3.44] Derive the profit function for a firm with the Cobb-Douglas technology $y = x_1^{\alpha} x_2^{\beta}$.

$$\pi(p, \vec{w}) \equiv \max_{(\vec{x}, y) \geq 0} \quad p \cdot y - \vec{w} \cdot \vec{x} \quad \text{s.t. } f(\vec{x}) \geq y$$

$$\mathcal{L} \equiv p \cdot y - (w_1 x_1 + w_2 x_2) - \lambda [y - f(\vec{x})]$$

$$\equiv p \cdot y - (w_1 x_1 + w_2 x_2) - \lambda [y - x_1^{\alpha} x_2^{\beta}]$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -w_1 + \lambda \alpha x_1^{\alpha - 1} x_2^{\beta} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -w_2 + \lambda x_1^{\alpha} \beta x_2^{\beta - 1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = p - \lambda = 0 \qquad \Rightarrow \quad p = \lambda > 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -[y - x_1^{\alpha} x_2^{\beta}] = 0 \qquad \Rightarrow \quad y = x_1^{\alpha} x_2^{\beta}$$

$$\Rightarrow \quad w_1 = \lambda \alpha x_1^{\alpha - 1} x_2^{\beta}$$

$$w_2 = \lambda \beta x_1^{\alpha} x_2^{\beta - 1}$$

$$\boxed{\frac{w_1}{w_2} = \frac{\alpha}{\beta} \frac{x_2}{x_1}} \qquad \boxed{\frac{x_2}{x_1} = \frac{\beta}{\alpha} \frac{w_1}{w_2}}$$

$$y = x_1^{\alpha} x_2^{\beta} = x_1^{\alpha}\left(x_1 \frac{\beta}{\alpha}\frac{w_1}{w_2}\right)^{\beta}$$

$$= x_1^{\alpha+\beta}\left(\frac{w_1}{\alpha}\right)^{\beta}\left(\frac{w_2}{\beta}\right)^{-\beta}$$

$$x_1^{\alpha+\beta} = y\left(\frac{w_1}{\alpha}\right)^{-\beta}\left(\frac{w_2}{\beta}\right)^{\beta}$$

$$\boxed{X_1 = y^{\frac{1}{\alpha+\beta}}\left(\frac{w_1}{\alpha}\right)^{\frac{-\beta}{\alpha+\beta}}\left(\frac{w_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} = x_1(y, w_1, w_2)}$$ conditional input demand

$$X_2 = x_1 \frac{\beta}{\alpha}\frac{w_1}{w_2} = y^{\frac{1}{\alpha+\beta}}\left(\frac{w_1}{\alpha}\right)^{\frac{-\beta}{\alpha+\beta}}\left(\frac{w_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\frac{w_1}{\alpha}\left(\frac{w_2}{\beta}\right)^{-1}$$

$$\boxed{X_2 = y^{\frac{1}{\alpha+\beta}}\left(\frac{w_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{w_2}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}}$$ conditional input demand

$$w_1 = \lambda\alpha x_1^{\alpha-1}x_2^{\beta} \qquad p = \lambda$$

$$w_1 = p\alpha\left[y^{\frac{1}{\alpha+\beta}}\left(\frac{w_1}{\alpha}\right)^{\frac{-\beta}{\alpha+\beta}}\left(\frac{w_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\right]^{\alpha-1} \cdot \left[y^{\frac{1}{\alpha+\beta}}\left(\frac{w_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{w_2}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}\right]^{\beta}$$

$$w_1 = p\cdot\alpha\cdot y^{\frac{\alpha+\beta-1}{\alpha+\beta}}\cdot\left(\frac{w_1}{\alpha}\right)^{\frac{\beta}{\alpha+\beta}}\left(\frac{w_2}{\beta}\right)^{\frac{-\beta}{\alpha+\beta}}$$

