# Advanced microeconomics problem set 6

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## 1 Exercise 5.10

### 1.1 Original question

In a two-person, two-good exchange economy with strictly increasing utility functions, it is easy to see that an allocation $\bar{x} \in F(\mathbf{e})$ is Pareto efficient if and only if $\bar{x}^i$ solves the problem

$$\max_{\mathbf{x}^i} u^i(\mathbf{x}^i) \quad \text{s.t.} \tag{1}$$

$$u^j(\mathbf{x}^j) \geq u^j(\bar{\mathbf{x}}^j) \tag{2}$$

$$x_1^1 + x_1^2 = e_1^1 + e_1^2 \tag{3}$$

$$x_2^1 + x_2^2 = e_2^1 + e_2^2 \tag{4}$$

for $i = 1, 2$ and $i \neq j$[1].

(a) Prove the claim.

(b) Generalise this equivalent definition of a Pareto-efficient allocation to the case of $n$ goods and $I$ consumers. Then prove the general claim.

### 1.2 Solution

(a) Recall some definitions. An allocation is *feasible* if[2]

$$F(\mathbf{e}) = \left\{ \mathbf{x} \middle| \sum_{i \in \mathcal{I}}^{n} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i \right\} \tag{5}$$

A feasible allocation $x \in F(\mathbf{e})$ is Pareto efficient if there is no other feasible allocation such that $y^i \succsim^i x^i$ for all consumers $i \in \mathcal{I}$ with at least one preference strict.

**(⇐)**: Suppose some allocation $\bar{\mathbf{x}}$ solves the problem, from equation (3) and (4) we see that the allocation is feasible.

Then, suppose the allocation is not Pareto efficient, we will get a contradiction: If $\bar{\mathbf{x}}$ is not Pareto efficient, then there exist another allocation $\mathbf{y}$ that not only is feasible,

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[1] Notice that this means actually 2 problems, one for each agent.

[2] see equation 5.1 on textbook

but also makes one agent strictly better off[3]. Denote the better off agent as agent $i$, then we have:

$$u^i(\mathbf{y}^i) > u^i(\mathbf{x}^i) \quad s.t. \quad \text{equation (3), (4)} \tag{6}$$

contradicting the assumption that $\bar{\mathbf{x}}^i$ is the optimal solution for the problem.

**(⇒)**: Suppose the allocation $\bar{\mathbf{x}}$ is feasible and Pareto optimal, then from the definition of feasibility we see constraints in equation (3), (4) are satisfied.

Then, suppose for some agent $i$ that $\bar{\mathbf{x}}^i$ is not the solution to the problem, we will also get a contradiction: If $\bar{\mathbf{x}}^i$ is not the solution, then there exist $\mathbf{y}^i$, $\mathbf{y}^j = \bar{\mathbf{x}}^j$ such that $u^i(\mathbf{y}^i) > u^i(\bar{\mathbf{x}}^i)$, and $u^j(\mathbf{y}^i) = u^j(\bar{\mathbf{x}}^j)$, and since $\mathbf{y}$ is the solution to the problem, it naturally satisfies the feasibility constraint implied by equation (3), (4). So $\mathbf{y}$ is a Pareto improvement to $\bar{\mathbf{x}}$, contradicting the assumption that $\bar{\mathbf{x}}$ is Pareto efficient.

(b) **Generalized claim**:

**Theorem 1.** *An allocation $\bar{\mathbf{x}} = (\bar{\mathbf{x}}^1, \bar{\mathbf{x}}^2, \bar{\mathbf{x}}^3, ..., \bar{\mathbf{x}}^I) \in F(\mathbf{e})$ is Pareto efficient iff[4] for each agent $i$, $\bar{\mathbf{x}}^i$ solves the problem:*

$$\max_{\mathbf{x}^i} u^i(\mathbf{x}^i) \quad s.t. \tag{7}$$

$$u^j(\mathbf{x}^j) \geq u^j(\bar{\mathbf{x}}^j) \quad \textit{for all} \quad j \neq i \tag{8}$$

$$\sum_{k=1,2,...,I} \mathbf{x}^k = \sum_{k=1,2,....I} \mathbf{e}^k \tag{9}$$

Notice that $\bar{\mathbf{x}}^k$ is a $n$-dimensional vector, corresponding to $n$ types of goods.

