Lecture on “Endogenous Growth”

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Introduction

This lecture is based on

- Paul Segerstrom (1998), "Endogenous Growth Without Scale Effects," *American Economic Review*. [1343 Google Scholar citations]
- This paper presents a model of economic growth with two attractive properties: endogenous technological change and no scale effects.

Endogenous Technological Change

- In the earliest models of economics growth, the Solow (1956) growth model and the neoclassical growth model developed by Cass (1965) and Koopmans (1965), the rate of technological change is assumed to be given and constant over time (exogenous technological change).
- In a model with endogenous technological change, one solves for the rate of technological change when one solves the model and it typically depends on the incentives that profit-maximizing firms have to engage in investment/R&D activities.
- I think it is important to study economic growth in a setting where technological change is endogenous because otherwise one ends up taking technological change for granted. The rate of technological change has varied a lot over time and across countries. It doesn't just automatically happen.

Scale Effects

- In the 1980s and early 1990s, new models of economic growth were developed with endogenous technological change [Romer (1990, JPE), Segerstrom, Anant and Dinopoulos (1990, AER), Grossman and Helpman (1991, RES), Aghion and Howitt (1992, Econometica)]. Paul Romer recieved the 2018 economics Nobel prize
- All of these models have the "scale effect" property that a larger economy grows faster.
- For example, in the Grossman and Helpman (1991, RES) model, when the labor force L doubles, the economic growth rate g almost doubles and when the aggregate R&D labor L_I doubles, the economic growth rate g exactly doubles.
- Initially, the scale effect property was viewed as a virtue of the new "endogenous growth theory."

- But then Jones (1995, JPE) pointed out a problem: In the United States, the number of scientists and engineers engaged in R&D has increased from $\approx$ 200,000 in 1950 to $\approx$ 1,000,000 in 1993.
- Have we seen a 5-fold increase in the rate of economic growth? No.
- Have we seen **any** increase in the US TFP (total factor productivity) growth rate from 1950-1993? No.
- One does not have to do any sophisticated econometric analysis to realise that a central implication of the early endogenous growth models is clearly at odds with the empirical evidence.
- This has led economists [beginning with Jones (1995, JPE)] to try to develop endogenous growth models without the counter-factual scale effect property.

- The Segerstrom (1998, AER) model of "endogenous growth without scale effects" should be viewed as a model of a single country in autarky (no international trade).
- After solving the model, in the next lecture I will add to the model a second structurally identical country, allow for international trade between the two countries, and then study the implications of lowering the costs of international trade (symmetric trade liberalization).

The Model

- Consider an economy with a continuum of industries indexed by $\omega \in [0, 1]$.
- In each industry, firms are distinguished by the quality j of the products they produce.
- Higher values of j denote higher quality and j is restricted to taking on integer values.
- At time $t = 0$, the state-of-the-art quality in each industry is $j = 0$, that is, some firm in each industry knows how to produce a $j = 0$ quality product and no firm knows how to produce any higher quality product.

- To learn how to produce higher quality products, firms in each industry engage in R&D (research and development).
- In general, when the state-of-the-art quality in an industry is j, the next winner of a R&D race becomes the sole producer of a $j+1$ quality product.
- Thus, over time, products improve as innovations push each industry up its "quality ladder," as in Grossman and Helpman (1991, RES).
- Because of the nature of innovation and because there are many industries where R&D takes place, this model is commonly referred to as a "quality ladders endogenous growth model."

Consumers and Workers

- The economy has a fixed number of identical households that provide labor services in exchange for wages, and save by holding assets of firms engaged in R&D.
- Each individual member of a household lives forever and is endowed with one unit of labor, which is inelastically supplied.
- The number of members in each family grows over time at the exogenous rate $n > 0$. We normalize the total number of individuals in the economy at time 0 to equal unity. Then the population of workers in the economy at time t is $L_t = e^{nt}$. $[L_t = L(t)$, etc.$]$
- Each household is modelled as a dynastic family which maximizes the discounted utility

$$U \equiv \int_0^{\infty} e^{nt} e^{-\rho t} \ln u(t) \, dt$$

where $\rho > n$ is the common subjective discount rate and $u(t)$ is the static utility per person at time t.

- The static utility per person at time t is given by

$$\ln u(t) \equiv \int_0^1 \ln \left[\sum_j \lambda^j d_{jt}(\omega) \right] d\omega$$

  where $d_{jt}(\omega)$ denotes the quantity consumed of a product of quality j produced in industry ω at time t, and $\lambda > 1$ represents the extent to which higher quality products improve upon lower quality products.
- Because λ^j is increasing in j, this equation captures as simply as possible the idea that consumers prefer higher quality products.
- Since consumers cannot buy products that have not been invented yet, at any point in time t, $d_{jt}(\omega)$ is constrained to equal zero for all sufficiently large j.
- This static utility function was introduced in Segerstrom, Anant and Dinopoulos (1990, AER).

- The typical household utility maximization problem is solved in three steps.
- First, we solve for how given consumer expenditure is allocated across products of differing qualities within an industry at a point in time (the static within-industry problem).
- Second, we solve for how given consumer expenditure is allocated across products in different industries at a point in time (the static across-industry problem).
- Finally, we solve for how consumer expenditure is allocated across time (the dynamic problem), taking into account the results derived in the first two steps.

- Turning to the first step of solving the static within-industry problem, let $p_{jt}(\omega)$ denote the price of the product with quality j produced in industry ω at time t and let $c_t(\omega)$ denote the consumer's expenditure on products in industry ω at time t (which we take as given).
- We claim that, in any industry ω at time t, the consumer only buys the product(s) with the lowest quality-adjusted price $\frac{p_{jt}(\omega)}{\lambda^j}$.
- To understand why, it is helpful to consider a simple special case of the consumer's optimization problem.
- Suppose that in industry ω at time t, only $j = 0$ and $j = 1$ quality products are available for sale.

- Looking at the bracketed expression in

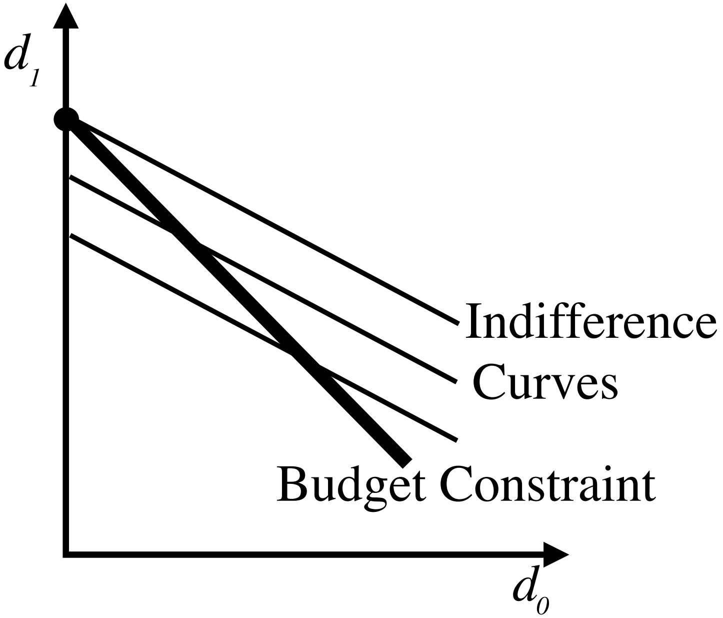
$$\ln u(t) \equiv \int_0^1 \ln \left[\sum_j \lambda^j d_{jt}(\omega) \right] d\omega,$$

the consumer wants to maximize $\lambda^0 d_{0t}(\omega) + \lambda^1 d_{1t}(\omega)$ subject to the budget constraint $p_{0t}(\omega) d_{0t}(\omega) + p_{1t}(\omega) d_{1t}(\omega) = c_t(\omega)$, or simplifying the notation, the consumer's problem can be written as

$$\boxed{\begin{array}{cc} \max_{d_0 \geq 0, d_1 \geq 0} & d_0 + \lambda d_1 \\ \text{subject to} & p_0 d_0 + p_1 d_1 = c. \end{array}}$$

