Applied Econometric Time Series – Problem Set 3

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1. (a) To show that $Cov(y_t, \varepsilon_{zt}) \neq 0$. we can express the covariance as expectations and perform the calculations with substitutions from the bivariate system:

$$Cov(y_t, \varepsilon_{zt}) = E[[y_t - E[y_t]] \cdot [\varepsilon_{zt} - E[\varepsilon_{zt}]]$$

We can express substitute the expression of y_t from the bivariate system. The expectation of all factors that are multiplied by ε_{zt} and ε_{yt} goes to zero by the white-noise process assumption, and we are left with:

$$Cov(y_t, \varepsilon_{zt}) = Cov(\frac{\varepsilon_{yt} - b_{12}\varepsilon_{zt}}{1 - b_{12}b_{21}}, \varepsilon_{zt})$$

$$Cov(y_t, \varepsilon_{zt}) = Cov(\frac{-b_{12}\varepsilon_{zt}}{1 - b_{12}b_{21}}, \varepsilon_{zt}) = \frac{-b_{12}}{1 - b_{12}b_{21}} \cdot \sigma_z^2$$

We can now conclude that the covariance will be different from zero as long as $b_{12} \neq 0$ and $1 - b_{12}b_{21} \neq 0$.

(b) From the derivations in part a) we can conclude that the $Cov(y_t, \varepsilon_{zt}) \neq 0$. would not hold if $b_{12} = 0$. Setting $b_{12} = 0$ removes the direct link between z_t and y_t hence also removing the pathway between ε_{zt} and y_t.

(c) We begin with the following set of equations:

$$y_t = -b_{12}z_t + \gamma_{11}y_{t-1} + \gamma_{12}z_{t-1} + \epsilon_{yt}$$
$$z_t = -b_{21}y_t + \gamma_{21}y_{t-1} + \gamma_{22}z_{t-1} + \epsilon_{zt}$$

We use matrix algebra to write the system in a compact form:

$$X_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix}$$
$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$$
$$\epsilon_t = \begin{bmatrix} \epsilon_{yt} \\ \epsilon_{zt} \end{bmatrix}$$
$$Thus,$$
$$BX_t = \Gamma.X_{t-1} + \epsilon_t$$

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Multiplying by B^{-1},

$$\begin{aligned} B^{-1}BX_t &= B^{-1}\Gamma.X_{t-1} + B^{-1}\epsilon_t \\ X_t &= A_1X_{t-1} + e_t....(1) \end{aligned}$$

where,

$$\begin{aligned} A_1 &= B^{-1}\Gamma \\ e_t &= B^{-1}\epsilon_t \end{aligned}$$

(1) is known as the reduced form representation. It can also be expressed as:

$$\begin{aligned} y_t &= a_{11}y_{t-1} + a_{12}z_{t-1} + e_{yt} \\ z_t &= a_{21}y_{t-1} + a_{22}z_{t-1} + e_{zt} \end{aligned}$$

(d) We know that:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.50 & 0.10 \\ 0.40 & 0.20 \end{bmatrix} \\ \Sigma_e &= \begin{bmatrix} 1 & 0.50 \\ 0.50 & 2 \end{bmatrix} \\ b_{12} &= 0 \end{aligned}$$

We start with the A_1 matrix,

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = B^{-1}\Gamma$$

$$B^{-1}\Gamma = \frac{1}{\delta}\begin{bmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{bmatrix}\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}$$

$$\delta = |B| = \begin{vmatrix} 1 & b_{12} \\ b_{21} & 1 \end{vmatrix} = 1 - b_{12}b_{21} = 1$$

(as $b_{12} = 0$)
Thus,

$$A_1 = B^{-1}\Gamma = \begin{bmatrix} \gamma_{11} - b_{12}\gamma_{21} & \gamma_{12} - b_{12}\gamma_{22} \\ \gamma_{21} - b_{21}\gamma_{11} & \gamma_{22} - b_{21}\gamma_{12} \end{bmatrix}$$

