Advanced microeconomics problem set 3

Zhaoqin Zhu *

October 12, 2023

1 Exercise 3.11

1.1 Original question

Calculate σ for the Cobb-Douglas production function $y = Ax_1^{\alpha}x_2^{\beta}$, where $A > 0$, $\alpha > 0$, and $\beta > 0$.

1.2 Solution

First, we calculate the marginal rate of technical substitution:

$$MRTS_{12}(\mathbf{x}) = \frac{\partial f(\mathbf{x})/\partial x_1}{\partial f(\mathbf{x})/\partial x_2} = \frac{A\alpha x_1^{\alpha-1}x_2^{\beta}}{Ax_1^{\alpha}\beta x_2^{\beta-1}} = \frac{\alpha x_2}{\beta x_1} \tag{1}$$

Take logs of both sides we have: $\ln MRTS_{12} = \ln(\frac{\alpha}{\beta}) + \ln(\frac{x_2}{x_1})$, and denote $r = \ln\frac{x_2}{x_1}$[1], we thus have:

$$\ln MRTS_{12} = \ln(\frac{\alpha}{\beta}) + r \tag{2}$$

Finally, we take derivatives against r (which is $\ln\frac{x_2}{x_1}$), and have:

$$\sigma_{12} = \frac{dr}{d\ln MRTS_{12}} = \frac{dr}{d\ln(\frac{\alpha}{\beta}) + r} = 1 \tag{3}$$

2 Exercise 3.17

2.1 Original question

For the CES technology in the preceding exercise, prove the following statements made in the text

a) $\lim_{\rho\to 0} y = \left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} = \prod_{i=1}^{n} x_i^{\alpha_i}$

b) $\lim_{\rho\to -\infty} y = \left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} = \min\{\mathbf{x}\}$

*zhaoqin.zhu@phdstudent.hhs.se

[1] This step may not be necessary here, but proves to be useful when the production function is more complicated

2.2 Solution

(a) Observe that we can rewrite the production function as follows

$$\lim_{\rho\to 0}\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} = \lim_{\rho\to 0}\exp\left\{\ln\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}}\right\} \tag{4}$$

$$= \lim_{\rho\to 0}\exp\left\{\rho^{-1}\ln\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)\right\} \tag{5}$$

Now we can apply Theorem A1.9 on page 520 in the Mathematical Appendix. The theorem states that if $D \subseteq \mathbb{R}^n$ and $f : D \mapsto \mathbb{R}^m$, then f being continuous is equivalent to saying that whenever a sequence $\{\mathbf{x}^k\}_{k=1}^{\infty}$ in D converges to $\mathbf{x} \in D$, then $\{f(\mathbf{x}^k)\}_{k=1}^{\infty}$ converges to $f(\mathbf{x})$. Now note that $\exp\{\cdot\}$ is continuous on $\mathbb{R}$. Therefore, the theorem allows us to write

$$\lim_{\rho\to 0}\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} = \exp\left\{\lim_{\rho\to 0}\rho^{-1}\ln\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)\right\} \tag{6}$$

Now observe that the two sides of the fraction in (6) satisfy

$$\lim_{\rho\to 0}\rho^{-1} = \infty \qquad \lim_{\rho\to 0}\ln\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right) = 0 \tag{7}$$

In order to make progress we need to apply L'Hopitals rule (see page 186 in Pemberton & Rau (2016)). This result tells us roughly speaking that if $f(a) = g(a) = 0$ and $g'(x) \neq 0$ if x is closed but not equal to a, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{8}$$

In our case, the limit of the derivative in the denominator is easy to compute

$$\lim_{\rho\to 0}\left(\frac{\partial\rho}{\partial\rho}\right) = 1 \tag{9}$$

The derivative for the numerator takes a little bit more effort. We directly calculate

$$\frac{\partial}{\partial\rho}\ln\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right) = \frac{1}{\sum_{i=1}^{n}\alpha_i x^{\rho}}\frac{\partial\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)}{\partial\rho}$$

$$= \frac{1}{\sum_{i=1}^{n}\alpha_i x^{\rho}}\left(\sum_{i=1}^{n}\alpha_i\ln(x_i)x^{\rho}\right)$$

