Final Examination: 5330 Advanced Microeconomic Theory

January 12, 2024, 14:00-18:00

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Instructions: Answer all questions.

1. (10 points) Suppose that the direct utility function of a consumer takes the CES form:

$$u(x_1, x_2) \equiv (x_1^\rho + x_2^\rho)^{1/\rho}$$

where $-\infty < \rho < 1$ and $\rho \neq 0$. Letting $r \equiv \rho/(\rho - 1)$, solve the expenditure minimization problem and show that the Hicksian demand function for good 2 is

$$x_2^h(\mathbf{p}, u) = u\, p_2^{r-1}\, (p_1^r + p_2^r)^{(1/r)-1}\,.$$

2. (9 points) For a firm with the Cobb-Douglas technology $y = x_1^\alpha x_2^\beta$, solve for the conditional input demand functions $x_1(y, w_1, w_2)$ and $x_2(y, w_1, w_2)$. Then solve for the output supply function $y(p, w_1, w_2)$. What restrictions on α and β are required to ensure that these functions is well-defined? Explain.

3. (8 points) Consider an example of Cournot oligopoly in the market for some homogeneous good. Suppose there are J identical firms, that entry by additional firms is effectively blocked and that each firm j has identical costs, with cost function $c(q^j) = k + cq^j$, where $k > 0$ is fixed cost and $c > 0$ is marginal cost. Firms sell output on a common market, so market

price depends on the total output sold by all firms in the market. Let inverse market demand be the linear form,

$$p = a - b\left(\sum_{j=1}^{J} q^j\right),$$

where $a > 0$, $b > 0$ and we require $a > c$.

(a) What will be the Cournot equilibrium price, individual firm output, market output, and firm profits with J firms in the market?

(b) With free entry and exit, what will be the long-run equilibrium number of firms in the market?

4. (13 points) A monopolist faces linear demand $p = \alpha - \beta q$ and has cost $C = cq + F$, where all parameters are positive, $\alpha > c$, and $(\alpha - c)^2 > 4\beta F$.

(a) Solve for the monopolist's output, price, and profits.

(b) Calculate the deadweight loss and show that it is positive.

(c) If the government requires this firm to set the price that maximizes the sum of consumer and producer surplus, and to serve all buyers at that price, what is the price the firm must charge? Show that the firm's profits are negative under this regulation, so that this form of regulation is not sustainable in the long run.

5. (12 points) Suppose that in a single-consumer "Robinson Crusoe" economy, the consumer is endowed with none of the consumption good, y, and 24 hours of time, h, so that $\mathbf{e} = (24, 0)$. Suppose as well that preferences are defined over $\mathbb{R}^2_+$ and represented by $u(h, y) = hy$, and production possibilities are

$$Y = \{(-h, y) \mid 0 \leq h \leq b \text{ and } 0 \leq y \leq h^{1/2}\},$$

where b is some very large positive number. Let p_y and p_h be prices of the consumption good and leisure, respectively.

(a) State the firm's profit maximization problem. Solve for the firm's input demand function $h^f(p_h, p_y)$, the firm's output supply function $y^f(p_h, p_y)$ and the firm's profit function $\pi^f(p_h, p_y)$.

(b) State the consumer's utility maximization problem taking into account that it owns the firm. Solve the consumer's problem to obtain the demand function for leisure $h^c(p_h, p_y)$ and the demand function for the consumption good $y^c(p_h, p_y)$.

6. (13 points) Consider a two-consumer, two-good exchange economy. Utility functions and endowments are

$$u^1(x_1, x_2) = (x_1 x_2)^2 \qquad \text{and} \qquad \mathbf{e}^1 = (18, 4),$$

$$u^2(x_1, x_2) = \ln(x_1) + 2\ln(x_2) \qquad \text{and} \qquad \mathbf{e}^2 = (3, 6).$$

Find a Walrasian equilibrium and compute the Walrasian equilibrium allocation (WEA).

7. (10 points) One of the most important results in general equilibrium theory is the Second Welfare Theorem, which states that under certain conditions, *any* Pareto-efficient allocation can be achieved by competitive markets and *some* initial endowments.