Now we substitute for e_t,

$$e_t = B^{-1}\epsilon_t = \frac{1}{\delta}\begin{bmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{bmatrix}\begin{bmatrix} \epsilon_{yt} \\ \epsilon_{zt} \end{bmatrix}$$

$$\begin{bmatrix} e_{yt} \\ e_{zt} \end{bmatrix} = \begin{bmatrix} \epsilon_{yt} - b_{12}\epsilon_{zt} \\ \epsilon_{zt} - b_{21}\epsilon_{yt} \end{bmatrix}$$

We know that the $Cov(e_{yt}, e_{zt}) \neq 0$. Substituting for these terms from the derived value of e_t,

$$\begin{aligned} Cov(e_{yt}, e_{zt}) &= Cov(\epsilon_{yt} - b_{12}\epsilon_{zt}, \epsilon_{zt} - b_{21}\epsilon_{yt}) \\ &= Cov(\epsilon_{yt}, \epsilon_{zt}) - Cov(\epsilon_{yt}, b_{21}\epsilon_{yt}) - Cov(b_{12}\epsilon_{zt}, \epsilon_{zt}) + Cov(b_{12}\epsilon_{zt}, b_{21}\epsilon_{yt}) \\ &= Cov(\epsilon_{yt}, \epsilon_{zt}) - b_{21}Cov(\epsilon_{yt}, \epsilon_{yt}) - b_{12}Cov(\epsilon_{zt}, \epsilon_{zt}) + b_{12}b_{21}Cov(\epsilon_{zt}, \epsilon_{yt}) \end{aligned}$$

We know that $Cov(X, X) = Var(X) = \sigma_X^2$.
Since ϵ_t follows a Vector White Noise process, $Cov(\epsilon_{yt}, \epsilon_{zt}) = 0$.

$$Cov(e_{yt}, e_{zt}) = 0 - b_{21}\sigma_{yt}^2 - b_{12}\sigma_{zt}^2 + 0$$

As $b_{12} = 0$

$$b_{21} = \frac{-Cov(e_{yt}, e_{zt})}{\sigma_{yt}^2}$$

$$b_{21} = \frac{\Sigma_{12}}{\Sigma_{11}} = \frac{-0.5}{1} = -0.5$$

$$B = \begin{bmatrix} 1 & 0 \\ -0.50 & 1 \end{bmatrix}$$

As we know B, we can find the value of Γ by using the value of A_1,

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.50 & 0.10 \\ 0.40 & 0.20 \end{bmatrix} = \begin{bmatrix} \gamma_{11} - b_{12}\gamma_{21} & \gamma_{12} - b_{12}\gamma_{22} \\ \gamma_{21} - b_{21}\gamma_{11} & \gamma_{22} - b_{21}\gamma_{12} \end{bmatrix} \\
0.50 &= \gamma_{11} - 0\gamma_{21} \\
&= \gamma_{11} \\
0.10 &= \gamma_{12} - 0\gamma_{22} \\
&= \gamma_{12} \\
0.40 &= \gamma_{21} - (-0.50 * 0.50) \\
&= \gamma_{21} + 0.25 \\
\gamma_{21} &= 0.15 \\
0.20 &= \gamma_{22} - (-0.50 * 0.10) \\
&= \gamma_{22} + 0.05 \\
\gamma_{22} &= 0.15 \\
\Gamma &= \begin{bmatrix} 0.50 & 0.10 \\ 0.15 & 0.15 \end{bmatrix}
\end{aligned}
$$

Lastly, we find the value of Σ_ϵ,

$$
\begin{aligned}
\begin{bmatrix} e_{yt} \\ e_{zt} \end{bmatrix} &= \begin{bmatrix} \epsilon_{yt} - b_{12}\epsilon_{zt} \\ \epsilon_{zt} - b_{21}\epsilon_{yt} \end{bmatrix} \\
e_{yt} &= \epsilon_{yt} - 0 \\
\Sigma_{e_{11}} &= \Sigma_{\epsilon_{11}} = 1
\end{aligned}
$$

$$
\begin{aligned}
e_{zt} &= \epsilon_{zt} - b_{21}\epsilon_{yt} \\
2 &= \epsilon_{zt} - (-0.50 * 1) \\
\epsilon_{zt} &= 1.50
\end{aligned}
$$

$$
\Sigma_\epsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1.50 \end{bmatrix}
$$

