Inequality, Household Behavior, & the Macroeconomy
Problem Set 2

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Part 1

1 Risk aversion and precautionary premium

In the lecture, we saw that in the case of the CRRA utility function, risk aversion is lower than prudence. In this exercise, you need to understand the economic intuition of this statement, through an example.

Utility comes from consumption now and in the next period. Income now is 1, but income in the next period (y) is uncertain, with mean 1, i.e. $\mathbb{E}[y] = 1$. This is because your salary depends on the performance of your firm. You can save, but the interest rate is 0. Per-period utility function is of the CRRA kind, and there is no impatience. Therefore, you need to maximize

$$u(c_0) + \mathbb{E}[u(c_1)]$$

where $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, subject to the constraints

$$c_0 = 1 - s$$
$$c_1 = y + s$$

1. Suppose you figure out that your optimal saving level is s^*, therefore you expect $\mathbb{E}[u(y + s^*)]$ utility in the next period. However, your boss offers you an insurance policy: instead of your risky future income (y), you can get $1 - \tau$ for sure (τ can be interpreted as the price of this insurance). From now on suppose that
$$u(1 - \tau + s^*) > \mathbb{E}[u(y + s^*)]$$
holds. Should you take this insurance?

Based on the above, at savings rate $s*$ we have

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$$U^*_{uninsured} = u(1 - s^*) + \mathbb{E}(u(y + s^*))$$

$$U^*_{insured} = u(1 - s^*) + u(1 - \tau + s^*)$$

Since we assume $u(1 - \tau + s^*) > \mathbb{E}(u(y + s^*))$, $U_{insured} > U_{uninsured}$ and you should take the insurance.
Note that even if s^* is not the optimal savings rate when insured, for the optimal savings rate s^o, the utility from being insured $U^o_{insured} > U^*_{insured}$ by definition of optimality, and you should still choose the insurance.

2. What is the relation between τ and the risk premium?

The general espression for the risk premium is

$$u(\mathbb{E}[x] - RP) = \mathbb{E}[u(x)]$$

Considering this for the consumption in period 1:

$$u(\mathbb{E}[c_1] - RP) = \mathbb{E}[u(c_1)]$$

$$u(1 + s^* - RP) = \mathbb{E}[u(y + s^*)]$$

Since from above we have $u(1 - \tau + s^*) > \mathbb{E}(u(y + s^*))$, then we can say $u(1 - \tau + s^*) > u(1 + s^* - RP)$.
Then

$$\frac{(1 + s^* - \tau)^{1-\gamma} - 1}{1 - \gamma} > \frac{(1 + s^* - RP)^{1-\gamma} - 1}{1 - \gamma}$$

$$\iff \tau < RP$$

3. If you take the insurance, how should you change your optimal saving level relative to s^*? Increase, decrease, or it is impossible to tell without knowing more?

We will show that the optimal savings level under insurance, denoted $\hat{s}$ is smaller than s^*.
First, note that utility under the insured case is

$$U_{ins} = \frac{(1 - \hat{s})^{1-\gamma} - 1}{1 - \gamma} + \frac{(1 - \tau + \hat{s})^{1-\gamma} - 1}{1 - \gamma}$$

Which agents maximize by choosing $\hat{s}$ such that

$$\frac{\partial U_{ins}}{\partial \hat{s}} = (1 - \hat{s})^{-\gamma} - (1 - \tau + \hat{s})^{-\gamma} = 0 \implies \hat{s} = \frac{\tau}{2}$$

Next, in the uncertain case, we have from the Euler equation that

$$u'(c_0) = E[u'(c_1)] \iff (1 - s^*)^{-\gamma} = E[(y + s^*)^{-\gamma}]$$

And for the precautionary premium we have the formula

$$u'(E[y+s^*]-PP) = E[u'(y+s^*)] \iff (1+s^*-PP)^{-\gamma} = E[(y+s^*)^{-\gamma}]$$

Thus from the above two expressions we can conclude that

$$(1-s^*)^{-\gamma} = (1+s^*-PP)^{-\gamma} \implies PP = 2s^*$$

We also know that

$$PP = -\frac{u'''(E[y+s^*])}{u''(E[y+s^*])}\frac{var(y+s^*)}{2}$$

Thus (additionally noting $var(y+s^*) = var(y)$)

$$-\frac{-(-\gamma-1)(1+s^*)^{-\gamma-2}}{-\gamma(1+s^*)^{-\gamma-1}}\frac{var(y)}{2} = 2s^* \implies \frac{var(y)}{2} = \frac{2s^*(1+s^*)}{\gamma+1}$$

