Lecture on “Optimal Control Theory”

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Introduction

This lecture is based on

- Malcolm Pemberton and Nicholas Rau (2023), *Mathematics for Economists*, chapter 30, chapter 13, chapter 14 and chapter 15.
- Robert Dorfman (1969), "An Economic Interpretation of Optimal Control Theory," *American Economic Review*.

- One of the most significant advances in mathematics occurred in the 1950s.
- Optimal control theory was developed by a team of Russian mathematicians lead by L. S. Pontryagin.
- Back in the 1660s, Isaac Newton discovered a mechanical procedure for finding where functions are maximized (differential calculus).
- Building on Newton's discovery, Pontryagin's team discovered a mechanical procedure for finding where control processes are maximized (optimal control theory).
- This mechanical procedure was explained in their book *The Mathematical Theory of Optimal Processes* and has turned out to be extremely useful for solving dynamic economic models.

- The *Fundamental Problem of Optimal Control* is

$$\boxed{\begin{array}{ll} \max_{I(\cdot)} & \int_0^T F(K(t), I(t), t)\, dt \\ \text{subject to} & \dot{K}(t) = G(K(t), I(t), t) \\ & K(0) = K_0 > 0 \text{ given}, K(T) \geq 0 \end{array}}$$

  where K is called the *state variable* and I is called the *control variable*.
- Thus the problem is to find the control function $I(\cdot)$ that maximizes the integral $\int_0^T F(K(t), I(t), t)\ dt$ subject to the constraint that the state variable K satisfies the differential equation $\dot{K}(t) = G(K(t), I(t), t)$ for all $t \in [0, T]$, the initial condition $K(0) = K_0$ and the terminal condition $K(T) \geq 0$.

- $$\boxed{\begin{array}{ll} \max_{I(\cdot)} & \int_0^T F(K(t), I(t), t)\, dt \\ \text{subject to} & \dot{K}(t) = G(K(t), I(t), t) \\ & K(0) = K_0 > 0 \text{ given}, K(T) \geq 0 \end{array}}$$
- The decisionmaker takes as given the real numbers K_0 and T as well as the functions $F(\cdot)$ and $G(\cdot)$.
- We assume that F and G are continuously differentiable throughout this lecture.
- The solution $I(\cdot)$ to this constrained maximization problem is called the *optimal control*.

- Before we proceed further, a word is in order about notational issues.
- The notation $K(t)$ indicates that K is a function of t.
- For convenience, we will often replace this functional notation by the more compact subscript notation K_t, or omit the t's when they are not particularly needed and leave the functional dependence on t implicit.
- Using the more compact subscript notation, the fundamental problem of optimal control is

$$\boxed{\begin{array}{ll} \max_{I(\cdot)} & \int_0^T F(K_t, I_t, t)\, dt \\ \text{subject to} & \dot{K}_t = G(K_t, I_t, t) \\ & K_0 > 0 \text{ given}, K_T \geq 0 \end{array}}$$

- This optimal control problem has a natural economic interpretation.
- Consider a firm that wants to maximize its discounted profits over some time period.
- With t denoting time, $t = 0$ represents the beginning of the time period and $t = T$ represents the end of the time period, the *terminal time* when the firm plans to go out of business.
- K_t denotes the firm's capital stock at time t, I_t denotes the firm's investment expenditure at time t and K_0 denotes how much capital the firm starts off with at time $t = 0$.
- The firm directly controls (and can immediately change) its investment expenditure I_t whereas its capital stock K_t is a state variable that only gradually changes over time.

- The *state equation* $\dot{K} = G(K_t, I_t, t)$ describes how the change $\dot{K}$ in the firm's capital stock at time t depends on how much capital K_t the firm has, how much it chooses to invest I_t and the time t at which this investment is taking place.
- The terminal condition $K_T \geq 0$ just rules out the possibility that the firm is in debt (has a negative capital stock) at the end of the planning horizon.
- The firm's profits at time t are denoted by $F(K_t, I_t, t)$. The firm's profits also depend on how much capital K_t the firm has, how much it chooses to invest I_t and the time t at which this investment is taking place.
- Assuming that the profits earned by the firm at time t are appropriately discounted back to time $t = 0$, then $\int_0^T F(K_t, I_t, t)\, dt$ represents the firm's discounted profits from $t = 0$ to $t = T$.
- Thus, the optimal control problem specifies that the firm chooses its investment expenditures over time so as to maximize its discounted profits.

The Mechanical Procedure For Finding A Solution

- To solve an optimal control problem, we first construct a Hamiltonian function using information about the functions $F(\cdot)$ and $G(\cdot)$ in the optimal control problem.
- The *Hamiltonian function* H is defined by

$$H(K(t), I(t), \lambda(t), t) \equiv F(K(t), I(t), t) + \lambda(t)G(K(t), I(t), t)$$

  where the new term λ is called the *costate variable* and $\lambda(t)$ is called the *costate function*.
- After constructing the Hamiltonian function for the problem, we use the main result in optimal control theory:

- **Theorem 1** If $I^*(t)$ is the control function that solves the optimal control problem, then there exists a continuous costate function $\lambda(t)$ such that, for each $t \in [0, T]$, the following five conditions are satisfied:

  (i) *Pontryagin's Maximum Principle:* the control maximizes the Hamiltonian

$$H(K(t), I^*(t), \lambda(t), t) = \max_{I(t)} H(K(t), I(t), \lambda(t), t),$$

  (ii) *Costate Equation:* the costate function $\lambda(t)$ satisfies the differential equation

$$\dot{\lambda}(t) = -\frac{\partial H(K(t), I^*(t), \lambda(t), t)}{\partial K},$$

- (iii) *State Equation:* the state function $K(t)$ satisfies the differential equation
$$\dot{K}(t) = G(K(t), I^*(t), t),$$
(iv) *Initial Condition:* $K(0) = K_0$,
(v) *Transversality Condition:* $K(T) \geq 0$, $\lambda(T) \geq 0$ and $K(T)\lambda(T) = 0$.
- These conditions are called *necessary* conditions for a solution to the optimal control problem.
- That is, if the problem has a solution, it must satisfy each of these five conditions.
- Note that Theorem 1 does not say that the optimal control problem has a solution.

- To be certain that what satisfies these five necessary conditions really is a solution to the optimal control problem, we use a second result in optimal control theory:
- **Theorem 2** If there exists a control function $I^*(t)$ and a costate function $\lambda(t)$ for the optimal control problem such that, for each $t \in [0, T]$, conditions (i)-(v) in Theorem 1 are satisfied, and

  (vi) *Concavity Condition:* the maximized Hamiltonian, $\tilde{H}(K, \lambda, t) \equiv \max_I H(K, I, \lambda, t)$, is a concave function of K for given λ and t,

  then $I^*(t)$ is the optimal control function.

An Example

- To illustrate how this solution procedure works, consider the following example:

$$\begin{aligned} \max_{I(\cdot)} \quad & \int_0^1 [K(t) - I(t)^2] dt \\ \text{subject to} \quad & \dot{K}(t) = I(t) \\ & K(0) = 2, K(1) \geq 0 \end{aligned}$$

- The first step is to form the Hamiltonian function, which for this optimal control problem is

$$H \equiv F + \lambda G = K - I^2 + \lambda I.$$

- Next, we will go through sequentially and check each of the conditions in Theorem 1.
- Since these are *necessary* conditions, at any solution to the optimal control problem, all of these conditions must satisfied.
- The first necessary condition to check is Pontryagin's Maximum Principle, which states that the optimal control maximizes the Hamiltonian function $H = K - I^2 + \lambda I$ at each point in time.
- Since $\frac{\partial^2 H}{\partial I^2} = -2 < 0$, the Hamiltonian function has the right curvature with respect to I and the control I that maximizes the Hamiltonian function must satisfy the first order condition

$$\frac{\partial H}{\partial I} = -2I + \lambda = 0.$$

- This yields

$$I(t) = \frac{\lambda(t)}{2}.$$

- Given $H = K - I^2 + \lambda I$, the second necessary condition to check is the costate equation

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial K} = -1.$$

- Normally the costate equation and the state equation yield two differential equations that must be solved simultaneously (as we will see later) but for this particular problem, the costate equation can be solved immediately.
- This simple differential equation has as its general solution $\lambda(t) = c_1 - t$ where c_1 is an arbitrary constant of integration.

- The third necessary condition to check is the state equation

$$\dot{K}(t) = I(t) = \frac{\lambda(t)}{2} = \frac{c_1}{2} - \frac{1}{2}t.$$

- The general solution to this simple differential equation is

$$K(t) = \frac{c_1}{2}t - \frac{1}{4}t^2 + c_2$$

where c_2 is a second arbitrary constant of integration.

