Advanced microeconomics problem set 6

Hugo, Luis, Marek*

October 5, 2023

1 Exercise 5.10

1.1 Original question

In a two-person, two-good exchange economy with strictly increasing utility functions, it is easy to see that an allocation $\bar{x} \in F(e)$ is Pareto efficient if and only if $\bar{x}^i$ solves the problem

$$\begin{aligned} \max_{x^i} u^i(x^i) \text{ s.t. } & u^j(x^j) \geq u^j(\bar{x}^j), \\ & x_1^1 + x_1^2 = e_1^1 + e_1^2, \\ & x_2^1 + x_2^2 = e_2^1 + e_2^2 \end{aligned}$$

for $i = 1, 2$ and $i \neq j$.

(a) Prove the claim.

(b) Generalise this equivalent definition of a Pareto-efficient allocation to the case of n goods and I consumers. Then prove the general claim.

1.2 Solution

(a) Suppose each consumer's preferences can be represented by the a utility function u^i. It follows that, for a given i, an allocation x satisfies the problems first constraint if and only if it lies in the the upper contour sets of u belonging to the other agent. This means that agent j will not accept a bundle that gives him a lower utility than the one he got from $\bar{x}^j$. The second constraint says simply that a solution needs to be feasible. For the sake of contradiction, suppose that $\bar{x}$ solves the problem but is not Pareto efficient. Then there has to be a bundle $y \in F(e)$ that still fulfills $u^j(y^j) \geq u^j(\bar{x}^j)$.

*42597@student.hhs.se, 42635@student.hhs.se, 42624@student.hhs.se

But, because the utility functions are strictly increasing, y gives consumer i a higher utility ($u^i(y^i) > u^i(\bar{x}^i)$). Consequently, $\bar{x}$ was no solution in the first place.

Now suppose $\bar{x}$ does not solve the problem, but is Pareto-efficient. If $\bar{x}$ does not solve the problem, there has to be be a bundle that achieves a higher value than $u^i(\bar{x}^i)$ but still fulfills the first constraint. Name this bundle y. Then $u^i(y^i) > u^i(\bar{x}^i)$ and $u^j(y^j) \geq u^j(\bar{x}^j)$. Consequently, $\bar{x}$ is not Pareto-efficient.

(b) The constrained optimization problem for the n consumer with I goods is given by

$$\begin{aligned}
&\max_{x^i} u^i(x^i) \\
&\text{s.t. } u^j(x^j) \geq u^j(\bar{x}^j) \forall j \in \mathcal{I} \setminus \{i\}, \\
&\sum_{i \in \mathcal{I}} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i \\
&\text{for all } i \in \mathcal{I}.
\end{aligned}$$

An allocation is feasible if $F(e) = \{\mathbf{x} : \sum_{i \in \mathcal{I}} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i\}$ A feasible allocation, is Pareto efficient if there is no other feasible allocation, $y \in F(e)$, such that $y^i \succsim^i x^i$ for all consumers, i, with at least one preference strict. Define as $C_i(x)$ the upper contour set of agent i. $C_i(x) = \{\mathbf{y} \in X : u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i)\}$ $C_i(x)$ is always non-empty, since$x \in C_i(x)$. For a given i, an allocation x satisfies the first constraint of the problem if and only if $\mathbf{x} \in \cap_{j \neq i} C_j(\bar{x})$. An allocation $\bar{x}$ solves the constrained optimization problem subject to lying in the intersection of the set of feasible allocations and the intersection of upper contours sets for all other agents.

