Advanced microeconomics problem set 7

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1 Exercise 5.32

1.1 Original question

Consider a simple economy with two consumers, a single consumption good x, and two time periods, Consumption of the good in period t is denoted x_t for $t = 1, 2$. Intertemporal utility functions for the two consumers are,

$$u_i(x_1, x_2) = x_1 x_2, \quad i = 1, 2 \tag{1}$$

and endowments are $e^1 = (19, 1)$ and $e^2 = (1, 9)$. To capture the idea that the good is perfectly storable, we introduce a firm producing storage services. The firm can transform one unit of the good in period one into one unit of the good in period 2. Hence, the production set Y is the set of all vectors $(y1, y2) \in \mathbb{R}^2$ such that $y_1 + y_2 \leq 0$ and $y_1 \leq 0$. Consumer 1 is endowed with a 100 per cent ownership share of the firm.

a) Suppose the two consumers cannot trade with one another. That is, suppose that each consumer is in a Robinson Crusoe economy and where consumer 1 has access to his storage firm. How much does each consumer consume in each period? How well off is each consumer? How much storage takes place?

b) Now suppose the two consumers together with consumer 1's storage firm constitute a competitive production economy. What are the Walrasian equilibrium prices, p_1 and p_2? How much storage takes place now?

c) Interpret p_1 as a spot price and p_2 as a futures price.

d) Repeat the exercise under the assumption that storage is costly, i.e., that Y is the set of vectors $(y_1 . y_2) \in \mathbb{R}^2$ such that $\delta y_1 + y_2 \leq 0$ and $y_1 \leq 0$, where $\delta \in [0, 1)$. Show the existence of spot and futures markets now makes both consumers strictly better off.

1.2 Solution

(a) Isolated islands

Consumer 2 has no access to storage, and no access to markets. Without any formal derivation, it is easy to see that his choice in that situation will be to consume all of his

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endowments, i.e. $\mathbf{x}^2 = \mathbf{e}^2$ and therefore $u_2(\mathbf{x}^2) = 1 \times 9 = 9$.
The situation for consumer 1 is a bit different. I will offer two conceptualizations of this situation.

First solution Assume that the consumer's problem is given by

$$\max_{\mathbf{x} \in \mathbb{R}^2_+} u(\mathbf{x}^1) \quad \text{s.t.} \quad x_1 \leq 19 \tag{2}$$

$$x_2 \leq 20 - x_1 \tag{3}$$

Let λ_1 and λ_2 be nonnegative multipliers associated with the constraints. We form the Lagrangian

$$\mathcal{L} = u(\mathbf{x}^1) + \lambda_1(19 - x_1^1) + \lambda_2(20 - x_1^1 - x_2^1) \tag{4}$$

An interior solution $\mathbf{x}^1 \gg 0$ satisfies the Kuhn-Tucker conditions

$$\frac{\partial \mathcal{L}}{\partial x_1^1} = u_1(\mathbf{x}^1) - \lambda_1 - \lambda_2 = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial x_2^1} = u_2(\mathbf{x}^1) - \lambda_2 = 0 \tag{6}$$

$$x_1^1 \leq 19 \tag{7}$$

$$x_2^1 \leq 20 - x_1^1 \tag{8}$$

$$\lambda_1(x_1^1 - 19) = 0 \tag{9}$$

$$\lambda_2(20 - x_1^1 - x_2^1) = 0 \tag{10}$$

If $u(\cdot)$ is strictly increasing on $\mathbb{R}^2_{++}$, then (6) implies that $\lambda_2 > 0$. Hence, (10) implies that in optimum the second constraint binds

$$x_1^1 = 20 - x_2^1 \tag{11}$$

Now impose functional forms and distinguish the following cases regarding the first multiplier: Assume that $\lambda_1 > 0$. Then from the complementary slackness condition in (9), it follows that $x_1^1 = 19$. Since the second constraint needs to bind ($\lambda_2 > 0$), this implies $x_2^1 = 1$. This allocation attains a utility of $u(19, 1) = 19$. Now assume that $\lambda_1 = 0$. Then (5) and (6) imply that $u_1 = u_2$, which means $x_1^1 = x_2^1$. Then (11) implies $x_1^1 = x_2^1 = 10$, resulting in a utility of $u(10, 10) = 100$. Hence, $\mathbf{x}^1 = (10, 10)$ constitutes the optimal choice for consumer 2.

Alternative solution to question (a) Now let us study this through the lense of the Robinson Crusoe economy. We essentially separate the production technology, put it in control of a profit-maximizing firm and endow consumer 1 with the full ownership of that firm. Consumer 1 solves a consumer's problem

$$\mathbf{x}^1(\mathbf{p}, \mathbf{e}) \in \arg\max_{\mathbf{x}^1 \in \mathbb{R}^2_+} u(\mathbf{x}^1) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x}^1 \leq \mathbf{p} \cdot \mathbf{e}^1 + \Pi(\mathbf{p}) \tag{12}$$

Let λ be a nonnegative multiplier associated with the budget constraint. We form the Lagrangian

$$\mathcal{L} = u(\mathbf{x}^1) + \lambda(\mathbf{p} \cdot \mathbf{e}^1 + \Pi(\mathbf{p}) - \mathbf{p} \cdot \mathbf{x}^1) \tag{13}$$

