5303 Advanced Macroeconomics - Group 3
Assignment 4

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Homework # 4

① $u(c_t, h_t) = \gamma_{1t} c_t^{\alpha_1} - \gamma_{2t} h_t^{\alpha_2} \qquad 0 < \alpha_1 < 1, \ \alpha_2 > 1$

lifetime BC: $A_0 + \sum_{t=0}^{T} \frac{1}{(1+r)^t} w_t h_t = \sum_{t=0}^{T} \frac{1}{(1+r)^t} c_t$

ⓐ Agents' problem:

$$\max_{\{c_t, h_t\}} \sum_{t=0}^{T} \beta^t (\gamma_{1t} c_t^{\alpha_1} - \gamma_{2t} h_t^{\alpha_2}) \text{ s.t. } A_0 + \sum_{t=0}^{T} \frac{1}{(1+r)^t} w_t h_t = \sum_{t=0}^{T} \frac{1}{(1+r)^t} c_t$$

Lagrangian:

$$\mathcal{L}: \sum_{t=0}^{T} \beta^t (\gamma_{1t} c_t^{\alpha_1} - \gamma_{2t} h_t^{\alpha_2}) + \mu \left(A_0 + \sum_{t=0}^{T} \frac{1}{(1+r)^t} w_t h_t - \sum_{t=0}^{T} \frac{1}{(1+r)^t} c_t\right)$$

FOCs: $\forall t \geq 0$

w.r.t: h_t: $-\beta^t \gamma_{2t} \alpha_2 h_t^{\alpha_2 - 1} + \mu \frac{1}{(1+r)^t} w_t = 0$

w.r.t c_t: $\beta^t \gamma_{1t} \alpha_1 c_t^{\alpha_1 - 1} - \mu \frac{1}{(1+r)^t} = 0$

w.r.t μ: $A_0 + \sum_{t=0}^{T} \frac{1}{(1+r)^t} w_t h_t - \sum_{t=0}^{T} \frac{1}{(1+r)^t} c_t = 0$

ⓑ Deriving the labor supply function:

Using the first FOC:

$$h_t = \left(\frac{\mu \frac{1}{(1+r)^t} w_t}{\beta^t \gamma_{2t} \alpha_2}\right)^{\frac{1}{\alpha_2 - 1}}$$

ⓒ Marginal utility of wealth

By the Envelope Theorem, the marginal utility of wealth is

$$\frac{\partial \mathcal{L}}{\partial A_0} = \mu.$$

ⓓ derive ies:

We first take logs of both sides of the labor supply function:

$$\ln h_t = \frac{1}{\alpha_2 - 1}\left[\ln \mu + t \ln\left(\frac{1}{1+r}\right) + \ln w_t - t \ln \beta - \ln \gamma_{2t} - \ln \alpha_2\right]$$

And $\text{ies} = \frac{\partial \ln h_t}{\partial \ln w_t} = \frac{1}{\alpha_2 - 1}$

the i.e.s. measures how agents respond to small increases in their wage (here, since $\alpha_2 > 1$, we say agents will increase their hours worked in response to increased wages). Importantly, the i.e.s. holds wealth constant!

(e) To estimate the parameters of the labor supply function, it is useful to define $\beta = \frac{1}{1+\rho}$.

Then from (d) we have

$$\ln h_t = \frac{1}{\alpha_2 - 1}\left[\ln \mu + t \ln \frac{1}{1+r} + \ln w_t + t\ln(1+\rho) - \ln \gamma_{2t} - \ln \alpha_2\right]$$

Taking the first difference yields

$$\Delta \ln h_t = \frac{1}{\alpha_2 - 1}\left[\ln \frac{1}{1+r} + \ln(1+\rho) - \Delta \ln \gamma_{2t}\right]$$

And we can approximate $\ln(1+\rho) - \ln(1+r) \approx \rho - r$

$$\Delta \ln h_t = \frac{1}{\alpha_2 - 1}\left[\rho - r - \Delta \ln \gamma_{2t}\right].$$

Now, it would be possible to gather data to use in a regression to estimate the parameters! It would be wise to consider an IV model to deal with measurement error, or consider that estimates at the end may suffer from measurement bias. MaCurdy's instruments include family background, education and age.

(f) Estimating in this way yields small estimates for i.e.s. between ~.10 and ~.20 in MaCurdy's paper. The estimates are small because the slope of the wage path is three times that of hours. Among other critiques, Domeij and Floden (2006) show that the estimates are biased downward because borrowing constraints are not accounted for. Similarly, omitting human capital accumulation (Wallenius 2011) and ignoring extensive supply margins (Rogerson and Wallenius 2009) similarly bias estimates downward.

2. Consider the following two-period OLG model with human capital. The number of young agents at any date t is fixed at N.

Agent preferences are described by $u_t^h = \ln c_t^h(t) + \ln c_t^h(t+1)$

The production function is given by $Y_t = S_t^\gamma L_t^{1-\gamma}$

where L_t is the labour of young (unskilled) workers and S_t is the labour of the old (skilled) workers.