- The fourth necessary condition to check is the initial condition $K(0) = 2$. This condition implies that $c_2 = 2$ and thus pins down the second constant of integration.

- The fifth and final necessary condition to check is the transversality condition $K(1) \geq 0$, $\lambda(1) \geq 0$ and $K(1)\lambda(1) = 0$.
- This step is tricky since there are two possible ways of satisfying the transversality condition: $K(1) = 0$ and $\lambda(1) = 0$.
- First, consider the case where $K(1) = 0$.
- Given $K(t) = \frac{c_1}{2}t - \frac{1}{4}t^2 + 2$, then $K(1) = \frac{c_1}{2} - \frac{1}{4} + 2 = 0$ imply that $c_1 = -\frac{7}{2}$.
- It immediately follows that $\lambda(t) = c_1 - t = -\frac{7}{2} - t$ and $\lambda(1) = -\frac{9}{2} < 0$.
- Thus the transversality condition $\lambda(1) \geq 0$ cannot be satisfied when $K(1) = 0$.

- Second, consider the case where $\lambda(1) = 0$.
- Given $\lambda(t) = c_1 - t$, then $\lambda(1) = c_1 - 1 = 0$ implies that $c_1 = 1$ and it immediately follows from $K(t) = \frac{c_1}{2}t - \frac{1}{4}t^2 + 2$ that $K(1) = \frac{1}{2} - \frac{1}{4} + 2 = \frac{9}{4} > 0$.
- Thus the transversality condition is satisfied in this second case.

- To summarize, checking all the necessary conditions has yielded one candidate solution to the optimal control problem:

$$I^*(t) = \frac{\lambda(t)}{2} = \frac{1}{2} - \frac{1}{2}t.$$

- Furthermore the paths of the costate and state variables associated with this candidate solution are

$$\lambda(t) = c_1 - t = 1 - t$$

and

$$K(t) = \frac{c_1}{2}t - \frac{1}{4}t^2 + c_2 = \frac{1}{2}t - \frac{1}{4}t^2 + 2.$$

- To determine whether the candidate solution really is the solution to the optimal control problem, we use Theorem 2.
- The maximized Hamiltonian function is obtained by plugging $I = \frac{\lambda}{2}$ back into the Hamiltonian function $H = K - I^2 + \lambda I$:

$$\tilde{H}(K, \lambda, t) \equiv \max_{I} H(K, I, \lambda, t) = K + \frac{\lambda^2}{4}$$

- Since $\frac{\partial \tilde{H}}{\partial K} = 1$ and $\frac{\partial^2 \tilde{H}}{\partial K^2} = 0$, the maximized Hamiltonian is linear in K and therefore also concave in K.
- Thus, the candidate solution is the solution to the optimal control problem.

- $$\begin{aligned} \max_{I(\cdot)} \quad & \int_0^1 [K(t) - I(t)^2] dt \\ \text{subject to} \quad & \dot{K}(t) = I(t) \\ & K(0) = 2, K(1) \geq 0 \end{aligned}$$
- Note that the optimal control $I^*(t) = \frac{1}{2} - \frac{1}{2}t$ has intuitively reasonable properties.
- Investment I declines over time and converges to zero as the firm approaches the terminal time $T = 1$.
- This seems reasonable since only profits earned before time $T = 1$ matter for the firm and as the firm approaches the terminal time, the benefits from investing become progressively weaker.

An Intuitive Explanation for Optimal Control Theory

- Although the above-described mechanical procedure for solving optimal control problems works, it is not obvious why it works.
- It is also not obvious what economic interpretation can be given to the new costate variable.
- In a classic paper "An Economic Interpretation of Optimal Control Theory" published in the *American Economic Review* in 1969, the Harvard University economist Robert Dorfman developed a heuristic proof of the main theorem of optimal control theory.
- This heuristic proof is presented next and sheds considerable light on why the mechanical procedure works.
- It also yields an economic interpretation of the costate variable.

- Returning to the optimal control problem

$$\boxed{\begin{array}{rl} \max_{I(\cdot)} & \int_0^T F(K(t), I(t), t)\, dt \\ \text{subject to} & \dot{K}(t) = G(K(t), I(t), t) \\ & K(0) = K_0 > 0 \text{ given}, K(T) \geq 0, \end{array}}$$

suppose that this problem has the same economic interpretation as was given earlier, namely, that a firm chooses its investment expenditure over time so as to maximize its discounted profits.

- To determine the optimal time path of investment $I(\cdot)$, we first define a *value function* for the firm:

$$V(K_t, I(\cdot), t) \equiv \int_t^T F(K(\tau), I(\tau), \tau)\ d\tau.$$

- The value function $V(K_t, I(\cdot), t)$ gives the discounted profits of the firm from time t to the terminal time T when the firm starts off with the capital stock K_t and follows the investment path $I(\cdot)$ from time t on.
- We can break this value function into two parts by considering a time interval Δ following t and then the rest of the time period.
- This yields

$$V(K_t, I(\cdot), t) = \int_t^{t+\Delta} F(K(\tau), I(\tau), \tau)\ d\tau + \int_{t+\Delta}^T F(K(\tau), I(\tau), \tau)\ d\tau.$$

- $$V(K_t, I(\cdot), t) = \int_t^{t+\Delta} F(K(\tau), I(\tau), \tau)\, d\tau + \int_{t+\Delta}^{T} F(K(\tau), I(\tau), \tau)\, d\tau.$$

- Now if the interval Δ is sufficiently short so that K and I do not change much over the interval, then $F(\cdot)$ is roughly constant over the interval and the value function can be written more simply as

$$V(K_t, I(\cdot), t) \cong F(K_t, I_t, t)\Delta + \int_{t+\Delta}^{T} F(K(\tau), I(\tau), \tau)\, d\tau.$$

- It immediately follows that the value function satisfies the recursive equation

$$V(K_t, I(\cdot), t) \cong F(K_t, I_t, t)\Delta + V(K_{t+\Delta}, I(\cdot), t + \Delta).$$

- Since the firm maximizes discounted profits, it obviously maximizes discounted profits from any point in time on.
- So let

$$V^*(K_t, t) \equiv \max_{I(\cdot)} V(K_t, I(\cdot), t)$$

denote the maximized discounted profits from time t on given that the firm starts off with the capital stock K_t at time t.

- Then it immediately follows from the recursive equation

$$V(K_t, I(\cdot), t) \cong F(K_t, I_t, t)\Delta + V(K_{t+\Delta}, I(\cdot), t + \Delta).$$

that

$$V^*(K_t, t) \cong \max_{I_t} F(K_t, I_t, t)\Delta + V^*(K_{t+\Delta}, t + \Delta).$$

- Keep in mind that given the initial capital stock K_t at time t, the investment choice I_t during the following time interval Δ determines the capital stock $K_{t+\Delta}$ that the firm inherits at time $t + \Delta$.
- Differentiating the right hand side (henceforth abbreviated RHS) of

$$V^*(K_t, t) \cong \max_{I_t} F(K_t, I_t, t)\Delta + V^*(K_{t+\Delta}, t + \Delta).$$

with respect to I_t yields the first order condition

$$\frac{\partial F(K_t, I_t, t)}{\partial I_t}\Delta + \frac{\partial V^*(K_{t+\Delta}, t + \Delta)}{\partial K_{t+\Delta}} \cdot \frac{\partial K_{t+\Delta}}{\partial I_t} = 0.$$

- But
$$K_{t+\Delta} \cong K_t + \dot{K}(t)\Delta = K_t + G(K_t, I_t, t)\Delta,$$
where the first equality follows from the theory of first degree Taylor polynomial approximations and the second equality is implied by the state equation $\dot{K}(t) = G(K_t, I_t, t)$.
- Thus
$$\frac{\partial K_{t+\Delta}}{\partial I_t} \cong \frac{\partial G(K_t, I_t, t)}{\partial I_t}\Delta.$$
- Furthermore, let
$$\lambda(t) \equiv \frac{\partial V^*(K_t, t)}{\partial K_t}$$
be the marginal value of capital at time t.