Need $\alpha+\beta<1$ to get $p\uparrow \Rightarrow y\uparrow$ holding w_1, w_2 fixed
<u>upward sloping firm supply.</u>

$$y^{\frac{1-\alpha-\beta}{\alpha+\beta}} = \left(\frac{p\alpha}{w_1}\right)\left(\frac{w_1}{\alpha}\right)^{\frac{\beta}{\alpha+\beta}}\left(\frac{w_2}{\beta}\right)^{\frac{-\beta}{\alpha+\beta}}$$

$$\boxed{y = \left(\frac{p\alpha}{w_1}\right)^{\frac{\alpha+\beta}{1-\alpha-\beta}}\left(\frac{w_1}{\alpha}\right)^{\frac{\beta}{1-\alpha-\beta}}\left(\frac{w_2}{\beta}\right)^{\frac{-\beta}{1-\alpha-\beta}}}$$

output supply function

$y = y(p, w_1, w_2)$

We need $\alpha+\beta<1$ to get well-defined output supply function

$$\boxed{y = p^{\frac{\alpha+\beta}{1-\alpha-\beta}}\left(\frac{w_1}{\alpha}\right)^{\frac{-\alpha}{1-\alpha-\beta}}\left(\frac{w_2}{\beta}\right)^{\frac{-\beta}{1-\alpha-\beta}}}$$

4.11 $c(q) = k + c \cdot q \quad k > 0.$

(a) Suppose J firms, all symmetric, Cournot equilibrium

$$p = a - b\left(\sum_{j=1}^{J} q^j\right)$$

$$\pi^j = p \cdot q^j - (k + c \cdot q^j)$$

$$= \left[a - b\left(\sum_{k=1}^{J} q^k\right)\right] q^j - k - c q^j$$

$$\frac{\partial \pi^j}{\partial q^j} = \left(a - b\sum_{k=1}^{J} q^k\right) + q^j(-b) - c$$

$$= a - c - b\sum_{k \neq j} q^k - 2b q^j \stackrel{?}{=} 0$$

$$\Longrightarrow \quad 2b q^j = a - c - b\sum_{k \neq j} q^k$$

$$2b q^j = a - c - b(J-1) q^j$$

$$\left[2b + b(J-1)\right] q^j = a - c$$

$$\boxed{q^j = \frac{a-c}{b\left[2 + J - 1\right]} = \frac{a-c}{b(J+1)} > 0 \quad \text{firm output}}$$

$$J \cdot q^j = q = J \frac{a-c}{b(J+1)} = \left(\frac{a-c}{b}\right)\frac{J}{J+1} > 0$$

market output

$$p = a - bq = a - b\left(\frac{a-c}{b}\right)\frac{J}{J+1}$$

$$p = a - (a-c)\frac{J}{J+1}$$

price $$p = \frac{(J+1)a - Ja + cJ}{J+1} = \frac{a + cJ}{J+1}$$

$$\pi^j = p \cdot q^j - k - cq^j$$

$$= (p-c)q^j - k$$

$$= \left[\frac{a+cJ}{J+1} - c\right] \cdot \frac{a-c}{b(J+1)} - k$$

$$= \frac{a + cJ - c(J+1)}{J+1} \cdot \frac{a-c}{b(J+1)} - k$$

$$\pi^j = \frac{(a-c)^2}{b(J+1)^2} - k$$ firm profits

(b) what will be the long-run equilibrium number of firms

$\pi^j > 0 \Rightarrow$ firms enter

$\pi^j < 0 \Rightarrow$ firms exit

$$\pi^j = 0 = \frac{(a-c)^2}{b(J+1)^2} - k \Rightarrow \frac{(a-c)^2}{b(J+1)^2} = k$$

$$\frac{(a-c)^2}{bk} = (J+1)^2$$

$$\left[\frac{(a-c)^2}{bk}\right]^{\frac{1}{2}} = J+1 \quad \Rightarrow \boxed{J = \frac{a-c}{(bk)^{1/2}} - 1}$$

long run equilibrium number of firms.

