5303 - Advanced Macroeconomics
Assignment 3 Solutions

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Contents

Exercise 1 **2**

Exercise 2 **6**

Exercise 3 **10**

Exercise 4 **14**

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Chapter 9 - Neoclassical Growth

Some definitions:

- People endowed with labour, not goods.
- Goods produced by combining labour (L) and forgone consumption, which we call capital (K): $Y(t) = F(K(t), L(t))$
- Labour endowmentes denoted by $\Delta_t^h = [\Delta_t^h(t), \Delta_t^h(t+1)]$
- Total amount of labour at date t is $L(t) = \sum_{h=1}^{N(t)} \Delta_t^h(t) + \sum_{h=1}^{N(t-1)} \Delta_{t-1}^h(t)$

Exercise 1

Consider an overlapping generations economy with a neoclassical production function. The size of each generation is $N(t) = 1, \forall t$. Preferences are represented by:

$$u_t^h = \ln c_t^h(t) + \frac{3}{5} \ln c_t^h(t+1)$$

and endowments of labour are given by: $\Delta_t^h = \left[\frac{4}{5}, \frac{1}{5}\right]$. Suppose that firms have access to the following production function:

$$Y(t) = K(t)^{\frac{1}{2}} L(t)^{\frac{1}{2}}$$

where $K(t)$ is the period t capital stock and $L(t)$ is the period t supply of labour. Depreciation is total; once capital has been used in production in one period, it vanishes.

(a) Find the stationary competitive equilibrium (with positive capital).

(b) Find the golden rule level of capital.

(c) Can you conclude anything about the Pareto optimality of the stationary competitive equilibrium?

Answer:

(a) Let's start by solving the household's problem. In a production economy, the budget constraints of the young and of the old are given by:

$$\begin{cases} \text{Young:} & \rightarrow c_t^h(t) \leq w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) \\ \text{Old:} & \rightarrow c_t^h(t+1) \leq w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) \end{cases}$$

where $w(t)$ and $R(t)$ are the wage rate and the rental rate of capital at time t, respectively. The household's problem is thus given by:

$$\begin{aligned} \max_{\{c_t^h(t), c_t^h(t+1), l^h(t), k^h(t+1)\}} \quad & \ln c_t^h(t) + \frac{3}{5}\ln c_t^h(t+1) \\ \text{s.t.} \quad & c_t^h(t) \leq w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) \\ & c_t^h(t+1) \leq w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) \end{aligned}$$

And our Lagrangian:

$$\begin{aligned} \mathcal{L} = {} & \ln c_t^h(t) + \frac{3}{5}\ln c_t^h(t+1) + \mu(t)\Big[w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) - c_t^h(t)\Big] \\ & + \mu(t+1)\Big[w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) - c_t^h(t+1)\Big] \end{aligned}$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the FOC:

$$\begin{aligned} \mathbf{c_t^h(t)}: & \ \frac{1}{c_t^h(t)} - \mu(t) = 0, \forall t \geq 1 \\ \mathbf{c_t^h(t+1)}: & \ \frac{3}{5}\frac{1}{c_t^h(t+1)} - \mu(t+1) = 0, \forall t \geq 1 \\ \mathbf{l^h(t)}: & \ -\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1 \\ \mathbf{k^h(t+1)}: & \ -\mu(t) + \mu(t+1)R(t+1) = 0, \forall t \geq 1 \end{aligned}$$

From which we can obtain the following equilibrium conditions:

$$\begin{aligned} R(t+1) &= r(t) \\ c_t^h(t+1) &= \frac{3}{5}r(t)c_t^h(t) \end{aligned}$$

Where the first condition is the no arbitrage condition between private borrowing and lending and holding capital. Substituting in the lifetime budget constraint:

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r(t)} - k^h(t+1)\Big[1 - \frac{R(t+1)}{r(t)}\Big]$$
$$c_t^h(t) = \frac{5}{8}\Big[w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r(t)}\Big]$$

And savings are given by:

$$s_t^h(t) = w(t)\Delta_t^h(t) - c_t^h(t) = \frac{3}{8}w(t)\Delta_t^h(t) - \frac{5}{8}\frac{w(t+1)\Delta_t^h(t+1)}{r(t)}$$

Since $N(t) = 1, \forall t$, we know that aggregate savings are equal to individuals savings, i.e. $S_t(r(t)) = s_t^h(t)$.

