5330 Advanced Microeconomic Theory

Lecture: The Competitive Market System

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Introduction

This lecture is mainly based on

- Geoffrey Jehle and Philip Reny (2011), *Advanced Microeconomic Theory*, chapter 5, and any good mathematics textbook.

- Many scholars trace the birth of economics to the publication of Adam Smith's *The Wealth of Nations* (1776).
- Behind the superficial chaos of countless interdependent market actions by selfish agents, Smith saw a harmonizing force serving society.
- This *Invisible Hand* guides the market system to an equilibrium that Smith believed possessed certain socially desirable characteristics.
- Is Adam Smith's vision of a smoothly functioning system composed of many self-interested individuals buying and selling on impersonal markets – with no regard for anything but their personal gain – a logically coherent vision at all?

- In a discussion of import tariffs, Adam Smith (1776) wrote that:

  “Every individual necessarily labours to render the annual revenue of the society as great as he can... He is in this, as in many other ways, led by an invisible hand to promote an end which was no part of his intention... By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it.”
- Was Adam Smith right?

Equilibrium in Exchange

- We begin by exploring the basic economic problem of distribution in a very simple society. There is no production. Commodities exist, but for now we do not ask how they came to be.
- Instead, we merely assume each consumer is "endowed" by nature with a certain amount of a finite number of consumable goods.
- Each consumer has preferences over the available commodity bundles, and each cares only about his or her individual well-being.
- Agents may consume their endowment of commodities or may engage in barter exchange with others.

- Suppose there are only two consumers in this society, consumer 1 and consumer 2, and only two goods, x_1 and x_2.
- Let $\mathbf{e}^1 \equiv (e_1^1, e_2^1)$ denote the endowment of the two goods owned by consumer 1, and $\mathbf{e}^2 \equiv (e_1^2, e_2^2)$ the endowment of consumer 2.
- The total amount of each good available in this society then can be summarized by the vector $\mathbf{e}^1 + \mathbf{e}^2 \equiv (e_1^1, e_2^1) + (e_1^2, e_2^2)$.
- The essential aspects of this economy can be analyzed with the **Edgeworth box**. This box diagram provides a complete picture of every feasible distribution of existing commodities between consumers.

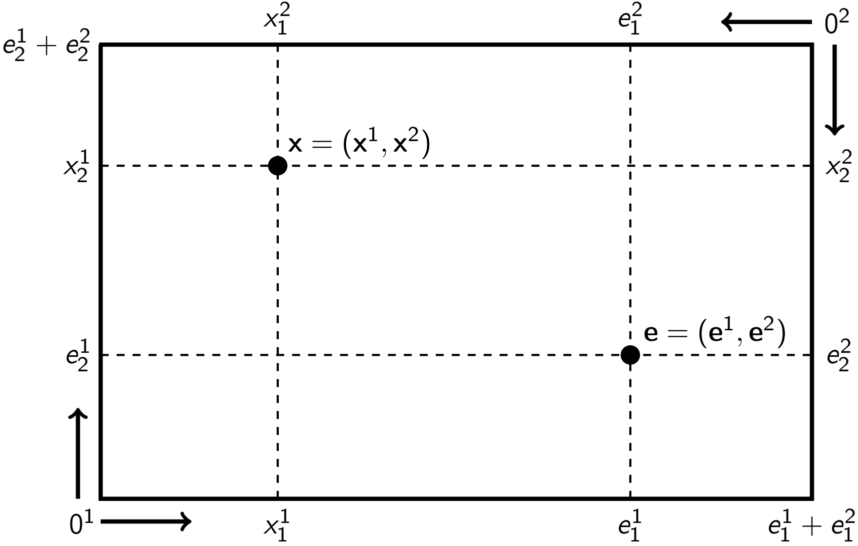

Figure: The Edgeworth box.

- Suppose each consumer has preferences represented by a usual, convex indifference map.
- One indifference curve for each consumer passes through every point in the box.
- The curve labeled CC is the subset of allocations where the consumers' indifference curves though the point are tangent to each other, and it is called the **contract curve**.

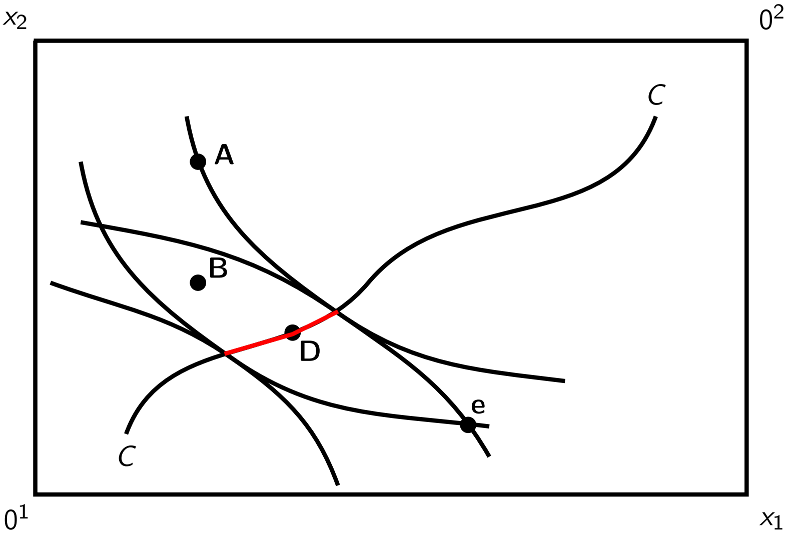

Figure: Equilibrium in two-person exchange.

- Given initial endowments at **e**, which allocations will be barter equilibria in this exchange economy?
- Suppose a redistribution from **e** to point A were proposed. This would be refused or "blocked" by consumer 2 and so could not arise as a barter equilibrium given the initial endowment.
- Suppose a redistribution to point B were proposed. Both consumers will be strictly better off than they are at **e**. However, because B is off the contract curve, both consumers can gain by further trade (to D).
- The only points of equilibrium are on the contract curve (shaded in red) and each point of equilibrium in exchange is Pareto efficient.

- Consider now the case of many consumers and many goods. Let

$$\mathcal{I} = \{1, \ldots, I\}$$

  index the set of consumers, and suppose there are n goods.
- Each consumer $i \in \mathcal{I}$ has a preference relation $\succsim^i$ and is endowed with a nonnegative vector of the n goods, $\mathbf{e}^i = (e_1^i, \ldots, e_n^i)$.
- Altogether, the collection $\mathcal{E} = (\succsim^i, \mathbf{e}^i)_{i \in \mathcal{I}}$ defines an **exchange economy**.

- What conditions characterize barter equilibria in this exchange economy?

- Let
$$\mathbf{e} \equiv (\mathbf{e}^1, \ldots, \mathbf{e}^I)$$
denote the economy's endowment vector, and define an **allocation** as a vector
$$\mathbf{x} \equiv (\mathbf{x}^1, \ldots, \mathbf{x}^I)$$
where $\mathbf{x}^i \equiv (x_1^i, \ldots, x_n^i)$ denotes consumer i's bundle according to the allocation. The set of **feasible allocations** in this economy is given by
$$F(\mathbf{e}) \equiv \left\{ \mathbf{x} \,|\, \sum_{i \in \mathcal{I}} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i \right\}.$$
- The first requirement for a barter equilibrium is that $\mathbf{x} \in F(\mathbf{e})$.

- Now in the two-consumer case, we noted that if both consumers could be made better off by trading with one another, then we could not yet be at a barter equilibrium.
- **DEFINITION (Pareto-Efficient Allocations)**: A feasible allocation $\mathbf{x} \in F(\mathbf{e})$ is Pareto-efficient if there is no other feasible allocation $\mathbf{y} \in F(\mathbf{e})$ such that $\mathbf{y}^i \succsim^i \mathbf{x}^i$ for all consumers i, with at least one preference strict.
- Only Pareto-efficient allocations are candidates for barter equilibrium, and whenever a Pareto-efficient allocation is reached, it will indeed be an equilibrium of our process of voluntary exchange.

- When there are more than two consumers, no equilibrium allocation can make any consumer worse off than he would be consuming his endowment.
- In fact there are now additional reasons you might refuse to trade to some Pareto-efficient allocation.
- **DEFINITION (Blocking Coalitions)**: Let $S \subset \mathcal{I}$ denote a coalition of consumers. We say that S blocks $\mathbf{x} \in F(\mathbf{e})$ if there is an allocation $\mathbf{y}$ such that:

  1. $\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$.
  2. $\mathbf{y}^i \succsim^i \mathbf{x}^i$ for all $i \in S$, with at least one preference strict.

- We say that an allocation is "unblocked" if no coalition can block it. Our final requirement for equilibrium then is that the allocation be unblocked.
- An allocation $\mathbf{x} \in F(\mathbf{e})$ is an equilibrium in the exchange economy with endowment $\mathbf{e}$ if $\mathbf{x}$ is not blocked by any coalition of consumers.
- **DEFINITION (The Core of an Exchange Economy)**: The core of an exchange economy with endowment $\mathbf{e}$, denoted $C(\mathbf{e})$, is the set of all unblocked feasible allocations.

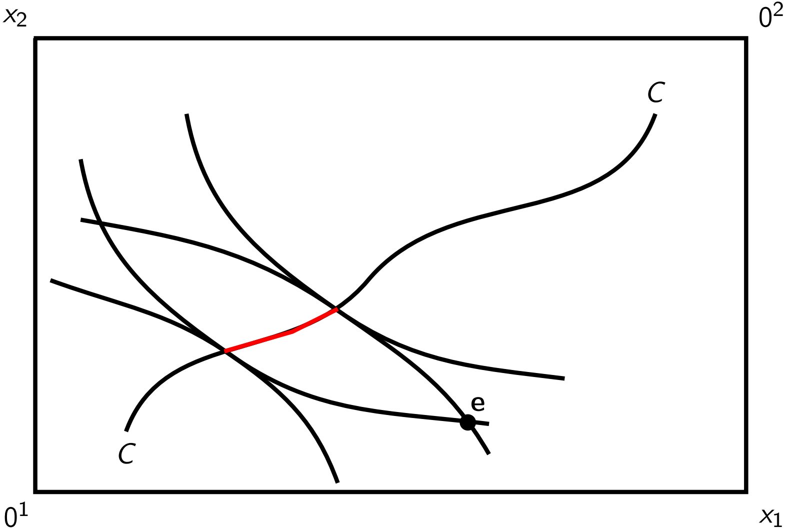

Figure: The core of a two-consumer, two-good exchange economy.

- Points in the core seem very far from becoming a reality in a real-world economy. After all, most of us have very little or no direct contact with the vast majority of other consumers.
- Consequently, one would be quite surprised were there not substantial gains from trade left unrealized, regardless of how the economy were organized – centrally planned, market-based, or otherwise.

- Prepare for a surprise.

Equilibrium In Competitive Market Systems

- We just examined a very primitive economic system based wholly on voluntary barter exchange. Now we look at questions of equilibrium and distribution in a more sophisticated economic system.
- In a **perfectly competitive market system**, all transactions between individuals are mediated by impersonal markets. Equilibrium in the market system is achieved when the demands of buyers match the supplies of sellers at prevailing prices in every market simultaneously.
- For simplicity, let us first consider an economy without the complications of production in the model. Again let $i \in \mathcal{I}$ index the set of consumers and assume that each is endowed with a nonnegative vector $\mathbf{e}^i$ of n goods.

- We will further suppose each consumer's preferences on the consumption set $\mathbb{R}^n_+$ can be represented by a utility function u^i satisfying the following:

- **ASSUMPTION (Consumer Utility)**: Utility u^i is continuous, strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.

- **DEFINITION (Increasing Functions)**: Let $f : D \longrightarrow \mathbb{R}$, where D is a subset of $\mathbb{R}^n$. Then,

  f is increasing if $f(\mathrm{x}^0) \geq f(\mathrm{x}^1)$ whenever $\mathrm{x}^0 \geq \mathrm{x}^1$,
  f is strictly increasing if $f(\mathrm{x}^0) > f(\mathrm{x}^1)$ whenever $\mathrm{x}^0 \gg \mathrm{x}^1$ and f is increasing,
  f is strongly increasing if $f(\mathrm{x}^0) > f(\mathrm{x}^1)$ whenever x^0 and x^1 are distinct and $\mathrm{x}^0 \geq \mathrm{x}^1$.

- On competitive markets, each consumer takes prices as given, whether acting as a buyer or a seller. If $\mathbf{p} \equiv (p_1, \ldots, p_n) \gg \mathbf{0}$ is the vector of market prices, then each consumer solves

$$\max_{\mathbf{x}^i \in \mathbb{R}^n_+} u^i(\mathbf{x}^i) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i.$$

- **THEOREM (Basic Properties of Demand)**: If u^i satisfies the previous assumptions (continuous, strongly increasing, strictly quasiconcave), then for each $\mathbf{p} \gg \mathbf{0}$, the consumer's problem has a unique solution $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$. In addition, $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ is continuous in $\mathbf{p}$ on $\mathbb{R}^n_{++}$.

- Recall that existence of a solution follows because $\mathbf{p} \gg \mathbf{0}$ implies that the budget set is bounded, and uniqueness follows from the strict quasiconcavity of u^i. Continuity in $\mathbf{p}$ follows from the Theorem of the Maximum, and this requires $\mathbf{p} \gg \mathbf{0}$.

- In the earliest analysis of market systems, undertaken by Leon Walras (1874, *Elements of Pure Economics*), each market is described by separate demand and supply functions. Today, largely as a matter of convenience and notational simplicity, it is more common to describe each separate market by a single excess demand function.
- **DEFINITION (Excess Demand)**: The excess demand function for market k is the real-valued function

$$z_k(\mathbf{p}) \equiv \sum_{i \in \mathcal{I}} x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - \sum_{i \in \mathcal{I}} e_k^i$$

Aggregate excess demand is the vector-valued function

$$\mathbf{z}(\mathbf{p}) \equiv (z_1(\mathbf{p}), \ldots, z_n(\mathbf{p})).$$

- **THEOREM (Properties of Excess Demand Functions)**: If for each consumer i, u^i satisfies the previous assumptions (continuous, strongly increasing, strictly quasiconcave), then for all $\mathbf{p} \gg \mathbf{0}$,

  1. Continuity: $\mathbf{z}(\cdot)$ is continuous in $\mathbf{p}$.
  2. Homogeneity: $\mathbf{z}(\lambda\mathbf{p}) = \mathbf{z}(\mathbf{p})$ for all $\lambda > 0$.
  3. Walras' law: $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$.

