Inequality, Household Behavior, & the Macroeconomy
Problem Set 3

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Part 1

1 Identification based on covariances

In the lecture, we saw that STY identified the parameters of their income process based on the relation of age and the cross-sectional variance of the log income residual. In this exercise, you need to explore an alternative method, relying on the covariance structure of log income residuals.

Similarly to STY, assume that

$$u_{i,h} = \alpha_i + \epsilon_{i,h} + z_{i,h} \text{ and } z_{i,h} = \rho z_{i,h-1} + \eta_{i,h}$$

$$\eta_{i,h} \sim \mathcal{N}(0, \sigma_\eta^2) \quad \epsilon_{i,h} \sim \mathcal{N}(0, \sigma_\epsilon^2) \quad \alpha_i \sim \mathcal{N}(0, \sigma_\alpha^2)$$

where u denotes the log income residual and h is age, expressed in years. Assume that $z_{i,22} = 0$, so the persistent income component of 22-year-old individuals is 0.

1. Express the following quantities as formulas containing the four parameters $(\sigma_\alpha, \sigma_\epsilon, \sigma_\eta, \rho)$: $Var(u_{i,60})$, $Cov(u_{i,60}, u_{i,58})$, $Cov(u_{i,60}, u_{i,56})$, $Cov(u_{i,60}, u_{i,30})$

From the slides, we have

$$Var(u_{i,60}) = \sigma_\alpha^2 + \sigma_\epsilon^2 + \sigma_\eta^2 \sum_{j=22}^{59} \rho^{2(j-22)}$$

And

$$cov(u_{i,60}, u_{i,58}) = cov(\alpha_i + \epsilon_{i,60} + z_{i,60}, \alpha_i + \epsilon_{i,58} + z_{i,58}) = \sigma_\alpha^2 + cov(z_{i,60}, z_{i,58})$$

Note $z_{i,60} = \rho(\rho z_{i,58} + \eta_{i,59}) + \eta_{i,60}$
And thus $cov(z_{i,60}, z_{i,58}) = cov(\rho^2 z_{i,58}, z_{i,58}) = \rho^2 var(z_{i,58}) = \rho^2 \sigma_\eta^2 \sum_{j=22}^{57} \rho^{2(j-22)}$

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So we have

$$cov(u_{i,60}, u_{i,58}) = \sigma_\alpha^2 + \rho^2\sigma_\eta^2 \sum_{j=22}^{57} \rho^{2(j-22)}$$

By the same process, we find

$$cov(u_{i,60}, u_{i,56}) = \sigma_\alpha^2 + \rho^4\sigma_\eta^2 \sum_{j=22}^{55} \rho^{2(j-22)}$$

and

$$cov(u_{i,60}, u_{i,30}) = \sigma_\alpha^2 + \rho^{30}\sigma_\eta^2 \sum_{j=22}^{29} \rho^{2(j-22)}$$

2. Write a function in Julia, that takes a 4-element candidate vector of parameters and computes the corresponding moments from 1.1.

Please refer to the .jl file for the code.

3. Compute these same moments in PSID. (Hint: It's simplest to create residuals as in 40_*Income_empirics.jl* up to line 69. Then proceed as in the following 10 lines, but use the 'age' variable instead of 'year' (and make other changes that you see necessary). If unstack complains about duplicate rows, you might need to run it with an extra keyword argument to drop all but the last one, like this:

$unstack(psid\_residual, "id\_ind", "age", "income\_residual", combine = last))$

$$cov(u_{i,60}, u_{i,60}) = Var(u_{i,60}) = 0.878$$

$$cov(u_{i,60}, u_{i,58}) = 0.563$$

$$cov(u_{i,60}, u_{i,56}) = 0.509$$

$$cov(u_{i,60}, u_{i,30}) = 0.245$$

4. Find parameters that make the theoretical moments equal to the empirical ones. (Hint: relying on what you have from 1.2 and 1.3, you can write a function that to any parameter guess returns the vector of differences between the implied theoretical moments and the empirical ones. This new function you can pass to nlsolve. You can find some examples of using NLsolve in 7_*numerics.ipynb*.)

Please refer to the .jl file for the code.