If we take the limit we obtain

$$\lim_{\rho\to 0}\frac{\partial}{\partial\rho}\ln\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right) = \frac{1}{\sum_{i=1}^{n}\alpha_i}\sum_{i=1}^{n}\alpha_i\ln(x_i) \tag{10}$$

$$= \sum_{i=1}^{n}\alpha_i\ln(x_i) \tag{11}$$

$$= \sum_{i=1}^{n}\ln(x_i^{\alpha_i}) \tag{12}$$

$$= \ln\left(\prod_{i=1}^{n}x_i^{\alpha_i}\right) \tag{13}$$

The second equality follows from our assumption on the coefficients, which we require to sum up to one, $\sum_{i=1}^{n} \alpha_i = 1$. Equations (8), (9) and (13) imply

$$\lim_{\rho \to 0} \rho^{-1} \ln \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho} \right) = \ln \left(\prod_{i=1}^{n} x_i^{\alpha_i} \right) \tag{14}$$

If we substitute for the limit on the right-hand side of (6) we obtain

$$\lim_{\rho \to 0} \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho} \right)^{\frac{1}{\rho}} = \prod_{i=1}^{n} x_i^{\alpha_i} \tag{15}$$

which is the familiar expression for the Cobb-Douglas technology with constant returns.

(b) The CES function is defined on an n-dimensional commodity space $X \subseteq \mathbb{R}^n_+$. So for any $\mathbf{x} \in X$, the vector $\mathbf{x}$ contains a minimal element, denoted as $\underline{\mathbf{x}} = \min(\mathbf{x})$. The algebra of limits (see p.93 for the properties of limits) allows us to write

$$\lim_{\rho \to -\infty} \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho} \right)^{\frac{1}{\rho}} = \underline{\mathbf{x}} \lim_{\rho \to -\infty} \left(\sum_{i=1}^{n} \alpha_i \left(\frac{x_i}{\underline{\mathbf{x}}} \right)^{\rho} \right)^{\frac{1}{\rho}} \tag{16}$$

Let us define $r_i = x_i / \underline{\mathbf{x}}$. Then the limit can be written as:

$$\lim_{\rho \to -\infty} \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho} \right)^{\frac{1}{\rho}} = \underline{\mathbf{x}} \lim_{\rho \to -\infty} \left(\sum_{i=1}^{n} \alpha_i \left(r_i \right)^{\rho} \right)^{\frac{1}{\rho}} \tag{17}$$

Note that for $x_i > 0$ for all i, so it holds by definition of the minimum operator that

$$r_i = \frac{x_i}{\underline{\mathbf{x}}} \geq 1 \quad \forall i \in \mathcal{I} \tag{18}$$

In order to write the expression in the limit as a fraction, we apply the same steps as in the previous exercise and write

$$\lim_{\rho \to -\infty} \left(\sum_{i=1}^{n} \alpha_i r_i^{\rho} \right)^{\frac{1}{\rho}} = \lim_{\rho \to -\infty} \exp \left\{ \rho^{-1} \ln \left(\sum_{i=1}^{n} \alpha_i r_i^{\rho} \right) \right\} \tag{19}$$

$$= \exp \left\{ \rho^{-1} \ln \left(\lim_{\rho \to -\infty} \sum_{i=1}^{n} \alpha_i r_i^{\rho} \right) \right\} \tag{20}$$

Note that for $x_i > 0$ for all i, so it holds by definition of the minimum operator that

$$r_i = \frac{x_i}{\underline{\mathbf{x}}} \geq 1 \quad \forall i \in \mathcal{I} \tag{21}$$