- Substituting these expressions $\partial K_{t+\Delta}/\partial I_t \cong \partial G(K_t, I_t, t)/\partial I_t \cdot \Delta$ and $\lambda(t) \equiv \partial V^*(K_t, t)/\partial K_t$ back into the first order condition

$$\frac{\partial F(K_t, I_t, t)}{\partial I_t}\Delta + \frac{\partial V^*(K_{t+\Delta}, t+\Delta)}{\partial K_{t+\Delta}} \cdot \frac{\partial K_{t+\Delta}}{\partial I_t} = 0,$$

we obtain

$$\frac{\partial F(K_t, I_t, t)}{\partial I_t}\Delta + \lambda(t+\Delta)\frac{\partial G(K_t, I_t, t)}{\partial I_t}\Delta \cong 0$$

- This equation can be further simplified by dividing both sides by Δ and then recognizing that as Δ converges to zero, $\lambda(t+\Delta)$ converges to $\lambda(t)$.

- Taking the limit as Δ converges to zero yields the first order condition for profit maximization

$$\frac{\partial F(K_t, I_t, t)}{\partial I_t} + \lambda_t \frac{\partial G(K_t, I_t, t)}{\partial I_t} = 0.$$

- Given the definition of the Hamiltonian function $H \equiv F(K_t, I_t, t) + \lambda_t G(K_t, I_t, t)$,

$$\frac{\partial H}{\partial I_t} = 0$$

  coincides with this equation.
- Thus, we have derived the first order condition for *Pontryagin's maximum principle*. Note that this equation must hold for all time t if the investment path $I(\cdot)$ is profit-maximizing.

- Equation

$$\frac{\partial F(K_t, I_t, t)}{\partial I_t} + \lambda_t \frac{\partial G(K_t, I_t, t)}{\partial I_t} = 0$$

implies that Pontryagin's maximum principle has a natural economic interpretation.

- $\partial F(K_t, I_t, t)/\partial I_t$ represents the marginal contribution of investment to the firm's current profits (at time t).
- λ_t is the marginal value of capital at time t.
- $\partial G(K_t, I_t, t)/\partial I_t = \partial \dot{K}(t)/\partial I_t$ is the marginal contribution of investment to the growth of the capital stock.

- $$\frac{\partial F(K_t, I_t, t)}{\partial I_t} + \lambda_t \frac{\partial G(K_t, I_t, t)}{\partial I_t} = 0$$
- Normally, we would expect $\partial F(K_t, I_t, t)/\partial I_t < 0$ since investment is costly and the benefits from current investment come in the future, $\lambda_t > 0$ since capital has positive value and $\partial G(K_t, I_t, t)/\partial I_t > 0$ since it is by investing that the capital stock grows.
- Thus Pontryagin's maximum principle states that the marginal cost of investment $-\partial F(K_t, I_t, t)/\partial I_t$ must equal the marginal benefit of investment $\lambda_t \cdot \partial G(K_t, I_t, t)/\partial I_t$ at each point in time.
- Investment on the margin benefits a firm by contributing to the growth in the value of the capital stock an instant later.

$$-MC_I + MB_I = 0 \quad \Longrightarrow \quad MC_I = MB_I$$

- The next step in Dorfman's heuristic proof is a derivation of the costate equation.
- Equation

$$V^*(K_t, t) \cong \max_{I_t} F(K_t, I_t, t)\Delta + V^*(K_{t+\Delta}, t + \Delta).$$

implies that the optimal investment path $I(\cdot)$ satisfies the recursive equation

$$V^*(K_t, t) \cong F(K_t, I_t, t)\Delta + V^*(K_{t+\Delta}, t + \Delta).$$

- Note that this equation holds for all t and for all K_t. Thus, we can differentiate both sides with respect to K_t and this yields

$$\frac{\partial V^*(K_t, t)}{\partial K_t} \cong \cdot \frac{\partial F(K_t, I_t, t)}{\partial K_t}\Delta + \frac{\partial V^*(K_{t+\Delta}, t + \Delta)}{\partial K_{t+\Delta}} \cdot \frac{\partial K_{t+\Delta}}{\partial K_t}.$$

- Since $\partial V^*(K_t, t)/\partial K_t = \lambda(t)$ is the marginal value of capital at time t,
$$\partial V^*(K_{t+\Delta}, t+\Delta)/\partial K_{t+\Delta} = \lambda(t+\Delta)$$
is the marginal value of capital an instant later.
- Using the earlier Taylor polynomial approximation $K_{t+\Delta} \cong K_t + \dot{K}(t)\Delta = K_t + G(K_t, I_t, t)\Delta$, it immediately follows that
$$\frac{\partial K_{t+\Delta}}{\partial K_t} \cong 1 + \frac{\partial G(K_t, I_t, t)}{\partial K_t}\Delta.$$

- Substituting these expressions and the Taylor polynomial approximation $\lambda(t+\Delta) \cong \lambda(t) + \dot{\lambda}(t)\Delta$ back into the previous equation

$$\frac{\partial V^*(K_t, t)}{\partial K_t} \cong \cdot \frac{\partial F(K_t, I_t, t)}{\partial K_t}\Delta + \frac{\partial V^*(K_{t+\Delta}, t+\Delta)}{\partial K_{t+\Delta}} \cdot \frac{\partial K_{t+\Delta}}{\partial K_t}$$

yields

$$\lambda(t) \cong \frac{\partial F(K_t, I_t, t)}{\partial K_t}\Delta + \left(\lambda(t) + \dot{\lambda}(t)\Delta\right)\left(1 + \frac{\partial G(K_t, I_t, t)}{\partial K_t}\Delta\right)$$

or when the terms in parenthesis are multiplied out

$$\lambda(t) \cong \frac{\partial F(K_t, I_t, t)}{\partial K_t}\Delta + \lambda(t) + \lambda(t)\frac{\partial G(K_t, I_t, t)}{\partial K_t}\Delta$$

$$+\dot{\lambda}(t)\Delta + \dot{\lambda}(t)\Delta\frac{\partial G(K_t, I_t, t)}{\partial K_t}\Delta.$$

- $$\lambda(t) \cong \frac{\partial F(K_t, I_t, t)}{\partial K_t}\Delta + \lambda(t) + \lambda(t)\frac{\partial G(K_t, I_t, t)}{\partial K_t}\Delta$$
$$+\dot{\lambda}(t)\Delta + \dot{\lambda}(t)\Delta\frac{\partial G(K_t, I_t, t)}{\partial K_t}\Delta.$$
- Canceling the $\lambda(t)$ terms on both sides, then dividing both sides by Δ and letting Δ converge to zero yields
$$-\dot{\lambda}(t) = \frac{\partial F(K_t, I_t, t)}{\partial K_t} + \lambda(t)\frac{\partial G(K_t, I_t, t)}{\partial K_t}.$$

- $$-\dot{\lambda}(t) = \frac{\partial F(K_t, I_t, t)}{\partial K_t} + \lambda(t)\frac{\partial G(K_t, I_t, t)}{\partial K_t}.$$
- Given the definition of the Hamiltonian function $H \equiv F(K_t, I_t, t) + \lambda_t G(K_t, I_t, t)$,

  $$-\dot{\lambda} = \frac{\partial H}{\partial K_t}$$

  coincides with this equation.
- Thus, we have derived the *costate equation*.
- Note that this equation must hold for all time t if the investment path $I(\cdot)$ is profit-maximizing.

- Like Pontryagin's maximum principle, the costate equation has a natural economic interpretation.
- Normally, one expects each unit of capital to lose value as the terminal time T approaches, that is, $\dot{\lambda}(t) < 0$. Thus $-\dot{\lambda}(t) > 0$ is the rate at which a unit of capital depreciates.
- $\partial F(K_t, I_t, t)/\partial K_t$ is the marginal contribution of capital to the firm's current profits and $\partial G(K_t, I_t, t)/\partial K_t = \partial \dot{K}(t)/\partial K_t$ is the marginal contribution of capital to the growth rate of the capital stock.
- Thus the costate equation

$$-\dot{\lambda}(t) = \frac{\partial F(K_t, I_t, t)}{\partial K_t} + \lambda(t)\frac{\partial G(K_t, I_t, t)}{\partial K_t}.$$

states that along the optimal investment path $I(\cdot)$, at each point in time t, capital is depreciating at the same rate at which it is giving off value by (i) contributing to current profits and (ii) enhancing the value of the capital stock (future profits).

- The only remaining condition to derive is the transversality condition.
- But this condition follows immediately from our earlier analysis, in particular, given that λ is the marginal value of capital.
- At the terminal time T, the total capital stock that remains must have zero value ($\lambda_T K_T = 0$) since the firm will not be earning any more profits from this capital after time T.
- The firm goes out of business at time T and ceases to exist.
- If the firm could sell its remaining capital stock to another firm, then the capital stock would have positive value at time T but we are implicitly assuming that this is not possible in the fundamental problem of optimal control.

