5303 Advanced Macroeconomics - Group 3
Assignment 3

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Assignment #3

1) $N(t)=1$, $u_t^h(t) = \ln c_t^h(t) + \frac{3}{5}\ln c_t^h(t+1)$ $\quad \Delta_t^h\,[\frac{4}{5},\frac{1}{5}]$ $\quad Y(t)=K(t)^{1/2}L(t)^{1/2}$

(a) Find stationary eq. with pos. capital.

All consumers maximize:

$$\max \quad u_t^h(t) \quad \text{s.t.} \quad \begin{aligned} c_t^h(t) &= w(t)\Delta_t^h(t) - k^h(t+1) - \ell^h(t) \\ c_t^h(t+1) &= w(t+1)\Delta_t^h(t+1) + r(t)\ell(t) + R(t+1)k^h(t+1) \end{aligned}$$

We can combine the budget constraints to yield:

$$c_t^h(t+1) = \underbrace{w(t+1)\Delta_t^h(t+1) + r(t)[w(t)\Delta_t^h(t) - k^h(t+1) - c_t^h(t)] + R(t+1)k^h(t+1)}_{1}$$

And substitute to get an unconstrained maximization problem

$$\max_{\{c_t^h(t),\,k^h(t+1)\}} \ln c_t^h(t) + \frac{3}{5}\ln[\boxed{1}]$$

FOC's:

(2) $c_t^h(t)$: $\dfrac{1}{c_t^h(t)} + \dfrac{3}{5}\cdot\dfrac{-r(t)}{c_t^h(t+1)} = 0 \Rightarrow c_t^h(t+1) = \frac{3}{5}r(t)c_t^h(t)$

(3) $k^h(t+1)$: $\frac{3}{5}\cdot\dfrac{1}{c_t^h(t+1)}\left[(-r(t)) + R(t+1)\right] = 0 \Rightarrow R(t+1) = r(t)$

Substituting the above into the combined BC:

$$\frac{3}{5}r(t)c_t^h(t) = w(t+1)\Delta_t^h(t+1) + r(t)[w(t)\Delta_t^h(t) - k^h(t+1) - c_t^h(t)] + r(t)k^h(t+1)$$

$$\Leftrightarrow \frac{3}{5}r(t)c_t^h(t) + r(t)c_t^h(t) = w(t+1)\Delta_t^h(t+1) + r(t)[w(t)\Delta_t^h(t)]$$

$$\Leftrightarrow c_t^h(t) = \left[\frac{w(t+1)\Delta_t^h(t+1)}{r(t)} + w(t)\Delta_t^h(t)\right]\frac{5}{8} = \frac{5}{8}\left[\frac{w(t+1)\frac{1}{5}}{r(t)} + w(t)\frac{4}{5}\right] = \frac{w(t+1)}{r(t)8} + \frac{w(t)}{2}$$

Savings: endowment − cons

$$s_t^h(t) = w(t)\Delta_t^h(t) - c_t^h(t) = \frac{4w(t)}{5} - \frac{w(t+1)}{r(t)\cdot 8} - \frac{w(t)}{2} = \frac{3}{10}w(t) - \frac{w(t+1)}{r(t)\cdot 8} = S_t(r(t))$$

since $N(t)=1$ $\forall t$.

The firm maximizes

$$\max_{\{K(t),\,L(t)\}} K(t)^{1/2}L(t)^{1/2} - R(t)K(t) - w(t)L(t)$$

FOCs:

$K(t)$: $\frac{1}{2}K(t)^{-1/2}L(t)^{1/2} - R(t) = 0 \Leftrightarrow R(t) = \frac{1}{2}\left(\frac{L(t)}{K(t)}\right)^{1/2}$

$L(t)$: $\frac{1}{2}K(t)^{1/2}L(t)^{-1/2} - w(t) = 0 \Leftrightarrow w(t) = \frac{1}{2}\left(\frac{K(t)}{L(t)}\right)^{1/2}$

In each period, we have only two agents alive with ~~and~~ labor endowments $\frac{4}{5}, \frac{1}{5}$. So $L(t) = \frac{4}{5} + \frac{1}{5} = 1 \quad \forall t$.