**Proof**: Similar to question (a), we stick to the definition of Pareto improvement

**(⇐)**: Suppose $\bar{\mathbf{x}}$ solves the problem. Similar to question (a), we see that the allocation is feasible[5]. Suppose $\bar{\mathbf{x}}$ is not Pareto efficient, then there exist $\mathbf{y} \in F(\mathbf{e})$ such that $u^i(\mathbf{y}^i) > u^i(\bar{\mathbf{x}}^i)$ for one agent $i$ and $u^j(\mathbf{y}^j) = u^j(\bar{\mathbf{x}}^j)$. So we see $\bar{\mathbf{x}}^i$ is not the solution of the problem, contradicting with our initial assumption.

**(⇒)**: Suppose $\bar{\mathbf{x}} \in F(\mathbf{e})$ is Pareto optimal, but $\mathbf{y} \neq \bar{\mathbf{x}}$ is the solution to the problem. From the problem we know $\mathbf{y} \in F(\mathbf{e})$, which means $\mathbf{y}$ is feasible. Moreover, since $\mathbf{y}$ is solution but $\bar{\mathbf{x}}$ is not, we have $u^i(\mathbf{y}^i) > u^i(\bar{\mathbf{x}}^i)$ for some agents and $u^j(\mathbf{y}^j) = u^j(\bar{\mathbf{x}}^j)$ for other agents. So $\mathbf{y}$ becomes a Pareto improvement to $\bar{\mathbf{x}}$, contradicting with our assumption that $\bar{\mathbf{x}}$ is Pareto efficient.

# 2 Exercise 5.11

## 2.1 Original question

Consider a two-consumer, two-good exchange economy. Utility functions and endowments are

$$u^1(x_1, x_2) = (x_1 x_2)^2 \quad \text{and} \quad \mathbf{e}^1 = (18, 4) \tag{10}$$

$$u^2(x_1, x_2) = \ln(x_1) + 2\ln(x_2) \quad \text{and} \quad \mathbf{e}^2 = (3, 6) \tag{11}$$

for $i = 1, 2$ and $i \neq j$.

[3]A more "Pareto" improvement that everyone is better off may exist, but we only need to find some improvement to complete the proof

[4]if and only if, sometimes people write it this way

[5]By now we should see that equation (9) is the definition of feasibility

a) Characterise the set of Pareto-efficient allocations as completely as possible.

b) Characterise the core of this economy.

c) Find a Walrasian equilibrium and compute the WEA.

d) Verify that the WEA you found in part (c) is in the core.

## 2.2 Solution

**a)** The trick is to observe that at any point on the Pareto-efficient allocations, the utility functions of two agents are tangent. If there are more than 2 agents, then we see the ratio of marginal utility is equal to relative prices. Also, the allocations must exhaust all endowments, so first we have:

$$x_1^1 + x_1^2 = 18 + 3 \tag{12}$$
$$x_2^1 + x_2^2 = 4 + 6 \tag{13}$$

The tangent condition is:

$$\frac{\frac{\partial u^1}{\partial x_1^1}}{\frac{\partial u^1}{\partial x_2^1}} = \frac{\frac{\partial u^2}{\partial x_1^2}}{\frac{\partial u^2}{\partial x_2^2}} = \frac{p_1}{p_2} \tag{14}$$

Denote $x_1 = x_1^1, x_2 = x_2^1$, and $x_1^2 = 21 - x_1^1 = 21 - x_1, x_2^2 = 10 - x_2^1 = 10 - x_2$, plugging these into the tangent condition we have:

$$\frac{2x_1(x_2)^2}{2(x_1)^2 x_2} = \frac{10 - x_2}{2 * (21 - x_1)} \tag{15}$$

which simplifies to

$$x_2 = \frac{10x_1}{42 - x_1} \tag{16}$$

In other words, we know that allocations that has $x_2^1 = \frac{10x_1^1}{42-x_1^1}$[6] are Pareto efficient

**b)** The core consists of all allocations $x \in X$ which are i) Pareto-efficient, ii) unblocked by all coalitions. In particular, in a two person economy, there are only two cases where an allocation is blocked: if there is a Pareto improvement, or if one agent prefers to stay autarky.