Recall from question 1.2 that $\tau < RP$. Then:

$$RP = -\frac{u''(E[y+s^*])}{u'(E[y+s^*])}\frac{var(y+s^*)}{2} > \tau$$

Which implies that

$$-\frac{\gamma(1+s^*)^{-\gamma-1}}{-(1+s^*)^{-\gamma}}\frac{var(y)}{2} > \tau \implies \frac{\gamma}{(1+s^*)}\frac{var(y)}{2} > \tau$$

Using our previous expression for $\frac{var(y)}{2}$, we have

$$\frac{\gamma}{(1+s^*)}\frac{2s^*(1+s^*)}{\gamma+1} > \tau \implies s^*\frac{\gamma}{1+\gamma} > \frac{\tau}{2}$$

Recall that $\hat{s} = \frac{\tau}{2}$. Thus $s^*\frac{\gamma}{1+\gamma} > \hat{s}$. Since we have $\gamma > 0$, then $\frac{\gamma}{1+\gamma} < 1$. Thus we can say that $\hat{s} < s^*$, meaning that in the insured case, the optimal level of savings is lower than in the uncertain case. This indicates that some of the consumers' savings in the uncertain case is precautionary in nature.

Part 2

1. After loading 30_*ageprofile.jl* with *include*, simulate the average paths of income, consumption and wealth over age in the three economies. (You can use directly the *solve_simul* function for this.) Explain the economic intuition behind the patterns you observe. In particular, discuss how the introduction of borrowing constraints and uncertainty affect the results. In addition, explain how the new income profile changes findings relative to the flat income profile used in class.

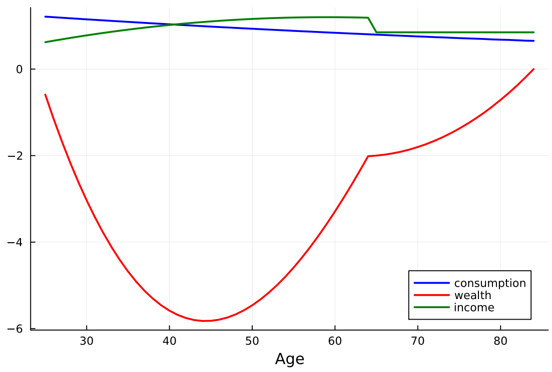

Figure 1: Economy A

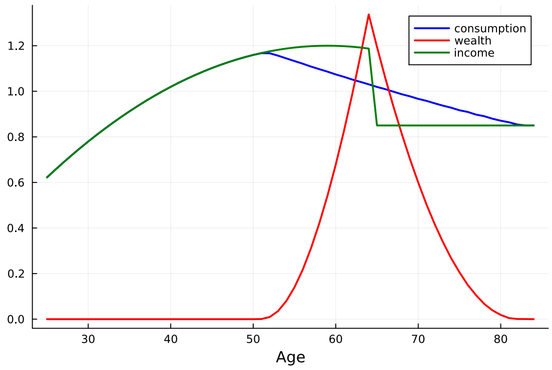

Figure 2: Economy B

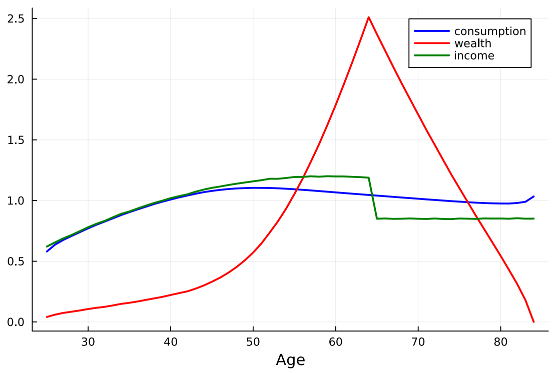

Figure 3: Economy C

Borrowing Constrains
Figures 1 to 3 show the average paths of income, consumption, and wealth over age in the three economies. Figure 1 simulates the paths with a low borrowing limit (i.e., agents are allowed to borrow a multiple of their income), while figures 2 and 3 simulate economies not allowing for borrowing. Therefore, the average wealth path in the first economy is negative for most of the average agent's life as they borrow in order to achieve a smooth, slightly downward sloping (due to relative impatience, $\beta(1+r) < 1$) consumption path. In contrast, the agents in the latter two models are unable to save, and thus consume their

income in the beginning, then are able to build up wealth by accumulating saved income in preparation for retirement.

Uncertainty
Model C, in contrast to the model B, introduces income uncertainty. On average, agents appear to start consuming less than their income, thus building up wealth, significantly earlier than their income-certain counterparts in figure 2. Intuitively, if income in future periods is uncertain, agents will start saving for retirement earlier than under full income certainty, which can be seen in the simulated data. Further, comparing the wealth in models B and C, we observe that, on average, agents in model C have higher wealth compared to agents in model B of the same age. This can be explained by precautionary savings due to income uncertainty (as the expected value of income is the same across the two models).