- **Proof**: Continuity follows from the previous theorem about continuity of demand functions.

  Looking at
$$z_k(\mathbf{p}) \equiv \sum_{i\in\mathcal{I}} x_k^i(\mathbf{p}, \mathbf{p}\cdot\mathbf{e}^i) - \sum_{i\in\mathcal{I}} e_k^i,$$
  individual demand functions are homogeneous of degree zero in prices. It follows immediately that aggregate excess demand is also homogeneous of degree zero in prices.

  The third property (Walras' law $\mathbf{p}\cdot\mathbf{z}(\mathbf{p}) = 0$) is important. It says that the value of aggregate excess demand will always be zero for any set of positive prices. Walras' law follows because when u^i is strongly increasing, each consumer's budget constraint holds with equality.

- When each consumer's budget constraint holds with equality, $\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i$ becomes $\mathbf{p} \cdot \mathbf{x}^i = \mathbf{p} \cdot \mathbf{e}^i$ or

$$\sum_{k=1}^{n} p_k[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i] = 0.$$

Summing over individuals gives

$$\sum_{i \in \mathcal{I}} \sum_{k=1}^{n} p_k[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i] = 0.$$

Because the order of summation does not matter, we can reverse it

$$\sum_{k=1}^{n} \sum_{i \in \mathcal{I}} p_k[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i] = 0.$$

- $$\sum_{k=1}^{n} \sum_{i \in \mathcal{I}} p_k [x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i] = 0$$

$$\sum_{k=1}^{n} p_k \left(\sum_{i \in \mathcal{I}} x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - \sum_{i \in \mathcal{I}} e_k^i \right) = 0$$

$$\sum_{k=1}^{n} p_k z_k(\mathbf{p}) = \mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0,$$

and the claim is proved.

$\square$

- Walras' law has some interesting implications. For example, consider a two-good economy and suppose that prices are strictly positive. Then

$$p_1 z_1(\mathbf{p}) + p_2 z_2(\mathbf{p}) = \mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0.$$

- If there is excess demand in market 1, so that $z_1(\mathbf{p}) > 0$, we know immediately that we must have $z_2(\mathbf{p}) < 0$ or excess supply in market 2.
- Similarly, if market 1 is in equilibrium at $\mathbf{p}$, so that $z_1(\mathbf{p}) = 0$, then market 2 is also in equilibrium with $z_2(\mathbf{p}) = 0$.

- In the case of n markets, if at some set of prices $n-1$ markets are in equilibrium, Walras' law ensures the nth market is also in equilibrium. This is often quite useful to remember.

$$\sum_{k=1}^{n} p_k z_k(\mathbf{p}) = 0.$$

- Prices that equate demand and supply in every market are called **Walrasian**.
- **DEFINITION (Walrasian Equilibrium)**: A vector $\mathbf{p}^* \in \mathbb{R}^n_{++}$ is called a Walrasian equilibrium if $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.
- We now turn to the question of existence of Walrasian equilibrium. This is indeed an important question because it speaks directly to the logical coherence of Adam Smith's vision of a market economy.
- One certainly cannot explore sensibly the social and economic properties of equilibria in market economies without full confidence that they exist, and without full knowledge of the circumstances under which they can be expected to exist.

- This central question in economic theory has attracted the attention of a great many theorists over time.
- Walras (1874) was the first to attempt an answer to the question of existence by reducing it to a question of whether a system of market demand and market supply equations possessed a solution.
- However, Walras cannot be credited with providing a satisfactory answer to the question, because his conclusion rested on the fallacious assumption that any system of equations with as many unknowns as equations always possesses a solution.
- Abraham Wald (1936) was the first to point to Walras' error by offering a simple counter example: the two equations in two unknowns $x^2 + y^2 = 0$ and $x^2 - y^2 = 1$ have no solution (where $(x, y) \in \mathbb{R}^2_+$), as you can easily verify.

- Lionel McKenzie (1954, *Econometrica*), and Kenneth Arrow and Gerard Debreu (1954, *Econometrica*) were the first to offer reasonably general proofs of existence.
- Each framed their search for market-clearing prices as the search for a fixed point to a carefully chosen mapping and employed powerful fixed-point theorems to reach their conclusion.
- In what follows, we will also employ the same method (following closely Arrow and Debreu). Arrow was awarded the economics Nobel prize in 1972 (together with John Hicks) and Debreu was awarded the economics Nobel prize in 1983.

- **THEOREM (Aggregate Excess Demand and Walrasian Equilibrium)**: Suppose $\mathbf{z}(\mathbf{p})$ satisfies the following three conditions.

  1. $\mathbf{z}(\cdot)$ is continuous on $\mathbb{R}^n_{++}$.

  2. $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ for all $\mathbf{p} \gg \mathbf{0}$.

  3. If $\{\mathbf{p}^m\}$ is a sequence of price vectors in $\mathbb{R}^n_{++}$ converging to $\bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some good k, then for some good k' with $\bar{p}_{k'} = 0$, the associated sequence of excess demands in the market for good k', $\{z_{k'}(\mathbf{p}^m)\}$, is unbounded above.

  Then there is a price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

- The third condition is new and says roughly that if the prices of some but not all goods is arbitrarily close to zero, then the excess demand for at least one of those goods is arbitrarily high.

- **Proof**: For each good k, let

$$\bar{z}_k(\mathbf{p}) \equiv \min\left[z_k(\mathbf{p}), 1\right] \quad \text{for all} \quad \mathbf{p} \gg \mathbf{0}$$

  and let $\bar{\mathbf{z}}(\mathbf{p}) \equiv (\bar{z}_1(\mathbf{p}), \ldots, \bar{z}_n(\mathbf{p}))$. Then we are assured that $\bar{z}_k(\mathbf{p})$ is bounded above by 1.

- Now, fix $\epsilon \in (0,1)$ and let

$$S_\epsilon \equiv \left\{ \mathbf{p} \,\middle|\, \sum_{k=1}^{n} p_k = 1 \quad \text{and} \quad p_k \geq \frac{\epsilon}{1+2n} \quad \text{for all } k \right\}.$$

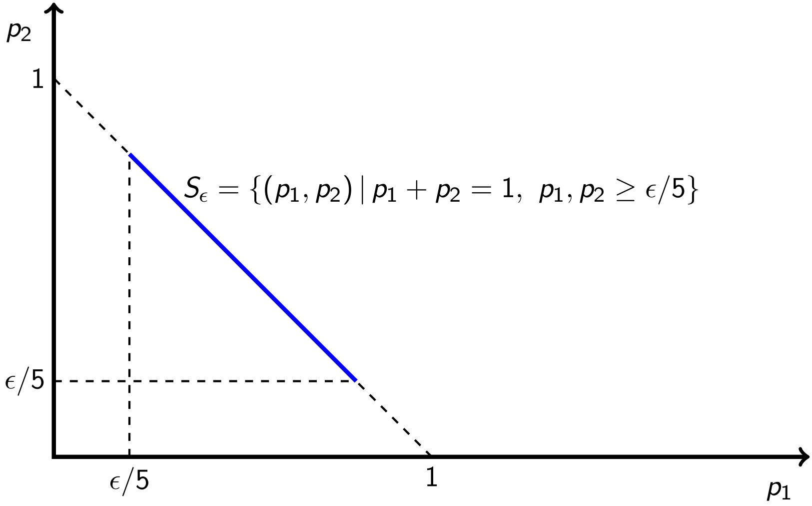

Figure: The set S_ϵ in $\mathbb{R}^2_+$.

- In searching for $\mathbf{p}^*$ satisfying $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$, we shall begin by restricting our search to the set S_ϵ.
- Note how prices on and near the boundary of the nonnegative orthant are excluded from S_ϵ.
- Note also that as ϵ is allowed to approach zero, S_ϵ includes more and more prices. Thus we can expand the scope of our search by letting ϵ tend to zero.
- We shall do so a little later. For now, however, ϵ remains fixed.

- $$S_\epsilon \equiv \left\{ \mathbf{p} \,\middle|\, \sum_{k=1}^{n} p_k = 1 \quad \text{and} \quad p_k \geq \frac{\epsilon}{1+2n} \quad \text{for all } k \right\}.$$

  Note the following properties of the set S_ϵ. It is compact, convex and nonempty.
- Compactness follows because it is both closed and bounded, and convexity can be easily checked. To see that it is nonempty, note that the price vector with each component equal to

  $$p_k = \frac{2+1/n}{1+2n}$$

  is always a member because $\epsilon < 1$.

  $$\sum_{k=1}^{n} p_k = n\frac{2+1/n}{1+2n} = \frac{2n+1}{1+2n} = 1 \qquad p_k = \frac{2+1/n}{1+2n} > \frac{\epsilon}{1+2n}$$

- For each good k and every $\mathbf{p} \in S_\epsilon$, define $f_k(\mathbf{p})$ as follows:

$$f_k(\mathbf{p}) \equiv \frac{\epsilon + p_k + \max\left[0, \bar{z}_k(\mathbf{p})\right]}{n\epsilon + 1 + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p})\right]}$$

and let $f(\mathbf{p}) \equiv (f_1(\mathbf{p}), \ldots, f_n(\mathbf{p}))$. Consequently

$$\sum_{k=1}^{n} f_k(\mathbf{p}) = \frac{\sum_{k=1}^{n} \epsilon + \sum_{k=1}^{n} p_k + \sum_{k=1}^{n} \max\left[0, \bar{z}_k(\mathbf{p})\right]}{n\epsilon + 1 + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p})\right]} = 1.$$

Because $\bar{z}_m(\mathbf{p}) \equiv \min\left[z_m(\mathbf{p}), 1\right] \leq 1$ and $\epsilon < 1$,

$$f_k(\mathbf{p}) \geq \frac{\epsilon}{n\epsilon + 1 + \sum_{m=1}^{n} 1} = \frac{\epsilon}{n\epsilon + 1 + n} \geq \frac{\epsilon}{1 + 2n}.$$

- It follows that $f(\mathbf{p}) \equiv (f_1(\mathbf{p}), \ldots, f_n(\mathbf{p})) \in S_\epsilon$ since

$$S_\epsilon \equiv \left\{ \mathbf{p} \;\middle|\; \sum_{k=1}^{n} p_k = 1 \quad \text{and} \quad p_k \geq \frac{\epsilon}{1+2n} \quad \text{for all } k \right\}.$$

Therefore

$$f : S_\epsilon \longrightarrow S_\epsilon.$$

- Note now that each

$$f_k(\mathbf{p}) \equiv \frac{\epsilon + p_k + \max\left[0, \bar{z}_k(\mathbf{p})\right]}{n\epsilon + 1 + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p})\right]}$$

  is continuous on S_ϵ, because by condition 1 of the theorem [$\mathbf{z}(\cdot)$ is continuous on $\mathbb{R}^n_{++}$], $z_k(\cdot)$ is continuous on S_ϵ and therefore $\bar{z}_k(\cdot)$ is continuous on S_ϵ, so both the numerator and denominator defining f_k are continuous on S_ϵ.
- Moreover, the denominator is bounded away from zero because it always takes on a value of at least 1.
- Therefore f is a continuous function mapping the nonempty, compact, convex set S_ϵ into itself.

- **THEOREM (The Brouwer Fixed-Point Theorem, proved by the Dutch mathematician Bertus Brouwer in 1910)**:
  Let $S \subset \mathbb{R}^n$ be a nonempty compact and convex set. Let $f : S \to S$ be a continuous function. Then there exists at least one fixed point of f in S. That is, there exists at least one $\mathbf{x}^* \in S$ such that $\mathbf{x}^* = f(\mathbf{x}^*)$.

- The phrase "fixed point" is used because, should such a point exist, it will be one that is left undisturbed or "unmoved" by the function in going from the domain to the range. The function f merely takes $\mathbf{x}^*$ and maps it right back into itself.

- Returning to our proof, since f is a continuous function mapping the nonempty, compact, convex set S_ϵ into itself, we may appeal to Brouwer's fixed-point theorem to conclude that there exists a fixed-point, a price vector $\mathbf{p}^\epsilon \in S_\epsilon$ such that $f(\mathbf{p}^\epsilon) = \mathbf{p}^\epsilon$.
- Or equivalently, that $f_k(\mathbf{p}^\epsilon) = p_k^\epsilon$ for every $k = 1, \ldots, n$.
- Going back to the definition,

$$f_k(\mathbf{p}) \equiv \frac{\epsilon + p_k + \max\left[0, \bar{z}_k(\mathbf{p})\right]}{n\epsilon + 1 + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p})\right]}$$

this means that for every k,

$$f_k(\mathbf{p}^\epsilon) \equiv \frac{\epsilon + p_k^\epsilon + \max\left[0, \bar{z}_k(\mathbf{p}^\epsilon)\right]}{n\epsilon + 1 + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^\epsilon)\right]} = p_k^\epsilon.$$

- For every k,

$$\frac{\epsilon + p_k^\epsilon + \max\left[0, \bar{z}_k(\mathbf{p}^\epsilon)\right]}{n\epsilon + 1 + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^\epsilon)\right]} = p_k^\epsilon$$

implies that

$$p_k^\epsilon \left(n\epsilon + 1 + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^\epsilon)\right] \right) = \epsilon + p_k^\epsilon + \max\left[0, \bar{z}_k(\mathbf{p}^\epsilon)\right]$$

or

$$p_k^\epsilon \left(n\epsilon + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^\epsilon)\right] \right) = \epsilon + \max\left[0, \bar{z}_k(\mathbf{p}^\epsilon)\right]$$

So, up to this point, we have shown that for every $\epsilon \in (0, 1)$, there exists a price vector in S_ϵ satisfying the previous equation.