Assume that $\bar{x}$ solves the problem. For the sake of contradiction, suppose that it is not Pareto efficient. Then there must exist an $y \in F(e)$ such that and (at least) one index $m \in I$ such that

$$\begin{aligned}
u^i(y^i) &\geq u^i(\bar{x}^i), \forall i \in \mathcal{I} \\
u^m(y^m) &> u^m(\bar{x}^m)
\end{aligned}$$

The first statement of these two implies that y is in the intersection of the set of feasible allocations and the intersection of upper contours sets for all other agents. But then there exists some y that also lies in this set such that $\bar{x}$ is not a maximum on the set, a contradiction. Now suppose that $\bar{x}$ is Pareto efficient, but does not solve the problem. Then there must exist some m $\in \mathcal{I}$ such that there exists some y in the intersection of the set of feasible allocations and the intersection of upper contours sets for all other agents such that $u^m(y^m) > u^m(\bar{x}^m)$. The fact that y lies in the above mentioned set implies that y is feasible, and weakly preferred to all to $\bar{x}$ by all consumers $i \neq m$. This contradicts that $\bar{x}$ is Pareto-efficient.

Exercise 5.11

Consider a two-consumer, two-good exchange economy. Utility functions and endowments are

$$
\begin{aligned}
u^1(x_1, x_2) &= (x_1 x_2)^2 && \text{and } e^1 = (18, 4),\\
u^2(x_1, x_2) &= ln(x_1) + 2ln(x_2) && \text{and } e^2 = (3, 6).
\end{aligned}
$$

(a) Characterise the set of Pareto-efficient allocations as completely as possible.

(b) Characterise the core of this economy.

(c) Find a Walrasian equilibrium and compute the WEA.

(d) Verify that the WEA you found in part (c) is in the core.

Solution

(a) In a two-person setting, the set of Pareto optimal allocations are all $x^1 \in \mathbb{R}^2$ such that

$$
\begin{aligned}
\max_{x \in \mathbb{R}^n_+} \quad & u^1(x^1)\\
\text{s.t} \quad & u^2(x^2) \geq v, \\
& x^1 + x^2 = e^1 + e^2.
\end{aligned} \tag{M}
$$

The Lagrangian for the maximization problem is

$$\mathcal{L}(x^1, x^2) = u^1(x^1) + \lambda_1(e^1_1 + e^2_1 - x^1_1 - x^2_1) + \lambda_2(e^2_2 + e^2_2 - x^1_2 - x^2_2) + \mu(u^2(x^2) - v).$$

All restrictions bind due to u^1 and u^2 being increasing functions, hence $\lambda_1, \lambda_2, \mu > 0$. The necessary first order conditions for an optimal point are

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^1_1} &= \frac{1}{2x^1_1} - \lambda_1 = 0,\\
\frac{\partial \mathcal{L}}{\partial x^1_2} &= \frac{1}{2x^1_2} - \lambda_2 = 0,\\
\frac{\partial \mathcal{L}}{\partial x^2_1} &= \frac{\mu}{x^2_1} - \lambda_1 = 0,\\
\frac{\partial \mathcal{L}}{\partial x^2_2} &= \frac{\mu}{2x^2_2} - \lambda_2 = 0.
\end{aligned}
$$

Solving each for λ_i, dividing (1) with (2) and (3) with (4), then setting them equal gives us the expression

$$\frac{x^1_2}{x^1_1} = \frac{x^2_2}{2x^2_1}.$$

Substitute the resource equations for x_1^1 and x_2^1, this yields

$$\frac{(e_2 - x_2^2)2x_2^2}{e_1 - x_1^2} = \frac{x_2^2}{2x_1^2},$$

$$x_2^2 = 2e_2\left(\frac{x_1^2}{e_1 + x_1^2}\right).$$

We conclude that all points (x_1^2, x_2^2) such that $x_2^2 = 2e_2x_1^2/(e_1 + x_1^2)$ constitute the set of pareto optimal allocations.

