(c) Compute the expenditure function

$$e(\vec{p},u) \equiv \min_{\vec{x} \in \mathbb{R}^n_+} \vec{p}\cdot\vec{x} \quad \text{s.t.} \quad u(\vec{x}) \geqq u$$

$$\text{or} \quad u - u(\vec{x}) \leqq 0$$

Convert to maximization problem

$$-e(\vec{p},u) \equiv \max_{\vec{x} \in \mathbb{R}^n_+} -\vec{p}\cdot\vec{x} \quad \text{s.t.} \quad u - u(\vec{x}) \leqq 0$$

Because $u(\cdot)$ is continuous and strictly increasing and $\vec{p} >> \vec{0}$, The constraint must be binding

$$\Rightarrow \quad u - u(\vec{x}) = 0.$$

Lagrangian $\mathcal{L} \equiv -\vec{p}\cdot\vec{x} - \lambda\left[u - u(\vec{x})\right]$

$$\frac{\partial \mathcal{L}}{\partial x_i} = -p_i + \lambda \frac{\partial u}{\partial x_i} = 0$$

$$= -p_i + \lambda A \alpha_i x_i^{\alpha_i - 1} \prod_{j \neq i} x_j^{\alpha_j} = 0$$

$$= -p_i + \lambda A \alpha_i x_i^{-1} \prod_{j=1}^{n} x_j^{\alpha_j} = 0$$

$$\Rightarrow \quad p_i = \lambda A \frac{\alpha_i}{x_i} \prod_{j=1}^{n} x_j^{\alpha_j} \qquad i = 1, 2, \cdots, n$$

$$u = u(\vec{x}) = A \prod_{j=1}^{n} x_j^{\alpha_j}$$

$$\Rightarrow \quad p_i = \lambda \frac{\alpha_i}{x_i} u$$

$$\Rightarrow \quad x_i = \lambda \frac{\alpha_i u}{p_i}$$

$$u = A \prod_{j=1}^{n} x_j^{\alpha_j} = A \prod_{j=1}^{n} \left(\frac{\lambda \alpha_j u}{p_j} \right)^{\alpha_j}$$

$$= A \lambda^{\sum \alpha_j} u^{\sum \alpha_j} \prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}$$

$$= A \lambda u \prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}$$

$$\frac{1}{\lambda} = A \prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}$$

$$\Rightarrow \quad \boxed{x_i^h = \lambda \frac{\alpha_i}{p_i} u = \frac{\frac{\alpha_i}{p_i} u}{A \prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j}}}$$

Hicksian demand for good i

$i = 1, 2, \ldots, n$

$$e(\vec{p},u) = \vec{p}\cdot\vec{x}^h$$

$$= \sum_{i=1}^{n} p_i x_i^h$$

$$= \sum_{i=1}^{n} p_i \left(\frac{\frac{\alpha_i}{p_i} u}{A \prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}} \right)$$

$$\boxed{e(\vec{p},u) = \frac{u}{A \prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}} \quad \text{expenditure function}}$$

4.27

monopolist, $q = p^{-\varepsilon}$ $\quad \varepsilon > 1$

constant average costs $\quad C = c \cdot q$

per-unit tax $t > 0$.

$$q^{-\frac{1}{\varepsilon}} = p$$

$$\pi = p \cdot q - C - t \cdot q$$

$$= q^{-\frac{1}{\varepsilon}} \cdot q - c \cdot q - t \cdot q$$

$$\frac{d\pi}{dq} = \left(1 - \frac{1}{\varepsilon}\right) q^{-\frac{1}{\varepsilon}} - c - t \stackrel{?}{=} 0$$

$$\left(1 - \frac{1}{\varepsilon}\right) q^{-\frac{1}{\varepsilon}} = c + t$$

$$\left(1 - \frac{1}{\varepsilon}\right) p = c + t$$

$$\frac{\varepsilon - 1}{\varepsilon} p = c + t$$

$$p^* = \frac{\varepsilon}{\varepsilon - 1}(c + t)$$

$$\frac{dp^*}{dt} = \frac{\varepsilon}{\varepsilon - 1} > 1$$

$\Rightarrow$ The monopolist will increase price by more than the amount of the per-unit tax.