If $u(\cdot)$ is strictly increasing, an interior optimum solves the first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_1^1} = u_1(\mathbf{x}^1) - p_1\lambda = 0 \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial x_2^1} = u_2(\mathbf{x}^1) - p_2\lambda = 0 \tag{15}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mathbf{p}\cdot\mathbf{e}^1 + \Pi(\mathbf{p}) - \mathbf{p}\cdot\mathbf{x}^1 = 0 \tag{16}$$

We obtain the familiar optimality conditions

$$\frac{u_1(\mathbf{x}^1)}{u_2(\mathbf{x}^1)} = \frac{p_1}{p_2} \tag{17}$$

$$\mathbf{p}\cdot\mathbf{x}^1 = \mathbf{p}\cdot\mathbf{e}^1 + \Pi(\mathbf{p}) \tag{18}$$

Imposing our functional form for $u(\cdot)$ in (17) and rearranging yields

$$p_1 x_1^1 = p_2 x_2^1 \tag{19}$$

We can then solve for the Marshallian demand function using the budget constraint in (17)

$$\mathbf{x}_1^1(\mathbf{p},\mathbf{e}) = \frac{\mathbf{p}\cdot\mathbf{e}^1 + \Pi(\mathbf{p})}{2p_1} \tag{20}$$

$$\mathbf{x}_2^1(\mathbf{p},\mathbf{e}) = \frac{\mathbf{p}\cdot\mathbf{e}^1 + \Pi(\mathbf{p})}{2p_2} \tag{21}$$

Firm We now study the firm's behavior. The firm solves the profit maximization problem

$$\Pi(\mathbf{p}) = \max_{\mathbf{y}\in Y} \mathbf{p}\cdot\mathbf{y} \tag{22}$$

where the production set is defined through

$$Y = \{\mathbf{y}\in\mathbb{R}^2 : y_1 + y_2 \le 0 \quad \& \quad y_1 \le 0\} \tag{23}$$

I find it more convenient to work with the following notation: let $y_1^d = -y_1$ denote the input demand of the firm. Then the firm solves

$$\Pi(\mathbf{p}) = \max_{(y_1^d, y_2)\in\mathbb{R}_+^2} p_2 y_2 - p_1 y_1^d \quad \text{s.t.} \quad y_2 \le y_1^d \tag{24}$$

It is easy to show that if $\mathbf{p} \gg 0$, then it will never be optimal to let any stored goods go to waste, hence $y_2 = y_1^d$ in optimum. The firm's problem can then be rewritten as

$$\Pi(\mathbf{p}) = \max_{y_2\in\mathbb{R}_+} (p_2 - p_1) y_2 \quad \text{s.t.} \quad y_2 \le y_1^d \tag{25}$$

Now distinguish three cases:

(1) If $p_2 > p_1$, then the firm makes a constant profit on each unit produced, hence it will be optimal for y_2 to go to infinity. This will imply that firm's input demand y_1^d will go to infinity, which will violate eventually violate feasibility (the aggregate endowment of today's goods is finite). So this will not turn out to be consistent with a competitive market equilibrium.

(2) If $p_2 < p_1$, the firm makes a constant loss on each unit produced. Since it can transform today's goods into tomorrow's goods, but not tomorrow's goods into today's (formally this is represented by the restriction that $y_1 \leq 0$), it will choose $y_2 = y_1^d = 0$ given those prices. There will be no storage.

(3) If $p_2 = p_1$, then the firm makes zero profit regardless of the amount of storage it provides. Hence, its demand is perfectly elastic (any amount of storage that is requested is provided by the firm).

Based on the preceding argumentation, we write the firm's optimal production plan as

$$y_2(\mathbf{p}) = \begin{cases} 0 & \text{if} \quad p_2 < p_1 \\ [0, \infty) & \text{if} \quad p_2 = p_1 \\ \infty & \text{if} \quad p_2 > p_1 \end{cases} \tag{26}$$

Likewise, the conditional input demand of the firm is given by

$$y_1^d(\mathbf{p}) = \begin{cases} 0 & \text{if} \quad p_2 < p_1 \\ [0, \infty) & \text{if} \quad p_2 = p_1 \\ \infty & \text{if} \quad p_2 > p_1 \end{cases} \tag{27}$$

Profits satisfy

$$\Pi(\mathbf{p}) = \begin{cases} 0 & \text{if} \quad p_2 \leq p_1 \\ \infty & \text{if} \quad p_2 > p_1 \end{cases} \tag{28}$$