(b) The core is the points that are Pareto optimal and unblocked. Consumer 1 will block any allocation x such that $u^1(x^1) < u^1(e^1)$ and consumer 2 will block any allocation $u^2(x^2) < u^2(e^2)$. This implies that the core is equal

$$C(e) = \{x : x_2^2 = 2e_2\left(\frac{x_1^2}{e_1 + x_1^2}\right) \quad \text{and} \quad u^1(x^1) \geq u(e^1) \quad \text{and} \quad u^2(x^2) \geq u(e^2)\}$$

(c) Each individual i maximizes u^i subject to the value of its endowment y^i. Transformation $f(x) = x^{1/4}$ on u^1 and $g(x) = e^{x/3}$ on u^2, reveals both consumers have Cobb-Douglas utility and therefore demand $x^i(y^i, p)$ equal to

$$x_1^1 = \frac{y^1}{2p_1}. \quad x_2^1 = \frac{y^1}{2p_2}, \quad x_1^2 = \frac{y^2}{3p_1}, \quad x_2^2 = \frac{y^2 2}{3p_2}.$$

Normalize the price vector $p = (1, p_2)$. The vector must clear markets, hence

$$x_1^1 + x_1^2 = e_1^2 + e_1^2,$$

$$\frac{18 + 4p_2}{2} + \frac{3 + 6p_2}{3} = 18 + 3,$$

$$54 + 12p_2 + 6 + 12p_2 = 126,$$

$$p_2 = \frac{66}{24} = \frac{11}{4}.$$

Any αp with $\alpha > 0$ will generate the same equilibrium. Let the new price vector be $p = (4, 11)$, the value of each endowments are $y^1 = 4 \cdot 18 + 11 \cdot 4 = 116$, $y^2 = 4 \cdot 3 + 11 \cdot 6 = 78$. The demand and walrasian equilibrium allocation is

$$x_1^1 = \frac{116}{2 \cdot 4} = 14.5, \quad x_2^1 = \frac{116}{2 \cdot 11} = 5.27, \quad x_1^2 = \frac{78}{3 \cdot 4} = 6.5, \quad x_2^2 = \frac{78 \cdot 2}{3 \cdot 11} = 4.727.$$

(d) The utility at the endowment bundles are

$$u^1(e^1) = (18 \cdot 4)^2 = 5184,$$

$$u^2(e^2) = \ln 3 + 2\ln 6 = 4.6.$$

The utility at the new bundles are

$$u^1(x) = (14.5 \cdot 5.27)^2 = 5839,$$

$$u^2(x) = \ln 6.5 + 2\ln 4.72 = 4.97.$$

The new allocation is therefore unblocked. Its feasible since

$$x^1 + x^2 = (21, 10) \quad \text{and} \quad e^1 + e^2 = (21, 10).$$

The point $x^2 = (6.5, 4.7267)$ is on the contract curve, shown by

$$x_2^2 = 2e_2\left(\frac{x_1^2}{e_1 + x_1^2}\right) = 2 \cdot 10\left(\frac{6.5}{21 + 6.5}\right) = 4.727,$$

and is therefore pareto optimal. We conclude that the walrasian equilibrium allocation point is in the core set.

2 Exercise 5.17

5.17 Consider an exchange economy with two identical consumers. Their common utility function is $u^i(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ for $0 < \alpha < 1$. Society has 10 units of x_1 and 10 units of x_2 in all. Find endowments $\mathbf{e}^1$ and $\mathbf{e}^2$, where $\mathbf{e}^1 \neq \mathbf{e}^2$, and Walrasian equilibrium prices that will 'support' as a WEA the equal-division allocation giving both consumers the bundle (5, 5).

$$e_x^1 + e_x^2 = 10$$
$$e_y^1 + e_y^2 = 10$$

$$u^1(x,y) = x^{\alpha} \cdot y^{1-\alpha}$$

$$MU_x^1 = \alpha \cdot \left(\frac{y^1}{x^1}\right)^{1-\alpha}$$

$$MU_y^1 = (1-\alpha) \cdot \left(\frac{x^1}{y^1}\right)^{\alpha} \quad \Rightarrow MRS^1 = \frac{\alpha}{1-\alpha} \cdot \frac{\left(\frac{y^1}{x^1}\right)^{1-\alpha}}{\left(\frac{x^1}{y^1}\right)^{\alpha}} = \frac{\alpha}{1-\alpha} \cdot \frac{y^1}{x^1}$$