Because there exists a minimum in $\mathbf{x}$, there must be at least one $i \in \mathcal{I}$ such that $x_i = \min(\mathbf{x})$. Let us define $S = \{i \in \mathcal{I} : x_i = \min(\mathbf{x})\}$. S is the set of indicators which are equal to the minimum value in $\mathbf{x}$. By construction, $r_i = 1$ for all $i \in S$. Furthermore, let us define $T = X \setminus S$. We are collecting in T all the indices of goods that are not equal to (strictly larger) the minimum. Because $\mathcal{I} = S \cup T$ by construction, and using the algebra of limits (we are allowed to write the limit of a sum as the sum of its limits, see p.93 in Pemberton & Rau (2016)), we can write

$$\lim_{\rho\to-\infty}\sum_{i=1}^{n}\alpha_i r_i^{\rho} = \lim_{\rho\to-\infty}\sum_{i\in S}\alpha_i r_i^{\rho} + \lim_{\rho\to-\infty}\sum_{i\in T}\alpha_i r_i^{\rho} \tag{22}$$

Exploiting the algebra of limits further, we can rewrite the expression as

$$\lim_{\rho\to-\infty}\sum_{i=1}^{n}\alpha_i r_i^{\rho} = \sum_{i\in S}\alpha_i\left(\lim_{\rho\to-\infty} r_i^{\rho}\right) + \sum_{i\in T}\alpha_i\left(\lim_{\rho\to-\infty} r_i^{\rho}\right) \tag{23}$$

Because $r_i = 1$ for all $i \in S$, we can ignore the exponent ρ and therefore the limit operator in the first term on the right-hand side of (24). Hence,

$$\lim_{\rho\to-\infty}\sum_{i=1}^{n}\alpha_i r_i^{\rho} = \sum_{i\in S}\alpha_i + \sum_{i\in T}\alpha_i\left(\lim_{\rho\to-\infty} r_i^{\rho}\right) \tag{24}$$

Now consider two cases:
If $T = \emptyset$, then $x_i = \min(\mathbf{x})$ for all $i \in \mathcal{I}$. Then all of the fractions are equal to one, and . In this case

$$\lim_{\rho\to-\infty}\sum_{i=1}^{n}\alpha_i r_i^{\rho} = \sum_{i=1}^{n}\alpha_i\left(\lim_{\rho\to-\infty} r_i^{\rho}\right) = \sum_{i\in S}^{n}\alpha_i = 1 \tag{25}$$

where the second equality follows from our parametric assumption that all α_i must sum up to one.

Now consider the second case, where there exists at least one $i \in \mathcal{I}$ such that $x_i > \min(\mathbf{x})$ so $T \neq \emptyset$. S can never be empty, because of the existence of a minimum in $\mathbf{x}$. Because $r_i > 1$ for all $i \in T$, the second term in (24) vanishes as $\rho \to -\infty$[2]. But because $\alpha_i > 0$ for all $i \in \mathcal{I}$ and because $S \neq \emptyset$, the first sum on the right-hand side of (24) is always strictly larger than zero.

$$\lim_{\rho\to-\infty}\sum_{i=1}^{n}\alpha_i r_i^{\rho} = \sum_{i\in S}\alpha_i > 0 \tag{26}$$

Hence, we can bound the term in the log-operator

$$0 < \lim_{\rho\to-\infty}\sum_{i=1}^{n}\alpha_i r_i^{\rho} < 1 \tag{27}$$

It follows that the counter in (20) is well-defined[3]

$$-\infty < \lim_{\rho\to-\infty}\ln\left(\sum_{i=1}^{n}\alpha_i r_i^{\rho}\right) < 0 \tag{28}$$

Therefore, the fraction in (20) must tend to zero as $\rho \to -\infty$, that is

$$\lim_{\rho\to-\infty}\exp\left\{\rho^{-1}\ln\left(\sum_{i=1}^{n}\alpha_i r_i^{\rho}\right)\right\} = 1 \tag{29}$$