General equilibrium Having determined firm's and consumer's optimal decision rules, we need to determine the market-clearing prices. In an economy with production, market clearing requires

$$x_1^1(\mathbf{p}, \mathbf{e}^2) + y_1^d(\mathbf{p}) = e_1^1 \tag{29}$$

$$x_2^1(\mathbf{p}, \mathbf{e}^2) = e_2^1 + y_2(\mathbf{p}) \tag{30}$$

Market clearing on today's goods market requires that the amount of today's goods that is consumed plus the amount of goods that are stored by the firm (the factor demand of the firm) is equal to the economy's endowment. Market clearing on tomorrow's goods market requires that the amount of goods consumed tomorrow, is equal to the endowment of tomorrow's good (think the fruits that will be ripe tomorrow) plus the amount of goods that are stored (the fruits that are harvested today, and are stored until tomorrow).
If we rewrite the first market clearing condition slightly:

$$y_1^d(\mathbf{p}) = e_1^1 - x_1^1(\mathbf{p}, \mathbf{e}^1) \tag{31}$$

We see that in equilibrium, demand for goods and supply for goods today (the fruits that are harvested but not eaten) have to meet. Plugging in equation (20) gives

$$y_1^d(\mathbf{p}) = \left(\frac{1}{2}\right) e_1^1 - \left(\frac{p_2}{2p_1}\right) e_2^1 \tag{32}$$

We have previously ruled out $p_2 > p_1$. If $p_2 > p_1$, the left-hand side explodes whereas excess supply is bounded above by $e_1^2/2$. Now consider $p_2 < p_1$. Then the firm finds it unprofitable to store any goods, i.e. $y_1^d = 0$. An equilibrium requires that

$$\left(\frac{1}{2}\right) e_1^1 = \left(\frac{p_2}{2p_1}\right) e_2^1 \iff \frac{e_1^1}{e_2^1} = \frac{p_2}{p_1} < 1 \tag{33}$$

So an equilibrium with zero storage is possible if $e_1^1 < e_2^1$. The intuition behind this is simple: if the harvest is meager today, but plentiful tomorrow, the agent has no demand for storage (rather she would like to borrow, which is not possible in this economy).
Now consider $p_2 = p_1$. Then $y_1^d(\mathbf{p})$ adjusts freely to meet any demand for storage in the economy.

$$y_1^d(\mathbf{p}) = \left(\frac{1}{2}\right)\left(e_1^1 - e_2^1\right) \tag{34}$$

which is strictly positive if $e_1^1 > e_2^1$. The nature of the equilibrium that we are seeking depends on the underlying economic environment.

For our given parameter values $\mathbf{e}^1 = (19, 1)$, there is demand for storage. The competitive market equilibrium features a price vector satisfying $p_2 = p_1$. The firm will produce $y_2 = 18/2 = 9$ units of stored goods. Since only relative prices matter, we can normalize $p_1 = 1$ and therefore $p_2 = 1$. So far we have made use only of the market clearing condition for today's good (essentially invoking Walras' Law). You can verify for yourself that the market for stored goods also clears at this price vector.

Conclusion A competitive market equilibrium exists. It consists of a price vector $p = \{1, 1\}$, a consumption bundle $\mathbf{x}^1 = \{10, 10\}$ and a production plan for the storage firm $\mathbf{y} = \{-9, 9\}$.

(b) Connecting the islands

Firm: First, we look at the firm's problem.

$$\max_{y_1, y_2} \quad p_1 y_1 + p_2 y_2 \quad s.t. \quad y_1 + y_2 = 0, y_2 \geq 0 \tag{35}$$

Which is equivalent to:

$$\max_{y_2} \quad -p_1 y_2 + p_2 y_2 \quad s.t. \quad y_2 \geq 0 \tag{36}$$

We see the solution is:

$$y_2 = \begin{cases} +\infty \quad \text{if} \quad p_2 > p_1 \\ [0, +\infty)] \quad \text{if} \quad p_2 = p_1 \\ 0 \quad \text{if} \quad p_2 < p_1 \end{cases} \tag{37}$$

and the corresponding profit $\Pi(\mathbf{p}) = y_2 p_2 + y_1 p_1 = y_2(p_2 - p_1)$ is:

$$\Pi(\mathbf{p}) = \begin{cases} +\infty \quad \text{if} \quad p_2 > p_1 \\ 0 \quad \text{if} \quad p_2 = p_1 \\ 0 \quad \text{if} \quad p_2 < p_1 \end{cases} \tag{38}$$

Here we can see that under certain prices, the firm's demand for y_1 can be infinity, which is typically not feasible. It is very likely that such prices are not equilibrium prices.
Households Then we look at household problems: For consumer 1 it is:

$$\max_{x_1^1, x_2^1 \geq 0} x_1^1 x_2^1 \quad s.t. \quad p_1 x_1^1 + p_2 x_2^1 \leq p_1 * 19 + p_2 * 1 + \Pi(p_1, p_2) \tag{39}$$

If $p_2 > p_1$, we have $\Pi(\mathbf{p}) = +\infty$, and with unlimited budget the choice would be infinity: $x_1^1 = x_2^1 = +\infty$.
If $p_2 \le p_1$, $\Pi(\mathbf{p}) = 0$, and we then form the Lagrangian:

$$L = x_1^1 x_2^1 - \lambda[p_1 x_1^1 + p_2 x_2^1 - p_1 * 19 - p_2 * 1] \tag{40}$$

