Lecture on "The Melitz Model and Trade"

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Introduction

This lecture is based on

- Marc Melitz (2003), "The Impact of Trade on Intra-Industry Reallocations and Aggregate Industry Productivity, *Econometrica*. [18,726 Google Scholar citations]
- Daniel Trefler (2004), "The Long and Short of the Canada-U.S. Free trade Agreement," *American Economic Review*.
- Paul Segerstrom and Yoichi Sugita (2015), "The Impact of Trade Liberalization on Industrial Productivity," *Journal of the European Economic Association*.
- Paul Segerstrom and Yoichi Sugita (2018), "A Solution to the Melitz-Trefler Puzzle," Stockholm School of Economics working paper.

- The Melitz (2003, *Econometrica*) model is the currently most influential model of international trade, with almost 19,000 Google Scholar citations. This model has profoundly changed how economists think about international trade.
- In this lecture, I present all the calculations that are done to solve the Melitz model and fully explain the intuition behind the model's properties. The original paper is hard to read because most of the calculations that Melitz did to solve the model are left to the reader to figure out. Just the most important equations are presented in the paper.

Empirical evidence that motivated Melitz (2003)

- large and persistent productivity differences among firms in narrowly defined industries
- within so-called export sectors, many firms do not export their products
- it is the most productive firms that tend to export [Bernard and Jensen (1999) for US, Aw, Chung and Roberts (2000) for Taiwan, Clerides, Lack and Tybout (1998) for Columbia, Mexico and Morrocco]
- exposure to international trade forces the least productive firms to exit and contributes to productivity growth [Aw, Chung and Roberts (2000), Pavcnik (2002), Bernard and Jensen (1999)]
- Melitz (2003, *Econometrica*) developed a model to explain this empirical evidence.

The Melitz Model

- Consider a global economy consisting of $n \geq 2$ countries with $m \geq 2$ sectors (or industries).
- Subscripts i and j denote countries ($i, j \in \{1, 2, \ldots, n\}$) and subscript s denotes sectors ($s \in \{1, 2, \ldots, m\}$).
- Though the model has infinitely many periods, there is no means for saving over periods.
- We focus on a stationary steady state equilibrium where aggregate variables do not change over time and omit notation for time periods.

Consumer Preferences

- The representative consumer in country i has a two-tier (Cobb-Douglas plus CES) utility function:

$$U_i \equiv \prod_{s=1}^{m} (C_{is})^{\alpha_s} \text{ where } C_{is} \equiv \left[\int_{\omega \in \Omega_{is}} q_{is}(\omega)^{\rho} d\omega \right]^{1/\rho} \text{ and } \sum_{s=1}^{m} \alpha_s = 1.$$

- In this equation, $q_{is}(\omega)$ is the country i's consumption of a product variety ω produced in sector s, Ω_{is} is the set of available varieties in sector s and ρ measures the degree of product differentiation.
- We assume that products within a sector are closer substitutes than products across sectors, which implies that a within-sector elasticity of substitution $\sigma \equiv 1/(1-\rho)$ satisfies $\sigma > 1$.
- Given that $\sum_{s=1}^{m} \alpha_s = 1$, α_s represents the share of consumer expenditure on sector s products.

- First, we solve the within-sector consumer optimization problem

$$\max_{q_{is}(\cdot)} C_{is} \equiv \left[\int_{\omega\in\Omega_{is}} q_{is}(\omega)^{\rho}\, d\omega\right]^{1/\rho} \text{ s.t. } \int_{\omega\in\Omega_{is}} p_{is}(\omega)q_{is}(\omega)\, d\omega = E_{is}$$

  where $q_{is}(\omega)$ is quantity demanded for variety ω in country i and sector s, $p_{is}(\omega)$ is the price of variety ω and E_{is} is consumer expenditure on sector s products.
- This problem of maximizing a CES utility function subject to a budget constraint can be rewritten as the optimal control problem

$$\max_{q_{is}(\cdot)} \int_{\omega\in\Omega_{is}} q_{is}(\omega)^{\rho}\, d\omega \text{ s.t. } \dot{y}_{is}(\omega) = p_{is}(\omega)q_{is}(\omega),\ y_{is}(0) = 0,$$

  and $y_{is}(+\infty) = E_{is}$, where $y_{is}(\omega)$ is a new state variable and $\dot{y}_{is}(\omega)$ is the derivative of y_s with respect to ω. $[y_{is}(x) \equiv \int_0^x p_{is}(\omega)q_{is}(\omega)\, d\omega]$

- The Hamiltonian function for this optimal control problem is

$$H = q_{is}(\omega)^{\rho} + \xi(\omega) p_{is}(\omega) q_{is}(\omega)$$

where $\xi(\omega)$ is the costate variable.

- The costate equation $\partial H / \partial y_{is} = 0 = -\dot{\xi}(\omega)$ implies that $\xi(\omega)$ is constant across ω.

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$$\partial H / \partial q_{is} = \rho q_{is}(\omega)^{\rho - 1} + \xi \cdot p_{is}(\omega) = 0$$

implies that

$$q_{is}(\omega) = \left(\frac{-\rho}{\xi \cdot p_{is}(\omega)} \right)^{1/(1-\rho)} .$$

- Substituting this back into the budget constraint yields

$$
\begin{aligned}
E_{is} &= \int_{\omega\in\Omega_{is}} p_{is}(\omega) q_{is}(\omega)\, d\omega = \int_{\omega\in\Omega_{is}} p_{is}(\omega) \left(\frac{-\rho}{\xi\cdot p_{is}(\omega)}\right)^{1/(1-\rho)} d\omega \\
&= \left(\frac{-\rho}{\xi}\right)^{1/(1-\rho)} \int_{\omega\in\Omega_{is}} p_{is}(\omega)^{\frac{1-\rho-1}{1-\rho}} d\omega.
\end{aligned}
$$

- Now $\sigma \equiv \frac{1}{1-\rho}$ implies that $1-\sigma = \frac{1-\rho-1}{1-\rho} = \frac{-\rho}{1-\rho}$, so

$$
\frac{E_{is}}{\int_{\omega\in\Omega_{is}} p_{is}(\omega)^{1-\sigma} d\omega} = \left(\frac{-\rho}{\xi}\right)^{1/(1-\rho)}.
$$

- It immediately follows that the consumer demand function is

$$
q_{is}(\omega) = \left(\frac{-\rho}{\xi\cdot p_{is}(\omega)}\right)^{1/(1-\rho)} = \frac{p_{is}(\omega)^{-\sigma} E_{is}}{P_{is}^{1-\sigma}}
$$

where $P_{is} \equiv \left[\int_{\omega\in\Omega_{is}} p_{is}(\omega)^{1-\sigma} d\omega\right]^{1/(1-\sigma)}$ is the price index for country i and sector s.

- Substituting this consumer demand function back into the CES utility function yields

$$C_{is} = \left[\int_{\omega\in\Omega_{is}} q_{is}(\omega)^{\rho}\, d\omega\right]^{1/\rho} = \left[\int_{\omega\in\Omega_{is}} \frac{p_{is}(\omega)^{-\sigma\rho} E_{is}^{\rho}}{P_{is}^{(1-\sigma)\rho}}\, d\omega\right]^{1/\rho}$$

$$= \frac{E_{is}}{P_{is}^{1-\sigma}} \left[\int_{\omega\in\Omega_{is}} p_{is}(\omega)^{-\sigma\rho}\, d\omega\right]^{1/\rho}.$$

- Taking into account that $-\sigma\rho = \frac{-\rho}{1-\rho} = 1-\sigma$, the CES utility can be simplified further to

$$C_{is} = \frac{E_{is}}{P_{is}^{1-\sigma}} \left[\int_{\omega\in\Omega_{is}} p_{is}(\omega)^{1-\sigma}\, d\omega\right]^{1/\rho} = \frac{E_{is}}{P_{is}^{1-\sigma}} \left[P_{is}^{1-\sigma}\right]^{1/\rho}$$

$$= \frac{E_{is}}{P_{is}^{1-\sigma}} P_{is}^{-\sigma} = \frac{E_{is}}{P_{is}}.$$

- Thus, we can write the across-sector consumer optimization problem as

$$\max_{E_{is}} U_i \equiv \prod_{s=1}^{m} (C_{is})^{\alpha_s} = \prod_{s=1}^{m} \left(\frac{E_{is}}{P_{is}} \right)^{\alpha_s} \qquad \text{s.t.} \qquad \sum_{s=1}^{m} E_{is} = E_i$$

where E_i is consumer expenditure on products in all sectors combined.

- To solve the across-sector consumer optimization problem, we first write down the Lagrangean function

$$\mathcal{L}_i \equiv \prod_{s=1}^{m} \left(\frac{E_{is}}{P_{is}} \right)^{\alpha_s} - \lambda \left(\sum_{s=1}^{m} E_{is} - E_i \right)$$

where λ is the Lagrange multiplier.