Thus the firm FOCs give us:

$$R(t) = \frac{1}{2}\left(\frac{1}{K(t)}\right)^{1/2} \Rightarrow R(t+1) = \frac{1}{2}K(t+1)^{-1/2} = r(t) \text{ by } ③$$

$$w(t) = \frac{1}{2}K(t)^{1/2} \Rightarrow w(t+1) = \frac{1}{2}K(t+1)^{1/2}$$

Substituting into aggregate savings:

$$S_t(r(t)) = \frac{3}{10}w(t) - \frac{w(t+1)}{8r(t)} = \frac{3}{10}\cdot\frac{1}{2}K(t)^{1/2} - \frac{\frac{1}{8}\cdot\frac{1}{2}K(t+1)^{1/2}}{\frac{1}{2}K(t+1)^{-1/2}}$$

$$= \frac{3}{20}K(t)^{1/2} - \frac{1}{8}K(t+1)$$

By p. 238 in MW, in equilibrium we have, for $K(1) > 0$.

$$S_t(r(t)) = K(t+1)$$
$$r(t) = R(t+1)$$
$$w(t) = \partial[\gamma(t)F(L(t), K(t))]/\partial L(t)$$
$$R(t) = \partial[\gamma(t)F(L(t), K(t))]/\partial K(t)$$

~~Then~~ Since we're looking at a stationary eq., let $K(t) = K \in \mathbb{R} \; \forall t$

$$S_t(r(t)) = \frac{3}{20}K^{1/2} - \frac{1}{8}K = K \Rightarrow \frac{3}{20}K^{1/2} = \frac{9}{8}K \Rightarrow K^{1/2} = \frac{15}{2}K$$

~~$\Rightarrow \frac{15}{2}K^2 - K = 0 \Rightarrow K = 0$ or $K = \frac{2}{15}$. We want positive capital, so take $K = \frac{2}{15}$.~~

~~Let $r(t) = r \; \forall t$, $w(t) = w \; \forall t$.~~

~~Then $w = \frac{1}{2}K^{1/2} = \frac{1}{2}\left(\frac{2}{15}\right)^{1/2} \approx 0.183$~~

~~$r = \frac{1}{2}K^{-1/2} = \frac{1}{2}\left(\frac{15}{2}\right)^{1/2} \approx 1.369$~~

~~$c_t^h(t) = \frac{w}{8r} + \frac{w}{2} = \frac{1}{8}\cdot\frac{1}{15} + \frac{1}{4}\left(\frac{2}{15}\right)^{1/2}$~~

~~$c_t^h(t+1) = \frac{3}{5}r\,c_t^h(t) = \frac{3}{5}\left(\frac{1}{2}\left(\frac{2}{15}\right)^{1/2}\right)($~~

Then we have potential solutions $K = 0$, $K = \frac{4}{225}$. We want $K > 0$, so take $K = \frac{4}{225}$.

Let $r(t) = r$ and $w(t) = w \quad \forall t$.

Then
$$w = \frac{1}{2}K^{1/2} = \frac{1}{2}\cdot\left(\frac{4}{225}\right)^{1/2} = \frac{1}{2}\cdot\frac{2}{15} = \frac{1}{15}.$$
$$r = \frac{1}{2}K^{-1/2} = \frac{1}{2}\left(225/4\right)^{1/2} = \frac{15}{4}$$
$$c_t^h(t) = \frac{w}{8r} + \frac{w}{2} = \frac{1/15}{8\cdot\frac{15}{4}} + \frac{1/15}{2} = \frac{8}{225}$$
$$c_t^h(t+1) = \frac{3}{5}r(t)c_t^h(t) = \frac{3}{5}\cdot\frac{15}{4}\cdot\frac{8}{225} = \frac{2}{25}$$

This allocation is feasible. By MW p. 236, feasibility at equality is $C(t) + K(t+1) = \gamma(t)F(L(t), K(t))$. Here, we have

$$\left(\frac{2}{25} + \frac{8}{225}\right) + \frac{4}{225} = K^{1/2}\cdot 1^{1/2} = (4/225)^{1/2} = 2/15.$$

(b) K_{gold}:

We just showed that $F(K,L) = C + K$.