(e) The system is stationary if the unit roots of the characteristic polynomial lay outside the unit circle. For a system of equations, the characteristic polynomial is defined by

$$\det(I - Az) = 0.$$

This gives us roots

$$(1 - a_{11}z)(1 - a_{22}z) - (a_{12}a_{a21}z^2) = 0,$$

$$1 - 0.8z + 0.07z^2 = 0,$$

$$z = \frac{40 \pm \sqrt{900}}{7},$$

$$z_1 = 10, \quad z_2 = \frac{10}{7}.$$

Since they both are outside the unit circle the system is stationary.

The impulse response of a unit shock can be calculated by recursive substitution. However, a much easier way of computing the impulse responses is to use the VMA representation and note that

$$\frac{\partial x_t}{\partial \epsilon_{t-s}} = A^s.$$

The four period sequence after a unit shock to y_t is,

$$\{y_t\}_{t=0}^{4} = \{1, 0.5, 0.33, 0.229, 0.16\},$$

$$\{z_t\}_{t=0}^{4} = \{0, 0.4, 0.32, 0.228, 0.16\}.$$

The four period sequence after a unit shock to z_t is,

$$\{y_t\}_{t=0}^{4} = \{0.25, 0.325, 0.24, 0.171, 0.12\},$$

$$\{z_t\}_{t=0}^{4} = \{1, 0.4, 0.25, 0.172, 0.12\}.$$

The simplest way of introducing unit roots is setting one coefficient in each row equal to 1. Two examples are

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The impulse response in a unit root system will not die out, illustrated by the following 4-period impulse responses for A_1 and A_2. A unit increase in ϵ_y for the system described by A_1 yields

$$\{y_t\}_{t=0}^{4} = \{1, 1, 1, 1, 1\},$$

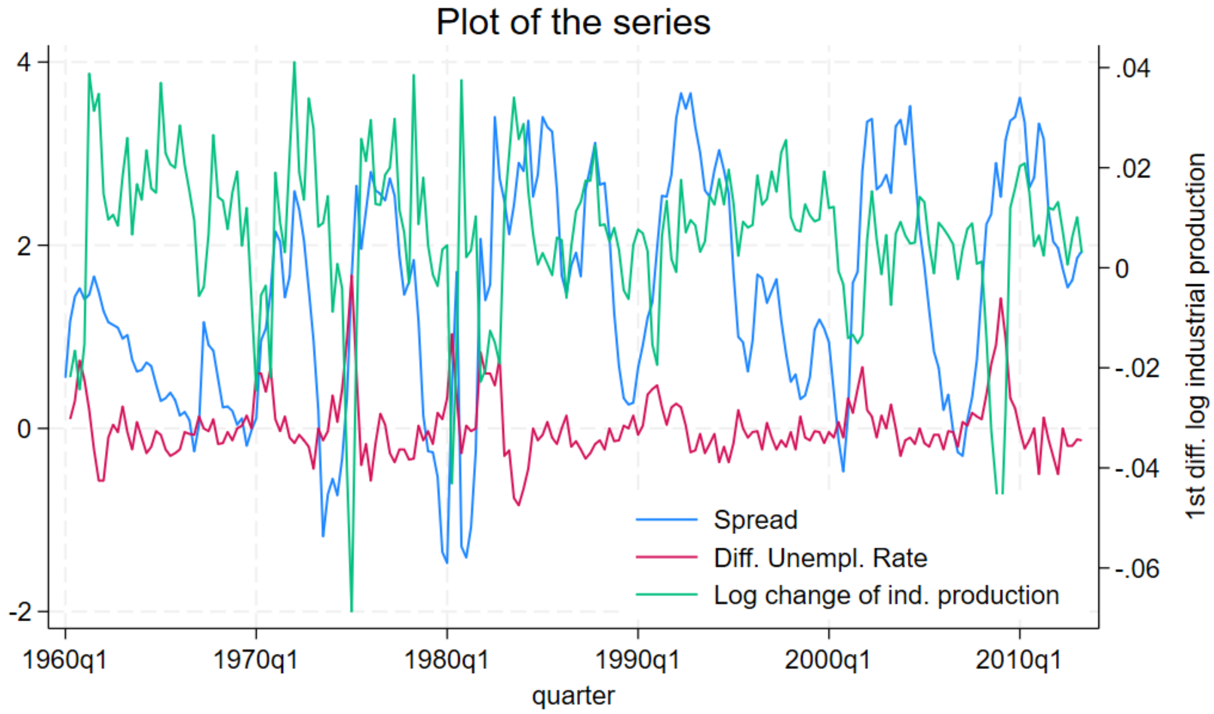
$$\{z_t\}_{t=0}^{4} = \{0, 0, 0, 0, 0\},$$