4.15

$$q^j = (p^j)^{-2}\left(\sum_{\substack{i=1\\i\neq j}}^{J}(p^i)^{-1/2}\right)^{-2} \qquad j=1,\dots,J$$

$$c(q) = c\cdot q + k \qquad c>0\ ,\ k>0$$

(a) $p^j \uparrow \Rightarrow (p^j)^{-2}\downarrow \Rightarrow q^j \downarrow$

each firm's demand is negatively sloped.

price elasticity $\varepsilon \equiv \dfrac{\partial q^j}{\partial p^j}\dfrac{p^j}{q^j}$

$$\varepsilon = -2\,(p^j)^{-3}\left(\sum_{\substack{i=1\\i\neq j}}^{J}(p^i)^{-\frac{1}{2}}\right)^{-2}\frac{p^j}{(p^j)^{-2}\left(\sum_{\substack{i=1\\i\neq j}}^{J}(p^i)^{-\frac{1}{2}}\right)^{-2}}$$

$= -2 \qquad \Rightarrow$ price elasticity ε is constant

If two goods are substitutes then if the price of one good falls, the demand for the other good falls also.

$$\frac{\partial q^j}{\partial p^i} = (p^j)^{-2}(-2)\left(\sum_{\substack{i=1\\i\neq j}}^{J}(p^i)^{-\frac{1}{2}}\right)^{-3}\left(-\frac{1}{2}\right)(p^i)^{-\frac{1}{2}-1}$$

$$= (p^j)^{-2}\left(\sum_{\substack{i=1\\i\neq j}}^{J}(p^i)^{-\frac{1}{2}}\right)^{-3}(p^i)^{-\frac{3}{2}} > 0$$

$p^i\downarrow \Rightarrow q^j\downarrow \qquad \Rightarrow$ all goods are substitutes for each other.

(b) Show that if all firms raise their prices proportionately, the demand for any given good declines.

$$q^j = (p^j)^{-2}\left(\sum_{\substack{i=1\\ i\neq j}}^{J} (p^i)^{-\frac{1}{2}}\right)^{-2}$$

$$p^j \longrightarrow \lambda p^j \qquad \lambda > 1$$

$$\text{new } q^j = (\lambda p^j)^{-2}\left(\sum_{\substack{i=1\\ i\neq j}}^{J} (\lambda p^i)^{-\frac{1}{2}}\right)^{-2}$$

$$= \lambda^{-2}(p^j)^{-2}\left(\sum_{\substack{i=1\\ i\neq j}}^{J} \lambda^{-\frac{1}{2}}(p^i)^{-\frac{1}{2}}\right)^{-2}$$

$$= \lambda^{-2}(p^j)^{-2}\left(\lambda^{-\frac{1}{2}}\right)^{-2}\left(\sum_{\substack{i=1\\ i\neq j}}^{J} (p^i)^{-\frac{1}{2}}\right)^{-2}$$

$$= \lambda^{-1}(p^j)^{-2}\left(\sum_{\substack{i=1\\ i\neq j}}^{J} (p^i)^{-\frac{1}{2}}\right)^{-2} < \text{old } q^j$$

since $\lambda > 1$ and $\lambda^{-1} < 1$

(c) Find the long run Nash equilibrium numbers of firms.