$$\Rightarrow MRS^2 = \frac{\alpha}{1-\alpha} \cdot \frac{y^2}{x^2}$$

$$MRS^1 = \frac{p_x}{p_y} = MRS^2 \quad \Rightarrow x^1 = \frac{\alpha}{1-\alpha} \cdot \frac{p_y}{p_x} \cdot y^1$$

$$\Rightarrow x^2 = \frac{\alpha}{1-\alpha} \cdot \frac{p_y}{p_x} \cdot y^2$$

Budget constraints:

$$10 = e_x^1 + e_x^2 = x^1 + x^2$$

$$10 = e_y^1 + e_y^2 = y^1 + y^2$$

$$p_x \cdot e_x^1 + p_y \cdot e_y^1 = p_x \cdot x^1 + p_y \cdot y^1$$

$$p_x \cdot e_x^2 + p_y \cdot e_y^2 = p_x \cdot x^2 + p_y \cdot y^2$$

Plug in: $p_x \cdot e_x^1 + p_y \cdot e_y^1 = \frac{\alpha}{1-\alpha} \cdot p_y \cdot y^1 + p_y \cdot y^1$

$$\Leftrightarrow p_x \cdot e_x^1 + p_y \cdot e_y^1 = \frac{1}{1-\alpha} \cdot p_y \cdot y^1$$

$$\Leftrightarrow \boxed{y^1 = (1-\alpha) \cdot \left(\frac{p_x}{p_y} \cdot e_x^1 + e_y^1\right)}$$

analog: $p_x \cdot e_x^2 + p_y \cdot e_y^2 = \frac{1}{1-\alpha} \cdot p_y \cdot y^2$

$$\boxed{y^2 = (1-\alpha) \cdot \left(\frac{p_x}{p_y} \cdot e_x^2 + e_y^2\right)}$$

$$\Rightarrow x^1 = \frac{\alpha}{1-\alpha} \cdot \frac{p_y}{p_x} \cdot (1-\alpha) \cdot \left(\frac{p_x}{p_y} \cdot e_x^1 + e_y^1\right) = \boxed{\alpha \cdot \left(e_x^1 + \frac{p_y}{p_x} \cdot e_y^1\right) = x^1}$$

analog: $\boxed{x^2 = \alpha \cdot \left(e_x^2 + \frac{p_y}{p_x} \cdot e_y^2\right)}$

Plug y^1, y^2 into budget constraint:

$$\Rightarrow e_y^1 + e_y^2 = (1-\alpha) \cdot \left(\frac{p_x}{p_y} \cdot (e_x^1 + e_x^2) + e_y^1 + e_y^2\right)$$

$$\Leftrightarrow \frac{1}{1-\alpha}(e_y^1 + e_y^2) - (e_y^1 + e_y^2) = \frac{p_x}{p_y} \cdot (e_x^1 + e_x^2)$$

$$\Leftrightarrow \frac{\alpha}{1-\alpha} \cdot \frac{e_y^1 + e_y^2}{e_x^1 + e_x^2} = \frac{p_x}{p_y}$$

$$\Rightarrow \text{analog: } e_x^1 + e_x^2 = \alpha \cdot \left(e_x^1 + e_x^2 + \frac{p_y}{p_x} \cdot (e_y^1 + e_y^2)\right)$$

$$\Rightarrow \frac{p_y}{p_x} = \frac{1-\alpha}{\alpha} \cdot \frac{e_x^1 + e_x^2}{e_y^1 + e_y^2} \Rightarrow \boxed{\frac{p_y}{p_x} = \frac{1-\alpha}{\alpha}}$$

Plug $\frac{p_y}{p_x}$ in function for x^1, x^2.