4.22 monopolist

$p = \alpha - \beta q$ $\qquad C = c \cdot q + F$

$\alpha > c \quad (\alpha - c)^2 > 4\beta F$

(a) $\pi = p \cdot q - C$

$= (\alpha - \beta q) \cdot q - (c \cdot q + F)$

$\pi = (\alpha - \beta q - c) q - F$

$$\frac{d\pi}{dq} = \alpha - \beta q - c + q(-\beta)$$

$$= \alpha - 2\beta q - c = 0$$

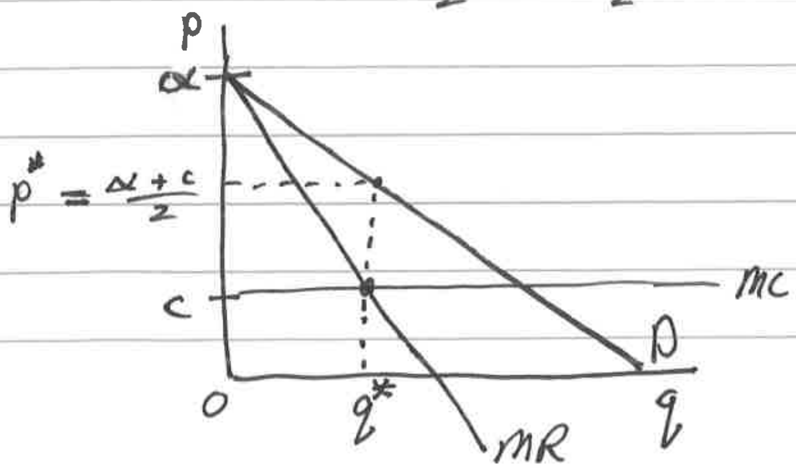
$$\frac{d^2\pi}{dq^2} = -2\beta < 0$$ second order condition satisfied.

$\Rightarrow \quad \alpha - c = 2\beta q \quad \Rightarrow \quad \boxed{\frac{\alpha - c}{2\beta} = q > 0}$

monopolist's output

$$p = \alpha - \beta q = \alpha - \beta\left(\frac{\alpha - c}{2\beta}\right)$$

$$= \alpha - \frac{\alpha}{2} + \frac{c}{2} = \frac{\alpha + c}{2} > 0$$ monopolist's price



$$\pi = p \cdot q - C$$

$$= \frac{\alpha + c}{2}\left(\frac{\alpha - c}{2\beta}\right) - \left(c\,\frac{\alpha - c}{2\beta} + F\right)$$

$$= \left(\frac{\alpha}{2} - \frac{c}{2}\right)\left(\frac{\alpha - c}{2\beta}\right) - F$$

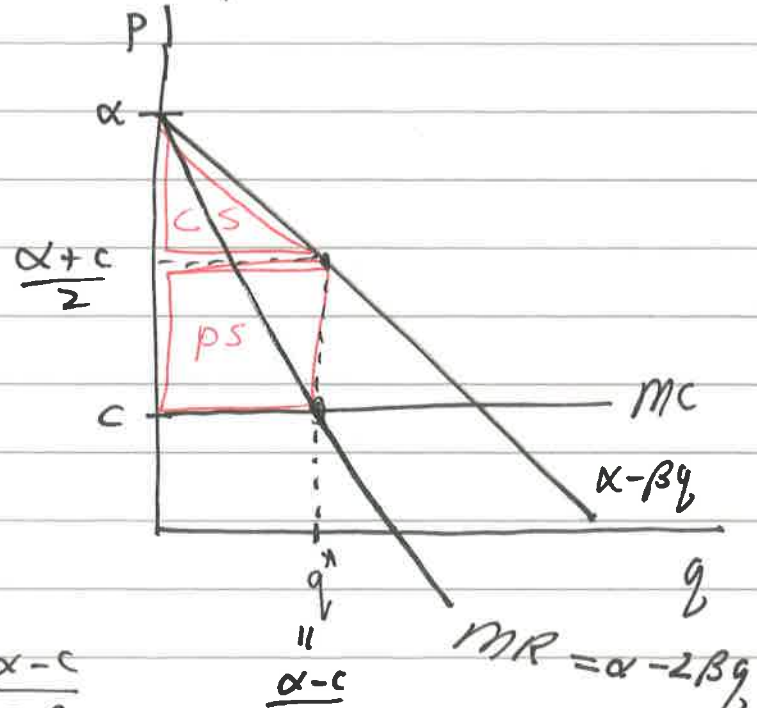
$$= \frac{(\alpha - c)^2}{4\beta} - F \overset{?}{>} 0$$ monopolist's profits

given assumption $(\alpha - c)^2 > 4\beta F$

(b) Calculate the deadweight loss

Given monopoly pricing,

$W = CS + PS$



$$CS = \frac{\left[\alpha - \frac{\alpha + c}{2}\right]\frac{\alpha - c}{2\beta}}{2}$$

$$= \frac{\left[\frac{\alpha}{2} - \frac{c}{2}\right]\left[\frac{\alpha - c}{2\beta}\right]}{2} = \frac{(\alpha - c)^2}{8\beta}$$