- For a given constant $k > 0$, the level set of the objective function $d_0 + \lambda d_1 = k$ defines a straight line "indifference curve" in (d_0, d_1) space with slope $-\frac{1}{\lambda}$. $d_1 = (k/\lambda) - (1/\lambda) d_0$
- The budget constraint $p_0 d_0 + p_1 d_1 = c$ defines a straight line in (d_0, d_1) space with slope $-\frac{p_0}{p_1}$. $d_1 = (c/p_1) - (p_0/p_1) d_0$
- It follows that when $\frac{1}{\lambda} < \frac{p_0}{p_1}$ (or $\frac{p_1}{\lambda} < p_0$, which means that good 1 has the lower quality-adjusted price), the budget constraint has a steeper slope than any indifference curve and we have a corner solution where $d_0 = 0$ and $d_1 = c/p_1$.
- This case is illustrated in Figure 1.

Figure 1



- On the other hand, when $\frac{1}{\lambda} > \frac{p_0}{p_1}$ (or $\frac{p_1}{\lambda} > p_0$, which means that good 0 has the lower quality-adjusted price), the budget constraint has a flatter slope than any indifference curve and we have a corner solution where $d_0 = c/p_0$ and $d_1 = 0$.
- Finally, when $\frac{1}{\lambda} = \frac{p_0}{p_1}$ (or $\frac{p_1}{\lambda} = p_0$, which means that both goods have the lowest quality-adjusted price), all consumption choices on the budget line are utility-maximizing.
- Thus, the consumer only buys a good if it has the lowest quality-adjusted price.
- This conclusion obviously generalizes to cases involving more than two different quality levels.

- The second step is to solve for how given consumer expenditure is allocated across products in different industries at a point in time.
- Having established that consumers only buy goods with the lowest quality-adjusted prices, let $h = h_t(\omega)$ denote the index of the product with the lowest quality-adjusted price in industry ω at time t (in the event of ties, pick the largest j).
- Then the consumer's static across-industry optimization problem can be expressed more simply as maximizing

$$\ln u(t) \equiv \int_0^1 \ln \left[\sum_j \lambda^j d_{jt}(\omega) \right] d\omega = \int_0^1 \ln[\lambda^h d_{ht}(\omega)] d\omega$$

subject to the budget constraint $\int_0^1 p_{ht}(\omega) d_{ht}(\omega) d\omega = c_t$, where c_t is the consumer's expenditure on all products at time t.

- The integral constraint

$$\int_0^1 p_{ht}(\omega)d_{ht}(\omega)d\omega = c_t$$

is equivalent to the three conditions $\dot{y}(\omega) = p_{ht}(\omega)d_{ht}(\omega)$, $y(0) = 0$ and $y(1) = c_t$ where y is a new state variable.

Hint: let $\quad y(\omega) \equiv \int_0^\omega p_{ht}(\hat{\omega})d_{ht}(\hat{\omega})d\hat{\omega} \quad$ and use Leibnitz Rule.

- With this reformulation, the consumer's static optimization problem becomes the optimal control problem

$$\boxed{\begin{array}{ll} \max_{d_{ht}(\cdot)} & \int_0^1 \ln[\lambda^h d_{ht}(\omega)]d\omega \\ \text{subject to} & \dot{y}(\omega) = p_{ht}(\omega)d_{ht}(\omega) \\ & y(0) = 0, y(1) = c_t. \end{array}}$$

- To solve this optimal control problem, we begin by forming the Hamiltonian function

$$H \equiv \ln[\lambda^h d_{ht}(\omega)] + \mu(\omega) p_{ht}(\omega) d_{ht}(\omega).$$

- Next, we check the costate equation

$$\dot{\mu} = -\frac{\partial H}{\partial y} = 0,$$

  which implies that μ is a constant function [$\mu(\omega) = \mu$ for all ω].

- Then applying Pontryagin's Maximum Principle yields the first-order condition

$$\frac{\partial H}{\partial d_{ht}} = \frac{1}{d_{ht}} + \mu p_{ht} = 0,$$

  which implies that $d_{ht} = -1/\mu p_{ht}$.

- Plugging this result back into the budget constraint in integral form gives

$$\int_0^1 p_{ht}(\omega)\frac{1}{-\mu p_{ht}(\omega)}d\omega = \frac{1}{-\mu} = c_t,$$

from which we conclude that

$$d_{ht}(\omega) = \frac{-1}{\mu p_{ht}} = \frac{c_t}{p_{ht}(\omega)}.$$

- This is the simplest possible consumer demand function and it implies that consumer demand in each industry is unit elastic for the product with the lowest quality-adjusted price. The quantity demanded for all other products is zero.
- To break ties, we have assumed that when quality-adjusted prices are the same for two products of different qualities, a consumer only buys the higher quality product.

- The third and final step to solving the representative household's utility maximization problem is to solve the dynamic problem of how to allocate the consumer's expenditure across time.
- Substituting the consumer's demand function $d_{ht}(\omega) = c_t / p_{ht}(\omega)$ into

$$U \equiv \int_0^\infty e^{nt} e^{-\rho t} \ln u(t) dt$$

using

$$\ln u(t) \equiv \int_0^1 \ln \left[\sum_j \lambda^j d_{jt}(\omega) \right] d\omega,$$

we obtain the household's discounted utility

$$U = \int_0^\infty e^{nt} e^{-\rho t} \left(\int_0^1 \ln \left[\lambda^h \frac{c_t}{p_{ht}(\omega)} \right] d\omega \right) dt.$$

- Using the additive property of the natural logarithm function $[\ln(a \cdot b) = \ln(a) + \ln(b)]$, discounted utility

$$U = \int_0^\infty e^{nt} e^{-\rho t} \left(\int_0^1 \ln \left[\lambda^h \frac{c_t}{p_{ht}(\omega)} \right] d\omega \right) dt$$

can be broken up into two parts:

$$U = \int_0^\infty e^{(n-\rho)t} \ln c_t \, dt + \int_0^\infty e^{(n-\rho)t} \left(\int_0^1 \ln \left[\frac{\lambda^h}{p_{ht}(\omega)} \right] d\omega \right) dt.$$

- The individual household has no ability to influence the value of the second integral since each consumer takes market prices of products as given as well as the rate of technological change in the economy.

- The representative household choices the path of consumer expenditure c_t over time to maximize discounted utility U, which is equivalent to maximizing

$$\int_0^{\infty} e^{(n-\rho)t} \ln c_t \, dt$$

subject to the intertemporal budget constraint

$$\dot{a}_t = w_t + r_t a_t - c_t - n a_t,$$

where a_t denotes the per capita financial assets, w_t is the wage income of the representative household member, and r_t is the market interest rate at time t.

