EC38002 Econometrics I, Spring 2024

Problem Set 6: Marek Chadim[1]

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1. Given moment equations $g_i(\beta) = Z_i(Y_i - X_i'\beta)$, the criterion can be written as

$$J(\beta) = (Z'Y - Z'X\beta)'W(Z'Y - Z'X\beta).$$

The constrained problem is equivalent to finding the saddle point of the Lagrangian $L = \frac{1}{2}J(\beta) + \lambda'(R'\beta - c)$. The first order conditions are

$$\frac{\partial}{\partial\beta}L(\tilde{\beta}_{\text{cmm}}, \tilde{\lambda}_{\text{cmm}}) = -X'ZW(Z'Y - Z'X\tilde{\beta}_{\text{cmm}}) + R\tilde{\lambda}_{\text{cmm}} = 0 \quad (1)$$

$$\frac{\partial}{\partial\lambda}L(\tilde{\beta}_{\text{cmm}}, \tilde{\lambda}_{\text{cmm}}) = R'\tilde{\beta}_{\text{cmm}} - c = 0 \quad (2)$$

Premultiplying (1) by $R'(X'ZWZ'X)^{-1}$

$$R'(X'ZWZ'X)^{-1}X'ZWZ'Y - R'(X'ZWZ'X)^{-1}X'ZWZ'X\tilde{\beta}_{\text{cmm}} = R'(X'ZWZ'X)^{-1}R\tilde{\lambda}_{\text{cmm}}$$

Recognizing the unrestricted GMM estimator $\hat{\beta}_{\text{gmm}}$, imposing (2) and assuming full rank condition for invertibility to solve for $\tilde{\lambda}_{\text{cmm}}$

$$\tilde{\lambda}_{\text{cmm}} = \left[R'(X'ZWZ'X)^{-1}R\right]^{-1}(R'\hat{\beta}_{\text{gmm}} - c).$$

Substituting this expression into (1) and solving for $\tilde{\beta}_{\text{cmm}}$

$$\tilde{\beta}_{\text{cmm}} = \hat{\beta}_{\text{gmm}} - (X'ZWZ'X)^{-1}R\left[R'(X'ZWZ'X)^{-1}R\right]^{-1}(R'\hat{\beta}_{\text{gmm}} - c).$$

2. (a) The criterion function is

$$J(\beta) = \frac{1}{n}e'X\hat{\Omega}^{-1}X'e$$

rewrite using $\hat{V}_\beta = \left(\frac{1}{n}X'X\right)^{-1}\hat{\Omega}\left(\frac{1}{n}X'X\right)^{-1}$

$$J(\beta) = n\left((X'X)^{-1}X'e\right)'\hat{V}_\beta^{-1}\left((X'X)^{-1}X'e\right) = n(\beta - \hat{\beta})'\hat{V}_\beta^{-1}(\hat{\beta} - \beta),$$

since $(X'X)^{-1}X'e = (X'X)^{-1}X'(y - X'\beta) = \hat{\beta} - \beta$.

(b) Setting the efficient weight matrix from unconstrained estimation the constrained GMM estimator can be written as efficient minimum distance

$$\tilde{\beta} = \hat{\beta} - \hat{V}_\beta R\left(R'\hat{V}_\beta R\right)^{-1}(R'\hat{\beta} - 0)$$

Therefore

$$\begin{aligned} D &= n(\hat{V}_\beta(R'\hat{V}_\beta)^{-1}R'\beta)'\hat{V}_\beta^{-1}(\hat{V}_\beta(R'\hat{V}_\beta)^{-1}R'\beta) = n(\hat{V}_\beta(R'\hat{V}_\beta)^{-1}R'\beta)'R(R'\hat{V}_\beta)^{-1}R'\beta \\ &= n\beta'R(R'\hat{V}_\beta)^{-1}R'\hat{V}_\beta(R'\hat{V}_\beta)^{-1}R'\beta = n(R'\beta)(R'\hat{V}_\beta)^{-1}(R'\beta) = W_n \end{aligned}$$

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3. (a)
```
use AJR2001.dta
gen logmort2 = logmort0*logmort0
ivregress gmm loggdp (risk=logmort0 logmort2)
```

loggdp	Coefficient	std. err.
risk	.7278459	.0900634
_cons	3.336234	.6138203

(b)
```
estat overid
Hansen's J chi2(1) = 3.74788 (p = 0.0529)
```

(c)
```
ivregress 2sls loggdp (risk=logmort0 logmort2)
```

loggdp	Coefficient	Std. err.
risk	.7722554	.1130303
_cons	3.018849	.7434204

The GMM and 2SLS estimates are similar.