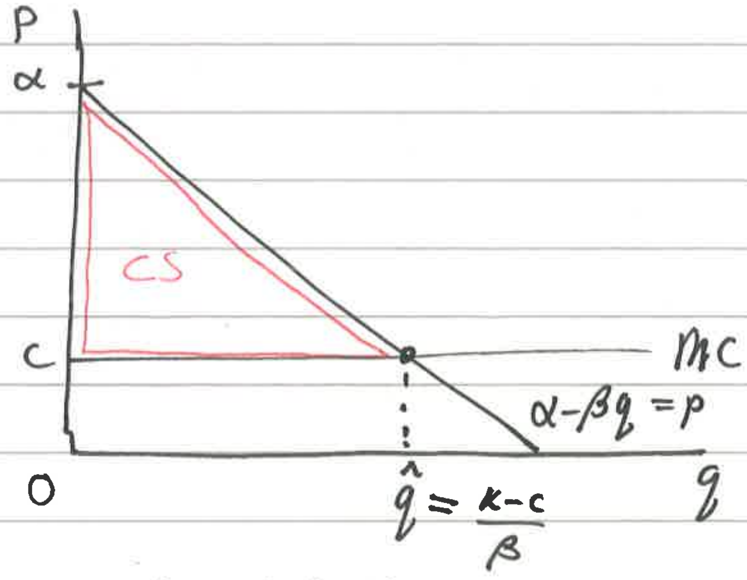
$$ps = \left[\frac{\alpha + c}{2} - c\right]\frac{\alpha - c}{2\beta} = \left[\frac{\alpha}{2} - \frac{c}{2}\right]\left[\frac{\alpha - c}{2\beta}\right]$$

$$= \frac{(\alpha - c)^2}{4\beta}$$

$$\Rightarrow W = cs + ps = \frac{(\alpha - c)^2}{8\beta} + \frac{(\alpha - c)^2}{4\beta}$$

$$W_m = \frac{(\alpha - c)^2 + 2(\alpha - c)^2}{8\beta} = \frac{3(\alpha - c)^2}{8\beta}$$ welfare under monopoly

$cs + ps$ is maximized when $p = mc$



$$\alpha - \beta\hat{q} = c$$
$$\alpha - c = \beta\hat{q}$$
$$\hat{q} = \frac{\alpha - c}{\beta}$$

$$cs = \frac{(\alpha - c)\left(\frac{\alpha - c}{\beta}\right)}{2} = \frac{(\alpha - c)^2}{2\beta}$$ $ps = 0$

$$W_{opt.} = cs + ps = \frac{(\alpha - c)^2}{2\beta} + 0 = \frac{(\alpha - c)^2}{2\beta}$$

dead weight loss $= W_{opt.} - W_m = \dfrac{(\alpha - c)^2}{2\beta} - \dfrac{3(\alpha - c)^2}{8\beta}$

$$= \frac{4 - 3}{8}\,\frac{(\alpha - c)^2}{\beta} = \frac{(\alpha - c)^2}{8\beta} > 0.$$

(c) The firm must charge the price $p = c$ to maximize $CS + PS$

$$\Rightarrow p = \alpha - \beta q = c \Rightarrow \alpha - c = \beta q$$

$$\Rightarrow \frac{\alpha - c}{\beta} = q$$

$$\Rightarrow \pi = (\alpha - \beta q - c)q - F$$

$$= \left(\alpha - \beta\left[\frac{\alpha-c}{\beta}\right] - c\right)\frac{\alpha-c}{\beta} - F$$

$$= (\alpha - \alpha + c - c)\frac{\alpha-c}{\beta} - F = -F < 0$$

Profits are negative under this regulation, so that this form of regulation is not sustainable in the long run.

5.26 Single consumer economy

$\vec{e} = (24, 0) = (h, y)$

$u(h,y) = h \cdot y$

$Y = \{(-h, y) \mid 0 \leqq h \leqq b \text{ and } 0 \leqq y \leqq \sqrt{h}\}$

$b > 0$ given

prices p_y p_h

(a) Find equilibrium $\frac{p_y}{p_h}$

The consumer's problem is

$$\max_{\vec{x}^i \in \mathbb{R}^n_+} u^i(\vec{x}^i) \text{ s.t. } \vec{p} \cdot \vec{x}^i \leqq m^i(\vec{p}) \equiv \vec{p} \cdot \vec{e}^i + \sum_{j \in J} \theta^{ij} \pi^j(\vec{p})$$

$$\vec{p} \cdot \vec{x}^i = (p_h, p_y) \cdot (h, y) = p_h \cdot h + p_y \cdot y$$