[2]Take an example: $2^{-4} = 1/16$, while $2^{-10} = 1/1024$, a lot smaller

[3]Meaning this number is finite. In this case, it is the sum of some α_i

It follows from equation (20) that

$$\lim_{\rho\to-\infty}\left(\sum_{i=1}^{n}\alpha_i r_i^{\rho}\right)^{\frac{1}{\rho}} = \lim_{\rho\to-\infty}\exp\left\{\rho^{-1}\ln\left(\sum_{i=1}^{n}\alpha_i r_i^{\rho}\right)\right\} = 1 \tag{30}$$

Plugging this into (16), one obtains

$$\lim_{\rho\to-\infty}\left(\sum_{i=1}^{n}\alpha_i x_i^{\rho}\right)^{\frac{1}{\rho}} = \underline{\mathbf{x}}\lim_{\rho\to-\infty}\left(\sum_{i=1}^{n}\alpha_i\left(\frac{x_i}{\underline{\mathbf{x}}}\right)^{\rho}\right)^{\frac{1}{\rho}} = \underline{\mathbf{x}} \tag{31}$$

3 Exercise 3.26

3.1 Original question

Calculate the cost function and conditional input demands for the Leontief production function in exercise 3.10 [4].

3.2 Solution

First, observe that we can write the Leontief technology as follows

$$f(\mathbf{x}) = \begin{cases} \alpha x_1 & \text{if} \quad \alpha x_1 < \beta x_2 \\ \alpha x_1 = \beta x_2 & \text{if} \quad \alpha x_1 = \beta x_2 \\ \beta x_2 & \text{if} \quad \alpha x_1 > \beta x_2 \end{cases} \tag{32}$$

So we see that for every bit of input to be used in production, we need the ratio between inputs to be exactly as $x_2/x_1 = \alpha/\beta$. Since cost minimization requires zero waste in money, every bit of input purchased must be used in production. So, regardless of the price ratio, a producer faces with Leontief technology will always choose x_1 and x_2 in a fixed ratio that satisfies $x_2/x_1 = \alpha/\beta$. Therefore, production will satisfy $y = \alpha x_1 = \beta x_2$ and conversely, the inputs satisfy $x_1 = \alpha^{-1}y$ and $x_2 = \beta^{-1}y$. So the firm's conditional input demands are

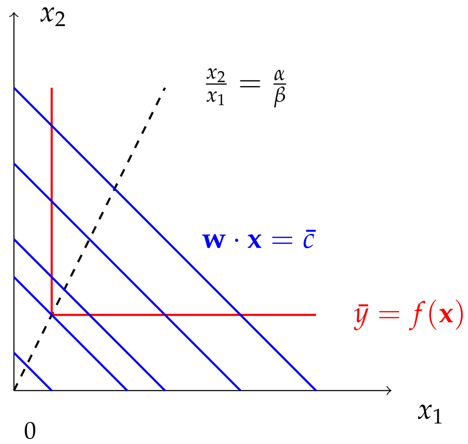
$$x_1(w,y) = \alpha^{-1}y \qquad x_2(w,y) = \beta^{-1}y \tag{33}$$

Hence, the cost function is given by

$$c(w,y) = w_1\alpha^{-1}y + w_2\beta^{-1}y \tag{34}$$

We can also observe the fact by looking at the isoquant curve for production function:

[4] There is a typo in the original text, where it said "exercise 3.8"



The red L shaped curve is the isoquant representing all input combination which can produce the same amount of output $\bar{y}$- $\{x \in \mathbb{R}^2_+ | \bar{y} = \min\{\alpha x_1, \beta x_2\}\}$.

The negatively sloped blue lines show input combinations which cost the same- $\{x \in \mathbb{R}^2_+ | \mathbf{w} \cdot \mathbf{x} = \bar{c}\}$.