Then consumption is $C = F(K,L) - K$. We maximize consumption at the FOC with respect to capital (note we showed earlier that $L(t)=1\ \forall t$):

$$\text{FOC: } F'(K,L) - 1 = 0 \Rightarrow F'(K,L) = 1 \Rightarrow \tfrac{1}{2}K^{-1/2}L^{1/2} = 1 \Rightarrow K^{-1/2} = 2$$
$$\Rightarrow K = 2^{-2} = \tfrac{1}{4}.$$

So $K_{gold} = \frac{1}{4} > K$ from part A

(c) Since $K_{gold} > K^*$, we know this equilibrium is ~~dynamically inefficient~~ dynamically efficient.
~~We could increase total consumption by moving from K^* to K_{gold}~~
But would this be a Pareto improvement?

At K_{gold}, we would have:

$$w = \tfrac{1}{2}K^{1/2} = \tfrac{1}{4}$$
$$r = \tfrac{1}{2}K^{-1/2} = 1$$

$$c_e^n(t) = \frac{1}{8}\cdot\frac{w}{r} + \frac{w}{2} = \frac{1}{8}\cdot\frac{1}{4} + \frac{1}{8} = \frac{5}{32}$$
$$c_e^n(t+1) = \tfrac{3}{5}r\,c_e^n(t) = \frac{3}{5}\cdot 1\cdot\frac{5}{32} = \frac{3}{32}.$$

Then $u_{gold} = \log\frac{5}{32} + \frac{3}{5}\log\frac{3}{32} \approx -3.277$

While $u^* = \log\frac{8}{225} + \frac{3}{5}\log\left(\frac{2}{25}\right) \approx -4.85$

Thus $\forall\, t \geq 1$, $u_{gold} > u^*$. The initial old also prefer consuming $c_e^n(t+1)_{gold}$ in time period 1 since $3/32 > 2/25$.

Thus all agents strictly prefer K_{gold} to K^*, so the stationary equilibrium cannot be Pareto Optimal.

(We should also check the K_{gold} allocation is feasible:

$$C(t) + K(t+1) = L^{1/2}K^{1/2}$$

$$\left(\frac{5}{32} + \frac{3}{32}\right) + \frac{1}{4} = 1^{1/2}\left(\frac{1}{4}\right)^{1/2} = \frac{1}{2} \quad \text{so it's feasible)}$$

2. Consider a 2-period OLG model with production. N=1 young agents are born every period. All agents live for two periods.

Preferences are described by $u_t = \ln(c_t(t) - \theta a(t)) + \beta \ln c_t(t+1)$, where $\theta > 0$ and $\beta > 0$ are parameters. $a(t) = \bar{c}_{t-1}(t-1)$ is the average consumption of the young at t-1. The idea is that agents want to "keep up" with the consumption levels of their parent's generation.

Young agents are endowed with 1 unit of time and no capital. The initial old are endowed with S_0 units of capital. Thus, the young work and receive a wage of $w(t)$ (the old do not work).

Output is produced according to $Y(t) = K(t)^\alpha L(t)^{1-\alpha}$. Capital depreciates fully so $Y(t) = C(t) + K(t+1)$.

(a) State the household's problem and solve for the savings function.

(b) Define a competitive equilibrium.

(c) Derive the equilibrium laws of motion for $k(t+1)$ and $a(t+1)$ (i.e., an expression for $k(t+1)$ as a function of $k(t)$ and $a(t)$, and an expression for $a(t+1)$ as a function of $a(t)$ and $k(t)$).

(d) How does the steady state k respond to θ? Provide intuition.