- Now allow ϵ to approach zero and consider the associated sequence of price vectors $\{\mathbf{p}^\epsilon\}$ satisfying the previous equation.
- Note that the price sequence is bounded, because $\mathbf{p}^\epsilon \in S_\epsilon$ implies that the price in every market always lies between zero and one.
- Consequently by the following theorem

  THEOREM (On Bounded Sequences):
  Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

  some subsequence of $\{\mathbf{p}^\epsilon\}$ must converge.
- To keep the notation simple, let us suppose that we were clever enough to choose this convergent subsequence right from the start so $\{\mathbf{p}^\epsilon\}$ itself converges to $\mathbf{p}^*$.

- So $\{\mathbf{p}^\epsilon\}$ converges to $\mathbf{p}^*$.
- Of course, $\mathbf{p}^* \geq \mathbf{0}$ and $\mathbf{p}^* \neq \mathbf{0}$ because its components sum to 1. We would like to argue that in fact $\mathbf{p}^* \gg \mathbf{0}$. This is where condition 3 enters the picture.
- Let us argue by way of contradiction. So suppose it is not the case that $\mathbf{p}^* \gg \mathbf{0}$. Then for some $\bar{k}$, we must have $p^*_{\bar{k}} = 0$.
- But condition 3 of the statement of the theorem then implies that there must be some good k' with $p^*_{k'} = 0$ such that $z_{k'}(\mathbf{p}^\epsilon)$ is unbounded above as ϵ tends to zero.

- But note that because $\mathbf{p}^\epsilon \longrightarrow \mathbf{p}^*$, $p^*_{k'} = 0$ implies that $p^\epsilon_{k'} \longrightarrow 0$.
- Consequently, the left-hand side of

$$p^\epsilon_k \left(n\epsilon + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^\epsilon)\right] \right) = \epsilon + \max\left[0, \bar{z}_k(\mathbf{p}^\epsilon)\right]$$

for $k = k'$ must tend to zero, because the term in parenthesis is bounded above by the definition of $\bar{z}_k$:

$$\bar{z}_k(\mathbf{p}) \equiv \min\left[z_k(\mathbf{p}), 1\right] \quad \text{for all} \quad \mathbf{p} \gg \mathbf{0}.$$

- However, the right-hand side apparently does not tend to zero, because the unboundedness above of $z_{k'}(\mathbf{p}^\epsilon)$ implies that $\bar{z}_{k'}(\mathbf{p}^\epsilon)$ assumes its maximum value of 1 infinitely often.

- Of course, this is a contradiction because the two sides are equal for all values of ϵ. We conclude, therefore, that $\mathbf{p}^* \gg \mathbf{0}$.

- Thus $\mathbf{p}^\epsilon \longrightarrow \mathbf{p}^* \gg \mathbf{0}$ as $\epsilon \longrightarrow 0$. Because $\bar{\mathbf{z}}(\cdot)$ inherits continuity on $\mathbb{R}^n_{++}$ from $\mathbf{z}(\cdot)$, we may take the limit as $\epsilon \longrightarrow 0$ in

$$p_k^\epsilon \left(n\epsilon + \sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^\epsilon)\right] \right) = \epsilon + \max\left[0, \bar{z}_k(\mathbf{p}^\epsilon)\right]$$

  to obtain

$$p_k^* \left(\sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^*)\right] \right) = \max\left[0, \bar{z}_k(\mathbf{p}^*)\right] \quad \text{for all} \quad k = 1, \ldots, n.$$

- $$p_k^* \left(\sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^*)\right] \right) = \max\left[0, \bar{z}_k(\mathbf{p}^*)\right] \quad \text{for all} \quad k = 1, \ldots, n.$$

  Multiplying both sides by $z_k(\mathbf{p}^*)$ and summing over k yields

  $$\sum_{k=1}^{n} p_k^* \, z_k(\mathbf{p}^*) \left(\sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^*)\right] \right) = \sum_{k=1}^{n} z_k(\mathbf{p}^*) \max\left[0, \bar{z}_k(\mathbf{p}^*)\right]$$

  or

  $$\mathbf{p}^* \cdot \mathbf{z}(\mathbf{p}^*) \left(\sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^*)\right] \right) = \sum_{k=1}^{n} z_k(\mathbf{p}^*) \max\left[0, \bar{z}_k(\mathbf{p}^*)\right].$$

- $$\mathbf{p}^* \cdot \mathbf{z}(\mathbf{p}^*) \left(\sum_{m=1}^{n} \max\left[0, \bar{z}_m(\mathbf{p}^*)\right] \right) = \sum_{k=1}^{n} z_k(\mathbf{p}^*) \max\left[0, \bar{z}_k(\mathbf{p}^*)\right].$$

  Now, condition 2 of the theorem (Walras' law, $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ for all $\mathbf{p} \gg \mathbf{0}$) implies that $\mathbf{p}^* \cdot \mathbf{z}(\mathbf{p}^*) = 0$, so the left-hand side is zero and therefore the right-hand side is zero.

- But because the sign of $\bar{z}_k(\mathbf{p}^*)$ is the same as that of $z_k(\mathbf{p}^*)$ from the definition

  $$\bar{z}_k(\mathbf{p}) \equiv \min\left[z_k(\mathbf{p}), 1\right] \quad \text{for all} \quad \mathbf{p} \gg \mathbf{0},$$

  the right-hand side can be zero only if $\bar{z}_k(\mathbf{p}^*) \leq 0$ for all k. This, together with $\mathbf{p}^* \gg \mathbf{0}$ and Walras' law implies that each $z_k(\mathbf{p}^*) = 0$, as desired.

$\square$

- Thus, as long as aggregate excess demand is continuous on $\mathbb{R}^n_{++}$, satisfies Walras' law, and is unbounded above as some, but not all, prices approach zero, a Walrasian equilibrium (with the price of every good strictly positive) is guaranteed to exist.
- We already know that when each consumer's utility function satisfies standard assumptions (continuous, strongly increasing and strictly quasiconcave), conditions 1 and 2 of the theorem will hold. It remains to show when condition 3 holds. We do so now.

- **THEOREM (Utility and Aggregate Excess Demand)**:
  If each consumer's utility function satisfies the standard assumptions (continuous, strongly increasing and strictly quasiconcave) and if the aggregate endowment of each good is strictly positive ($\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$), then aggregate excess demand satisfies conditions 1 through 3 of the previous theorem.

- **Proof**: Conditions 1 and 2 have already been established. Thus, it remains to verify condition 3.
- Consider a sequence of strictly positive price vectors $\{\mathbf{p}^m\}$ converging to $\bar{\mathbf{p}} \neq \mathbf{0}$, such that $\bar{p}_k = 0$ for some good k. Because $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, we must have $\bar{\mathbf{p}} \cdot \sum_{i=1}^{I} \mathbf{e}^i > 0$.
- Consequently, $\bar{\mathbf{p}} \cdot \sum_{i=1}^{I} \mathbf{e}^i = \sum_{i=1}^{I} \bar{\mathbf{p}} \cdot \mathbf{e}^i > 0$, so that there must be at least one consumer i for whom $\bar{\mathbf{p}} \cdot \mathbf{e}^i > 0$.

- Consider this consumer i's demand $\mathbf{x}^i(\mathbf{p}^m, \mathbf{p}^m \cdot \mathbf{e}^i)$ along the sequence of prices $\{\mathbf{p}^m\}$.
- We want to find a sequence of excess demands for a good that is unbounded above. So let us suppose, by way of contradiction, that this sequence of demand vectors is bounded. Then, using the following theorem:

  THEOREM (On Bounded Sequences):
  Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

  there must be a convergent subsequence.
- So we may assume without any loss (by reindexing the subsequence) that the original sequence of demands converges to $\mathbf{x}^*$. That is, $\mathbf{x}^i(\mathbf{p}^m, \mathbf{p}^m \cdot \mathbf{e}^i) \longrightarrow \mathbf{x}^*$.

- To simplify notation, let $\mathbf{x}^m \equiv \mathbf{x}^i(\mathbf{p}^m, \mathbf{p}^m \cdot \mathbf{e}^i)$ for every m.
- Now because $\mathbf{x}^m$ maximizes u^i subject to i's budget constraint given the prices $\mathbf{p}^m$, and because u^i is strongly increasing, the budget constraint must be satisfied with equality:

  $$\mathbf{p}^m \cdot \mathbf{x}^m = \mathbf{p}^m \cdot \mathbf{e}^i \quad \text{for every } m.$$

  Taking the limit as $m \longrightarrow \infty$ yields

  $$\bar{\mathbf{p}} \cdot \mathbf{x}^* = \bar{\mathbf{p}} \cdot \mathbf{e}^i > 0,$$

  where the strict inequality follows from our choice of consumer i.

- Now let $\hat{\mathbf{x}} \equiv \mathbf{x}^* + (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 occurs in the kth position (a good where $\bar{p}_k = 0$). Then because u^i is strongly increasing on $\mathbb{R}^n_+$

$$u^i(\hat{\mathbf{x}}) > u^i(\mathbf{x}^*).$$

In addition, because $\bar{p}_k = 0$,

$$\bar{\mathbf{p}} \cdot \hat{\mathbf{x}} = \bar{\mathbf{p}} \cdot \mathbf{x}^* = \bar{\mathbf{p}} \cdot \mathbf{e}^i > 0.$$

So because u^i is continuous, there is a $t \in (0, 1)$ such that

$$u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^*) \qquad \bar{\mathbf{p}} \cdot t\hat{\mathbf{x}} < \bar{\mathbf{p}} \cdot \mathbf{e}^i.$$

Now because $\mathbf{p}^m \longrightarrow \bar{\mathbf{p}}$, $\mathbf{x}^m \longrightarrow \mathbf{x}^*$ and u^i is continuous, this implies that for m large enough,

$$u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^m) \qquad \mathbf{p}^m \cdot t\hat{\mathbf{x}} < \mathbf{p}^m \cdot \mathbf{e}^i.$$

- But
$$u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^m) \qquad \mathbf{p}^m \cdot t\hat{\mathbf{x}} < \mathbf{p}^m \cdot \mathbf{e}^i$$
contradicts the fact that $\mathbf{x}^m$ solves the consumer's problem at prices $\mathbf{p}^m$. We conclude therefore that consumer i's sequence of demand vectors must be unbounded.
- Now because i's sequence of demand vectors $\{\mathbf{x}^m\}$ is unbounded yet nonnegative, there must be some good k' such that $\{x_{k'}^m\}$ is unbounded above.
- But because i's income converges to $\bar{\mathbf{p}} \cdot \mathbf{e}^i > 0$, the sequence of i's income $\{\mathbf{p}^m \cdot \mathbf{e}^i\}$ is bounded. Consequently, we must have $p_{k'}^m \longrightarrow 0$, because this is the only way that the demand for good k' can be unbounded above and affordable.

- Finally, note that because the aggregate supply of good k' is fixed and equal to the total endowment of it, and all consumers demand a nonnegative amount of good k', the fact that i's demand for good k' is unbounded above implies that the aggregate excess demand for good k' is unbounded above.
- Consequently, beginning with the assumption that $\mathbf{p}^m \longrightarrow \bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some k, we have shown that there exists some good k' with $\bar{p}_{k'} = 0$, such that the aggregate excess demand for good k' is unbounded above along the sequence of prices $\{\mathbf{p}^m\}$, as desired in condition 3.

$\square$

- We can now summarize what we have just shown with

  THEOREM (Existence of Walrasian Equilibrium):
  If each consumer's utility function satisfies the standard assumptions (continuous, strongly increasing and strictly quasiconcave) and $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

- We have worked hard to prove this theorem but now you understand the Nobel prize winning contributions of Kenneth Arrow (1972) and Gerard Debreu (1983), and have seen how fixed-point theorems are used in economics! This is the beginning of modern economics because it helped economists to see how computers can potentially be used to solve general equilibrium models.

Example: Solving for a Walrasian Equilibrium

- Let us take a simple two-person economy and solve for a Walrasian equilibrium. Let consumers 1 and 2 have identical CES utility functions,
$$u^i(x_1, x_2) = x_1^\rho + x_2^\rho, \qquad i = 1, 2,$$
where $0 < \rho < 1$.
- Let there be 1 unit of each good and suppose each consumer owns all of one good, so initial endowments are $\mathbf{e}^1 = (1, 0)$ and $\mathbf{e}^2 = (0, 1)$.
- Because the aggregate endowment of each good is strictly positive, $\mathbf{e}^1 + \mathbf{e}^2 = (1, 1)$, and the CES form of utility is continuous, strongly increasing, and strictly quasiconcave on $\mathbb{R}^2_+$ when $0 < \rho < 1$, the requirements of the previous theorem are satisfied, so we know a Walrasian equilibrium exists in this economy.

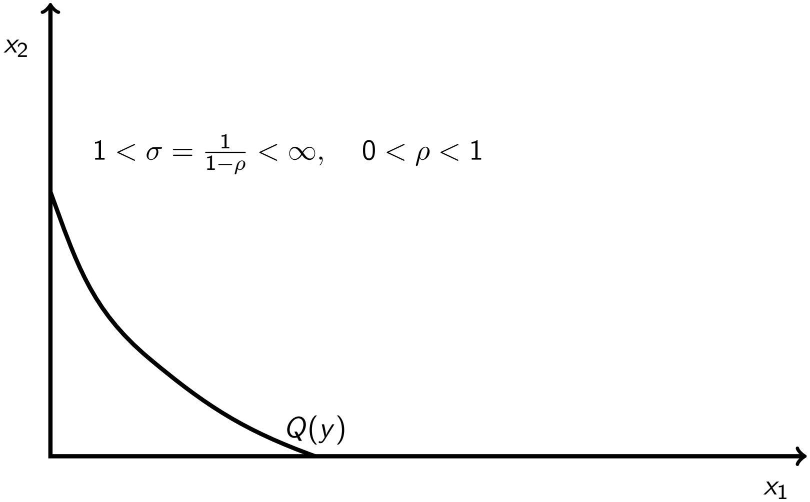

Figure: σ is finite but larger than one, indicating less that perfect substitutability. $\sigma = +\infty$ corresponds to downward-sloping linear indifference curves.