No matter the slope of the isocost curves, as long as $\mathbf{w} >> 0$, cost is minimized at the point where $\bar{y} = \alpha x_1 = \beta x_2$.

4 Exercise 3.29

4.1 Original question

In Figure 3.8 (check in the textbook yourself, p.158), the cost functions of firms A and B are grpahed against the input price w_1 for fixed values w_2 and y.

(a) At wage rate w_1^0, which firm uses more of input 1? At w_1^1? Explain.

(b) Which firm's production function has the higher elasticity of substitution? Explain.

4.2 Solution

(a) Looking at Figure 3.8, we observe four different facts. Firstly, the derivative of the cost function is constant for all $\mathbf{w}$, as the cost function is linear. Secondly, at $\mathbf{w}^0$, the derivative of the $c^B(\mathbf{w}, y)$ with respect to good w_1 is greater than the derivative of $c^A(\mathbf{w}, y)$. At $\mathbf{w}^1$ this relationship is reversed. At $\mathbf{w}^1$ firm B's derivative of the cost function with respect to w_1 is lower than at $\mathbf{w}^0$. Thirdly, at both w^0 and w^1, the two cost functions are equal.

$$c_1^A(\mathbf{w}^0, y) < c_1^B(\mathbf{w}^0, y) \tag{35}$$
$$c_1^A(\mathbf{w}^1, y) > c_1^B(\mathbf{w}^1, y) \tag{36}$$
$$c_1^A(\mathbf{w}^0, y) = c_1^A(\mathbf{w}^1, y) \tag{37}$$
$$c_1^B(\mathbf{w}^0, y) > c_1^B(\mathbf{w}^1, y) \tag{38}$$

What can we do with these facts? The key is to use Shephard's Lemma for cost-minimization (see p.138), which states that $c_i(\mathbf{w}, y) = x_i(\mathbf{w}, y)$ and allows us to convert the derivatives of the cost-function into input quantities. Apply the lemma to equations (35) through (38) gives

$$x_1^A(\mathbf{w}^0, y) < x_1^B(\mathbf{w}^0, y) \tag{39}$$
$$x_1^A(\mathbf{w}^1, y) > x_1^B(\mathbf{w}^1, y) \tag{40}$$
$$x_1^A(\mathbf{w}^0, y) = x_1^A(\mathbf{w}^1, y) \tag{41}$$
$$x_1^B(\mathbf{w}^0, y) > x_1^B(\mathbf{w}^1, y) \tag{42}$$

So we can infer from the graph that if costs are rising more strongly in the price of good 1 for firm B than for firm A at a certain point, then firm B must use a greater amount of good 1 in production firm A. Equations (39) and (40) give the answer to our first question. At $\mathbf{w}^0$, firm B uses a higher amount of input x_1 than firm A, while the reverse holds at $\mathbf{w}^1$.

(b) The fact that firm A has a cost function that is linear in w, suggests that it is limited in its ability to substitute input x_1 for x_2 as the price of the former increases. We have already seen one case of linear cost functions in conjunction with the Leontief technology, which is the limit of the CES function as the elasticity of substitution tends to zero.
Meanwhile, for firm B the input composition is quite sensitive to changes in prices. As x_1 becomes more expensive, the firm can minimize costs by reducing its use of x_1 and increasing its use of x_2. It cannot be that the function reduces x_2 at the same time as its decreasing x_1, as under normal assumptions this would mean that we no longer move along a given isoquant (producing y), but produce strictly less than y. Hence, the ratio of inputs x_1 over x_2 must change as prices change for firm B. We conclude that the elasticity of substitution must be higher for firm B than for firm A.

5 Exercise 3.32

5.1 Original question

Show that when average cost is declining, marginal cost must be less than average cost; when average cost is constant, marginal cost must equal average cost; and when average cost is increasing, marginal cost must be greater than average cost.