Autonomous Optimal Control Problems

- A special type of optimal control problem that frequently arises in economic applications is the *autonomous* optimal control problem

$$\boxed{\begin{array}{ll} \max_{I(\cdot)} & \int_0^T F(K(t), I(t)) e^{-\rho t} dt \\ \text{subject to} & \dot{K}(t) = G(K(t), I(t)) \\ & K(0) = K_0 > 0 \text{ given}, K(T) \geq 0 \end{array}}$$

- What is special about autonomous optimal control problems is that time t does not appear as a separate argument in the $G(\cdot)$ function and only appears separately inside the integral in the form of the discounting term $e^{-\rho t}$.

- $$\boxed{\begin{array}{ll} \max_{I(\cdot)} & \int_0^T F(K(t), I(t)) e^{-\rho t} dt \\ \text{subject to} & \dot{K}(t) = G(K(t), I(t)) \\ & K(0) = K_0 > 0 \text{ given}, K(T) \geq 0 \end{array}}$$
- Since any autonomous optimal control problem is a special case of the fundamental problem of optimal control, the previously developed techniques can be used to solve any autonomous optimal control problem.
- However there is a short-cut way of solving autonomous optimal control problems that exploits their special structure (this short-cut should always be used when it can be used).

- From before, the *Hamiltonian function* (or more precisely the *present-valued Hamiltonian function*) is defined by

$$H \equiv F(K(t), I(t))e^{-\rho t} + \lambda(t)G(K(t), I(t))$$

where $\lambda(t)$ is the *present-valued costate variable*.
- Instead of valuing outcomes relative to time 0, it is convenient in autonomous optimal control problems to value outcomes relative to the current time t.
- To do this, we first let

$$\mu(t) \equiv \lambda(t)e^{\rho t}$$

denote the *current-valued costate variable* (pronounced "mu") and let

$$\mathcal{H} \equiv He^{\rho t} = F(K(t), I(t)) + \mu(t)G(K(t), I(t))$$

denote the *current-valued Hamiltonian function*.

- $$H \equiv F(K(t), I(t))e^{-\rho t} + \lambda(t)G(K(t), I(t))$$
  $$\mathcal{H} \equiv He^{\rho t} = F(K(t), I(t)) + \mu(t)G(K(t), I(t))$$
- Whereas $\lambda(t)$ represents the marginal value of capital at time t discounted back to time 0, $\mu(t)$ represents the undiscounted marginal value of capital at time t.
- $\lambda(t)$ is the answer to the question, how much would the firm be willing to pay at time 0 for one unit of capital at time t? $\mu(t)$ is the answer to the question, how much would the firm be willing to pay at time t for one unit of capital at time t?

- $$H \equiv F(K(t), I(t))e^{-\rho t} + \lambda(t)G(K(t), I(t))$$
$$\mathcal{H} \equiv He^{\rho t} = F(K(t), I(t)) + \mu(t)G(K(t), I(t))$$
- From the definition of the current-valued Hamiltonian function, it follows that $\partial H/\partial I = 0$ if and only if
$$\frac{\partial \mathcal{H}}{\partial I} = 0.$$
- Thus, as an alternative to maximizing the present-valued Hamiltonian with respect to the control at each point in time, we can solve autonomous optimal control problems by maximizing the current-valued Hamiltonian with respect to the control at each point in time. This is Pontryagin's Maximum Principle restated.

- Finding a condition that corresponds to the earlier costate equation

$$\dot{\lambda} = -\frac{\partial H}{\partial K}$$

takes a bit more work.

- Differentiating $\mathcal{H} = He^{\rho t}$ with respect to K yields

$$\frac{\partial \mathcal{H}}{\partial K} = \frac{\partial H}{\partial K} e^{\rho t}$$

and differentiating $\mu(t) = \lambda(t)e^{\rho t}$ with respect to t yields

$$\dot{\mu}(t) = \dot{\lambda}(t)e^{\rho t} + \rho\lambda(t)e^{\rho t} = \dot{\lambda}(t)e^{\rho t} + \rho\mu(t).$$

- Thus

$$\dot{\lambda}(t) = [\dot{\mu}(t) - \rho\mu(t)]\, e^{-\rho t} = -\frac{\partial H}{\partial K} = -\frac{\partial \mathcal{H}}{\partial K} e^{-\rho t}.$$

- $$\dot{\lambda}(t) = [\dot{\mu}(t) - \rho\mu(t)]\, e^{-\rho t} = -\frac{\partial H}{\partial K} = -\frac{\partial \mathcal{H}}{\partial K} e^{-\rho t}.$$

- Since the $e^{-\rho t}$ terms on both sides cancel, we obtain a *current-valued costate equation*

$$\dot{\mu}(t) - \rho\mu(t) = -\frac{\partial \mathcal{H}}{\partial K}$$

  that corresponds to the present-valued costate equation

$$\dot{\lambda} = -\frac{\partial H}{\partial K}.$$

- Since $\mu(t) = \lambda(t)e^{\rho t}$ implies that $\lambda(t) = \mu(t)e^{-\rho t}$ and

$$K(T)\lambda(T) = K(T)\mu(T)e^{-\rho T} = 0,$$

  the transversality condition $K(T)\lambda(T) = 0$ holds if and only if $K(T)\mu(T) = 0$.

- We are ready to state a new theorem for autonomous optimal control problems:

Theorem 3 If $I^*(t)$ is the control function that solves the autonomous optimal control problem, then there exists a continuous costate function $\mu(t)$ such that, for each $t \in [0, T]$, the following five conditions are satisfied:
(i) *Pontryagin's Maximum Principle:* the control maximizes the current-valued Hamiltonian

$$\mathcal{H}(K(t), I^*(t), \mu(t)) = \max_{I(t)} \mathcal{H}(K(t), I(t), \mu(t)),$$

- (ii) *Costate Equation:* the costate function $\mu(t)$ satisfies the differential equation

$$\dot{\mu}(t) - \rho\mu(t) = -\frac{\partial\mathcal{H}(K(t), I^*(t), \mu(t))}{\partial K},$$

  (iii) *State Equation:* the state function $K(t)$ satisfies the differential equation

$$\dot{K}(t) = G(K(t), I^*(t)),$$

  (iv) *Initial Condition:* $K(0) = K_0$,
  (v) *Transversality Condition:* $K(T) \geq 0$, $\mu(T) \geq 0$ and $K(T)\mu(T) = 0$.

- Theorem 3 gives necessary conditions for the optimal control.
- Earlier, the corresponding Theorem 2 with sufficient conditions for the optimal control specified that the maximized Hamiltonian function be concave with respect to the state variable, holding time and the costate variable fixed.
- The same concavity condition carries over to the maximized current-valued Hamiltonian:

- **Theorem 4** If there exists a control function $I^*(t)$ and a costate function $\mu(t)$ for the autonomous optimal control problem such that, for each $t \in [0, T]$, conditions (i)-(v) in Theorem 3 are satisfied, and

  (vi) *Concavity Condition:* the maximized current-valued Hamiltonian, $\tilde{\mathcal{H}}(K, \mu) \equiv \max_I \mathcal{H}(K, I, \mu)$, is a concave function of K for given μ,

  then $I^*(t)$ is the optimal control function.

- The question naturally arises, where is the simplification? How do these new results for solving autonomous optimal control problems represent a short-cut compared to the earlier results?
- On the surface, it seems like the "current-valued" formulation is a step backwards since the current-valued costate equation

$$\dot{\mu}_t - \rho\mu_t = -\frac{\partial \mathcal{H}}{\partial K}$$

is more complicated than the original costate equation

$$\dot{\lambda} = -\frac{\partial H}{\partial K}.$$

- Surface appearances, however, are misleading and the "current-valued" formulation really does represent a simplification, as we will now explain.

- In an autonomous optimal control problem, the first order condition

$$\frac{\partial \mathcal{H}}{\partial I} = \frac{\partial F(K, I)}{\partial I} + \mu \frac{\partial G(K, I)}{\partial I} = 0$$

  implicitly defines the optimal control I as a function of the state and costate variables, that is, $I = \phi(K, \mu)$, where ϕ is pronounced "phi."
- The important thing to emphasize is that time t is not a separate argument in the ϕ function.
- In contrast, in a more general (or non-autonomous) optimal control problem, the first order condition

$$\frac{\partial H}{\partial I} = \frac{\partial F(K, I, t)}{\partial I} + \lambda \frac{\partial G(K, I, t)}{\partial I} = 0$$

  implicitly defines the optimal control I as a function of the state and costate variables and time, that is, $I = \phi(K, \lambda, t)$.