Furthermore, the firm's problem is given by:

$$\max_{\{K(t),L(t)\}} K(t)^{\frac{1}{2}}L(t)^{\frac{1}{2}} - R(t)K(t) - w(t)L(t)$$

Taking the FOC:

$\mathbf{K(t)}$: $R(t) = \frac{1}{2}\left(\frac{K(t)}{L(t)}\right)^{-\frac{1}{2}}, \forall t \geq 1$

$\mathbf{L(t)}$: $w(t) = \frac{1}{2}\left(\frac{K(t)}{L(t)}\right)^{\frac{1}{2}}, \forall t \geq 1$

Note that labour endowments are fixed, meaning that the total labour supply is also fixed for every period t and given by:

$$L(t) = \sum_{h=1}^{N(t)} \Delta_t^h(t) + \sum_{h=1}^{N(t-1)} \Delta_{t-1}^h(t) = \frac{4}{5} \times 1 + \frac{1}{5} \times 1 = 1$$

So we have that:

$$R(t) = \frac{1}{2}K(t)^{-\frac{1}{2}} \Rightarrow R(t+1) = \frac{1}{2}K(t+1)^{-\frac{1}{2}} = r(t)$$
$$w(t) = \frac{1}{2}K(t)^{\frac{1}{2}}$$

And aggregate savings are thus given by:

$$\begin{aligned} S_t(r(t)) &= \frac{3}{8}w(t)\Delta_t^h(t) - \frac{5}{8}\frac{w(t+1)\Delta_t^h(t+1)}{r(t)} \\ &= \frac{3}{8} \times \frac{1}{2}K(t)^{\frac{1}{2}} \times \frac{4}{5} - \frac{5}{8} \times \frac{\frac{1}{2}K(t+1)^{\frac{1}{2}} \times \frac{1}{5}}{\frac{1}{2}K(t+1)^{-\frac{1}{2}}} \\ &= \frac{3}{20}K(t)^{\frac{1}{2}} - \frac{1}{8}K(t+1) \end{aligned}$$

Definition 1. [1] A perfect foresight competitive equilibrium for an economy with labour endowments and a production function of $\gamma(t)F(K(t), L(t))$ is a sequence of $K(t)$, $r(t)$, $w(t)$ and $R(t)$ for $t \geq 1$ such that, given an initial $K(1) > 0$,

$$\begin{aligned} S_t(r(t)) &= K(t+1) \\ r(t) &= R(t+1) \\ w(t) &= \frac{\partial[\gamma(t)F(K(t), L(t))]}{\partial L(t)} \\ R(t) &= \frac{\partial[\gamma(t)F(K(t), L(t))]}{\partial K(t)} \end{aligned}$$

hold for all $t \geq 1$.

Making use of definition 1 above, we know that in a perfect foresight competitive equilbrium, all savings go to capital accumulation, such that:

$$S_t(r(t)) = K(t{+}1) \Leftrightarrow \frac{3}{20}K(t)^{\frac{1}{2}} - \frac{1}{8}K(t{+}1) = K(t{+}1) \Leftrightarrow K(t{+}1) = \frac{2}{15}K(t)^{\frac{1}{2}}$$

And since a stationary equilibrium is such that $K(t+1) = K(t) = K$, we have that:

$$\begin{aligned} K = \frac{2}{15}K^{\frac{1}{2}} \Leftrightarrow K &= \frac{4}{225} \\ r &= \frac{1}{2}K^{-\frac{1}{2}} = \frac{15}{4} \\ w &= \frac{1}{2}K^{\frac{1}{2}} = \frac{1}{15} \end{aligned}$$