$$\pi^j = q^j \cdot p^j - (cq^j + k)$$

$$= (p^j)^{-2}\left(\sum_{\substack{i=1\\ i\neq j}}^{J} (p^i)^{-\frac{1}{2}}\right)^{-2} p^j - c(p^j)^{-2}\left(\sum_{\substack{i=1\\ i\neq j}}^{J} (p^i)^{-\frac{1}{2}}\right)^{-2} - k$$

$$\pi^j = (p^j - c)(p^j)^{-2}\left(\sum_{\substack{i=1\\i\neq j}}^{J}(p^i)^{-\frac{1}{2}}\right)^{-2} - k$$

$$\frac{\partial \pi^j}{\partial p^j} = (p^j)^{-2}\left(\sum_{\substack{i=1\\i\neq j}}^{J}(p^i)^{-\frac{1}{2}}\right)^{-2} + (p^j - c)(-2)(p^j)^{-3}\left(\sum_{\substack{i=1\\i\neq j}}^{J}(p^i)^{-\frac{1}{2}}\right)^{-2}$$

$$\stackrel{?}{=} 0$$

$$\Rightarrow (p^j)^{-2} + -2(p^j - c)(p^j)^{-3} = 0$$

$$1 = \frac{2(p^j - c)}{p^j}$$

$$p^j = 2p^j - 2c$$

$\boxed{2c = p^j}$ profit-maximizing price for every firm j

$$\Rightarrow \text{equilibrium } q^j = (p^j)^{-2}\left(\sum_{\substack{i=1\\i\neq j}}^{J}(p^i)^{-\frac{1}{2}}\right)^{-2}$$

$$= (2c)^{-2}\left(\sum_{\substack{i=1\\i\neq j}}^{J}(2c)^{-\frac{1}{2}}\right)^{-2}$$

$$= (2c)^{-2}\left((J-1)(2c)^{-\frac{1}{2}}\right)^{-2}$$

$$= (2c)^{-2}(J-1)^{-2}\,2c$$

$$\boxed{q^j = (2c)^{-1}(J-1)^{-2}}$$

$$\pi^j = q^j \cdot p^j - (cq^j + k)$$

$$= (p^j - c)q^j \quad - k$$

$$= (2c - c)\frac{1}{2c}\frac{1}{(J-1)^2} = \frac{1}{2(J-1)^2} - k$$

$$-k$$

J increases until $\pi^j = 0$

$$\Rightarrow \frac{1}{2(J-1)^2} - k = 0$$

$$\Rightarrow \frac{1}{2(J-1)^2} = k \quad \Rightarrow \frac{1}{2k} = (J-1)^2$$

$$\Rightarrow \left(\frac{1}{2k}\right)^{\frac{1}{2}} = J - 1$$

$$\Rightarrow \boxed{J = 1 + \left(\frac{1}{2k}\right)^{\frac{1}{2}}}$$

long run Nash equilibrium number of firms

$k\downarrow \Rightarrow RHS\uparrow$

$\Rightarrow J\uparrow$

$k \to 0 \Rightarrow J \to +\infty$

4.8 Cournot oligopoly, $J=2$

$0 \leq c^1 < c^2$ marginal costs

Show that firm 1 will have greater profits and produce a greater share of the market output than firm 2

$$\pi^1 = \left(a - b\sum_{k=1}^{2} q^k\right) q^1 - c^1 q^1$$

$$\pi^2 = \left(a - b\sum_{k=1}^{2} q^k\right) q^2 - c^2 q^2$$

$$\frac{\partial \pi^1}{\partial q^1} = a - b\sum_{k=1}^{2} q^k + q^1(-b) - c^1 \stackrel{?}{=} 0$$

$$a - bq^2 - bq^1 - bq^1 - c^1 = 0$$

$$a - bq^2 - c^1 = 2bq^1$$

$$q^1 = \frac{a - bq^2 - c^1}{2b} \quad \text{firm 1's reaction curve}$$

$$\Rightarrow \quad q^2 = \frac{a - bq^1 - c^2}{2b} \quad \text{firm 2's reaction curve}$$