- The fact that the optimal control $I = \phi(K, \mu)$ does not directly depend on t in autonomous optimal control problems has important implications.
- (Even in autonomous optimal control problems, the optimal control is indirectly a function of t because K and μ are functions of t.)
- Substituting the optimal control function $I = \phi(K, \mu)$ into the state equation $\dot{K} = G(K, I)$ yields

$$\dot{K} = G(K, \phi(K, \mu)),$$

that is, $\dot{K}$ only depends on the current values of K and μ.

- Furthermore, since

$$\frac{\partial \mathcal{H}}{\partial K} = \frac{\partial F(K, \phi(K, \mu))}{\partial K} + \mu \frac{\partial G(K, \phi(K, \mu))}{\partial K},$$

$\partial \mathcal{H} / \partial K$ only depends on the current values of K and μ, it follows from the costate equation

$$\dot{\mu} = \rho\mu - \frac{\partial \mathcal{H}}{\partial K}$$

that $\dot{\mu}$ only depends on the current values of K and μ.

- Thus,
$$\dot{K} = G(K, \phi(K, \mu))$$
and
$$\dot{\mu} = \rho\mu - \frac{\partial \mathcal{H}}{\partial K}$$
represent a system of 2 autonomous differential equations in 2 variables, K and μ.
- This is why the "current-valued" formulation really does represent a simplification compared to the original formulation.
- Autonomous differential equation systems are considerably easier to solve than the non-autonomous differential equation systems that typically arise when solving more general optimal control problems.
- With autonomous differential equation systems, one can use phase diagram solution techniques.

Infinite-Horizon Optimal Control Problems

- Up to now, we have assumed a finite terminal time T. While some problems in economics require finite-horizon optimal control, it is the infinite-horizon ($T = +\infty$) results that are most often used in economic applications.
- To understand some of the new issues that arise with infinite-horizon problems, it is helpful to study the following economic model [taken from Robert Barro and Xavier Sala-i-Martin *Economic Growth* (1995)]:
- Imagine an economy where households provide labor services in exchange for wages, receive interest income on assets, purchase goods for consumption, and save by accumulating additional assets. (first 2 models)
- If we normalize the number of adults at time 0 to unity, then family size at time t – which corresponds to the adult population – is $L(t) = e^{nt}$, where $n > 0$ is the exogenous population growth rate.

- If $C(t)$ is total consumption at time t, then $c(t) \equiv C(t)/L(t)$ is consumption per adult person.
- Each household wishes to maximize overall utility, U, as given by

$$U = \int_0^\infty u[c(t)] \cdot e^{nt} \cdot e^{-\rho t}\, dt$$

where $u[c(t)]$ is static utility per person, $\rho > 0$ is the subjective discount rate and $\rho > n$ is assumed to guarantee that discounted utility U is finite.

- The real rate of return (or interest rate) at time t is $r(t)$. Let $A(t)$ denote total assets and $a(t) \equiv A(t)/L(t)$ denote assets per person.
- Total assets evolve over time according to the differential equation

$$\dot{A}(t) = w(t)L(t) + r(t)A(t) - C(t)$$

where $w(t)$ is the wage rate at time t.

- Dropping the function of time notation, $\dot{A} = wL + rA - C$. Since $A = aL$,

$$\dot{A} = wL + rA - C = \dot{a}L + \dot{L}a = \dot{a}L + nLa = wL + raL - cL.$$

- Dividing by L, we obtain $\dot{a} + na = w + ra - c$ or

$$\dot{a} = w + ra - c - na$$

  This is the flow budget constraint for the household. The equation says that assets per person rise with per capita income $w + ra$, fall with per capita consumption c and fall because of expansion of the population in accordance with the term na.

- If each household can borrow an unlimited amount at the going interest rate $r(t)$, then it has an incentive to pursue a form of chain letter or Ponzi game.
- The household can borrow, say \$1, to finance current consumption and then use future borrowings to roll over the principal and pay all of the interest. In this case, the household's debt grows forever at the rate of interest $r(t)$.
- Since no principal ever gets repaid, today's added consumption of \$1 is effectively free. Thus, a household that can borrow in this manner would be able to finance an arbitrarily high level of consumption in perpetuity.

- To rule out chain-letter possibilities, we assume that the credit market imposes a constraint on the amount of borrowing. The appropriate restriction turns out to be that the present value of assets must be asymptotically nonnegative:

$$\lim_{t\to\infty}\left\{A(t)\exp\left[-\int_0^t r(\tau)d\tau\right]\right\}\geq 0$$

  where we intentionally allow the interest rate $r(t)$ to vary over time.
- Note that if the interest rate is constant over time, $\int_0^t r\,d\tau = r\,t$ and the present value of assets is $A(t)e^{-r\,t}$.
- Since $A(t) = a(t)L(t) = a(t)e^{nt}$, the no Ponzi game restriction can written alternatively as

$$\lim_{t\to\infty}\left\{a(t)\exp\left[-\int_0^t [r(\tau)-n]d\tau\right]\right\}\geq 0.$$

- The household's optimization problem is to maximize discounted utility U subject to the budget constraint, the stock of initial assets $a(0)$ and the limitation on borrowing given by the no Ponzi game restriction.
- The relevant present-value Hamiltonian for this optimal control problem is

$$H = u(c)e^{-(\rho-n)t} + \lambda[w + ra - c - na]$$

and the first order conditions are

$$\frac{\partial H}{\partial c} = u'(c)e^{-(\rho-n)t} - \lambda = 0 \quad \text{and} \quad \dot{\lambda} = -\frac{\partial H}{\partial a} = -(r-n)\lambda.$$

- For the finite time horizon problem, the relevant transversality condition is $\lambda(T)a(T) = 0$. For the infinite time horizon problem, it turns out that the relevant transversality condition is

$$\lim_{t\to\infty} \{\lambda(t)a(t)\} = 0.$$

- The transversality condition $\lim_{t\to\infty}\{\lambda(t)a(t)\} = 0$ says that the value of the household's assets must approach 0 at time approaches infinity.
- If we think of infinity loosely as the end of the planning horizon, then the intuition is that optimizing agents do not want to have any valuable assets left over at the end. Utility would increase if the assets, which are effectively being wasted, were used instead to raise consumption at some dates in finite time.
- The costate equation $\dot{\lambda} = -\frac{\partial H}{\partial a} = -(r-n)\lambda$ implies that $\dot{\lambda}/\lambda = -(r-n)$. Integrating both sides, we obtain

$$\int_0^t \frac{\dot{\lambda}}{\lambda} d\tau = \ln \lambda(\tau)\Big|_0^t = \ln \lambda(t) - \ln \lambda(0) = \ln \frac{\lambda(t)}{\lambda(0)} = -\int_0^t [r(\tau)-n]d\tau$$

or

$$\lambda(t) = \lambda(0)\exp\left[-\int_0^t [r(\tau)-n]d\tau\right]$$

- Leaving out the constant term $\lambda(0)$, it follows that the transversality condition $\lim_{t\to\infty}\{\lambda(t)a(t)\} = 0$ becomes

$$\lim_{t\to\infty}\left\{a(t)\exp\left[-\int_0^t [r(\tau)-n]d\tau\right]\right\} = 0.$$

- The no Ponzi game restriction is $\lim_{t\to\infty}\left\{a(t)\exp\left[-\int_0^t [r(\tau)-n]d\tau\right]\right\} \geq 0$. Thus, if the transversality condition is satisfied, then the no Ponzi game restriction is automatically satisfied. In order to borrow on a perpetual basis, households would have to find willing lenders; that is, other households that were willing to hold positive assets that grew at the rate r or higher. But we know from the above-mentioned transversality condition that these other households will be unwilling to absorb assets asymptotically at this high a rate. Therefore, in equilibrium, each household will be unable to borrow in a chain-letter fashion.