5. Compare your parameters to the ones obtained by STY. Compute the theoretical cross-sectional variances for each age group implied by your parameters, and compare the pattern with the empirical one (obtained in line 79 of $40\_Income\_empirics.jl$). Discuss the differences.

We get estimated parameters $\sigma_\alpha = 0.471, \sigma_\epsilon = 0.501, \sigma_\eta = 0.253, \rho = 0.918$. In comparison, Storesletten et al. find parameters $\sigma_\alpha = 0.459, \sigma_\epsilon = 0.251, \sigma_\eta = 0.129, \rho = 0.999$. This means we find similar values for σ_α and ρ while we estimate higher values for σ_ϵ and σ_{eta}, meaning we estimate higher variance for both persistent and transitory shocks in our model.

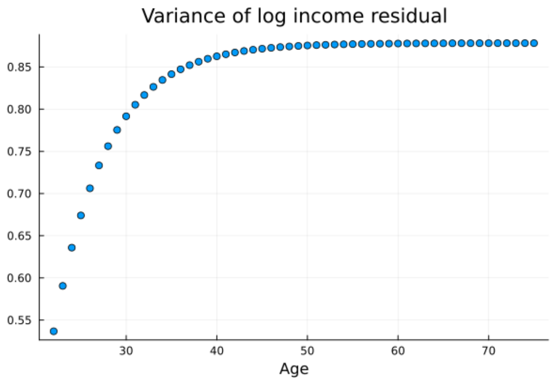

Figure 1: Theoretical Values for Cross-Sectional Variance

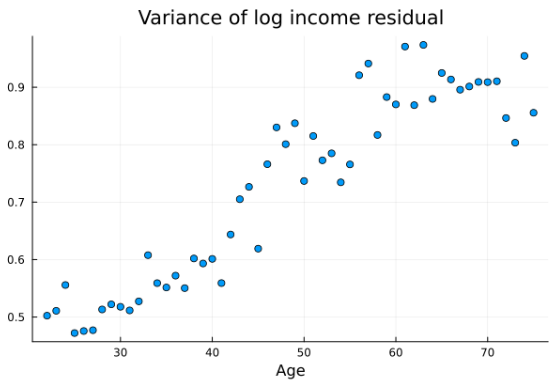

Figure 2: Empirical Values for Cross-Sectional Variance

Looking at the cross-sectional variations in the log income residual, we see that our theoretical values, variance increases sharply before flattening out, whereas in the data, cross-sectional variances increase fairly steadily across time.
We see that variance at age 60 is reasonably matched since this is one of the moments that we used to determine our theoretical parameters. However, for most lower ages, our parameters overestimate the variances in comparison to the PSID data. One possible explanation for this is that idiosyncratic shocks follow the same distribution for all ages ($\epsilon_{i,h} \sim N(0, \sigma_\epsilon^2) \quad \forall h$), whereas we might expect that young agents can only experience relatively small idiosyncratic shocks to income in comparison to older agents.

Part 2

2 Luxury bequests

When there was no bequest motive, the optimal decision of an agent in the final time-period was trivial. Not anymore. As in class, the oldest agents solve

$$\max_{a\geq 0} \frac{(x-a)^{1-\gamma}-1}{1-\gamma} + \Phi_1\left(1+\frac{a[1+r(1-\tau_c)](1-\tau_b)}{\Phi_2}\right)^{1-\gamma}$$

1. Derive the optimal saving level as a function of x (cash-on-hand).

The consumer's optimal savings condition is

$$\frac{\partial U}{\partial a} = -(x-a)^{-\gamma} + \Phi_1(1-\gamma)\left(1+a\frac{[1+r(1-\tau_c)](1-\tau_b)}{\Phi_2}\right)^{-\gamma}\left(\frac{[1+r(1-\tau_c)](1-\tau_b)}{\Phi_2}\right) = 0$$