and vice versa with a shock to ϵ_z. For A_2, a unit shock to ϵ_y gives us

$$\{y_t\}_{t=0}^{4} = \{1, 0, 1, 0, 1\},$$

$$\{z_t\}_{t=0}^{4} = \{0, 1, 0, 1, 0\},$$

and vice versa for a shock to ϵ_z.



2. (a) The test rejects the hypothesis (and thus finds evidence of Granger causality) if the statistic is larger than the critical value (if the p-value is small) and fails to reject the hypothesis (and thus finds no evidence of causality) if the statistic is smaller than the critical value. We find that the F-statistic is 2.40 with a p-value of 0.0691. We therefore fail to reject null hypothesis of s_t not being causally prior to Δlip_t at at the 5% significance level.

```
var dlip dur spread, lags(1/3)
Sample: 4 thru 213                   Number of obs    =        210
Log likelihood =   569.5957          AIC              =  -5.139007
FPE            =   1.18e-06          HQIC             =  -4.945705
Det(Sigma_ml)  =   8.84e-07          SBIC             =  -4.660848

Equation           Parms      RMSE     R-sq      chi2     P>chi2
-----------------------------------------------------------------
dlip                 10     .01236    0.4043   142.5076   0.0000
dur                  10    .237422    0.5213   228.7073   0.0000
spread               10    .506265    0.8367    1075.81   0.0000

test [dlip]L1.spread [dlip]L2.spread [dlip]L3.spread, df(180)
[dlip]L.spread = [dlip]L2.spread = [dlip]L3.spread = 0
      F(  3,   180) =    2.40     Prob > F =    0.0691
```

(b) The corresponding F-statistic is 6.83 with a p-value of 0.0002. This is statistically significant at any conventional level so we can conclude that the s_t series has a predictively causal impact on $\Delta unemp_t$.

```
test [dur]L1.spread [dur]L2.spread [dur]L3.spread, df(180)
[dur]L.spread = [dur]L2.spread = [dur]L3.spread = 0
      F(  3,   180) =    6.83      Prob > F =    0.0002
```

(c) The correlation coefficient between e_{1t} and e_{2t} is -0.719 between e_{1t} and e_{3t} is $-.186$ and between e_{2t} and e_{3t} is .178. Unless the errors are uncorrelated different orderings will lead to different impulse response functions and forecast error decompositions. In order for impulse responses and forecast error decompositions to be interpreted causally the orthogonalization must be identified by the user based on a structural economic argument. There is no data-dependent choice. The results should therefore be compared the to those obtained by changing the ordering as a sensitivity analysis.

```
matrix sig_var = e(Sigma), matrix corr_var = corr(sig_var)
matrix list corr_var
                   dlip          dur        spread
      dlip            1
       dur   -.71936136            1
    spread   -.18584509    .17848363             1
```