Let us compute the utility of each agent at autarky

$$u^1(\mathbf{e}^1) = (18 * 4)^2 \qquad u^2(\mathbf{e}^2) = \ln 3 + 2\ln 6 \tag{17}$$

So the solution is first feasible, meaning that $x_1^2 = 21 - x_1^1, x_2^2 = 10 - x_2^1$. Moreover, it satisfies:

$$(x_1^1 * x_2^1)^2 \geq (18 * 4)^2 \tag{18}$$
$$\ln(21 - x_1^1) + 2\ln(10 - x_2^1) \geq \ln 3 + 2\ln 6 \tag{19}$$

[6] Notice here superscripts means the first agent, not raised to the power of 1

Replacing each inequality with equal sign, we get the boundary of the core:

$$(x_1^1 * x_2^1)^2 = (18 * 4)^2 \tag{20}$$
$$\ln(21 - x_1^1) + 2\ln(10 - x_2^1) = \ln 3 + 2\ln 6 \tag{21}$$

From the previous problem, we have the set of Pareto efficient allocations, so the core is the intersection of three sets (I inherit the notation from question a), which gives a segment of a curve in Edgeworth box:

$$core = A \cap B \cap C \tag{22}$$
$$A = \{(x_1, x_2) | x_1 * x_2 \geq (18 * 4)\} \tag{23}$$
$$B = \{(x_1, x_2) | \ln(21 - x_1) + 2\ln(10 - x_2) \geq \ln 3 + 2\ln 6\} \tag{24}$$
$$C = \{(x_1, x_2) | x_2 = \frac{10x_1}{42 - x_1}\} \tag{25}$$

**c)** A Walrasian equilibrium consists of a price vector $\mathbf{p}$ and an allocation $\mathbf{x}$ such that

1. Given prices, the allocation solves the consumer's problem

$$\max_{\mathbf{x}_i \in \mathbb{R}_+^n} u^i(\mathbf{x}^i) \quad \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{e} \tag{26}$$

for all $i \in \mathcal{I}$.

2. Markets clear

$$\sum_{i \in \mathcal{I}} (\mathbf{e}_k^i - \mathbf{x}_k^i) = 0 \quad \forall k \tag{27}$$

Let $\lambda$ be a nonnegative multiplier associated with agent i's budget constraint. We form the Lagrangian

$$\mathcal{L} = u^i(\mathbf{x}^i) + \lambda[\mathbf{p} \cdot \mathbf{e}^i - \mathbf{p} \cdot \mathbf{x}^i] \tag{28}$$

An interior solution satisfies the first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_k^i} = u_k^i(\mathbf{x}^i) - \lambda p_k = 0 \tag{29}$$

If $\mathbf{p} \gg 0$, then at an interior solution $\lambda > 0$ and the budget constraint must bind

$$\mathbf{p} \cdot \mathbf{e}^i = \mathbf{p} \cdot \mathbf{x}^i \tag{30}$$

Furthermore, by substituting for $\lambda$, we obtain that in optimum marginal rates of substitution must satisfy

$$\frac{u_k^i(\mathbf{x}^i)}{u_h^i(\mathbf{x}^i)} = \frac{p_k}{p_h} \quad \forall k, h \in \{1, ..., n\} \tag{31}$$

for all $i \in \mathcal{I}$.
Let us set $n = 2$ and impose our functional forms for the utility functions. If $n = 2$, there are exactly two conditions characterizing an optimum for each agent. In the case of agent 1 condition (31) becomes

$$x_2^1 = \left(\frac{p_1}{p_2}\right) x_1^1 \tag{32}$$

By substituting for $x_2$ in the budget constraint (30), we can solve for the Marshallian demand functions

$$x_1^1 = \frac{\mathbf{p} \cdot \mathbf{e}^1}{2p_1} \qquad x_2^1 = \frac{\mathbf{p} \cdot \mathbf{e}^1}{2p_2} \tag{33}$$

In the case of agent 2 condition (31) becomes

$$x_2^2 = 2\left(\frac{p_1}{p_2}\right) x_1^2 \tag{34}$$

Together with the binding budget constraint this implies the Marshallian demand functions

$$x_1^2 = \frac{\mathbf{p} \cdot \mathbf{e}^2}{3p_1} \qquad x_2^2 = \frac{2(\mathbf{p} \cdot \mathbf{e}^2)}{3p_2} \tag{35}$$

