Statistics

2023 Lectures
Part 3 - Multivariate Distributions and Transformations

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Example of pair of random variables

- in many practical situations we are often required to deal with several (possibly dependent) random variables
- analysis of rv's can be extended to the case of several rv's, i.e., to the analysis of random vectors, of the joint effects

Example 20: Let the experiment consist of three tosses of a coin, and let X = "number of heads in all three tosses" and Y = "number of tails in the last two tosses".

- To simplify the notation, we write $P(X = x, Y = y)$ instead of $P(\{X = x\} \cap \{Y = y\})$ or $P(\{s \in S|X(s) = x\} \cap \{s \in S|Y(s) = y\})$.

Joint discrete and continuous distribution

Definition 9: We say that the pair (X, Y) of random variables has a discrete distribution if there exist finite or countable sets A and B such that

$$P((X, Y) \in A \times B) = \sum_{x \in A, y \in B} P(X = x, Y = y) = 1.$$

Example 20 (cont.): Clearly, we have $A = \{0, 1, 2, 3\}$ and $B = \{0, 1, 2\}$.

Definition 10: Random variables (X, Y) are jointly continuous if there exists a function $f(x, y), x, y \in \mathbb{R}$, such that for every

$$C = \{(x, y) | a \leq x \leq b, c \leq y \leq d\},$$

where $-\infty \leq a < b \leq \infty, -\infty \leq c < d \leq \infty$, we have

$$P((X, Y) \in C) = \iint_{(x,y) \in C} f(x, y) \mathrm{d}(x, y).$$

Fubini theorem for integral

Theorem 8c: (without proof) (Fubini for Lebesgue integral)
Suppose that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R}^m), B \in \mathcal{B}(\mathbb{R}^n)$. If

$$\iint_{A\times B} |f(x,y)|\mathrm{d}(x,y) < \infty,$$

i.e. f is absolutely integrable, then

$$\iint_{A\times B} f(x,y)\mathrm{d}(x,y) = \int_A \left(\int_B f(x,y)\mathrm{d}y\right)\mathrm{d}x = \int_B \left(\int_A f(x,y)\mathrm{d}x\right)\mathrm{d}y.$$

- In Definition 10, f, called joint density function of (X,Y), is nonnegative almost everywhere. Thus the double integral and both iterated integrals are either all infinite or all equal to the same finite number.

Fubini theorem for sums

Theorem 8d: (without proof) (Fubini for sums)
Suppose that $a_{ij}, i, j \in \mathbb{N}$, is absolutely summable, i.e.

$$\sum_{i,j \in \mathbb{N}} |a_{ij}| < \infty.$$

Then

$$\sum_{i,j \in \mathbb{N}} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

- we consider a_{ij}'s nonnegative since they represent probabilities (see Definition 9). Hence the double sum and both iterated sums are either all infinite or all equal to the same finite number.

Role of Fubini theorem and marginal density functions

- With the use of Fubini theorem

$$P((X,Y)\in C)=\iint_{(x,y)\in C} f(x,y)\mathrm{d}(x,y)=$$

$$=\int_a^b\left[\int_c^d f(x,y)\mathrm{d}y\right]\mathrm{d}x=\int_c^d\left[\int_a^b f(x,y)\mathrm{d}x\right]\mathrm{d}y.$$

- In general

$$\iint_C f(x,y)\mathrm{d}(x,y)=\int_{C_2}\varphi(y)\mathrm{d}y=\int_{C_1}\psi(x)\mathrm{d}x,$$

where

$$C_1=\{x|(x,y)\in C\},C_2=\{y|(x,y)\in C\}.$$

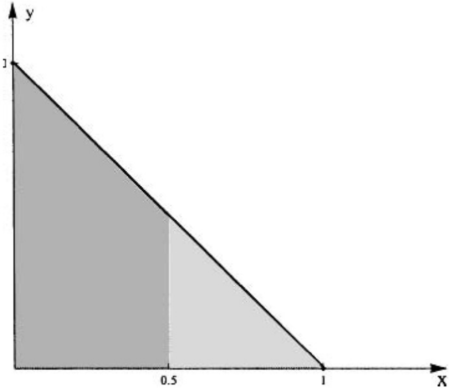
- We call φ and ψ marginal density functions of (X,Y).