$$x^1 = \frac{\alpha}{1-\alpha} \cdot \frac{p_y}{p_x} \cdot y^1 = \frac{\alpha}{1-\alpha} \cdot \frac{1-\alpha}{\alpha} \cdot y^1 = y^1$$

$$x^2 = \frac{\alpha}{1-\alpha} \cdot \frac{p_y}{p_x} \cdot y^2 = \frac{\alpha}{1-\alpha} \cdot \frac{1-\alpha}{\alpha} \cdot y^2 = y^2$$

Given by the exercise : $\boxed{x^1 = y^1 = x^2 = y^2 = 5}$

$$p_x \cdot e_x^1 + p_y \cdot e_y^1 = p_x \cdot x^1 + p_y \cdot y^1$$

$$p_x \cdot e_x^2 + p_y \cdot e_y^2 = p_x \cdot x^2 + p_y \cdot y^2$$

Accordingly, the right side of the two equations above are equal

Set $p = \binom{p_x}{p_y} = \binom{1}{p_y}$ because only relative prices matter

Plug in p and set the left sides equal:

$$e_x^1 + p_y \cdot e_y^1 = e_x^2 + p_y \cdot e_y^2 \qquad \text{use } p_y = \frac{1-\alpha}{\alpha}$$

$$\Rightarrow \quad e_x^1 - e_x^2 = \left(\frac{1-\alpha}{\alpha}\right) \cdot \left(e_y^2 - e_y^1\right)$$

Note that both consumers have to receive a bundle that is equally valuable given the prices p_x and p_y.

Additionally, we know:

$$e_x^1 + e_x^2 = 10 \iff e_x^1 = 10 - e_x^2$$

$$e_y^1 + e_y^2 = 10 \iff e_y^1 = 10 - e_y^2$$

So, for the given problem, the endowments have to lie on the budget line dependent on α (slope $-\frac{\alpha}{\alpha+1}$) going through $(5,5)$.

Plug in: $10 - 2e_x^2 = \left(\frac{1-\alpha}{\alpha}\right)\left(2e_y^2 - 10\right)$

Solve for e_x^2:

$$e_x^2 = \frac{1}{2} \cdot \left(10 - \left(\frac{1-\alpha}{\alpha}\right) \cdot \left(2 \cdot e_y^2 - 10\right)\right) = 5 - \left(\frac{1-\alpha}{\alpha}\right) \cdot \left(e_y^2 - 5\right)$$

$$e_x^1 = 10 - e_x^2 = 5 + \left(\frac{1-\alpha}{\alpha}\right) \cdot \left(e_y^2 - 5\right)$$

$$e_y^1 = 10 - e_y^2$$

Make sure that e_x^2, e_x^1, e_y^1 are ≥ 0

$$e_x^2 = 5 - \left(\frac{1-\alpha}{\alpha}\right)\cdot\left(e_y^2 - 5\right) \overset{!}{\geq} 0$$

$$\Leftrightarrow \quad 5 \geq \left(\frac{1-\alpha}{\alpha}\right)\cdot\left(e_y^2 - 5\right)$$

$$\Leftrightarrow \quad 5\cdot\left(\frac{\alpha}{1-\alpha}\right) \geq e_y^2 - 5$$

$$\Leftrightarrow \quad e_y^2 \leq 5\left(1 + \frac{\alpha}{1-\alpha}\right)$$

$$\Leftrightarrow \quad e_y^2 \leq \frac{5}{1-\alpha}$$

$$e_x^1 = 5 + \left(\frac{1-\alpha}{\alpha}\right)\cdot\left(e_y^2 - 5\right) \overset{!}{\geq} 0$$

$$\Leftrightarrow \quad e_y^2 - 5 \geq -5\cdot\left(\frac{\alpha}{1-\alpha}\right)$$

$$\Leftrightarrow \quad e_y^2 \geq 5\left(1 - \frac{\alpha}{1-\alpha}\right)$$

$$\Leftrightarrow \quad e_y^2 \geq 5\left(\frac{1-2\alpha}{1-\alpha}\right)$$

$$e_y^1 = 10 - e_y^2 \overset{!}{\geq} 0$$

$$\Leftrightarrow \quad e_y^2 \leq 10$$

$\Rightarrow$ WEA exists for $p = \begin{pmatrix} 1 \\ p_y \end{pmatrix}$, $p_y = \frac{1-\alpha}{\alpha}$

under the following condition:

