Differential and difference equations - application in simple economic models

Josef Strásky
josef.strasky@gmail.com

October 24, 2013

Differential vs. difference

Some similarities (not equivalencies) of differential and difference equations:

Differential	Difference
• continuous time	• discrete time
• $y(t)$	• y_t
• $\dot{y}(t)$	• $y_{t+1} - y_t$
• $e^{\lambda t}$	• λ^t
• phase diagram:	• phase diagram:
• $\dot{y}(t)$ vs. $y(t)$	• y_{t+1} vs. y_t

Harrod-Domar continuous I

- The most simple model ever developed
- Simple growth model
- Asumptions:

$$S = sY \tag{1}$$

- savings are proportional to income Y
- s is exogenous propensity to save

$$I = \dot{K} = v\dot{Y} \tag{2}$$

- change in capital (i.e. investment) is proportional to change in income Y
- v is exogenous 'propensity to invest'

$$I = S \tag{3}$$

- investment is equal to savings

Harrod-Domar continuous II

Once again:

$$S = sY \tag{4}$$

$$I = \dot{K} = v\dot{Y} \tag{5}$$

$$I = S \tag{6}$$

It clearly follows that:

$$v\dot{Y} = sY \tag{7}$$

$$\dot{Y} - \frac{s}{v}Y = 0 \tag{8}$$

Initial condition (formally):

$$Y(0) = Y_0 \tag{9}$$

This is linear, homogenous differential equation of the first order, with constant coefficients...

Harrod-Domar continuous III

$$\dot{Y}(t) - \frac{s}{v} Y(t) = 0 \tag{10}$$

Please solve either by

- guess (recommended)
- integration factor method
- characteristic polynomial method

Harrod-Domar continuous IV

$$\dot{Y}(t) - \frac{s}{v}Y(t) = 0 \tag{11}$$

$$Y(t) = Y_0 e^{\frac{s}{v}t} \tag{12}$$

- explosive path
- no equilibrium

Harrod-Domar continuous V

Question 1: Given the parameters, how much is the yearly growth?
Question 2: In which units should we measure time?
Answer1: Not provided in presentation.
Answer2: Parameter v scales with time.

$$I = \dot{K} = v\dot{Y} \tag{13}$$

Income Y is 'something per time period' (time unit), whereas capital K is only 'something'. Therefore time units are incorporated in Y (not in K) and parameter v must then be dependent on time unit choice.

Harrod-Domar discrete I

Very similar assumptions:

$$S_t = sY_t \tag{14}$$
$$I_t = v\left(Y_t - Y_{t-1}\right) \tag{15}$$
$$I_t = S_t \tag{16}$$

Parameter v is referred to as **acceleration** factor since it is responsible for growth of output without any improvement in technology (compare to Sollow model).
Show your algebraic manipulation skill and derive equation for Y_t in a nice form.

Harrod-Domar discrete II

$$Y_t - \left(\frac{v}{v-s}\right) Y_{t-1} = 0 \tag{17}$$

Please solve either by

- guess (recommended)
- characteristic polynomial method

Harrod-Domar discrete III

$$Y_t = \left(\frac{v}{v-s}\right)^t Y_0 \tag{18}$$

Harrod-Domar discrete IV

$$Y_t = \left(\frac{v}{v-s}\right)^t Y_0 \tag{19}$$

Discussion:

- We always assume $v > 0$, $s > 0$
- $v > s \rightarrow \frac{v}{v-s} > 1$ thus solution is explosive, non oscillatory
- $s > v$ and $s > 2v \rightarrow -1 < \frac{v}{v-s} < 0$ oscillatory but damped
- $s > v$ and $s = 2v \rightarrow \frac{v}{v-s} = -1$ oscillatory
- $s > v$ and $s < 2v \rightarrow \frac{v}{v-s} < -1$ oscillatory and explosive

Sollow continuous I

$$\frac{Y}{L} = \frac{F(K,L)}{L} = F\left(\frac{K}{L},1\right) = F(k,1) = f(k) \tag{20}$$

$$y = f(k) \tag{21}$$

$$f(0) = 0, f'(k) > 0, f''(k) < 0, k > 0 \tag{22}$$

$$\tag{23}$$

$$\dot{L} = nL \tag{24}$$

$$\dot{K} = I - \delta K \tag{25}$$

$$S = sY \tag{26}$$

$$I = S \tag{27}$$

$$\dot{K} = sY - \delta K \tag{28}$$

$$\tag{29}$$

Sollow continuous II

$$k(t) = \frac{K(t)}{L(t)} \tag{30}$$

Now, compute:

$$\dot{k} = \frac{d\left(\frac{K(t)}{L(t)}\right)}{dt} = \dot{\left(\frac{K(t)}{L(t)}\right)} \tag{31}$$

Hints:

$$\frac{K(t)}{L(t)} = K(t).\frac{1}{L(t)} \tag{32}$$

$$\dot{(f(t).g(t))} = \dot{f}(t).g(t) + f(t).\dot{g}(t) \tag{33}$$

$$\dot{(f(g(t)))} = \frac{df(t)}{dg(t)}.\dot{g}(t) \tag{34}$$

$$\left(\frac{1}{x}\right)' = (x^{-1})' = -x^{-2} = -\frac{1}{x^2} \tag{35}$$

Sollow continuous III

Thus:

$$\dot{k} = k\left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right) \tag{36}$$

