# Consumption-based Capital Asset Pricing
## Motivation

- Large differences in the average return to stocks vs. bonds
  - Depends on how you compute it, but often people cite an equity premium of roughly 5%
- Did individuals anticipate these returns in making their consumption, savings and portfolio decisions?
- Will these differences in average returns persist?

# Consumption-based Capital Asset Pricing

## Motivation

- As economist, hard to believe that people systematically misperceive average return over long periods of time
- Given that differences in returns were anticipated, a natural answer to explain this is to appeal to differences in riskiness
  - Stocks probably have a higher return to compensate the holder for bearing additional risk

# Consumption-based Capital Asset Pricing

## Motivation

- Goal is to understand basic consumption-based capital asset pricing model
  - Extent to which model can account for data
  - And what additional features might bring model more in line with data

# Consumption-based Capital Asset Pricing

## Environment

- Population is constant, $N(t) = N = 100$ for all $t$
- Economy can be in two different states of the world
  - Denote first state $B$ (a boom)
  - Denote second state $R$ (a recession)
- Endowments of young agents given by $\omega_t = \omega_1$ in all periods
- Endowments of old agents uncertain, can take two values
  - If economy is in a boom $\omega_{t+1} = \omega_2^B$
  - If economy is in a recession $\omega_{t+1} = \omega_2^R < \omega_2^B$
- Assume that, independent of state today, equally likely that economy tomorrow will be in a boom or a recession

# Consumption-based Capital Asset Pricing

## Environment

- Two assets in this economy
  1. Private borrowing and lending paying a certain rate of return $r(t)$ no matter what state of the world economy is in
  2. $A = 100$ units of land, yielding an uncertain crop

$$d(t) = d + \varepsilon(t)$$

where

$$\varepsilon(t) \in \begin{cases} \sigma^B & \text{if economy is in a boom (state } B) \\ \sigma^R & \text{if economy is in a recession (state } R) \end{cases}$$

# Consumption-based Capital Asset Pricing

## Environment

- Assume the following preferences for all $h, t$ :

$$u_t^h = \log c_t^h(t) + \beta \log c_t^h(t+1)$$

# Consumption-based Capital Asset Pricing

## Individual's Problem

- When return on land stochastic, maximize *expected utility*
- That is, agents solve

$$\max_{\{a^h(t), l^h(t)\}} E_t \left\{ \log c_t^h(t) + \beta \log c_t^h(t+1) \right\}$$

$c_t^h(t),\ c_t^h(t+1)$

subject to the budget constraints:

$$c_t^h(t) = \omega_t - l^h(t) - p(t)a^h(t)$$

$$c_t^h(t+1) = \omega_{t+1} + r(t)l^h(t) + [p(t+1) + d + \varepsilon(t+1)]\, a^h(t)$$

- Expectations operator $E_t$ is expectation conditional on information at time $t$, i.e., expectation of $\varepsilon(t+1)$ and $\omega_{t+1}$

# Consumption-based Capital Asset Pricing

## Individual's Problem

- For loans we have:

$$0 = E_t \left\{ -\frac{\partial u_t^h}{\partial c_t^h(t)} + \beta r(t) \frac{\partial u_t^h}{\partial c_t^h(t+1)} \right\}$$

or

$$0 = E_t \left\{ -\frac{1}{c_t^h(t)} + \beta r(t) \frac{1}{c_t^h(t+1)} \right\}$$

or since $r(t)$ and $c_t^h(t)$ are known

$$\frac{1}{r(t)} = E_t \left\{ \beta \frac{c_t^h(t)}{c_t^h(t+1)} \right\} \tag{1}$$