- From our earlier calculations, consumer i's demand for good j at prices $\mathbf{p}$ and income y^i will be

$$x_j^i(\mathbf{p}, y^i) = \frac{p_j^{r-1} y^i}{p_1^r + p_2^r} \quad \text{where} \quad r = \frac{\rho}{\rho - 1}.$$

  Here, income is equal to the market value of the endowment, so $y^1 = \mathbf{p} \cdot \mathbf{e}^1 = p_1$ and $y^2 = \mathbf{p} \cdot \mathbf{e}^2 = p_2$.
- Because only relative prices matter (excess demand functions are homogeneous of degree zero, $\mathbf{z}(\lambda \mathbf{p}) = \mathbf{z}(\mathbf{p})$ for all $\lambda > 0$) and all prices are strictly positive in equilibrium, we can choose a convenient normalization to simplify the calculations. Let $p_2 = 1$, so we only need to solve for $p_1 > 0$.

- Now consider the market for good 1. Assuming an interior equilibrium where total quantity demanded equals total quantity supplied,

$$x_1^1 + x_1^2 = \frac{p_1^{r-1} y^1}{p_1^r + p_2^r} + \frac{p_1^{r-1} y^2}{p_1^r + p_2^r} = e_1^1 + e_1^2$$

$$\frac{p_1^{r-1} p_1}{p_1^r + 1} + \frac{p_1^{r-1} 1}{p_1^r + 1} = 1$$

$$p_1^r + p_1^{r-1} = p_1^r + 1$$

$$p_1^{r-1} = 1$$

Solving, we obtain $p_1^* = 1$. We conclude that any vector $\mathbf{p}^*$ where $p_1^* = p_2^*$ equates demand and supply in market 1. By Walras' law, those same prices must equate demand and supply in market 2, so we are done. The equilibrium quantities are $x_1^1 + x_1^2 = 0.5 + 0.5 = 1$.

Efficiency

- We can adapt the Edgeworth box description of a two-person economy to gain useful perspective on the nature of Walrasian equilibrium.
- We do some calculations to determine the slope of the budget constraint:

$$p_1 x_1 + p_2 x_2 = I$$

$$p_2 x_2 = I - p_1 x_1$$

$$x_2 = \frac{I}{p_2} - \frac{p_1}{p_2} x_1$$

$$\text{slope} = -\frac{p_1}{p_2}$$

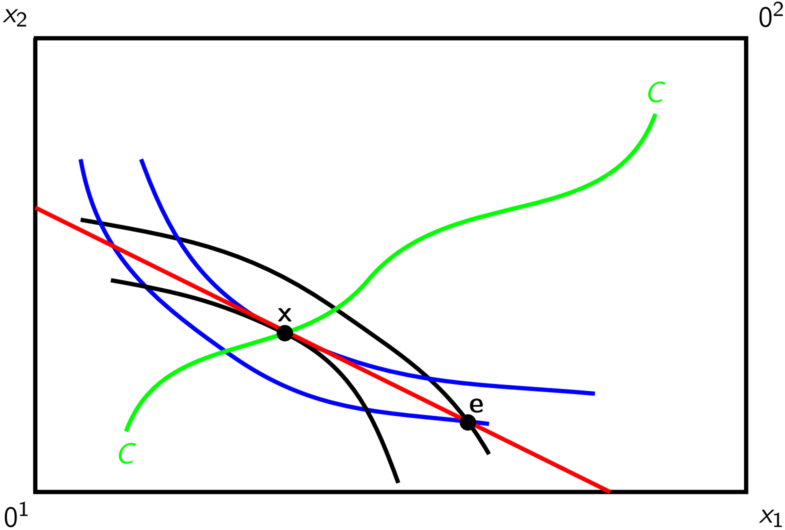

Figure: Walrasian equilibrium in the Edgeworth box.

- Walrasian equilibrium is characterized by a tangency of the consumers' indifference curves through their respective demanded bundles.
- Thus, having begun with some initial distribution of the goods given by $\mathbf{e}$, the maximizing actions of self-interested consumers on impersonal markets has led to a redistribution of goods (to $\mathbf{x}$) that is both 'inside the lens' formed by the indifference curves of each consumer through their respective endowments and 'on the contract curve.'
- The allocation $\mathbf{x}$ resulting from Walrasian equilibrium prices is in the core, at least for the Edgeworth box economy. Does this hold more generally?

- **DEFINITION (Walrasian Equilibrium Allocations)**:
  Let $\mathbf{p}^*$ be a Walrasian equilibrium for some economy with initial endowments $\mathbf{e}$, and let

$$\mathbf{x}(\mathbf{p}^*) \equiv (\mathbf{x}^1(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^1), \ldots, \mathbf{x}^I(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^I)),$$

  where component i gives the n-vector of goods demanded and received by consumer i at prices $\mathbf{p}^*$. Then $\mathbf{x}(\mathbf{p}^*)$ is called a Walrasian equilibrium allocation, or WEA.

- **LEMMA**: Let $\mathbf{p}^*$ be a Walrasian equilibrium for some economy with initial endowments $\mathbf{e}$. Let $\mathbf{x}(\mathbf{p}^*)$ be the associated WEA. Then $\mathbf{x}(\mathbf{p}^*) \in F(\mathbf{e})$.

- **Proof**: Lemma follows from staring at the definitions and is left as an exercise.

- **LEMMA**: Suppose that u^i is strictly increasing on $\mathbb{R}^n_+$, that consumer i's demand is well-defined at $\mathbf{p} \geq \mathbf{0}$ and equal to $\hat{\mathbf{x}}^i$, and that $\mathbf{x}^i \in \mathbb{R}^n_+$.

  i. If $u^i(\mathbf{x}^i) > u^i(\hat{\mathbf{x}}^i)$, then $\mathbf{p} \cdot \mathbf{x}^i > \mathbf{p} \cdot \hat{\mathbf{x}}^i$.

  ii. If $u^i(\mathbf{x}^i) \geq u^i(\hat{\mathbf{x}}^i)$, then $\mathbf{p} \cdot \mathbf{x}^i \geq \mathbf{p} \cdot \hat{\mathbf{x}}^i$.

- **Proof**: To show (i), suppose it is not true. Then $u^i(\mathbf{x}^i) > u^i(\hat{\mathbf{x}}^i)$ and $\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \hat{\mathbf{x}}^i$. Then $\mathbf{x}^i$ is affordable at prices $\mathbf{p}$ and generates higher utility, so the consumer is not maximizing utility by choosing $\hat{\mathbf{x}}^i$. Contradiction.

- Next suppose that (i) holds but (ii) does not. Then $u^i(\mathbf{x}^i) \geq u^i(\hat{\mathbf{x}}^i)$, and $\mathbf{p} \cdot \mathbf{x}^i < \mathbf{p} \cdot \hat{\mathbf{x}}^i$. Consequently, beginning at $\mathbf{x}^i$, we can increase the amount of every good consumed slightly so that the resulting bundle $\bar{\mathbf{x}}^i$ remains affordable: $u^i(\bar{\mathbf{x}}^i) > u^i(\hat{\mathbf{x}}^i)$ and $\mathbf{p} \cdot \bar{\mathbf{x}}^i < \mathbf{p} \cdot \hat{\mathbf{x}}^i$. But this contradictions (i) with $\mathbf{x}^i$ replaced by $\bar{\mathbf{x}}^i$.

$\square$

- **DEFINITION (The Set of WEAs)**:
  For any economy with endowments $\mathbf{e}$, let $W(\mathbf{e})$ denote the set of Walrasian equilibrium allocations.

- **THEOREM (Core and Equilibria in Competitive Economies)**:
  Consider an exchange economy $(u^i, \mathbf{e}^i)_{i\in\mathcal{I}}$. If each consumer's utility function u^i is strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation is in the core. That is,

$$W(\mathbf{e}) \subset C(\mathbf{e}).$$

- **Proof**: The theorem claims that if $\mathbf{x}(\mathbf{p}^*)$ is a WEA for equilibrium prices $\mathbf{p}^*$, then $\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$. To prove it, suppose $\mathbf{x}(\mathbf{p}^*)$ is a WEA and assume that $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$.
- Because $\mathbf{x}(\mathbf{p}^*)$ is a WEA, we know from the earlier Lemma that $\mathbf{x}(\mathbf{p}^*) \in F(\mathbf{e})$, so $\mathbf{x}(\mathbf{p}^*)$ is feasible.
- However, because $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$, we can find a coalition S and another allocation $\mathbf{y}$ such that

$$\sum_{i \in S} \mathbf{y}^i = \sum_{i \in S} \mathbf{e}^i$$

and

$$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)) \quad \text{for all } i \in S,$$

with at least one inequality strict.

- Multiplying both sides of the summation by $\mathbf{p}^*$ yields

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i$$

- From

$$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)) \quad \text{for all } i \in S$$

  and the last Lemma, we know that

$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) = \mathbf{p}^* \cdot \mathbf{e}^i,$$

  with at least one inequality strict.

- Given
$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) = \mathbf{p}^* \cdot \mathbf{e}^i,$$
summing over all consumers in S, we obtain
$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i,$$
which contradicts the earlier equation
$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i.$$
Since we assumed $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$ and obtained a contradiction, it must be that $\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$.

$\square$

- Because all core allocations are Pareto efficient, so too must be all Walrasian equilibrium allocations. Although we have proven more, this alone is quite remarkable.
- Imagine being charged with allocating all the economy's resources, so that in the end, the allocation is Pareto efficient. To keep you from giving all the resources to one person, let us also insist that in the end, every consumer must be at least as well off at they would have been just consuming their endowment.

- Think about how you might accomplish this. You might start by trying to gather information about the preferences of all consumers in the economy. What a task that would be! Only then could you attempt to redistribute goods in a manner that left no further gains from trade (assuming that people did not lie in reporting their preferences!).
- As incredibly difficult as this task is, the competitive market mechanism achieves it, and more. To emphasise the fact that competitive outcomes are Pareto efficient, we state it as a theorem.

- **THEOREM (First Welfare Theorem)**:
  Consider an exchange economy $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$. If each consumer's utility function u^i is strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation is Pareto efficient.

- **Proof**: From the previous theorem, $W(\mathbf{e}) \subset C(\mathbf{e})$. Next, note that all core allocations are Pareto efficient (if they were not Pareto efficient, they would be blocked by the grand coalition).

$\square$

- The First Welfare Theorem provides support for Adam Smith's contention that society's interests are served by an economic system where self-interested actions of individuals are mediated by impersonal markets.
- If conditions are sufficient to ensure that Walrasian equilibria exist, then regardless of the initial allocation of resources, the allocation realized in market equilibrium will be Pareto efficient.

- It can be shown that under certain conditions, *any* Pareto-efficient allocation can be achieved by competitive markets and *some* initial endowments.

- **THEOREM (Second Welfare Theorem)**:
Consider an exchange economy $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$ with aggregate endowment $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, and with each consumer's utility function u^i satisfying the standard assumptions (continuous, strongly increasing and strictly quasiconcave). Suppose that $\bar{\mathbf{x}}$ is a Pareto-efficient allocation for $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$ and that endowments are redistributed so that the new endowment vector is $\bar{\mathbf{x}}$. Then $\bar{\mathbf{x}}$ is a Walrasian equilibrium allocation of the resulting exchange economy $(u^i, \bar{\mathbf{x}}^i)_{i \in \mathcal{I}}$.

- **Proof**: Because $\bar{\mathbf{x}}$ is Pareto-efficient, it is feasible. Hence,

$$\sum_{i=1}^{I} \bar{\mathbf{x}}^i = \sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}.$$

Consequently, we may apply the earlier theorem

THEOREM (Existence of Walrasian Equilibrium):
If each consumer's utility function satisfies the standard assumptions (continuous, strongly increasing and strictly quasiconcave) and $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

to conclude that the exchange economy $(u^i, \bar{\mathbf{x}}^i)_{i \in \mathcal{I}}$ possesses a Walrasian equilibrium allocation $\hat{\mathbf{x}}$. It only remains to show that $\hat{\mathbf{x}} = \bar{\mathbf{x}}$.

- Now in the Walrasian equilibrium, each consumer's demand is utility maximizing subject to her budget constraint. Consequently, because i demands $\hat{\mathsf{x}}^i$ and has endowment $\bar{\mathsf{x}}^i$, we must have

$$u^i(\hat{\mathsf{x}}^i) \geq u^i(\bar{\mathsf{x}}^i) \quad \text{for all } i \in \mathcal{I}.$$

But because $\hat{\mathsf{x}}$ is an equilibrium allocation, it must be feasible for the economy $(u^i, \bar{\mathsf{x}}^i)_{i\in\mathcal{I}}$. Consequently,

$$\sum_{i=1}^{I} \hat{\mathsf{x}}^i = \sum_{i=1}^{I} \bar{\mathsf{x}}^i = \sum_{i=1}^{I} \mathsf{e}^i \gg \mathbf{0},$$

so that $\hat{\mathsf{x}}$ is feasible for the original economy $(u^i, \mathsf{e}^i)_{i\in\mathcal{I}}$ as well.

- Thus, by
$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\bar{\mathbf{x}}^i) \quad \text{for all } i \in \mathcal{I},$$
$\hat{\mathbf{x}}$ is feasible for the original economy $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$ and makes no consumer worse off than the Pareto-efficient (for the original economy) allocation $\bar{\mathbf{x}}$.
- Therefore $\hat{\mathbf{x}}$ cannot make anyone strictly better off; otherwise $\bar{\mathbf{x}}$ would not be Pareto efficient. It follows that
$$u^i(\hat{\mathbf{x}}^i) = u^i(\bar{\mathbf{x}}^i) \quad \text{for all } i \in \mathcal{I}.$$

- To see now that $\hat{\mathbf{x}}^i = \bar{\mathbf{x}}^i$ for every i, note that if for some consumer this were not the case, so $\hat{\mathbf{x}}^i \neq \bar{\mathbf{x}}^i$ for some i, then in the Walrasian equilibrium of the new economy, that consumer could afford the average of the bundles $\hat{\mathbf{x}}^i$ and $\bar{\mathbf{x}}^i$ and strictly increase his utility (by strict quasiconcavity).