- The Hamiltonian for this optimal control problem is

$$H \equiv e^{(n-\rho)t} \ln c_t + \lambda_t \left[w_t + r_t a_t - c_t - n a_t \right].$$

- From the costate equation

$$-\dot{\lambda}_t = \frac{\partial H}{\partial a_t} = \lambda_t \left[r_t - n \right].$$

- The first order condition for Pontryagin's Maximum Principle is

$$\frac{\partial H}{\partial c_t} = e^{(n-\rho)t} \frac{1}{c_t} - \lambda_t = 0$$

or

$$\lambda_t = e^{(n-\rho)t} \frac{1}{c_t}$$

- Taking natural logarithms of both sides of $\lambda_t = e^{(n-\rho)t}/c_t$ yields

$$\ln \lambda_t = (n-\rho)t - \ln c_t.$$

- Now differentiating both sides with respect to t yields

$$\frac{\dot{\lambda}_t}{\lambda_t} = n - \rho - \frac{\dot{c}_t}{c_t} = n - r_t$$

where the second equality follows from the costate equation.

- Rearranging terms, we obtain the intertemporal consumer optimization condition

$$\frac{\dot{c}_t}{c_t} = r_t - \rho.$$

- This equation must hold for any consumer expenditure path c_t over time that maximises the household's discounted utility subject to the intertemporal budget constraint.

- $$\dot{c}_t/c_t = r_t - \rho.$$
- This equation states that the growth rate of consumer expenditure $\dot{c}_t/c_t$ at each point in time t is an increasing function of the currently prevailing market interest rate r_t.
- When the market interest rate is relatively high ($r_t > \rho$), it is optimal for the consumer to take advantage of the high return on savings by saving more now and spending more later ($\dot{c}_t/c_t > 0$).
- On the other hand, when the market interest rate is relatively low ($r_t < \rho$), it is optimal for the consumer to "borrow against the future" by saving less now and cutting back on spending later ($\dot{c}_t/c_t < 0$).
- Only when the market interest rate r_t equals ρ is it optimal for the consumer to maintain a constant expenditure path over time ($\dot{c}_t = 0$).

Product Markets

- In each industry and at each point in time, firms compete in prices.
- Labor is the only input in production and there are constant returns to scale.
- One unit of labor is required to produce one unit of output, regardless of quality.
- The labor market is perfectly competitive and labor is used as the numeraire, so the wage of workers is set equal to one.
- Consequently, each firm has a constant marginal cost of production equal to one (MC=1).

- When a firm innovates and becomes a quality leader, it receives a patent to exclusively produce the new product.
- Patent enforcement is perfect so firms never have to worry about other firms copying their products.
- To determine static Nash equilibrium prices and profits, consider an industry $\omega \in [0, 1]$ where there is one quality leader and one follower firm (one step down in the quality ladder).
- This is the only type of industry configuration that occurs in equilibrium.

- With the follower firm charging a price of one, the lowest price it can charge and not lose money, the quality leader earns the profit flow

$$\pi(p) = (p-1)c_t L_t / p$$

from charging the price p if $p \leq \lambda$, and zero profits otherwise.

- These profits are maximized by choosing the limit price $p = \lambda > 1$.
- Thus the quality leader earns the profit flow

$$\pi^L \equiv \left(\frac{\lambda - 1}{\lambda}\right) c_t L_t$$

at time t and none of the other firms in the industry can do better than break even (by selling nothing at all).

- In this model, new products drive old products from the market.
- The fact that quality leaders charge a price markup over marginal cost is an essential feature of the model.
- Quality leaders must earn positive profits if they are to recover their upfront outlays in R&D.
- Some imperfect competition is necessary to support private investments in new technologies.

R&D

- Labor is the only input used to do R&D in any industry, is perfectly mobile across industries and between production and R&D activities.
- There is free entry into R&D activities (anyone can do it) and all firms in an industry have the same R&D technology.
- A firm i that hires ℓ_i units of R&D labor in industry ω at time t is successful in discovering the next higher quality product with instantaneous probability (or Poisson arrival rate)

$$\frac{A\ell_i}{X(\omega, t)},$$

where $X(\omega, t)$ is a R&D difficulty index and $A > 0$ is a given technology parameter. [see chapter 21 in Pemberton and Rao for more about Poisson arrival rates.]

- By instantaneous probability, we mean that

$$\frac{A\ell_i}{X(\omega, t)}dt$$

is the probability that the firm will innovate by time $t + dt$ conditional on not having innovated by time t, where dt is an infinitesimal increment of time.

- The returns to engaging in R&D are independently distributed across firms, across industries, and over time.
- Thus, the industry-wide instantaneous probability of innovative success at time t is simply

$$I(\omega, t) \equiv \sum_i \frac{A\ell_i}{X(\omega, t)} = \frac{AL_I(\omega, t)}{X(\omega, t)}$$

where $\sum_i \ell_i = L_I$ is the industry-wide employment of labor in R&D.

- This R&D technology is new to the endogenous growth literature and the novel feature is the $X(\omega, t)$ expression in the denominator.
- I assume that R&D starts off being equally difficult in all industries [$X(\omega, 0) = X_0$ for all ω, $X_0 > 0$ given] and that R&D difficulty grows in each industry as firms do more R&D:

$$\frac{\dot{X}(\omega, t)}{X(\omega, t)} = \mu I(\omega, t),$$

  where $\mu > 0$ is exogenously given.
- With this formulation, I capture in a simple way the idea that as the economy grows and $X(\omega, t)$ increases over time, innovating becomes more difficult. The most obvious ideas are discovered first, making it harder to find new ideas subsequently.
- Grossman and Helpman (1991, RES) assumed that $\mu = 0$.

- $$I(\omega, t) \equiv \frac{AL_I(\omega, t)}{X(\omega, t)}$$

  also implies that a constant innovation rate I can be consistent with positive growth in R&D labor L_I when X grows over time.
- Thus, this model has the potential to explain the evidence cited earlier: growth in the number of scientists and engineers engaged in R&D without accelerating economic growth.
- In the United States, the number of scientists and engineers engaged in R&D has increased from $\approx$ 200,000 in 1950 to $\approx$ 1,000,000 in 1993 without generating any increase in the US TFP (total factor productivity) growth rate from 1950-1993.

- After reading Jones (1995, JPE), I immediately began to think, what can be done to the Grossman-Helpman (1991, RES) model to get rid of the scale effect property?
- There is a 5-fold increase in US R&D employment from 1950-1993: why did the economic growth rate not go up?
- During this time period, patent statistics were roughly constant in the US and many other countries. This suggests that there is a problem with the G-H equation $I = AL_I/X_0$.
- In the US, it is remarkable how little the number of patents granted has changed over time. The number of patents granted in 1993 (53,200) was only slightly different from the number of patents granted in 1966 (54,600).

- Then it occurred to me, maybe innovating is becoming progressively more difficult over time. [L_I increases over time, I does not]
- Maybe in each industry the most obvious ideas are discovered first, making it harder to find new ideas subsequently.
- In a history of technology book called *The Microprocessor: A Biography*, Malone (1985) describes the evolution of R&D difficulty in the microprocessor industry, and writes,

"But miracles, by definition, aren't easy and in the microprocessor business they get harder all the time. The challenges seem to grow with the complexity of the devices. . . that is, exponentially."

