Systems of differential equations

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Systems of differential equations

- cookbook approach
- systems of **two equations**, **first order**, **linear** and **autonomous**
- graphical analysis
- fixed points (equilibria)
- matrix notation
- exact solving
- stability analysis

Definition

System of two first-order differential equations

$$\frac{dx}{dt} = \dot{x} = f(x, y, t) \tag{1}$$

$$\frac{dy}{dt} = \dot{y} = g(x, y, t) \tag{2}$$

Solution of this system is a pair of functions $x(t)$ and $y(t)$

It is often necessary to impose initial condition, which is in the form:

$$x_0 = x(t_0) \tag{3}$$

$$y_0 = y(t_0) \tag{4}$$

Simplifications

Autonomous:

$$\dot{x} = f(x, y) \tag{5}$$
$$\dot{y} = g(x, y) \tag{6}$$

allows for graphical analysis

Linear, with constant coefficients:

$$\dot{x} = ax + by + e \tag{7}$$
$$\dot{y} = cx + dy + f \tag{8}$$

allows (us) to solve the system
most important version for applications in economics

Moreover **homogenous**:

$$\dot{x} = ax + by \tag{9}$$
$$\dot{y} = cx + dy \tag{10}$$

Example #1 (I)

$$\dot{x} = 2x \tag{11}$$

$$\dot{y} = y \tag{12}$$

$$x(0) = 2 \quad and \quad y(0) = 3 \tag{13}$$

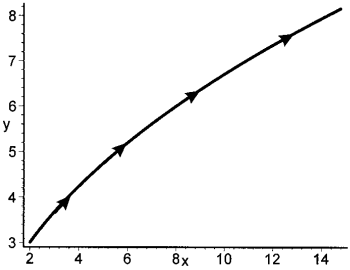
- cheating - it is no system, just two equations
- we can solve

$$x(t) = 2e^{2t} \quad and \quad y(t) = 3e^{t} \tag{14}$$

We can now elimnate variable t (because the original 'system' is autonomous):

$$y = \sqrt{\frac{9x}{2}} \tag{15}$$

Example #1 (II)



- graphical representation of solution of previous example
- this curve is called **trajectory, path, orbit**
- (x, y) plane is **phase plane**
- for autonomous system there is uniquely defined trajectory for given initial condition

Fixed point - equilibrium

Considering autonoumous system:

$$\dot{x} = f(x,y) \tag{16}$$

$$\dot{y} = g(x,y) \tag{17}$$

The fixed point (x^*, y^*) is defined by:

$$f(x^*, y^*) = 0 \tag{18}$$

$$g(x^*, y^*) = 0 \tag{19}$$

This obviously implies:

$$\dot{x} = \dot{y} = 0 \tag{20}$$

For autonomous system fixed point does not change over time (does not hold generally). Recall homogenous system:

$$\dot{x} = ax + by \tag{21}$$

$$\dot{y} = cx + dy \tag{22}$$

One fixed point is (always for homogenous) $(x^*, y^*) = (0, 0)$

Stability analysis - definitions (I)

Our main interest is stability of fixed points - i.e. whether the equilibrium is attracting the solutions or repelling them, however not so simple:

Equilibrium is said to be **stable** or **attracting** (slightly misleading) if any solution that appears (e.g. starts) 'close to' fixed point remains 'close'.

More properly: Fixed point is said to be stable if for any $\varepsilon > 0$ exists δ such that any solution in δ-neighbourhood around fixed point never leave ε-neighbourhood. (ε-neighbourhood is ball with center (x^*, y^*) and radius ε).

Remark: This definition does not say that solution must converge (run to) fixed point, it only has to stay 'close to' fixed point and never leave.

Stability analysis - definitions (II)

A fixed point that is not stable is said to be **unstable** or **repelling**.

A fixed point is **asymptotically stable** if it is stable and moreover solution approaches (x^*, y^*) for $t \to \infty$.

Remarks:

- System may have several equlibria that may be of different types
- Equlibrium is either stable or unstable
- Solution close to given fixed point does not necessarily approach this equlibirum even if the equilibrium is stable
- previous holds only for asymptotically stable equilibrium
- equlibrium that is stable but not asymptotically stable is sometimes called **limit cycle**

Stability analysis - definitions (III)

If fixed point (x^*, y^*) is asymptotically stable and moreover if every trajectory (for any initial condition) approaches this fixed point it is said to be **asymptotically stable**.