By Walras' Law, it suffices to find prices such that one market clears. Furthermore, since the demand functions are homogeneous of degree one, we can normalize prices and set $p_2 = 1$.

$$\begin{aligned}
\frac{\mathbf{p} \cdot \mathbf{e}^1}{2} + \frac{2(\mathbf{p} \cdot \mathbf{e}^2)}{3} &= \sum_{i \in \mathcal{I}} e_2^i \\
\iff 3\mathbf{p} \cdot \mathbf{e}^1 + 4\mathbf{p} \cdot \mathbf{e}^2 &= 6\left(\sum_{i \in \mathcal{I}} e_2^i\right) \\
\iff 3(p_1 e_1^1 + e_2^1) + 4(p_1 e_1^2 + e_2^2) &= 6\left(\sum_{i \in \mathcal{I}} e_2^i\right) \\
\iff p_1 e_1^1 + 4p_1 e_1^2 &= 6\left(\sum_{i \in \mathcal{I}} e_2^i\right) - 3e_2^1 - 4e_2^2 \\
\iff p_1 &= \left(3e_1^1 + 4p_1 e_1^2\right)^{-1}\left[6\left(\sum_{i \in \mathcal{I}} e_2^i\right) - 3e_2^1 - 4e_2^2\right]
\end{aligned}$$

At the endowment $\mathbf{e}^1 = (18, 4)$ and $\mathbf{e}^2 = (3, 6)$, the price $p_1$ that clears markets is given by $p_1 = 4/11$. Therefore, the Walrasian (market-clearing) price vector is $\mathbf{p}^\star = (4/11, 1)$. In order to compute the equilibrium allocation it is useful to compute the market value of each agents endowments

$$\mathbf{p}^\star \cdot \mathbf{e}^1 = 18\left(\frac{4}{11}\right) + 4 = 29\left(\frac{4}{11}\right) \qquad \mathbf{p}^\star \cdot \mathbf{e}^2 = 3\left(\frac{4}{11}\right) + 6 = 13\left(\frac{3}{2}\right)\left(\frac{4}{11}\right) \tag{36}$$

Then the Walrasian equilibrium allocation is given by

$$x_1^1(\mathbf{e}^1, \mathbf{p}^\star) = \left(\frac{1}{2}\right)\left(\frac{11}{4}\right)\left(29\left(\frac{4}{11}\right)\right) = \frac{29}{2} \tag{37}$$

$$x_2^1(\mathbf{e}^1, \mathbf{p}^\star) = \left(\frac{1}{2}\right)\left(29\left(\frac{4}{11}\right)\right) = \frac{58}{11} \tag{38}$$

$$x_1^2(\mathbf{e}^2, \mathbf{p}^\star) = \left(\frac{1}{3}\right)\left(\frac{11}{4}\right)\left(13\left(\frac{3}{2}\right)\left(\frac{4}{11}\right)\right) = \frac{13}{2} \tag{39}$$

$$x_2^2(\mathbf{e}^2, \mathbf{p}^\star) = \left(\frac{2}{3}\right)\left(13\left(\frac{3}{2}\right)\left(\frac{4}{11}\right)\right) = \frac{52}{11} \tag{40}$$

**d)** Let $x^\star$ denote the WEA. Then in this equilibrium, agents' utility levels are given by

$$u^1(x^\star) = (\frac{29}{2}\frac{58}{11})^2 > (18 * 4)^2 \qquad u^2(x^\star) = \ln\left(\frac{13}{2}\right) + 2\ln\left(\frac{52}{11}\right) > \ln(3) + 2\ln(6) \quad (41)$$

We see that both agents are better off in the Walrasian equilibrium than in autarky. Hence the Walrasian equilibrium allocation, being a Pareto efficient allocation, must lie in the core.

# 3 Exercise 5.17

## 3.1 Original question

Consider an exchange economy with two identical consumers. Their common utility function is $u_i(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ for $0 < \alpha < 1$. Society has 10 units of $x_1$ and 10 units of $x_2$ in all. Find endowments $e_1$ and $e_2$, where $e_1 \neq e_2$, and Walrasian equilibrium prices that will 'support' as a WEA the equal-division allocation giving both consumers the bundle $(5, 5)$.

