Answers to Advanced Microeconomics Problem Set 1

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1 Exercise 1.17

1.1 Original question

Suppose that preferences are convex but not strictly convex. Give a clear and convincing argument that a solution to the consumer's problem still exists, but that it need not be unique. Illustrate your argument with a two-good example.

1.2 Solution

Suppose the consumer's preference relation $\succsim$ is complete, transitive, continuous, strictly monotonic, and convex on R^n_+. Therefore, by Theorems 1.1 and 1.3 it can be represented by a real-valued utility function, u, that is continuous, strictly increasing, and quasiconcave on R^n_+. Under the assumptions on preferences, the utility function $u(x)$ is real- valued and continuous. The budget set B is a non-empty, closed, bounded, and thus compact subset of R^n. By the Weierstrass theorem, a maximum of $u(x)$ over B exists. However, it need not be unique. Proof: Since B is convex, any convex combination x^t of $x^1, x^2 \in B$ is also in B and $u(x^t) \geq min\left[u(x^1), u(x^2)\right]$ by quasiconcavity of u. Denote u^* the maximum of $u(x)$ over B. If $u(x^1)$=$u(x^2) = u^*$, then also $u(x^t) = u^*$. This holds, for example, if the indifference curves are linear and have the same slope as the budget constraint, as illustrated by Figure 1:

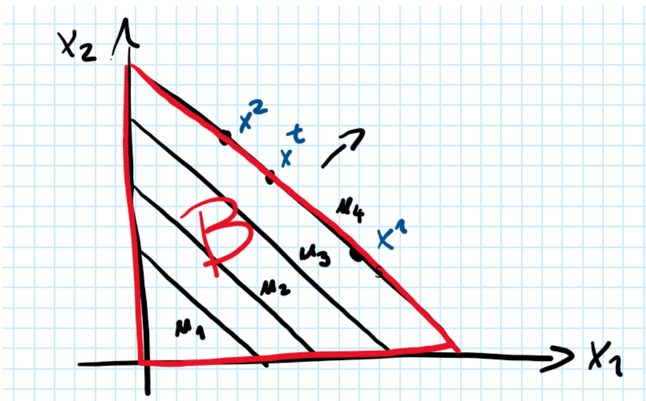

Figure 1: Non-uniqueness solution

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2 Exercise 1.20

2.1 Original question

Suppose preferences are represented by the Cobb-Douglas utility function $u(x_1, x_2) = Ax_1^{\alpha}x_2^{1-\alpha}$, $0 < \alpha < 1$, and $A > 0$. Assuming an interior solution, solve for the Marshallian demand functions.

2.2 Solution

Cobb-Douglas utility function: $u(x_1,x_2) = Ax_1^{\alpha}x_2^{1-\alpha}$ $0 < \alpha < 1$ and $A > 0$

Assumption: interior solution

Using the Lagrangian:

$$\mathcal{L} = Ax_1^{\alpha}x_2^{1-\alpha} + \lambda(y - p_1x_1 - p_2x_2)$$

$p_1x_1 + p_2x_2 = y$

$y = - p_1x_1 - p_2x_2$

Finding the first order conditions; settin them equal to zero:

$$\frac{\partial\mathcal{L}}{\partial x_1} = \alpha A x_1^{\alpha-1}x_2^{1-\alpha} - \lambda p_1 = 0$$

$$\frac{\partial\mathcal{L}}{\partial x_2} = (1-\alpha)Ax_1^{\alpha}x_2^{-\alpha} - \lambda p_2 = 0$$

$$\frac{\partial\mathcal{L}}{\partial \lambda} = y - p_1x_1 - p_2x_2 = 0$$

$$\frac{\frac{\partial\mathcal{L}}{\partial x_1}}{\frac{\partial\mathcal{L}}{\partial x_2}} = \frac{\alpha A x_1^{\alpha-1}x_2^{1-\alpha}}{(1-\alpha)Ax_1^{\alpha}x_2^{-\alpha}} = \frac{\lambda p_1}{\lambda p_2}$$