Some remarks:

- there is no more than one trajectory trhrough any point in the phase plane
- fixed point is never reached in finite time (except if initial point is already fixed point)
- no trajectory can cross itself unless it is a close curve (periodic solution)

Consider following system:

$$\dot{x} = x - 3y \tag{23}$$

$$\dot{y} = -2x + y \tag{24}$$

$$x_0 = 4 \quad \textit{and} \quad y_0 = 5 \tag{25}$$

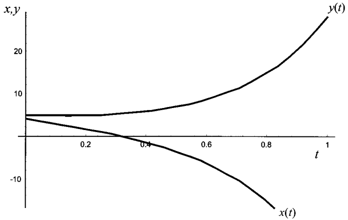
Solution (black box):

$$x(t) = \frac{8e^{(1-\sqrt{6})t} + 5\sqrt{6}e^{(1-\sqrt{6})t} + 8e^{(1+\sqrt{6})t} - 5\sqrt{6}e^{(1+\sqrt{6})t}}{4} \tag{26}$$

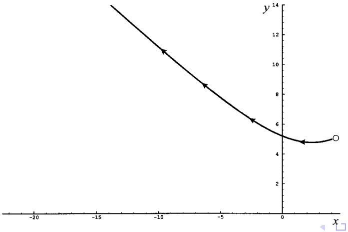
$$y(t) = \frac{15e^{(1-\sqrt{6})t} + 4\sqrt{6}e^{(1-\sqrt{6})t} + 15e^{(1+\sqrt{6})t} - 4\sqrt{6}e^{(1+\sqrt{6})t}}{6} \tag{27}$$

Example #2 (II)

Solution of system - functions $x(t)$ and $y(t)$



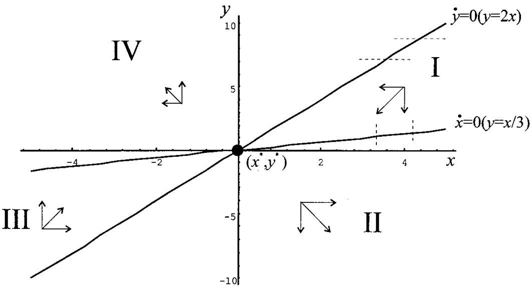
One solution trajectory (solution for given initial condition) in phase plane



Fixed point and equilibrium lines (should not be black box anymore):

$$(x^*, y^*) = (0, 0) \tag{28}$$

$$y = \frac{x}{3} \quad \text{and} \quad y = 2x \tag{29}$$



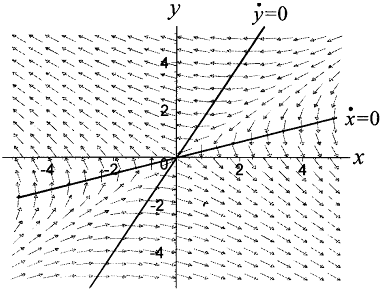
Example #2 (IV)

$$y < \frac{x}{3} \quad \longrightarrow \quad \dot{x} > 0 \tag{30}$$

$$y > \frac{x}{3} \quad \longrightarrow \quad \dot{x} < 0 \tag{31}$$

$$y < 2x \quad \longrightarrow \quad \dot{y} < 0 \tag{32}$$

$$y > 2x \quad \longrightarrow \quad \dot{y} > 0 \tag{33}$$



Matrix notation (I)

Recall (autonomus, linear, constant coefficients) **homogenous** system:

$$\dot{x}(t) = ax(t) + by(t) \tag{34}$$
$$\dot{y}(t) = cx(t) + dy(t) \tag{35}$$

This can be equivalently written in the form:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{36}$$

Similarly we can write for nonhomogenous system:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} d \\ e \end{pmatrix} \tag{37}$$

Matrix notation (II)

Let us define:

$$\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} d \\ e \end{pmatrix} \tag{38}$$

Thus we can write for homogenous and nonhomogenous respectively:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{39}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \tag{40}$$

Considering notation for the fixed point:

$$\boldsymbol{x}^* = \begin{pmatrix} x^* \\ y^* \end{pmatrix} \tag{41}$$

We can write (from the definition of fixed point):

$$\boldsymbol{0} = \boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{b} \tag{42}$$

This equation can be used for determining equilibrium.