Relation to flat income profile
In comparison to the flat income profile that we saw in class, in figure 1 agents are also able to smooth their consumption fully, but require a kinked wealth path to achieve it because the negative income shock of retirement causes them to start repaying their debt earlier. Setting the borrowing limit to zero makes all the agents with a flat income profile hand to mouth consumers. These consumers have no incentive to save as they do not need to smooth consumption around the income shock of retirement, unlike the consumers in figures 2 and 3.

2. Simulate all three models, and for each, perform the following exercise: for each simulated individual, compute their correlation between consumption and income changes over their working age (i.e. ignoring everything after period 40). How do the distributions of the obtained correlations compare across the three models? Discuss the economic intuition.

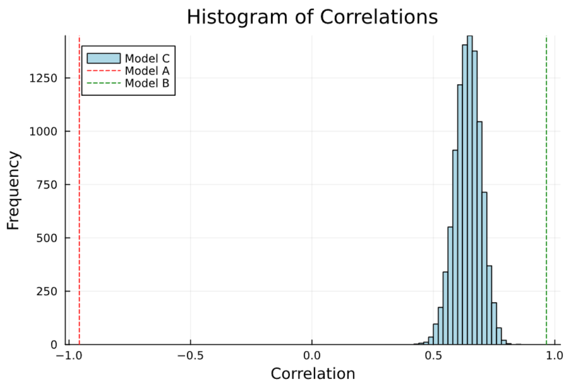

Figure 4: Correlations between changes in consumption and changes in income

Figure 4 illustrates the correlation between consumption and income changes over their working age for individuals in the three models. First, we note that models A and B only have one value each, as all individuals in these economies are identical (since there is no income uncertainty or other variation). In model C, income uncertainty yields different outcomes for different agents.
In model A, we find that the correlation between changes in income and changes in consumption is close to -0.96. Looking at figure 1, we see that as the relatively impatient

consumers are not borrowing constrained, they are able to smooth their consumption to slightly decrease over time independent of their income path. The income path, however, turns out to be on average increasing over their lifetime. This results in an average negative correlation very close to 1.
In model B, we find a correlation of approximately 0.97. Figure 2 shows why the correlation might be so high: in this model, consumers are hand-to-mouth from ages 25 to just after age 50, during which time the correlation between income growth and consumption growth is 1. After this point, income falls sharply due to retirement and then remains constant, while consumption trends downward, explaining why the final correlation is less than 1.
In Model C, we find individual-level correlations are roughly normally distributed around 0.65. Looking at Figure 3, consumers' consumption roughly follows their income until age 45 (with the exception of some precautionary saving) after which consumption begins to decrease as they save for retirement. Thus on average, positive income shocks are accompanied by positive, but smaller consumption shocks, while some of the income is instead saved, resulting in a lower correlation than in model B. Variation in the correlation is a result of the heterogeneity caused by the different income paths that agents in the simulation are faced with.

3. Comparing the three models regarding
   - the relation between average income and consumption paths over age;
   - and the individual-specific correlations you computed in point 2,

   which of the three models seems to fit best the stylized facts in the data?

Relation between average income & consumption: The three estimated models show somewhat different average income and consumption paths. Model A, allowing for borrowing, has a decreasing consumption over time, while income grows up to retirement age, before dropping and remaining on a constant level. For models B and C, not allowing for borrowing, consumption overlaps with income up to a certain point, where the agents start to consume less (relative impatience) and build wealth. Introducing income uncertainty in model C, we observe that agents start saving (i.e., consuming less than their income) at an earlier point in time, thus building up more wealth, than agents in model B.
Correlations & real-world data: In real-world data, aggregate consumption appears to follow aggregate income closely, while on the individual level, the connection between the two is not observed. Comparing the correlations in figure 4, it is apparent that, in our simulated data for model C, changes in consumption are correlated with changes in income, even on the individual level, while models A & B show (almost) perfect negative and positive correlations. Therefore, model C fits the stylized facts in the data the best.

Part 3

1. Write a code that (keeping the defaults for all other economic parameters) computes the budget-balancing level of benefit for any given tax rate τ. You might want to make a copy of 31_$infinite_horizon.jl$ and amend it at the right places[1], following the steps below:

Please refer to .jl file for our solution along with comments.

2. Choose a range/vector of various candidates for τ between 0 and 1. (Computing results for many τs might take long, but use at least 10 candidates so that you can see patterns. Using more points close to 0 might make sense if more action seems to be going on there, based on your finding below.) Compute the corresponding b for each of these τs and comment on your findings.