$$= \frac{\alpha}{1-\alpha}\cdot\frac{A}{A}\cdot\frac{x_1^{\alpha-1}}{x_1^{\alpha}}\cdot\frac{x_2^{1-\alpha}}{x_2^{-\alpha}} = \frac{p_1}{p_2}$$

$$= \frac{\alpha}{1-\alpha}\cdot 1\cdot\frac{1}{x_1}\cdot x_2 = \frac{p_1}{p_2}$$

$$= \frac{\alpha x_2}{(1-\alpha)x_1} = \frac{p_1}{p_2}$$

Expression x_2: $\alpha x_2 = \dfrac{p_1(1-\alpha)x_1}{p_2}$

$$x_2 = \frac{p_1(1-\alpha)x_1}{\alpha p_2}$$

rearrange:

$$x_2 = \frac{(1-\alpha)p_1x_1}{\alpha p_2}$$

Expression x_1:

$$p_1(1-\alpha)x_1 = \alpha x_2 p_2$$

$$x_1 = \frac{\alpha x_2 p_2}{(1-\alpha)p_1}$$

Substituting x_1 into constraint $p_1x_1 + p_2x_2 = y$ to express marshallian demand functions:

$$y = p_1\left(\frac{\alpha x_2 p_2}{(1-\alpha)p_1}\right) + p_2x_2$$

$$y = \frac{\alpha x_2 p_2}{(1-\alpha)} + p_2x_2$$

$$(1-\alpha)y = \alpha x_2 p_2 + p_2x_2(1-\alpha)$$

$$\frac{(1-\alpha)y}{p_2} = \alpha x_2 + x_2(1-\alpha)$$

$$\frac{(1-\alpha)y}{p_2} = x_2$$

$$y = p_1x_1 + p_2\left(\frac{(1-\alpha)y}{p_2}\right)$$

$$y = p_1x_1 + (1-\alpha)y$$

$$y = p_1x_1 + y - \alpha y$$

$$1 = \frac{p_1x_1}{y} + 1 - \alpha$$

$$\alpha = \frac{p_1x_1}{y}$$

$$\frac{\alpha y}{p_1} = x_1$$

Answer:
The two Marshallian demand functions are

$$x_1 = \frac{\alpha y}{p_1}$$

$$x_2 = \frac{(1-\alpha)y}{p_2}$$

3 Exercise 1.23

3.1 Original question

Prove Theorem 1.3.

3.2 Solution

Considering the consumer's preference relation is represented by u: $\mathbb{R}^n_+ \to \mathbb{R}$. And assuming that for all $x^0, x^1 \in \mathbb{R}^n_+$, $x^0 < x^1$. Then, the following properties will be proved:

a) $u(\mathbf{x})$ is strictly increasing if and only if $\succeq$ is strictly monotonic.

- Part 1: Considering $u(\mathbf{x})$ is strictly increasing,

  $$u(\mathbf{x^1}) > u(\mathbf{x^0})$$

  Following the Theorem 1.1 (*Existence of a Real-Valued Function Representing the Preference Relation*),

  $$u(\mathbf{x^1})e \sim x^1$$
  $$u(\mathbf{x^0})e \sim x^0$$

  Then,

  $$u(\mathbf{x^1})e \sim x^1 > u(\mathbf{x^0})e \sim x^0$$

  And, as $u(\mathbf{x})$ is a function representing the consumer's preferences, then

  $$x^1 \succ x^0$$

  Hence, it is shown that the consumer's preference relation is strictly monotonic, as $x^1 > x^0$.

- Part 2: Considering $\succeq$ is strictly monotonic,

  $$x^1 \succ x^0$$

  Following the Theorem 1.1 (*Existence of a Real-Valued Function Representing the Preference Relation*),

  $$u(\mathbf{x^1})e \sim x^1 \succ u(\mathbf{x^0})e \sim x^0$$

  And following the transitivity axioms of $\sim$ and $\succ$,

  $$u(\mathbf{x^1}) \succ u(\mathbf{x^0})$$

  because the preference relation is strictly monotonic. Thus

  $$u(\mathbf{x^1}) > u(\mathbf{x^0})$$

  Hence, it is shown that the utility function is strictly increasing.

$\square$

b) $u(\mathbf{x})$ is quasiconcave if and only if $\succeq$ is convex.