And consumption and savings are given by:

$$\begin{aligned} c_t^h(t) &= \frac{5}{8}\left[\frac{1}{15} \times \frac{4}{5} + \frac{\frac{1}{15} \times \frac{1}{5}}{\frac{15}{4}}\right] = \frac{8}{225} \\ c_t^h(t+1) &= \frac{3}{5}r(t)c_t^h(t) = \frac{3}{5} \times \frac{15}{4} \times \frac{8}{225} = \frac{2}{25} \\ s_t^h(t) &= \frac{3}{20} \times \left(\frac{4}{225}\right)^{\frac{1}{2}} - \frac{1}{8} \times \left(\frac{4}{225}\right) = \frac{4}{225} \end{aligned}$$

Notice that savings are equal to aggregate capital, as we imposed in the equilibrium condition. We also need to check whether this allocation is feasible:

Definition 2. [2] An allocation is feasible if there is a sequence $K(t)$ such

[1] Page 238 of the course textbook.
[2] Page 236 of the course textbook.

that $K(t) \geq 0$ and

$$Y(t) \geq C(t) + K(t+1)$$

Making use of definition 2 above, we have that:

$$\left(\frac{4}{225}\right)^{\frac{1}{2}} \geq \frac{8}{225} + \frac{2}{25} + \frac{4}{225} \Leftrightarrow \frac{2}{15} \geq \frac{2}{15}$$

Meaning that this allocation is feasible.

(b) Let's refer to the following definition:

Definition 3. The golden rule level of capital is the steady state level of capital that maximizes consumption.

Let's make use of the resource constraint (remember that $L(t) = 1$):

$$C + K = Y = F(K, 1) \Leftrightarrow C = F(K, 1) - K$$

Taking the FOC with respect to capital:

$$F'(K, 1) = 1 \Leftrightarrow \frac{1}{2}K^{-\frac{1}{2}} = 1 \Leftrightarrow K = \frac{1}{4}$$

which is our golden rule level of capital. Notice that the golden rule level of capital its higher than the stationary equilibrium level of capital, as expected.

(c) Since the stationary capital stock is below the golden rule level of capital, the economy is dynamically efficient. Furthermore, it does not satisfy a sufficient condition for it not to be Pareto optimal, since $r = \frac{15}{4} > (1+n) = 1$, i.e. it is not dynamically inefficient.[3]

Exercise 2

Consider a 2-period OLG model with production. $N = 1$ young agents are born every period. All agents live for two periods.

Preferences are described by $u_t^h = \ln(c_t^h(t) - \theta a(t)) + \beta \ln c_t^h(t+1)$, where $\theta > 0$ and $\beta > 0$ are parameters. $a(t) = \bar{c}_{t-1}(t-1)$ is the average consumption of the young at $t-1$. The idea is that agents want to "keep up" with the consumption levels of their parent's generation.

[3] Please refer to the discussion in pages 29-30 of the course textbook and to handout 5.

Young agents are endowed with 1 unit of time and no capital. The initial old are endowed with S_0 units of capital. Thus, the young work and receive a wage of $w(t)$ (the old do not work).

Output is produced according to $Y(t) = K(t)^\alpha L(t)^{1-\alpha}$. Capital depreciates fully so $Y(t) = C(t) + K(t+1)$.

State the household's problem and solve for the savings function.

Define a competitive equilibrium.

Derive the equilibrium laws of motion for $k(t+1)$ and $a(t+1)$ (i.e. an expression for $k(t+1)$ as a function of $k(t)$ and $a(t)$, and an expression for $a(t+1)$ as a function of $a(t)$ and $k(t)$).

How does the steady state k respond to θ? Provide intuition.

Answer:

Let's start by writing down the budget constraints of the young and of the old:

$$\begin{cases} \text{Young:} \ \rightarrow c_t^h(t) \leq w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) \\ \text{Old:} \ \rightarrow c_t^h(t+1) \leq w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) \end{cases}$$

where $w(t)$ and $R(t)$ are the wage rate and the rental rate of capital at time t, respectively. The household's problem is thus given by[4]:

$$\begin{aligned} \max_{\{c_t^h(t), c_t^h(t+1), l^h(t), k^h(t+1)\}} \quad & \ln(c_t^h(t) - \theta a(t)) + \beta \ln c_t^h(t+1) \\ \text{s.t.} \quad & c_t^h(t) \leq w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) \\ & c_t^h(t+1) \leq w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) \end{aligned}$$

And our Lagrangian:

$$\begin{aligned} \mathcal{L} = {} & \ln(c_t^h(t) - \theta a(t)) + \beta \ln c_t^h(t+1) + \mu(t)\Big[w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) - c_t^h(t)\Big] \\ & + \mu(t+1)\Big[w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) - c_t^h(t+1)\Big] \end{aligned}$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the

[4] Note that we are ignoring the condition $c_t^h(t) \geq \theta a(t)$ since we are mostly interested in the cases where this condition does not bind, i.e. $c_t^h(t) > \theta a(t)$, which would set the corresponding Lagrangian multiplier to zero.