$$2b\,q^1 = a - bq^2 - c^1 = a - b\left[\frac{a - bq^1 - c^2}{2b}\right] - c^1$$

$$= (a - c^1) - \frac{1}{2}(a - c^2 - bq^1)$$

$$= \frac{a}{2} + \frac{c^2}{2} - c^1 + \frac{b}{2}q^1$$

$$\left(2b - \frac{b}{2}\right) q^1 = \frac{a}{2} + \frac{c^2}{2} - c^1$$

$$q^1 = \frac{\frac{a}{2} + \frac{c^2}{2} - c^1}{2b - \frac{b}{2}} = \frac{\frac{a}{2} + \frac{c^2}{2} - c^1}{b\frac{3}{2}}$$

$$q^1 = \frac{\frac{a+c^2}{3} - \frac{2c^1}{3}}{b} = \frac{a+c^2-2c^1}{3b}$$

$$q^2 = \frac{a-c^2}{2b} - \frac{1}{2}q^1 = \frac{a-c^2}{2b} - \frac{1}{2}\left[\frac{a+c^2-2c^1}{3b}\right]$$

$$q^2 = \frac{\frac{3}{2}(a-c^2) - \frac{1}{2}\left[a+c^2-2c^1\right]}{3b}$$

$$q^2 = \frac{a-2c^2+c^1}{3b}$$

Want to show $q^1 > q^2$ given $c^1 < c^2$

$\Longleftrightarrow \quad \frac{a+c^2-2c^1}{3b} \overset{?}{>} \frac{a+c^1-2c^2}{3b}$

$\Longleftrightarrow \quad c^2 - 2c^1 > c^1 - 2c^2$

$\Longleftrightarrow \quad 3c^3 > 3c^1 \quad \Longleftrightarrow \quad c^2 > c^1$ which is true

Thus $q^1 > q^2$ holds.

$$q^1 = \frac{a + c^2 - 2c^1}{3b} \qquad q^2 = \frac{a + c^1 - 2c^2}{3b}$$

$$p = a - b\sum_{k=1}^{2} q^k = a - b\left[\frac{a + c^2 - 2c^1 + a + c^1 - 2c^2}{3b}\right]$$

$$= a - \left[\frac{2a - c^2 - c^1}{3}\right]$$

$$p = \frac{a}{3} + \frac{c^1}{3} + \frac{c^2}{3} = \frac{a + c^1 + c^2}{3}$$

$$\pi^1 = \frac{a + c^1 + c^2}{3} \cdot \left[\frac{a + c^2 - 2c^1}{3b}\right] - c^1\left[\frac{a + c^2 - 2c^1}{3b}\right]$$

$$= \left[\frac{a + c^2 - 2c^1}{3b}\right]\left[\frac{a + c^1 + c^2}{3} - c^1\right]$$

$$= \left[\frac{a + c^2 - 2c^1}{3b}\right]\left[\frac{a}{3} + \frac{c^2}{3} - \frac{2c^1}{3}\right]$$

$$\pi^1 = \frac{(a + c^2 - 2c^1)^2}{9b} > 0$$

$$\Rightarrow \pi^2 = \frac{(a + c^1 - 2c^2)^2}{9b} > 0$$

profits cannot be negative because a firm can guarantee zero profits by choosing output equal to zero

$$\pi^1 > \pi^2 \iff (a + c^2 - 2c^1)^2 > (a + c^1 - 2c^2)^2$$

$$\tilde{\pi}^1 > \tilde{\pi}^2 \iff a + c^2 - 2c^1 > a + c^1 - 2c^2$$

$$\iff 3c^2 > 3c^1$$

$$\iff c^2 > c^1 \quad \text{which is true.}$$

5.19)