In Segerstrom (1998, AER), I present a model where

- aggregate R&D employment increases over time,
- industry-level R&D employment increases over time,
- the patents per researcher ratio decreases over time, and
- the total number of patents granted per year does not change over time.

The driving assumption behind these properties is that R&D becomes progressively more difficult over time ($\mu > 0$).

- Let $v(\omega, t)$ denote the expected discounted profit or reward for winning a R&D race in industry ω at time t (the *value* of being a quality leader) and let s_R denote the fraction of each firm's R&D costs paid by the government (the rate at which the government subsidies the R&D expenditures of firms).
- I will assume that the government finances the chosen R&D subsidy s_R using lump-sum taxation.
- Then at each point in time t, a firm i chooses it's R&D labor input ℓ_i to maximize its expected profits

$$v(\omega, t)\frac{A\ell_i}{X(\omega, t)} - \ell_i(1 - s_R).$$

- If $v(\omega, t) > X(\omega, t)(1 - s_R)/A$, then $\ell_i = +\infty$ is profit maximizing and if $v(\omega, t) < X(\omega, t)(1 - s_R)/A$, then $\ell_i = 0$ is profit maximizing.

- Thus it is only profit-maximising for firms to devote a positive (finite) amount of labor to R&D if

$$v(\omega, t) = \frac{X(\omega, t)(1 - s_R)}{A}.$$

- This equation is the equilibrium implication of free entry into R&D activities. It tells us that when R&D is more difficult [$X(\omega, t)$ is higher] or when the government subsidizes R&D less [s_R is lower], the reward for winning a R&D race must be larger to induce positive R&D effort.
- When equality holds, firms are globally indifferent concerning their choice of R&D effort. Given the symmetric structure of the model, I focus on equilibrium behavior where the R&D intensity $I(\omega, t)$ is the same in all industries ω at time t and is strictly positive. Thus the ω argument of functions is dropped in the remaining analysis.

- The reward for innovating $v(t)$ at time t can be directly determined using the usual "no arbitrage" reasoning.
- There exists a stock market in the economy that channels consumer savings to firms engaged in R&D and helps households to diversify the risk of holding stocks issued by these firms. Consider a shareholder that owns the stock of a quality leader.
- Over a time interval dt, the shareholder receives a dividend $\pi^L(t)\,dt$, and the value of the quality leader appreciates by $\dot{v}(t)\,dt$.
- Because the quality leader is targeted by other firms that conduct R&D to discover the next higher quality product, the shareholder suffers a loss of $v(t)$ if further innovation occurs. This event occurs with probability $I(t)\,dt$, whereas no innovation occurs with probability $1 - I(t)\,dt$.

- No arbitrage in financial markets requires that the expected rate of return from holding the stock of a quality leader is equal to the risk-free rate of return $r(t)\,dt$ that can be obtained through complete diversification:

$$\frac{\pi^L}{v}dt + \frac{\dot{v}}{v}(1 - I\,dt)\,dt - \left[\frac{v-0}{v}\right] I\,dt = r\,dt.$$

  Dividing both sides by dt and then taking limits as dt approaches zero, I obtain:

$$\frac{\pi^L}{v} + \frac{\dot{v}}{v} - I = r \quad \text{or} \quad v(t) = \frac{\pi^L(t)}{r(t) + I(t) - \frac{\dot{v}(t)}{v(t)}}.$$

- The profit flow earned by each quality leader π^L is appropriately discounted using the interest rate r and the instantaneous probability I of being driven out of business by further innovation.

- $$v(t) = \frac{\pi^L(t)}{r(t) + I(t) - \frac{\dot{v}(t)}{v(t)}}.$$

  The fact that profit flows grow over time is captured by the capital gains term $\dot{v}(t)/v(t)$.

- R&D profit maximization implies that the discounted marginal revenue product of an idea $v(t)$ must equal its marginal cost $X(t)(1 - s_R)/A$ at each point in time, that is,

  $$X(t)(1 - s_R)/A = \frac{\left(\frac{\lambda - 1}{\lambda}\right) c_t L_t}{r(t) + I(t) - \frac{\dot{v}(t)}{v(t)}}.$$

- Note that $\frac{\dot{v}(t)}{v(t)} = \frac{\dot{X}(t)}{X(t)} = \mu I(t)$.

- Thus

$$X(t)(1 - s_R)/A = \frac{\left(\frac{\lambda-1}{\lambda}\right) c_t L_t}{r(t) + I(t) - \frac{\dot{v}(t)}{v(t)}}.$$

becomes the R&D condition

$$\frac{x(t)(1 - s_R)}{A} = \frac{\left(\frac{\lambda-1}{\lambda}\right) c_t}{r(t) + (1-\mu)I(t)},$$

where $x(t) \equiv \frac{X(t)}{L(t)}$ is a measure of relative R&D difficulty.

- The R&D condition has a natural economic interpretation:

$$\frac{x_t(1 - s_R)}{A} = \frac{\left(\frac{\lambda - 1}{\lambda}\right) c_t}{r_t + (1 - \mu) I_t}.$$

- Since the LHS corresponds to the market size-adjusted marginal cost of innovating and the RHS corresponds to the market size-adjusted marginal benefit from innovating, the equation states that firms do R&D up to the point where the marginal cost equals the marginal benefit.
- The market size-adjusted marginal benefit from innovating is the profit flow per consumer $\left(\frac{\lambda-1}{\lambda}\right) c_t$ appropriately discounted using the market interest rate r_t and the instantaneous probability I_t that the firm will be driven out of business by further innovation.
- The market size-adjusted marginal cost of innovating is higher when R&D is relatively more difficult (higher x_t) and when R&D is less subsidized (lower s_R).

The Labor Market

- From before, the population of workers in the economy at time t (or labor supply) is $L_t = L(t) = e^{nt}$.
- In each industry ω at time t, consumers only buy from the current quality leader and pay the equilibrium price λ.
- In each industry, individual consumer demand $d_{ht} = c_t / p_{ht}(\omega)$ implies that $c(t)L(t)/\lambda$ workers are employed in production.
- In each industry, the R&D technology $I(t) = AL_I(t)/X(t)$ implies that $L_I(t) = I(t)X(t)/A$ workers are employed in R&D.

- Thus, with a measure one of industries ($\omega \in [0,1]$), full employment of workers is satisfied when

$$L(t) = \frac{c(t)L(t)}{\lambda} + \frac{I(t)X(t)}{A} \quad \text{holds for all } t.$$

- Labor market clearing requires that employment in manufacturing plus employment in R&D equals the available labor supply $L(t)$.
- Dividing both sides by $L(t)$ and using the relative R&D difficulty definition $x(t) \equiv X(t)/L(t)$ yields the resource condition

$$1 = \frac{c(t)}{\lambda} + \frac{I(t)x(t)}{A}.$$

  At any point in time t, given the level of relative R&D difficulty $x(t)$, more consumption $c(t)$ comes at the expense of less innovation $I(t)$.
- This completes the description of the model.