Pick any $e_y^2 \in \left[5\cdot\frac{1-2\alpha}{1-\alpha}, \min\left\{10, \frac{5}{1-\alpha}\right\}\right]$

then e_x^1, e_x^2, e_y^1 are defined as:

$$e_x^1 = 5 + \left(\frac{1-\alpha}{\alpha}\right)\cdot\left(e_y^2 - 5\right)$$

$$e_x^2 = 5 - \left(\frac{1-\alpha}{\alpha}\right)\cdot\left(e_y^2 - 5\right)$$

$$e_y^1 = 10 - e_y^2$$

3 Exercise 5.19

An exchange economy has three consumers and three goods. Consumers' utility functions and initial endowments are as follows: $u^1 = \min(x_1, x_2)$, $e^1 = (1, 0, 0)$, $u^2 = \min(x_2, x_3)$, $e^2 = (0, 1, 0)$, $u^3 = \min(x_1, x_3)$, $e^3 = (0, 0, 1)$. Find a Walrasian eq. and the associated WEA.

For person 1, in the optimum $x_1^1 = x_2^1$, $x_3^1 = 0$

Budget$_1$: $\underbrace{p_1 \cdot e_1^1}_{Y} \overset{!}{=} p_1 \cdot x_1^1 + p_2 \cdot x_2^1 = (p_1 + p_2) \cdot x_1^1 = (p_1 + p_2) \cdot x_2^1$

$$\Leftrightarrow \quad x_2^1 = x_1^1 = \frac{p_1 \cdot e_1^1}{p_1 + p_2} = \frac{p_1}{p_1 + p_2}$$

Analogically: $x_2^2 = x_3^2$, $x_1^2 = 0$, $x_1^3 = x_3^3$, $x_2^3 = 0$

Budget$_2$: $x_2^2 = x_3^2 = \frac{p_2}{p_2 + p_3}$

Budget$_3$: $x_1^3 = x_3^3 = \frac{p_3}{p_1 + p_3}$

$$x_1^1 + x_1^3 = 1 \quad \Leftrightarrow \quad \frac{p_1}{p_1 + p_2} + \frac{p_3}{p_1 + p_3} = 1$$

$$z_1 = \frac{p_1}{p_1 + p_2} + \frac{p_3}{p_1 + p_3} - 1$$

$$z_2 = \frac{p_2}{p_2 + p_3} + \frac{p_1}{p_1 + p_2} - 1$$

$$z_3 = \frac{p_2}{p_2 + p_3} + \frac{p_3}{p_1 + p_3} - 1$$

Walras' Law:

(i) $p_1 \cdot z_1 = p_1 \cdot \left(\frac{p_1}{p_1 + p_2} + \frac{p_3}{p_1 + p_3} - 1 \right) \overset{!}{=} 0 \quad \Leftrightarrow \quad \frac{p_1}{p_1 + p_2} + \frac{p_3}{p_1 + p_3} = 1$

(ii) $p_2 \cdot z_2 = 0 \quad \Leftrightarrow \quad \frac{p_2}{p_2 + p_3} + \frac{p_1}{p_1 + p_2} = 1$

(iii) $p_3 \cdot z_3 = 0 \quad \Leftrightarrow \quad \frac{p_2}{p_2 + p_3} + \frac{p_3}{p_1 + p_3} = 1$