Exchange economy with 3 consumers and 3 goods

$$u^1(x_1, x_2, x_3) = \min(x_1, x_2) \qquad \vec{e}^{\,1} = (1, 0, 0)$$

$$u^2(x_1, x_2, x_3) = \min(x_2, x_3) \qquad \vec{e}^{\,2} = (0, 1, 0)$$

$$u^3(x_1, x_2, x_3) = \min(x_1, x_3) \qquad \vec{e}^{\,3} = (0, 0, 1)$$

Find a Walrasian equilibrium and WEA for this economy

Consumer 1 chooses always $x_1 = x_2$ and $x_3 = 0$

$$\Rightarrow p_1 x_1 + p_2 x_2 + p_3 x_3 = y$$

$$\Rightarrow p_1 x_1 + p_2 x_1 + 0 = y \Rightarrow x_1(p_1 + p_2) = y$$

$$\Rightarrow \boxed{x_1 = \frac{y^1}{p_1 + p_2}}$$ Marshallian demand

$$\boxed{x_2 = \frac{y^1}{p_1 + p_2}} \quad \boxed{x_3 = 0}$$

For consumer 2

$$x_2 = \frac{y^2}{p_2 + p_3} \qquad x_3 = \frac{y^2}{p_2 + p_3} \qquad x_1 = 0$$

For consumer 3,

$$x_1 = \frac{y^3}{p_1 + p_3} \qquad x_3 = \frac{y^3}{p_1 + p_3} \qquad x_2 = 0$$

$$y^1 = p_1 \cdot 1 = p_1$$
$$y^2 = p_2 \cdot 1 = p_2$$
$$y^3 = p_3 \cdot 1 = p_3$$

Since only relative prices matter, set $p_3 = 1$.

Excess demand for good 1 is

$$z_1(\vec{p}) = \frac{y^1}{p_1+p_2} + \frac{y^3}{p_1+p_3} - 1 \overset{?}{=} 0$$

$$= \boxed{\frac{p_1}{p_1+p_2} + \frac{1}{p_1+1} - 1 = 0} \qquad (1)$$

Excess demand for good 2 is

$$z_2(\vec{p}) = \frac{y^1}{p_1+p_2} + \frac{y^2}{p_2+p_3} - 1 \overset{?}{=} 0$$

$$= \boxed{\frac{p_1}{p_1+p_2} + \frac{p_2}{p_2+1} - 1 = 0} \qquad (2)$$

Excess demand for good 3 is

$$z_3(\vec{p}) = \frac{y^2}{p_2+p_3} + \frac{y^3}{p_1+p_3} - 1 \overset{?}{=} 0$$

$$= \boxed{\frac{p_2}{p_2+1} + \frac{1}{p_1+1} - 1 = 0} \qquad (3)$$

$$\frac{p_1}{p_1+p_2} + \frac{1}{p_1+1} = 1 \qquad (1)$$

and

$$\frac{p_1}{p_1+p_2} + \frac{p_2}{p_2+1} = 1 \qquad (2)$$

$$\Rightarrow \quad p_1+1 = \frac{p_2+1}{p_2} = 1 + \frac{1}{p_2}$$

$$\Rightarrow \quad \boxed{p_1 = \frac{1}{p_2}}$$

$$\frac{p_1}{p_1+p_2} + \frac{p_2}{p_2+1} = 1 \qquad (2)$$

and

$$\frac{p_2}{p_2+1} + \frac{1}{p_1+1} = 1 \qquad (3)$$

$$\Rightarrow \quad \frac{p_1+p_2}{p_1} = \frac{p_1+1}{1} = 1 + \frac{p_2}{p_1}$$