FOC:

$\mathbf{c_t^h(t)}$: $\frac{1}{c_t^h(t)-\theta a(t)} - \mu(t) = 0, \forall t \geq 1$

$\mathbf{c_t^h(t+1)}$: $\frac{\beta}{c_t^h(t+1)} - \mu(t+1) = 0, \forall t \geq 1$

$\mathbf{l^h(t)}$: $-\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1$

$\mathbf{k^h(t+1)}$: $-\mu(t) + \mu(t+1)R(t+1) = 0, \forall t \geq 1$

From which we can obtain the following equilibrium conditions:

$$R(t+1) = r(t)$$
$$c_t^h(t+1) = \beta r(t)[c_t^h(t) - \theta a(t)]$$

Where the first condition is the no arbitrage condition between private borrowing and lending and holding capital. Substituting in the lifetime budget constraint:

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r(t)} - k^h(t+1)\Big[1 - \frac{R(t+1)}{r(t)}\Big]$$
$$c_t^h(t) = \Big(\frac{1}{1+\beta}\Big)\Big[w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r(t)}\Big] + \Big(\frac{\beta}{1+\beta}\Big)\theta a(t)$$

And savings are given by:

$$s_t^h(t) = w(t)\Delta_t^h(t) - c_t^h(t) = \Big(\frac{\beta}{1+\beta}\Big)[w(t)\Delta_t^h(t) - \theta a(t)] - \Big(\frac{1}{1+\beta}\Big)\frac{w(t+1)\Delta_t^h(t+1)}{r(t)}$$

Since $N(t) = 1, \forall t$, we know that aggregate savings are equal to individuals savings, i.e. $S_t(r(t)) = s_t^h(t)$.

Furthermore, the firm's problem is given by:

$$\max_{\{K(t),L(t)\}} K(t)^{\alpha}L(t)^{1-\alpha} - R(t)K(t) - w(t)L(t)$$

Taking the FOC:

$\mathbf{K(t)}$: $R(t) = \alpha\Big(\frac{K(t)}{L(t)}\Big)^{\alpha-1}, \forall t \geq 1$

$\mathbf{L(t)}$: $w(t) = (1-\alpha)\Big(\frac{K(t)}{L(t)}\Big)^{\alpha}, \forall t \geq 1$

Since labour endowments are fixed, total labour supply will also be fixed for every period t and given by:

$$L(t) = \sum_{h=1}^{N(t)} \Delta_t^h(t) + \sum_{h=1}^{N(t-1)} \Delta_{t-1}^h(t) = 1 \times 1 + 1 \times 0 = 1$$

So we have that:

$$\begin{aligned} &R(t) = \alpha K(t)^{\alpha-1} \Rightarrow R(t+1) = \alpha K(t+1)^{\alpha-1} = r(t) \\ &w(t) = (1-\alpha)K(t)^\alpha \end{aligned}$$

And aggregate savings are thus given by:

$$\begin{aligned} S_t(r(t)) &= \Big(\frac{\beta}{1+\beta}\Big)[w(t)\Delta_t^h(t) - \theta a(t)] - \Big(\frac{1}{1+\beta}\Big)\frac{w(t+1)\Delta_t^h(t+1)}{r(t)} \\ &= \Big(\frac{\beta}{1+\beta}\Big) \times [(1-\alpha)K(t)^\alpha \times 1 - \theta a(t)] - \Big(\frac{1}{1+\beta}\Big)\frac{(1-\alpha)K(t+1)^\alpha \times 0}{\alpha K(t+1)^{\alpha-1}} \\ &= \Big(\frac{\beta}{1+\beta}\Big)\big[(1-\alpha)K(t)^\alpha - \theta a(t)\big] \end{aligned}$$

We are now able to define a competitive equilibrium:

Definition 4. A competitive equilibrium is a sequence of allocations $\{c_t^h(t), c_t^h(t+1), l^h(t), k^h(t)\}_{t=1}^\infty$ and prices $\{w(t), R(t), r(t)\}_{t=1}^\infty$ such that:

(i) Given prices, the allocation solves the individual's utility maximization problem.