Which gives the FOC[1]

$$\frac{\partial L}{\partial x_1^1} = x_2^1 - \lambda p_1 = 0 \tag{41}$$

$$\frac{\partial L}{\partial x_2^1} = x_1^1 - \lambda p_2 = 0 \tag{42}$$

$$\frac{\partial L}{\partial \lambda} = -[p_1 x_1^1 + p_2 x_2^1 - p_1 * 19 - p_2 * 1] = 0 \tag{43}$$

$$\tag{44}$$

Dividing the first equation against the second gives:

$$\frac{x_2^1}{x_1^1} = \frac{p_1}{p_2} \tag{45}$$

Combine equation (45) and (43) gives the conditional demand for consumer 1:

$$x_1^1 = \frac{19p_1 + p_2}{2p_1} \quad x_2^1 = \frac{19p_1 + p_2}{2p_2} \quad \text{with restriction that} \quad p_2 \le p_1 \tag{46}$$

For consumer 2 it is:

$$\max_{x_1^2, x_2^2 \ge 0} x_1^2 x_2^2 \quad s.t. \quad p_1 x_1^2 + p_2 x_2^2 \le p_1 * 1 + p_2 * 9 \tag{47}$$

With the exact same way as in consumer 1's problem, we get the conditional demand:

$$x_1^2 = \frac{p_1 + 9p_2}{2p_1} \quad x_2^2 = \frac{p_1 + 9p_2}{2p_2} \tag{48}$$

General equilibrium: So the equilibrium requires market clearing, the aggregate excess demand for good 1 is:

$$z_1(p_1, p_2) = \frac{19p_1 + p_2}{2p_1} + \frac{p_1 + 9p_2}{2p_1} - y_1(p_1) - (19 + 1) \tag{49}$$

Then we require $z_1(\mathbf{p}) = 0$, which is clearly not possible if $p_2 > p_1$.
If $p_2 < p_1$, then $y_1(p_1) = 0$. To simplify, we can normalize $p_2 = 1$, and thus $p_1 > 1$. $z_1(\mathbf{p}) = \frac{19p_1+1}{2p_1} + \frac{p_1+9}{2p_1} - 20$, which gives a solution $p_1 = \frac{1}{2}$. However, we just assumed $p_1 > 1$, so this is a contradiction, and $p_2 < p_1$ will not lead to equilibrium.
If $p_2 = p_1$, then to simplify we can denote $p_1 = p_2 = 1$, and get $z_1(\mathbf{p}) = 10 + 5 - y_1 - 20$. If $z_1(\mathbf{p}) = 0$, then $y_1 = -5$. By production technology, $y_2 = -y_1 = 5$.
In summary, a Walrasian equilibrium exists where $p_1 = p_2, y_1 = -5, y_2 = 5, \Pi(\mathbf{p}) = 0$. The allocation is $\mathbf{x}^1 = (10, 10), \mathbf{x}^2 = (5, 5)$. The storage is 5.

[1] first order condition

(c) Spot price and futures price

p_1 can be interpreted as the price at time $t = 1$ of 1 unit of the consumption good x at time $t = 1$. That is the spot price.
p_2 can be interpreted as the price at time $t = 1$ of 1 unit of the consumption good x at time $t = 2$. That is the futures price[2].

(d) Storage with depreciation

Isolated island: First, we assume the consumers cannot trade with each other. For the second individual this is straightforward: $\mathbf{x}^2 = e^2 = (1, 9)$, $u^2(\mathbf{p}) = 1 * 9 = 9$.
The problem for consumer 1 is:

$$\max_{\mathbf{x}^1 \in \mathbb{R}^2_+} u(\mathbf{x}^1) \quad \text{s.t.} \quad x_1 = 19 + y_1 \tag{50}$$

$$x_2 = 1 + y_2 \tag{51}$$

$$y_1 \leq 0 \tag{52}$$

$$y_2 = -\delta y_1 \tag{53}$$

We can use the constraints in this problem and express other variables in terms of y_2, then we get:

$$\max_{y_2 \geq 0} u^1 = (19 - \frac{y_2}{\delta})(1 + y_2) \tag{54}$$

So $\frac{\partial u^1}{\partial y_2} = 19 - \frac{1+2y_2}{\delta}$, and the maximum u^1 is where $\frac{\partial u^1}{\partial y_2} = 0$, we get $y_2 = \frac{19\delta - 1}{2}$. Here we add the constraint that $19\delta - 1 > 0$, otherwise the solution simply becomes: no storage and consume what is given.
With storage plan solved, the consumption plan for agent 1 is $\mathbf{x}^1 = (19 - \frac{19\delta-1}{2\delta}, 1 + \frac{19\delta-1}{2})$
Plug in the consumption plan we get the utility for consumer 1:

$$u^1 = x_1^1 x_2^1 = \frac{(19\delta + 1)^2}{4\delta} \tag{55}$$

And to summarize we write the utility for consumer 2 again

$$u^2(\mathbf{p}) = 1 * 9 = 9 \tag{56}$$

Connected islands:
The solution is similar to question (b):
Firm: First, we look at the firm's problem.

$$\max_{y_1, y_2} \quad p_1 y_1 + p_2 y_2 \quad s.t. \quad \delta y_1 + y_2 = 0, y_2 \geq 0 \tag{57}$$

Which is equivalent to:

$$\max_{y_2} \quad -p_1 \frac{y_2}{\delta} + p_2 y_2 \quad s.t. \quad y_2 \geq 0 \tag{58}$$

[2] this is just like people abstain consumption today to save money, and use this saving to go to a summer trip in the future