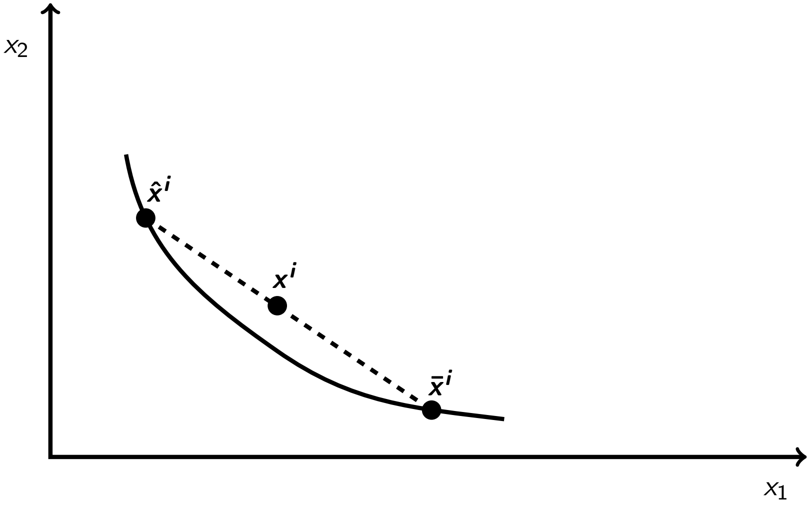

Figure: Preferences that are represented by a strictly quasiconcave utility function.

- Since that consumer could afford the average of the bundles $\hat{\mathsf{x}}^i$ and $\bar{\mathsf{x}}^i$ and strictly increase his utility (by strict quasiconcavity), this contradictions the fact that $\hat{\mathsf{x}}^i$ is utility-maximizing in the Walrasian equilibrium.
- We get a contradiction assuming that $\hat{\mathsf{x}}^i \neq \bar{\mathsf{x}}^i$ for some i. Therefore, it must be that $\hat{\mathsf{x}}^i = \bar{\mathsf{x}}^i$ for all i. It follows that $\bar{\mathsf{x}}$ is a Walrasian equilibrium allocation of the exchange economy $(u^i, \bar{\mathsf{x}}^i)_{i \in \mathcal{I}}$.

$\square$

- Is a system that depends on decentralised, self-interested decision-making by a large number of consumers capable of sustaining the socially 'best' allocation of resources $\bar{\mathbf{x}}$ (assuming that we could agree on what that allocation is)?
- The Second Welfare Theorem says yes, as long as socially 'best' requires Pareto efficiency (which seems like a reasonable requirement).
- The theorem says that there are Walrasian equilibrium prices $\bar{\mathbf{p}}$ such that, when the endowment allocation is $\bar{\mathbf{x}}$, each consumer i will maximize $u^i(\mathbf{x}^i)$ subject to $\bar{\mathbf{p}} \cdot \mathbf{x}^i \leq \bar{\mathbf{p}} \cdot \bar{\mathbf{x}}^i$ by choosing $\mathbf{x}^i = \bar{\mathbf{x}}^i$.

- So, starting from any initial endowments $\mathbf{e} \gg \mathbf{0}$, redistribution to the socially 'best' allocation $\bar{\mathbf{x}}$ yields $\bar{\mathbf{x}}$ as a Walrasian equilibrium allocation.
- It should be clear that $\bar{\mathbf{x}}$ will be a WEA for market prices $\bar{\mathbf{p}}$ under a redistribution of initial endowments to *any* point along the budget line through $\bar{\mathbf{x}}$:
- **THEOREM (Another look at the Second Welfare Theorem)**: Under the assumptions of the Second Welfare Theorem, if $\bar{\mathbf{x}}$ is Pareto-efficient, then $\bar{\mathbf{x}}$ is a WEA for some Walrasian equilibrium prices $\bar{\mathbf{p}}$ after redistribution of initial endowments to any allocation $\mathbf{e}^* \in F(\mathbf{e})$, such that $\bar{\mathbf{p}} \cdot \mathbf{e}^{*i} = \bar{\mathbf{p}} \cdot \bar{\mathbf{x}}^i$ for all $i \in \mathcal{I}$.

Equilibrium in Production

- Now we expand our description of the economy to include production as well as consumption (and we allow consumers/workers to earn wage income by providing labor services to firms).
- We will find that most of the important properties of competitive market systems uncovered earlier continue to hold. However, production brings with it several new issues that must be addressed.

- For example, the profits earned by firms must be distributed back to the consumers who own them.
- Also, in a single firm, the distinction between what constitutes an input and what constitutes an output is usually quite clear. This distinction becomes blurred when we look across firms and view the production side of the economy as a whole. An input for one firm may well be the output of another.

Producers

- To describe the production sector, we suppose there is a fixed number J of firms that we index by the set

$$\mathcal{J} = \{1, \dots, J\}.$$

  We now let $\mathbf{y}^j \in \mathbb{R}^n$ be a production plan for some firm, and observe the convention of writing $y_k^j < 0$ if commodity k is an input used in the production plan and $y_k^j > 0$ if it is an output produced from the production plan.
- If, for example, there are two commodities and $\mathbf{y}^j = (-7, 3)$, then the production plan requires 7 units of commodity one as an input, to produce 3 units of commodity two as an output.

Example

- Intel described by (labor, chip, computer)= (-2, 4, 0)
- Apple described by (labor, chip, computer)= (-7, -3, 2)
- Intel+Apple described by (-2, 4, 0)+(-7, -3, 2)= (-9,1,2)

- We suppose each firm possesses a production possibility set Y^j, $j \in \mathcal{J}$.
- **ASSUMPTION (The Individual Firm)**:

  1. $\mathbf{0} \in Y^j \subset \mathbb{R}^n$.

  2. Y^j is closed and bounded.

  3. Y^j is strongly convex. That is, for all distinct $\mathbf{y}^1, \mathbf{y}^2 \in Y^j$ and all $t \in (0,1)$, there exists $\bar{\mathbf{y}} \in Y^j$ such that $\bar{\mathbf{y}} \geq t\mathbf{y}^1 + (1-t)\mathbf{y}^2$ and equality does not hold.

- The closedness part of the second condition imposes continuity. It says that the limits of possible production plans are themselves possible production plans.
- The boundedness part of this condition is very restrictive and is made only to keep the analysis simple to follow. Regard it as a simplifying yet dispensable assumption.
- The third assumption, strong convexity, is new. Strong convexity rules out constant and increasing returns to scale in production and ensures that the firm's profit-maximizing production plan is unique.

- Each firm faces fixed commodity prices $\mathbf{p} \geq \mathbf{0}$ and chooses a production plan to maximize profit. Thus, each firm solves the problem

$$\max_{\mathbf{y}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j.$$

  Note how our sign convention ensures that inputs are accounted for in profits as costs and outputs as revenues.

- Because the objective function is continuous and the constraint set is closed and bounded, a maximum of firm profit will exist. So, for all $\mathbf{p} \geq \mathbf{0}$, let

$$\pi^j(\mathbf{p}) \equiv \max_{\mathbf{y}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j$$

  denote firm j's profit function.

- $$\pi^j(\mathbf{p}) \equiv \max_{\mathbf{y}^j \in Y^j} \mathbf{p} \cdot \mathbf{y}^j$$
- By the Theorem of the Maximum, $\pi^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_+$.
- Strong convexity ensures that the profit-maximizing production plan $\mathbf{y}^j(\mathbf{p})$ will be unique whenever $\mathbf{p} \gg \mathbf{0}$.

  Y^j is strongly convex. That is, for all distinct $\mathbf{y}^1, \mathbf{y}^2 \in Y^j$ and all $t \in (0, 1)$, there exists $\bar{\mathbf{y}} \in Y^j$ such that $\bar{\mathbf{y}} \geq t\mathbf{y}^1 + (1-t)\mathbf{y}^2$ and equality does not hold.
- By the Theorem of the Maximum, $\mathbf{y}^j(\mathbf{p})$ will be continuous on $\mathbb{R}^n_{++}$.

- **THEOREM (Basic Properties of Supply and Profits)**:
  If Y^j satisfies the previously stated assumptions (closed, bounded, strongly convex), then for every price $\mathbf{p} \gg \mathbf{0}$, the solution to the firm's problem is unique and denoted by $\mathbf{y}^j(\mathbf{p})$. Moreover, $\mathbf{y}^j(\mathbf{p})$ is continuous on $\mathbb{R}^n_{++}$. In addition, $\pi^j(\mathbf{p})$ is well-defined and continuous on $\mathbb{R}^n_+$.

- Next we consider aggregate production possibilities economy-wide. We suppose there are no externalities in production between firms, and define the aggregate production possibilities set

$$Y \equiv \left\{ \mathbf{y} \,|\, \mathbf{y} = \sum_{j \in \mathcal{J}} \mathbf{y}^j, \quad \text{where } \mathbf{y}^j \in Y^j \right\}.$$

The set Y will inherit all the properties of the individual production sets.

THEOREM (Properties of Y):
If each Y^j satisfies the previously stated assumptions (closed, bounded, strongly convex), then the aggregate production possibility set Y also satisfies the previously stated assumptions.

- Now consider the problem of maximizing aggregate profits. Under the previous theorem, a maximum of $\mathbf{p} \cdot \mathbf{y}$ over the aggregate production set will exist and be unique when $\mathbf{p} \gg \mathbf{0}$.
- In addition, the aggregate profit-maximizing production plan $\mathbf{y}(\mathbf{p})$ will be a continuous function of $\mathbf{p}$. Furthermore, we can show that

  THEOREM (Aggregate Profit Maximization): For any prices $\mathbf{p} \geq \mathbf{0}$, we have

  $$\mathbf{p} \cdot \bar{\mathbf{y}} \geq \mathbf{p} \cdot \mathbf{y} \quad \text{for all } \mathbf{y} \in Y$$

  if and only if, for some $\bar{\mathbf{y}}^j \in Y^j$, $j \in \mathcal{J}$, we may write $\bar{\mathbf{y}} = \sum_{j \in \mathcal{J}} \bar{\mathbf{y}}^j$ and

  $$\mathbf{p} \cdot \bar{\mathbf{y}}^j \geq \mathbf{p} \cdot \mathbf{y}^j \quad \text{for all } \ \mathbf{y}^j \in Y^j, \ j \in \mathcal{J}.$$

- **Proof**: Let $\bar{\mathbf{y}} \in Y$ maximize aggregate profits at price $\mathbf{p}$. Suppose that $\bar{\mathbf{y}} = \sum_{j \in \mathcal{J}} \bar{\mathbf{y}}^j$ for $\bar{\mathbf{y}}^j \in Y^j$.
- If $\bar{\mathbf{y}}^k$ does not maximize profits for firm k, then there exists some other $\tilde{\mathbf{y}}^k \in Y^k$ that gives firm k higher profits.
- But then the aggregate production vector $\tilde{\mathbf{y}} \in Y$ composed of $\tilde{\mathbf{y}}^k$ and the sum of the $\bar{\mathbf{y}}^j$ for $j \neq k$ must give higher aggregate profits than the aggregate vector $\bar{\mathbf{y}}$, contradicting the assumption that $\bar{\mathbf{y}}$ maximizes aggregate profits at price $\mathbf{p}$.

- Next, suppose feasible production plans $\bar{\mathbf{y}}^1, \ldots, \bar{\mathbf{y}}^J$ maximizes profits at price $\mathbf{p}$ for the individual firms in $\mathcal{J}$ and $\bar{\mathbf{y}} \equiv \sum_{j \in \mathcal{J}} \bar{\mathbf{y}}^j$. Then

$$\mathbf{p} \cdot \bar{\mathbf{y}}^j \geq \mathbf{p} \cdot \mathbf{y}^j \quad \text{for all } \ \mathbf{y}^j \in Y^j,\ j \in \mathcal{J}.$$

Summing over all firms yields

$$\sum_{j \in \mathcal{J}} \mathbf{p} \cdot \bar{\mathbf{y}}^j \geq \sum_{j \in \mathcal{J}} \mathbf{p} \cdot \mathbf{y}^j \quad \text{for all } \ \mathbf{y}^j \in Y^j,\ j \in \mathcal{J}$$

$$\mathbf{p} \cdot \sum_{j \in \mathcal{J}} \bar{\mathbf{y}}^j \geq \mathbf{p} \cdot \sum_{j \in \mathcal{J}} \mathbf{y}^j \quad \text{for all } \ \mathbf{y}^j \in Y^j,\ j \in \mathcal{J}$$

$$\mathbf{p} \cdot \bar{\mathbf{y}} \geq \mathbf{p} \cdot \mathbf{y} \quad \text{for all } \ \mathbf{y} \in Y.$$

So $\bar{\mathbf{y}}$ maximizes aggregate profits at price $\mathbf{p}$.

$\square$

Consumers

- The description of consumers is just as it has always been. However, we need to modify some details to account for the distribution of firm profits because firms are owned by consumers.
- We let
$$\mathcal{I} \equiv \{1, \ldots, I\}$$
index the set of consumers and let u^i denote i's utility function over the consumption set $\mathbb{R}^n_+$.
- Labor services are easily included by endowing the consumer with a fixed number of hours that are available for consumption. Those that are not consumed as "leisure" are then supplied as labour services to firms.

