Problem Set 5

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Exercise 1

$$\max f(x_1, x_2) = \sqrt{(x_1+1)(x_2+1)}$$
$$\text{s. t. } x_2 - (x_1-1)^2 \leq 0,\ x_1 + x_2 \leq 7,\ x_1, x_2 \geq 0$$

Goal function f is a composition of continuous functions, hence continuous. The feasible set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\} \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\} \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \leq 7\} \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 - (x_1-1)^2 \leq 0\}$ is nonempty: it contains $(0,0)$, closed, since it is the intersection of four closed sets, and bounded: $(0,0) \leq (x_1, x_2) \leq (7,7)$. So by the Heine-Borel theorem, the feasible set is compact. By the Extreme Value Theorem, a maximum exists. The inequalities in standard form are $h_1(x) = -x_1, h_2(x) = -x_2, h_3(x) = x_1 + x_2 - 7$ and $h_4(x) = x_2 - (x_1-1)^2$ with gradients $\nabla h_1(x) = (-1, 0), \nabla h_2(x) = (0, -1), \nabla h_3(x) = (1, 1), \nabla h_4(x) = (-2(x_1-1), 1)$, note that only $\nabla h_4(x)$ depends on x.

First, we find the set X_{LD} of feasible points where the gradients of the binding constraints are linearly dependent. In feasible points where one constraint h_j is binding, the corresponding gradient $\nabla h_i(x)$ is distinct from the zero vector, so the set $\nabla h_i(x)$ is linearly independent: no such point belongs to X_{LD}. In feasible points where any two of the constraints $h_1(x), h_2(x), h_3(x)$ are binding, the set $\nabla h_i(x), \nabla h_j(x)$ is linearly independent: no such point belongs to X_{LD}. Now, consider feasible points where $h_4(x)$ and any of the constraints $h_1(x), h_2(x), h_3(x)$ are binding. If $h_4(x)$ and $h_1(x)$ are binding, $x_1 = 0, x_2 = 1 : \nabla h_4(x) = (2, 1)$, if $h_4(x)$ and $h_2(x)$ are binding, $x_1 = 1, x_2 = 0 : \nabla h_4(x) = (0, 1)$, and if $h_4(x)$ and $h_3(x)$ are binding, $x_1 = 3, x_2 = 4 : \nabla h_4(x) = (-8, 1)$. The set of gradients of the binding constraints $\nabla h_i(x), \nabla h_4(x)$, $i = 1, 3$ is linearly independent, since solving $\alpha \nabla h_i(x) + \beta \nabla h_4(x) = (0, 0)$ gives $\alpha = \beta = 0$. At $x = (1, 0)$ is the set of gradients of h_2, h_4 linearly dependent, since $(0, -1) + (0, 1) = (0, 0)$. So $(1, 0) \in X_{LD}$. If more than two constraints to bind, the feasible set is empty: there are no points X_{LD} in such cases. Conclude: only the point $(1, 0)$, where gradients h_2, h_4 are binding, belongs to X_{LD}.

Next, we find X_{KKT} , the set which satisfies the KKT conditions. The Lagrangian is

$$\mathcal{L}(x,\mu) = \sqrt{(x_1+1)(x_2+1)} - \mu_1(-x_1) - \mu_2(-x_2) - \mu_3(x_1+x_2-7) - \mu_4(x_2-(x_1-1)^2)$$

In a local maximum x, the following KKT-conditions must hold:

$$\frac{\sqrt{x_2+1}}{2\sqrt{x_1+1}} + \mu_1 - \mu_3 + 2\mu_4(x_1-1) = 0 \quad (1)$$
$$\frac{\sqrt{x_1+1}}{2\sqrt{x_2+1}} + \mu_2 - \mu_3 - \mu_4 = 0 \quad (2)$$
$$-x_1 \le 0 \quad (3)$$
$$-x_2 \le 0 \quad (4)$$
$$x_1 + x_2 - 7 \le 0 \quad (5)$$
$$x_2 - (x_1-1)^2 \le 0 \quad (6)$$
$$\mu_1, \mu_2, \mu_3, \mu_4 \ge 0 \quad (7)$$
$$-\mu_1 x_1 = 0 \quad (8)$$
$$-\mu_2 x_2 = 0 \quad (9)$$
$$\mu_3(x_1+x_2-7) = 0 \quad (10)$$
$$\mu_4(x_2-(x_1-1)^2) = 0 \quad (11)$$

Distinguish cases based on whether the nonnegativity constraints h_1, h_2 are binding:

A) $-x_1 = 0$

I. $-x_2 = 0$
By (10), (11) $\mu_3 = \mu_4 = 0$, and by (1), (2) $\mu_1 = \mu_2 = -\frac{1}{2}$, contradicting (7).

II. $-x_2 < 0$, $\mu_2 = 0$
By (10), (11) either $\mu_3 > 0$ or $\mu_4 > 0$: either $x_1 + x_2 = 7$ or $x_2 = (x_1-1)^2$. $x_1 + x_2 = 7 \implies x_2 = 7, \mu_4 = 0$. By (1), (2) $\mu_3 = \frac{\sqrt{2}}{8}$, $\mu_1 = -\frac{7\sqrt{2}}{8}$, contradicting (7). $x_2 = (x_1-1)^2 \implies x_2 = 1, \mu_3 = 0$. By (2) $\mu_4 = \frac{\sqrt{2}}{4}$, by (1) $\mu_1 = 0$. Candidate $x = (0,1) \in X_{KKT}$.