leisure hours

$$\vec{p} \cdot \vec{e}^i = (p_h, p_y) \cdot (24, 0) = p_h \cdot 24$$

Since the one consumer receives a 100 percent share of the one firm's profits π, the consumer's budget line is

$$p_h \cdot h + p_y \cdot y = p_h \cdot 24 + \pi$$

$$\Rightarrow \quad p_y y = p_h(24 - h) + \pi$$

To determine the firm's supply function,

$$y = \sqrt{h} \qquad \max_{h \geq 0} \ \pi = p_y \sqrt{h} - p_h \cdot h$$

$$\pi'(h) = p_y \tfrac{1}{2} h^{-\frac{1}{2}} - p_h \overset{?}{=} 0$$

$$h^{-\frac{1}{2}} = \frac{2p_h}{p_y} \qquad h^{\frac{1}{2}} = \frac{p_y}{2p_h}$$

$$h^f = \left(\frac{p_y}{2p_h}\right)^2 \quad \text{The firm's input demand}$$

$$y^f = (h^f)^{\frac{1}{2}} = \frac{p_y}{2p_h} \quad \text{The firm's output supply.}$$

$$\pi = p_y h^{\frac{1}{2}} - p_h \cdot h$$

$$\pi = p_y \left(\frac{p_y}{2p_h}\right) - p_h \left(\frac{p_y}{2p_h}\right)^2$$

$$\pi(p_h, p_y) = \frac{(p_y)^2}{2p_h} - \frac{(p_y)^2}{4p_h} = \frac{(p_y)^2}{4p_h} \quad \text{The firm's profit function}$$

Next we turn to the consumer's problem.

$$\max_{\vec{x}^i \in \mathbb{R}^n_+} u^i(\vec{x}^i) \ \text{ s.t. } \ \vec{p} \cdot \vec{x}^i \leq m^i(\vec{p}) \equiv \vec{p} \cdot \vec{e}^i + \sum_{j \in J} \theta^{ij} \pi^j(\vec{p})$$

$$\max_{(h,y)\in\mathbb{R}_+^2} \quad h\cdot y \quad \text{s.t.} \quad p_h\cdot h + p_y\cdot y \leq p_h 24 + \pi(p_h, p_y)$$

$$\mathcal{L} \equiv h\cdot y - \lambda\left[p_h\cdot h + p_y\cdot y - p_h 24 - \pi(p_h,p_y)\right]$$

$$\frac{\partial \mathcal{L}}{\partial h} = y - \lambda p_h = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = h - \lambda p_y = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -\left[p_h\cdot h + p_y\cdot y - p_h 24 - \pi(p_h,p_y)\right] = 0$$

$$\Rightarrow \quad \frac{y}{h} = \frac{\lambda p_h}{\lambda p_y} = \frac{p_h}{p_y}$$

$$\Rightarrow \boxed{p_y\cdot y = p_h\cdot h}$$

let I = consumer's income

$$p_h\cdot h + p_y\cdot y = I$$

$$p_h\cdot h + p_h\cdot h = 2p_h\cdot h = I \quad \Rightarrow p_h\cdot h = \frac{I}{2}$$

The consumer spends half of income on h (leisure) and half of income on y.

$$\Rightarrow p_y\cdot y = \frac{I}{2}$$

$$h^c = \frac{I}{2p_h} = \frac{p_h \cdot 24 + \pi(p_h, p_y)}{2p_h}$$

consumer demands

$$y^c = \frac{I}{2p_y} = \frac{p_h \cdot 24 + \pi(p_h, p_y)}{2p_y}$$

p_h such that

(c) Find a Walrasian equilibrium and compute the WEA.

For consumer 1,

$$\max_{\vec{x}} \; x_1 \cdot x_2 \quad \text{s.t.} \quad p_1 \cdot x_1 + p_2 x_2 \leq m$$

$$p_1 \cdot x_1 + p_2 \cdot x_2 - m \leq 0$$

$$\mathcal{L} = x_1 x_2 - \lambda [p_1 x_1 + p_2 x_2 - m]$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_2 - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = x_1 - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -[p_1 x_1 + p_2 x_2 - m] = 0$$

$$x_2 = \lambda p_1 \qquad x_1 = \lambda p_2 \qquad p_1 x_1 + p_2 x_2 = m$$

$$\frac{x_2}{x_1} = \frac{p_1}{p_2} \qquad x_2 = \frac{p_1}{p_2} x_1$$

$$p_1 x_1 + p_2 \left(\frac{p_1}{p_2} x_1 \right) = m \qquad p_1 x_1 + p_1 x_1 = m$$