(a) AGENT

$$\max_{\{c_t,\, c_t(t+1),\, \ell(t),\, k_t(t+1)\}} u_t$$

s. t.

$$c_t(t) \leq w(t)\Delta_t(t) - \ell(t) - k(t+1)$$
$$c_t(t+1) \leq w(t)\Delta_t(t+1) + r\,\ell(t) + R(t+1)k(t+1)$$

$$\mathcal{L} \equiv \ln(c_t(t) - \theta a(t)) + \beta \ln(c_t(t+1)) + \mu(t)\left[w(t)\Delta_t(t) - \ell(t) - k(t+1) - c_t(t)\right] + \mu(t+1)\left[w(t+1)\Delta_t(t+1) + r(t)\ell(t) + R(t+1)k(t+1) - c_t(t+1)\right]$$

FOC $\forall t \geq 1$

$c_t(t)$: $(c_t(t) - \theta a(t))^{-1} - \mu(t) = 0$ (1)

$c_t(t+1)$: $\beta(c_t(t+1))^{-1} - \mu(t+1) = 0$ (2)

$\ell(t)$: $-\mu(t) + \mu(t+1)r(t) = 0$ (3)

$k(t+1)$: $-\mu(t) + \mu(t+1)R(t+1) = 0$ (4)

(3), (4) ⇒ $\dfrac{\mu(t)}{\mu(t+1)} = r(t) = R(t+1)$

(1), (2) ⇒ $\dfrac{\mu(t)}{\mu(t+1)} = \dfrac{c_t(t+1)}{\beta(c_t(t) - \theta a(t))}$

$\Big\}\ c_t(t+1) = \beta r(t)\left[c_t(t) - \theta a(t)\right]$

Lifetime budget constraints: $c_t(t) + \dfrac{c_t(t+1)}{r(t)} = w(t)\Delta_t(t) + \dfrac{w(t+1)\Delta_t(t+1)}{r(t)} - k(t+1)\left[1 - \dfrac{R(t+1)}{r(t)}\right]$

(5) ⇒ $c_t(t) = \dfrac{1}{1+\beta}\left[w(t)\Delta_t(t) + \dfrac{w(t+1)\Delta_t(t+1)}{r(t)} + \beta\theta a(t)\right]$

$s_t(t) = w_t(t) - c_t(t) = \dfrac{\beta}{1+\beta}\left[w(t)\Delta_t(t)\right] - \dfrac{1}{1+\beta}\left[\dfrac{w(t+1)\Delta_t(t+1)}{r(t)} + \beta\theta a(t)\right]$

$\Delta_t = [1, 0]$, $N(t) = 1$ ⇒ $S_t(r(t)) = s_t(t) = \dfrac{\beta}{1+\beta}\left(w(t) - \theta a(t)\right)$

$= \dfrac{\beta}{1+\beta}\left[(1-\alpha)k^\alpha(t) - \theta a(t)\right] \quad \forall t \geq 1$

FIRM

$$\max_{\{K(t), L(t)\}} K(t)^\alpha L(t)^{1-\alpha} - R(t)K(t) - w(t)L(t)$$

FOC $\forall t \geq 1$

$K(t)$: $\alpha K(t)^{\alpha-1}L(t)^{1-\alpha} = R(t)$ (6)

$L(t)$: $(1-\alpha)K^\alpha(t)L(t)^{-\alpha} = w(t)$ (7)

$\Delta_t = [1, 0]$, $N(t) = 1$ ⇒ $L(t) = 1$, $k(t) \equiv \dfrac{K(t)}{N(t)} = k(t)$

(6) ⇒ $R(t+1) = \alpha k(t)^{\alpha-1} = r(t)$

(7) ⇒ $w(t) = (1-\alpha)k^\alpha(t)$

(b) **Definition** *A perfect foresight competitive equilibrium for an economy with labor endowments and a production function of $\gamma(t)F(L(t), K(t))$ is a sequence of $K(t)$, $r(t)$, wage(t), and rental(t) for $t \geq 1$ such that, given an initial $K(1) > 0$,*

$$S_t(r(t)) = K(t+1),$$

$$r(t) = \text{rental}(t+1),$$

$$\text{wage}(t) = \frac{\partial[\gamma(t)F(L(t), K(t))]}{\partial L(t)},$$

and $$\text{rental}(t) = \frac{\partial[\gamma(t)F(L(t), K(t))]}{\partial K(t)},$$

hold for all $t \geq 1$.

