1 + 18)^{\frac{1}{2}}(16x_2 + 32)^{\frac{1}{4}}$
   - Find demand functions $x_1(y, w_1, w_2)$ and $x_2(y, w_1, w_2)$ which minimize costs.
     A: $x_1 = \left(\dfrac{y}{6}\right)^{\frac{4}{3}}\left(\dfrac{2w_2}{w_1}\right)^{\frac{1}{3}} - 2$; $x_2 = \left(\dfrac{y}{6}\right)^{\frac{4}{3}}\left(\dfrac{w_1}{2w_2}\right)^{\frac{2}{3}} - 2$

- Find optimal demands x_1 and x_2 given required output level $y_0 = 18$ and input prices $w_1 = 6$ and $w_2 = 1$ which minimize costs.
  A: $x_1 = 1$, $x_2 = 7$
- Express cost function $c(y, w_1, w_2)$.
  A: $c(w_1, w_2, y) = w_1 \left[\left(\frac{y}{6}\right)^{\frac{4}{3}} \left(\frac{2w_2}{w_1}\right)^{\frac{1}{3}} - 2 \right] + w_2 \left[\left(\frac{y}{6}\right)^{\frac{4}{3}} \left(\frac{w_1}{2w_2}\right)^{\frac{2}{3}} - 2 \right]$