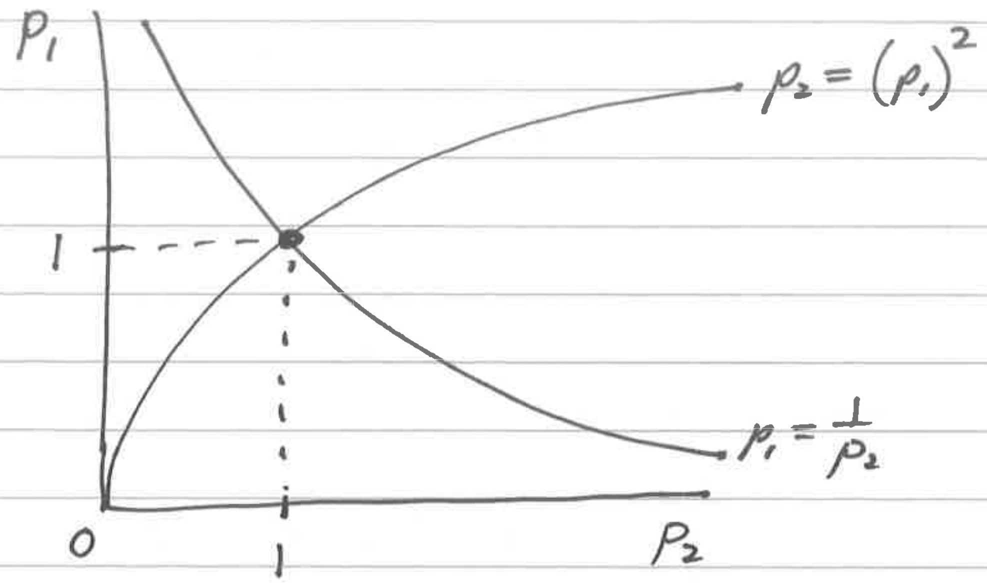
$$\Rightarrow \quad p_1 = \frac{p_2}{p_1} \quad \Rightarrow \boxed{(p_1)^2 = p_2}$$

$$\frac{p_1}{p_1+p_2} + \frac{1}{p_1+1} = 1 \qquad (1)$$

$$\frac{p_2}{p_2+1} + \frac{1}{p_1+1} = 1 \qquad (3)$$

$$\Rightarrow \quad \frac{p_1+p_2}{p_1} = \frac{p_2+1}{p_2} = 1 + \frac{p_2}{p_1} = 1 + \frac{1}{p_2}$$

$$\Rightarrow \frac{p_2}{p_1} = \frac{1}{p_2} \Rightarrow \boxed{p_1 = (p_2)^2}$$



First 2 equations have a unique solution.

$$p_1 = \frac{1}{p_2} = \frac{1}{(p_1)^2} \Rightarrow (p_1)^3 = 1$$

$$\Rightarrow \boxed{p_1 = 1}$$

$$\Rightarrow \boxed{p_2 = \frac{1}{p_1} = \frac{1}{1} = 1}$$

These are the Walrasian equilibrium prices.

$$x_1^1 = \frac{y^1}{p_1 + p_2} = \frac{p_1}{p_1 + p_2} = \frac{1}{1+1} = \frac{1}{2}$$

$$x_2^1 = \frac{1}{2} \qquad x_3^1 = 0 \quad \Rightarrow \vec{x}^1 = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

$$x_1^2 = 0 \qquad x_2^2 = x_3^2 = \frac{y^2}{p_2 + p_3} = \frac{p_2}{p_2 + p_3} = \frac{1}{1+1} = \frac{1}{2}$$

$$\Rightarrow \quad \vec{x}^{2} = \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$x_1^3 = x_3^3 = \frac{y^3}{p_1 + p_3} = \frac{1}{p_1 + 1} = \frac{1}{1+1} = \frac{1}{2} \qquad x_2^3 = 0$$

$$\Rightarrow \quad \vec{x}^{3} = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right)$$

$$\vec{x}^{1} = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right), \quad \vec{x}^{2} = \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right), \quad \vec{x}^{3} = \left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right)$$

is the WEA for this economy.

Note that $\vec{x}^{1} + \vec{x}^{2} + \vec{x}^{3} = \left(\tfrac{1}{2} + \tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{2}\right)$

$$= (1, 1, 1)$$

$$= \vec{e}^{1} + \vec{e}^{2} + \vec{e}^{3}$$

$$= (1, 0, 0) + (0, 1, 0) + (0, 0, 1)$$