We see the solution is:

$$y_2 = \begin{cases} +\infty & \text{if} \quad p_2 > \frac{p_1}{\delta} \\ [0, +\infty)] & \text{if} \quad p_2 = \frac{p_1}{\delta} \\ 0 & \text{if} \quad p_2 < \frac{p_1}{\delta} \end{cases} \tag{59}$$

and the corresponding profit $\Pi(\mathbf{p}) = y_2 p_2 + y_1 p_1 = y_2(p_2 - p_1)$ is:

$$\Pi(\mathbf{p}) = \begin{cases} +\infty & \text{if} \quad p_2 > \frac{p_1}{\delta} \\ 0 & \text{if} \quad p_2 = \frac{p_1}{\delta} \\ 0 & \text{if} \quad p_2 < \frac{p_1}{\delta} \end{cases} \tag{60}$$

Here we can see that under certain prices, the firm's demand for y_1 can be infinity, which is typically not feasible. It is very likely that such prices are not equilibrium prices.
Households Then we look at household problems: For consumer 1 it is:

$$\max_{x_1^1, x_2^1 \geq 0} x_1^1 x_2^1 \quad s.t. \quad p_1 x_1^1 + p_2 x_2^1 \leq p_1 * 19 + p_2 * 1 + \Pi(p_1, p_2) \tag{61}$$

If $p_2 > \frac{p_1}{\delta}$, we have $\Pi(\mathbf{p}) = +\infty$, and with unlimited budget the choice would be infinity: $x_1^1 = x_2^1 = +\infty$.
If $p_2 \leq \frac{p_1}{\delta}$, $\Pi(\mathbf{p}) = 0$, and we then form the Lagrangian:

$$L = x_1^1 x_2^1 - \lambda[p_1 x_1^1 + p_2 x_2^1 - p_1 * 19 - p_2 * 1] \tag{62}$$

Which gives the FOC[3]

$$\frac{\partial L}{\partial x_1^1} = x_2^1 - \lambda p_1 = 0 \tag{63}$$

$$\frac{\partial L}{\partial x_2^1} = x_1^1 - \lambda p_2 = 0 \tag{64}$$

$$\frac{\partial L}{\partial \lambda} = -[p_1 x_1^1 + p_2 x_2^1 - p_1 * 19 - p_2 * 1] = 0 \tag{65}$$

$$\tag{66}$$

Dividing the first equation against the second gives:

$$\frac{x_2^1}{x_1^1} = \frac{p_1}{p_2} \tag{67}$$

Combine equation (67) and (65) gives the conditional demand for consumer 1:

$$x_1^1 = \frac{19p_1 + p_2}{2p_1} \quad x_2^1 = \frac{19p_1 + p_2}{2p_2} \quad \text{with restriction that} \quad p_2 \leq \frac{p_1}{\delta} \tag{68}$$

For consumer 2 it is:

$$\max_{x_1^2, x_2^2 \geq 0} x_1^2 x_2^2 \quad s.t. \quad p_1 x_1^2 + p_2 x_2^2 \leq p_1 * 1 + p_2 * 9 \tag{69}$$

With the exact same way as in consumer 1's problem, we get the conditional demand:

$$x_1^2 = \frac{p_1 + 9p_2}{2p_1} \quad x_2^2 = \frac{p_1 + 9p_2}{2p_2} \tag{70}$$

[3] first order condition

General equilibrium: So the equilibrium requires market clearing, the aggregate excess demand for good 1 is:

$$z_1(p_1, p_2) = \frac{19p_1 + p_2}{2p_1} + \frac{p_1 + 9p_2}{2p_1} - y_1(p_1) - (19 + 1) \tag{71}$$

Then we require $z_1(\mathbf{p}) = 0$, which is clearly not possible if $p_2 > \frac{p_1}{\delta}$.

If $p_2 < \frac{p_1}{\delta}$, then $y_1(p_1) = 0$. To simplify, we can normalize $p_2 = 1$, and thus $p_1 > 1$. $z_1(\mathbf{p}) = \frac{19p_1+1}{2p_1} + \frac{p_1+9}{2p_1} - 20$ should be zero, which gives a solution $p_1 = \frac{1}{2}$. **This is possible if** $\delta < \frac{1}{2}$**, which is not seen in question (b)**. The equilibrium is then $y_1 = y_2 = 0, p_1 = \frac{1}{2}, p_2 = 1$, and the allocation is $\mathbf{x}^1 = (\frac{21}{2}, \frac{21}{4}), \mathbf{x}^2 = (\frac{19}{2}, \frac{19}{4})$. The corresponding utilities are:

$$u^1(\mathbf{x}^1) = \frac{21^2}{8} = 55.125 \quad u^2(\mathbf{x}^2) = \frac{19}{2}\frac{19}{4} = 45.125 \tag{72}$$

Then we look back into the case without trade, we had the utility for agent 1 and utility 2 as (equation (55, 56)):

$$u^1|_{\delta=\frac{1}{2}} = \frac{(19\delta+1)^2}{4\delta}|_{\delta=\frac{1}{2}} = 55.125 \tag{73}$$

$$u^1|_{\delta=\frac{1}{19}} = \frac{(19\delta+1)^2}{4\delta}|_{\delta=\frac{1}{19}} = 19 \tag{74}$$

$$u^2|_{\delta} = 1 * 9|_{\delta} = 9 \tag{75}$$

So consumer 2 is definitely better with trade. Consumer 1 is strictly better as long as $\frac{1}{19} < \delta < \frac{1}{2}$ in this case. There is no storage in this case.