- The first order conditions

$$\frac{\partial \mathcal{L}_i}{\partial E_{is}} = \alpha_s \frac{E_{is}^{\alpha_s - 1}}{P_{is}^{\alpha_s}} \prod_{k=1,\, k \neq s}^{m} \left(\frac{E_{ik}}{P_{ik}} \right)^{\alpha_k} - \lambda = 0 \qquad \text{for all } s$$

can be written more simply as $(\alpha_s / E_{is}) U_i = \lambda$ or $\alpha_s U_i / \lambda = E_{is}$ for all s.

- Substituting this back into the constraint yields

$$\sum_{s=1}^{m} E_{is} = \sum_{s=1}^{m} \alpha_s \frac{U_i}{\lambda} = \frac{U_i}{\lambda} \sum_{s=1}^{m} \alpha_s = \frac{U_i}{\lambda} = E_i$$

and it follows that the solution to this problem is $E_{is} = \alpha_s U_i / \lambda = \alpha_s E_i$.

Workers

- Country i is endowed with L_i unit of labor as the only factor of production.
- Labor is inelastically supplied and workers in country i earn the competitive wage rate w_i, so total wage income that can be spent on products is $w_i L_i$.
- Given free entry, there are no profits earned from entering markets, so consumers spend exactly what they earn in wage income. It follows that

$$E_{is} = \alpha_s E_i = \alpha_s w_i L_i.$$

- We measure all prices relative to the price of labor in country n by setting $w_n = 1$.

Firms

- Firms are risk neutral and maximize expected profits.
- In each time period, the measure M_{ise} of firms choose to enter in country i and sector s.
- Each firm uses f_{ise} units of labor to enter and incurs the fixed entry cost $w_i f_{ise}$.
- Each firm then independently draws its productivity φ from a Pareto distribution. The cumulative distribution function $G(\varphi)$ is given by

$$G(\varphi) = 1 - \left(\frac{b}{\varphi}\right)^{\theta} \quad \text{for} \quad \varphi \in [b, \infty),$$

where $\theta > 0$ and $b > 0$ are the shape and scale parameters of the distribution.

- A firm with productivity φ uses $1/\varphi$ units of labor to produce one unit of output and has constant marginal cost w_i/φ in country i.
- This firm must use f_{ij} units of domestic labor and incur the fixed "marketing" cost $w_i f_{ij}$ to sell in country j.
- Denoting $f_{ii} = f_d$ and $f_{ij} = f_x$ for $i \neq j$, we assume that exporting requires higher fixed costs than local selling ($f_x > f_d$).
- There are also iceberg trade costs associated with shipping products across countries: a firm that exports from country i to country $j \neq i$ in sector s needs to ship $\tau_{ijs} > 1$ units of a product in order for one unit to arrive at the foreign destination (if $j = i$, then $\tau_{iis} = 1$).

The Returns to Scale in R&D

- We assume that individual firms take f_{ise} as given but at the aggregate level, entry costs satisfy

$$f_{ise} = F \cdot M_{ise}^{\zeta} \qquad \text{where} \qquad \zeta \geq 0,$$

  that is, entry costs go up ($\zeta > 0$) or remain constant ($\zeta = 0$) as more firms choose to enter.

- Melitz (2003) assumed that $\zeta = 0$ so we are studying a more general model that includes the Melitz model as a special case.

- Since M_{ise} is the number of firms that enter and $F \cdot M_{ise}^{\zeta}$ is the labor used per firm, the total labor used for R&D in country i and sector s is $L_{ise} \equiv F \cdot M_{ise}^{1+\zeta}$.
- Solving this expression for M_{ise} yields

$$M_{ise} = (L_{ise}/F)^{1/(1+\zeta)},$$

  where M_{ise} can be thought of as the flow of new products developed by researchers and L_{ise} is the sector level of R&D labor.
- If $\zeta > 0$, we obtain decreasing returns to scale in R&D at the sector level: when R&D input L_{ise} is doubled, R&D output M_{ise} less than doubles.
- If $\zeta = 0$ (the Melitz model case), we obtain constant returns to scale in R&D at the sector level: when R&D input L_{ise} is doubled, R&D output M_{ise} doubles.

$$M_{ise} = (L_{ise}/F)^{1/(1+\zeta)}$$

- A large empirical literature on patents and R&D has shown that R&D is subject to significant decreasing returns at the sector level ($\zeta > 0$).
- The patents per R&D worker ratio has declined for most of the 20th century (Griliches, 1994). This trend holds across countries (Evenson, 1984) and across industries (Kortum, 1993).
- According to Kortum (1993), point estimates of $1/(1+\zeta)$ lie between 0.1 and 0.6, which corresponds to ζ values between 0.66 and 9. Blundell, Griffith and Windmeijer (2002, Journal of Econometrics) find that the long-run elasticity of patents to R&D is 0.5, which corresponds to $1/(1+\zeta) = 0.5$ or $\zeta = 1$.
- The Melitz model case where $\zeta = 0$ is outside the range of empirical estimates.

Profit Maximization

- Let $p_{ijs}(\varphi)$ denote the price charged by a firm in country i with productivity φ from selling to country j in sector s.
- Let $q_{ijs}(\varphi)$ denote the quantity that consumers in country j buy from this firm and let $r_{ijs}(\varphi) \equiv p_{ijs}(\varphi)\, q_{ijs}(\varphi)$ denote the corresponding firm revenue.
- Consumer optimization calculations imply that consumer demand and the corresponding firm revenue are

$$q_{ijs}(\varphi) = \frac{p_{ijs}(\varphi)^{-\sigma} \alpha_s w_j L_j}{P_{js}^{1-\sigma}} \quad \text{and} \quad r_{ijs}(\varphi) = \frac{p_{ijs}(\varphi)^{1-\sigma} \alpha_s w_j L_j}{P_{js}^{1-\sigma}}.$$

- A firm with productivity φ from country i earns gross profits $\pi_{ijs}(\varphi)$ from selling to country j in sector s (not including fixed costs). It follows that

$$\begin{aligned}\pi_{ijs}(\varphi) &= r_{ijs}(\varphi) - \frac{w_i \tau_{ijs}}{\varphi} q_{ijs}(\varphi) \\ &= \frac{p_{ijs}(\varphi)^{1-\sigma} \alpha_s w_j L_j}{P_{js}^{1-\sigma}} - \frac{w_i \tau_{ijs}}{\varphi} \frac{p_{ijs}(\varphi)^{-\sigma} \alpha_s w_j L_j}{P_{js}^{1-\sigma}}.\end{aligned}$$

- We obtain the price that maximizes gross profits by solving the first order condition

$$\begin{aligned}\frac{\partial \pi_{ijs}(\varphi)}{\partial p_{ijs}(\varphi)} &= \frac{(1-\sigma)p_{ijs}(\varphi)^{-\sigma}\alpha_s w_j L_j}{P_{js}^{1-\sigma}} + \frac{w_i \tau_{ijs}\sigma p_{ijs}(\varphi)^{-\sigma-1}\alpha_s w_j L_j}{\varphi P_{js}^{1-\sigma}} \\ &= \frac{p_{ijs}(\varphi)^{-\sigma}\alpha_s w_j L_j}{P_{js}^{1-\sigma}}\left[1-\sigma+\frac{w_i \tau_{ijs}\sigma}{\varphi p_{ijs}(\varphi)}\right] = 0\end{aligned}$$

which yields

$$\sigma - 1 = \frac{w_i\, \tau_{ijs}\, \sigma}{\varphi\, p_{ijs}(\varphi)}.$$

- Taking into account that $\frac{\sigma}{\sigma-1} = \frac{1}{1-\rho} / \frac{1-(1-\rho)}{1-\rho} = \frac{1}{\rho}$, we obtain the profit-maximizing price

$$p_{ijs}(\varphi) = \frac{w_i\, \tau_{ijs}}{\rho\, \varphi}.$$

which is a fixed markup over its marginal cost $w_i\, \tau_{ijs}/\varphi$.

- Substituting $\rho\, p_{ijs}(\varphi) = w_i\, \tau_{ijs}/\varphi$ back into gross profits, we obtain

$$\begin{aligned}
\pi_{ijs}(\varphi) &= r_{ijs}(\varphi) - \frac{w_i\, \tau_{ijs}}{\varphi} q_{ijs}(\varphi) \\
&= r_{ijs}(\varphi) - \rho\, p_{ijs}(\varphi) q_{ijs}(\varphi) \\
&= r_{ijs}(\varphi)\, [1 - \rho] \\
&= \frac{r_{ijs}(\varphi)}{\sigma}
\end{aligned}$$

since $\sigma = \frac{1}{1-\rho}$ implies that $1 - \rho = \frac{1}{\sigma}$.