Balanced Growth Equilibrium

- I now solve the model for a balanced growth (or steady-state) equilibrium where all endogenous variables grow at constant (not necessarily the same) rates and firms invest in R&D [$I(t) > 0$ for all t].
- Given $\mu > 0$,

$$\frac{\dot{X}(\omega, t)}{X(\omega, t)} = \mu I(\omega, t),$$

implies that I must be a constant over time. It then follows from

$$1 = \frac{c(t)}{\lambda} + \frac{I(t)x(t)}{A}.$$

that c and x must also be constants.
- Thus, any balanced growth equilibrium must involve c, x and I taking on constant values over time.

- Taking logs and then differentiating

$$I(\omega, t) \equiv \frac{AL_I(\omega, t)}{X(\omega, t)}$$

with respect to time using $\dot{X}(\omega, t)/X(\omega, t) = \mu I(\omega, t)$, yields

$$\frac{\dot{I}}{I} = \frac{\dot{L}_I}{L_I} - \frac{\dot{X}}{X} = \frac{\dot{L}_I}{L_I} - \mu I = 0.$$

- $$I = \frac{AL_I(t)}{X(t)} \qquad \text{and} \qquad 1 = \frac{c}{\lambda} + \frac{Ix}{A}.$$

  together imply that in any balanced growth equilibrium, employment in the R&D sector $L_I(t)$ must grow at the same rate as the population n.

  $L_I(t) = IX(t)/A \Longrightarrow L_I(t)/L(t) = Ix/A \Longrightarrow \dot{L}_I/L_I = \mu I = \dot{L}/L = n$

- Thus $\mu I = n$ and there is a unique balanced growth innovation rate

$$I = \frac{n}{\mu}.$$

- The innovation rate is completely determined by the exogenous rate of population growth $n > 0$ and the R&D difficulty growth parameter $\mu > 0$.
- The innovation rate is higher when the population of consumers grows more rapidly or when R&D difficulty increases more slowly over time.
- Note that if there is no increase in R&D difficulty over time ($\mu \leq 0$), then there is no balanced growth equilibrium. Instead, the growth rate of the economy increases without bound over time as L_t increases.

Implications

$$I = \frac{n}{\mu}.$$

- 1. Population growth is good for innovation in the long run. More people implies more potential innovators.
- 2. Without increasing R&D difficulty ($\mu > 0$), would get explosive technological change in the long run ($I = n/0 = +\infty$).
- 3. Public policies like R&D subsidies have no effect on innovation in the long run! ($I = n/\mu$ does not depend on s_R)

- With I given by $I = n/\mu$, and $\dot{c}_t/c_t = r_t - \rho = 0$ implying that the equilibrium interest rate is $r_t = \rho$,

$$\frac{x_t(1 - s_R)}{A} = \frac{\left(\frac{\lambda-1}{\lambda}\right) c_t}{r_t + (1 - \mu)I_t},$$

yields a balanced growth R&D condition

$$\frac{(1 - s_R)x}{A} = \frac{\left(\frac{\lambda-1}{\lambda}\right) c}{\rho + \frac{n}{\mu} - n},$$

and

$$1 = \frac{c(t)}{\lambda} + \frac{I(t)x(t)}{A}.$$

yields a balanced growth resource condition

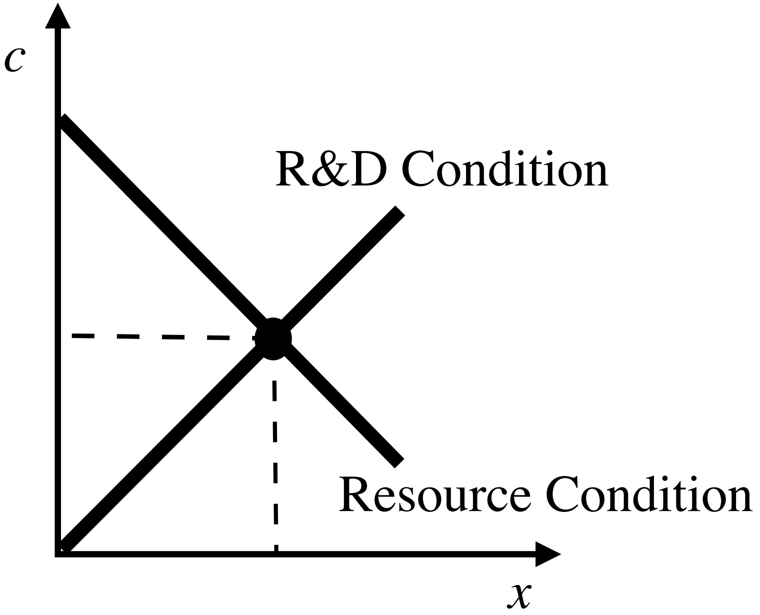
$$1 = \frac{c}{\lambda} + \frac{nx}{A\mu}.$$

- Both balanced growth conditions

$$\frac{(1-s_R)x}{A} = \frac{\left(\frac{\lambda-1}{\lambda}\right)c}{\rho + \frac{n}{\mu} - n} \qquad \text{and} \qquad 1 = \frac{c}{\lambda} + \frac{nx}{A\mu}$$

  are illustrated in Figure 1.
- The vertical axis measures consumption per capita c and the horizontal axis measures relative R&D difficulty x.
- The R&D condition is upward sloping in (x, c) space, indicating that when R&D is relatively more difficult, consumer expenditure must be higher to justify positive R&D effort by firms.
- The resource condition is downward sloping in (x, c) space, indicating that when R&D is relatively more difficult and more resources are used in the R&D sector to maintain the balanced growth innovation rate I, less resources are available to produce goods for consumers, so individual consumers must buy less.

Figure 1



- Both conditions are linear in (x, c) space, the R&D condition goes through the origin and the resource condition has a positive vertical intercept, so there is a unique intersection.
- This unique intersection between the R&D and resource conditions determines the balanced growth values of consumption per capita $\hat{c}$ and relative R&D difficulty $\hat{x}$.
- Thus, the model as a unique balanced growth (or steady-state) equilibrium.

The steady-state equilibrium

- The economist Joseph Schumpeter proposed in his classic book *Capitalism, Socialism and Democracy* (1942) that capitalism (or the free market system) is best understood as an evolutionary process.
- According to Schumpeter, in capitalist reality as distinguished from its textbook picture, the kind of competition that counts is not price competition within a rigid pattern of invariant conditions but "competition from the new commodity, the new technology, the new source of supply, the new type of organization...–competition which commands a decisive cost or quality advantage and which strikes not at the margins of the profits and the outputs of the existing firms but at their foundations and their very lives."
- Schumpeter coined a term to describe this kind of competition, where firms are constantly striving to drive other firms out of business and capture monopoly profits. He called it "the process of creative destruction."

- The steady-state equilibrium illustrated in Figure 1 has a distinctively Schumpeterian flavor: Firms in all industries are doing R&D to discover new higher quality consumer goods, so they can drive the incumbent industry leaders out of business and earn monopoly profits.
- All industry leaders realize that the profits they earn are temporary in nature and only last until the next innovation comes along.
- Because of the randomness of R&D success, some firms that innovate earn monopoly profits for long time periods while other firms that innovate only earn monopoly profits briefly.
- In spite of this uncertainty at the industry level, the model has a unique steady-state equilibrium. What is constant over time in this steady-state equilibrium is the *rate* I at which new higher quality products are discovered in each industry.