(d) We verify that the forecast error variance decompositions are:

```
irf set res, irf create res, replace, irf table fevd, noci
```

dlip impulse	dlip	dur	spread
1	1	.517481	.034538
4	.965395	.655408	.155866
8	.920089	.587014	.288695

.

dur impulse	dlip	dur	spread
1	0	.482519	.004158
4	.01492	.3245	.016408
8	024996	.29426	.016913

spread impulse	dlip	dur	spread
1	0	0	.961303
4	.019685	.020092	.827726
8	.054915	.118726	.694391

(e) We estimate the five lag lip_t, ur_t, s_t VAR model and report diagnostic tests.

```
var lip urate spread, lags(1/5)
Sample: 5 thru 213                   Number of obs      =        209
Log likelihood =    585.9788         AIC                =  -5.148122
FPE            =    1.17e-06         HQIC               =   -4.83777
Det(Sigma_ml)  =    7.37e-07         SBIC               =  -4.380505
Equation               Parms     RMSE      R-sq        chi2     P>chi2
------------------------------------------------------------------------
lip                      16    .012012    0.9992    249997.1    0.0000
urate                    16    .235503    0.9806    10577.52    0.0000
spread                   16    .503414    0.8442    1132.134    0.0000
------------------------------------------------------------------------
varstable  All the eigenvalues lie inside the unit circle.
           VAR satisfies stability condition.
```

Lag length of five is appropriate in terms of stability. Yet, remaining serial correlation in residuals suggests model misspecification. $urate_t$ disturbances display a significant departure from normality, the $\Delta urate_t$ transformation is thus likely to be desirable.

```
varlmar , mlag(12)
Lagrange-multiplier test , H0: no autocorrelation at lag order
              | lag |      chi2     df    Prob > chi2 |
              |  1  |   27.4988     9      0.00116    |
              |  4  |   16.7246     9      0.05321    |
              |  8  |   24.1334     9      0.00410    |
varnorm
        Jarque-Bera test      chi2    df   Prob > chi2 |
                 urate |             0.461    2     0.79406   |
```

We conclude that the model is not entirely suitable and would rather opt for optimal information criteria lag lengths: AIC selects three, and the BIC selects two lags.

```
varsoc lip urate spread , maxlag(5)  Sample: 1961q2 thru 2013q2
 N = 209      | Lag |     LL          AIC          SBIC       |
              |   0 | -789.703     7.58568      7.63365     |
              |   1 |  486.745    -4.54301     -4.35111     |
              |   2 |  564.226    -5.19834      -4.8625*    |
              |   3 |  576.175    -5.22655*    -4.74679     |
```

The forecast error decomposition are comparable to to those above, indicating that shocks in industrial production contribute the most towards the fluctuations of the other variables in the system.

```
irf set res2 , irf create res2 , replace , irf table fevd , noci
```