## 3.2 Solution

The Marshallian demand functions for each consumer $i \in \{1, 2\}$, are the solutions to the following problem-

$\max_{\mathbf{x} \in \mathbb{R}^2_+} x_1^\alpha x_2^{1-\alpha}$ subject to $p_1x_1 + p_2x_2 \leq y$

Setting up the Lagrangian function-

$\mathcal{L} = x_1^\alpha x_2^{1-\alpha} - \lambda(p_1x_1 + p_2x_2 - y)$

The first order conditions are-

$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\alpha-1}x_2^{1-\alpha} - \lambda p_1 = 0$

$\frac{\partial \mathcal{L}}{\partial x_2} = (1-\alpha)x_1^\alpha x_2^{-\alpha} - \lambda p_2 = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = -(p_1x_1 + p_2x_2 - y) = 0$

The utility function is strictly increasing, so we can conclude that the budget constraint will bind. The first order conditions can be rewritten as-

$\frac{\alpha x_2}{(1-\alpha)x_1} = \frac{p_1}{p_2}$

$p_1x_1 + p_2x_2 = y$

Solving these equations simultaneously, we get the following expressions-

$x_1^i(\mathbf{p}, y) = \frac{\alpha y^i}{p_1}$

$x_2^i(\mathbf{p}, y) = \frac{(1-\alpha)y^i}{p_2}$
In a Walrasian equilibrium, excess demand will be equal to zero in each market-

$x_1^1 + x_1^2 - e_1^1 - e_1^2 = \frac{\alpha(y^1+y^2)}{p_1} - 10 = 0$

$\implies \frac{\alpha}{p_1}((p_1e_1^1 + p_2e_2^1) + (p_1e_1^2 + p_2e_2^2)) = 10$

Denote $\frac{p_2}{p_1} = p$, and rewrite the above equation as-

$\alpha(e_1^1 + pe_2^1 + e_1^2 + pe_2^2) = 10$

$\implies \alpha((e_1^1 + e_1^2) + p(e_2^1 + e_2^2)) = 10$

$\implies \alpha(1+p) = 1$

$\implies p = \frac{1-\alpha}{\alpha}$[7]

Similarly for the second good we can write-

$\frac{(1-\alpha)(y^1+y^2)}{p_2} - 10 = 0$

$(1-\alpha)(\frac{1}{p} \cdot e_1^1 + e_2^1 + \frac{1}{p} \cdot e_1^2 + e_2^2) = 10$

$\implies (1-\alpha)(\frac{1}{p}(e_1^1 + e_1^2) + (e_2^1 + e_2^2)) = 10$

$\implies (1-\alpha)(\frac{1}{p} + 1) = 1$

$\implies p = \frac{1-\alpha}{\alpha}$

We have the equilibrium relative price now. The problem requires that each consumer demand 5 units of each commodity, given their income and $p$.

$x_1^1 = 5$

$\implies x_1^1 = \frac{\alpha y^1}{p_1} = \frac{\alpha(p_1e_1^1+p_2e_2^1)}{p_1} = \alpha(e_1^1 + pe_2^1) = 5$

$\implies \alpha e_1^1 + (1-\alpha)e_2^1 = 5$[8]

Similarly from $x_1^2 = x_2^2 = 5$ we get-

$\alpha e_1^2 + (1-\alpha)e_2^2 = 5$
We also know that the total endowment of each commodity is equal to 10-

$e_1^1 + e_1^2 = 10$

[7] A simpler way is to abuse the fact that at equilibrium the ratio of marginal utility is the same as the relative price, like what we did in exercise 5.11(a). Here we already know the consumption plan is (5,5) for both agents, and thus the ratio of marginal utility can be directly calculated.