Matrix notation (III)

The big magic how to produce homogenous from nonhomogenous, recall:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \tag{43}$$

$$\boldsymbol{0} = \boldsymbol{A}\boldsymbol{x}^* + \boldsymbol{b} \tag{44}$$

Now we subtract these equations and get:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{x}^*) \tag{45}$$

$$\tag{46}$$

Thus by substitution $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{x}^*$ we make **homogenous system**. We can solve this system for $\boldsymbol{x}'$.
In the end we just use backward substitution $\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{x}^*$ and we have solution of original system.
Thus from now on we focus only on homogenous system.

Recall:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{47}$$

Let us define determinant and trace:

$$det(\boldsymbol{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{48}$$

$$tr(\boldsymbol{A}) = a + d \tag{49}$$

Consider eigenvalue problem (will be found as very important for solving systems of equations):

$$\boldsymbol{Ax} = \lambda \boldsymbol{x} \tag{50}$$

where λ is a number (might more such numbers). We search for all λ's that solve this equation (we want to find eigenvalues of matrix $\boldsymbol{A}$).

Matrix notation (V)

Recall and derive:

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{51}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0} \tag{52}$$

where $\boldsymbol{I}$ is unity matrix (ones on the diagonal and zeros elsewhere).
But the last equation can be solved only for such λ for that holds:

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0 \tag{53}$$

Thus:

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a + d)\lambda + ad - bc = 0 \tag{54}$$

Matrix notation (VI)

Recall:

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^2 - (a + d)\lambda + ad - bc = \lambda^2 - tr(\boldsymbol{A})\lambda + det(\boldsymbol{A}) \tag{55}$$

We have thus two roots of this quadratic equation λ_1 and λ_2, sometimes denoted as λ_r and λ_s or even r and s.
It might be easily proven that the roots are:

- real and distinct if $tr(\boldsymbol{A})^2 > 4det(\boldsymbol{A})$
- real and equal if $tr(\boldsymbol{A})^2 = 4det(\boldsymbol{A})$
- complex conjugate if $tr(\boldsymbol{A})^2 < 4det(\boldsymbol{A})$

Matrix notation (VII)

We now know how to find **eigenvalues** of matrix $\boldsymbol{A}$.
Now we want to find so called **eigenvectors** $\boldsymbol{v}^r$ and $\boldsymbol{v}^s$ (there is separate eigenvector for each distinct eigenvalue):

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0} \tag{56}$$

For given λ this describes two equations and we need to find two coordinates of vector $\boldsymbol{v}$.
However the vector is usually defined only up to constant (only mutual relationship between two coordinates is defined). In this often case we just take any vector that corresponds to given relationship (we take the simplest possible).

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \tag{57}$$

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = 0 \tag{58}$$

Then $\lambda_1 = r = 3$ and $\lambda_2 = s = 2$.
Let us compute eigenvector $\boldsymbol{v^r}$:

$$\boldsymbol{A} = \begin{pmatrix} 1-3 & 1 \\ -2 & 4-3 \end{pmatrix} \begin{pmatrix} v_1^r \\ v_2^r \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} v_1^r \\ v_2^r \end{pmatrix} = 0 \tag{59}$$

We thus have single condition $-2v_1^r + v_2^r = 0$. We can thus compute the eigenvector as:

$$\boldsymbol{v^r} = c \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{60}$$

Solution to the homogenous system (I)

Recall that we have our system in the form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} \tag{61}$$

Let us try solution in the form:

$$\boldsymbol{x} = e^{\lambda t}\boldsymbol{v} \tag{62}$$

We substitute this solution into system of differential equations and get:

$$\lambda e^{\lambda t}\boldsymbol{v} = \boldsymbol{A}e^{\lambda t}\boldsymbol{v} \tag{63}$$

Eliminating the $e^{\lambda t}$ term we get:

$$\lambda\boldsymbol{v} = \boldsymbol{A}\boldsymbol{v} \tag{64}$$

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v} = 0 \tag{65}$$

From this condition it follows that λ must be eigenvalue of matrix $\boldsymbol{A}$ and $\boldsymbol{v}$ its eigenvector.

Let us assume we found two eigenvalues of matrix $\boldsymbol{A}$ and denoted them r and s, moreover we found a pair of eigenvectors $\boldsymbol{v^r}$ and $\boldsymbol{v^s}$.