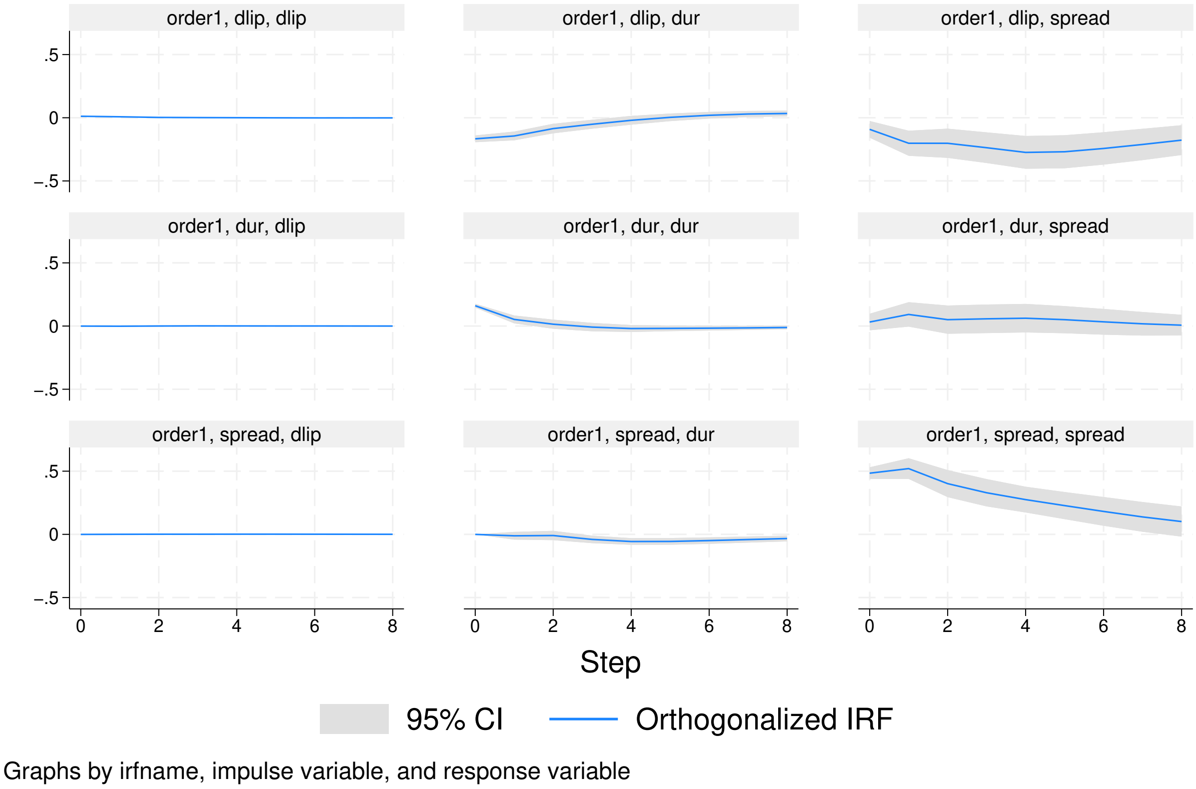
lip impulse	lip	ur	spread
1	1	.519842	.032788
4	.982678	.714612	.137568
8	.904433	.682112	.241657

ur impulse	lip	ur	spread
1	0	.480158	.006493
4	.004757	.284058	.031282
8	.003655	.256073	.048229

spread impulse	lip	ur	spread
1	0	0	.960719
4	.012565	.001329	.83115
8	.091912	.061815	.710113

(f) We obtain the orthogonolized impulse response functions. Modelling a positive shock to Δlip_t induces a decline in Δur_t that lasts for six quarters. Then, Δur_t overshoots its long-run level before returning to zero.

```
order(dlip dur spread) irf table oirf , impulse(dlip) response(dur)
```



order1		oirf	Lower	Upper
step	0	−.166676	−.193657	−.139696
	1	−.144006	−.178638	−.109374
	2	−.085494	−.123272	−.047717
	3	−.051388	−.086891	−.015886
	4	−.020162	−.055092	.014767
	5	.003389	−.026572	.033351
	6	.019609	−.006806	.046024
	7	.029682	.005463	.053901
	8	.033817	.010866	.056767

(g) When using the Cholesky decomposition the recursive structure is determined by the ordering of the variables in the system. The order matters and is the key identifying assumption. The second recursive structure excludes shocks in Δur_t and Δlip_t contemporeneously affecting s_t and, as illustrated below, restricts the contemporaneous impact of a shock in Δlip_t on Δur_t to zero as well.

```
order(spread dur dlip), irf table oirf, impulse(dlip) response(dur)
```

order2		oirf	Lower	Upper
step	0	0	0	0
	1	−.062949	−.094589	−.03131
	2	−.049399	−.086031	−.012766
	3	−.044768	−.081021	−.008515
	4	−.032753	−.063085	−.002421
	5	−.01535	−.038091	.00739
	6	−.002566	−.021993	.016861
	7	.006721	−.011259	.024701
	8	.012646	−.004359	.029651