- In a *private ownership economy*, consumers own shares in firms and firm profits are distributed to shareholders.
- Consumer i's shares in firm j entitle him to some proportion $\theta^{ij} \in [0,1]$ of the profits of firm j. Thus

$$0 \leq \theta^{ij} \leq 1 \quad \text{for all } i \in \mathcal{I} \text{ and } j \in \mathcal{J},$$

where

$$\sum_{i \in \mathcal{I}} \theta^{ij} = 1 \quad \text{for all } j \in \mathcal{J},$$

- If $\mathbf{p} \geq \mathbf{0}$ is the vector of market prices, the consumer's budget constraint is

$$\mathbf{p} \cdot \mathbf{x}^i \leq \mathbf{p} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij} \pi^j(\mathbf{p}).$$

- The consumer's problem is

$$\max_{\mathbf{x}^i \in \mathbb{R}^n_+} u^i(\mathbf{x}^i) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x}^i \leq m^i(\mathbf{p}) \equiv \mathbf{p} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij} \pi^j(\mathbf{p}).$$

- Now, under the previous assumptions about production, each firm will earn non-negative profits because each can always choose the zero production vector ($\mathbf{0} \in Y^j \subset \mathbb{R}^n$). Consequently, $m^i(\mathbf{p}) \geq 0$ because $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{e}^i \geq \mathbf{0}$.
- Therefore, given the assumptions about consumers, a solution to the consumer's problem will exist and be unique whenever $\mathbf{p} \gg \mathbf{0}$. We denote it by $\mathbf{x}^i(\mathbf{p}, m^i(\mathbf{p}))$ where $m^i(\mathbf{p})$ is the consumer's income.

- **THEOREM (Basic Property of Demand with Profit Shares)**: If each Y^j and each u^i satisfies the previously stated assumptions, then a solution to the consumer's problem exists and is unique for all $\mathbf{p} \gg \mathbf{0}$. Denoting it by $\mathbf{x}^i(\mathbf{p}, m^i(\mathbf{p}))$, we have furthermore that $m^i(\mathbf{p})$ in continuous on $\mathbb{R}^n_+$ and $\mathbf{x}^i(\mathbf{p}, m^i(\mathbf{p}))$ is continuous in $\mathbf{p}$ on $\mathbb{R}^n_{++}$.

- This completes the description of the economy. Altogether, we can represent it as the collection $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i \in \mathcal{I}, j \in \mathcal{J}}$

Welfare with Production

- **DEFINITION (WEAs in a Production Economy)**:
  Let $\mathbf{p}^* \gg \mathbf{0}$ be a Walrasian equilibrium for the economy $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i \in \mathcal{I}, j \in \mathcal{J}}$. Then the pair $(\mathbf{x}(\mathbf{p}^*), \mathbf{y}(\mathbf{p}^*))$ is a Walrasian equilibrium allocation (WEA) where $\mathbf{x}(\mathbf{p}^*)$ denotes the vector $(\mathbf{x}^1, \dots, \mathbf{x}^I)$, whose ith entry is the utility-maximizing bundle demanded by consumer i facing prices $\mathbf{p}^*$ and income $m^i(\mathbf{p}^*)$; and where $\mathbf{y}(\mathbf{p}^*)$ denotes the vector $(\mathbf{y}^1, \dots, \mathbf{y}^J)$ of profit-maximizing production vectors at prices $\mathbf{p}^*$.

  Note that because $\mathbf{p}^*$ is a Walrasian equilibrium,

$$\sum_{i \in \mathcal{I}} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i + \sum_{j \in \mathcal{J}} \mathbf{y}^j.$$

- An allocation $(\mathbf{x}, \mathbf{y}) = ((\mathbf{x}^1, \ldots, \mathbf{x}^I), (\mathbf{y}^1, \ldots, \mathbf{y}^J))$ of bundles to consumers and production plans to firms is **feasible** if $\mathbf{x}^i \in \mathbb{R}^n_+$ for all i, $\mathbf{y}^j \in Y^j$ for all j, and

$$\sum_{i \in \mathcal{I}} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i + \sum_{j \in \mathcal{J}} \mathbf{y}^j.$$

- **DEFINITION (Pareto-Efficient Allocations with Production)**: The feasible allocation $(\mathbf{x}, \mathbf{y})$ is Pareto efficient if there is no other feasible allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ such that $u^i(\bar{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i)$ for all $i \in \mathcal{I}$ with at least one strict inequality.

- **THEOREM (First Welfare Theorem with Production)**:
  If each consumer's utility function u^i is strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation is Pareto efficient.

- **Proof**: We suppose $(\mathbf{x}, \mathbf{y})$ is a WEA at prices $\mathbf{p}^*$, but is not Pareto efficient, and derive a contradiction.
- Because $(\mathbf{x}, \mathbf{y})$ is a WEA, it is feasible, so

$$\sum_{i \in \mathcal{I}} \mathbf{x}^i = \sum_{i \in \mathcal{I}} \mathbf{e}^i + \sum_{j \in \mathcal{J}} \mathbf{y}^j.$$

- Because $(\mathbf{x}, \mathbf{y})$ is not Pareto efficient, there exists some feasible allocation $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ such that

$$u^i(\hat{\mathbf{x}}^i) \geq u^i(\mathbf{x}^i), \quad i \in \mathcal{I},$$

with at least one strict inequality. Given the last Lemma, this implies that

$$\mathbf{p}^* \cdot \hat{\mathbf{x}}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i, \quad i \in \mathcal{I},$$

with at least one strict inequality. Summing over all consumers and rearranging gives

$$\mathbf{p}^* \cdot \sum_{i \in \mathcal{I}} \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \sum_{i \in \mathcal{I}} \mathbf{x}^i.$$

- Now
$$\mathbf{p}^* \cdot \sum_{i\in\mathcal{I}} \hat{\mathbf{x}}^i > \mathbf{p}^* \cdot \sum_{i\in\mathcal{I}} \mathbf{x}^i$$
together with
$$\sum_{i\in\mathcal{I}} \mathbf{x}^i = \sum_{i\in\mathcal{I}} \mathbf{e}^i + \sum_{j\in\mathcal{J}} \mathbf{y}^j$$
and the feasibility of $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ imply that
$$\mathbf{p}^* \cdot \left(\sum_{i\in\mathcal{I}} \mathbf{e}^i + \sum_{j\in\mathcal{J}} \hat{\mathbf{y}}^j \right) > \mathbf{p}^* \cdot \left(\sum_{i\in\mathcal{I}} \mathbf{e}^i + \sum_{j\in\mathcal{J}} \mathbf{y}^j \right)$$
so
$$\mathbf{p}^* \cdot \sum_{j\in\mathcal{J}} \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \sum_{j\in\mathcal{J}} \mathbf{y}^j.$$

- $$\mathbf{p}^* \cdot \sum_{j \in \mathcal{J}} \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \sum_{j \in \mathcal{J}} \mathbf{y}^j$$

  However, this means that $\mathbf{p}^* \cdot \hat{\mathbf{y}}^j > \mathbf{p}^* \cdot \mathbf{y}^j$ for some firm j, where $\hat{\mathbf{y}}^j \in Y^j$.
- This contradictions the fact that in the Walrasian equilibrium, $\mathbf{y}^j$ maximizes firm j's profit at prices $\mathbf{p}^*$.

$\square$

- It would be quite a task indeed to attempt to allocate resources in a manner that was Pareto efficient.
- Not only would you need information on consumer preferences, you would also require detailed knowledge of the technologies of all firms and the productivity of all inputs.
- In particular, you would have to assign individuals with particular skills to the firms that require those skills.
- It would be a massive undertaking. And yet, with apparently no central direction, the allocations obtained as Walrasian equilibria are Pareto efficient.

Evidence about Economic Freedom

In their book *The Poverty of Nations: A Sustainable Solution*, Wayne Grudem and Barry Asmus (2013) write:

- A free-market system is one in which economic production and consumption are determined by the free choices of individuals rather than governments, and this process is grounded in private ownership of the means of production.
- In very simple, practical terms, a free-market system means that people, not the government, owns the firms, businesses and properties in a nation.
- In addition, the people "own" themselves in the sense that they have the freedom to choose where to work. The government does not decide where they work (as in slavery or in communism).

- The genius of a free-market system is that it does not try to compel people to work. It rather leaves people free to choose to work, and it rewards that work by letting people keep the fruits of their labor. In a free market, no government officials have to force people to work. The government simply has to get out of the way and let the free market work all by itself (with some appropriate restraints on crime).
- The rule of law is important to a free-market system. It prevents thieves and other criminals from taking away people's economic freedom by taking their property through fraud, deceit, or force.
- Some laws are necessary to protect the idea of a free market, because the idea of free, voluntary exchanges is violated when people steal from, cheat, or deceive others, or when people do not have the information they need to make informed decisions. Therefore, a proper understanding of a free market includes laws against theft, fraud, the violation of contracts, and the sale of defective and dangerous products.

- From time to time when we mention free markets in our seminars, someone in the audience objects: "There is no such thing as a free-market system today, because all economies have a mixture of private ownership and government ownership and control." The discussion then becomes confused because the relabeling implies that different economic systems are mostly the same.
- We think it does make sense [to talk about a "free-market system"] because there are real differences between economic systems from country to country. The national economies of the world can be numerically arranged along a scale from "free" to "unfree." One such ranking has been published annually for the last eighteen years by the Heritage Foundation and *The Wall Street Journal*.

- In order to determine how free an economy is, each year the researchers score 179 countries according to ten factors grouped into four categories:
  A. Rule of Law (1. Property Rights 2. Freedom from corruption)
  B. Limited Government (3. Fiscal freedom 4. Government Spending)
  C. Regulatory efficiency (5. Business freedom 6. Labor freedom 7. Monetary freedom)
  D. Open markets (8. Trade freedom 9. Investment freedom 10. Financial freedom)
- Each country is ranked on an economic freedom scale from 0 to 100. In the *2012 Index of Economic Freedom*, Hong Kong scored the highest (89.9), then Singapore (87.5), Australia (83.1), New Zealand (82.1) and Switzerland (81.1). These five are considered "free" by this publication.

- The researchers then listed twenty-three countries that are “mostly free,” including Canada (79.9), Chile (78.3), Mauritius (77.0), Ireland (76.9), the United States (76.3), and eighteen others. For purposes of our study, these top twenty-eight countries may be considered to have free-market systems or mostly free-market systems.
- Sadly, eighty-eight countries still fall at the lower end of the scale: sixty countries rank as “mostly unfree,” and below them another twenty-eight rank as “repressed.” These countries score poorly in all ten categories of economic freedom. Such countries do not have free-market systems. The lowest ten are Equatorial Guinea (42.8), Iran (42.3), the Democratic Republic of the Congo (41.1), Myanmar (38.7), Venezuela (38.1), Eritrea (36.2), Libya (35.9), Cuba (28.3), Zimbabwe (26.3), and North Korea (1).

- In today's world, there is a strong correlation between economic freedom and prosperity. Those countries that have the highest levels of economic freedom (as measured by the *2012 Index of Economic Freedom*) also have the highest per capita incomes (GDP Per Capita, purchasing power parity).
- For Europe, the five most free countries have per capita income $46,593 and the five least free countries have per capita income $13,595.
- For Asia-Pacific, the five most free countries have per capita income $40,830 and the five least free countries have per capita income $3,400.

- For the Middle East and North Africa, the five most free countries have per capita income \$39,063 and the five least free countries have per capita income \$7,885.
- For the Americas, the five most free countries have per capita income \$25,198 and the five least free countries have per capita income \$8,243.
- For Sub-Suharan Africa, the five most free countries have per capita income \$8,989 and the five least free countries have per capita income \$1,514.

More Evidence about Economic Freedom

- Each year the Frazer Institute in Canada publishes a ranking of countries called *Economic Freedom of the World Index*. Milton Friedman (1976 economics Nobel Prize) was heavily involved in the design of this index.
- For 2016, 162 countries were ranked based on the degree of economic freedom in five broad areas:
  1. Size of Government,
  2. Legal System and Property Rights,
  3. Sound Money,
  4. Freedom to Trade Internationally,
  5. Regulation.

- 1. Size of Government: As government spending, taxation, and the size of government-controlled enterprises increases, government decision-making is substituted for individual choice and economic freedom is reduced.
- 2. Legal System and Property Rights: Protection of persons and their rightfully acquired property is a central element of both economic freedom and civil society. Indeed, it is the most important function of government.

- 3. Sound Money: Inflation erodes the value of rightfully earned wages and earnings. Sound money is thus essential to protecting property rights. When inflation is not only high but also volatile, it becomes difficult for individuals to plan for the future.
- 4. Freedom to Trade Internationally: Freedom in exchange – in its broadest sense, buying, selling. making contracts, and so on – is essential to economic freedom, which is reduced when freedom to exchange does not include businesses and individuals in other nations.
- 5. Regulation: Governments not only use a number of tools to limit the right to exchange internationally, they may also develop onerous regulations that limit the right to exchange, gain credit, hire or work for whom you wish, or freely operate your business.

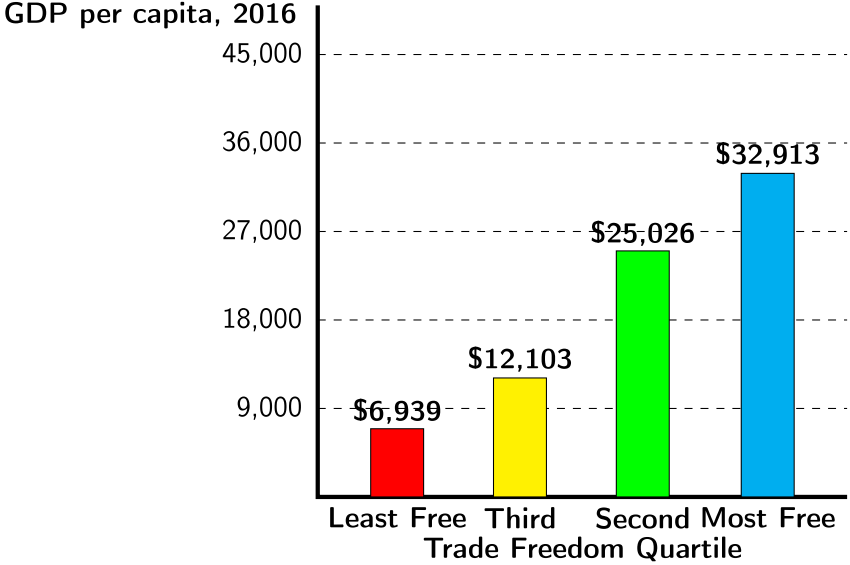

Figure: Trade Freedom and Income per Capita.

