**5314 Applied Econometric Time Series Spring 2024**

**Problem Set 2: Group Fast & Spurious**[1]

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1. (a)

$$y_t = \alpha + \beta.t + u_t$$
$$u_t = \sum_{j=1}^{4} a_j u_{t-j} + \epsilon_t$$

The given model can be written as:

$$u_t = y_t - \alpha - \beta.t$$

$$\begin{aligned}
y_t - \alpha - \beta.t &= \sum_{j=1}^{4} a_j y_{t-j} + \epsilon_t \\
y_t &= \alpha + \beta.t + \sum_{j=1}^{4} a_j y_{t-j} + \epsilon_t \\
&= \alpha + \beta.t + \sum_{j=1}^{4} a_j (y_{t-j} - \alpha - \beta(t-j)) + \epsilon_t \\
&= \alpha + \beta.t + \sum_{j=1}^{4} a_j y_{t-j} - \sum_{j=1}^{4} a_j \alpha - \sum_{j=1}^{4} a_j \beta.t + \beta \sum_{j=1}^{4} a_j j + \epsilon_t \\
&= \alpha + \beta.t(1 - \sum_{j=1}^{4} a_j) + \sum_{j=1}^{4} a_j y_{t-j} + \beta \sum_{j=1}^{4} a_j j + \epsilon_t \\
y_t - \sum_{j=1}^{4} a_j y_{t-j} &= \alpha + \beta.t(1 - \sum_{j=1}^{4} a_j) + \beta \sum_{j=1}^{4} a_j j + \epsilon_t
\end{aligned}$$

Using the lag operator,

$$\begin{aligned}
y_t(1 - \sum_{j=1}^{4} a_j L^j) &= \alpha + \beta.t(1 - \sum_{j=1}^{4} a_j) + \beta \sum_{j=1}^{4} a_j j + \epsilon_t \\
y_t A(L) &= \alpha + \beta.t(1 - \sum_{j=1}^{4} a_j) + \beta \sum_{j=1}^{4} a_j j + \epsilon_t
\end{aligned}$$

A trend stationary process is one which contains a time trend along with a stationary component. In the given model, $\beta.t$ represents the time trend. The process $u_t$ is said to be stationary if the roots of the lag polynomial lie outside the unit circle, i.e. $(1 - a_1 z_1 + a_2 z^2 - a_3 z^3 - a_4 z^4 = 0)$ lies outside the unit circle. If $u_t$ is stationary, then $y_t$ would be called as a trend stationary process.

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Any sequence that contains one or more characteristic roots that equal unity is called a unit root process. Thus, if $(1-\sum_{j=1}^{4} a_j L^j) = A(L) = 0$, then the process can be called as a unit root process.

(b) Substituting the expression for $u_t$, we get

$$y_t = \alpha + \beta t + a_1 u_{t-1} + a_2 u_{t-2} + a_3 u_{t-3} + a_4 u_{t-4} + \epsilon_t.$$

This expression can be written in terms of lagged $y_t$ by further substituting the lagged $u_t$ terms. This yields

$$y_t = \alpha + \beta t a_1(y_{t-1} - \alpha - \beta(t-1)) + a_2(y_{t-2} - \alpha - \beta(t-2)) + \alpha_3(y_{t-3} - \alpha - \beta(t-3)) + \alpha_4(y_{t-4} - \alpha - \beta(t-4)$$

$$= \alpha(1 - a_1 - a_2 - a_3 - a_4) + \beta t(1 - a_1 - a_2 - a_3 - a_4) +$$

$$+\beta(a_1 + 2a_2 + 3a_3 + 4a_4) + a_1 y_{t-1} + a_2 y_{t-2} + a_3 y_{t-3} + a_4 y_{t-4} + \epsilon_t.$$