# Consumption-based Capital Asset Pricing

## Individual's Problem

- For land we have:

$$0 = E_t \left\{ -p(t) \frac{\partial u_t^h}{\partial c_t^h(t)} + \beta \left[ p(t+1) + d + \varepsilon(t+1) \right] \frac{\partial u_t^h}{\partial c_t^h(t+1)} \right\}$$

or

$$0 = E_t \left\{ -p(t) \frac{1}{c_t^h(t)} + \beta \left[ p(t+1) + d + \varepsilon(t+1) \right] \frac{1}{c_t^h(t+1)} \right\}$$

or since $p(t)$ and $c_t^h(t)$ are known

$$1 = E_t \left\{ \beta \underbrace{\frac{c_t^h(t)}{c_t^h(t+1)}}_{x} \underbrace{\frac{\left[ p(t+1) + d + \varepsilon(t+1) \right]}{p(t)}}_{y} \right\} \qquad (2)$$

- Equations (1) and (2) are called *the fundamental asset pricing equations*

# Consumption-based Capital Asset Pricing

## Which assets get high returns?

- To find out which assets have a high or low expected return, make use of covariance:

$$cov(x, y) = E(xy) - E(x)E(y)$$

or

$$E(xy) = cov(x, y) + E(x)E(y)$$

- Let $x = \beta \frac{c_t^h(t)}{c_t^h(t+1)}$ and $y = \frac{[p(t+1)+d+\varepsilon(t+1)]}{p(t)}$

# Consumption-based Capital Asset Pricing

## Which assets get high returns?

- Rewrite second fundamental asset pricing equation (2) as

$$
\begin{aligned}
1 &= E_t\left\{\beta \frac{c_t^h(t)}{c_t^h(t+1)} \frac{[p(t+1)+d+\varepsilon(t+1)]}{p(t)}\right\} \qquad (3)\\
&= cov\left\{\beta \frac{c_t^h(t)}{c_t^h(t+1)}, \frac{[p(t+1)+d+\varepsilon(t+1)]}{p(t)}\right\}\\
&+ E_t\left\{\beta \frac{c_t^h(t)}{c_t^h(t+1)}\right\} E_t\left\{\frac{[p(t+1)+d+\varepsilon(t+1)]}{p(t)}\right\}
\end{aligned}
$$

# Consumption-based Capital Asset Pricing

## Which assets get high returns?

- Now solve for expected return to buying land (uncertain asset): $E\left\{\frac{[p(t+1)+d+\varepsilon(t+1)]}{p(t)}\right\}$
- Substitute in $\frac{1}{r(t)} = E_t\left\{\beta \frac{c_t^h(t)}{c_t^h(t+1)}\right\}$ from first fundamental asset pricing equation (1), so that

$$
\begin{aligned}
1 &= cov\left\{\beta \frac{c_t^h(t)}{c_t^h(t+1)}, \frac{[p(t+1)+d+\varepsilon(t+1)]}{p(t)}\right\} \\
&+ \frac{1}{r(t)} E_t\left\{\frac{[p(t+1)+d+\varepsilon(t+1)]}{p(t)}\right\}
\end{aligned}
$$

# Consumption-based Capital Asset Pricing

## Which assets get high returns?

Rearrange so that

$$E_t \left\{ \frac{[p(t+1) + d + \varepsilon(t+1)]}{p(t)} \right\} = r(t)$$

$$-r(t) cov \left\{ \beta \frac{c_t^h(t)}{c_t^h(t+1)}, \frac{[p(t+1) + d + \varepsilon(t+1)]}{p(t)} \right\}$$

# Consumption-based Capital Asset Pricing

## Which assets get high returns?

- Expected equity premium given by

$$\hat{r} = E_t \left\{ \frac{p(t+1) + d + \varepsilon(t+1)}{p(t)} \right\} - r(t)$$

$$= -r(t) cov \left\{ \beta \frac{c_t^h(t)}{c_t^h(t+1)}, \frac{[p(t+1) + d + \varepsilon(t+1)]}{p(t)} \right\} \quad (4)$$

- Since $r(t)$ known, only thing that matters for equity premium is covariance: $cov \left\{ \beta \frac{c_t^h(t)}{c_t^h(t+1)}, \frac{[p(t+1)+d+\varepsilon(t+1)]}{p(t)} \right\}$