(ii) Given wages and the rental rate of capital, the firm's profit function is maximized.

(iii) All markets clear.

Making use of definition 1 above, we know that in a perfect foresight competitive equilbrium, all savings go to capital accumulation, such that:

$$S_t(r(t)) = K(t+1) \Leftrightarrow \Big(\frac{\beta}{1+\beta}\Big)\big[(1-\alpha)K(t)^\alpha - \theta a(t)\big] = K(t+1)$$

which is the law of motion for $K(t+1)$. The law of motion for $a(t+1)$ is given by:

$$
\begin{aligned}
a(t+1) &= c_t^h(t) \\
a(t+1) &= \Big(\frac{1}{1+\beta}\Big)\Big[w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r(t)}\Big] + \Big(\frac{\beta}{1+\beta}\Big)\theta a(t) \\
a(t+1) &= \Big(\frac{1}{1+\beta}\Big)(1-\alpha)K(t)^\alpha + \Big(\frac{\beta}{1+\beta}\Big)\theta a(t)
\end{aligned}
$$

And since a stationary equilibrium is such that $K(t+1) = K(t) = K$ and $a(t+1) = a(t) = a$, we have that:

$$
\begin{aligned}
a &= \Big(\frac{1}{1+\beta}\Big)(1-\alpha)K^\alpha + \Big(\frac{\beta}{1+\beta}\Big)\theta a \\
a &= \Big(\frac{1-\alpha}{1+\beta-\beta\theta}\Big)K^\alpha
\end{aligned}
$$

which can be substituted in the stationary law of motion for capital

$$
\begin{aligned}
K &= \Big(\frac{\beta}{1+\beta}\Big)\big[(1-\alpha)K^\alpha - \theta a\big] \\
K &= \Big(\frac{\beta}{1+\beta}\Big)\Big[(1-\alpha)K^\alpha - \theta\Big(\frac{1-\alpha}{1+\beta-\beta\theta}\Big)K^\alpha\Big] \\
K &= \Big[\frac{\beta(1-\alpha)(1-\theta)}{1+\beta(1-\theta)}\Big]^{\frac{1}{1-\alpha}} \\
K &= \Big[\frac{\beta(1-\alpha)}{\frac{1}{1-\theta}+\beta}\Big]^{\frac{1}{1-\alpha}}
\end{aligned}
$$

It is clear from above that $\frac{\partial K}{\partial \theta} < 0$. The intution for this is twofold: on the one hand, a higher θ raises the marginal utility when young relative to old, which reduces savings; on the other hand, with a higher θ, young agents care more about keeping up with the previous generations' consumption, thus reducing savings.

Exercise 3

Consider the following overlapping generations environment.

Population: All agents live for two periods. There is an equal number of young and old alive initially, and there is no population growth.

Preferences: Household has preferences over consumption of a non-storable good and labour supply given by:

$$u_t^h = c_t^h(t) - \frac{\gamma}{2}\left(l_t^h(t)\right)^2 + \ln c_t^h(t+1)$$

with constraints on consumption $c_t^h(t) \geq 0$, $c_t^h(t+1) \geq 0$ and labour supply $l_t^h(t) \in [0,1]$. Old agents do not work.

Technology: $Y(t) = AL(t)$

Assume $A^2 > \gamma > A > 0$.

Government: Suppose the government implements a pay-as-you-go social security system (proportional tax at rate τ on the labour earnings of the young used to fund a lump-sum transfer b to the old).

(a) Define a competitive equilibrium with social security.

(b) Characterize the competitive equilbrium. In other words, what are the equilibrium allocations, prices and policy in terms of parameters?