Recall:

$$\frac{\dot{L}}{L} = \frac{nL}{L} = n \tag{37}$$

$$\dot{K} = sY - \delta K \tag{38}$$

$$\frac{\dot{K}}{K} = \frac{sY - \delta K}{K} = \frac{sY}{L}\frac{L}{K} - \delta = \frac{sf(k)}{k} - \delta \tag{39}$$

$$\tag{40}$$

Finally:

$$\dot{k} = sf(k) - \delta k - nk = sf(k) - (n + \delta)k \tag{41}$$

Sollow continuous IV

Let us introduce Cobb-Douglas function

$$Y = aK^{\alpha}L^{1-\alpha} \tag{42}$$

$$\frac{Y}{L} = y = f(k) = a\left(\frac{K}{L}\right)^{\alpha} = ak^{\alpha} \tag{43}$$

$$\dot{k} = sak^{\alpha} - (n+\delta)k \tag{44}$$

Solution of this differential equation is not easy (why?).

Solution, with initial condition $k(0) = k_0$.

$$k(t) = \left[\frac{as}{n+\delta} + e^{(1-\alpha)(n+\delta)t} \left(k_0^{1-\alpha} - \frac{as}{n+\delta} \right) \right]^{\frac{1}{1-\alpha}} \tag{45}$$

Question: Why does not work for $k_0 < 0$?

Multiplier - accelerator model I

- Developed by Samuelson in 1939
- Based on multiplier mechanism by Keynes and accelerator concept by Harrod
- The aim was to develop simple model allowing cyclical behaviour
- Model behaves cyclically for reasonable choice of parameters but it is either damping or explosive (except for specific parameters combination)

Keynesian consumption function with lag

$$C_t = C_0 + cY_{t-1} \tag{46}$$

- consumption is proportional to lagged value of output Y
- $c > 0$ is marginal propensity to consume - MULTIPLIER
- C_0 is autonomous consumption

Multiplier - accelerator model II

$$I_t = I_0 + v\left(C_t - C_{t-1}\right) \tag{47}$$

- investment is induced by changes in consumption demands (ACCELERATION principle)
- $v > 0$
- I_0 is autonomous consumption

Aggregate demand is assumed (without government and foreign sector):

$$Y_t^d = C_t + I_t \tag{48}$$

Finally we assume equilibrium in goods market:

$$Y_t^d = Y_t \tag{49}$$

Let us put the equations together - substituting for I_t:

$$Y_t = C_t + I_t = C_t + I_0 + v\left(C_t - C_{t-1}\right) \tag{50}$$

Now we can also substitute for C_t.

Multiplier - accelerator model I

$$Y_t - (1+v).c.Y_{t-1} + v.c.Y_{t-2} = I_0 + C_0 \tag{51}$$

This is second-order linear autonomous difference equation with constant coefficients.
Show that equilibrium is:

$$Y^* = \frac{C_0 + I_0}{1-c} \tag{52}$$

Characteristic polynomial:

$$\lambda^2 - (1+v).c.\lambda + v.c = 0 \tag{53}$$

$$\lambda_1 = \frac{1}{2}\left(c + cv + \sqrt{c^2 + 2c^2v + c^2v^2 - 4vc}\right) \tag{54}$$

$$\lambda_2 = \frac{1}{2}\left(c + cv - \sqrt{c^2 + 2c^2v + c^2v^2 - 4vc}\right) \tag{55}$$

$$\lambda_1 = \frac{1}{2}c\left(1 + v + \sqrt{(1+v)^2 - 4\frac{v}{c}}\right) \tag{56}$$

$$\lambda_2 = \frac{1}{2}c\left(1 + v - \sqrt{(1+v)^2 - 4\frac{v}{c}}\right) \tag{57}$$

Non-cyclic solutions:

$$(1+v)^2 - 4\frac{v}{c} > 0 \tag{58}$$
$$c(1+v)^2 - 4v > 0 \tag{59}$$
$$c > \frac{4v}{(1+v)^2} \tag{60}$$

λ_1 and λ_2 are then real and moreover positive since $v > 0$, $c > 0$
Solution is explosive if larger root is bigger than one.
Note that

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \tag{61}$$
$$\lambda_1\lambda_2 = vc \tag{62}$$

Therefore if $vc > 1$ then solution is explosive (either λ_1 or λ_2 must be > 1).
Solution is stable damped if $vc < 1$, which is possible only if $v < 1$ because $c > \frac{4v}{(1+v)^2}$.

Cyclic solutions:

$$(1+v)^2 - 4\frac{v}{c} < 0 \tag{63}$$

$$c < \frac{4v}{(1+v)^2} \tag{64}$$

Solution is damped if $R = \sqrt{\alpha^2 + \beta^2} < 1$

$$\alpha = \frac{1}{2}(c + cv) \tag{65}$$

$$\beta = \frac{1}{2}\sqrt{4vc - c^2 - 2c^2v - c^2v^2} \tag{66}$$

$$R = \sqrt{\frac{(c+vc)^2}{4} + \frac{4vc - (c+vc)^2}{4}} = \sqrt{vc} \tag{67}$$

- If $v.c < 1$ the solution is damped (converging)
- If $v.c > 1$ then sollution is explosive
- If $v.c = 1$ the solution is cyclic (by definition stable, but not converging).