5.2 Solution

Let $C(\mathbf{w}, y)$ denote the cost function. Then average cost is total cost divided by the output level:

$$AC(\mathbf{w}, y) = \frac{C(\mathbf{w}, y)}{y} \tag{43}$$

Differentiate the above with respect to y to get:

$$\frac{\partial AC}{\partial y} = \frac{1}{y}\frac{\partial C(\mathbf{w}, y)}{\partial y} - \frac{C(\mathbf{w}, y)}{y^2} = \frac{1}{y}\left[\frac{\partial C(\mathbf{w}, y)}{\partial y} - AC(\mathbf{w}, y)\right] \tag{44}$$

By definition of marginal cost, this equation can be written as:

$$\frac{\partial AC}{\partial y} = \frac{1}{y}[\frac{\partial C(\mathbf{w}, y)}{\partial y} - AC(\mathbf{w}, y)] = \frac{1}{y}[MC(\mathbf{w}, y) - AC(\mathbf{w}, y)] \tag{45}$$

Then, if average cost is increasing, i.e. $\frac{\partial AC}{\partial y} > 0$, then from the equation above we know $MC > AC$.
If average cost is constant, i.e. $\frac{\partial AC}{\partial y} = 0$, then from the equation above we know $MC = AC$.
If average cost is increasing, i.e. $\frac{\partial AC}{\partial y} < 0$, then from the equation above we know $MC < AC$.

6 Exercise 3.36, not in the homework, only for reference

6.1 Original question

Consider the two-input, constant returns, Cobb Douglas technology- $y = f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 < \alpha < 1$. Do the following-

(a) Derive the cost function

(b) Fix one input, and derive the short run cost function.

(c) Show that the long-run average and long-run marginal cost are constant and equal.

(d) Show that for every level of fixed input, short-run average cost and the long-run average cost are equal at the minimum level of short-run average cost.

6.2 Solution

(a) Consider the following minimization problem-

$\min\limits_{\mathbf{x} \in \mathbb{R}^2_+} w_1x_1 + w_2x_2$ subject to $x_1^\alpha x_2^{1-\alpha} \geq y$

Note that the constraint of this problem will always bind. The cost is strongly increasing in $\mathbf{x}$ while the production function is strictly increasing and continuous in $\mathbf{x}$. If $f(x_1, x_2) > y$, expenditure can be lowered by reducing the inputs a little while still satisfying the constraint of the problem, implying that at the solution we must have $f(x_1, x_2) = y$.

Set up the Lagrange function-

$\mathcal{L} = w_1x_1 + w_2x_2 - \lambda(x_1^\alpha x_2^{1-\alpha} - y)$

The first order conditions are-

$\frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda\alpha x_1^{\alpha-1}x_2^{1-\alpha} = 0$

$\frac{\partial \mathcal{L}}{\partial x_2} = w_2 - \lambda(1-\alpha)x_1^\alpha x_2^{-\alpha} = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x_1^{\alpha} x_2^{1-\alpha} - y) = 0$

The first two equations can be re-written as-

$\frac{w_1}{w_2} = \frac{\alpha x_2}{(1-\alpha)x_1}$

Using the above and the constraint, solving simultaneously gives us-

$$x_1 = y[\frac{\alpha w_2}{(1-\alpha)w_1}]^{1-\alpha}$$

$$x_2 = y[\frac{(1-\alpha)w_1}{(\alpha)w_2}]^{\alpha}$$

The cost function will be-

$C(\mathbf{w}, y) = w_1 y[\frac{\alpha w_2}{(1-\alpha)w_1}]^{1-\alpha} + w_2 y[\frac{(1-\alpha)w_1}{(\alpha)w_2}]^{\alpha}$

-which simplifies to-

$$C(\mathbf{w}, y) = \frac{y w_1 w_2}{(w_1(1-\alpha))^{1-\alpha}(w_2\alpha)^{\alpha}} \tag{46}$$

(b) Fix x_2 in the short run and denote the fixed amount by $\bar{x}_2$.