Case 1 - the easiest one
r and s are real and distinct
Then the solution of the system is:

$$\boldsymbol{x} = c_1 e^{rt} \boldsymbol{v^r} + c_2 e^{st} \boldsymbol{v^s} \tag{66}$$

Case 2 - the hardest to remember and proceed
r and s are real but $r = s$
We must distinct two different subcases (next slide):

Solution to the homogenous system (III)

Case 2a - the easy one

$r = s$, we can find two linearly independent eigenvectors $\boldsymbol{v^1}$ and $\boldsymbol{v^2}$, the solution is then simply:

$$\boldsymbol{x} = c_1 e^{rt} \boldsymbol{v^1} + c_2 e^{rt} \boldsymbol{v^2} \tag{67}$$

Case 2b - the horrible one

$r = s$, there is only one indpendent eigenvector $\boldsymbol{v^1}$, we then have to take solution in this form:

$$\boldsymbol{x} = c_1 e^{rt} \boldsymbol{v^1} + c_2 (t e^{rt} \boldsymbol{v^1} + e^{rt} \boldsymbol{v^2}) \tag{68}$$

However, we do not know that $\boldsymbol{v^2}$. The most straightforward way to find out is to take the second part of solution:

$$\boldsymbol{x} = t e^{rt} \boldsymbol{v^1} + e^{rt} \boldsymbol{v^2} \tag{69}$$

and substitute this to the original system with coordinates of $\boldsymbol{v^2}$ as unknown variables. By some manipulation (differentiating etc.) we find condition for $\boldsymbol{v^2}$, which we then can finaly substitute into the whole solution.

Solution to the homogenous system (IV)

Case 3 - the last one
r and s are complex conjugate:

$$r = \alpha + \beta i \tag{70}$$
$$s = \alpha - \beta i \tag{71}$$

This implies that eigenvectors might also be complex (have complex coordinates).
The solution can be still written in the form:

$$\boldsymbol{x} = c_1 e^{rt} \boldsymbol{v^r} + c_2 e^{st} \boldsymbol{v^s} \tag{72}$$

However, the final solution must be real, thus we need some furhter manipulation.
Note that v^r and v^s might be complex. But moreover, it can be prooved that v^s is always complex conujugate to v^r. (If $\boldsymbol{v^r} = \boldsymbol{a} + i\boldsymbol{b}$ then $\boldsymbol{v^s} = \boldsymbol{a} - i\boldsymbol{b}$.)

Solution to the homogenous system (V)

Before we proceed further we mention Euler's formula for exponential of complex number:

$$e^{i\beta} = \cos\beta + i\sin\beta$$

Similarly:

$$e^{-i\beta} = \cos(-\beta) + i\sin(-\beta) = \cos\beta - i\sin\beta$$

Let us take the vector v^r (generally with complex coordinates) and denote its coordinates:

$$v_1^r = a_1 + b_1 i, \qquad v_2^r = a_2 + b_2 i$$

Since we take eigenvector associated with $r = \alpha + \beta i$, one of the solutions can be written as:

$$e^{(\alpha+i\beta)t}\begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{pmatrix}$$

Solution to the homogenous system (VI)

Recall solution asscoiated to $r = \alpha + \beta i$:

$$e^{(\alpha+i\beta)t}\begin{pmatrix} a_1 + b_1 i \\ a_2 + b_2 i \end{pmatrix}$$

Focusing on first row only:

$$e^{(\alpha+i\beta)t}(a_1 + b_1 i) = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(a_1 + b_1 i) =$$
$$= e^{\alpha t}\left[(a_1\cos(\beta t) - b_1\sin(\beta t)) + i(b_1\cos(\beta t) + a_1\sin(\beta t))\right]$$

Thus second row:

$$e^{(\alpha+i\beta)t}(a_2 + b_2 i)$$
$$= e^{\alpha t}\left[(a_2\cos(\beta t) - b_2\sin(\beta t)) + i(b_2\cos(\beta t) + a_2\sin(\beta t))\right]$$

So the solution for r and associated $\boldsymbol{v^r}$ is:

$$e^{(\alpha+i\beta)t}\begin{pmatrix} a_1+b_1i \\ a_2+b_2i \end{pmatrix} =$$

$$\begin{pmatrix} e^{\alpha t}\left[(a_1\cos(\beta t)-b_1\sin(\beta t))+i(b_1\cos(\beta t)+a_1\sin(\beta t))\right] \\ e^{\alpha t}\left[(a_2\cos(\beta t)-b_2\sin(\beta t))+i(b_2\cos(\beta t)+a_2\sin(\beta t))\right] \end{pmatrix}$$