Joint distribution function

Example 21: Let

$$f(x,y) = \begin{cases} cx(x+y) & x \geq 0, y \geq 0, x+y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find $P(X \leq 1/2)$.



Joint distribution function

Definition 11: Function $F_{X,Y}(x,y) = F(x,y) = P(X \leq x, Y \leq y)$ is called the cumulative distribution function (cdf) of (X,Y).

- in discrete case $F(x,y) = \sum_{x_i \leq x, y_i \leq y} P(X = x_i, Y = y_i)$;
- in continuous case $F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y) \mathrm{d}(x,y)$.

Theorem 9: The cdf F has the following properties:

i) $\lim_{x,y\to\infty} F(x,y) = 1$.

ii) For every y, $F(\cdot, y)$ is nondecreasing and continuous from the right.

iii) For every x, $F(x, \cdot)$ is nondecreasing and continuous from the right.

iv) $\lim_{x\to-\infty} F(x,y) = 0 \ \ \forall y; \lim_{y\to-\infty} F(x,y) = 0 \ \ \forall x$.

v) For all $x_1 < x_2$ and $y_1 < y_2$

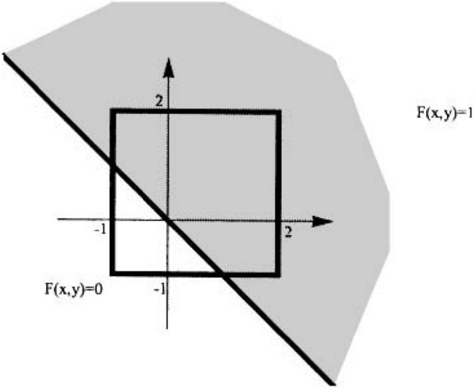
$$F(x_2,y_2) - F(x_2,y_1) - F(x_1,y_2) + F(x_1,y_1) \geq 0.$$

Joint distribution function

Example 22: Let F be defined by the formula

$$F(x,y) = \begin{cases} 1 & \text{for } x+y \geq 0; \\ 0 & \text{for } x+y < 0. \end{cases}$$

Consider e.g. $(x_1, x_2) = (-1, 2)$ and $(y_1, y_2) = (-1, 2)$.



Marginal distributions

- given the bivariate distribution we are able to recover both univariate marginal distributions of X and Y, but not conversely
- discrete case: $A = \{x_1, x_2, \dots\}, B = \{y_1, y_2, \dots\}$ sets of values of X and Y
  Let $p_{ij} = P(X = x_i, Y = y_j)$ then since $\{X = x_i\} = \cup_j(\{X = x_i, Y = y_j\})$

$$P(X = x_i) = \sum_j p_{ij} := p_{i\cdot}, \quad P(Y = y_j) = \sum_i p_{ij} := p_{\cdot j}.$$

- continuous case:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\mathrm{d}y, \; f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\mathrm{d}x$$

are called marginal densities of X and Y.

Independence of marginal rv's

Definition 12: We say that random variables X and Y are independent if events $\{X \in A\}, \{Y \in B\}$ are independent for all sets A, B. The latter holds if and only if

$$F(x,y) = F_X(x) \cdot F_Y(y) \quad \text{for every } x \text{ and } y.$$

- discrete case: $p_{ij} = p_{i\cdot} \cdot p_{\cdot j} \;\; \forall i,j$
- continuous case: $f(x,y) = f_X(x) \cdot f_Y(y)$ for almost all x and y

Example 23:

$$f(x,y) = \begin{cases} cxy^2, & x \geq 0, y \geq 0, x+y \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Determine whether X and Y are independent.

Application of independence

- formula in Definition 12 is useful not only to determine whether or not two variables are independent. One can also use it to find a joint distribution of independent variables X and Y.