$$2 p_1 x_1 = m$$

$$\Rightarrow \boxed{x_1^1 = \frac{m}{2p_1}} \qquad x_2 = \frac{p_1}{p_2} x_1 = \frac{p_1}{p_2} \frac{m}{2p_1}$$

$$\boxed{x_2^1 = \frac{m}{2p_2}}$$

For consumer 2,

$$\max_{\vec{x}} \ln x_1 + 2\ln x_2 \quad \text{s.t. } p_1x_1 + p_2x_2 - m \leqq 0$$

$$\mathcal{L} = \ln x_1 + 2\ln x_2 - \lambda[p_1x_1 + p_2x_2 - m]$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{2}{x_2} - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -[p_1x_1 + p_2x_2 - m] = 0$$

$$\Rightarrow \frac{1}{x_1} = \lambda p_1 \quad \frac{2}{x_2} = \lambda p_2 \quad p_1x_1 + p_2x_2 = m$$

$$\frac{\frac{1}{x_1}}{\frac{2}{x_2}} = \frac{\lambda p_1}{\lambda p_2} = \frac{p_1}{p_2} = \frac{x_2}{2x_1}$$

$$x_2 = 2x_1\frac{p_1}{p_2} \qquad p_1x_1 + p_2\left(2x_1\frac{p_1}{p_2}\right) = m$$

$$p_1 \cdot x_1 + 2x_1p_1 = 3x_1p_1 = m$$

$$x_2 = 2x_1\frac{p_1}{p_2} = 2\left(\frac{m}{3p_1}\right)\frac{p_1}{p_2}$$

$$\Rightarrow \boxed{x_1^2 = \frac{m}{3p_1}}$$

$$\boxed{x_2^2 = \frac{2m}{3p_2}}$$

Since only relative prices matter, set $p_2 = 1$ and $p_1 = p$

We try to solve for p.

$$m^1 = 18 \cdot p + 4 \cdot 1 = 18p + 4$$

$$m^2 = 3 \cdot p + 6 \cdot 1 = 3p + 6$$

$$x_1^1 + x_1^2 = 21 = \frac{m^1}{2p_1} + \frac{m^2}{3p_1} = \frac{18p+4}{2p} + \frac{3p+6}{3p}$$

$$x_2^1 + x_2^2 = 10 = \frac{m^1}{2} + \frac{2m^2}{3 \cdot 1} = \frac{18p+4}{2} + \frac{2(3p+6)}{3}$$

$$10 = \frac{18p+4}{2} + \frac{2}{3}(3p+6) = 9p + 2 + 2p + 4$$

$$= 11 \cdot p + 6$$

$$\Rightarrow \quad 4 = 11 \cdot p \quad \Rightarrow \quad \boxed{p = \frac{4}{11} = p_1}$$ Walrasian equilibrium price

<u>Check</u> $$21 = \frac{18p+4}{2p} + \frac{3p+6}{3p}$$

$$21 = \frac{18\frac{4}{11}+4}{2\frac{4}{11}} + \frac{3 \cdot \frac{4}{11}+6}{3\frac{4}{11}}$$

$$21 = \frac{10.5\overline{45}}{0.\overline{72}} + \frac{7.0\overline{9}}{1.0\overline{9}}$$

$$21 \cong 14{,}50 + 6{,}5$$ checks.

By Walras law, we knew that this had to hold.

$$m^1 = 18p + 4 = 18 \cdot \frac{4}{11} + 4 = 10.\overline{54}$$

$$m^2 = 3p + 6 = 3\,\frac{4}{11} + 6 = 7.\overline{09}$$

$$\left.\begin{aligned}
x_1^1 &= \frac{m^1}{2p_1} = \frac{10.\overline{54}}{2\,\frac{4}{11}} = 14.5 \\
x_2^1 &= \frac{m^1}{2p_2} = \frac{10.\overline{54}}{2} = 5.\overline{27} \\
x_1^2 &= \frac{m^2}{3p_1} = \frac{7.\overline{09}}{3\,\frac{4}{11}} = 6.5 \\
x_2^2 &= \frac{2m^2}{3p_2} = \frac{2}{3}\,\frac{7.\overline{09}}{1} = 4.\overline{72}
\end{aligned}\right\}$$

This is The Walrasian Equilibrium Allocation