- Because of the fixed marketing costs, there exist productivity cut-off levels φ^*_{ijs} such that only firms with $\varphi \geq \varphi^*_{ijs}$ sell products from country i to country j in sector s.
- We solve the model for an equilibrium where all countries produce goods in all sectors, and the more productive firms export ($\varphi^*_{iis} < \varphi^*_{ijs}$). Firms with $\varphi \geq \varphi^*_{ijs}$ export to country j and sell domestically, firms with $\varphi \in [\varphi^*_{iis}, \varphi^*_{ijs})$ only sell domestically and firms with $\varphi < \varphi^*_{iis}$ exit.
- A firm with cut-off productivity φ^*_{ijs} just breaks even from selling to country j:

$$\frac{r_{ijs}(\varphi^*_{ijs})}{\sigma} = \frac{\alpha_s w_j L_j}{\sigma} \left(\frac{p_{ijs}(\varphi^*_{ijs})}{P_{js}} \right)^{1-\sigma} = w_i f_{ij}.$$

A firm from country i needs to have a productivity $\varphi \geq \varphi^*_{ijs}$ to justify paying the fixed "marketing" cost $w_i f_{ij}$ of serving the country j market in sector s.

Firm Exit

- In each period, there is an exogenous probability δ with which actively operating firms in country i and sector s die and exit.
- In a stationary steady state equilibrium, the mass of actively operating firms M_{is} and the mass of entrants M_{ise} in country i and sector s satisfy

$$\left[1 - G\left(\varphi_{iis}^{*}\right)\right] M_{ise} = \delta M_{is},$$

that is, firm entry in each time period is matched by firm exit.

The Price Index

- Next we solve for the value of the price index P_{js} for country j and sector s.
- Given the Pareto distribution function $G(\varphi) \equiv 1 - (b/\varphi)^{\theta}$, let $g(\varphi) \equiv G'(\varphi) = b^{\theta}\theta\varphi^{-\theta-1}$ denote the corresponding productivity density function. Let $\mu_{is}(\varphi)$ denote the equilibrium productivity density function for country i and sector s.
- Since only firms with productivity $\varphi \geq \varphi^*_{iis}$ produce in equilibrium, firm exit is uncorrelated with productivity and $\varphi^*_{iis} < \varphi^*_{ijs}$, the equilibrium productivity density function is given by

$$\mu_{is}(\varphi) \equiv \begin{cases} \frac{g(\varphi)}{1-G(\varphi^*_{iis})} & \text{if } \varphi \geq \varphi^*_{iis} \\ 0 & \text{otherwise.} \end{cases}$$

In deriving this equation, we have used Bayes' rule for calculating conditional probabilities, which states that $P(A|B) = P(A \cap B)/P(B)$.

- Using $P_{is} \equiv \left[\int_{\omega \in \Omega_{is}} p_{is}(\omega)^{1-\sigma} d\omega\right]^{1/(1-\sigma)}$, the price index P_{js} for country j and sector s satisfies

$$P_{js}^{1-\sigma} = \sum_{i=1}^{n} \int_{\varphi_{ijs}^{*}}^{\infty} p_{ijs}(\varphi)^{1-\sigma} M_{is} \mu_{is}(\varphi)\, d\varphi.$$

- It follows that the price index P_{js} satisfies

$$P_{js} = \left[\sum_{i=1}^{n} \int_{\varphi_{ijs}^{*}}^{\infty} p_{ijs}(\varphi)^{1-\sigma} M_{is} \mu_{is}(\varphi)\, d\varphi\right]^{1/(1-\sigma)}.$$

Comparing Cut-off Productivity Levels

- Comparing the cut-off productivity levels of domestic firms and foreign firms in country j, we find that

$$
\begin{aligned}
\frac{w_i f_{ij}}{w_j f_{jj}} &= \frac{r_{ijs}(\varphi^*_{ijs})/\sigma}{r_{jjs}(\varphi^*_{jjs})/\sigma} = \frac{\left(p_{ijs}(\varphi^*_{ijs})/P_{js}\right)^{1-\sigma} \alpha_s w_j L_j}{\left(p_{jjs}(\varphi^*_{jjs})/P_{js}\right)^{1-\sigma} \alpha_s w_j L_j} \\
&= \frac{\left(w_i \tau_{ijs}/\rho\varphi^*_{ijs}\right)^{1-\sigma}}{\left(w_j \tau_{jjs}/\rho\varphi^*_{jjs}\right)^{1-\sigma}} = \left(\frac{w_i \tau_{ijs}\varphi^*_{jjs}}{w_j \varphi^*_{ijs}}\right)^{1-\sigma} .
\end{aligned}
$$

Solving for φ^*_{ijs} yields

$$
\varphi^*_{ijs} = \tau_{ijs} \left(\frac{f_{ij}}{f_{jj}}\right)^{1/(\sigma-1)} \left(\frac{w_i}{w_j}\right)^{1/\rho} \varphi^*_{jjs}.
$$

This equation shows that the cut-off productivity levels of domestic and foreign firms in country j would be the same if it were not for differences in trade costs and labor costs.

Expected Profits

- In each time period, there is free entry by firms in each sector s and country i.
- Let $\bar{\pi}_{is}$ denote the average profits across all domestic firms in country i and sector s (including the fixed marketing costs).
- Let

$$\bar{v}_{is} \equiv \bar{\pi}_{is} + (1-\delta)\bar{\pi}_{is} + (1-\delta)^2\bar{\pi}_{is} + \cdots = \sum_{t=0}^{\infty}(1-\delta)^t\bar{\pi}_{is} = \bar{\pi}_{is}/\delta$$

denote the present value of average profit flows in country i and sector s, taking into account the rate δ at which firms exit in each time period.

- The average profits across all domestic firms (exporters and non-exporters) is given by

$$
\begin{aligned}
\bar{\pi}_{is} &= \frac{1}{M_{is}} \left\{ \sum_{j=1}^{n} \int_{\varphi^*_{ijs}}^{\infty} [\pi_{ijs}(\varphi) - w_i f_{ij}] M_{is} \mu_{is}(\varphi) \, d\varphi \right\} \\
&= \sum_{j=1}^{n} \int_{\varphi^*_{ijs}}^{\infty} \left[\frac{r_{ijs}(\varphi)}{\sigma} - w_i f_{ij} \right] \frac{g(\varphi)}{1 - G(\varphi^*_{iis})} \, d\varphi
\end{aligned}
$$

and rearranging yields

$$
[1 - G(\varphi^*_{iis})] \, \bar{\pi}_{is} = \sum_{j=1}^{n} \int_{\varphi^*_{ijs}}^{\infty} \left[\frac{r_{ijs}(\varphi)}{\sigma} - w_i f_{ij} \right] g(\varphi) \, d\varphi.
$$

- To evaluate the integrals, next note that

$$\frac{r_{ijs}(\varphi)}{r_{ijs}(\varphi^*_{ijs})} = \frac{(\alpha_s w_j L_j)\, p_{ijs}(\varphi)^{1-\sigma} / P_{js}^{1-\sigma}}{(\alpha_s w_j L_j)\, p_{ijs}(\varphi^*_{ijs})^{1-\sigma} / P_{js}^{1-\sigma}} = \left(\frac{p_{ijs}(\varphi)}{p_{ijs}(\varphi^*_{ijs})}\right)^{1-\sigma}$$

$$= \left(\frac{w_i \tau_{ijs}}{\rho\varphi}\frac{\rho\varphi^*_{ijs}}{w_i \tau_{ijs}}\right)^{1-\sigma} = \left(\frac{\varphi}{\varphi^*_{ijs}}\right)^{\sigma-1}.$$

- Using the cut-off productivity condition, it follows that

$$\frac{r_{ijs}(\varphi)}{\sigma} = \frac{r_{ijs}(\varphi^*_{ijs})}{\sigma}\left(\frac{\varphi}{\varphi^*_{ijs}}\right)^{\sigma-1} = \frac{\sigma w_i f_{ij}}{\sigma}\left(\frac{\varphi}{\varphi^*_{ijs}}\right)^{\sigma-1} = w_i f_{ij}\left(\frac{\varphi}{\varphi^*_{ijs}}\right)^{\sigma-1}.$$

- We can now evaluate the integrals:

$$\int_{\varphi^*_{ijs}}^{\infty} \left[\frac{r_{ijs}(\varphi)}{\sigma} - w_i f_{ij} \right] g(\varphi)\, d\varphi$$

$$= \int_{\varphi^*_{ijs}}^{\infty} \left[w_i f_{ij} \left(\frac{\varphi}{\varphi^*_{ijs}} \right)^{\sigma-1} - w_i f_{ij} \right] g(\varphi)\, d\varphi$$

$$= w_i f_{ij} \int_{\varphi^*_{ijs}}^{\infty} \left[\left(\frac{\varphi}{\varphi^*_{ijs}} \right)^{\sigma-1} - 1 \right] g(\varphi)\, d\varphi = w_i f_{ij} J(\varphi^*_{ijs})$$

where the function $J(\cdot)$ is given by

$$J(x) \equiv \int_{x}^{\infty} \left[\left(\frac{\varphi}{x} \right)^{\sigma-1} - 1 \right] g(\varphi)\, d\varphi.$$