Example 24: A man makes two attempts at some goal. His performance X at the first attempt, measured on scale from 0 to 1, is a random variable with density

$$f_1(x) = 12x^2(1-x),\ 0 \leq x \leq 1.$$

His performance Y at the second attempt is independent of X and its density is

$$f_2(y) = 6y(1-y),\ 0 \leq y \leq 1.$$

What is the probability that the man exceeds level 0.75 in the best of the two attempts?

General case

- consider general random vector $X = (X_1, \ldots, X_n)^\top$, i.e. a vector of random variables $X_1, \ldots, X_n$.
- in the discrete case the distribution of X is determined by

$$P_X(x) = P(X = x) = P(X_1 = x_1, \ldots, X_n = x_n),$$

  where $x = (x_1, \ldots, x_n)^\top$ and $\sum_x P_X(x) = 1.$
- in the continuous case the joint density of X is a nonnegative function f_X mapping $\mathbb{R}^n$ to $\mathbb{R}$ such that

$$P(X \in Q) = \int \cdots \int_{x \in Q} f(x_1, \ldots, x_n) \mathrm{d}(x_1 \ldots x_n)$$

  with

$$\int \cdots \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) \mathrm{d}(x_1 \ldots x_n) = 1.$$

General marginal distributions

- the notion of marginal distributions and densities remain the same, except that they can be multivariate as well

Definition 13: Let $X = (Y, Z)$, Y and Z random vectors. In the discrete case the marginal distribution of Y is given by

$$p_Y(y) = \sum_z P(Y = y, Z = z).$$

In the continuous case, the marginal density of Y is given by

$$f_Y(y) = \int \cdots \int_z f(y, z) \mathrm{d}z.$$

Example 25: Consider an experiment with 3 outcomes A, B and C with probabilities α, β and γ, respectively, $\alpha + \beta + \gamma = 1$, and consider X_1 as counts of A among n repetitions and X_2 as counts of B among n repetitions. Find (marginal) distributions of X_1 and X_2.

Multivariate Transformations

- To determine densities of bivariate/multivariate transformations is typically regarded as challenging.
- Follow the steps:

a) To a transformation function φ choose a "companion function" η such that

$$z = \varphi(x, y)$$
$$w = \eta(x, y)$$

can be solved as $x = \alpha(z, w)$ and $y = \beta(z, w)$.

b) Determine the image D of the support C of density f in (z, w)-plane under the transformation.

Multivariate Transformations

c) Find the Jacobian of the transformation, i.e. the determinant

$$J = \begin{vmatrix} \frac{\partial\alpha}{\partial z} & \frac{\partial\alpha}{\partial w} \\ \frac{\partial\beta}{\partial z} & \frac{\partial\beta}{\partial w} \end{vmatrix}$$

d) Determine the joint density of (Z, W), $Z = \varphi(X, Y), W = \eta(X, Y)$ by

$$g(z, w) = \begin{cases} f(\alpha(z, w), \beta(z, w))|J| & \text{for } (z, w) \in D, \\ 0 & \text{otherwise.} \end{cases}$$

e) Compute the density of Z as the marginal density of (Z, W)

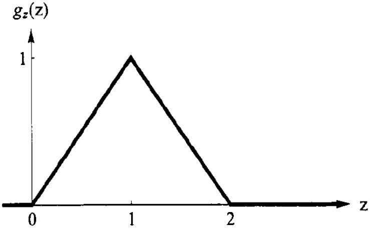
$$g_Z(z) = \int_{-\infty}^{\infty} g(z, w)\mathrm{d}w = \int_{D_z} f(\alpha(z, w), \beta(z, w))|J|\mathrm{d}w,$$

where $D_z = \{w|(z, w) \in D\}$.

Multivariate Transformations

Example 26: (sum of random variables)
Determine the general density function of the sum $Z = X + Y$ of continuous random variables X and Y.

Example 27: (triangular distribution)
Determine the distribution of $Z = X + Y$ when $X, Y \sim U(0, 1)$ independent.



Example 28: (product of random variables)
Determine the general density function of the product XY of continuous random variables X and Y.