When young, agents spend a fraction m_t of their time working and the remaining time $1 - m_t$ getting an education (acquiring human capital).

Their human capital accumulation is described by $h_{t+1} = h_t + (1 - m_t)\theta h_t$

whre h_t is human capital of the current old, h_{t+1} is human capital acquired by the current young, and $\theta > 1$ is a parameter.

The wage per unit of time worked when young is w_t. When young, agents cannot use the human capital they have acquired in that period.

When old, agents just work and earn a wage of v_t per unit of human capital they acquired when young. All markets are perfectly competitive.

(a) Describe the key feature(s) of the human capital accumulation equation.

Each young agent at time t acquires the human capital provided by the old agents at time and an additional fraction of the same old agents' human capital determined by the amount of time the individual spends on getting an education (learning from old agents). Agents can only use the human capital they acquired when young once they are old.

$$h_t(t) = h_{t-1}(t)\left[1 + (m_t(t) - 1)\,\theta\right] \qquad (1)$$

(b) Solve for the agent's optimal consumption, savings and human capital investment profile.

denote $x_t(t-1) \equiv x_{t-1}$, $x_t(t) \equiv x_t$, $x_t(t+1) \equiv x_{t+1}$ for all variables to simplify notation

$$\max_{\{c_t,\, c_{t+1},\, m_t\}} \ln c_t + \ln c_{t+1} \quad \text{s.t.} \quad c_t = w_t m_t, \quad c_{t+1} = v_{t+1} h_t$$

$$\mathcal{L} \equiv \ln c_t + \ln c_{t+1} + \mu_t (w_t m_t - c_t) + \mu_{t+1}(v_{t+1} h_t - c_{t+1})$$

FOC

c_t: $\quad c_t^{-1} - \mu_t = 0 \quad \forall t \geq 1$

c_{t+1}: $\quad c_{t+1}^{-1} - \mu_{t+1} = 0 \quad \forall t \geq 1$

m_t: $\quad \mu_t w_t - \mu_{t+1} v_{t+1} \theta h_{t-1} = 0 \quad \forall t \geq 1$ using (1)

$$\to \quad \frac{c_{t+1}}{c_t} = \frac{v_{t+1}}{w_t} \theta h_{t-1}$$

BC: $$\frac{v_{t+1} h_{t-1}[1 + (1 - m_t)\theta]}{w_t m_t} = \frac{v_{t+1} h_{t-1}}{w_t}\theta \quad / \cdot \frac{w_t m_t}{v_{t+1} h_{t-1}}$$

$$1 + \theta - \theta m_t = \theta m_t$$

$\to \forall t \geq 1$:

$$m_t = (1+\theta)/2\theta \qquad (2)$$
$$h_t = (1+\theta) h_{t-1}/2 \qquad (3)$$
$$c_t = (1+\theta) w_t / 2\theta$$
$$c_{t+1} = (1+\theta) v_{t+1} h_{t-1}/2$$

(c) What is the growth rate for output in this economy? (Hint: $S_t = N h_t$ and $L_t = N m_t$)

$$\frac{Y_{t+1}}{Y_t} = \underbrace{\left(\frac{N h_t}{N h_{t-1}}\right)^\gamma}_{\left(\frac{1+\theta}{2}\right)^\gamma \text{ by } (3)} \underbrace{\left(\frac{N m_{t+1}}{N m_t}\right)^{1-\gamma}}_{=1 \text{ by } (2)}$$

$$\to \quad \frac{Y_{t+1}}{Y_t} = \left(\frac{1+\theta}{2}\right)^\gamma \quad \forall t \geq 1$$

(d) Define a competitive equilibrium for this economy.

A competitive equilibrium is a sequence of allocations $\{c_{t(t)}, c_{t(t+1)}, m_{t(t)}, h_{t(t)}\}_{t=1}^{\infty}$ and prices $\{w(t), v(t)\}_{t=1}^{\infty}$ *such that:*

(i) Given prices and the human capital accumulation equation, the allocation solves the individual's utility maximization problem.

(ii) Given wages, the firm's profit function is maximized.

(iii) All markets clear.

(e) Suppose the government initiates a mandatory PAYG social security program, taxing the young a lump-sum amount T_1 and returning the old a lump-sum amount T_2. Will countries that run such a program invest more in education than before? Show all steps. Provide a clear intutition for your answer.

new BC: $$c'_t = c_t - T_1$$
$$c'_{t+1} = c_{t+1} + T_2$$

no population growth & balanced government budget in equilibrium $\implies T_1 = T_2 \equiv T$

from (b).

$$\frac{v_{t+1} h_{t-1}[1 + (1 - m_t)\theta] + T}{w_t m_t - T} = \frac{v_{t+1} h_{t-1}}{w_t}\theta$$

$$\to \quad m'_t = \frac{1+\theta}{2\theta} + T\left[\frac{w_t + v_{t+1}\theta h_{t-1}}{2 v_{t+1} h_{t-1} \theta w_t}\right] \quad \forall t \geq 1$$

$> m_t$ since $T, \theta > 0$ by assumption and $w_t, v_{t+1}, h_{t-1} > 0$ in equilibrium

$\to$ countries invest less in education

Intuition:

Agents smooth consumption, taxation scheme therefore disincentivises young to invest current labor hours into learning.