- $$\begin{aligned} J(x) &\equiv \int_x^\infty \left[\left(\frac{\varphi}{x}\right)^{\sigma-1} - 1\right] g(\varphi)\, d\varphi \\ &= \int_x^\infty \left(\frac{\varphi}{x}\right)^{\sigma-1} b^\theta \theta \varphi^{-\theta-1}\, d\varphi - \int_x^\infty g(\varphi)\, d\varphi \\ &= b^\theta \theta x^{1-\sigma} \int_x^\infty \varphi^{\sigma-1-\theta-1}\, d\varphi - [1 - G(x)] \\ &= b^\theta \theta x^{1-\sigma} \frac{x^{\sigma-1-\theta}}{\theta-\sigma+1} - \left(\frac{b}{x}\right)^\theta \\ &= \frac{\theta - (\theta-\sigma+1)}{\theta-\sigma+1} \left(\frac{b}{x}\right)^\theta = \frac{\sigma-1}{\theta-\sigma+1} \left(\frac{b}{x}\right)^\theta . \end{aligned}$$

- We assume that $\theta > \sigma - 1$ to guarantee that expected profits are finite. From the previous argument, it also follows that

$$\int_x^\infty \left(\frac{\varphi}{x}\right)^{\sigma-1} g(\varphi)\, d\varphi = \eta \left(\frac{b}{x}\right)^\theta \quad \text{where} \quad \eta \equiv \frac{\theta}{\theta-\sigma+1}.$$

The Market Access Index

- The expected profit of an entrant in country i from selling to country j in sector s (after the entrant has paid the entry cost $w_i f_{ise}$) is

$$\frac{[1-G(\varphi_{iis}^{*})]}{\delta}\int_{\varphi_{ijs}^{*}}^{\infty}\left[\frac{r_{ijs}(\varphi)}{\sigma_s}-w_i f_{ij}\right]\frac{g(\varphi)}{1-G(\varphi_{iis}^{*})}\,d\varphi$$

$$=\delta^{-1}\int_{\varphi_{ijs}^{*}}^{\infty}\left[\frac{r_{ijs}(\varphi)}{\sigma}-w_i f_{ij}\right]g(\varphi)\,d\varphi=\delta^{-1}w_i f_{ij}\frac{\sigma-1}{\theta-\sigma+1}\left(\frac{b}{\varphi_{ijs}^{*}}\right)^{\theta}.$$

- The expected profit of an entrant in country j from selling to country j in sector s (after the entrant has paid the entry cost $w_j f_{jse}$) is

$$\delta^{-1}w_j f_{jj}\frac{\sigma-1}{\theta-\sigma+1}\left(\frac{b}{\varphi_{jjs}^{*}}\right)^{\theta}.$$

- Thus the expected profit of an entrant in country i from selling to country j in sector s relative to that captured by an entrant in country j from selling to country j (or the relative expected profit) is given by

$$\phi_{ijs} \equiv \frac{\delta^{-1} w_i f_{ij} \frac{\sigma-1}{\theta-\sigma+1}\left(\frac{b}{\varphi^*_{ijs}}\right)^{\theta}}{\delta^{-1} w_j f_{jj} \frac{\sigma-1}{\theta-\sigma+1}\left(\frac{b}{\varphi^*_{jjs}}\right)^{\theta}}$$

$$= \frac{w_i f_{ij}}{w_j f_{jj}}\left(\frac{\varphi^*_{jjs}}{\varphi^*_{ijs}}\right)^{\theta} = \frac{w_i f_{ij}}{w_j f_{jj}}\left[\frac{1}{\tau_{ijs}}\left(\frac{f_{jj}}{f_{ij}}\right)^{1/(\sigma-1)}\left(\frac{w_j}{w_i}\right)^{1/\rho}\right]^{\theta}$$

or

$$\phi_{ijs} = \frac{1}{\tau^{\theta}_{ijs}}\left(\frac{f_{jj}}{f_{ij}}\right)^{(\theta-\sigma+1)/(\sigma-1)}\left(\frac{w_j}{w_i}\right)^{(\theta-\rho)/\rho}.$$

- $$\phi_{ijs} = \frac{1}{\tau_{ijs}^{\theta}} \left(\frac{f_{jj}}{f_{ij}}\right)^{(\theta-\sigma+1)/(\sigma-1)} \left(\frac{w_j}{w_i}\right)^{(\theta-\rho)/\rho}$$

  Variable ϕ_{ijs} is an index summarizing the degree of country i's market access to country j in sector s.
- Note that $\sigma = \frac{1}{1-\rho}$ implies that $\sigma - 1 = \frac{1}{1-\rho} - \frac{1-\rho}{1-\rho} = \frac{\rho}{1-\rho}$ and thus the assumption $\theta > \sigma - 1$ implies that $\theta > \frac{\rho}{1-\rho}$. Rearranging yields $\theta - \rho\theta > \rho$ or $\theta - \rho > \theta\rho > 0$.
- Since $\theta > \sigma - 1$ and $(\theta - \rho)/\rho > \theta$, the market access index ϕ_{ijs} decreases in variable trade costs τ_{ijs}, relative marketing costs f_{ij}/f_{jj}, and the relative wage w_i/w_j. As export barriers τ_{ijs} or f_{ij} increase to infinity, the market access index ϕ_{ijs} converges to zero.

The Price Integral

- From firm's pricing and the cutoff condition, we obtain

$$\begin{aligned}
\frac{p_{ijs}(\varphi^*_{ijs})}{p_{jjs}(\varphi^*_{jjs})} &= \frac{w_i \tau_{ijs}}{w_j} \left(\frac{\varphi^*_{jjs}}{\varphi^*_{ijs}} \right) \\
&= \frac{w_i \tau_{ijs}}{w_j} \left(\frac{1}{\tau_{ijs}} \left(\frac{f_{ij}}{f_{jj}} \right)^{1/(1-\sigma)} \left(\frac{w_i}{w_j} \right)^{-1/\rho} \right) \\
&= \left(\frac{w_i f_{ij}}{w_j f_{jj}} \right)^{1/(1-\sigma)}
\end{aligned}$$

since $1 - \sigma = \frac{-\rho}{1-\rho}$ implies that $\frac{1}{1-\sigma} = \frac{\rho - 1}{\rho}$.

- Using this result and $p_{ijs} = w_i \, \tau_{ijs} / \rho \, \varphi$, we can evaluate the price integral

$$\int_{\varphi^*_{ijs}}^{\infty} p_{ijs}(\varphi)^{1-\sigma} g(\varphi) \, d\varphi.$$

- $$\int_{\varphi^*_{ijs}}^{\infty} p_{ijs}(\varphi)^{1-\sigma} g(\varphi)\, d\varphi = \int_{\varphi^*_{ijs}}^{\infty} p_{ijs}(\varphi^*_{ijs})^{1-\sigma} \left(\frac{\varphi}{\varphi^*_{ijs}}\right)^{\sigma-1} g(\varphi)\, d\varphi$$

$$= p_{ijs}(\varphi^*_{ijs})^{1-\sigma} \int_{\varphi^*_{ijs}}^{\infty} \left(\frac{\varphi}{\varphi^*_{ijs}}\right)^{\sigma-1} g(\varphi)\, d\varphi$$

$$= p_{jjs}\left(\varphi^*_{jjs}\right)^{1-\sigma} \left(\frac{w_i f_{ij}}{w_j f_{jj}}\right) \eta \left(\frac{b}{\varphi^*_{ijs}}\right)^{\theta}$$

$$= \eta\, p_{jjs}\left(\varphi^*_{jjs}\right)^{1-\sigma} \left(\frac{w_i f_{ij}}{w_j f_{jj}}\right) \left(\frac{b}{\tau_{ijs}\left(f_{ij}/f_{jj}\right)^{1/(\sigma-1)}\left(w_i/w_j\right)^{1/\rho} \varphi^*_{jjs}}\right)^{\theta}$$

$$= \eta\, p_{jjs}\left(\varphi^*_{jjs}\right)^{1-\sigma} \left(\tau_{ijs}^{-\theta}\left(\frac{f_{ij}}{f_{jj}}\right)^{-(\theta-\sigma+1)/(\sigma-1)} \left(\frac{w_j}{w_i}\right)^{(\theta-\rho)/\rho}\right) \left(\frac{b}{\varphi^*_{jjs}}\right)^{\theta}$$

$$= \eta\, p_{jjs}\left(\varphi^*_{jjs}\right)^{1-\sigma} \left(\frac{b}{\varphi^*_{jjs}}\right)^{\theta} \phi_{ijs}$$