Present a proof of the following theorem (using any results from earlier theorems that are needed):

THEOREM (Second Welfare Theorem):

Consider an exchange economy $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$ with aggregate endowment $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, and with each consumer's utility function u^i satisfying the standard assumptions (continuous, strongly increasing and strictly quasiconcave). Suppose that $\bar{\mathbf{x}}$ is a Pareto-efficient allocation for $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$ and that endowments are redistributed so that the new endowment vector is $\bar{\mathbf{x}}$. Then $\bar{\mathbf{x}}$ is a Walrasian equilibrium allocation of the resulting exchange economy $(u^i, \bar{\mathbf{x}}^i)_{i \in \mathcal{I}}$.

Answers to the exam questions

1. (10 points) Suppose that the direct utility function takes the CES form: $u(x_1, x_2) \equiv (x_1^\rho + x_2^\rho)^{1/\rho}$ where $-\infty < \rho < 1$ and $\rho \neq 0$.

Because preferences are strictly monotonic, we can formulate the expenditure minimization problem as

$$\max_{x_1, x_2} -(p_1 x_1 + p_2 x_2) \ \text{ s.t. } \ u - (x_1^\rho + x_2^\rho)^{1/\rho} = 0, \ x_1 \geq 0, \ x_2 \geq 0,$$

and its Lagrangian,

$$\mathcal{L}(x_1, x_2, \lambda) = -(p_1 x_1 + p_2 x_2) - \lambda \left[u - (x_1^\rho + x_2^\rho)^{1/\rho}\right]$$

Assuming an interior solution in both goods, the first-order conditions for a maximum subject to the constraint ensure that the solution values x_1, x_2 and λ satisfy the equations:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= -p_1 + \lambda \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{(1/\rho)-1} \rho x_1^{\rho-1} = 0 \\
\frac{\partial \mathcal{L}}{\partial x_2} &= -p_2 + \lambda \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{(1/\rho)-1} \rho x_2^{\rho-1} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= -u + (x_1^\rho + x_2^\rho)^{1/\rho} = 0
\end{aligned}$$

By eliminating λ, these can be reduced to two equations in two unknowns:

$$\begin{aligned}
\frac{p_1}{p_2} &= \left(\frac{x_1}{x_2}\right)^{\rho-1} \\
u &= (x_1^\rho + x_2^\rho)^{1/\rho}
\end{aligned}$$

It follows that

$$(p_1/p_2)^{1/(\rho-1)} = x_1/x_2$$

$$u = (x_1^\rho + x_2^\rho)^{1/\rho} = \left(\left[x_2\left(\frac{p_1}{p_2}\right)^{1/(\rho-1)}\right]^\rho + x_2^\rho\right)^{1/\rho} = x_2\left(\left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)} + 1\right)^{1/\rho}$$

$$\left(\frac{u}{x_2}\right)^\rho = \left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)} + 1$$

$$\left(\frac{u}{x_2}\right)^\rho p_2^{\rho/(\rho-1)} = p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}$$

$$\left(\frac{x_2}{u}\right)^\rho = \frac{p_2^{\rho/(\rho-1)}}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}$$

$$\frac{x_2}{u} = \frac{p_2^{1/(\rho-1)}}{\left(p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\right)^{1/\rho}}$$

$$x_2 = u\,p_2^{1/(\rho-1)}\left(p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\right)^{-1/\rho}$$

Now using $r \equiv \rho/(\rho-1)$, which implies that

$$r - 1 = \frac{\rho - (\rho-1)}{\rho - 1} = \frac{1}{\rho-1} \qquad \frac{1}{r} = \frac{\rho-1}{\rho} = 1 - \frac{1}{\rho}$$

$$x_2 = x_2^h(\mathbf{p}, u) = u\,p_2^{r-1}\,(p_1^r + p_2^r)^{(1/r)-1}$$

We have derived the Hicksian demand function for good 2.

7. (10 points) **Proof**: Because $\bar{\mathbf{x}}$ is Pareto-efficient, it is feasible. Hence,

$$\sum_{i=1}^{I} \bar{\mathbf{x}}^i = \sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}.$$

Consequently, we may apply the earlier theorem

THEOREM (Existence of Walrasian Equilibrium):
If each consumer's utility function satisfies the standard assumptions (continuous, strongly increasing and strictly quasiconcave) and $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

to conclude that the exchange economy $(u^i, \bar{\mathbf{x}}^i)_{i \in \mathcal{I}}$ possesses a Walrasian equilibrium allocation $\hat{\mathbf{x}}$. It only remains to show that $\hat{\mathbf{x}} = \bar{\mathbf{x}}$.