To write the expression in first differences, subtract $y_{t-1}$ from both sides. Then progressively add and subtract lagged $y_t$ terms with the coefficients needed to build the lagged differences $\Delta y_t$, this gives

$$\Delta y_t = \gamma_0 + \gamma_1 y_{t-1} + \gamma_2 t + \beta_1 \Delta y_{t-1} + \beta_2 \Delta y_{t-2} + \beta_3 \Delta y_{t-3} + \epsilon_t,$$

where the coefficients are

$$\gamma_0 = \alpha(1 - a_1 - a_2 - a_3 - a_4) + \beta(a_1 + 2a_2 + 3a_3 + 4a_4),$$

$$\gamma_1 = a_1 + a_2 + a_3 + a_4 - 1,$$
$$\gamma_2 = \beta(1 - a_1 - a_2 - a_3 - a_4),$$
$$\beta_1 = -a_2 - a_3 - a_4,$$
$$\beta_2 = -a_3 - a_4,$$
$$\beta_3 = -a_4.$$

(c) In part a) we concluded that a unit root is implied by $1 = a_1 + a_2 + a_3 + a_4$. Looking at the expression for $\gamma_0$, we see that the unit root condition is equivalent to $\gamma_0 = 0$.

(d) If the null hypothesis is not rejected we estimate the model in first differences. The coefficient restriction $\gamma_1 = 0$, implies that $\gamma_2 = 0$ and therefore the time trend drops out too. The model is

$$\Delta y_t = \gamma_0 + \beta_1 \Delta y_{t-1} + \beta_2 \Delta y_{t-2} + \beta_3 \Delta y_{t-3} + \epsilon_t.$$

The parameters that need to be estimated are the coefficients $\gamma_0, \beta_1, \beta_2, \beta_3$ and the variance of the residual $\sigma_\epsilon^2$.

If we reject the null then the series is stationary. The estimation is then done by first regressing $y_t$ with an intercept and linear time trend. Then, using the residuals from the frist regression, we estimate the cyclical part as an AR(4) process. The estimated parameters are now the intercept and time trend $\alpha, \beta$ and the AR coefficients $a_1, a_2, a_3, a_4$ and the variance of the residual in the last regression $\sigma_\epsilon^2$.

(e) For the given model, we calculate $E[y_t]$ in both cases of the model being:
Case 1: Trend stationary (characteristic roots lie outside the unit circle)
The model can be written as:

$$y_t = \alpha_0 + \beta.t + u_t$$
$$E[y_t] = \alpha_0 + \beta.t$$

Case 2: Random walk with a drift (unit root process: characteristic roots equal one)
The model can be written as:

$$y_t = y_0 + \beta.t + \sum_{s=1}^{t} \epsilon_s$$
$$E[y_t] = E[y_0] + E[\beta.t] + E[\sum_{s=1}^{t} \epsilon_s]$$
$$E[y_t] = y_0 + \beta.t$$

No, the model will not revert around the mean i.e. $E[y_t]$ under a unit root condition. This is so because of the following reasons:
i) Unit roots imply non-stationarity. This indicates that they have a perfect memory process i.e. the effects of shocks to the economic process do not die out. As the effects of shocks do not die out, the process does not revert to its previous level.
ii) Non-stationarity implies that the mean changes over time. Therefore, it is difficult to revert to a long-term mean whose value remains the same. Mean reversion, on the other hand, implies that the series tends to move back towards a fixed mean over time, but with a unit root, there is no fixed mean to revert to.

(f) In his influential article "Efficient Capital Markets: A Review of Theory and Empirical Work" (1970), Eugene Fama argued that if the dynamics of stock prices are described by a random walk with drift, the stock markets are (weak form) efficient. Taking the basic pricing equation (m = stochastic discount factor, x = payoffs from the investment),

$$P_t = E[m_{t+1}x_{t+1}]$$

We rewrite it to predict future payoffs and then take a natural logarithm,

$$\frac{P_t}{E[m_{t+1}]} = E[x_{t+1}]$$
$$ln[E[x_{t+1}]] = ln[P_t] - ln[E[m_{t+1}]]$$

We incorporate an error term (which follows a White Noise process) to account for the unexpected return from the market,

$$ln[E[x_{t+1}]] = ln[P_t] - ln[E[m_{t+1}]] + \epsilon_t$$

A random walk with a drift model can be written as below with $\beta$ acting as the drift term.

$$y_t = y_{t-1} + \beta + \epsilon_t$$

Hence, we can observe that the equation derived from the pricing equation fits the random walk with drift model with $\beta$ or the drift term being represented by the negative natural log of the expected stochastic discount factor at time t+1.