For simplicity define $B = \frac{[1+r(1-\tau_c)](1-\tau_b)}{\Phi_2}$
Solving for a, we have

$$(x-a)^{-\gamma} = \Phi_1(1-\gamma)(1+aB)^{-\gamma}(B)$$

$$(x-a) = \Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(1+aB)(B)^{-\frac{1}{\gamma}}$$

$$(x-a) = \Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{-\frac{1}{\gamma}} + a\Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{\frac{-1+\gamma}{\gamma}}$$

$$a(1+\Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{\frac{-1+\gamma}{\gamma}}) = -\Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{-\frac{1}{\gamma}} + x$$

$$a = \frac{x-\Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{-\frac{1}{\gamma}}}{1+\Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{\frac{-1+\gamma}{\gamma}}}$$

2. Show that in this model leaving bequests is indeed a luxury good.

A good is a luxury good if its income elasticity of demand is greater than 1. In this case, the necessary condition is

$$\frac{\partial a}{\partial x} * \frac{x}{a} > 1$$

From 1.1 we can derive

$$\frac{\partial a}{\partial x} = \frac{1}{1+\Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{\frac{-1+\gamma}{\gamma}}}$$

$$\frac{x}{a} = x * \frac{1 + \Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{\frac{-1+\gamma}{\gamma}}}{x - \Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{-\frac{1}{\gamma}}}$$

Therefore,

$$\frac{\partial a}{\partial x} * \frac{x}{a} = \frac{x}{x - \Phi_1^{-\frac{1}{\gamma}}(1-\gamma)^{-\frac{1}{\gamma}}(B)^{-\frac{1}{\gamma}}}$$

Under reasonable assumptions about our parameters, this gives us $\frac{\partial a}{\partial x} * \frac{x}{a} > 1$ as desired.

3. Based on the solution you derived, interpret the two Φ parameters.

Based on the above, we see that as Φ_2 rises, the income elasticity of demand for bequests also rises, meaning that Φ_2 affects the extent to which bequests are a luxury good, while Φ_1 in the utility function given at the start weights how much a parent values bequesting to their children.

4. Which quantities are used to discipline the two Φ parameters in De Nardi (2004)? How does her approach conform to your findings?

De Nardi (2004) picks Φ_1 to match a transfer wealth share of 60% in the US and Φ_2 to match the ratio between the average bequest left by single descendants at the lowest 30th percentile and the median household income.
Based on our findings and interpretation of the parameters, this approach seems very fitting. Φ_1 has a linear relationship with the utility gained from bequesting and can be used to adjust the share of wealth transfered from one generation to another, and while Φ_1 cannot be observed in the individual, calibrating it over the whole data is very informative. Φ_2 affects the curvature of the utility from bequesting, and is matched by considering how much lower-income households bequest (scaled by median income), although it would be interesting to see how well this choice of Φ_2 matches the bequests observed at other percentiles.

Part 3

3 Capital tax and welfare

Consider the setting from 41_*Inequality_inheritence.jl* with bequest motive and ability transfer as your benchmark. As an alternative setting, consider an otherwise identical economy where capital income tax equals 0.

1. Compute the labor income tax level that balances the budget for your alternative model. (The balancing labor income tax for the benchmark setting should already be available from class.) Solve both models with the appropriate tax levels.

$$\tau_c = 0.3 \implies \tau_l = 0.1606 \implies \gamma_b = 2.93$$

$$\tau_c = 0.0 \implies \tau_l = 0.2093 \implies \gamma_a = 15.26$$

where γ is the wealth to income ratio. Note that the budget does not balance perfectly in either case, a small difference persists even after many iterations.

2. We want to figure out which model agents are happier in, depending on their wealth.

    (a) Write a function, that to any cash-on-hand level assigns the welfare gain of being in the alternative model instead of the benchmark one, for someone in the first period (age 22), with middle (so 0) z and α states. The welfare gain should be expressed as a percentage of consumption units.

    (b) Choose an appropriate grid for coh.

    (c) Plot your function over this grid and interpret the results.