The most useful infinite-horizon results

- In the previous model, the no Ponzi game restriction took the form $\lim_{t\to\infty}\left\{a(t)\exp\left[-\int_0^t[r(\tau)-n]d\tau\right]\right\}\geq 0$.
- More generally, we can write the no Ponzi game restriction as $\lim_{t\to\infty}\{K(t)b(t)\}\geq K_1$ where the function $b:\Re_+\to\Re_+$ that captures the time-varying interest rate satisfies $\lim_{t\to\infty} b(t)<\infty$.
- Economically interesting problems often take the following specific form:

$$\boxed{\begin{aligned} &\max_{I(\cdot)} \quad \int_0^\infty F(K(t),I(t))e^{-\rho t}\,dt \quad \text{with } \rho>0 \\ &\text{subject to} \quad \dot{K}(t)=G(t,K(t),I(t)) \\ &\qquad\qquad\quad K(0)=K_0>0 \text{ given, } \lim_{t\to\infty}\{K(t)b(t)\}\geq K_1. \end{aligned}}$$

- Infinite-horizon optimal control problems are solved in the same general way as finite-horizon optimal control problems with one twist that involves the transversality condition.
- The appropriate generalization of the earlier transversality condition $\lambda(T)K(T) = 0$ to the infinite time horizon is

$$\lim_{t\to\infty} \lambda(t)K(t) = \lim_{t\to\infty} e^{-\rho t}\mu(t)K(t) = 0,$$

  where $\lambda(t)$ is the present-valued costate variable and $\mu(t) \equiv \lambda(t)e^{\rho t}$ is the current-valued costate variable.
- The twist is that this infinite-horizon transversality condition is no longer a necessary condition for the optimal control. But it is still a sufficient condition and for practical purposes, that is usually all that we need to solve infinite-horizon optimal control problems.

- The current-valued Hamiltonian function is now

$$\mathcal{H} \equiv F(K(t), I(t)) + \mu(t) G(t, K(t), I(t))$$

Theorem 5 Suppose that $I^*(t)$ is a continuous interior optimal control function with corresponding continuous interior costate function $\mu(t)$ that solves the infinite-horizon optimal control problem. Then for each $t \in [0, \infty)$, the following four conditions are satisfied:

(i) *Pontryagin's Maximum Principle:* the control maximizes the current-valued Hamiltonian

$$\mathcal{H}(t, K(t), I^*(t), \mu(t)) = \max_{I(t)} \mathcal{H}(t, K(t), I(t), \mu(t)),$$

- (ii) *Costate Equation:* the costate function $\mu(t)$ satisfies the differential equation

$$\dot{\mu}(t) - \rho\mu(t) = -\frac{\partial \mathcal{H}(t, K(t), I^*(t), \mu(t))}{\partial K},$$

(iii) *State Equation:* the state function $K(t)$ satisfies the differential equation

$$\dot{K}(t) = G(t, K(t), I^*(t)),$$

(iv) *Boundary Conditions:* $K(0) = K_0$ and $\lim_{t\to\infty}\{K(t)b(t)\} \geq K_1$.

- Note that the transversality condition does not appear in the new list of necessary conditions. But it does in the next theorem.

- **Theorem 6** If there exists a control function $I^*(t)$ and corresponding costate function $\mu(t)$ for the infinite-horizon optimal control problem such that, for each $t \in [0, \infty)$, conditions (i)-(iv) in Theorem 5 are satisfied, and

  (v) *Transversality Condition:*

$$\lim_{t \to \infty} e^{-\rho t} \mu(t) K(t) = 0,$$

  (vi) *Concavity Condition:* the maximized current-valued Hamiltonian, $\tilde{\mathcal{H}}(t, K, \mu) \equiv \max_I \mathcal{H}(t, K, I, \mu)$, is a concave function of K for all t,

  then $I^*(t)$ is a solution to the optimal control problem.

  Furthermore, $I^*(t)$ is the unique solution if $\tilde{\mathcal{H}}$ is strictly concave.

- Theorem 6 is the most useful result in optimal control theory according to Daron Acemoglu (it is Theorem 7.14 in Acemoglu's 2009 textbook *Introduction to Modern Economic Growth*).
- Acemoglu writes, "While a number of problems in economics require finite-horizon optimal control, most economic problems are more naturally formulated as infinite-horizon problems. This is obvious in the context of economic growth, but is also the case in repeated games, political economy or industrial organization, where even if individuals may have finite expected lives, the end date of the game or of their lives may be uncertain. For this reason, the canonical model of optimization in economic problems is the infinite-horizon one."
- While Theorem 6 has broad applicability within economics, in this course we use it to solve models of international trade.

- Theorem 6 is particularly useful when, as will often occur in economic applications, the necessary conditions in Theorem 5 give rise to an autonomous differential equation system that has a unique saddle-point equilibrium.
- Then it is easy to see that the unique convergent path leading to the steady state must satisfy the transversality condition.
- With $K(t)$ and $\mu(t)$ converging over time to the steady state values $K^* > 0$ and $\mu^* > 0$, respectively, and $e^{-\rho t}$ converging over time to zero, clearly

$$\lim_{t \to \infty} e^{-\rho t} \mu(t) K(t) = 0 \cdot \mu^* \cdot K^* = 0.$$

- With the transversality condition satisfied by any path converging to a steady-state, one just has to check that the concavity condition is satisfied to conclude that the corresponding control is optimal. Furthermore, the optimal control is unique if strict concavity holds.

A Second Example

- To illustrate how Theorems 5 and 6 are used, consider the following autonomous infinite-horizon optimal control problem:

$$\max_{I(\cdot)} \quad \int_0^\infty \{\ln K(t) + \ln(1 - I(t)K(t))\} e^{-2t} dt$$

$$\text{subject to} \quad \dot{K}(t) = K(t)[3I(t) - 4]$$

$$K(0) = 1/10, \lim_{t\to\infty} e^{-2t} K(t) \geq 0.$$

- This is a typical example of the optimal control problems that are encountered when solving economic models. The example is a special case of the social planner's problem in Segerstrom (1998, AER).

- $$\begin{aligned} \max_{I(\cdot)} \quad & \int_0^\infty \{\ln K(t) + \ln(1 - I(t)K(t))\} e^{-2t} dt \\ \text{subject to} \quad & \dot{K}(t) = K(t)[3I(t) - 4] \\ & K(0) = 1/10, \lim_{t\to\infty} e^{-2t} K(t) \geq 0. \end{aligned}$$
- The integral being maximized is the discounted welfare of the representative household in an economy, $K(t)$ represents the economy's capital stock at time t, $I(t)$ is the economy's investment expenditure at time t, and the $-4K(t)$ term in the state equation reflects the fact that capital depreciates over time.
- The state equation implies that, in the absence of investment ($I = 0$), the economy's capital stock shrinks over time due to depreciation ($\dot{K} < 0$).

- $$\begin{aligned} \max_{I(\cdot)} \quad & \int_0^\infty \{\ln K(t) + \ln(1 - I(t)K(t))\} e^{-2t} dt \\ \text{subject to} \quad & \dot{K}(t) = K(t)[3I(t) - 4] \\ & K(0) = 1/10, \lim_{t\to\infty} e^{-2t} K(t) \geq 0. \end{aligned}$$
- The first step in solving the problem is to form the current-valued Hamiltonian function,

$$\mathcal{H} \equiv F + \mu G = \ln K + \ln(1 - IK) + \mu K(3I - 4).$$

- Next, we examine when each of the conditions in Theorem 5 are satisfied. Since these are *necessary* conditions, at any solution to the optimal control problem, all of these conditions must be satisfied.

- The first necessary condition to check is Pontryagin's Maximum Principle, which states that the optimal control maximizes the current-valued Hamiltonian function

$$\mathcal{H} = \ln K + \ln(1 - IK) + \mu K(3I - 4)$$

  at each point in time.
- Since the natural logarithm functions in the current-valued Hamiltonian are only well-defined when $K(t) > 0$ and $1 - I(t)K(t) > 0$, we will focus on finding an optimal control function $I^*(t)$ and corresponding costate $\mu(t)$ and state $K(t)$ functions such that $K(t) > 0$ and $1 - I^*(t)K(t) > 0$ for all $t \in [0, \infty)$.
- The first order condition for maximizing the current-valued Hamiltonian is

$$\frac{\partial \mathcal{H}}{\partial I} = -\frac{K}{1 - IK} + 3\mu K = 0.$$

- $$\frac{\partial \mathcal{H}}{\partial I} = -\frac{K}{1 - IK} + 3\mu K = 0,$$
  implies that $1 - IK = 1/(3\mu)$.
- Given that $K > 0$ and restricting attention to the range of I values where $1 - IK > 0$, it is clear that $\mu > 0$ for all t.
- Solving the first order condition for I yields the optimal control function
  $$I^*(t) = \frac{1}{K(t)}\left(1 - \frac{1}{3\mu(t)}\right).$$
- Since $\partial^2\mathcal{H}/\partial I^2 = -K^2/(1 - IK)^2 < 0$, the current-valued Hamiltonian function has the desired concave curvature with respect to I over the entire relevant range where $1 - IK > 0$, so we definitely have found the control that maximizes the current-valued Hamiltonian.