(g) Detrending a difference stationary model, which includes a unit root, fails to address the model's core non-stationary issue. The presence of a unit root indicates that statistical properties of the series, such as its mean and variance, do not remain constant over time. Therefore, even after detrending, the model retains its non-stationary nature, undermining the effectiveness of subsequent analyses due to the persistent problem of non-stationarity.

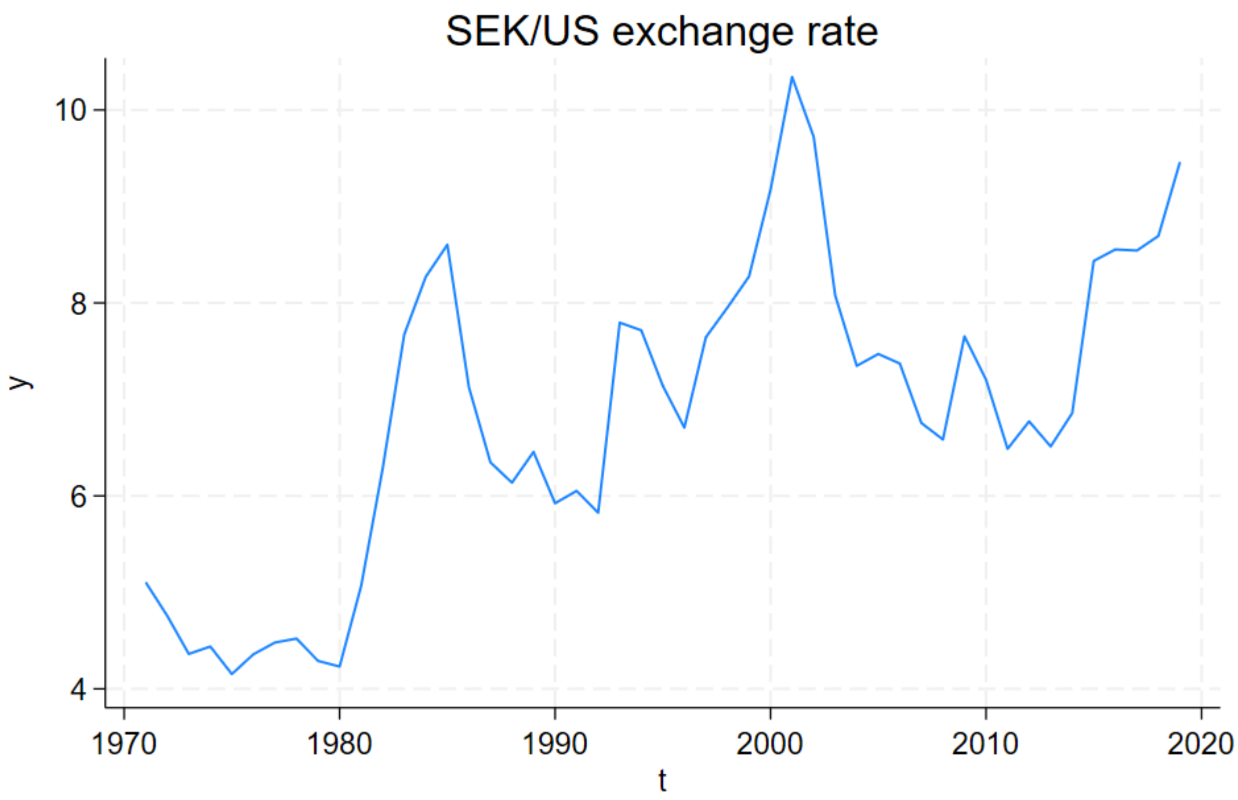
Secondly applying differencing to a trend stationary model does remove the deterministic trend, rendering the series stationary. This method is less efficient than detrending for such models. Differencing can increase the variance of the error term, decreasing the precision of the model's estimates. Additionally, it introduces an MA(1) term in the error, which complicates the model by adding a moving average component that induces additional serial correlation. This complexity can decrease the interpretability the model, which is a main drawback of differencing over detrending in the case of of trend stationary processes.