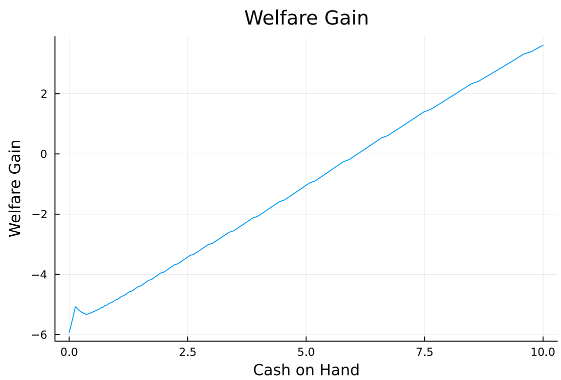

Figure 3: Welfare gain for middle z_i and α_i values

Figure 3 shows the welfare gain from being in the alternative setting (i.e. $\tau_c = 0$) instead of being in benchmark state (where $\tau_c = 0.3$). The figure shows that, for a higher value of cash on hand, the median 22-year-old agent prefers the no capital income tax setting (i.e., has a positive welfare gain). On the other hand, for low cash-on-hand values, agents prefer the benchmark model where capital income is taxed as the capital income tax revenue is distributed evenly across agents. Intuitively, as the value of cash on hand increases, future labor income becomes less important in relation to the future income from capital. Therefore, the agent would prefer a higher labor income tax and a lower capital income tax, as in the alternative setting.

3. Repeat the previous point with the lowest and highest values of α. Discuss the differences.

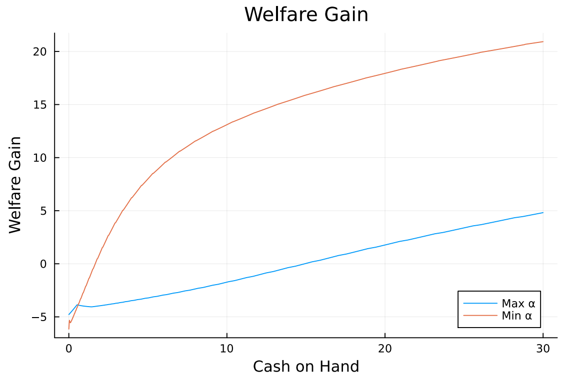

Figure 4: Welfare gain for middle z_i and min/max α_i values

Figure 4 illustrates the welfare gains of the agents of age 22 with middle z_i values and min/max α_i values. We note that in De Nardi (2004), $\alpha_i s$ abilities, correlated between parent and child. For higher α_i values, an agent's income is higher than for a lower α_i value, ceteris paribus. As noted in the previous part, also the low α_i agent derives a higher welfare gain from being in the alternative setting for higher cash on hand values as future labor income becomes less important relative to future capital gains income. Therefore the agent prefers the alternative setting with high τ_l and $\tau_c = 0$. The high α_i agent shows the same pattern, albeit with a slower progression as the cash on hand as a share of future labor income is lower than for the low α agent. Again, we observe that the welfare gain turns positive for higher cash on hand values where future labor income is even less important relative to capital income. Note that, if we extend the grid even further, the two curves converge.

4. To quantify the average gain or loss from being in the alternative world, compute the expected (average) value of age 22 agents in each economy. This is the simplest to do via simulation. Using these average values, again compute the welfare difference between the two settings expressed in percentages of consumption.

Simulating the data, we find that the welfare difference between the two settings is -3.81. See Julia file for code.

5. Our model is simpler than that of De Nardi (2004). How do each of the following simplifications affect your findings, in your opinion? Only qualitative reasoning is expected here.
   (a) We don't close the capital market, so capital is not used in production.
   (b) Our agents do not expect to receive bequests (even though they do receive them).

a) If we were to close the capital market e.g. with a representative firm as in De Nardi

(2004), we would see that wages (the marginal productivity of labor) would be influenced by the amount of capital invested. Thus, closing the capital market would mean that capital gains taxes influence labor income. Here, we would for example expect agents with less cash-on-hand to derive more value from a zero capital gains tax than they do in our model, as they would benefit from the increased savings in the economy leading to increased wages.

b) Not expecting to receive bequests means that agents in our model do not follow the optimal savings and consumption paths that they would choose if they expected a bequest. This means that values of a given cash on hand at age 22 (which we note is before most of the agents receive their bequest) must be lower than in the expected bequest case. This distortive effect is larger for those with a high α_i who on average receive higher bequests. If we were to model expected bequests, we would also find that those expecting higher bequests would be more negatively impacted by a high capital gains tax as their inheritance would be decreased by the tax. In sum, the welfare particularly of high-alpha individuals would be different across cases if expected bequests were to be modelled.