B) $-x_1 < 0$, $\mu_1 = 0$

I. $-x_2 = 0$
By (10), (11) either $\mu_3 > 0$ or $\mu_4 > 0$: either $x_1 + x_2 = 7$ or $x_2 = (x_1-1)^2$. $x_1 + x_2 = 7 \implies x_1 = 7, \mu_4 = 0$. By (1) $\mu_3 = \frac{\sqrt{2}}{8}$, by (2) $\mu_2 = -\frac{7\sqrt{2}}{8}$, contradicting (7). $x_2 = (x_1-1)^2 : x_1 = 1, \mu_3 = 0$. By (1) $0 = \frac{\sqrt{2}}{4}$, contradiction.

II. $-x_2 < 0$, $\mu_2 = 0$
$x_1 + x_2 = 7$. If $x < (x_1-1)^2, \mu_4 = 0$: By (1), (2) $x_1 = x_2 = \frac{7}{2}$ and $\mu_3 = \frac{1}{2}$. If $x_2 = (x_1-1)^2 : x_1 = 3, x_2 = 4$, by (1), (2) $\mu_4 = -\frac{1}{20\sqrt{5}}$, contradicting (7).

Thus for $x_2 = (x_1-1)^2, x_1 + x_2 < 7, \mu_3 = 0$. By (1), (2) $x_1 = \frac{2}{3}, x_2 = \frac{1}{9}, \mu_4 \frac{\sqrt{6}}{4}$. By (8), (9) $\mu_1 = \mu_2 = 0$. Candidates $\{(\frac{7}{2}, \frac{7}{2}), (\frac{2}{9}, \frac{1}{9})\} \in X_{KKT}$.

Comparing all solution candidates $X_{LD} = (1,0), X_{KKT} = \{(0,1), (\frac{7}{2}, \frac{7}{2}), (\frac{2}{3}, \frac{1}{9})\}$, we find maximal value 4.5 at $x^* = (x_1, x_2) = \left(\frac{7}{2}, \frac{7}{2}\right)$.

Exercise 2

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{t=0}^{T}(x(t)^2\sqrt{u(t)}) \\ \text{with} & u(t) \in [0,\beta],\, t = 0,1,\ldots,T \\ & x(t+1) = \alpha u(t)x(t),\, t = 0,1,\ldots,T-1 \\ & x(0) = x_0 \end{array}$$

(a) In the final period T , if the state x, then

$$J_T(x) = \sup_{u\in[0,\beta]} f(T,x,u) = \sup_{u\in[0,\beta]} x^2\sqrt{u} = x^2\sqrt{\beta}$$

with set of optimal controls $[0,\beta]$ if $x = 0$ and $\{\beta\}$ if $x \neq 0$. In period $T-1$

$$\begin{aligned} J_{T-1} &= \sup_{u\in[0,\beta]} f(T-1,x,u) + J_T(\alpha u(t)x(t)) \\ &= \sup_{u\in[0,\beta]} x^2\sqrt{u} + (\alpha u x)^2\sqrt{\beta} = \sup_{u\in[0,\beta]} x^2(\sqrt{u} + \alpha^2u^2\sqrt{\beta}) \\ &= x^2(\sqrt{\beta} + \alpha^2u^2\sqrt{\beta}) = x^2\sqrt{\beta}(1+(\alpha\beta)^2) \end{aligned}$$

with set of optimal controls $[0,\beta]$ if $x = 0$ and $\{\beta\}$ if $x \neq 0$. In period $T-2$

$$\begin{aligned} J_{T-2} &= \sup_{u\in[0,\beta]} f(T-2,x,u) + J_{T-1}(\alpha u(t)x(t)) \\ &= \sup_{u\in[0,\beta]} x^2\sqrt{u} + (\alpha u x)^2\sqrt{\beta}(1+\alpha^2u^2) = \sup_{u\in[0,\beta]} x^2(\sqrt{u} + (1+\alpha^2u^2)\alpha^2u^2\sqrt{\beta}) \\ &= x^2(\sqrt{\beta} + \alpha^2\beta^2\sqrt{\beta}(1+\alpha^2\beta^2) = x^2\sqrt{\beta}(1+(\alpha\beta)^2+(\alpha\beta)^4) \end{aligned}$$

with set of optimal controls $[0,\beta]$ if $x = 0$ and $\{\beta\}$ if $x \neq 0$.

(b) J_{T-t} is of the form $J_{T-t}(x) = x^2\sqrt{\beta}\sum_{s=0}^{t}(\alpha\beta)^{2s}$. From (a), this holds for $t = 0$. Induction step from t to $t+1$

$$\begin{aligned} J_{T-(t+1)} &= \sup_{u\in[0,\beta]} f(T-(t+1),x,u) + J_{T-t}(\alpha u(t)x(t)) \\ &= \sup_{u\in[0,\beta]} x^2\sqrt{u} + (\alpha u x)^2\sqrt{\beta}\sum_{s=0}^{t}(\alpha\beta)^{2s} = \sup_{u\in[0,\beta]} x^2(\sqrt{u} + \alpha^2u^2\sqrt{\beta}\sum_{s=0}^{t}(\alpha\beta)^{2s}) \\ &= x^2(\sqrt{\beta} + (\alpha\beta)^2\sqrt{\beta}\sum_{s=0}^{t}(\alpha\beta)^{2s}) = x^2\sqrt{\beta}(1+(\alpha\beta)^2\sum_{s=0}^{t}(\alpha\beta)^{2s}) = x^2\sqrt{\beta}\sum_{s=0}^{t+1}(\alpha\beta)^{2s} \end{aligned}$$

The optimal value of the goal function is $J_0(x_0) = x^2\sqrt{\beta}\sum_{s=0}^{T}(\alpha\beta)^{2s} = x_0^2\sqrt{\beta}\frac{1-(\alpha\beta)^{2(T+2)}}{1-(\alpha\beta)^2}$ and since $x_0 > 0 \implies \forall t : x(t) > 0$, the set of optimal controls is $\{\beta\}$ at all t.