2. (a) The different cases to which the augmented Dickey–Fuller test can be applied always consider the null hypothesis that the variable has a unit root. They differ in whether the null hypothesis includes a drift term and whether the regression used to obtain the test statistic includes a constant term and time trend. Deciding which case to use involves a combination of theory and visual inspection of the data. If economic theory favors a particular null hypothesis, the appropriate case can be chosen based on that. If a graph of the data shows an upward trend over time, then case with time trend may be preferred. While an exchange rate wouldn't be expected to contain a trend in theory, the time series appears to trend upward. The tests below therefore consider random walk with drift and random walk with drift and a time trend.

```
codebook, compact    N       Mean       Min      Max
------------------------------------------------------
SEKUSD              49   6.851322    4.1531   10.3425
```

To control for serial correlation, we rely on the Akaike information criterion and include lag length $k = 5$ for the difference augmentation. We proceed by computing the Ljung-Box test statistics and conclude that we cannot reject the null hypothesis of no serial correlation, supporting the white noise residuals assumption.

```
arimasoc D.y, maxar(6) maxma(0)
Sample: 1972 thru 2019          Number of obs = 48
Selected (min) AIC: ARMA(5,0)
-----------------------------------------------------
          Model |        LL          df        AIC
----------------+------------------------------------
      ARMA(5,0) |  -46.29799          7    106.596
```




```
corrgram ehat, lags(12) noplot
```

| LAG | AC | PAC | Q | Prob>Q |
|---|---|---|---|---|
| 1 | −0.0011 | −0.0010 | 5.5e−05 | 0.9941 |
| 4 | −0.0900 | −0.0974 | .57985 | 0.9653 |
| 8 | 0.0985 | 0.1252 | 4.1204 | 0.8461 |
| 12 | −0.0890 | −0.2305 | 5.8422 | 0.9238 |

Conducting a test for unit roots we find that in all cases, that the p value against the appropriate Dickey-Fuller distribution is above the 5% critical value. Hence we cannot reject the null hypothesis of the exchange rate series having a unit root. Next we consider the ERS test to see if there is a problem with the ADF test low power. The default selection including trend finds an optimal lag length of 4, at which the test statistic is −3.082, thus the corresponding critical values imply the null hypothesis is rejected at the 10% but not at the 5% level.

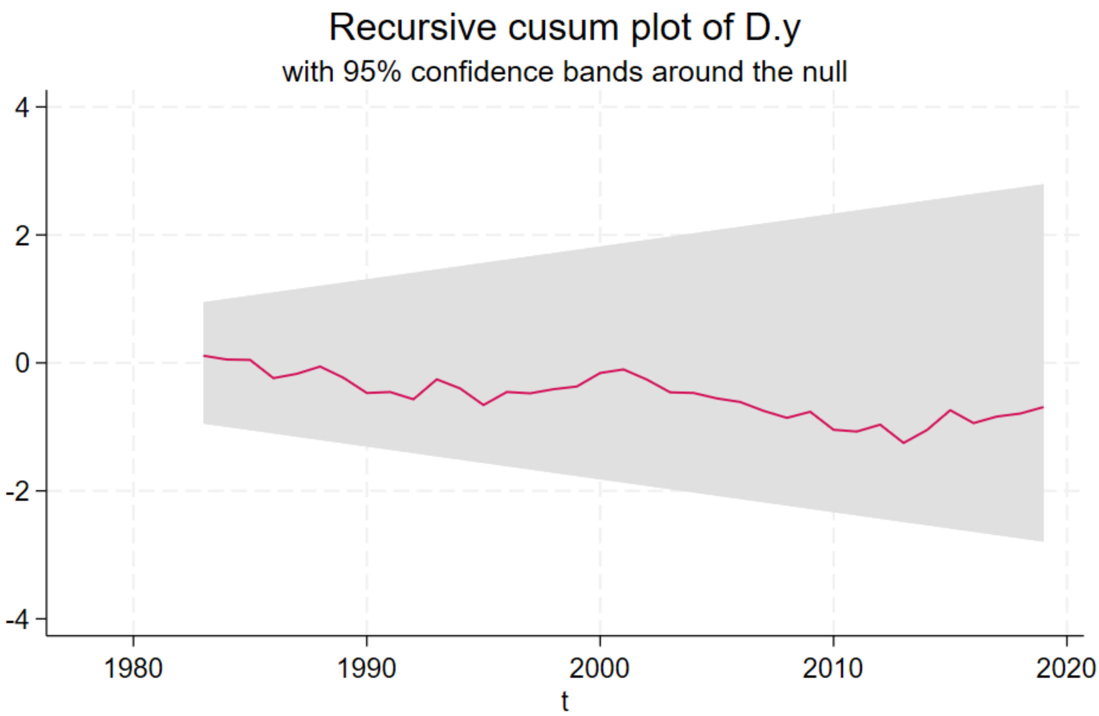
```
dfuller y, lags(5)                               Number of obs  = 43
H0: Random walk without drift, d = 0
```