- Substituting this back into the price index, we obtain

$$\begin{aligned}
P_{js}^{1-\sigma} &= \sum_{i=1}^{n} \int_{\varphi_{ijs}^{*}}^{\infty} p_{ijs}(\varphi)^{1-\sigma} M_{is} \mu_{is}(\varphi) d\varphi \\
&= \sum_{i=1}^{n} \frac{M_{ise}}{\delta} \int_{\varphi_{ijs}^{*}}^{\infty} p_{ijs}(\varphi)^{1-\sigma} g(\varphi)\, d\varphi \\
&= \frac{\eta}{\delta} p_{jjs} \left(\varphi_{jjs}^{*}\right)^{1-\sigma} \left(\frac{b}{\varphi_{jjs}^{*}}\right)^{\theta} \sum_{i=1}^{n} M_{ise} \phi_{ijs}.
\end{aligned}$$

Changing indexes yields

$$P_{is}^{1-\sigma} = \frac{\eta}{\delta} p_{iis} \left(\varphi_{iis}^{*}\right)^{1-\sigma} \left(\frac{b}{\varphi_{iis}^{*}}\right)^{\theta} \sum_{j=1}^{n} M_{jse} \phi_{jis}.$$

- To understand

$$P_{is}^{1-\sigma} = \frac{\eta}{\delta} p_{iis} \left(\varphi_{iis}^{*}\right)^{1-\sigma} \left(\frac{b}{\varphi_{iis}^{*}}\right)^{\theta} \sum_{j=1}^{n} M_{jse} \phi_{jis},$$

first consider the special case of autarky ($\phi_{jis} = 0$). Then, noting that $\phi_{iis} = 1$, this equation simplifies to

$$\begin{aligned} P_{is}^{1-\sigma} &= \frac{\eta}{\delta} p_{iis} \left(\varphi_{iis}^{*}\right)^{1-\sigma} \left(\frac{b}{\varphi_{iis}^{*}}\right)^{\theta} M_{ise} \\ &= \eta\, p_{iis} \left(\varphi_{iis}^{*}\right)^{1-\sigma} \left(\frac{b}{\varphi_{iis}^{*}}\right)^{\theta} \frac{M_{is}}{1 - G(\varphi_{iis}^{*})} = \eta\, p_{iis} \left(\varphi_{iis}^{*}\right)^{1-\sigma} M_{is}. \end{aligned}$$

The price index depends on the mass of domestic varieties and the distribution of prices. Under the Pareto distribution, the latter is summarized by the highest price set by the least productive firm.

- In the open economy with $\phi_{jis} > 0$, the price index also depends on the mass of entrants in each foreign country j (M_{jse}) and the degree of their market access (ϕ_{jis}).

The Domestic Productivity Cutoff

- Substituting the price index into the cutoff condition for country i, we obtain
$$\frac{r_{iis}(\varphi^*_{iis})}{\sigma} = w_i f_d = \frac{\alpha_s w_i L_i}{\sigma} \left(\frac{p_{iis}(\varphi^*_{iis})}{P_{is}} \right)^{1-\sigma}$$
$$\frac{\alpha_s L_i}{\sigma} \left[\frac{\eta}{\delta} \left(\frac{b}{\varphi^*_{iis}} \right)^{\theta} \sum_{j=1}^{n} M_{jse} \phi_{jis} \right]^{-1} = f_d$$
and solving for $\varphi^{*\theta}_{iis}$ yields
$$\varphi^{*\theta}_{iis} = \frac{\theta b^{\theta}}{\delta (\theta - \sigma + 1)} \frac{\sigma f_d}{\alpha_s L_i} \sum_{j=1}^{n} M_{jse} \phi_{jis}.$$

- $$\varphi_{iis}^{*\theta} = \frac{\theta b^{\theta}}{\delta(\theta - \sigma + 1)} \frac{\sigma f_d}{\alpha_s L_i} \sum_{j=1}^{n} M_{jse} \phi_{jis}.$$
- The domestic productivity cutoff φ_{11s}^{*} rises if and only if $\sum_{j=1}^{n} M_{jse}\phi_{jis}$ rises.
- If trade liberalization results in $\sum_{j=1}^{n} M_{jse}\phi_{jis}$ increasing, more firms are entering and **competition** is becoming tougher in country i and sector s.
- With tougher **competition**, firms need to have a higher productivity level to survive, so the domestic productivity cutoff φ_{iis}^{*} increases, and it follows that industrial productivity rises.

- $$\varphi_{iis}^{*\theta} = \frac{\theta b^{\theta}}{\delta(\theta - \sigma + 1)} \frac{\sigma f_d}{\alpha_s L_i} \sum_{j=1}^{n} M_{jse} \phi_{jis}.$$
- If trade liberalization results in $\sum_{j=1}^{n} M_{jse} \phi_{jis}$ decreasing, then fewer firms enter, **competition** becomes less tough, lower productivity firms can now survive and industrial productivity falls.
- For understanding the properties of the Melitz model, this is the most important equation.
- It implies that, for determining how trade liberalization impacts the domestic productivity cut-off and industrial productivity, it is sufficient to consider how the mass of entrants M_{jse} in each country j and each country's market access index ϕ_{jis} change.

The Free Entry Condition

- Free entry implies that the probability of successful entry times the expected profits earned from successful entry must equal the cost of entry, that is,

$$[1 - G(\varphi^*_{iis})]\bar{\pi}_{is}/\delta = w_i f_{ise}.$$

It follows that

$$[1 - G(\varphi^*_{iis})]\bar{\pi}_{is} = \sum_{j=1}^{n} \int_{\varphi^*_{ijs}}^{\infty} \left[\frac{r_{ijs}(\varphi)}{\sigma} - w_i f_{ij} \right] g(\varphi)\, d\varphi = \delta w_i f_{ise}.$$

- Making substitutions and rearranging terms, it follows that

$$\sum_{j=1}^{n} \int_{\varphi_{ijs}^*}^{\infty} \left[\frac{r_{ijs}(\varphi)}{\sigma} - w_i f_{ij} \right] g(\varphi)\, d\varphi = \delta w_i f_{ise}$$

$$\sum_{j=1}^{n} w_i f_{ij} J(\varphi_{ijs}^*) = \delta w_i f_{ise}$$

$$\sum_{j=1}^{n} f_{ij} J(\varphi_{ijs}^*) = \delta f_{ise}$$

$$\sum_{j=1}^{n} f_{ij} \frac{\sigma - 1}{\theta - \sigma + 1} \left(\frac{b}{\varphi_{ijs}^*} \right)^{\theta} = \delta f_{ise}$$

and the free entry condition is

$$\frac{1}{\delta} \left(\frac{\sigma - 1}{\theta - \sigma + 1} \right) \sum_{j=1}^{n} f_{ij} \left(\frac{b}{\varphi_{ijs}^*} \right)^{\theta} = f_{ise}.$$

The Mass of Entrants

- We use a three step argument to solve for the mass of entrants M_{ise}.
- First, we show that the fixed costs (the entry costs plus the marketing costs) are proportional to the mass of entrants in each country i and sector s.

$$w_i \left(M_{ise} f_{ise} + \sum_{j=1}^{n} \int_{\varphi_{ijs}^*}^{\infty} f_{ij} M_{is} \mu_{is}(\varphi)\, d\varphi \right)$$

$$= w_i \left(M_{ise} f_{ise} + \sum_{j=1}^{n} \int_{\varphi_{ijs}^*}^{\infty} f_{ij} \frac{M_{ise}}{\delta} g(\varphi)\, d\varphi \right)$$

$$= w_i \left(M_{ise} f_{ise} + \frac{M_{ise}}{\delta} \sum_{j=1}^{n} f_{ij} [1 - G(\varphi_{ijs}^*)] \right)$$

$$= w_i \left(M_{ise} f_{ise} + \frac{M_{ise}}{\delta} \sum_{j=1}^{n} f_{ij} \left(\frac{b}{\varphi_{ijs}^*} \right)^{\theta} \right)$$

- $$\begin{aligned} &= w_i \left(M_{ise} f_{ise} + \frac{M_{ise}}{\delta} \sum_{j=1}^{n} f_{ij} \left(\frac{b}{\varphi_{ijs}^*} \right)^{\theta} \right) \\ &= w_i \left(M_{ise} f_{ise} + \frac{M_{ise}}{\delta} \delta f_{ise} \frac{\theta - \sigma + 1}{\sigma - 1} \right) \\ &= w_i M_{ise} f_{ise} \left(\frac{\sigma - 1 + \theta - \sigma + 1}{\sigma - 1} \right) \end{aligned}$$

  from which it follows that

  $$w_i \left(M_{ise} f_{ise} + \sum_{j=1}^{n} \int_{\varphi_{ijs}^*}^{\infty} f_{ij} M_{is} \mu_{is}(\varphi)\, d\varphi \right) = w_i M_{ise} \left(\frac{\theta f_{ise}}{\sigma - 1} \right).$$