- Part 1: Considering $u(\mathbf{x})$ is quasiconcave, then, for all $t \in [0,1]$,

$$u(x^t) \geq min[u(x^1), u(x^2)]$$

where $x^t = tx^1 + (1-t)x^2$. Assuming $x^2 \geqslant x^1$, following the Theorem 1.1 (*Existence of a Real-Valued Function Representing the Preference Relation*),

$$u(\mathbf{x^2}) \geqslant u(\mathbf{x^t}) \geqslant u(\mathbf{x^1})$$

$$u(x^2)e \sim x^2 \geqslant u(x^t)e \sim x^t \geqslant u(x^1)e \sim x^1$$

Following the transitivity axiom and that preferences are strictly monotonic,

$$x^2 \geqslant x^t \geqslant x^1$$

$$x^2 \succeq x^t \succeq x^1$$

As we stated that $u(x)$ is quasiconcave, then

$$x^2 \succeq tx^1 + (1-t)x^2 \succeq x^1$$

Which demonstrates convexity in the consumer's preferences, as the consumption bundle x^t is at least as good as x^1.

- Part 2: Considering that the consumer's preference relation is convex, then:

$$tx^1 + (1-t)x^2 \succeq x^1, \text{for all } t \in [0,1]$$

As $x^2 \geqslant x^1$, following strict monotonicity and based on the transitivity axiom,

$$x^2 \succeq tx^1 + (1-t)x^2 \succeq x^1$$

As $u(\mathbf{x})$ is a function that represents the consumer's preferences (Theorem 1.1),

$$u(x^2)e \sim x^2 \geqslant u(tx^1 + (1-t)x^2)e \sim (tx^1 + (1-t)x^2) \geqslant u(x^1)e \sim x^1$$

$$u(x^2) \geqslant u(tx^1 + (1-t)x^2) \geqslant u(x^1)$$

Hence, it is shown that the utility function is quasiconcave, as the consumer's satisfaction with the consumption bundle created as a linear combination of x^1 and x^2 is greater or equal to the minimum utility between x^1 and x^2.

□

c) $u(\mathbf{x})$ is strictly quasiconcave if and only if $\succeq$ is strictly convex.

- Part 1: Considering $u(\mathbf{x})$ is strictly quasiconcave, then, for all $t \in (0,1)$,

$$u(x^t) > min[u(x^1), u(x^2)]$$

where $x^t = tx^1 + (1-t)x^2$. Assuming $x^2 > x^1$, following the Theorem 1.1 (*Existence of a Real-Valued Function Representing the Preference Relation*),

$$u(\mathbf{x^2}) > u(\mathbf{x^t}) > u(\mathbf{x^1})$$

$$u(x^2)e \sim x^2 > u(x^t)e \sim x^t > u(x^1)e \sim x^1$$

Following the transitivity axiom and that preferences are strictly monotonic,

$$x^2 > x^t > x^1$$

$$x^2 \succ x^t \succ x^1$$

As we stated that $u(x)$ is strictly quasiconcave, then

$$x^2 \succ tx^1 + (1-t)x^2 \succ x^1$$

This demonstrates strictly convexity in the consumer's preferences, as the consumption bundle x^t is better than x^1.

- Part 2: Considering that the consumer's preference relation is strictly convex, then:

  $$tx^1 + (1-t)x^2 \succ x^1, \text{for all } t \in (0,1)$$

  As $u(\mathbf{x})$ is a function that represents the consumer's preferences (Theorem 1.1),

  $$u(tx^1 + (1-t)x^2)e \sim (tx^1 + (1-t)x^2) > u(x^1)e \sim x^1$$

  Following the transitivity axiom,

  $$u(tx^1 + (1-t)x^2) > u(x^1) \text{ .... (P1)}$$

  And considering that $u(\mathbf{x})$ is strictly monotonic, it can be inferred that $u(\mathbf{x^2}) > u(\mathbf{x^1})$. Hence, it is possible to rephrase the previous expression (P1) as the following

  $$u(tx^1 + (1-t)x^2) > min[u(\mathbf{x^1}), u(\mathbf{x^2})]$$

  This shows that the utility function is strictly quasiconcave, as the consumer's satisfaction with the consumption bundle created as a linear combination of x^1 and x^2 is greater thant the minimum utility between x^1 and x^2.