| | Test statistic | Dickey−Fuller critical value 1% | 5% | 10% |
|---|---|---|---|---|
| Z(t) | −1.691 | −3.628 | −2.950 | −2.608 |

```
MacKinnon approximate p−value for Z(t) = 0.4356.
```

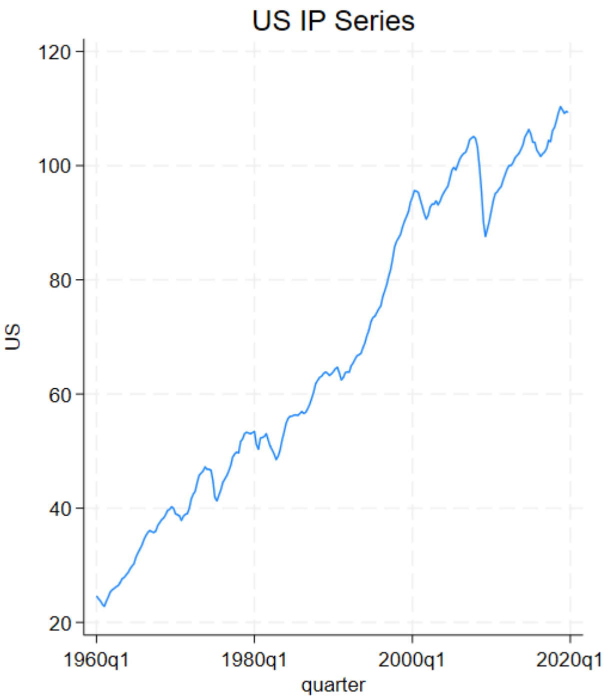
```
dfuller y, lags(5) trend
H0: Random walk with or without drift
                                       Dickey-Fuller
                  Test          -------- critical value ---------
             statistic            1%           5%           10%
---------------------------------------------------------------
 Z(t)           -2.255        -4.214       -3.528        -3.197
---------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.4591.

dfgls y, ers maxlag(6)                  Elliott-Rothenberg-Stock
Opt lag(Ng-Perron seq t)= 4     -------- critical value ---------
 [lags]       DF-GLS tau          1%           5%           10%
---------------------------------------------------------------
     4          -3.082        -3.770       -3.190        -2.890
---------------------------------------------------------------
```

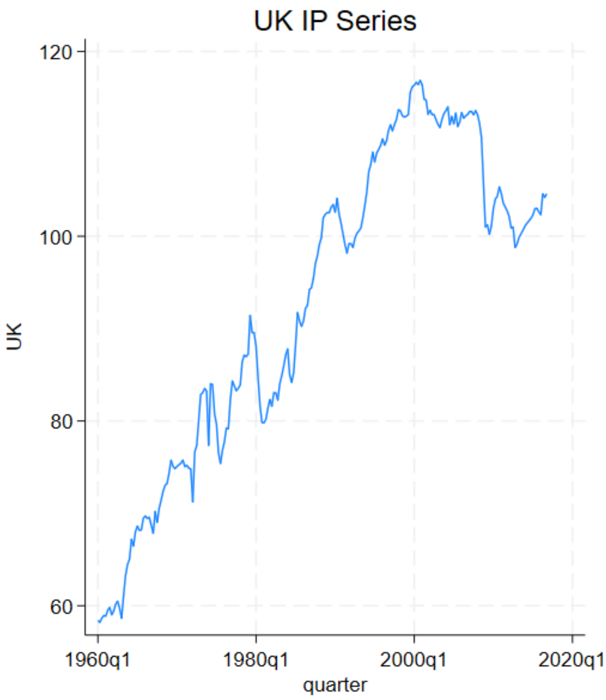
The sample size is likely to be a problem for both tests. 49 total observations may cause both tests to have low power. Structural change may be problematic as well since it would bias the test results towards the null. While the CUSUM test does not indicate a break the relatively short observed period makes it hard to detect one.