# Consumption-based Capital Asset Pricing

## Interpretation

- From now on, focus on stationary equilibrium
  - Thus, let $p(t) = p$ and $r(t) = r$ for all $t \geq 1$
- Suppose there exists three different kinds of land (uncertain equity)
  1. Asset pays out a lot in a boom (when endowment high) and very little in a recession (when endowment low); e.g., procyclical equity
  2. Asset pays out very little in a boom and a lot in a recession; e.g., life-insurance pays out a lot when really needed but its expected return is typically negative
  3. Asset not correlated with consumption; doesn't provide positive or negative insurance (same return as certain asset)

# Consumption-based Capital Asset Pricing

## How big is the equity premium of an asset?

- To find the expected equity premium of an asset, first solve for the competitive equilibrium
  - Need to find prices, $p$ and $r$

- Since all individuals $h \in \{1, 2, ..., N(t)\}$ in any generation $t$ are identical (same endowments and same preferences), their optimal demand for lending, $l^h(t)$, and land, $a^h(t)$, must be the same for all $h$
- Hence, in any competitive equilibrium it must be that asset demands for all $h$ are given by

$$l^h(t) = 0 \tag{5}$$

$$a^h(t) = \frac{A}{N(t)} = \frac{A}{N} = \frac{100}{100} = 1 \tag{6}$$

This implies that consumption when young is given by

$a(t) = 1$

$$c_t^h(t) = \omega_1 - p \tag{7}$$

which means that consumption when old is given by

$$\begin{aligned} c_t^h(t+1) &= \omega_{t+1} + [p + d + \varepsilon(t+1)] \qquad (8) \\ &= \begin{cases} \omega_2^B + \left[p + d + \sigma^B\right] & \text{if in a boom (state } B\text{)} \\ \omega_2^R + \left[p + d + \sigma^R\right] & \text{if in a recession (state } R\text{)} \end{cases} \end{aligned}$$

- Substitute eq. (7) and (8) into eq. (1) and (2) and take expectations
- We get

$$
\begin{aligned}
\frac{1}{r} &= \beta E_t \left\{ \frac{c_t^h(t)}{c_t^h(t+1)} \right\} \qquad (9) \\
&= \beta E_t \left\{ \frac{\omega_1 - p}{\omega_{t+1} + [p + d + \varepsilon(t+1)]} \right\} \\
&= \beta \frac{1}{2} \frac{\omega_1 - p}{\omega_2^B + [p + d + \sigma^B]} + \beta \frac{1}{2} \frac{\omega_1 - p}{\omega_2^R + [p + d + \sigma^R]}
\end{aligned}
$$

# Consumption-based Capital Asset Pricing

## Solving for Competitive Equilibrium

and

$$
\begin{aligned}
1 &= E_t\left\{\beta \frac{c_t^h(t)}{c_t^h(t+1)} \frac{[p(t+1)+d+\varepsilon(t+1)]}{p(t)}\right\} \\
&= \beta E_t\left\{\frac{\omega_1 - p}{\omega_{t+1} + [p+d+\varepsilon(t+1)]} \frac{[p+d+\varepsilon(t+1)]}{p}\right\} \\
&= \beta \frac{1}{2}\left\{\frac{\omega_1 - p}{\omega_2^B + [p+d+\sigma^B]} \frac{[p+d+\sigma^B]}{p}\right\} \\
&+ \beta \frac{1}{2}\left\{\frac{\omega_1 - p}{\omega_2^R + [p+d+\sigma^R]} \frac{[p+d+\sigma^R]}{p}\right\} \qquad (10)
\end{aligned}
$$

- Now can use equation (10) to find $p$ and then equation (9) to find $r$
- But this requires a lot of algebra, so let's look at some numerical examples