$\square$

4 Exercise 1.24

4.1 Original question

Let $u(x)$ represent some consumer's monotonic preferences over $x \in R^n_+$. For each of the functions $f(x)$ that follow, state whether or not f also represents the preferences of this consumer. In each case, be sure to justify your answer with either an argument or a counterexample.

a) $f(x) = u(x) + (u(x))^3$

b) $f(x) = u(x) - (u(x))^2$

c) $f(x) = u(x) + \sum_{i=1}^{n} x_i$

4.2 Solution

Invoking the invariance of the utility function to positive monotonic transforms, if f is strictly increasing on the set of values taken on by u (where $u = u(x)$), it represents the same preferences. Denote $f'(u)$ the derivative of f with respect to u. Then f represents the same preferences as u as long as $f'(u) > 0$ for all possible values of u.

a)
$$f' = 1 + 3u^2$$

$\forall u : 3u^2 \geq 0 \implies f'(u) > 0 \implies f$ represents the same preferences as u.

Since $f'(u)$ is positive, $f(u)$ is a strictly increasing function, same as u, then $f(u)$ represents the consumer's preference relation.

b)
$$f' = 1 - 2u$$

For all cases when $u > 0.5$, then $f'(u) < 0$. Hence, $f(x) = u(x) - (u(x))^2$ is not a strictly increasing function, therefore it does not represent the preferences of the consumer.

c)
$$f' = 1$$

As $\frac{d}{du}(\sum_{i=1}^{n} x_i) = 0 \implies \forall u : f'(u) > 0 \implies f$ represents the same preferences as u.

Since $f'(u)$ is positive (a positive constant), $f(u)$ is a strictly increasing function, same as u. Then $f(u)$ represents the consumer's preference relation.

5 Exercise 1.29

5.1 Original question

An infinitely lived agent owns 1 unit of a commodity that he consumes over his lifetime. The commodity is perfectly storable and he will receive no more than he has now. Consumption of the commodity in period t is denoted x_t, and his lifetime utility function is given by

$$u(x_0, x_1, x_2, ...) = \sum_{t=0}^{\infty} \beta^t ln(x_t)$$

where $0 < \beta < 1$. Calculate his optimal level of consumption in each period.

5.2 Solution

$$max_{x_0, x_1, x_2...} \sum_{t=0}^{\infty} \beta^t ln(x_t)$$

$$s.t. \sum_{t=0}^{\infty} x_t = 1$$

Then setting up the lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t ln(x_t) - \lambda(\sum_{t=0}^{\infty} x_t - 1)$$

Next step is finding the F.O.C: $\frac{\partial \mathcal{L}}{\partial x_t} = \frac{\beta^t}{x_t} - \lambda = 0$ - from which we can express x_t as: $x_t = \frac{\beta^t}{\lambda}$
Now we can put the expression for x_t into $\frac{\partial \mathcal{L}}{\partial \lambda}$ (also the constraint expression):

$$\sum_{t=0}^{\infty} x_t = 1$$

Then this becomes:

$$\sum_{t=0}^{\infty} \frac{\beta^t}{\lambda} = 1$$

By using the geometric series formula we can then rewrite the summation of β^t ($\sum_{t=0}^{\infty} \beta^t$) as: $\beta^t = \frac{1}{1-\beta}$. We are then left with the equation $\frac{1}{\lambda(1-\beta)} = 1$

Rearranging the equation to formulate an expression for λ:

$$\lambda = \frac{1}{1-\beta}$$

Now we have all the components to formulate an expression stating the optimal consumption in each period.

We place the expression for λ into $x_t = \frac{\beta^t}{\lambda}$, which becomes $x_t = \frac{\beta^t}{\frac{1}{1-\beta}}$.

By simplifying the function we get that the agent's optimal level of consumption in each period is:

$$x_t = \beta^t(1-\beta).$$

To conclude, the agent's optimal level of consumption in each period is:

$$x_t = \beta^t(1-\beta).$$