(c) A Laffer curve is defined as the relationship between revenue and tax rates, i.e. the function $b_t = f(\tau)$. Solve for the Laffer curve. What tax rate maximizes tax revenue? Show that the tax rate that maximizes a representative generations' welfare is different from that which maximizes tax revenue.

Answer:

(a) Please refer to the definition below:

Definition 5. Given a social security policy $\{\tau, b(t)\}_{t=1}^{\infty}$, a competitive equilibrium with social security is an allocation $\{c_t^h(t), c_t^h(t+1), l_t^h(t)\}_{t=1}^{\infty}$ and wages $\{w(t)\}_{t=1}^{\infty}$ such that:

(i) Given wages and the social security policy, the allocation maximizes the individual's utility.

(ii) Given wages, the firm's profit function is maximized.

(iii) Goods and labour markets clear.

(iv) The government budget constraint is at equilibrium.

(b) Since there is no capital in the production function, we know that there will not exist a capital market. Therefore, we can ignore the financial markets since nobody will want to borrow or lend in equilibrium. The budget constraints of the young and of the old are thus given by (remember that old agents do not work):

$$\begin{cases} \text{Young: } \rightarrow c_t^h(t) \leq (1-\tau)w(t)l_t^h(t) \\ \text{Old: } \rightarrow c_t^h(t+1) \leq b(t) \end{cases}$$

And the household's problem is given by:

$$\begin{aligned} \max_{\{c_t^h(t), c_t^h(t+1), l_t^h(t)\}} \quad & c_t^h(t) - \frac{\gamma}{2}\left(l_t^h(t)\right)^2 + \ln c_t^h(t+1) \\ \text{s.t. } & c_t^h(t) \leq (1-\tau)w(t)l_t^h(t) \\ & c_t^h(t+1) \leq b(t) \end{aligned}$$

And our Lagrangian:

$$\begin{aligned} \mathcal{L} = c_t^h(t) - \frac{\gamma}{2}\left(l_t^h(t)\right)^2 + \ln c_t^h(t+1) + \mu(t)\Big[(1-\tau)w(t)l_t^h(t) - c_t^h(t)\Big] \\ + \mu(t+1)\Big[b(t) - c_t^h(t+1)\Big] \end{aligned}$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the FOC:

$$\begin{aligned} \mathbf{c_t^h(t)}\text{: } & 1 - \mu(t) = 0, \forall t \geq 1 \\ \mathbf{c_t^h(t+1)}\text{: } & \tfrac{1}{c_t^h(t+1)} - \mu(t+1) = 0, \forall t \geq 1 \\ \mathbf{l_t^h(t)}\text{: } & -2 \times \tfrac{\gamma}{2} l_t^h(t) + \mu(t)(1-\tau)w(t) = 0, \forall t \geq 1 \end{aligned}$$

From where it follows that

$$l_t^h(t) = \frac{(1-\tau)w(t)}{\gamma}$$

We now need to solve for the firm's problem:

$$\max_{\{L(t)\}} AL(t) - w(t)L(t)$$

From where it follows that $w(t) = A$, i.e. wages are constant. This means that equilibrium profits are null and labour supply and consumption are thus given by:

$$\begin{aligned} l_t^h(t) &= \frac{(1-\tau)A}{\gamma} \\ c_t^h(t) &= (1-\tau)Al_t^h(t) = \frac{(1-\tau)^2A^2}{\gamma} \\ c_t^h(t+1) &= b(t) \end{aligned}$$

Remember that our equilibrium definition above requires the government budget constraint to be balanced, i.e. the total transfer given to generation $t-1$ individuals at time t must equal the total amount of taxes collected from generation t individuals at time t. Since we have no population growth, it follows that at equilibrium, we have that:

$$N(t-1)b(t) = N(t)\tau w(t)l_t^h(t) \Leftrightarrow b(t) = \frac{\tau(1-\tau)A^2}{\gamma}, \forall t$$