Now in the Walrasian equilibrium, each consumer's demand is utility maximizing subject to her budget constraint. Consequently, because i demands $\hat{\mathbf{x}}^i$ and has endowment $\bar{\mathbf{x}}^i$, we must have

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\bar{\mathbf{x}}^i) \quad \text{for all } i \in \mathcal{I}.$$

But because $\hat{\mathbf{x}}$ is an equilibrium allocation, it must be feasible for the economy $(u^i, \bar{\mathbf{x}}^i)_{i \in \mathcal{I}}$. Consequently,

$$\sum_{i=1}^{I} \hat{\mathbf{x}}^i = \sum_{i=1}^{I} \bar{\mathbf{x}}^i = \sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0},$$

so that $\hat{\mathbf{x}}$ is feasible for the original economy $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$ as well.

Thus, by

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\bar{\mathbf{x}}^i) \quad \text{for all } i \in \mathcal{I},$$

$\hat{\mathbf{x}}$ is feasible for the original economy $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$ and makes no consumer worse off than the Pareto-efficient (for the original economy) allocation $\bar{\mathbf{x}}$.

Therefore $\hat{\mathbf{x}}$ cannot make anyone strictly better off; otherwise $\bar{\mathbf{x}}$ would not be Pareto efficient. It follows that

$$u^i(\hat{\mathbf{x}}^i) = u^i(\bar{\mathbf{x}}^i) \quad \text{for all } i \in \mathcal{I}.$$

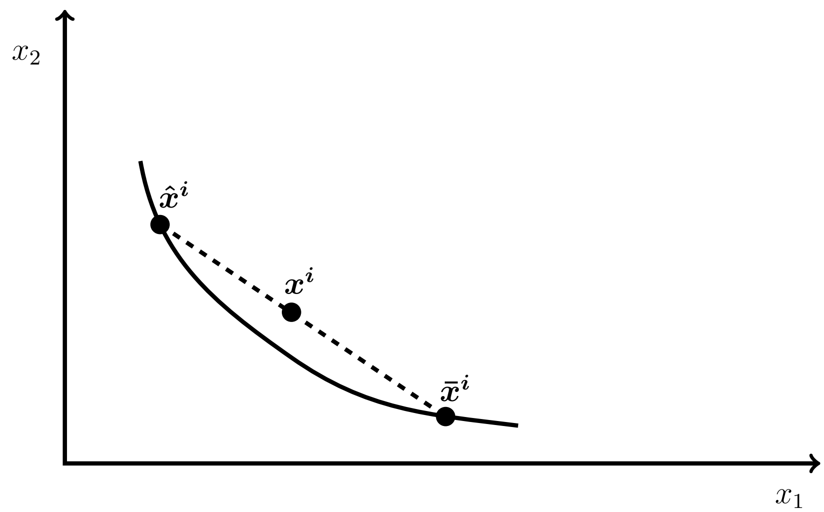

Figure 1: Preferences that are represented by a strictly quasiconcave utility function.

To see now that $\hat{\mathbf{x}}^i = \bar{\mathbf{x}}^i$ for every i, note that if for some consumer this were not the case, so $\hat{\mathbf{x}}^i \neq \bar{\mathbf{x}}^i$ for some i, then in the Walrasian equilibrium of the new economy, that consumer could afford the average of the bundles $\hat{\mathbf{x}}^i$ and $\bar{\mathbf{x}}^i$ and strictly increase his utility (by strict quasiconcavity).

Since that consumer could afford the average of the bundles $\hat{\mathbf{x}}^i$ and $\bar{\mathbf{x}}^i$ and strictly increase his utility (by strict quasiconcavity), this contradictions the fact that $\hat{\mathbf{x}}^i$ is utility-maximizing in the Walrasian equilibrium.

We get a contradiction assuming that $\hat{\mathbf{x}}^i \neq \bar{\mathbf{x}}^i$ for some i. Therefore, it must be that $\hat{\mathbf{x}}^i = \bar{\mathbf{x}}^i$ for all i. It follows that $\bar{\mathbf{x}}$ is a Walrasian equilibrium allocation of the exchange economy $(u^i, \bar{\mathbf{x}}^i)_{i \in \mathcal{I}}$.

$\square$