If $p_2 = \frac{p_1}{\delta}$, then from the optimal production plan (equation (59)) we see that y_1 can be any value smaller than zero, giving us much freedom to find the equilibrium. To simplify we can denote $p_1 = \delta, p_2 = 1$, and get $z_1(\mathbf{p}) = \frac{19\delta+1}{2\delta} + \frac{\delta+9}{2\delta} - y_1 - 20$. If $z_1(\mathbf{p}) = 0$, then $y_1 = \frac{10}{2\delta} - 10$. By production technology, $y_2 = -y_1 = 10 - \frac{10}{2\delta}$. Since by the restriction of production function, $y_2 \geq 0, y_1 \geq 0$, this case is only possible if $\delta \geq \frac{1}{2}$[4]. From the demands calculated before, the allocation plan is:

$$x_1^1 = \frac{19p_1 + p_2}{2p_1} = \frac{19}{2} + \frac{1}{2\delta} \tag{76}$$

$$x_2^1 = \frac{19}{2}\delta + \frac{1}{2} \tag{77}$$

$$x_1^2 = \frac{1}{2} + \frac{9}{2\delta} \tag{78}$$

$$x_2^2 = \frac{\delta}{2} + \frac{9}{2} \tag{79}$$

The utility of both consumers are:

$$u^1 = \frac{(19\delta+1)^2}{4\delta} \quad u^2 = \frac{(\delta+9)^2}{4\delta} \quad \delta \geq \frac{1}{2} \tag{80}$$

So consumer 1 has the same utility level as in the autarky case, and consumer 2 benefits from the trade.

[4]Interestingly, we just covered every possible state of δ with the discussion above

2 Exercise 5.39

2.1 Original question

(Cornwall) In an economy with two types of consumer, each type has the respective utility function and endowments:

$$u^{1q}(x_1.x_2) = x_1x_2 \quad \text{and} \quad e^1 = (8,2) \tag{81}$$

$$u^{2q}(x_1.x_2) = x_1x_2 \quad \text{and} \quad e^2 = (2,8) \tag{82}$$

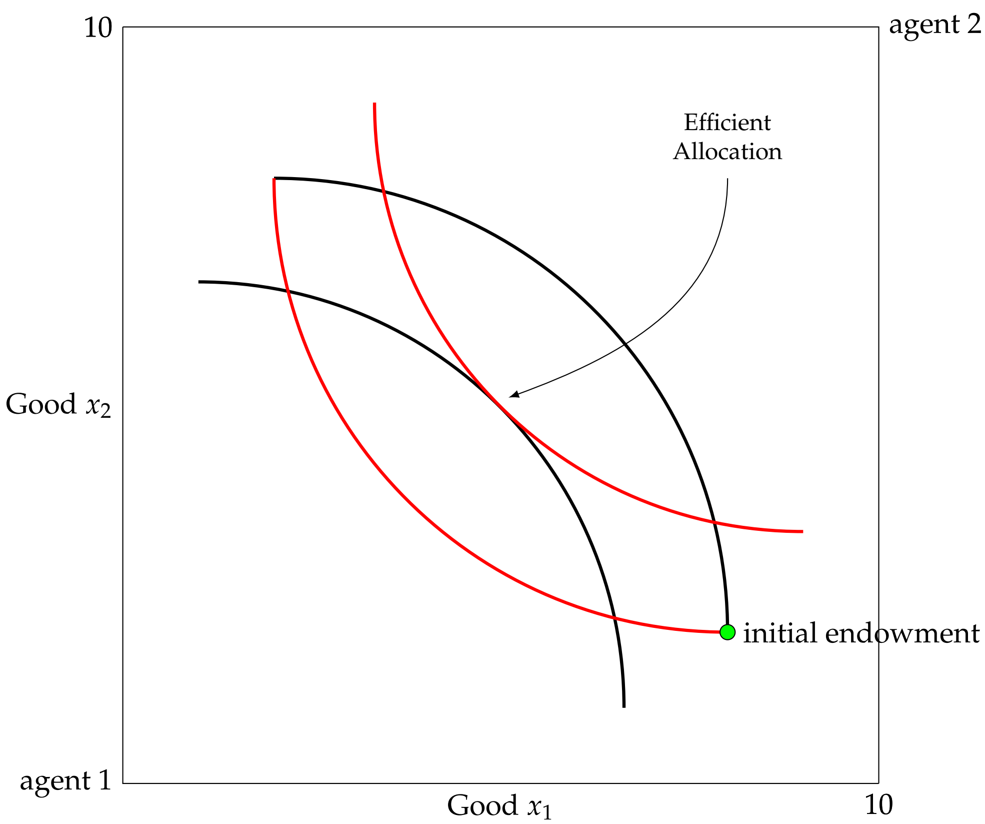
(a) Draw an Edgeworth box for this economy when there is one consumer of each type.