(c) Making use of our previous results, we know that the Laffer curve is given by:

$$b(\tau) = \frac{\tau(1-\tau)A^2}{\gamma}$$

The tax rate τ can only take values between 0 and 1, meaning that the Laffer Curve is only defined in $]0,1[$. The extrema cannot be part of the equilibrium since $\tau = 0 \Rightarrow c_t^h(t+1) = 0$, which will generate infinite negative utility, whereas $\tau = 1 \Rightarrow c_t^h(t) = 0$, and we can increase utility by merely decreasing τ a bit. Furthermore, the tax rate τ^* that maximizes the government's tax revenue is the tax rate rate that maximizes the Laffer Curve. Taking the FOC, we have that:

$$\frac{\partial}{\partial\tau}b(\tau) = \frac{A^2}{\gamma}(1-2\tau) = 0 \Leftrightarrow \tau^* = \frac{1}{2}$$

We also need to check whether this is a maximum. Taking the SOC:

$$\frac{\partial^2}{\partial\tau^2}b(\tau) = -\frac{2A^2}{\gamma} < 0, \forall\tau \in]0,1[$$

Meaning that the tax rate $\tau^* = \frac{1}{2}$ maximizes government income.

On the other hand, we know that representative generations' welfare is given by:

$$W(\tau) = \frac{(1-\tau)^2A^2}{\gamma} - \frac{\gamma}{2}\Big(\frac{(1-\tau)A}{\gamma}\Big)^2 + \ln\Big(\frac{\tau(1-\tau)A^2}{\gamma}\Big)$$

Taking the FOC:

$$\frac{\partial}{\partial \tau} W(\tau) = -\frac{A^2}{\gamma}(1-\tau) + \frac{1}{\tau} - \frac{1}{1-\tau} = 0$$

And the SOC is given by:

$$\frac{\partial^2}{\partial \tau^2} W(\tau) = \frac{A^2}{\gamma} - \frac{1}{\tau^2} - \frac{1}{(1-\tau)^2}$$

In order to have a maximum, we must have:

$$\frac{A^2}{\gamma} < \frac{1}{\tau^2} + \frac{1}{(1-\tau)^2}$$

Notice that the RHS of the inequality above is minimized for $\tau = \frac{1}{2}$. However, the RHS is greater for any other $\tau \in]0,1[$. Futhermore, the FOC above does not hold for $\tau = \frac{1}{2}$ since $\frac{\partial}{\partial \tau} W(\tau)|_{\tau=\frac{1}{2}} < 0$. Therefore, $\tau^* = \frac{1}{2}$ cannot be a solution and the optimal tax rate that maximizes a representative generations' welfare must be lower than $\frac{1}{2}$.

Exercise 4

Consider the following OLG model where people live for two periods. Each generation has the same number of people. Preferences are given by $\ln c_t^h(t) + \ln c_t^h(t+1)$. Each agent is endowed with e_1 units of output when young and e_2 units of output when old.

(a) Solve for the equilibrium where there is no government policy. When is the net interest rate negative? Provide intuition.

(b) Assume that the condition from part (a) is met, so that the equilibrium with no government policy is characterized by a negative net interest rate. Show how a pay-as-you-go system in the form of a lump-sum tax and transfer can improve the welfare of each generation by allowing for perfect consumption smoothing.

(c) Suppose that the endowment good is storable. In other words, a young person can put a unit of the endowment good in the refrigerator and when he/she is old, the good is still there. In this world, would anyone benefit from the government policy in part (b)?

Answer:

(a) The budget constraints of the young and of the old are given by:

$$\begin{cases} \text{Young:} \ \rightarrow c_t^h(t) \leq e_1 - l^h(t) \\ \text{Old:} \ \rightarrow c_t^h(t+1) \leq e_2 + r(t)l^h(t) \end{cases}$$

We can now solve the following household problem:

$$\begin{aligned} \max_{\{c_t^h(t), c_t^h(t+1), l^h(t)\}} \quad & \ln c_t^h(t) + \ln c_t^h(t+1) \\ \text{s.t.} \ & c_t^h(t) \leq e_1 - l^h(t) \\ & c_t^h(t+1) \leq e_2 + r(t)l^h(t) \end{aligned}$$

And our Lagrangian:

$$\begin{aligned} \mathcal{L} = \ln c_t^h(t) + \ln c_t^h(t+1) + \mu(t)\Big[e_1 - l^h(t) - c_t^h(t)\Big] \\ + \mu(t+1)\Big[e_2 + r(t)l^h(t) - c_t^h(t+1)\Big] \end{aligned}$$