# Consumption-based Capital Asset Pricing

## Example

- For simplicity assume that $d = 0$, so there is no certain component to dividends
- Assume that if in a boom $\omega_{t+1} = \omega_2^B = 1$ and if in a recession $\omega_{t+1} = \omega_2^R = 0$
- Set $\beta = 2/3$

# Consumption-based Capital Asset Pricing

## Example: Procyclical Assets

- Assume that $\omega_1 = 2.94$
- Consider an asset that pays

$$\varepsilon_t \in \begin{cases} 1/5 & \text{if economy is in a boom (state } B\text{)} \\ -1/5 & \text{if economy is in a recession (state } R\text{)} \end{cases}$$

# Consumption-based Capital Asset Pricing

## Example: Procyclical Assets

- Guess that price $p = 1$ (turns out to be correct in this case)
- Can then solve for $r$ using equation (9)

$$\begin{aligned} \frac{1}{r} &= \beta \frac{1}{2} \frac{\omega_1 - p}{\omega_2^B + [p + d + \sigma^B]} + \beta \frac{1}{2} \frac{\omega_1 - p}{\omega_2^R + [p + d + \sigma^R]} \\ &= \frac{2}{3} \frac{1}{2} \frac{2.94 - 1}{1 + 1 + 1/5} + \frac{2}{3} \frac{1}{2} \frac{2.94 - 1}{1 - 1/5} = 1.103 \end{aligned}$$

so that

$$r = 0.907$$

# Consumption-based Capital Asset Pricing

## Example: Procyclical Assets

- The equity premium is then

$$
\begin{aligned}
\hat{r} &= E\left\{\frac{p+d+\varepsilon(t+1)}{p} - r\right\} \\
&= \frac{1}{2}\frac{\left[p+d+\sigma^B\right]}{p} + \frac{1}{2}\frac{\left[p+d+\sigma^R\right]}{p} - r \\
&= \frac{1}{2}\frac{[1+1/5]}{1} + \frac{1}{2}\frac{[1-1/5]}{1} - 0.907 \\
&= 0.093
\end{aligned}
$$

So, the equity premium is 9 percent

# Consumption-based Capital Asset Pricing

## Example: Procyclical Assets

- This is an asset that pays out a lot in a boom when consumption is high

$$c_t^h(t+1) = \omega_2^B + \left[p + d + \sigma^B\right] = 1 + 1 + 1/5 = 2.2$$

and marginal utility of consumption low $(1/c_t^h(t+1) = 0.45)$

- Similarly, it pays out little in a recession when consumption is low

$$c_t^h(t+1) = \omega_2^R + \left[p + d + \sigma^R\right] = 1 - 1/5 = 0.8$$

and marginal utility of consumption high $(1/c_t^h(t+1) = 1.25)$

- Thus, the covariance is negative

asset provides no insurance against consumption fluctuations → only reason to hold is if return high

# Consumption-based Capital Asset Pricing

## Example: "Unemployment" Insurance

- Assume that $\omega_1 = 3.077$
- Consider an asset that pays

$$\varepsilon_t \in \begin{cases} -1/5 & \text{if economy is in a boom (state } B\text{)} \\ 1/5 & \text{if economy is in a recession (state } R\text{)} \end{cases}$$

# Consumption-based Capital Asset Pricing

## Example: "Unemployment" Insurance

- Guess that price $p = 1$ (turns out to be correct in this case)
- Can then solve for $r$ using equation (9)

$$\begin{aligned} \frac{1}{r} &= \beta \frac{1}{2} \frac{\omega_1 - p}{\omega_2^B + [p + d + \sigma^B]} + \beta \frac{1}{2} \frac{\omega_1 - p}{\omega_2^R + [p + d + \sigma^R]} \\ &= \frac{2}{3} \frac{1}{2} \frac{3.077 - 1}{1 + 1 - 1/5} + \frac{2}{3} \frac{1}{2} \frac{3.077 - 1}{1 + 1/5} = 0.962 \end{aligned}$$

so that

$$r = 1.04$$

- The equity premium is then

$$
\begin{aligned}
\hat{r} &= E\left\{\frac{p+d+\varepsilon(t+1)}{p} - r\right\} \\
&= \frac{1}{2}\frac{\left[p+d+\sigma^B\right]}{p} + \frac{1}{2}\frac{\left[p+d+\sigma^R\right]}{p} - r \\
&= \frac{1}{2}\frac{[1-1/5]}{1} + \frac{1}{2}\frac{[1+1/5]}{1} - 1.04 \\
&= -0.04
\end{aligned}
$$