The Economic Growth Rate

- The question naturally arises, what is the relationship between the steady-state innovation rate $I = n/\mu$ and the economic growth rate g of the economy?
- I define the economic growth rate of the economy as the growth rate of static utility for the representative consumer, that is,

$$g \equiv \frac{d \ln u(t)}{dt} = \frac{1}{u(t)} \frac{du(t)}{dt} = \frac{\dot{u}(t)}{u(t)},$$

(and will argue shortly that this is a reasonable way of measuring economic growth).

- Since along any steady-state equilibrium path, consumers only buy state-of-the-art quality products,

$$\ln u(t) \equiv \int_0^1 \ln \left[\sum_j \lambda^j d_{jt}(\omega) \right] d\omega$$

can be rewritten as

$$\ln u(t) = \int_0^1 \ln \left(\lambda^j \frac{c}{\lambda} \right) d\omega = \int_0^1 \ln \lambda^j d\omega + \ln \left(\frac{c}{\lambda} \right),$$

where $j = j(\omega, t)$ is the state-of-the-art quality level in industry ω at time t and c is the steady-state consumer expenditure of the representative consumer.
- In order to solve for the economic growth rate g, I first need to calculate the value of the integral $\int_0^1 \ln \lambda^j d\omega$.

- In each industry, since the probability of innovation during a time interval of length dt is $I\,dt$, the exact number m of innovations that occur during a time interval of length τ has a Poisson distribution with parameter $I\tau$. [see pages 477-8 in Pemberton and Rao (2023)]
- It follows that in each industry the probability of exactly m improvements during a time interval of length τ is
$f(m,\tau) = (I\tau)^m e^{-I\tau}/m!$ for $m = 0, 1, 2, 3, \ldots$

$$P(X = k) = \lambda^k e^{-\lambda}/k! \quad \text{in Pemberton and Rao}$$

- Since this hold for each industry and there is a measure one of industries ($\omega \in [0, 1]$), $f(m,\tau)$ also represents the measure of products that are improved exactly m times during a time interval of length τ.
- Without loss of generality, we assume the initial condition $j(\omega, 0) = 0$ for all ω.

- Then, we can calculate the value of relevant integral at time t as follows:

$$
\begin{aligned}
\int_0^1 \ln \lambda^j d\omega = \sum_{m=0}^{\infty} \frac{(It)^m e^{-It}}{m!} \ln \lambda^m &= It(\ln \lambda)e^{-It} \sum_{m=0}^{\infty} \frac{(It)^{m-1}}{m!} m \\
It(\ln \lambda)e^{-It}\left[0 + \frac{1}{1} + \frac{It}{1} + \frac{(It)^2}{2!} + \cdots\right] &= It(\ln \lambda)e^{-It} \sum_{m=0}^{\infty} \frac{(It)^m}{m!} \\
It(\ln \lambda)e^{-It}e^{It} &= It \ln \lambda
\end{aligned}
$$

where we have used the definition of the factorial function $[m! = m(m-1)(m-2)\cdots 1$ for $m = 1, 2, 3, \ldots$ and $0! = 1]$ and the Taylor series expansion of the exponential function around 0 $[e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ implies that $e^{It} = 1 + It + \frac{(It)^2}{2!} + \cdots]$.

- Referring back to

$$\ln u(t) = \int_0^1 \ln\left(\lambda^j \frac{c}{\lambda}\right) d\omega = \int_0^1 \ln \lambda^j d\omega + \ln\left(\frac{c}{\lambda}\right),$$

since consumer expenditure c is constant over time in a steady-state equilibrium, it immediately follows that the economic growth rate on the steady-state equilibrium path is

$$g \equiv \frac{d \ln u(t)}{dt} = \frac{d}{dt}[It \ln \lambda] = I \ln \lambda.$$

- In spite of all the uncertainty at the micro level, the model generates steady economic growth at the macro level. The equation

$$g \equiv \frac{d \ln u(t)}{dt} = I \ln \lambda.$$

implies that the steady-state equilibrium rate of economic growth only depends on the innovation rate I in each industry and the innovation size parameter λ.

- Other things being equal, economic growth is faster when firms innovate more frequently and when innovations represent bigger improvements in product quality.
- Taking λ as given, there is a one-to-one relationship between the innovation rate in each industry I and the economic growth rate of the economy g.

- In solving for the balanced growth equilibrium, I have let the wage rate for labor be the numeriare ($w(t) = 1$ for all t).
- Thus I have implicitly assumed that the nominal wage is constant over time.
- Since quality-adjusted prices are falling over time due to innovation, the real wage $W(t)$ must be rising and we can calculate the long-run equilibrium growth rate of the real wage g_w.

- In the typical industry, an innovation occurs every $1/I$ years and leads to a proportional real wage increase of λ (the innovation size).

  [see page 506 in Pemberton and Rao (2023), where λ is the Poisson arrival rate of calls,

  $$\text{average number of calls per minute} \quad X_L/L \approx \lambda \quad \text{if } L \text{ is large}$$

  $$\implies \quad 1/\lambda \approx L/X_L \quad \text{average number of minutes per call]}$$

- It follows that $W(\frac{1}{I}) = \lambda W(0) = W(0)e^{g_w \cdot 1/I}$ and solving for g_w yields $\lambda = e^{g_w/I}$, $\ln \lambda = g_w/I$ or $g_w = I \ln \lambda$.
- Thus the utility growth rate $g = I \ln \lambda$ is also the real wage growth rate $g_w = I \ln \lambda$ in this model!

Transitional dynamic properties

- The unique intersection between the R&D and resource conditions determines the balanced growth values of consumption per capita $\hat{c}$ and relative R&D difficulty $\hat{x}$.
- If $x = \hat{x}$ at time $t = 0$, then an immediate jump to the balanced growth path can occur.
- Otherwise, it is imperative to investigate the transitional dynamic properties of the model.

- Taking logs of relative R&D difficulty $x(t) \equiv X(t)/L(t)$ yields $\ln x(t) = \ln X(t) - \ln L(t)$ and then differentiating with respect to time using
$$\frac{\dot{X}(t)}{X(t)} = \mu I(t),$$
yields
$$\frac{\dot{x}(t)}{x(t)} = \frac{\dot{X}(t)}{X(t)} - \frac{\dot{L}(t)}{L(t)} = \mu I(t) - n.$$
- Solving the resource condition
$$1 = \frac{c(t)}{\lambda} + \frac{I(t)x(t)}{A}$$
for $I(t)$ yields
$$I(t) = \frac{A}{x(t)}\left(1 - \frac{c(t)}{\lambda}\right)$$

- Substituting this expression for

$$I(t) = \frac{A}{x(t)} \left(1 - \frac{c(t)}{\lambda}\right)$$

back into

$$\frac{\dot{x}(t)}{x(t)} = \mu I(t) - n$$

yields one differential equation that must be satisfied along any equilibrium path for the economy:

$$\dot{x}(t) = \mu A \left(1 - \frac{c(t)}{\lambda}\right) - nx(t)$$

- $$\dot{x}(t) = \mu A \left(1 - \frac{c(t)}{\lambda}\right) - nx(t)$$
- Since the RHS is decreasing in both x and c, $\dot{x}(t) = 0$ defines a downward-sloping curve. [$\dot{x}(t)$ is linear in x and c]
- Starting from any point on this curve, an increase in x leads to $\dot{x} < 0$ and a decrease in x leads to $\dot{x} > 0$, as is illustrated by the horizontal arrows in the next figure.