(c) from (b) follows $k(t+1) = \dfrac{\beta}{1+\beta}(1-\alpha)k^\alpha(t) - \dfrac{\beta}{1+\beta}\theta a(t) \quad \forall t \geq 1$

by definition $a(t+1) = \bar{c}_t(t) = \dfrac{c_t(t)}{1} = \dfrac{1}{1+\beta}(1-\alpha)k^\alpha(t) + \dfrac{\beta}{1+\beta}\theta a(t) \quad \forall t \geq 1$

(d) in steady state $a(t) = a(t+1) \equiv a$

⇒ $a = \dfrac{1-\alpha}{1+\beta}k^\alpha(t) + \dfrac{\beta\theta}{1+\beta}a$

⇔ $\dfrac{\beta}{1+\beta}(1-\alpha)k^\alpha(t) = \beta a\left(1 - \dfrac{\theta}{1+\beta}\right)$ (*)

and $k(t) = k(t+1) \equiv k$

⇒ $k = \dfrac{\beta}{1+\beta}(1-\alpha)k^\alpha - \dfrac{\beta\theta}{1+\beta}a$

(*) ⇒ $k = \begin{cases} \beta a\left(1 - \dfrac{\theta}{1+\beta} - \dfrac{1}{1+\beta}\right) = \beta a(1-\theta), & \theta \in (0, 1); \\ 0 & \text{else.} \end{cases}$

$\dfrac{\partial k}{\partial \theta} = \begin{cases} -\beta a < 0 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \end{cases}$ $\Bigg\}$ For positive $\bar{c}_{t-1}(t-1)$, the steady state k decreases as θ increases on $(0, 1)$ and $\theta > 1$ implies $k = 0$.

θ measures the eagerness of the young to "keep up" with consumption levels of previous generation. Higher θ therefore causes the young to consume more when young and, by reducing the capital that they would otherwise save, to consume less when old (since lifetime income is fixed).

Question 3. Consider the following overlapping generations enviorment. All agents live for two periods, there is an equal number of young alive and no population growth. Households has preferences over consumption of a non-storable good and labour supply gien by

$$u_t^h = c_t^h(t) - \frac{\gamma}{2}(l_t^h(t))^2 + \ln c_t^h(t+1),$$

with constraints on consumption $c_t^h \geq 0, c_t^h(t+1) \geq 0$ and labour supply $0 \leq l_t^h \leq 1$. Technology is given by $Y(t) = AL(t)$ and assume $A^2 \geq \gamma A \geq 0$. Government has a pay as you go social security system.

1. Define a competitve equilibrium with social security.
2. Characterize the competitive equilibrium allocation, prices and policy.
3. A laffer curve is defined as the relationship between revenue and tax rates. Solve for the Laffer curve. What tax rate maximizes tax revenue? Show that the tax rate that maximizes the representative generations welafre is diffirent fom that which maximizes tax revenue.

Solution. a) An economy is in competitive equilibrium if it satisifies

1. The allocation maximizes consumer utility
2. The firms profits are maximized
3. Both goods and labour markets clear.
4. The government budget constraint is in equilibrium.

b) The budget constraint in each period for each individual is

$$c_t^h(t) \leq (1-\tau)w(t)l_t^h(t),$$

$$c^h(t+1) \leq b(t).$$

Household maximization problem is

$$\begin{aligned} \max \quad & c_t^h(t) - \frac{\gamma}{2}(l_t^h(t))^2 + \ln c_t^h(t+1) \\ \text{s.t} \quad & c_t^h(t) \leq (1-\tau)w(t)l_t^h(t) \\ & c^h(t+1) \leq b(t). \end{aligned}$$

The problem lagrangian is

$$L = c_t^h(t) - \frac{\gamma}{2}(l_t^h(t))^2 + \ln c_t^h(t+1) + \mu_t(c_t^h(t) \leq (1-\tau)w(t)l_t^h(t) - c_t^h(t)) + \mu_{t+1}(b(t) - c_t^h(t+1)).$$