$\boxed{\text{Set } p_1 = 1}$

Use (i),(ii): $$\frac{p_1}{p_1+p_2} + \frac{p_3}{p_1+p_3} = \frac{p_2}{p_2+p_3} + \frac{p_1}{p_1+p_2}$$

$$\Leftrightarrow \frac{1}{1+p_2} + \frac{p_3}{1+p_3} = \frac{p_2}{p_2+p_3} + \frac{1}{1+p_2}$$

$$\Leftrightarrow (p_2+p_3)\cdot p_3 = p_2\cdot(1+p_3)$$

$$\Leftrightarrow p_2 - p_2(1+p_3) = -p_3^2$$

$$\Leftrightarrow p_2(-p_3) = -p_3^2$$

$$\boxed{\Leftrightarrow p_2 = p_3}$$

Plug in (iii): $$\frac{p_3}{p_3+p_3} + \frac{p_3}{1+p_3} = 1$$

$$\Leftrightarrow \frac{1}{2} + \frac{p_3}{1+p_3} = 1$$

$$\Leftrightarrow p_3 = \frac{1}{2} + \frac{1}{2}p_3$$

$$\boxed{\Leftrightarrow p_3 = 1}$$

$$\boxed{\Rightarrow p_2 = 1}$$

$$\Rightarrow p^* = (1,1,1)$$

Plug in Budget$_i$:

$$\Rightarrow \left.\begin{aligned} x^1 &= \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right) \\ x^2 &= \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right) \\ x^3 &= \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right) \end{aligned}\right\}$$ WEA for this economy with prices $p^* = (1,1,1)$

4 Execise 5.21

4.1 Original question

Consider an exchange economy with the two consumers. Consumer 1 has utility function $u^1(x_1, x_2) = x_2$ and endowment $e^1 = (1, 1)$ and consumer 2 has utility function $u^2(x_1, x_2) = x_1 + x_2$ and endowment $e^2 = (1, 0)$.

(a) Which of the hypotheses of Theorem 5.4 fail in this example?

(b) Show that there does not exist a Walrasian equilibrium in this exchange economy.

4.2 Solution

(a) Utility u^1 is not strongly increasing since $u^1(x_0) = u^1(x_1)$ if $x_0 = (1, 1)$ and $x_1 = (0, 1)$. Further, neither utility is strictly quasiconcave as their functional form is linear.

(b) Consider the excess demand for good two $z_2 = \sum x_2^i(p, p \cdot e) - \sum e_2^i$.The functional forms of the utility functions imply that consumer one demands $x_2^1 = \frac{y^1}{p_2}$ of good one and consumer two demands $x_2^1 = \frac{y^2}{p_1+p_2}$ of good two. Thus

$$\begin{aligned} z_2 &= \sum x_2^i(p, p \cdot e) - \sum e_2^i \\ &= \frac{p_2}{p_2} + \frac{p_2}{p_1 + p_2} - 1 - 0 \\ &= \frac{p_2}{p_1 + p_2}, \end{aligned}$$

and consider the excess demand for good one $z_1 = \sum x_1^i(p, p \cdot e) - \sum e_1^i$. The functional forms of the utility functions imply that consumer one demands $x_1^1 = 0$ of good one and consumer two demands $x_2^1 = \frac{y^2}{p_1+p_2}$ of good one. Thus

$$\begin{aligned} z_1 &= \sum x_1^i(p, p \cdot e) - \sum e_1^i \\ &= 0 + \frac{p_1}{p_1 + p_2} - 1 - 1 \\ &= \frac{p_1}{p_1 + p_2} - 2, \end{aligned}$$

when the price of good one is positive:

$$\mathbf{z} = z_1 + z_2 = \frac{p_1 + p_2}{p_1 + p_2} - 2 = -1 \neq 0,$$

and when the price of good one is zero:

$$\mathbf{z} = z_1 + z_2 = \frac{p_2}{p_2} - 2 = -1 \neq 0.$$

Conclude: there does not exist a Walrasian equilibrium in this exchange economy.