[8] we can also get this using $x_2^1 = 5$

$e_2^1 + e_2^2 = 10$

These equations can be summarized as follows-

$$\begin{pmatrix} \alpha & (1-\alpha) & 0 & 0 \\ 0 & 0 & \alpha & (1-\alpha) \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1^1 \\ e_2^1 \\ e_1^2 \\ e_2^2 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 10 \\ 10 \end{pmatrix}$$

Notice that the columns of the first matrix are not linearly independent.

$$\begin{pmatrix} \alpha \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\alpha}{1-\alpha} \begin{pmatrix} 1-\alpha \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha \\ 1 \\ 0 \end{pmatrix} - \frac{\alpha}{1-\alpha} \begin{pmatrix} 0 \\ 1-\alpha \\ 0 \\ 1 \end{pmatrix}$$

Therefore, this system of equations does not have a unique solution. Any endowment vector $\mathbf{e} \geq \mathbf{0}$ which satisfies these conditions will give us the required Walrasian equilibrium allocation.

# 4 Exercise 5.19

## 4.1 Original question

(Scarf) An exchange economy has three consumers and three goods. Consumers' utility functions and initial endowments are as follows[9] :

$$u^1(x_1, x_2, x_3) = min(x_1, x_2) \quad e^1 = (1, 0, 0) \tag{42}$$

$$u^2(x_1, x_2, x_3) = min(x_2, x_3) \quad e^2 = (0, 1, 0) \tag{43}$$

$$u^3(x_1, x_2, x_3) = min(x_3, x_1) \quad e^3 = (0, 0, 1) \tag{44}$$

Find a Walrasian equilibrium and the associated WEA for this economy.

## 4.2 Solution

First, notice that consumer 1 chooses to have $x_1^1 = x_2^1$ and $x_3^1 = 0$ as optimal consumption plan. With budget constraint $p_1x_1^1 + p_2x_2^1 + p_3x_3^1 = y^1$, his Marshallian demand is:

$$x_1^1 = x_2^1 = \frac{y^1}{p_1 + p_2} \quad x_3^1 = 0 \tag{45}$$

Similarly, for consumer 2 and 3 we have:

$$x_2^2 = x_3^2 = \frac{y^2}{p_2 + p_3} \quad x_1^2 = 0 \tag{46}$$

$$x_1^3 = x_3^3 = \frac{y^3}{p_1 + p_3} \quad x_2^3 = 0 \tag{47}$$

[9]There is a typo in the book: it should not be $e^1$ in the third equation

We also have $y^1 = p_1 * 1 = p_1, y^2 = p_2, y^3 = p_3$. Since only relative price matters, we set $p_3 = 1$, and we calculate the excess demand:

$$z_1(\mathbf{p}) = \frac{p_1}{p_1 + p_2} + \frac{p_3}{p_1 + p_3} - 1 \tag{48}$$

$$z_2(\mathbf{p}) = \frac{p_1}{p_1 + p_2} + \frac{p_2}{p_2 + p_3} - 1 \tag{49}$$

$$z_3(\mathbf{p}) = \frac{p_2}{p_2 + p_3} + \frac{p_3}{p_1 + p_3} - 1 \tag{50}$$

At equilibrium, excess demands should be zero, so we have:

$$0 = \frac{p_1}{p_1 + p_2} + \frac{p_3}{p_1 + p_3} - 1 \tag{51}$$

$$0 = \frac{p_1}{p_1 + p_2} + \frac{p_2}{p_2 + p_3} - 1 \tag{52}$$

$$0 = \frac{p_2}{p_2 + p_3} + \frac{p_3}{p_1 + p_3} - 1 \tag{53}$$

With our normalization $p_3 = 1$, we can subtract the second equation from the first one, and get $\frac{1}{p_1+1} = \frac{p_2}{p_2+1}$, which gives $p_1 = \frac{1}{p_2}$, subtract the second from the third equation, we get $\frac{p_1}{p_1+p_2} = \frac{p_3}{p_1+p_3}$, which gives $(p_1)^2 = p_2$. So we see that $p_1 = p_2 = 1$.

The price vector is thus $\mathbf{p} = (1, 1, 1)$. Plug it into Marshallian demand we just solved, we get the Walrasian equilibrium allocation:

$$x^1 = (\frac{1}{2}, \frac{1}{2}, 0) \tag{54}$$

$$x^2 = (0, \frac{1}{2}, \frac{1}{2}) \tag{55}$$

$$x^3 = (\frac{1}{2}, 0, \frac{1}{2}) \tag{56}$$

We can see that the allocation is feasible.