(b) Characterise as precisely as possible the set of allocations that are in the core of this two-consumer economy

(c) Show that the allocation giving $x^{11} = (4,4)$ and $x^{21} = (6,6)$ is in the core

(d) Now replicate this economy once so there are two consumers of each type, for a total of four consumers in the economy. Show that the double copy of the previous allocation, giving $x^{11} = x^{12} = (4,4)$ and $x^{21} = x^{22} = (6,6)$, is **not** in the core of the replicated economy.

2.2 Solution



(a)

(b) The core of the economy is a subset of the set of feasible Pareto efficient allocations. So, first characterize allocations which are feasible and Pareto efficient. Notice that if an allocation is Pareto efficient, then the consumers will not want to trade, which means their ratio of marginal utility between two goods should be the same:

$$\frac{dx_2^1}{dx_1^1} = \frac{dx_2^2}{dx_1^2}$$

$$\implies -\frac{x_2^1}{x_1^1} = -\frac{x_2^2}{x_1^2}$$

Imposing feasibility requirements on the equation above, we can write-

$\frac{x_2^1}{x_1^1} = \frac{10-x_2^1}{10-x_1^1}$

$\implies 10x_2^1 - x_1^1x_2^1 = 10x_1^1 - x_1^1x_2^1$

$\implies x_2^1 = x_1^1$

$\implies x_2^2 = x_1^2$

Therefore, a feasible allocation which exhausts all resources in the economy and in which each consumer has equal amounts of both goods, will be a Pareto efficient allocation.

Finally, each allocation in the core must be an unblocked feasible allocation. We have already taken care of blocks implementable by the grand coalition consisting of both consumers. A single consumer will choose to block an allocation if the utility they derive from their bundle is less than the utility they derive from their endowment. The following two restrictions ensure this is not the case-

$$x_1^1x_2^1 = (x_1^1)^2 \geq e_1^1e_2^1 = 16 \implies x_1^1 = x_2^1 \geq 4$$

$$x_1^2x_2^2 = (x_1^2)^2 \geq e_1^2e_2^2 = 16 \implies x_1^2 = x_2^2 \geq 4$$

We can summarize our characterization of the core as follows-

$$C((8,2),(2,8)) = \{((x_1^1, x_2^1),(x_1^2, x_2^2)) \mid \ (x_1^1 = x_2^1) \wedge (x_1^2 = x_2^2) \wedge (x_1^1 + x_1^2 = 10) \wedge$$

$$(x_2^1 + x_2^2 = 10) \wedge (x_1^1 \geq 4) \wedge (x_1^2 \geq 4)\}$$

(c) The given allocation satisfies all the properties of a core allocation-

- It is feasible: $(4,4) + (6,6) = (10,10)$
- The allocation is Pareto Efficient since each consumer has equal amounts of both goods. Since it is Pareto efficient, it is unblocked by the grand coalition.

- Consumer 1 derives utility equal to 16 from $(4,4)$ which is the same as from their endowment, while consumer 2 derives utility equal to 36 from $(6,6)$ which is greater than what they get from their endowment. Therefore, this allocation will not be blocked by either consumer 1 or consumer 2.

(d) To demonstrate that the given allocation is not in the core of the replica economy, it is enough to construct a blocking coalition.

Consider a coalition of three consumers- both consumers of type 1 (each has endowment $(8,2)$ and bundle $(4,4)$) and one consumer of type 2 (endowment $(2,8)$ and bundle $(6,6)$).

Total resources available to this coalition is the vector $(8,2)+(8,2)+(2,8) = (18,12)$.

The following alternative allocation is both feasible for these three consumers, and gives each of them strictly higher utility compared to the allocation given in the question-

$$((4.5,3.75),(4.5,3.75),(9,4.5))$$

- Verify that the alternative is feasible and exhausts the coalition's resources-

$$(4.5,3.75)+(4.5,3.75)+(9,4.5)=(18,12)$$

- The utility for each type 1 consumer is $4.5 \times 3.75 = 16.875 > 16$.
- The utility for the type 2 consumer is $9 \times 4.5 = 40.5 > 36$.

3 Exercise 5.40

3.1 Original question

In a pure exchange economy, consumer i *envies* consumer j if $\mathbf{x}^j \succ^i \mathbf{x}^i$. (Thus, i envies j if i likes j's bundle better than his own[5].) An allocation $\mathbf{x}$ is therefore *envy free* if $\mathbf{x}^i \succsim^i \mathbf{x}^j$ for all i and j. We know that envy-free allocations will always exist, because the equal-division allocation, $\bar{\mathbf{x}} = (1/I)e$, must be envy free. An allocation is called fair if it is both envy free and Pareto efficient.