4. $Y_0 = 0 \implies \text{Var}[Y_0] = 0$

$$
\begin{aligned}
\text{Var}[Y_1] &= \text{Var}[Y_0 + e_1] = \text{Var}[e_1] = 1, \\
\text{Var}[Y_2] &= \text{Var}[Y_1 + e_2] = \text{Var}[Y_0 + e_1 + e_2] = \text{Var}[e_1] + \text{Var}[e_2] = 2, \\
&\vdots \\
\text{Var}[Y_t] &= \sum_{j=1}^{t} \text{Var}[e_j] = t.
\end{aligned}
$$

The variance is not constant over time. Consequently the series is non-stationary.

5. (a) Set $e_t = 0$ for $t \neq 0$ and set $e_0 = 1$ to recursively calculate $b_j = \frac{\partial}{\partial e_t} Y_{t+j} = \frac{\partial}{\partial e_0} Y_j$.

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= \alpha_1 b_0 = \alpha_1 \\
b_2 &= \alpha_1 b_1 = \alpha_1^2 \\
&\vdots \\
b_j &= \alpha_1 b_{j-1} = \alpha_1^j.
\end{aligned}
$$

(b) $\hat{b}_j(\hat{\alpha}_1) = \hat{\alpha}_1^j$ using the least squares estimator

$$
\hat{\alpha} = \left(\sum_{t=2}^{n} X_t X_t' \right)^{-1} \left(\sum_{t=2}^{n} X_t Y_t \right),
$$

where $X_t = (1, Y_{t-1}, \ldots, Y_{t-2})'$.

(c) By the delta method as $n \to \infty$

$$
\sqrt{n}(\hat{b}_j - b_j) \xrightarrow{d} \mathcal{N}\left(0, j^2 (\hat{\alpha}_1)^{2(j-1)} s^2(\hat{\alpha}_1)\right)
$$

since $\frac{\partial}{\partial \alpha_1} b_j = j \alpha_1^{j-1}$. The 95% asymptotic confidence interval is

$$
\hat{b}_j \pm 1.96 \times j (\hat{\alpha}_1)^{(j-1)} s(\hat{\alpha}_1)
$$

6. (a)
```
use FRED-QD.dta
gen y = 100*(pnfix/L.pnfix-1)
```

(b) Under the assumption of correct specification covariance matrix estimation is identical to the cross- section case using conventional regression methods.

```
reg y L(1/4).y, r
Linear regression                    Number of obs = 231
F(4, 226) = 25.32                     Prob > F = 0.0000
---------------------------------------------------------
      y | Coefficient  std. err.       t     P>|t|
--------+------------------------------------------------
    L1. |    .501609   .0760312     6.60     0.000
    L2. |    .168321   .0702794     2.40     0.017
    L3. |  -.0261832     .06278    -0.42     0.677
    L4. |   -.068269   .0525659    -1.30     0.195
  _cons |   .4911459   .1477817     3.32     0.001
```

(c) Heteroskedasticity and Autocorrelation Consistent/Robust (HAC/HAR) covariance matrix estimators are appropriate under general dependence.

```
newey y L(1/4).y, lag(5)
Regression with Newey-West standard errors
Maximum lag = 5                      Number of obs = 231
F(4,226) = 32.25                      Prob > F = 0.0000
---------------------------------------------------------
      y | Coefficient  std. err.       t     P>|t|
--------+------------------------------------------------
    L1. |    .501609   .0843377     5.95     0.000
    L2. |    .168321   .0692569     2.43     0.016
    L3. |  -.0261832   .0654151    -0.40     0.689
    L4. |   -.068269    .051575    -1.32     0.187
  _cons |   .4911459   .1425101     3.45     0.001
```

(d) Since the standard errors are similar the model is likely to be correctly specified in the sense that it has unforecastable errors and serially uncorrelated regression scores.

(e)
```
matrix alpha = e(b)
gen IRF = 0
replace IRF = 1 in 4
forval j = 5/14 {
    replace IRF = alpha[1, 1] * l.IRF + alpha[1, 2] * l2.IRF
    + alpha[1, 3] * l3.IRF + alpha[1, 4] * l4.IRF in `j'
}
list IRF in 5/14
                                   5. |   .501609 | 10. |  .0164396 |
                                   6. |  .4199326 | 11. | -.0028752 |
                                   7. |  .2688901 | 12. | -.0087824 |
                                   8. |  .1241585 | 13. | -.0095729 |
                                   9. |  .0622994 | 14. | -.0073271 |
```