- The second necessary condition is the costate equation. Solving for this equation using $\mathcal{H} = \ln K + \ln(1 - IK) + \mu K(3I - 4)$ and $1 - IK = 1/(3\mu)$ yields

$$\begin{aligned} \dot{\mu} - 2\mu &= -\tfrac{\partial \mathcal{H}}{\partial K} \\ &= -\left[K^{-1} - I^*(1 - I^*K)^{-1} + \mu(3I^* - 4)\right] \\ &= -\left[K^{-1} - 3\mu I^* + 3\mu I^* - 4\mu\right] \\ &= -\left[K^{-1} - 4\mu\right] \end{aligned}$$

- Rearranging terms, the costate equation can be written as

$$\dot{\mu}(t) = 6\mu(t) - \frac{1}{K(t)}.$$

- The third necessary condition is the state equation

$$\dot{K}(t) = K(t)[3I^*(t) - 4].$$

- Substituting for the optimal control using

$$I^*(t) = \frac{1}{K(t)}\left(1 - \frac{1}{3\mu(t)}\right),$$

  the state equation becomes

$$\dot{K}(t) = 3 - \frac{1}{\mu(t)} - 4K(t).$$

- Normally the costate equation

$$\dot{\mu}(t) = 6\mu(t) - \frac{1}{K(t)}$$

and the state equation

$$\dot{K}(t) = 3 - \frac{1}{\mu(t)} - 4K(t)$$

are two differential equations that must be solved simultaneously and that is the case here.
- Because this differential equation system is nonlinear, it is not possible to write down a general closed form solution.
- However, because the differential equation system is *autonomous* (at any point in time, both $\dot{K}$ and $\dot{\mu}$ only depend on the current values of K and μ, not also on t), it is possible to analysis its qualitative properties using the phase diagram technique.

- Since the right-hand side of

$$\dot{\mu} = 6\mu - \frac{1}{K}$$

is increasing in both μ and K, $\dot{\mu} = 0$ defines a downward-sloping curve in (K, μ) space:

$$6\mu - \frac{1}{K} = 0 \quad \Longrightarrow \quad 6\mu = \frac{1}{K} \quad \text{or} \quad \mu K = \frac{1}{6}.$$

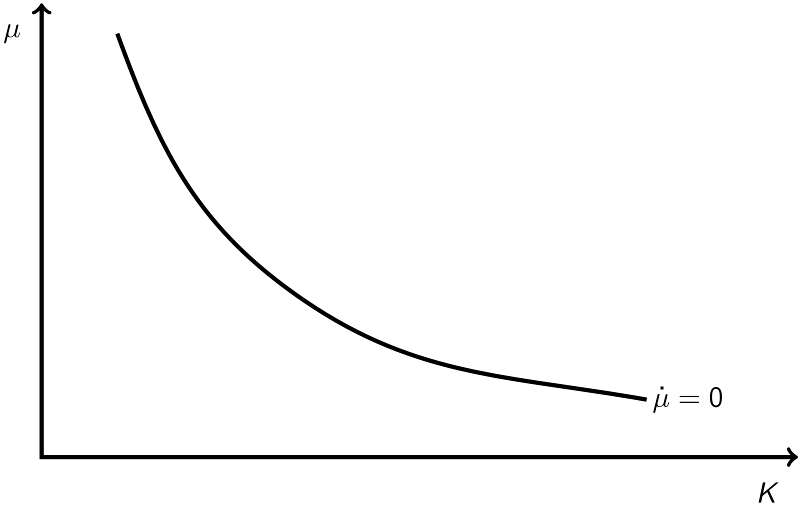

Figure: Drawing the Phase Diagram 1.

- Starting from any point on the curve

$$\dot{\mu} = 6\mu - \frac{1}{K} = 0,$$

an increase in μ leads to $\dot{\mu} > 0$ and a decrease in μ leads to $\dot{\mu} < 0$, as is illustrated by the vertical arrows in the next figure.

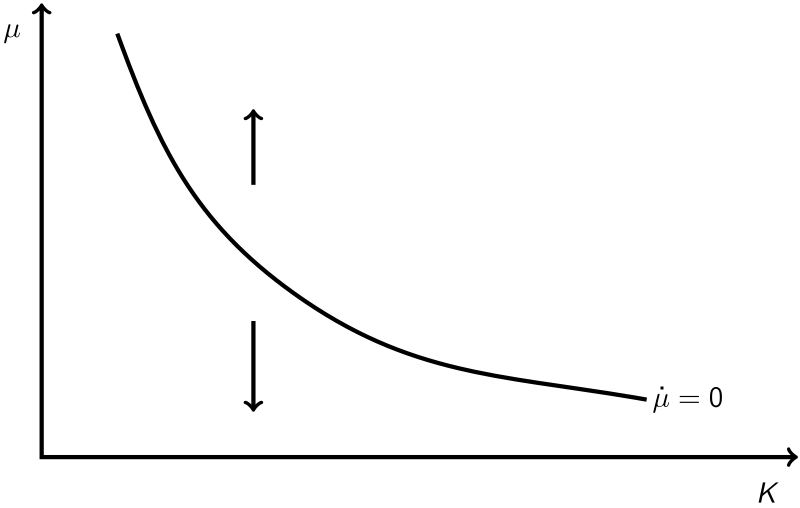

Figure: Drawing the Phase Diagram 2.

- Since the right-hand side of

$$\dot{K} = 3 - \frac{1}{\mu} - 4K$$

is increasing in μ and decreasing in K, $\dot{K} = 0$ defines a upward-sloping curve in (K, μ) space.

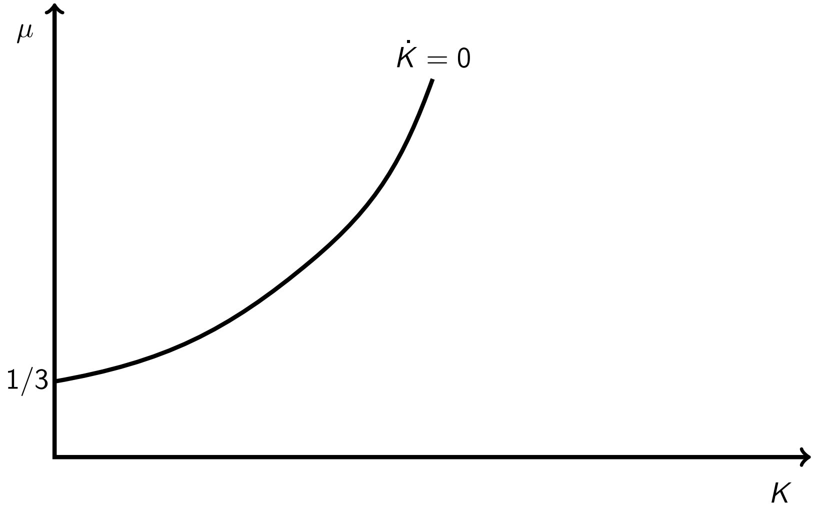

Figure: Drawing the Phase Diagram 3.

- Starting from any point on this curve

$$\dot{K} = 3 - \frac{1}{\mu} - 4K = 0,$$

an increase in K leads to $\dot{K} < 0$ and a decrease in K leads to $\dot{K} > 0$, as is illustrated by the horizontal arrows in the next figure.

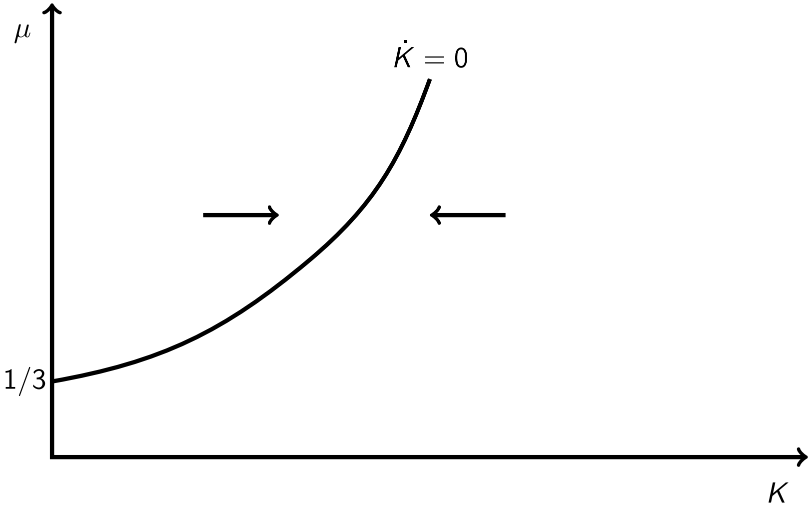

Figure: Drawing the Phase Diagram 4.