Let us denote:

$$\boldsymbol{u_1}=\begin{pmatrix} e^{\alpha t}(a_1\cos(\beta t)-b_1\sin(\beta t)) \\ e^{\alpha t}(a_2\cos(\beta t)-b_2\sin(\beta t)) \end{pmatrix}$$

$$\boldsymbol{u_2}=\begin{pmatrix} e^{\alpha t}(b_1\cos(\beta t)+a_1\sin(\beta t)) \\ e^{\alpha t}(b_2\cos(\beta t)+a_2\sin(\beta t)) \end{pmatrix}$$

We thus found that solution associated to r can be written as:

$$e^{(\alpha+i\beta)t}\boldsymbol{v^r}=\boldsymbol{u_1}+i\boldsymbol{u_2}$$

Note that $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are real!

Solution to the homogenous system (VIII)

Recall solution for r and $\boldsymbol{v^r}$:

$$e^{(\alpha+i\beta)t}\boldsymbol{v^r} = \boldsymbol{u_1} + i\boldsymbol{u_2}$$

It might be shown that for s and $\boldsymbol{v^s}$ the solution is very similar:

$$e^{(\alpha-i\beta)t}\boldsymbol{v^s} = \boldsymbol{u_1} - i\boldsymbol{u_2}$$

This is because you need to switch the sign in the Euler´s formula (because s is complex conjugate of r) and also switch sign of b_1 and b_2 (because $\boldsymbol{v^s}$ is complex conjugate of $\boldsymbol{v^r}$). Complete solution can be written as:

$$\boldsymbol{x} = k_1(\boldsymbol{u_1} + i\boldsymbol{u_2}) + k_2(\boldsymbol{u_1} - i\boldsymbol{u_2}) = (k_1 + k_2)\boldsymbol{u_1} + i(k_1 - k_2)\boldsymbol{u_2}$$

However, since constants in homogenous solution can be any (even complex) - we might define new constants and the final complete solution takes the form:

$$\boldsymbol{x} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}$$

Example #4 (I)

Let us have system with following matrix of coefficients:

$$\boldsymbol{A} = \begin{pmatrix} -3 & 4 \\ -2 & 1 \end{pmatrix} \tag{73}$$

We find that $r = -1 + 2i$, $s = -1 - 2i$. Eigenvectors can be found for example in this form (they must be complex conjugate). (Note that two conditions defining the eigenvector are always linearly dependent, thus it is enough to fulfil one of them - but always control the other one, it is a control of correct eigenvalues):

$$\boldsymbol{v}^r = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}, \quad \boldsymbol{v}^s = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$$

Following previous notation:

$$\begin{aligned} \alpha &= -1 & \beta &= 2 \\ a_1 &= 2 & a_2 &= 1 \\ b_1 &= 0 & b_2 &= 1 \end{aligned}$$

Example #4 (II)

Recall:

$$\boldsymbol{u_1} = \begin{pmatrix} e^{\alpha t}(a_1 \cos(\beta t) - b_1 \sin(\beta t)) \\ e^{\alpha t}(a_2 \cos(\beta t) - b_2 \sin(\beta t)) \end{pmatrix}$$

$$\boldsymbol{u_2} = \begin{pmatrix} e^{\alpha t}(b_1 \cos(\beta t) + a_1 \sin(\beta t)) \\ e^{\alpha t}(b_2 \cos(\beta t) + a_2 \sin(\beta t)) \end{pmatrix}$$

Thus the solution is:

$$\boldsymbol{x} = c_1 \begin{pmatrix} e^{-t}(2\cos(2t)) \\ e^{-t}(\cos(2t) - \sin(2t)) \end{pmatrix} + c_2 \begin{pmatrix} e^{-t}(2\sin(2t)) \\ e^{-t}(\cos(2t) + \sin(2t)) \end{pmatrix}$$

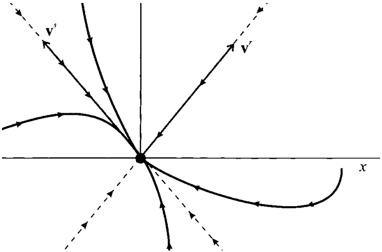
$$x = 2e^{-t}(c_1 \cos(2t) - c_2 \sin(2t))$$
$$y = e^{-t}(c_1(\cos(2t) - \sin(2t)) + c_2(\cos(2t) + \sin(2t)))$$

Stability of the system (I)

Case 1 - real distinct roots
Solution can then be written as

$$\boldsymbol{x} = c_1 e^{rt} \boldsymbol{v^r} + c_2 e^{st} \boldsymbol{v^s} \tag{74}$$