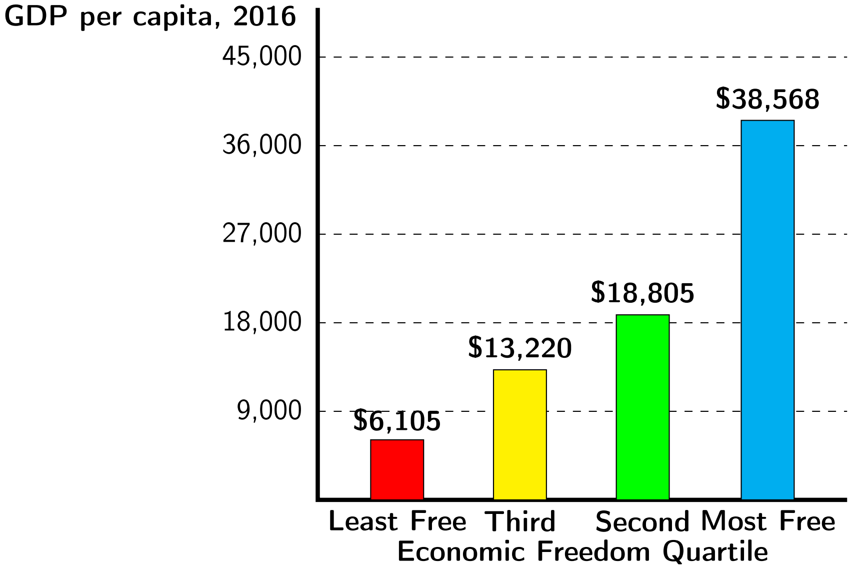

Figure: Economic Freedom and Income per Capita.

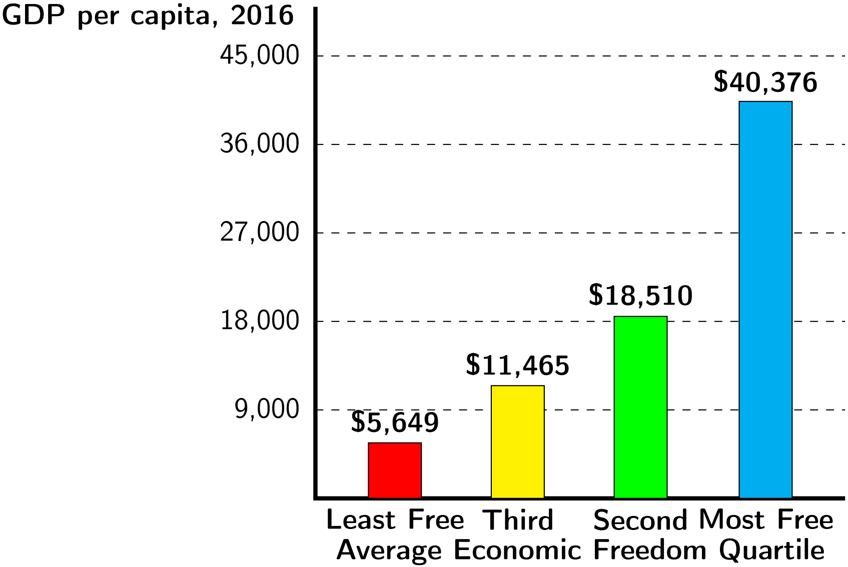

Figure: Average Economic Freedom During 1995-2016 and Income per Capita.

- Because persistence is important and the impact of economic freedom will be felt over a lengthy time period, it is better to use the average rating over a fairly long time span rather than the current rating to observe the impact of economic freedom on performance.

- The share of income earned by the poorest 10% of the population is unrelated to economic freedom:

Average Economic Freedom Score, 1995-2016, Quartile	Income share (Bottom 10%) 2016
Most Free	2.74%
Second	2.38%
Third	2.52%
Least Free	2.47%

- The amount of income, as opposed to the share, earned by the poorest 10% of the population is much higher in countries with higher economic freedom:

Average Economic Freedom Score, 1995-2016, Quartile	Annual income per capita of poorest 10% 2016
Most Free	$10,660
Second	$3,721
Third	$2,774
Least Free	$1,345

- Life expectancy is about 15 years longer in countries with the most economic freedom than in countries with the least:

Average Economic Freedom Score, 1995-2016, Quartile	Life expectancy at birth, total years, 2016
Most Free	79.45
Second	73.47
Third	70.76
Least Free	64.40

- Greater economic freedom is associated with more political rights. Countries with democratic institutions are likely to score much higher on economic freedom than countries ruled by dictators:

Average Economic Freedom Score, 1995-2016, Quartile	Political Rights, 1=best, 7=worst 2016
Most Free	1.98
Second	3.38
Third	3.49
Least Free	4.74

History of Economic Ideas

- Francis Edgeworth developed utility theory, including the indifference curve and the famous Edgeworth box in his 1881 book *Mathematical Psychics*. He is also known for the Edgeworth conjecture, which states that the core of an economy shrinks to the set of competitive equilibria as the number of agents in the economy gets larger.
- Edgeworth was not a supporter of the free market economic system. He set the utilitarian foundations for highly progressive taxation, arguing that the optimal distribution of taxes should be such that "the marginal disutility incurred by each taxpayer should be the same."

- Leon Walras (1874, *Elements of Pure Economics*) was the first to attempt to show that a competitive equilibrium exists, by reducing it to a question of whether a system of market demand and market supply equations possessed a solution. However, his reasoning was wrong that any system of equations with as many unknowns as equations always possesses a solution.
- Lionel McKenzie (1954, *Econometrica*), and Kenneth Arrow and Gerard Debreu (1954, *Econometrica*) were the first to offer reasonably general proofs of existence of competitive (or Walrasian) equilibria.

- The two welfare theorems have a long history. Vilfredo Pareto (1906) stated the idea of the first welfare theorem (without production) in words but could not offer a rigorous proof. Pareto also asks a question about distribution and his answer is an informal precursor of the second welfare theorem:

  "Having distributed goods according to the answer to the first problem, the state should allow the members of the collectivity to operate a second distribution, or operate it itself, in either case making sure that it is performed in conformity with the workings of free competition."

- Oscar Lange (1942, *Econometrica*) is the source of the now-traditional pairing of the two theorems, one governing markets, the other distribution. His reasoning is a mathematical translation (into Lagrange multipliers) of earlier graphical arguments. The second theorem does not take its familiar form in his hands; rather he simply shows that the optimization conditions for a genuine social utility function are similar to those for Pareto optimality.
- Paul Samuelson (1947, *Foundations of Economic Analysis*) brought Lange's second welfare theorem to approximately its modern form. He follows Lange in deriving a set of equations which are necessary for Pareto optimality, and then considers what additional constraints arise if the economy is required to satisfy a genuine social welfare function, finding a further set of equations from which if follows 'that all of the action necessary to achieve a given ethical desideratum may take the form of lump sum taxes or bounties'.

- Kenneth Arrow (1951) and Gerard Debreu (1951) sought to improve on the rigour of Lange's first theorem. Their proofs use dot products for both consumer and producer problems, and required a change of mathematical style from the calculus to convex set theory. Arrow motivated his paper by reference to the need to extend proofs to cover equilibria at the edge of the space, and Debreu by the possibility of indifference curves being non-differentiable. Modern texts follow their style of proof.
- Paul Samuelson was awarded the economics Nobel prize in 1970. Arrow was awarded the economics Nobel prize in 1972 (together with John Hicks) and Debreu was awarded the economics Nobel prize in 1983.

How economists thought about the welfare theorems

- Tom Bethell (1998) writes:

  Over one hundred years ago, the institution of private property fell into intellectual disrepute. Property's fall from grace was gradual, beginning early in the nineteenth century. But with the *Communist Manifesto* the war on property became overt, and finally it became respectable.

  In economics, best-selling textbooks by Paul Samuelson and others either skirted questions of ownership or relegated them to a paragraph under the rubric of "capitalist ideology." Since World War II, almost all such texts have argued that a more rapid growth could be attained with state ownership than with private property. Robert Solow of the Massachusetts Institute of Technology, winner of the Nobel Prize in economics in 1987, said: "I still believe the institution of private property has to keep proving itself." He referred to Proudhon's "insight" that "property is theft."

- Khrushchev promised in 1961 that communism would be "built" by 1980. By then, "public wealth" would "gush forth abundantly." GNP would have multiplied by five, production would be double that of the United States. Every Soviet family would have a free apartment and utilities, and "all that a man could reasonably want." CIA director Allen Dulles said in 1958 that the Soviet economy had been growing, and was expected to continue growing "at roughly twice that of the United States." These figures were widely repeated and believed.

  Paul Samuelson's *Economics* sold over three million copies and has been the most influential economics textbook since World War II. In successive editions, it included a graph showing the growth paths of Soviet and American GNP, the Soviet output starting from a lower level, but rising more steeply. Theirs was shown as overtaking ours approximately twenty years in the future.

- With each new edition the paths remained unchanged, but the date of intersection was advanced. But in 1980 the long-promised crossover was no nearer at hand, and in the 1985 edition the graph was at last eliminated.

- In the 1989 edition, Paul Samuelson stated, "The Soviet economy is proof that, contrary to what many skeptics had earlier believed, a socialist command economy can function and even thrive."

It is beyond the scope of this course but to understand that the Soviet Union did not have a model economy worth imitating, it is helpful to read

- Aleksandr Solzhenitsyn, *The Gulag Archipelago*, (1973-1978).
- Jung Chang and Jon Halliday, *Mao: The Unknown Story*, (2005).
- Richard Pipes, *Communism: A History*, (2001).

Does a Walrasian equilibrium with production exist?

- Aggregate excess demand for commodity k is

$$z_k(\mathbf{p}) \equiv \sum_{i\in\mathcal{I}} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{j\in\mathcal{J}} y_k^j(\mathbf{p}) - \sum_{i\in\mathcal{I}} e_k^i$$

and the aggregate excess demand vector is

$$\mathbf{z}(\mathbf{p}) \equiv (z_1(\mathbf{p}), \ldots, z_n(\mathbf{p})).$$

As before, a Walrasian equilibrium price vector $\mathbf{p}^* \gg \mathbf{0}$ clears all markets. That is, $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

- **THEOREM (Existence of Walrasian Equilibrium with Production)**: Consider the economy $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i\in\mathcal{I}, j\in\mathcal{J}}$. If each u^i and each Y^j satisfy the previously stated assumptions, and $\mathbf{y} + \sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y} \in \sum_{j\in\mathcal{J}} Y^j$, then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

- The proof is left as an exercise in the textbook. Here it is presented. We repeat the same arguments as before (for the exchange economy), making small adjustments where needed.

- **THEOREM (Properties of Excess Demand Functions with Production)**: If each u^i and each Y^j satisfy the previously stated assumptions, then for all $\mathbf{p} \gg \mathbf{0}$,

  1. Continuity: $\mathbf{z}(\cdot)$ is continuous in $\mathbf{p}$.
  2. Homogeneity: $\mathbf{z}(\lambda\mathbf{p}) = \mathbf{z}(\mathbf{p})$ for all $\lambda > 0$.
  3. Walras' law: $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$.

- **Proof**: Continuity of

$$z_k(\mathbf{p}) \equiv \sum_{i\in\mathcal{I}} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{j\in\mathcal{J}} y_k^j(\mathbf{p}) - \sum_{i\in\mathcal{I}} e_k^i$$

follows from the previous theorems about continuity of demand and supply functions.

Looking at

$$\pi^j(\mathbf{p}) \equiv \max_{\mathbf{y}^j\in Y^j} \mathbf{p}\cdot\mathbf{y}^j,$$

each supply function is homogeneous of degree zero in prices and each profit function is homogeneous of degree one in prices. Therefore

$$m^i(\mathbf{p}) \equiv \mathbf{p}\cdot\mathbf{e}^i + \sum_{j\in\mathcal{J}} \theta^{ij}\pi^j(\mathbf{p})$$

is homogeneous of degree one in prices.

- $$z_k(\mathbf{p}) \equiv \sum_{i \in \mathcal{I}} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{j \in \mathcal{J}} y_k^j(\mathbf{p}) - \sum_{i \in \mathcal{I}} e_k^i$$

  It follows that $x_k^i(\mathbf{p}, m^i(\mathbf{p}))$ is homogeneous of degree zero in prices, since $m^i(\mathbf{p})$ is homogeneous of degree one in prices. Because each supply function is homogeneous of degree zero in prices, $z_k(\mathbf{p})$ is homogeneous of degree zero in prices.

- It remains to prove the third property: Walras' law $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$. Walras' law follows because when u^i is strongly increasing, each consumer's budget constraint holds with equality.