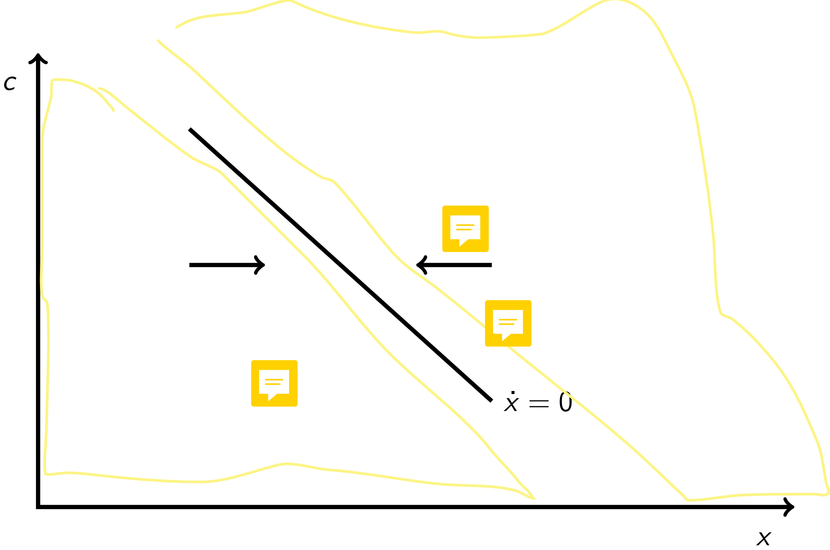


Figure: Drawing the Phase Diagram 1.

- Solving
$$\frac{x(t)(1-s_R)}{A} = \frac{\left(\frac{\lambda-1}{\lambda}\right)c(t)}{r(t)+(1-\mu)I(t)},$$
for $r(t)$ yields
$$r(t) = \frac{(\lambda-1)A}{\lambda(1-s_R)}\frac{c(t)}{x(t)} + (\mu-1)I(t)$$

- Then substituting into
$$\frac{\dot{c}(t)}{c(t)} = r(t) - \rho$$
using
$$I(t) = \frac{A}{x(t)}\left(1-\frac{c(t)}{\lambda}\right)$$
yields
$$\dot{c}(t) = c(t)\left[\frac{(\lambda-1)A}{\lambda(1-s_R)}\frac{c(t)}{x(t)} + \frac{(\mu-1)A}{x(t)}\left(1-\frac{c(t)}{\lambda}\right) - \rho\right].$$

- Thus we obtain a second differential equation that must be satisfied along any equilibrium path for the economy:

$$\dot{c}(t) = c(t)\left[\frac{(\lambda-1)A}{\lambda(1-s_R)}\frac{c(t)}{x(t)} + \frac{(\mu-1)A}{x(t)}\left(1-\frac{c(t)}{\lambda}\right) - \rho\right]$$

- If $\mu \leq 1$, then the $\dot{c}(t) = 0$ curve is definitely upward sloping in (x, c) space:

$$\frac{(\lambda-1)A}{\lambda(1-s_R)}c + (\mu-1)A\left(1-\frac{c}{\lambda}\right) = \rho x$$

- Starting from any point on this curve, an increase in x leads to $\dot{c} < 0$ and a decrease in x leads to $\dot{c} > 0$, implying that there exists an upward-sloping saddlepath.

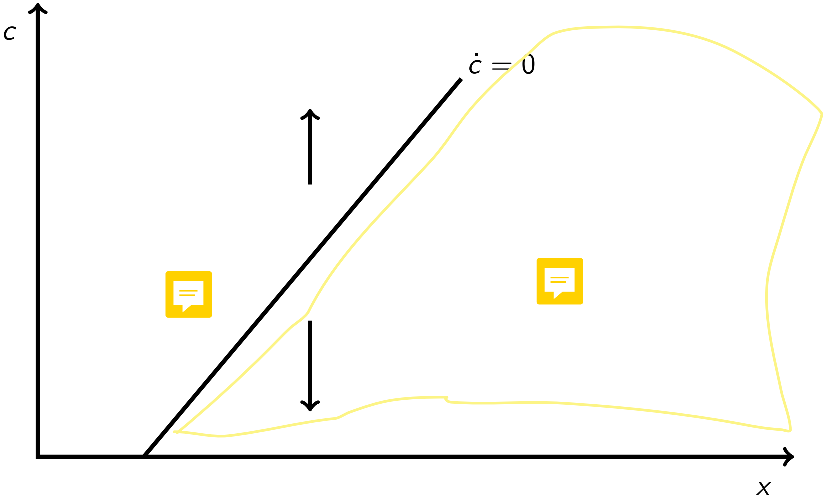
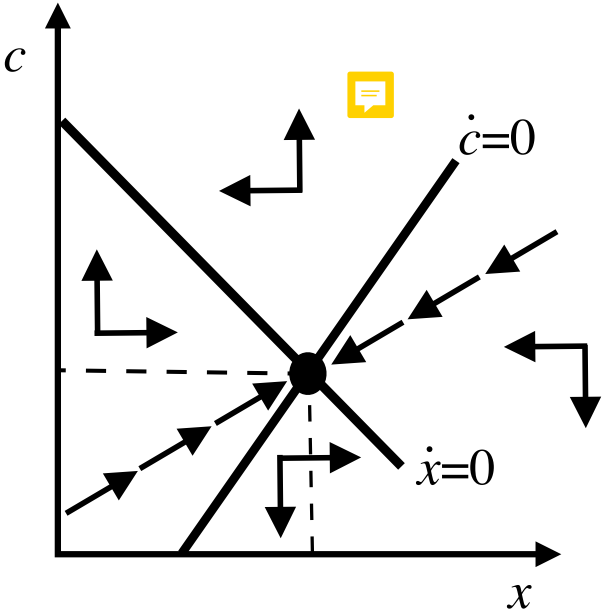


Figure: Drawing the Phase Diagram 2.

The Phase Diagram



- If μ is slightly greater than 1, then the $\dot{c}(t) = 0$ curve is still upward sloping in (x, c) space and there exists an upward sloping saddlepath.
- Even if μ is significantly greater than 1 and the $\dot{c}(t) = 0$ curve is downward sloping, there still exists an upward-sloping saddlepath going through the unique balanced growth equilibrium point.
- Thus the balanced growth equilibrium is saddlepath stable.
- By jumping onto this saddlepath and staying on it forever, convergence to the balanced growth equilibrium occurs, just like in the neoclassical growth model.

