Impulse Response Function Analysis

The VAR(1) model in (2, from the Structural Form and Reduced Form VAR document) has a so called Vector Moving Average infinity ($VMA(\infty)$) representation

$$X_t = \mu + \sum_{i=0}^{\infty} A_1^i e_{t-i} = \mu + \sum_{i=0}^{\infty} \phi_i e_{t-i} \tag{1}$$

where $\phi_i = A_1^i$. Impulse response analysis now means that we shall study the effects on y_t and z_t when a unit shock is sent into e_{1t} or into e_{2t} (but not at the same time), i.e. we shall analyze how this shock propagates through the system as time passes. A convenient tool for analyzing this is so called dynamic multipliers (or impact multipliers). In this case they are defined as:

$$\frac{\partial X_{t+s}}{\partial e_t'} = \begin{bmatrix} \frac{\partial y_{t+s}}{\partial e_{1t}} & \frac{\partial y_{t+s}}{\partial e_{2t}} \\ \frac{\partial z_{t+s}}{\partial e_{1t}} & \frac{\partial z_{t+s}}{\partial e_{2t}} \end{bmatrix} = \begin{bmatrix} \phi_{11,s} & \phi_{12,s} \\ \phi_{21,s} & \phi_{22,s} \end{bmatrix} = \phi_s, \quad s = 0, 1, 2, ...$$

To demonstrate further I let[1]

$$A_1 = \begin{bmatrix} -0.0175878 & 0.2828466 \\ 0.3082278 & -0.2086812 \end{bmatrix}$$

For this choice I obtain the following dynamic multipliers

[1] This is a result of estimating a VAR(1) process for West German Income (y_t) and Consumption (z_t). These series can be downloaded from the course web.

$$
\begin{aligned}
\frac{\partial X_t}{\partial e_t'} &= \phi_0 = A_1^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \\
\frac{\partial X_{t+1}}{\partial e_t'} &= \phi_1 = A_1^1 = \begin{bmatrix} -0.0175878 & 0.2828466 \\ 0.3082278 & -0.2086812 \end{bmatrix} \\
\frac{\partial X_{t+2}}{\partial e_t'} &= \phi_2 = A_1^2 = \begin{bmatrix} 8.7491 \times 10^{-2} & -6.3999 \times 10^{-2} \\ -6.9742 \times 10^{-2} & 0.13073 \end{bmatrix} \\
\frac{\partial X_{t+3}}{\partial e_t'} &= \phi_3 = A_1^3 = \begin{bmatrix} -2.1265 \times 10^{-2} & 3.8102 \times 10^{-2} \\ 4.1521 \times 10^{-2} & -4.7007 \times 10^{-2} \end{bmatrix} \\
\frac{\partial X_{t+4}}{\partial e_t'} &= \phi_4 = A_1^4 = \begin{bmatrix} 1.2118 \times 10^{-2} & -1.3966 \times 10^{-2} \\ -1.5219 \times 10^{-2} & 2.1554 \times 10^{-2} \end{bmatrix} \\
&\vdots
\end{aligned}
$$

and so forth (noticing that all elements will tend to zero as s tend to infinity).

Impulse response analysis is now conducted as follows: if we send a unit shock into e_{1t} ($e_{1t} = 1$, $e_{2t} = 0$) we obtain

$$\underbrace{I_2}_{\phi_0}\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\text{shock } e_{1t}} = \begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{matrix}y_t \to y_t\\ y_t \to z_t\end{matrix}$$

which are the instantaneous impacts on y_t and z_t caused by a shock *in* e_{1t}. Furthermore

$$\underbrace{\begin{bmatrix}-0.0175878 & 0.2828466\\ 0.3082278 & -0.2086812\end{bmatrix}}_{\phi_1}\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\text{shock } e_{1t}} = \begin{bmatrix}-1.7588\times10^{-2}\\ 0.30823\end{bmatrix} \qquad \begin{matrix}y_t \to y_{t+1}\\ y_t \to z_{t+1}\end{matrix}$$

which are the impacts on y_t and z_t after one time-period (caused by a shock *in* e_{1t}), and

$$\underbrace{\begin{bmatrix}8.7491\times10^{-2} & -6.3999\times10^{-2}\\ -6.9742\times10^{-2} & 0.13073\end{bmatrix}}_{\phi_2}\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\text{shock } e_{1t}} = \begin{bmatrix}8.7491\times10^{-2}\\ -6.9742\times10^{-2}\end{bmatrix} \qquad \begin{matrix}y_t \to y_{t+2}\\ y_t \to z_{t+2}\end{matrix}$$

are the impacts on y_t and z_t after two time-periods (caused by a shock *in* e_{1t}), and so forth.

Similarly, if we instead send a unit shock into e_{2t} ($e_{1t} = 0$, $e_{2t} = 1$) we obtain

$$\underbrace{I_2}_{\phi_0}\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\text{shock } e_{2t}} = \begin{bmatrix}0\\1\end{bmatrix} \qquad \begin{matrix}z_t \to y_t\\ z_t \to z_t\end{matrix}$$

which are the instantaneous impacts on y_t and z_t caused by a shock *in* e_{2t}. Furthermore

$$\underbrace{\begin{bmatrix}-0.0175878 & 0.2828466\\ 0.3082278 & -0.2086812\end{bmatrix}}_{\phi_1}\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\text{shock } e_{2t}} = \begin{bmatrix}0.2828466\\ -0.2086812\end{bmatrix} \qquad \begin{matrix}z_t \to y_{t+1}\\ z_t \to z_{t+1}\end{matrix}$$

which are the impacts on y_t and z_t after one time-period (caused by a shock *in* e_{2t}), and

$$\underbrace{\begin{bmatrix}8.7491\times10^{-2} & -6.3999\times10^{-2}\\ -6.9742\times10^{-2} & 0.13073\end{bmatrix}}_{\phi_2}\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\text{shock } e_{2t}} = \begin{bmatrix}-6.3999\times10^{-2}\\ 0.13073\end{bmatrix} \qquad \begin{matrix}z_t \to y_{t+2}\\ z_t \to z_{t+2}\end{matrix}$$

are the impacts on y_t and z_t after two time-periods (caused by a shock *in* e_{2t}), and so forth.

Remark. In PC exercise 3 we will replicate the above results.