- Second, we show that the fixed costs are equal to the gross profits in each country i and sector s.
- From the free entry condition, we obtain

$$\delta w_i f_{ise} = \sum_{j=1}^{n} \int_{\varphi^*_{ijs}}^{\infty} \left[\frac{r_{ijs}(\varphi)}{\sigma} - w_i f_{ij} \right] g(\varphi)\, d\varphi$$

$$w_i \left(\delta f_{ise} + \sum_{j=1}^{n} f_{ij} [1 - G(\varphi^*_{ijs})] \right) = \sum_{j=1}^{n} \int_{\varphi^*_{ijs}}^{\infty} \frac{r_{ijs}(\varphi)}{\sigma} g(\varphi)\, d\varphi$$

$$w_i \left(M_{ise} f_{ise} + \frac{M_{ise}}{\delta} \sum_{j=1}^{n} f_{ij} [1 - G(\varphi^*_{ijs})] \right)$$

$$= \frac{M_{ise}}{\delta} \sum_{j=1}^{n} \int_{\varphi^*_{ijs}}^{\infty} \frac{r_{ijs}(\varphi)}{\sigma} g(\varphi)\, d\varphi$$

$$w_i M_{ise} \left(\frac{\theta f_{ise}}{\sigma - 1} \right) = \frac{M_{is}}{1 - G(\varphi^*_{iis})} \sum_{j=1}^{n} \int_{\varphi^*_{ijs}}^{\infty} \frac{r_{ijs}(\varphi)}{\sigma} g(\varphi)\, d\varphi$$

- $$w_i M_{ise}\left(\frac{\theta f_{ise}}{\sigma-1}\right) = \frac{M_{is}}{1-G(\varphi^*_{iis})}\sum_{j=1}^{n}\int_{\varphi^*_{ijs}}^{\infty}\frac{r_{ijs}(\varphi)}{\sigma}g(\varphi)\,d\varphi$$
$$= \frac{1}{\sigma}\sum_{j=1}^{n}\int_{\varphi^*_{ijs}}^{\infty} r_{ijs}(\varphi)M_{is}\mu_{is}(\varphi)d\varphi$$
$$= \frac{1}{\sigma}\sum_{j=1}^{n} R_{ijs}$$
where $R_{ijs} \equiv \int_{\varphi^*_{ijs}}^{\infty} r_{ijs}(\varphi)M_{is}\mu_{is}(\varphi)d\varphi$ is the total revenue associated with shipments from country i to country j in sector s.
- Third, we show that the wage payments to labor equals the total revenue in each country i and sector s.

- Let L_{is} denote labor demand by all firms in country i and sector s.
- Firms use labor for market entry, for the production of goods sold to domestic consumers and for the production of goods sold to foreign consumers. Taking into account both the marginal and fixed costs of production, we obtain

$$
\begin{aligned}
w_i L_{is} &= w_i M_{ise} f_{ise} + w_i \sum_{j=1}^{n} \int_{\varphi_{ijs}^*}^{\infty} \left[f_{ij} + q_{ijs}(\varphi) \frac{\tau_{ijs}}{\varphi} \right] M_{is} \mu_{is}(\varphi)\, d\varphi \\
&= w_i \left(M_{ise} f_{ise} + \sum_{j=1}^{n} \int_{\varphi_{ijs}^*}^{\infty} f_{ij} M_{is} \mu_{is}(\varphi)\, d\varphi \right) \\
&\quad + \sum_{j=1}^{n} \int_{\varphi_{ijs}^*}^{\infty} q_{ijs}(\varphi) \frac{w_i \tau_{ijs}}{\rho \varphi} \rho M_{is} \mu_{is}(\varphi)\, d\varphi \\
&= w_i M_{ise} \left(\frac{\theta f_{ise}}{\sigma - 1} \right) + \rho \sum_{j=1}^{n} \int_{\varphi_{ijs}^*}^{\infty} r_{ijs}(\varphi) M_{is} \mu_{is}(\varphi)\, d\varphi
\end{aligned}
$$

- $$\begin{aligned} w_i L_{is} &= w_i M_{ise} \left(\frac{\theta f_{ise}}{\sigma - 1} \right) + \rho \sum_{j=1}^{n} \int_{\varphi^*_{ijs}}^{\infty} r_{ijs}(\varphi) M_{is} \mu_{is}(\varphi) \, d\varphi \\ &= \frac{1}{\sigma} \sum_{j=1}^{n} R_{ijs} + \rho \sum_{j=1}^{n} R_{ijs} \\ &= (1 - \rho + \rho) \sum_{j=1}^{n} R_{ijs} = \sum_{j=1}^{n} R_{ijs}. \end{aligned}$$

- The total revenue R_{ijs} can be rewritten as

$$\begin{aligned} R_{ijs} &\equiv \int_{\varphi^*_{ijs}}^{\infty} r_{ijs}(\varphi) M_{is} \mu_{is}(\varphi) \, d\varphi \\ &= \frac{M_{is}}{1 - G(\varphi^*_{iis})} \int_{\varphi^*_{ijs}}^{\infty} r_{ijs}(\varphi) g(\varphi) \, d\varphi \\ &= \frac{[1 - G(\varphi^*_{iis})] M_{ise}}{\delta [1 - G(\varphi^*_{iis})]} \int_{\varphi^*_{ijs}}^{\infty} p_{ijs}(\varphi) q_{ijs}(\varphi) g(\varphi) \, d\varphi \end{aligned}$$

-

$$
\begin{aligned}
R_{ijs} &= \frac{[1 - G(\varphi^*_{iis})]M_{ise}}{\delta[1 - G(\varphi^*_{iis})]} \int_{\varphi^*_{ijs}}^{\infty} p_{ijs}(\varphi) q_{ijs}(\varphi) g(\varphi)\, d\varphi \\
&= \frac{M_{ise}}{\delta} \int_{\varphi^*_{ijs}}^{\infty} \frac{p_{ijs}(\varphi)^{1-\sigma} \alpha_s w_j L_j}{P_{js}^{1-\sigma}} g(\varphi)\, d\varphi \\
&= \frac{\alpha_s w_j L_j}{P_{js}^{1-\sigma}} \frac{M_{ise}}{\delta} \int_{\varphi^*_{ijs}}^{\infty} p_{ijs}(\varphi)^{1-\sigma} g(\varphi)\, d\varphi \\
&= \alpha_s w_j L_j \frac{M_{ise}}{\delta} \frac{\eta\, p_{jjs} \left(\varphi^*_{jjs}\right)^{1-\sigma} \left(\frac{b}{\varphi^*_{jjs}}\right)^{\theta} \phi_{ijs}}{\frac{\eta}{\delta}\, p_{jjs} \left(\varphi^*_{jjs}\right)^{1-\sigma} \left(\frac{b}{\varphi^*_{jjs}}\right)^{\theta} \sum_{k=1}^{n} M_{kse} \phi_{kjs}} \\
&= \alpha_s w_j L_j \left(\frac{M_{ise} \phi_{ijs}}{\sum_{k=1}^{n} M_{kse} \phi_{kjs}} \right).
\end{aligned}
$$

- Substituting

$$R_{ijs} = \alpha_s w_j L_j \left(\frac{M_{ise}\phi_{ijs}}{\sum_{k=1}^{n} M_{kse}\phi_{kjs}} \right)$$

into $\frac{1}{\sigma}\sum_{j=1}^{n} R_{ijs} = w_i M_{ise}\left(\frac{\theta f_{ise}}{\sigma-1}\right)$, we obtain

$$\sum_{j=1}^{n} \alpha_s w_j L_j \left(\frac{\phi_{ijs}}{\sum_{k=1}^{n} M_{kse}\phi_{kjs}} \right) = w_i f_{ise}\left(\frac{\theta}{\rho}\right).$$

Since f_{ise} is a function of M_{ise} and ϕ_{ijs} is a function of the wage rates w_i, it is possible to express the mass of entrants M_{ise} as a function of the wage rates w_i.