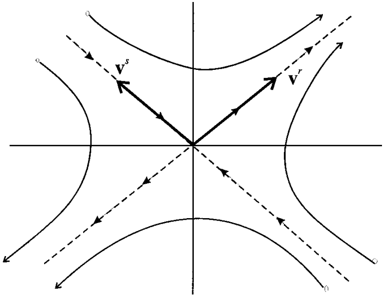
Case 1a $r > s > 0$ or $s > r > 0$
Due to explosive exponential function, the system is unstable.
Case 1b $r < s < 0$ or $s < r < 0$ (see figure below)
Both exponential functions are now converging (to zero), thus the system is stable.



Stability of the system (II)

Case 1c - real roots $r > 0$, $s < 0$

- the fixed point is **unstable** and is reffered to as **saddle**
- eigenvector $\boldsymbol{v}^r$ defines the direction of **unstable arm**
- whereas $\boldsymbol{v}^s$ represents **stable arm**
- if the system starts directly on stable arm, it converges to equilibrium
- in any other case the positive root dominates the system



Stability of the system (III)

Case 2 - real roots $r = s$

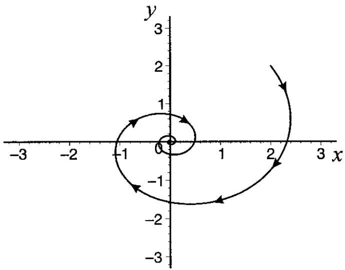
- any solution includes exponential part e^{rt}
- for $r < 0$ the system is stable
- for $r > 0$ the system is unstable
- the solution and thus its graphical representation heavily depends on structure of eigenvectors

Case 3 - complex roots $r = \alpha + \beta i$, $s = \alpha - \beta i$ the solution always insolve term $e^{\alpha t}$ which is responsible for eventual stability

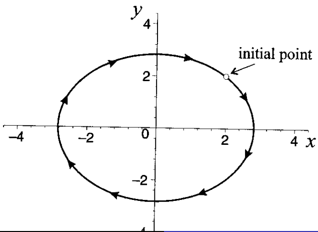
Case 3a - $\alpha > 0$

the system is unstable

Case 3b - $\alpha < 0$ - the system is stable



Case 3c - $\alpha = 0$ - periodic solution



Stability of the system (III)

Matrix and eigenvalues	Type of point	Type of stability
$\mathrm{tr}(\mathbf{A}) < 0$, $\det(\mathbf{A}) > 0$, $\mathrm{tr}(\mathbf{A})^2 > 4\det(\mathbf{A})$ $r < s < 0$	Improper node	Asymptotically stable
$\mathrm{tr}(\mathbf{A}) > 0$, $\det(\mathbf{A}) > 0$, $\mathrm{tr}(\mathbf{A})^2 > 4\det(\mathbf{A})$ $r > s > 0$	Improper node	Unstable
$\det(\mathbf{A}) < 0$ $r > 0,\ s < 0$ or $r < 0,\ s > 0$	Saddle point	Unstable saddle
$\mathrm{tr}(\mathbf{A}) < 0$, $\det(\mathbf{A}) > 0$, $\mathrm{tr}(\mathbf{A})^2 = 4\det(\mathbf{A})$ $r = s < 0$	Star node or proper node	Stable
$\mathrm{tr}(\mathbf{A}) > 0$, $\det(\mathbf{A}) > 0$, $\mathrm{tr}(\mathbf{A})^2 = 4\det(\mathbf{A})$ $r = s > 0$	Star node or proper node	Unstable
$\mathrm{tr}(\mathbf{A}) < 0$, $\det(\mathbf{A}) > 0$, $\mathrm{tr}(\mathbf{A})^2 < 4\det(\mathbf{A})$ $r = \alpha + \beta i,\ s = \alpha - \beta i,\ \alpha < 0$	Spiral node	Asymptotically stable
$\mathrm{tr}(\mathbf{A}) > 0$, $\det(\mathbf{A}) > 0$, $\mathrm{tr}(\mathbf{A})^2 < 4\det(\mathbf{A})$ $r = \alpha + \beta i,\ s = \alpha - \beta i,\ \alpha > 0$	Spiral node	Unstable
$\mathrm{tr}(\mathbf{A}) = 0$, $\det(\mathbf{A}) > 0$ $r = \beta i,\ s = -\beta i$	Centre	Stable