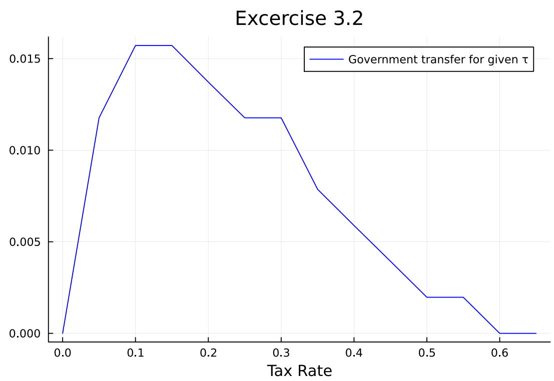

Figure 5: Tax Rate vs Budget-Balancing Transfer Size

Figure 5 shows the estimated government transfers (b) for an array of tax rates (τ). The values of τ were chosen at 5% intervals such that $\tau \in [0, 0.65]$. We observe that the transfer size increases in τ up to a 15% tax rate, then decreases continuously until until a tax rate of 60%, from where on the transfer will be zero. This implies that if the government taxes wealth to high, agents in our simulated economy are not incentivized to build up wealth as it's all taxed away.

3. For the τ candidates in 2, compute average wealth, the Gini index of wealth, and the share of wealth held by the richest 1% in the implied steady-state distribution. (To fight sampling errors, you might want to compute these statistics over all individuals AND over many time periods after the burn-in period.) Discuss.

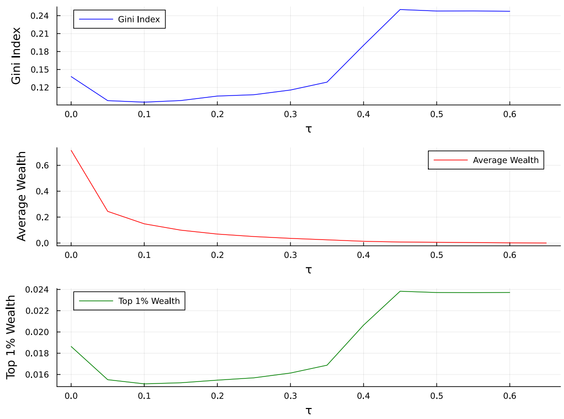

Figure 6: Illustrated solution for question 3.3

Figure 6 illustrates the Gini indices, average individual wealth, and share of wealth held by the richest 1% individuals in the economy for the same array of tax rates as described in 3.2. Where the Gini index is given by

$$G = \frac{1}{n}\left(n+1-2\left(\frac{\Sigma_{i=1}^{n}\left(n+1-i\right)y_i}{\Sigma_{i=1}^{n}y_i}\right)\right)$$

where n is the number of individuals in the economy, y_i is the non-decreasing array of wealth, and i is the index for the ith individual.[1]

The Gini index is higher with higher tax rates because, with a uniform tax rate τ, only wealthier individuals find it worthwhile to pay the tax in order to smooth consumption, and thus build up wealth more than poorer agents. Average wealth, as already implied in the previous part, decreases in τ and reaches zero for $\tau \geq 0.6$ as, given the high tax rate, a savings motive is no longer present. Lastly, the share of wealth held by the top 1% of the population remains fairly constant, however, showing a slight increase with tax rate (for the same reasons that the Gini index rises).

4. For the τ candidates in 2, compute the value of holding 1 unit of cash-on-hand. Comment on your results.

[1]Note that the .jl file submitted along with this exercise also contains a solution using the suggested function in *31_Wealth_indequality.jl*, yielding a similar result.

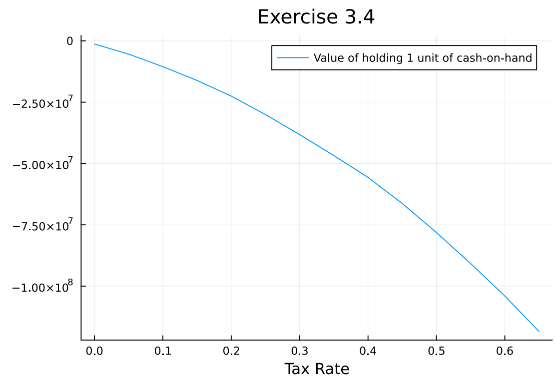

Figure 7: Illustrated solution for question 3.4

Figure 7 shows the solution value functions, for all ages, corresponding to holding 1 unit of cash-on-hand (the expected amount of exogenous income under the default parameterization) against an array of 14 evenly spaced tax rates $\tau \in [0, 0.65]$ and corresponding budget balancing benefits. We can see the value decreases monotonically in τ with a concave shape.

Intuitively, this indicates that the more agressive the tax-and-transfer scheme is, the lower the value of cash-on-hand for the agents. As τ increases, agents are disincentivized from saving and their ability to use a given amount of cash-on-hand to play their utility-maximizing strategy (e.g. to consumption smooth) is increasingly limited. Thus we see the value of cash-on-hand decrease in τ.