Balanced Growth Equilibrium Properties

- Along a balanced growth path, we can determine the fraction of the labor force devoted to R&D.
- $$I(t) \equiv \frac{AL_I(t)}{X(t)}$$

  implies that

  $$\frac{L_I(t)}{L(t)} = \frac{Ix}{A}.$$

- Solving the steady-state R&D condition

  $$\frac{(1-s_R)x}{A} = \frac{\left(\frac{\lambda-1}{\lambda}\right)c}{\rho + \frac{n}{\mu} - n}$$

  for c yields

  $$c = \frac{(1-s_R)x}{A}\left(\rho + \frac{n}{\mu} - n\right)\left(\frac{\lambda}{\lambda-1}\right).$$

- Substituting this value for c into the steady-state resource condition

$$1 = \frac{c}{\lambda} + \frac{lx}{A}$$

yields

$$1 = \frac{(1-s_R)x}{A}\left(\rho + \frac{n}{\mu} - n\right)\left(\frac{1}{\lambda - 1}\right) + \frac{lx}{A}$$

$$1 = \frac{x}{A}\left\{\frac{(1-s_R)}{(\lambda-1)}(l + \rho - n) + l\right\}$$

$$1 = \frac{lx}{A}\left\{\frac{(1-s_R)}{(\lambda-1)}\left[1 + \frac{\rho - n}{l}\right] + 1\right\}$$

from which it follows that

$$\frac{L_I}{L} = \frac{lx}{A} = 1/\left\{\frac{(1-s_R)}{(\lambda-1)}\left[1 + \frac{\rho - n}{l}\right] + 1\right\}.$$

- $$\frac{L_I}{L} = \frac{Ix}{A} = 1/\left\{\frac{(1-s_R)}{(\lambda-1)}\left[1+\frac{\rho-n}{I}\right]+1\right\}.$$
- Given $I = \ n/\mu$, the balanced growth fraction of the labor force devoted to R&D L_I/L is completely determined by parameter values.
- Interestingly, although a higher R&D subsidy s_R has no effect on the long-run innovation rate $I = \ n/\mu$, it does increase the fraction of workers in the economy doing R&D.
  $[s_R \uparrow \Longrightarrow 1 - s_R \downarrow \Longrightarrow L_I/L \uparrow]$

- Returning to
$$g \equiv \frac{\dot{u}}{u} = I \ln \lambda = \frac{n}{\mu} \ln \lambda,$$
individual utility growth depends on the rate at which new higher quality products are introduced.
- Treating individual utility growth g as our measure of long-run economic growth for the economy, this equation implies that a higher R&D subsidy s_R has no effect on the long-run economic growth rate g since it does not impact the long-run innovation rate $I = n/\mu$ in any industry.
- But a higher R&D subsidy s_R does increase the fraction of workers in the economy doing R&D. How can this happen?

Transitional dynamic effects of a higher R&D subsidy s_R

- $$\dot{x}(t) = \mu A \left(1 - \frac{c(t)}{\lambda}\right) - n x(t)$$

  implies that an increase in the R&D subsidy s_R has no effect on the $\dot{x}(t) = 0$ curve.

- $$\dot{c}(t) = c(t) \left[\frac{(\lambda - 1)A}{\lambda(1 - s_R)} \frac{c(t)}{x(t)} + \frac{(\mu - 1)A}{x(t)} \left(1 - \frac{c(t)}{\lambda}\right) - \rho \right]$$

  and

  $$\frac{(\lambda - 1)A}{\lambda(1 - s_R)} c + (\mu - 1)A \left(1 - \frac{c}{\lambda}\right) = \rho x$$

  implies that when s_R increases holding c fixed, x must increase to restore equality, so the $\dot{c}(t) = 0$ curve shifts to the right.

- Starting from a balanced growth equilibrium, when s_R is permanently increased, the $\dot{c}(t) = 0$ curve shifts to the right (redraw Figure 2).
- Thus a higher R&D subsidy leads to an immediate drop in consumption c and an immediate increase in R&D employment. $[L_I(t)/L(t) = I(t)x(t)/A \uparrow \Longrightarrow I(t) \uparrow]$
- Over time, $x(t) \equiv X(t)/L(t)$ increases due to faster than usual rates of technological change. $[I(t) \uparrow \Longrightarrow \dot{X}(t)/X(t) = \mu I(t) \uparrow]$
- But technological change gradually slows down (eventually returning to the initial steady-state level $I = n/\mu$) as researchers find themselves wrestling with increasingly difficult problems.
- Thus the higher R&D subsidy s_R has no effect on the long-run economic growth rate $g = I \ln \lambda$ but because relative R&D difficulty x permanently increases, the fraction of workers in the economy doing R&D $L_I/L = Ix/A$ permanently increases.

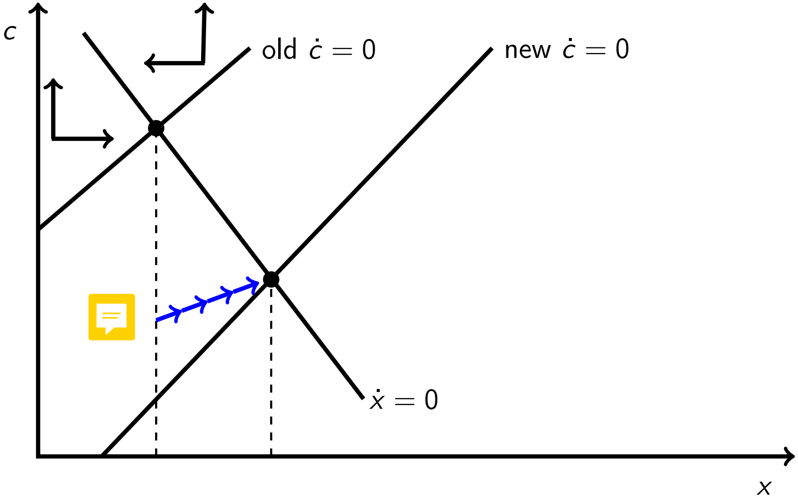


Figure: The saddle-path (in blue), following a permanent increase in s_R.

Related Literature

- Venturini (2012, EER) provides a useful discussion of the literature that is worth quoting at length. He writes,
- "The critique formulated by Jones (1995a) against the prediction of first-generation Schumpeterian growth models that productivity growth increases in the *level* of R&D resources, marked a change in the development of the endogenous growth theory. On a theoretical ground, alternative mechanisms have been suggested to eliminate what is known as the *scale effect* of R&D."

- “A first body of studies emphasizes that R&D spreads thinly across product varieties as the economy grows (Aghion and Howitt, 1998; Dinoupoulos and Thompson, 1998; Peretto, 1998; Howitt, 1999; Young, 1998). Due to population growth, product varieties have to be continuously expanded in order to satisfy consumer demand; therefore, increasing research resources is necessary to make R&D input per inhabitant stable over time. Along the steady-state growth path, the rate of economic growth depends on the proportion of factors allocated to R&D, as well as on the policies able to raise this value. These are usually referred to as *non-scale endogenous* growth models.”

- "A second direction has been explored by Jones (1995b), Kortum (1997) and Segerstrom (1998), who point out that a feature common to most innovation processes is the exhaustion of technological opportunities and the associated difficulty of doing research. This implies that R&D inputs have to be increased over time to maintain a constant rate of innovation. In this strand of the literature, long-run economic growth depends on population dynamics, while innovation policies are effective only along the transition to the steady-state. Consistently, these are defined as *semi-endogenous* growth models."

- Venturini (2012, EER) then reports his own research finding. He writes,
- "This paper assesses whether the most important R&D technologies at the roots of second-generation Schumpeterian growth theories are consistent with innovation statistics. Using US manufacturing industry data, we estimate some systems of simultaneous equations modeling the innovation functions based on variety expansion and diminishing technological opportunities. Our findings indicate that the framework characterized by the increasing difficulty of R&D fits US data better. The exhaustion of technological opportunities is the mechanism best matching the real dynamics of business innovation."