# 5 Exercise 5.21

## 5.1 Original question

Consider an exchange economy with the two consumers. Consumer 1 has utility function $u^1(x_1, x_2) = x_2$ and endowment $e^1 = (1, 1)$ and consumer 2 has utility function $u^2(x^1, x^2) = x^1 + x^2$ and endowment $e^2 = (1, 0)$.

(a) Which of the hypotheses of Theorem 5.4 fail in this example?

(b) Show that there does not exist a Walrasian equilibrium in this exchange economy.

## 5.2 Solution

**Definitions reviewed**:
**Strongly increasing function**: A real valued function $f$ is strongly increasing if:

$$\mathbf{x}^0 \geq \mathbf{x}^1, \mathbf{x}^0 \neq \mathbf{x}^1 \Rightarrow f(\mathbf{x}^0) > f(\mathbf{x}^1) \tag{57}$$

**Strictly quasiconcave function**: A real valued function $f$ is strictly quasiconcave if and only if: for all $\mathbf{x}^1, \mathbf{x}^2$ in the domain, $f(\mathbf{x}^t) > \min\{\mathbf{x}^1, \mathbf{x}^2\}$ for all $t \in (0,1)$

(a) $u^1(x_1, x_2) = x_2$ is neither strongly increasing nor strictly quasiconcave.

It is enough to provide suitable counter example. Consider the vector $(1,2)$ and $(2,2)$.

$(2,2) \geq (1,2)$ and $u^1(2,2) = u^1(1,2), \Rightarrow u^1$ is not strongly increasing.

$0.5 * (2,2) + 0.5 * (1,2) = (1.5,2)$, and $u^1(1.5,2) = 2 = \min\{u^1(2,2), u^1(1,2)\}$. So $u^1$ is not strictly quasiconcave.

(b) Denote the relative price between goods as $p = \frac{p_2}{p_1}$. Since the first consumer only values the second good, they will spend all their income on $x_2$. Therefore, the demand is

$$x_1^1 = 0, x_2^1 = \frac{y^1}{p_2} = \frac{p_1 + p_2}{p_2} = \frac{1}{p} + 1 \tag{58}$$

Now we look at consumer 2. Notice that the optimal decision is to get the most utility per "dollar"(unit of money), so we can summarize the demand as:

$$(x_1^2, x_2^2) = \begin{cases} (\frac{y^2}{p_1}, 0) & \text{if} \quad p > 1 \\ x_1^2 + x_2^2 = \frac{y^2}{p_1} & \text{if} \quad p = 1 \\ (0, \frac{y^2}{p_2}) & \text{if} \quad p < 1 \end{cases} \tag{59}$$

With $p = \frac{p_2}{p_1}$ and $y^2 = p_1 * 1$, we can further rewrite this equation as:

$$(x_1^2, x_2^2) = \begin{cases} (1, 0) & \text{if} \quad p > 1 \\ x_1^2 + x_2^2 = 1 & \text{if} \quad p = 1 \\ (0, \frac{1}{p}) & \text{if} \quad p < 1 \end{cases} \tag{60}$$

Notice that in total, the economy has endowment $(e_1^1 + e_1^2, e_2^1 + e_2^2) = (1+1, 1+0) = (2,1)$ units of goods. Then we can discuss cases based on the value of $p$:

If $p > 1$, then for good 1 we have excess supply: $z_1(\mathbf{p}) = 1 - 2 = -1$, and excess demand for good 2: $z_2(\mathbf{p}) = 1 + \frac{1}{p} - 1 = \frac{1}{p}$. So it cannot be an equilibrium.

If $p = 1$, then $z_1(\mathbf{p}) = x_1^2 - 2$, which cannot be greater than $1 - 2 = -1$ because $x_1^2 \leq 1$. For the second good $z_2(\mathbf{p}) = 2 + x_2^2 - 1 = 1 + x_2^2 > 0$. So we have excess supply in good 1 and excess demand in good 2, and it cannot be an equilibrium.

If $p < 1$, then $z_1(\mathbf{p}) = -2$ and $z_2(\mathbf{p}) = 1 + \frac{2}{p} - 1 = \frac{2}{p} > 0$. Still, excess supply for good 1 and excess demand for good 2. This cannot be an equilibrium.

In summary, there is no equilibrium.