Question 3. Each generaton consists of 50 type H individuals with a high earnings profile, $\omega = (3, 1)$ and 50 type L individuals with a low earnings profile, $\omega = (\frac{9}{4}, 1)$. Each individual has preferences

$$u = \ln c_t^h(t) + \beta c_t^h(t+1),$$

with $\beta = \frac{2}{3}$. When young, each individal can invest in accumulating human capital using a linear technology where 1 unit of endowment can be converted to 1 unit of human capital k_t^h. When old each individual can use his/her human capital to produce the consumption good according to the following production function,

$$y(t+1) = k_t^i k_t^j.$$

1. Sovle for the amount of human capital accumulated by each type of individual. Is there a coordination problem?
2. Suppose the government wants to increase the level of schooling and introduces a subsidy a rate $\theta = 0.25$ on human capital accumulation. Assume the subsidy is financed by a proportional tax τ on the endowments of the old. Did the government succeed in its goal of increasing the level of schooling?
3. Would a mandatory schooling system have a different effect?

Solution. a) To reduce the amount of calculations, we derive the optimal solution with the subsidy in question a) setting it equal to 0. Individuals of each type maximizes

$$\begin{aligned} \max \quad & \ln c_t^h(t) + \beta c_t^h(t+1) \\ \text{s.t} \quad & c_t^h(t) \leq \omega_t^h(t) - (1-\theta)k_t^h \\ & c^h(t+1) \leq (1-\tau)w_t^h(t+1) + k_t^h k_t^j \end{aligned}$$

The problem lagrangian is

$$L = \ln c_t^h(t) + \beta c_t^h(t+1) - \mu_t(c_t^h(t) - \omega_t^h(t) + (1-\theta)k_t^h) - \mu_{t+1}(c^h(t+1) - (1-\tau)w_t^h(t+1) - k_t^h k_t^j).$$

The consumer chooses period t consumption, period $t+1$ consumption and human capital investment k as to maximize lifetime utility. The first order conditions are

$$\frac{1}{c_t^h(t)} - \mu_t = 0,$$

$$\beta - \mu_{t+1} = 0,$$

$$-(1-\theta)\mu_t + \mu_{t+1}k_t^j = 0.$$

Note that k^j is not a decision variable for the individual but k^h is. First solve for μ_t and μ_{t+1} and substitute into the third equation, then substitute the resulting $c_t^h(t)$ expression into the period t budget constraint. This yields the equilibrium condition

$$\frac{1-\theta}{(\omega_t^h(t) - (1-\theta)k_t^h)k_t^j} = \beta.$$

The condition holds true for the high and low income individuals. Setting them equal results in

$$k_t^L(\omega_t^H(t) - (1-\theta)k_t^H) = k_t^H(\omega_t^L(t) - (1-\theta)K_t^L),$$

$$k_t^L = \frac{\omega_t^L(t)k_t^H}{\omega_t^H(t)}.$$

Substituting the equation into the equilibrium condition, plugging in values for β and ω, then solve, gives us two condition

$$k_t^L = \frac{3}{4} k_t^H,$$

$$(k_t^H)^2 - \frac{3}{1-\theta} k_t^H + 2 = 0.$$

Setting $\theta = 0$ we get

$$(k_t^H)^2 - 3k_t^H + 2 = 0.$$

Complete the square and solve for k_t^H this yields the two solutions pairs for (k_t^H, k_t^L) equal to $(1, \frac{3}{4})$ and $(2, \frac{3}{2})$. This becomes a coordination problem since the $(2, \frac{3}{2})$ solution results in higher utility for both types.

b) There is now a subsidy $\theta = \frac{1}{4}$ and a tax τ to finance it. A balanced government budget requires that the tax revenue equals the spending on the subsidies

$$\theta(k_t^H + k_t^L) = \tau(\omega_t^H(t) + \omega_t^L(t)).$$

Subsituting the expression for low income workers human capital investment and solving for the tax gives us

$$\theta(k_t^H + \frac{3}{4} k_t^H) = 2\tau$$

$$\tau = \frac{7}{8} k_t^H \theta.$$

For the budget to balance τ has to be set according to the above expression. The before derived human capital investment with $\theta = \frac{1}{4}$ is

$$(k_t^H)^2 - 4k^h t + 2 = 0.$$

It has the two solution $2 - \sqrt{2}$ and $2 + \sqrt{2}$, implying the low income people attain k_t^L either equal to $(6 - 3\sqrt{2})/4 \approx 0.43$ or $(6 + 3\sqrt{2})/4 \approx 2.56$. Comparing the result to the solution in part a) we can not conclude if the subsidy was successful in increasing the level of education.

c) If the government sets a mandatory k for both high and low income individuals equal to the high education equilibrium for the low income individuals, $\frac{3}{4}$ there would be an increase in schooling and a welfare improvement.