The first order conditions for an optimum are

$$1 - \mu_t = 0,$$

$$\frac{1}{c_t^h(t+1)} - \mu_{t+1} = 0,$$

$$-\gamma l_t^h(t) + \mu_t(1-\tau)w(t) = 0.$$

Substitute the expression for μ_t into the third equation and solve for $l_t^h(t)$, this gives us

$$l_t^h(t) = \frac{(1-\tau)w(t)}{\gamma}.$$

The firm maximization problem in a competitive economy with production function $AL(t)$ imply that wages equals the marginal product of labour, $w(t) = A$ and are therefore constant. Substituting the expression into the lending equation we find

$$l_t^h(t) = \frac{(1-\tau)A}{\gamma}.$$

Now we can derive an expression for period t and $t+1$ consumption, they are

$$c^h(t) = \frac{(1-\tau)^2 A^2}{\gamma},$$

$$c_t^h(t+1) = b(t).$$

The government has to balance its budget. This implies transfers from one generation has to equal taxes from the next,

$$N(t-1)b(t) = N(t)\tau w(t) l_t^h(t).$$

Solving for $b(t)$ we find lump-sum transfers equal to

$$b(t) = \frac{A^2\tau(1-\tau)}{\gamma}.$$

c) The laffer curve is equal to the above equation. It relates tax rates to tax revenue. The tax rate rate that maximizes the government revenue is found by taking the derivative with respect to the tax rate and setting it equal to zero, it is

$$\frac{\partial b(t)}{\partial \tau} = \frac{A^2(1-2\tau)}{\gamma} = 0,$$

$$\tau = \frac{1}{2}.$$

The function $b(t)$ is a second degree polynomial in τ. Since the highest degree exponent is 2 and with a negative coefficient, due to $A, \gamma > 0$, we know the function is concave and the point is a maximum.

However, this might not be the tax rate that maximizes welfare. Optimal utility is found by substituting the optimal consumption derived earlier, it is

$$U(\tau) = \frac{(1-\tau)^2 A^2}{\gamma} - \frac{\gamma}{2}(\frac{(1-\tau)A}{\gamma})^2 + \ln \frac{\tau(1-\tau)A^2}{\gamma}.$$

Find the maximimum by taking the derivative, simplifying and setting it equal to zero, this yields

$$\frac{\partial U}{\partial \tau} = -\frac{A^2}{\gamma}(1-\tau) + \frac{1}{\tau} - \frac{1}{1-\tau} = 0.$$

The point is a local maximum if the second derivative is negative, it is

$$\frac{\partial^2 U}{\partial^2 \tau} = \frac{A^2}{\gamma} - \frac{1}{\tau^2} - \frac{1}{(1-\tau)^2}.$$

this expression is greater than zero if

$$\frac{A^2}{\gamma} < \frac{1}{\tau^2} + \frac{1}{(1-\tau)^2}.$$

The right hand side of the expression is minimized when $\tau = \frac{1}{2}$,so the inequality holds for all $t \in (0,1)$. However, when we plug in $\tau = \frac{1}{2}$ the first derivative is negative and hence can not be an optimum of the individual utility function. The tax rate that maximizes utility must therefore be lower than $\frac{1}{2}$.

Question 4. Consider the following OLG model where people live for two periods. Each generation has the same number of people. Preferences are given by $\ln c_t^h(t) + \ln c_t^h(t+1)$. Each agent is endowe with e_1 units of ouput when young and e_2 units of output when old.

1. Solve for the equilibrium where there is no government policy. When is the net interest rate negative?
2. Assume that the condition form part (a) is met, so that the equilibrium with no government policy is characterized by a negative net interest rate. Show how a pay as you go system in the form of a lump-som tax and transfer can improve welfare of each generation by allowing for perfect consumption smoothing.
3. Suppose that the endowment good is storable. In other words, a young person can put a unit of the endowment good in the refrigerator and when he/she is old, the good is still there. In this world, would anyone benefit from the government policy in part (b)?

Solution.