The Labor Market Clearing Condition

- The labor market clearing conditions for all the countries together determine the wage rates w_i.
- Free entry implies that wage payments to labor equal total revenue in each country i and sector s, $w_i L_{is} = \sum_{j=1}^{n} R_{ijs}$, or

$$L_{is} = \frac{1}{w_i} \sum_{j=1}^{n} R_{ijs} = \frac{1}{w_i} w_i M_{ise} \left(\frac{\sigma\theta}{\sigma - 1} \right) f_{ise} = M_{ise}^{1+\zeta} \left(\frac{\theta F}{\rho} \right).$$

  Notice that labor demand L_{is} depends only on the mass of entrants M_{ise} and not on any cut-off productivity levels φ_{ijs}^{*}.
- The country i labor supply is given by L_i so the requirement that labor supply equal labor demand

$$L_i = \sum_{s=1}^{m} L_{is} = \frac{\theta F}{\rho} \sum_{s=1}^{m} M_{ise}^{1+\zeta}$$

  determines the equilibrium wage rates w_i.

How the Melitz Model is Solved

- We have now fully presented the Melitz model. And it is quite complicated!
- It is not at all obvious how one solves the Melitz model for a steady-state equilibrium.
- In general, with different countries choosing different trade policies ($\tau_{ijs} \neq \tau_{jis}$) or having different sizes ($L_i \neq L_j$), it is not possible to solve this model analytically and prove any theorems about the effects of trade policy changes.
- But the model can be solved numerically using a computer and computers are very good at solving this type of model. It takes a long time to write the computer code (using *Matlab*) but once done, a computer can solve the Melitz model (with all parameter values specified) for a steady-state equilibrium in less than 1 second.

- We use the following two step procedure to solve the model numerically when there are two industries ($m = 2$) and two countries ($n = 2$).
- First, we solve a system of 5 equations

$$\sum_{j=1}^{n} \alpha_s w_j L_j \left(\frac{\phi_{ijs}}{\sum_{k=1}^{n} M_{kse} \phi_{kjs}} \right) = w_i f_{ise} \left(\frac{\theta}{\rho} \right) \text{ for } is = 1A, 1B, 2A, 2B$$

$$L_i = \sum_{s=1}^{m} L_{is} = \frac{\theta F}{\rho} \sum_{s=1}^{m} M_{ise}^{1+\zeta} \text{ for } i = 1$$

that determine 5 unknowns [M_{ise} for $is = 1A, 1B, 2A, 2B$ and w_1].

- Then, using the equilibrium value of the wage rate w_1, we solve a second system of 8 equations

$$\varphi^*_{ijs} = \tau_{ijs} \left(\frac{f_{ij}}{f_{jj}} \right)^{1/(\sigma-1)} \left(\frac{w_i}{w_j} \right)^{1/\rho} \varphi^*_{jjs} \text{ for } ijs = 12A, 21A, 12B, 21B$$

$$\frac{1}{\delta} \left(\frac{\sigma - 1}{\theta - \sigma + 1} \right) \sum_{j=1}^{n} f_{ij} \left(\frac{b}{\varphi^*_{ijs}} \right)^{\theta} = f_{ise} \text{ for } is = 1A, 1B, 2A, 2B$$

  that determine 8 unknowns [φ^*_{ijs} for $ijs = 12A, 21A, 12B, 21B, 11A, 11B, 22A, 22B$].
- Solving for the remaining endogenous variables is straightforward.

Industrial Productivity

- To measure industrial labor productivity, we calculate the real industrial output per unit of labor:

$$\Phi_{is}^{L} \equiv \frac{\sum_{j=1}^{n} R_{ijs}}{\tilde{P}_{is} L_{is}} \text{ where } \tilde{P}_{is} \equiv \int_{\varphi_{iis}^{*}}^{\infty} p_{iis}(\varphi)\, \mu_{is}(\varphi) d\varphi$$

  is the simple average of prices set by domestic firms at the factory gate.
- The term $\sum_{j=1}^{n} R_{ijs}$ is the total revenue of firms in country i and sector s. Dividing by the price index $\tilde{P}_{is}$ gives a measure of the real output of sector s. Then dividing by the number of workers L_{is} gives a measure of real output per worker.
- This productivity measure is widely used in empirical studies, for example, Trefler (2004, AER).

- From $w_i L_{is} = \sum_{j=1}^{n} R_{ijs}$ and

$$\begin{aligned}
\tilde{P}_{is} \equiv \int_{\varphi_{iis}^*}^{\infty} p_{iis}\left(\varphi\right)\mu_{is}(\varphi) d\varphi &= \int_{\varphi_{iis}^*}^{\infty} \left(\frac{w_i}{\rho\varphi}\right) \frac{g(\varphi)}{1 - G(\varphi_{iis}^*)} d\varphi \\
= \frac{w_i}{\rho \left(b/\varphi_{iis}^*\right)^{\theta}} \int_{\varphi_{iis}^*}^{\infty} \frac{\theta b^{\theta}}{\varphi^{\theta+2}} d\varphi &= \frac{w_i \theta \varphi_{iis}^{*\theta}}{\rho} \left[\frac{-\varphi_{iis}^{*-(\theta+2)+1}}{-(\theta+2)+1}\right] \\
&= \frac{w_i}{\rho\varphi_{iis}^*} \left(\frac{\theta}{\theta+1}\right),
\end{aligned}$$

industrial labor productivity becomes

$$\Phi_{is}^{L} \equiv \frac{\sum_{j=1}^{n} R_{ijs}}{\tilde{P}_{is} L_{is}} = \frac{w_i}{\tilde{P}_{is}} = w_i / \left[\frac{w_i}{\rho\varphi_{iis}^*} \left(\frac{\theta}{\theta+1}\right)\right]$$

or

$$\Phi_{is}^{L} = \left(\frac{\theta+1}{\theta}\right) \rho\varphi_{iis}^*.$$

- A second measure is industrial labor productivity uses the "theoretically correct" price index P_{is} that we derived earlier:

$$\Phi_{is}^{W} \equiv \frac{\sum_{j=1}^{n} R_{ijs}}{P_{is} L_{is}}.$$

- Starting from the cut-off productivity condition,

$$\begin{aligned} \frac{r_{iis}(\varphi_{iis}^{*})}{\sigma} &= w_i f_{ii} \\ \alpha_s w_i L_i \frac{p_{iis}(\varphi_{iis}^{*})^{1-\sigma}}{P_{is}^{1-\sigma}} &= \sigma w_i f_{ii} \\ \alpha_s w_i L_i \left(\frac{w_i \tau_{iis}}{\rho \varphi_{iis}^{*} P_{is}} \right)^{1-\sigma} &= \sigma w_i f_{ii} \\ \left(\frac{w_i}{P_{is}} \right)^{1-\sigma} &= \frac{\sigma f_{ii}}{\alpha_s L_i} (\rho \varphi_{iis}^{*})^{1-\sigma} \end{aligned}$$

- $$\begin{aligned}\left(\frac{w_i}{P_{is}}\right)^{1-\sigma} &= \frac{\sigma f_{ii}}{\alpha_s L_i}(\rho\varphi^*_{iis})^{1-\sigma}\\ \frac{w_i}{P_{is}} &= \left(\frac{\sigma f_{ii}}{\alpha_s L_i}\right)^{1/(1-\sigma)}\rho\varphi^*_{iis}\end{aligned}$$

  and then using $w_i L_{is} = \sum_{j=1}^{n} R_{ijs}$, we obtain

  $$\Phi^W_{is} \equiv \frac{\sum_{j=1}^{n} R_{ijs}}{P_{is} L_{is}} = \frac{w_i}{P_{is}} = \left(\frac{\alpha_s L_i}{\sigma f_{ii}}\right)^{1/(\sigma-1)}\rho\varphi^*_{iis}.$$

- Thus, both productivity measures are increasing functions of the domestic productivity cut-off φ^*_{iis}.

- The second productivity measure Φ_{is}^{W} will be used in our welfare analysis.
- Consider the representative consumer in country i who supplies one unit of labor. Since her income is w_i, her aggregate consumption over varieties in sector s is
$$C_{is} = \frac{\alpha_s w_i}{P_{is}}.$$
From the utility function $U_i \equiv \prod_{s=1}^{m} (C_{is})^{\alpha_s}$ and $\Phi_{is}^{W} = w_i / P_{is}$, her utility can be written as:
$$U_i = \prod_{s=1}^{m} \left(\frac{\alpha_s w_i}{P_{is}} \right)^{\alpha_s} = \prod_{s=1}^{m} \left(\alpha_s \Phi_{is}^{W} \right)^{\alpha_s}.$$
Thus Φ_{is}^{W} is the productivity measure for industry s that is directly relevant for calculating consumer welfare U_i.

The Case of Symmetric Countries

- There is one case where the Melitz model can be solved analytically and that is the case where all countries are symmetric in every respect.
- This is the case that Melitz (2003, *Econometrica*) studied and his paper contains one beautiful theorem (the most important result in the international trade literature).
- We will now study the case of symmetric countries and prove the Melitz theorem (this is the high-point of the course!).