So, the equity premium is $-4$ percent

# Consumption-based Capital Asset Pricing

## Example: "Unemployment" Insurance

- This is an asset that pays out little in a boom when consumption is high

life insurance

$$c_t^h(t+1) = \omega_2^B + \left[p + d + \sigma^B\right] = 1 + 1 - 1/5 = 1.8$$

and marginal utility of consumption low $(1/c_t^h(t+1) = 0.55)$

- Similarly, it pays out more in a recession when consumption is low

$$c_t^h(t+1) = \omega_2^R + \left[p + d + \sigma^R\right] = 1 + 1/5 = 1.2$$

hold even if return lower than for safe asset

and marginal utility of consumption high $(1/c_t^h(t+1) = 0.8)$

- Thus, the covariance is positive

asset provides insurance against consumption fluctuations

# Consumption-based Capital Asset Pricing

## How big is the equity premium in general?

- To connect with data, assume

$$u_t^h = \frac{\left[c_t^h(t)\right]^{1-\mu}}{1-\mu} + \beta \frac{\left[c_t^h(t+1)\right]^{1-\mu}}{1-\mu}$$

  - $\mu \geq 0$ is coefficient of relative risk aversion
  - If $\mu = 0$ then utility is linear and agents are risk neutral
  - If $\mu > 0$ then agents are risk averse

- With this utility specification, equity premium on an asset depends positively on coefficient of relative risk aversion and and how strongly return on assets covaries with consumption growth

- Can model fit the data?

# Consumption-based Capital Asset Pricing

## Fitting Data?

- To match equity premium, coefficient of relative risk aversion would need to be huge
  - Not empirically plausible
- This is known as equity premium puzzle
  - First pointed out by Mehra and Prescott (1985)

# Consumption-based Capital Asset Pricing

## Bringing Model in Line with Data?

Additional features intended to bring model more in line with data

- Incomplete markets
- Transaction costs
- Life cycle
- Habit formation

Incomplete markets

- Risk that can't be fully insured against
- Induces precautionary savings motive
- Agents willing to hold risk free bonds at lower rate than in complete markets economy

Transaction costs

- Costly to trade stocks but not risk free asset
- No arbitrage implies that in expected terms return to bonds and return to stocks minus transaction costs must be the same
- For individuals to hold both equity and bonds, return to stocks must increase relative to world without transaction costs

# Consumption-based Capital Asset Pricing

## Life Cycle

- Attractiveness of equity depends on correlation between consumption and equity income
- Correlation not constant over life cycle
- Young individuals would like to hold more equity, but credit constrained
- Marginal investor is middle-aged

# Consumption-based Capital Asset Pricing

## Life Cycle

- Young individuals
  - Face lots of uncertainty regarding labor earnings, but less regarding equity income (since they have little equity )
  - As long as correlation between earnings and equity income low (true in data), makes sense to use equity income to insure wage fluctuations
  - Equity is then desired and individuals will buy it even at a fairly low return
- Middle-aged individuals
  - Wage uncertainty has been resolved
  - Consumption uncertainty then goes hand in hand with uncertainty regarding equity income
  - For these individuals to buy equity, must have high return

Habit formation

- Past consumption represents consumer's stock of habit
- Dislike variations in habit-adjusted consumption (rather than variations in consumption itself)
- A given percentage change in consumption produces much larger percentage change in habit-adjusted consumption than in consumption itself
- This type of utility function essentially makes people more risk averse