(b) We dont't transform the two series since their upward trend appears to be linear. For lag selection we include quarter dummies to account for the trend, and choose augmentation of order $k = 4$ for both series. We find no serial correlation in residuals.

```
codebook, compact    N      Mean        Min         Max
----------------------------------------------------------
UK                 228   92.56085       58.2    116.8667
US                 240   67.77603    22.8189    110.3249
```






```
arimasoc D.US quarter, maxar(6) maxma(0)
Sample: 1960q2 thru 2019q4     Number of obs = 239
Selected (min) AIC:  ARMA(4,0)
```

| Model | LL | df | AIC |
|---|---|---|---|
| ARMA(4,0) | −260.664 | 7 | 535.3281 |

```
corrgram ehatUS, lags(12) noplot
```

| LAG | AC | PAC | Q | Prob>Q |
|---|---|---|---|---|
| 1 | −0.0097 | −0.0097 | .02259 | 0.8805 |
| 4 | 0.0335 | 0.0339 | .34097 | 0.9870 |
| 8 | −0.2036 | −0.2144 | 11.737 | 0.1633 |
| 12 | −0.1349 | −0.1361 | 17.571 | 0.1294 |

```
arimasoc D.UK quarter, maxar(6) maxma(0)
Sample: 1960q2 thru 2016q4       Number of obs = 227
Selected (min) AIC:  ARMA(4,0)
```

| Model | LL | df | AIC |
|---|---|---|---|
| ARMA(4,0) | −397.0982 | 7 | 808.1963 |

```
corrgram ehatUK, lags(12) noplot
```

| LAG | AC | PAC | Q | Prob>Q |
|---|---|---|---|---|
| 1 | −0.0016 | −0.0016 | .0006 | 0.9805 |

```
4           0.0060    0.0061    .12006   0.9983
8           0.0237    0.0232    5.6052   0.6914
12         −0.0354   −0.0502    8.3717   0.7554
```