The budget constraints for each person when young and old is

$$c_t^h(t) \leq e_1 - l_t^h(t),$$

$$c_t^h(t+1) \leq e_2 r(t) l^h(t).$$

The houshold maximize utility subject to the constraints

$$\begin{aligned} \max \quad & \ln c_t^h(t) + \ln c_t^h(t+1) \\ \text{s.t} \quad & c_t^h(t) \leq e_1 - l^h(t) \\ & c^h(t+1) \leq e_2 + r(t) l^h(t). \end{aligned}$$

The lagrangian for the problem is

$$L = \ln c_t^h(t) + \ln c_t^h(t+1) + \mu_t(e_1 - l^h(t) - c_t^h(t)) + \mu_{t+1}(e_2 + r(t)l^h(t) - c_t^h(t+1)).$$

The first order conditions for a maximum is

$$\frac{1}{c_t^h(t)} - \mu_t = 0,$$

$$\frac{1}{c_t^h(t+1)} - \mu_{t+1} = 0,$$

$$-\mu_t + \mu_{t+1} r(t) = 0.$$

Substitute 1 and 2 into the third equation and solve for $c_t^h(t+1)$, this yields

$$c_t^h(t+1) = r(t) c_t^h(t).$$

Substitution into the budget constraint gives us consumption and savings

$$c_t^h(t) = \frac{e_1}{2} + \frac{e_2}{2r(t)},$$

$$s_t^h(t) = \frac{e_1}{2} - \frac{e_2}{2r(t)}.$$

All individuals are the same so there will be no lending/savings, they consume their endowments

$$c_t^h(t) = e_1,$$

$$c_t^h(t+1) = e_2.$$

Aggregate savings equal to zero imply

$$S(r(t)) = 0,$$

$$N(\frac{e_1}{2} - \frac{e_2}{2r(t)}) = 0,$$
$$r(t) = \frac{e_2}{e_1}.$$

The net interest rate is defined by $r(t) = 1 + i(t)$. It is negative if $\frac{e_2}{e_1}$ is less than 1, which happens if period two endowment is lower than period one. A negative net interest rate means it is costly to lend to the old.

b) We assume $e_1 > e_2$ and the government introduces a pay-as-you-go system which equalizes consumption in each period. To finance the system there is a lumpsum tax. The new constraints are

$$c_t^h(t) = e_1 - l^h(t) - \tau,$$
$$c_t^h(t+1) = e_2 + r(t)l^h(t) + \tau$$

The new maximization problem is

$$\begin{aligned} \max \quad & \ln c_t^h(t) + \ln c_t^h(t+1) \\ \text{s.t} \quad & c_t^h(t) \leq e_1 - l^h(t) - \tau \\ & c^h(t+1) \leq e_2 + r(t)l^h(t) + \tau. \end{aligned}$$

The lagrangian for this problem is

$$L = \ln c_t^h(t) + \ln c_t^h(t+1) + \mu_t(e_1 - l^h(t) - c_t^h(t) - \tau) + \mu_{t+1}(e_2 + r(t)l^h(t) - c_t^h(t+1) + \tau).$$

The first order conditions are the same as in the first part of the question. Substitution into the budget constraint and solving for $c_t^h(t)$ and then calculating savings gives us

$$c_t^h(t) = \frac{e_1 - \tau}{2} + \frac{e_2 + \tau}{2r(t)},$$
$$s_t^h(t) = \frac{e_1 - \tau}{2} - \frac{e_2 + \tau}{2r(t)}.$$

A negative net interest rate ensures no wants to lend. Therefore consumption will equal the endowment and the lump sum transfers, they are set as to smooth out consumption

$$e_1 - \tau = e_2 + \tau,$$
$$\tau = \frac{e_1 - e_2}{2}.$$

So consumption becomes

$$c_1 = \frac{e_1 + e_2}{2},$$
$$c_2 = \frac{e_1 + e_2}{2}.$$

Finally, the scheme is shown to be welfare improving by first calculating utility

$$u(c_1, c_2) = \ln \frac{e_1 + e_2}{2} + \ln \frac{e_1 + e_2}{2} = 2 \ln \frac{e_1 + e_2}{2},$$

then note that the log function is concave which implies

$$2 \ln \frac{e_1 + e_2}{2} > \ln e_1 + \ln e_2.$$

c) Given the opportunity to store consumption in period 1 for period 2, there is no need for the government policy. The higher period 1 endowment can be substituted for period 2 consumption through the storage technology at the same rate as the government can via transfers. The only generation which prefers the government system is the initial old who would consume their endowment and the transfer.