- Solving
$$\dot{\mu} = 6\mu - \frac{1}{K} = 0$$
and
$$\dot{K} = 3 - \frac{1}{\mu} - 4K = 0$$
simultaneously yields a unique *steady-state* $(K, \mu) = (3/10, 5/9)$ and this is illustrated in the next figure by the intersection of the two curves.

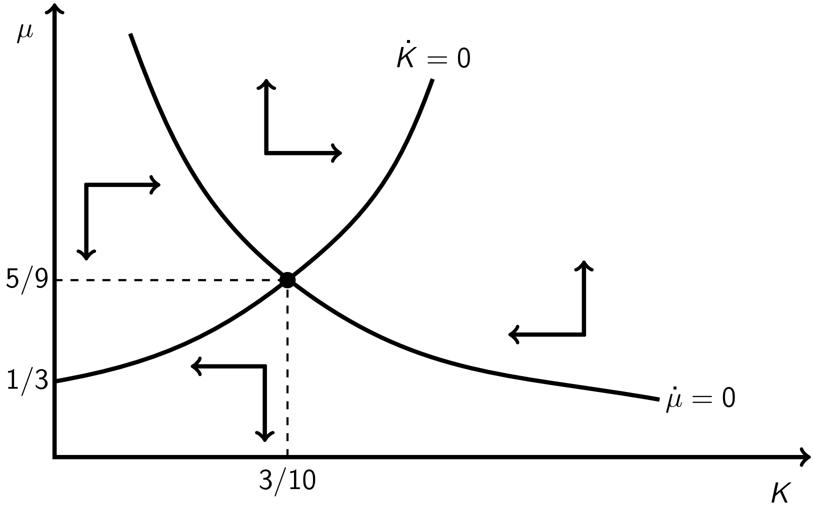

Figure: The Phase Diagram.

- Various possible paths for K and μ over time satisfy the differential equation system.

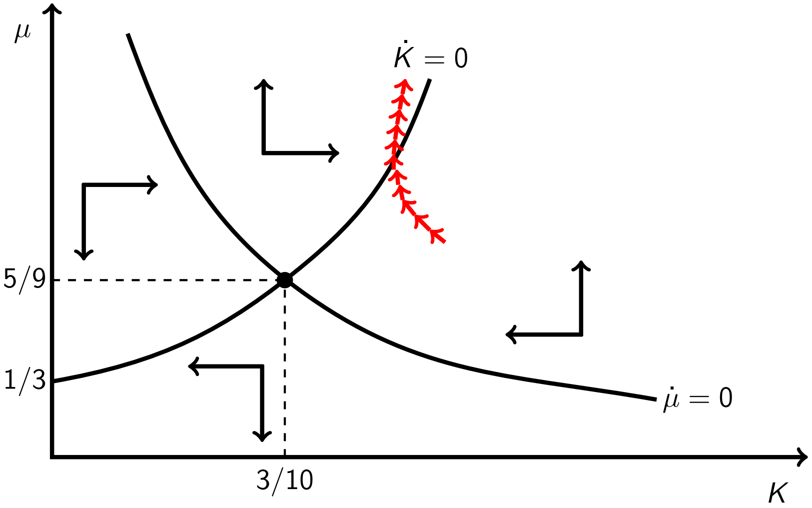

Figure: Possible Trajectory.

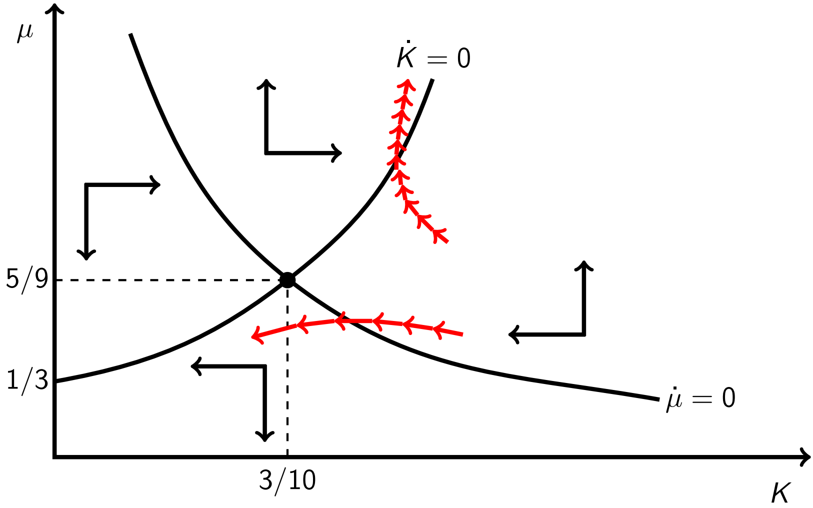

Figure: Possible Trajectories 2.

- Given the continuity of the differential equations, it is clear that, for each initial value of K, there must exist a corresponding initial value of μ such that K and μ converge over time to their steady-state values.
- Such a path is called a *saddle-path.*
- If we start off at a (K, μ) point on the saddle-path, then we remain on the saddle-path forever and there is gradual convergence to the steady-state point.
- A steady-state point with this property is called *saddle-path stable*.

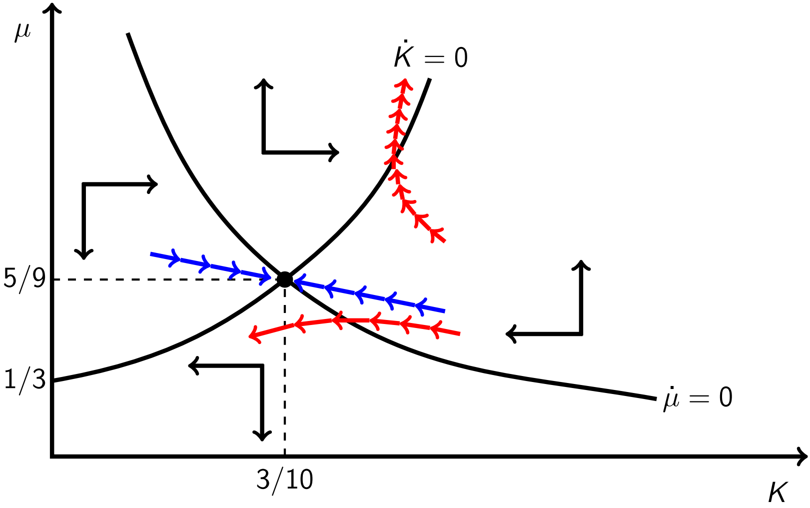

Figure: The saddle-path (in blue).

- From Theorem 5, the remaining necessary condition to check is the initial condition $K(0) = 1/10$.
- Thus, only those paths that begin with $K(0) = 1/10$ are possible candidates for optimal paths (that is, paths for K and μ consistent with the optimal control).
- There are an infinite number of paths satisfying the two differential equations and the initial condition $K(0) = 1/10$.
- Based on Theorem 5 alone, these are all possible optimal paths, candidate solutions to the problem.

- To say more, we turn to Theorem 6. The first condition in Theorem 6 to check is the concavity condition.
- For this problem, the maximized current-valued Hamiltonian is

$$\begin{aligned}\tilde{\mathcal{H}} &= \ln K + \ln(1 - I^*K) + \mu K(3I^* - 4) \\ &= \ln K + \ln\left(\frac{1}{3\mu}\right) + \mu K\left[3\frac{1}{K}\left(1 - \frac{1}{3\mu}\right) - 4\right] \\ &= \ln K - \ln(3\mu) + 3\mu - 1 - 4K\mu\end{aligned}$$

- It follows that

$$\frac{\partial \tilde{\mathcal{H}}}{\partial K} = K^{-1} - 4\mu \qquad \frac{\partial^2 \tilde{\mathcal{H}}}{\partial K^2} = -K^{-2} < 0$$

for all $K > 0$ and the maximized current-valued Hamiltonian is strictly concave over the relevant range.

- The second condition in Theorem 6 to check is the transversality condition.
- For the path that satisfies the initial condition and converges over time to the steady-state point, the transversality condition is definitely satisfied.
- With $K(t)$ and $\mu(t)$ converging over time to the steady state values $K = 3/10$ and $\mu = 5/9$, respectively, and e^{-2t} converging over time to zero, clearly

$$\lim_{t\to\infty} e^{-2t}K(t)\mu(t) = 0.$$

- Thus, Theorem 6 implies that this convergent path is optimal, or more precisely, the control function

$$I^*(t) = \frac{1}{K(t)}\left(1 - \frac{1}{3\mu(t)}\right)$$

associated with this path is optimal (it is also the unique solution given that the maximized Hamiltonian is strictly concave).