Taking the FOC:

$$\begin{aligned} \mathbf{c_t^h(t)}&: \ \tfrac{1}{c_t^h(t)} - \mu(t) = 0, \forall t \geq 1 \\ \mathbf{c_t^h(t+1)}&: \ \tfrac{1}{c_t^h(t+1)} - \mu(t+1) = 0, \forall t \geq 1 \\ \mathbf{l_t^h(t)}&: \ -\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1 \end{aligned}$$

And we get the usual equilibrium condition:

$$c_t^h(t+1) = r(t)c_t^h(t)$$

From where it follows:

$$\begin{aligned} c_t^h(t) &= \frac{1}{2}e_1 + \frac{1}{2}\frac{e_2}{r(t)} \\ s_t^h(t) &= \frac{1}{2}e_1 - \frac{1}{2}\frac{e_2}{r(t)} \end{aligned}$$

Since all individuals are the same and there is no government, no land, no production and no storage, we know that individuals will not save anything and they will consume their entire endowment, i.e. $c_t^h(t) = e_1$ and $c_t^h(t+1) = e_2$. Making use of the equilibrium condition:

$$S_t(r(t)) = 0 \Leftrightarrow N \times s_t^h(t) = 0 \Leftrightarrow r(t) = \frac{e_2}{e_1}$$

Recall the gross interest rate, $r(t)$, is equal to 1 plus the net interest rate $i(t)$, meaning that:

$$r(t) = 1 + i(t) = \frac{e_2}{e_1}$$

We can thus conclude that the net interest rate rate is negative iff $e_1 > e_2$, i.e. agents have a higher endowment when young. If agents have higher endowments when young, they may have incentives to save in order to smooth consumption over time. A negative net interest rate implies that the gross interest rate is less than 1, i.e. young agents would need to pay to save, thus reducing the incentives for saving to smooth consumption. A similar reasoning could be applied to the private lending market, where a negative net interest rate means that agents would need to pay to lend, thus reducing the incentives to do so. Since all agents are identical, there will be no private borrowing/lending.

(b) Suppose that $e_1 > e_2$. The government would implement a pay-as-you-go system to help smooth out the effects of the differences in endowment when young and old, such that, under perfect consumption smoothing, $c_t^h(t) = c_t^h(t+1) = c$. To do so, the government imposes a lump-sum tax τ on young agents and provides a lump-sum subsidy $-\tau$ to the old. The budget constraints are now given by:

$$\begin{cases} \text{Young:} \ \rightarrow c_t^h(t) \leq e_1 - l^h(t) - \tau \\ \text{Old:} \ \rightarrow c_t^h(t+1) \leq e_2 + r(t) l^h(t) + \tau \end{cases}$$

Solving the maximization problem, we get that:

$$c_t^h(t) = \frac{1}{2}(e_1 - \tau) + \frac{1}{2}\frac{e_2 + \tau}{r(t)}$$
$$s_t^h(t) = \frac{1}{2}(e_1 - \tau) - \frac{1}{2}\frac{e_2 + \tau}{r(t)}$$

Since $r(t) < 1$, individuals will be better off by participating in the government scheme than by participating in the private borrowing and lending market, meaning that no agent will want to borrow or lend. Therefore, under perfect consumption smoothing, we have that:

$$e_1 - \tau = e_2 + \tau \Leftrightarrow \tau = \frac{e_1 - e_2}{2}$$

$$c = \frac{e_1 + e_2}{2}$$

Furthermore, by the concavity of the logarithmic function, we have that:

$$u(c,c) = 2\ln\left(\frac{e_1 + e_2}{2}\right) > \ln e_1 + \ln e_2 = u(e_1, e_2)$$

Meaning that imposing the pay-as-you-go system is welfare improving.

(c) In such a framework, every generation $t \geq 1$ can achieve perfect consumption smoothing with both the government scheme and the storage technology. With a $\lambda = 1$, individuals will be indifferent between storing the good and participating in the scheme. However, the initial old are better off with both systems in place since they would consume their endowment, the initial exogenous amount of storage $K(1)$ and the transfer $-\tau$.