The cost minimization problem in the short run is-

$\min_{\mathbf{x} \in \mathbb{R}_+} w_1 x_1 + w_2 \bar{x}_2$ subject to $x_1^{\alpha} \bar{x}_2^{1-\alpha} \geq y$

Using the same arguments as before, we know that the constraint will bind.

Therefore, in the short run $x_1 = [\frac{y}{\bar{x}_2^{1-\alpha}}]^{\frac{1}{\alpha}}$

Set up the Lagrange function-

$\mathcal{L} = w_1 x_1 + w_2 \bar{x}_2 - \lambda(x_1^{\alpha} \bar{x}_2^{1-\alpha} - y)$

First order conditions are as follows-

$\frac{\partial \mathcal{L}}{\partial x_1} = w_1 - \lambda \alpha x_1^{\alpha-1} \bar{x}_2^{1-\alpha} = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = -(x_1^{\alpha} \bar{x}_2^{1-\alpha} - y) = 0$

At the optimal solution we should have $\lambda^* > 0$. Check if this is true for the optimal value of x_1 that we derived above-

$\lambda^* = \frac{w_1}{\alpha} \times [\frac{x_1}{\bar{x}_2}]^{1-\alpha} = \frac{w_1}{\alpha} \times [\frac{y}{\bar{x}_2}]^{\frac{1-\alpha}{\alpha}} > 0$

In the short run, the cost function will be

$$SC(\mathbf{w}, y, \bar{x}_2) = w_1[\frac{y}{\bar{x}_2^{1-\alpha}}]^{\frac{1}{\alpha}} + w_2\bar{x}_2 \tag{47}$$

(c) This part follows straight away from equation (46)-

$AC(\mathbf{w}, y) = \frac{C(\mathbf{w},y)}{y} = \frac{w_1 w_2}{(w_1(1-\alpha))^{1-\alpha}(w_2\alpha)^{\alpha}}$

$MC(\mathbf{w}, y) = \frac{\partial C(\mathbf{w},y)}{\partial y} = \frac{w_1 w_2}{(w_1(1-\alpha))^{1-\alpha}(w_2\alpha)^{\alpha}}$

(d) Using previous calculation, equation (47), the short run average cost function is

$SC(\mathbf{w}, y, \bar{x}_2)/y = w_1[\frac{y}{\bar{x}_2}]^{\frac{1-\alpha}{\alpha}} + \frac{w_2\bar{x}_2}{y}$

To find the minimum level of short run average cost, differentiate the above with respect to y

$\partial(SC(\mathbf{w}, y, \bar{x}_2)/y)/\partial y = w_1 \times [\frac{1}{\bar{x}_2}]^{\frac{1-\alpha}{\alpha}} \times (\frac{1-\alpha}{\alpha}) \times y^{\frac{1-2\alpha}{\alpha}} - \frac{w_2\bar{x}_2}{y^2}$

Setting[5] the above equal to zero we get-

$\bar{x}_2^* = y[\frac{w_1}{w_2}\frac{1-\alpha}{\alpha}]^{\alpha}$

Setting $\bar{x}_2$ equal to the above expression in the short run average cost we get

$SC(\mathbf{w}, y, \bar{x}_2^*)/y = \frac{w_1 w_2}{(w_1(1-\alpha))^{1-\alpha}(w_2\alpha)^{\alpha}} = AC(\mathbf{w}, y)$

[5] We should also check if the second order condition for minimization is satisfied. Taking the second derivative of the short run average cost function with respect to y, setting $\bar{x}_2$ at its optimal level that we find here, we get the following

$\frac{w_1 w_2(1-\alpha)}{y^2\alpha(w_1(1-\alpha))^{1-\alpha}(w_2\alpha)^{\alpha}} > 0$