- Melitz (2003, *Econometrica*) asked the question, what happens when all countries open up to trade in a symmetric way?
- Suppose that countries and sectors are initially symmetric, that is, $L_i = L$ for all i, $\alpha_s = 1/m$ for all s, $w_i = 1$ for all i, $f_{ii} = f_{jj} \equiv f_d$ for all i and j, $f_{ij} = f_{ji} \equiv f_x$ and $\tau_{ijs} = \tau$ for all $i \neq j$. We now study the implications of symmetric trade liberalization ($\tau \downarrow$).
- Then the market access index

$$\phi_{ijs} = \frac{1}{\tau_{ijs}^{\theta}} \left(\frac{f_{jj}}{f_{ij}}\right)^{(\theta-\sigma+1)/(\sigma-1)} \left(\frac{w_j}{w_i}\right)^{(\theta-\rho)/\rho}$$

simplifies to

$$\phi = \frac{1}{\tau^{\theta}} \left(\frac{f_d}{f_x}\right)^{(\theta-\sigma+1)/(\sigma-1)}. \qquad (\tau \downarrow \Rightarrow \phi \uparrow)$$

- The system of equations

$$\sum_{j=1}^{n} \alpha_s w_j L_j \left(\frac{\phi_{ijs}}{\sum_{k=1}^{n} M_{kse} \phi_{kjs}} \right) = w_i f_{ise} \left(\frac{\theta}{\rho} \right)$$

simplifies to one equation

$$\frac{L}{m} \frac{1}{M_{ise}(1 + (n-1)\phi)} + \frac{L}{m} \frac{(n-1)\phi}{M_{ise}(1 + (n-1)\phi)} = F\, M_{ise}^{\zeta} \left(\frac{\theta}{\rho} \right)$$

or

$$\frac{L}{m} \frac{(1 + (n-1)\phi)}{M_{ise}(1 + (n-1)\phi)} = \frac{L}{m\, M_{ise}} = F\, M_{ise}^{\zeta} \left(\frac{\theta}{\rho} \right)$$

and solving yields

$$M_{ise} = \left(\frac{L\, \rho}{m\, \theta\, F} \right)^{1/(1+\zeta)} .$$

- The equation

$$M_{ise} = \left(\frac{L\,\rho}{m\,\theta\,F} \right)^{1/(1+\zeta)}$$

has a remarkable implication: it implies that symmetric trade liberalization has no effect on the mass of firms that choose to enter in each country and sector ($\tau \downarrow \Rightarrow M_{ise}$ unchanged).
- This property of the Melitz model greatly simplifies the analysis of symmetric trade liberalization.
- From the domestic productivity cutoff condition

$$\varphi_{iis}^{*\theta} = \frac{\theta\,b^{\theta}}{\delta\,(\theta - \sigma + 1)} \frac{\sigma\,f_d}{\alpha_s\,L_i} \sum_{j=1}^{n} M_{jse}\phi_{jis},$$

$\tau \downarrow \Rightarrow \phi \uparrow$, M_{ise} unchanged $\Rightarrow M_{ise}(1 + (n-1)\phi) \uparrow \Rightarrow \varphi_{iis}^{*} \uparrow$.

- Because symmetric trade liberalization results in $\sum_{j=1}^{n} M_{jse}\phi_{jis}$ increasing, competition becomes tougher in each country and sector.
- Firms need to have a higher productivity level to survive, so the domestic productivity cutoff φ_{iis}^{*} increases.
- It follows that industrial productivity $\Phi_{is}^{W} = (\alpha_s L_i/\sigma f_{ii})^{1/(\sigma-1)} \rho\varphi_{iis}^{*}$ rises in each country and sector, and consumer welfare $U_i = \prod_{s=1}^{m} \left(\alpha_s \Phi_{is}^{W}\right)^{\alpha_s}$ unambiguously increases.
- We have proved the

Melitz Theorem [main result in Melitz (2003, *Econometrica*)]:
$\tau \downarrow \Rightarrow M_{ise}$ unchanged, $\phi_{ijs} \uparrow, \varphi_{iis}^{*} \uparrow, \Phi_{is}^{k} \uparrow$ and $U_i \uparrow$.

- This is arguably the most important theorem in the international trade field!

Empirical Evidence about Trade Liberalization

- Recent empirical research has established new gains from international trade.
- Trade liberalization improves industrial productivity by shifting resources from less to more productive firms within industries.
- A typical empirical strategy is to compare liberalized industries that experienced tariff cuts and other non-liberalized industries.
- Researchers find that when countries open up to trade, productivity increases more strongly in liberalized industries than non-liberalized industries.
- This holds for large trade liberalization episodes in Canada (Trefler, 2004), in Chile (Pavcnik, 2002), in Columbia (Eslava, Haltiwanger, Kugler and Kugler, 2012) and in India (Nataraji, 2011).

- By investigating the long-run impact of the Canada-USA free trade agreement on Canadian manufacturing industries, Trefler (2004, AER) found that industrial productivity increased more strongly in liberalized industries that experienced large Canadian tariff cuts than in non-liberalized industries, and that the rise in productivity was mainly due to the shift of resources from less to more productive firms.
- This empirical finding by Trefler (and others) has been widely accepted as evidence for the Melitz (2003, *Econometrica*) model.
- For example, Melitz and Trefler (2012, *Journal of Economic Perspectives*) write,

"Empirical confirmation of the gains from trade predicted by models with heterogeneous firms represents one of the truly significant advances in the field of international economics."

- But there is a problem.

- Melitz (2003, *Econometrica*) studied the effects of symmetric multilateral trade liberalization, where the tariff rate is the same across countries and when this tariff rate is lowered, it is lowered in a symmetric way.
- Empirical researchers typically study what happens when one country unilaterally reduces tariffs in some industries but not others (unilateral and non-uniform trade liberalization).
- What happens if we ask the same question of the Melitz model that empirical researchers ask of the data? What is the implication of the Melitz model for difference-in-differences estimates of the impact of tariff cuts on industrial productivity?

- In Segerstrom and Sugita (2015, JEEA), we show that under very general assumptions, a multi-industry version of the Melitz model predicts that industrial productivity increases more strongly in non-liberalized industries than in liberalized industries.
- This surprising result holds for all productivity distributions (not just Pareto), for all upper tier utility functions (not just Cobb-Douglas) and for any finite number of industries (not just the two industry case).
- The Trefler finding that is cited as evidence for the Melitz model in survey papers written by leading scholars (Bernard, Jensen, Redding and Schott, 2007, 2012; Helpman, 2011; Redding, 2011; Melitz and Trefler, 2012) is actually evidence against the Melitz model.
- This disconnect between theory and evidence we call the Melitz-Trefler puzzle.

Back to the Melitz Model

- We now restrict attention to the case where there are two country ($n = 2$) and two industries ($m = 2$).
- We focus on what happens when country 1 unilaterally opens up to trade in industry A (τ_{21A} decreases from 1.3 to 1.15).

- For the numerical results, we assume that countries and industries are symmetric before trade liberalization. Then there are only nine parameters that need to be chosen.
- $\sigma = 3.8$, $\delta = .025$, $b = .2$, $\theta = 4.582$, $F = 2$, $f_{ii} = .043$
  These parameter values come from Balistreri, Hillbery and Rutherford (2011, JIE), where a version of the Melitz model is calibrated to fit trade data.
- $L_i = 1$ is a convenient normalization given that an increase in country size L_i has no effect on the key endogenous variables that we are solving for (the relative wage w_1/w_2, productivity cutoff levels φ^*_{ijs} and industrial productivity levels Φ^L_{is}).
- $\tau_{ijs} = 1.3$ corresponds to a 30 percent tax on all traded goods.
- Finally, we chose $f_{ij} = .059$ to guarantee that 18 percent of firms export in the initial equilibrium, consistent with evidence for the United States (Bernard et al., 2007).

- We decrease τ_{21A} while holding τ_{12A}, τ_{12B}, and τ_{21B} fixed (unilateral and non-uniform trade liberalization).
- Estimates of ζ typically fall between 0.66 and 9 (Kortum, 1993).
- Blundell, Griffith and Windmeijer (2002, Journal of Econometrics) find that the long-run elasticity of patents to R&D is 0.5, which corresponds to $1/(1+\zeta) = 0.5$ or $\zeta = 1$.

Table 3: Effects of trade liberalization ($\tau_{21A} = 1.30 \to 1.15$) on industrial productivity when $n = 2$ and $m = 2$.

	$\zeta = 0$	$\zeta = .25$	$\zeta = 1$	$\zeta = 5$
Φ^L_{1A}	+0.4%	+1.2%	+1.8%	+2.1%
Φ^L_{1B}	+1.5%	+0.6%	-0.1%	-0.4%

Thanks to the US Constitution of 1787, the US became the world's first large free trade zone, since the Constitution prohibited tariffs on trade between US states.