- When each consumer's budget constraint holds with equality, $\mathbf{p} \cdot \mathbf{x}^i = m^i(\mathbf{p}) \equiv \mathbf{p} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij} \pi^j(\mathbf{p})$ or

$$\sum_{k=1}^{n} p_k[x_k^i(\mathbf{p}, m^i(\mathbf{p})) - e_k^i] = \sum_{j \in \mathcal{J}} \theta^{ij} \pi^j(\mathbf{p}).$$

Summing over individuals gives

$$\sum_{i \in \mathcal{I}} \sum_{k=1}^{n} p_k[x_k^i(\mathbf{p}, m^i(\mathbf{p})) - e_k^i] = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \theta^{ij} \pi^j(\mathbf{p}).$$

Because the order of summation does not matter, we can reverse it

$$\sum_{k=1}^{n} \sum_{i \in \mathcal{I}} p_k[x_k^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i) - e_k^i] = \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \theta^{ij} \pi^j(\mathbf{p}) = \sum_{j \in \mathcal{J}} \pi^j(\mathbf{p}).$$

- $$\sum_{k=1}^{n}\sum_{i\in\mathcal{I}} p_k[x_k^i(\mathbf{p},\mathbf{p}\cdot\mathbf{e}^i) - e_k^i] = \sum_{j\in\mathcal{J}} \pi^j(\mathbf{p}) = \sum_{j\in\mathcal{J}} \mathbf{p}\cdot\mathbf{y}^j(\mathbf{p})$$

$$\sum_{k=1}^{n}\sum_{i\in\mathcal{I}} p_k[x_k^i(\mathbf{p},\mathbf{p}\cdot\mathbf{e}^i) - e_k^i] - \sum_{k=1}^{n}\sum_{j\in\mathcal{J}} p_k y_k^j(\mathbf{p}) = 0$$

Now remember that

$$z_k(\mathbf{p}) \equiv \sum_{i\in\mathcal{I}} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{j\in\mathcal{J}} y_k^j(\mathbf{p}) - \sum_{i\in\mathcal{I}} e_k^i$$

It follows that

$$\sum_{k=1}^{n} p_k z_k(\mathbf{p}) = \mathbf{p}\cdot\mathbf{z}(\mathbf{p}) = 0,$$

and the claim is proved.

$\square$

- **THEOREM (Aggregate Excess Demand and Walrasian Equilibrium with Production)**: Suppose $\mathbf{z}(\mathbf{p})$ satisfies the following three conditions.

  1. $\mathbf{z}(\cdot)$ is continuous on $\mathbb{R}^n_{++}$.

  2. $\mathbf{p} \cdot \mathbf{z}(\mathbf{p}) = 0$ for all $\mathbf{p} \gg \mathbf{0}$.

  3. If $\{\mathbf{p}^m\}$ is a sequence of price vectors in $\mathbb{R}^n_{++}$ converging to $\bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some good k, then for some good k' with $\bar{p}_{k'} = 0$, the associated sequence of excess demands in the market for good k', $\{z_{k'}(\mathbf{p}^m)\}$, is unbounded above.

  Then there is a price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

- The statement and proof of this theorem is the same as before. There is nothing to change!

- **THEOREM (Utility and Aggregate Excess Demand with Production)**: If each u^i and each Y^j satisfy the previously stated assumptions, and $\mathbf{y} + \sum_{i \in \mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y} \in \sum_{j \in \mathcal{J}} Y^j$, then aggregate excess demand satisfies conditions 1 through 3 of the previous theorem.

- **Proof**: Conditions 1 and 2 have already been established. Thus, it remains to verify condition 3.
- Consider a sequence of strictly positive price vectors $\{\mathbf{p}^m\}$ converging to $\bar{\mathbf{p}} \neq \mathbf{0}$, such that $\bar{p}_k = 0$ for some good k. Because $\mathbf{y} + \sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$ for some possible $\mathbf{y}$, we must have

$$\bar{\mathbf{p}} \cdot \left(\mathbf{y} + \sum_{i=1}^{I} \mathbf{e}^i \right) > 0.$$

- Consequently, recalling that both $m^i(\mathbf{p})$ and $\pi^j(\mathbf{p})$ are well-defined for all $\mathbf{p} \geq \mathbf{0}$,

$$\begin{aligned}
\sum_{i \in \mathcal{I}} m^i(\bar{\mathbf{p}}) &= \sum_{i \in \mathcal{I}} \left(\bar{\mathbf{p}} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \theta^{ij} \pi^j(\bar{\mathbf{p}}) \right) \\
&= \sum_{i \in \mathcal{I}} \bar{\mathbf{p}} \cdot \mathbf{e}^i + \sum_{j \in \mathcal{J}} \pi^j(\bar{\mathbf{p}}) \\
&\geq \sum_{i \in \mathcal{I}} \bar{\mathbf{p}} \cdot \mathbf{e}^i + \bar{\mathbf{p}} \cdot \mathbf{y} \\
&= \bar{\mathbf{p}} \cdot \left(\sum_{i \in \mathcal{I}} \mathbf{e}^i + \mathbf{y} \right) > 0
\end{aligned}$$

Therefore, there must exist at least one consumer i whose income at prices $\bar{\mathbf{p}}$, $m^i(\bar{\mathbf{p}})$, is strictly positive.

- Consider this consumer i's demand $\mathbf{x}^i(\mathbf{p}^m, m^i(\mathbf{p}^m))$ along the sequence of prices $\{\mathbf{p}^m\}$.
- We want to find a sequence of excess demands for a good that is unbounded above. So let us suppose, by way of contradiction, that this sequence of demand vectors is bounded. Then, using the following theorem:

  THEOREM (On Bounded Sequences):
  Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

  there must be a convergent subsequence.
- So we may assume without any loss (by reindexing the subsequence) that the original sequence of demands converges to $\mathbf{x}^*$. That is, $\mathbf{x}^i(\mathbf{p}^m, m^i(\mathbf{p}^m)) \longrightarrow \mathbf{x}^*$.

- To simplify notation, let $\mathbf{x}^m \equiv \mathbf{x}^i(\mathbf{p}^m, m^i(\mathbf{p}^m))$ for every m.
- Now because $\mathbf{x}^m$ maximizes u^i subject to i's budget constraint given the prices $\mathbf{p}^m$, and because u^i is strongly increasing, the budget constraint must be satisfied with equality:

$$\mathbf{p}^m \cdot \mathbf{x}^m = m^i(\mathbf{p}^m) \quad \text{for every } m.$$

Taking the limit as $m \longrightarrow \infty$ yields

$$\bar{\mathbf{p}} \cdot \mathbf{x}^* = m^i(\bar{\mathbf{p}}) > 0,$$

where the strict inequality follows from our choice of consumer i.

- Now let $\hat{\mathbf{x}} \equiv \mathbf{x}^* + (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 occurs in the kth position (a good where $\bar{p}_k = 0$). Then because u^i is strongly increasing on $\mathbb{R}^n_+$

$$u^i(\hat{\mathbf{x}}) > u^i(\mathbf{x}^*).$$

In addition, because $\bar{p}_k = 0$,

$$\bar{\mathbf{p}} \cdot \hat{\mathbf{x}} = \bar{\mathbf{p}} \cdot \mathbf{x}^* = m^i(\bar{\mathbf{p}}) > 0.$$

So because u^i is continuous, there is a $t \in (0, 1)$ such that

$$u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^*) \qquad \bar{\mathbf{p}} \cdot t\hat{\mathbf{x}} < m^i(\bar{\mathbf{p}}).$$

Now because $\mathbf{p}^m \longrightarrow \bar{\mathbf{p}}$, $\mathbf{x}^m \longrightarrow \mathbf{x}^*$ and u^i is continuous, this implies that for m large enough,

$$u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^m) \qquad \mathbf{p}^m \cdot t\hat{\mathbf{x}} < m^i(\mathbf{p}^m).$$

- But
$$u^i(t\hat{\mathbf{x}}) > u^i(\mathbf{x}^m) \qquad \mathbf{p}^m \cdot t\hat{\mathbf{x}} < m^i(\mathbf{p}^m)$$
contradicts the fact that $\mathbf{x}^m$ solves the consumer's problem at prices $\mathbf{p}^m$. We conclude therefore that consumer i's sequence of demand vectors must be unbounded.
- Now because i's sequence of demand vectors $\{\mathbf{x}^m\}$ is unbounded yet nonnegative, there must be some good k' such that $\{x_{k'}^m\}$ is unbounded above.
- But because i's income converges to $m^i(\bar{\mathbf{p}}) > 0$, the sequence of i's income $\{m^i(\mathbf{p}^m)\}$ is bounded. Consequently, we must have $p_{k'}^m \longrightarrow 0$, because this is the only way that the demand for good k' can be unbounded above and affordable.

- $$z_k(\mathbf{p}) \equiv \sum_{i \in \mathcal{I}} x_k^i(\mathbf{p}, m^i(\mathbf{p})) - \sum_{j \in \mathcal{J}} y_k^j(\mathbf{p}) - \sum_{i \in \mathcal{I}} e_k^i$$

  Finally, note that because production sets are bounded and consumption is non-negative, the fact that i's demand for good k' is unbounded above implies that the aggregate excess demand for good k' is unbounded above.
- Consequently, beginning with the assumption that $\mathbf{p}^m \longrightarrow \bar{\mathbf{p}} \neq \mathbf{0}$ and $\bar{p}_k = 0$ for some k, we have shown that there exists some good k' with $\bar{p}_{k'} = 0$, such that the aggregate excess demand for good k' is unbounded above along the sequence of prices $\{\mathbf{p}^m\}$, as desired in condition 3.

$\square$

- We have just established

  THEOREM (Existence of Walrasian Equilibrium with Production): Consider the economy $(u^i, \mathbf{e}^i, \theta^{ij}, Y^j)_{i\in\mathcal{I}, j\in\mathcal{J}}$. If each u^i and each Y^j satisfy the previously stated assumptions, and $\mathbf{y} + \sum_{i\in\mathcal{I}} \mathbf{e}^i \gg \mathbf{0}$ for some aggregate production vector $\mathbf{y} \in \sum_{j\in\mathcal{J}} Y^j$, then there exists at least one price vector $\mathbf{p}^* \gg \mathbf{0}$ such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Can we relax the strong assumptions and still get existence of Walrasian equilibrium?

- We now return to the assumption of boundedness of the firms' production sets.
- The boundedness assumption is not needed.
- It turns out that a standard method of proving existence with unbounded production sets is to first prove it by placing artificial bounds on them (which is essentially what we have done) and then letting the artificial bounds become arbitrarily large (which we will not do).
- Under suitable conditions, this will yield a Walrasian equilibrium of the economy without bounded production sets.

- Also, strong convexity of the firm production possibility sets is more stringent than needed to prove existence of Walrasian equilibrium.
- If, instead, mere convexity of firm production possibility sets is assumed (which allows for constant returns to scale in production), existence can still be proved.
- The proof uses a generalized version of Brouwer's fixed point theorem due to Kakutani (1941). The details can be found in Debreu (1959, *Theory of Value*).

- **Brouwer fixed point theorem (1910)**: Let the set $K \subset \mathbb{R}^m$ be non-empty, compact and convex. Let $F : K \longrightarrow K$ be a continuous function. Then there exists at least one $\bar{x} \in K$ such that $\bar{x} = F(\bar{x})$.

- **Kakutani fixed point theorem (1941)**: Let the set $K \subset \mathbb{R}^m$ be non-empty, compact and convex. Let A be a closed set in $\mathbb{R}^m \times \mathbb{R}^m$ with the following two properties:

  (a) Let $A \subset K \times K$;

  (b) for all $x \in K$, the set $A_x \equiv \{y \in K | (x, y) \in A\}$ is non-empty and convex. Then there exists at least one $\bar{x} \in K$ such that $(\bar{x}, \bar{x}) \in A$.

Adam Smith, *The Wealth of Nations*

We conclude with a quotation from Adam Smith's *The Wealth of Nations*:

- The trade of the merchant carrier, or of the importer of foreign corn in order to export it again, contributes to the plentiful supply of the home market. It is not indeed the direct purpose of his trade to sell his corn there. But he will generally be willing to do so, and even for a good deal less money than he might expect in a foreign market; because he saves in this manner the expence of loading and unloading, of freight and insurance. The inhabitants of the country which, by means of the carrying trade, becomes the magazine and storehouse for the supply of other countries, can very seldom be in want themselves.

- The carrying trade was in effect prohibited in Great Britain upon all ordinary occasions, by the high duties [tariffs] upon the importation of foreign corn, of the greater part of which there was no drawback; and upon extraordinary occasions, when a scarcity made it necessary to suspend those duties [tariffs] by temporary statutes, exportation was always prohibited. By this system of laws, therefore, the carrying trade was in effect prohibited upon all occasions.
- That system of laws, therefore, which is connected with the establishment of the bounty [export subsidy], seems to deserve no part of the praise which has been bestowed upon it. The improvement and prosperity of Great Britain, which has been so often ascribed to those laws, may very easily be accounted for by other causes.

- That security which the laws in Great Britain give to every man that he shall enjoy the fruits of his own labour, is alone sufficient to make any country flourish, notwithstanding these and twenty other absurd regulations of commerce; and this security was perfected by the revolution [the Glorious Revolution of 1688], much about the same time that the bounty [export subsidy] was established.

- The natural effort of every individual to better his own condition, when suffered [allowed] to exert itself with freedom and security, is so powerful a principle, that it is alone, and without any assistance, not only capable of carrying on the society to wealth and prosperity, but of surmounting a hundred impertinent obstructions with which the folly of human laws too often incumbers its operations; though the effects of these obstructions is always more or less either to encroach upon its freedom, or to diminish its security. In Great Britain industry is perfectly secure; and though it is far from being perfectly free, it is as free or freer than in any other part of Europe.

- Though the period of the greatest prosperity and improvement of Great Britain, has been posterior to that system of laws which is connected with the bounty [export subsidy], we must not upon that account impute it to those laws. It has been posterior likewise to the national debt. But the national debt has most assuredly not been the cause of it.

- Though the system of laws which is connected with the bounty [export subsidy], has exactly the same tendency with the police of Spain and Portugal; to lower somewhat the value of the precious metals in the country where it takes place; yet Great Britain is certainly one of the richest countries in Europe, while Spain and Portugal are perhaps among the most beggarly. This difference of situation, however, may easily be accounted for from two different causes.

- First, the tax in Spain, the prohibition in Portugal of exporting gold and silver, and the vigilant police which watches over the execution of those laws, must, in two very poor countries, which between them import annually upwards of six millions sterling, operate, not only more directly, but much more forcibly in reducing the value of those metals there, than the corn laws can do in Great Britain.
- And, secondly, this bad policy is not in those countries counter-balanced by the general liberty and security of the people. Industry is there neither free nor secure, and the civil and ecclesiastical governments of both Spain and Portugal, are such as would alone be sufficient to perpetuate their present state of poverty, even though their regulations of commerce were as wise as the greater part of them are absurd and foolish.