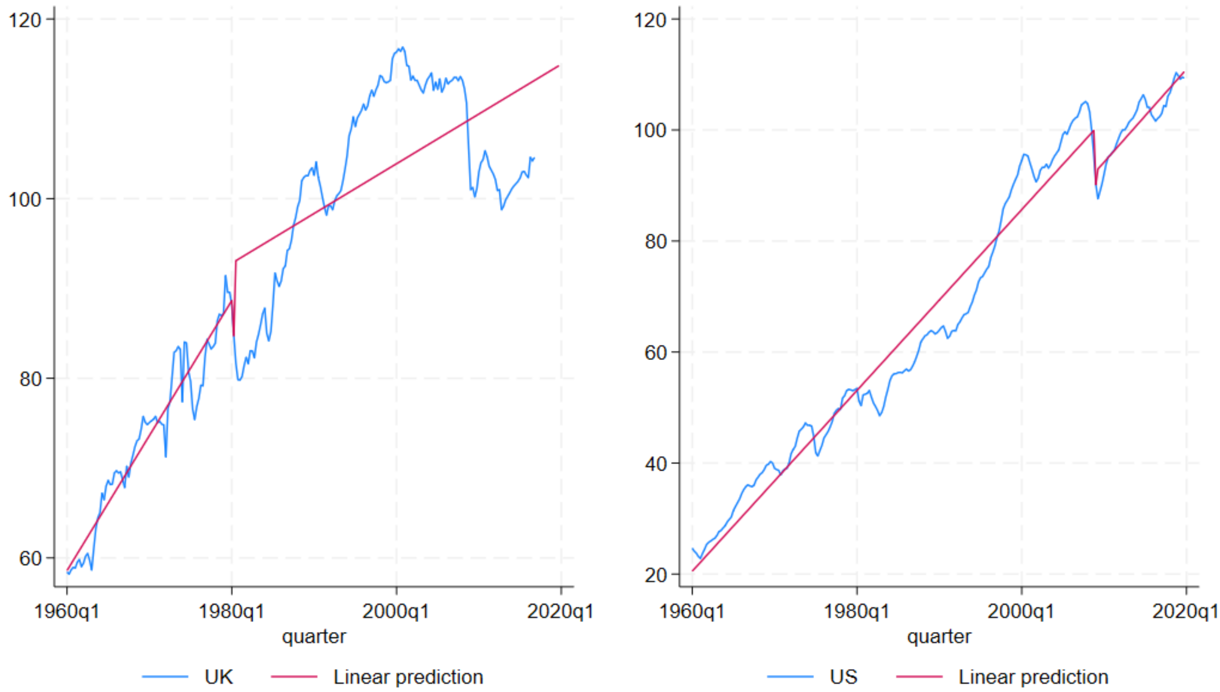
We fail to reject the null hypothesis of a unit root in either series under the trend specifications of the ADF and ERS tests, the latter selecting optimal $k = 4$ as well.

```
dfuller US, trend lags(4)           Number of obs   = 235
                                    Number of lags  =   4

H0: Random walk with or without drift
                                     Dickey−Fuller
                    Test     ———————— critical value ————————
               statistic          1%          5%          10%
─────────────────────────────────────────────────────────────
 Z(t)             −2.509      −3.995      −3.432      −3.132
─────────────────────────────────────────────────────────────
MacKinnon approximate p−value for Z(t) = 0.3236.

dfuller UK, trend  lags(4)            Number of obs   = 223
                                      Number of lags  =   4

H0: Random walk with or without drift
                                     Dickey−Fuller
                    Test     ———————— critical value ————————
               statistic          1%          5%          10%
─────────────────────────────────────────────────────────────
 Z(t)             −1.208      −3.999      −3.434      −3.134
─────────────────────────────────────────────────────────────
MacKinnon approximate p−value for Z(t) = 0.9089.

dfgls US, maxlag(6)                 Number of obs = 233
Opt lag (Ng−Perron seq t) = 4       with RMSE =  .714052
                             ———————— Critical value ————————
 [lags]       DF−GLS tau          1%          5%          10%
─────────────────────────────────────────────────────────────
     4            −2.518      −3.480      −2.898      −2.611
─────────────────────────────────────────────────────────────

dfgls UK, maxlag(6)                   Number of obs = 221
Opt lag (Ng−Perron seq t) = 4         with RMSE = 1.41472
                             ———————— Critical value ————————
 [lags]       DF−GLS tau          1%          5%          10%
─────────────────────────────────────────────────────────────
     4            −1.004      −3.480      −2.901      −2.614
─────────────────────────────────────────────────────────────
```

The Perron test a allows to incorporate a suggested break. We consider first quarters of 1980 (t=81) for UK and 2009 (t=196) for US. Our null is a unit root process with a change in both level and drift and the alternative is trend stationarity with a change in both intercept and slope. We estimate a the alternative regression using pulse, level and trend shift dummies and use the augmented Dickey–Fuller test methodology until the errors appear to be white noise (k=0 for UK and k=1 for US). The t-statistic for the null hypothesis (first lag coefficient = 1) can be compared to the critical values based on the proportion of observations before the break equal to 81/227 = 0.357 for UK and 196/238 = 0.824 for US. Both $t_{UK} = -1.5416082$ and $t_{US} = -3.0063474$ are well above critical values of conventional significance levels. Therefore we cannot reject the null of a unit root for either of the series.

```
reg UK L.UK t dp dl dt                  Number of obs    =        227
------------------------------------------------------------------------
          UK |  Coefficient   Std. err.
-------------+----------------------------------------------------------
         L1. |    .9677869    .0253712
```

```
reg US L.US DL.US t dp dl dt            Number of obs    =        238
------------------------------------------------------------------------
          US |  Coefficient   Std. err.
-------------+----------------------------------------------------------
         L1. |    .9682578    .0105584
```




(c) In addition to using Newey–West standard errors to account for serial correlation in the above mentioned, additional unit root tailored to data specifics exist, for example multiple structural breaks, seasonal unit root and nonlinear dynamics. Panel data tests are potentially more powerful as they pool the estimates from a number separate series and then test the pooled value. Another approach is the KPSS test which posits the null hypothesis of stationarity, with non-stationarity as the alternative.