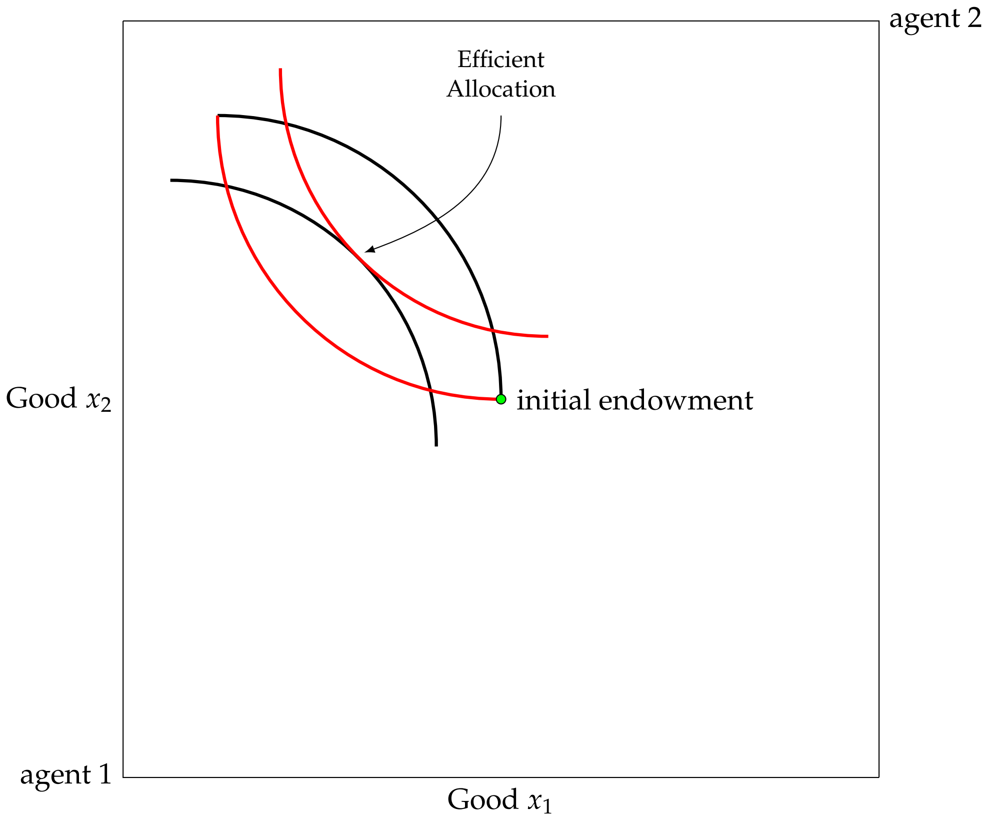
(a) In an Edgeworth box, demonstrate that envy-free allocations need not be fair.

(b) Under Assumption 5.1 on utilities, prove that every exchange economy having a strictly positive aggregate endowment vector possesses at least one fair allocation.

[5] So we are measuring on i's preference, that is why the preference is denoted as $\succ^i$

3.2 Solution

(a) The key to this problem is to allow different preferences: If one person likes apple more, and the others likes banana more, then it is likely that the fair outcome is that apple person has more apple, and banana person has more banana. However, an envy-free allocation can be that each person has half of the total apples and half of the total bananas.[6]



(b) Notice here we may have more than 2 people and more than 2 goods, so I try to give a general argument. Since we have a strictly positive total endowment, we can start our market by giving each agent $\frac{1}{I}$ of the total endowment $\mathbf{e}^1 = \frac{1}{I}\mathbf{e}$. This initial allocation is feasible.

Then we open the market and let agents trade with each other. By Theorem 5.5, given that each agent's utility function is continuous, strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$, and that the aggregate endowment is strictly positive, i.e. $\mathbf{e} \gg \mathbf{0}$, we know that a price vector $\mathbf{p} \gg 0$ exists such that $\mathbf{z}(\mathbf{p}) = \mathbf{0}$, and with this price vector there is a Walrasian equilibrium.

By the first welfare theorem (theorem 5.7), every Walrasian equilibrium allocation is Pareto efficient. So we only need to prove this particular WEA is envy-free: Look at the budget constraints for each consumer, and we see that they are exactly the same. After the trade an agent does not lose money, i.e. $\mathbf{p} \cdot \mathbf{x}^i = \mathbf{p} \cdot \mathbf{e}^i$. We also know that by design

[6] One can even do some "cheating" by giving the example where both people get nothing. On page 199 in the textbook, we see that "allocation" does not require using all of the total endowment, only "feasible allocation" requires us to allocate everything. Meanwhile, such tragedies happen in human history, so it is not useless imagination.

$\mathbf{e}^i = \mathbf{e}^j$. Then the consumer's problem can be written as:

$$\max \quad u^i(\mathbf{x}^i) \quad s.t. \quad \mathbf{p} \cdot \mathbf{x}^i = \mathbf{p} \cdot \mathbf{e}^i = \mathbf{p} \cdot \mathbf{e}^j = \mathbf{p} \cdot \mathbf{x}^j \tag{83}$$

Since $\mathbf{x}^i$ is the solution when the restriction is $pp \cdot \mathbf{x}^i = \mathbf{p} \cdot \mathbf{e}^i$, it will remain to be the solution when the restriction is $pp \cdot \mathbf{x}^i = \mathbf{p} \cdot \mathbf{x}^j$, and we see that consumer i will not envy j's consumption bundle. If i is forced to swap consumption bundle with j, he will then use this bundle $\mathbf{x}^j$ to trade on the market into his own bundle $\mathbf{x}^i$ again, because $\mathbf{x}^i$ is the exact solution to the problem we stated before (I repeat this again below):

$$\max \quad u^i(\mathbf{x}^i) \quad s.t. \quad \mathbf{p} \cdot \mathbf{x}^i = \mathbf{p} \cdot \mathbf{x}^j \tag{84}$$