Problem Set 3

Yifan Yang

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- Groups of up to four students may submit one set of solutions. Write each group member's name and student number clearly on the first page of your solutions. Group compositions are allowed to change from one home assignment to another.
- To obtain a good score, write clearly and logically, starting from the definitions and correctly deducing and motivating your answers.
- Please submit your answer **before 13:15 on September 22th**.

Exercise 1 Assume we are working in $\mathbb{R}$ with the usual distance. Which of the following sequences in $\mathbb{R}$ converges? Use $\epsilon - \delta$ definition to show whether they are convergent or not, and indicate the limit where applicable.

(1) $(x_k)_{k\in\mathbb{N}}$ with $x_k = \frac{(-1)^k}{k}$

(2) $(x_k)_{k\in\mathbb{N}}$ with $x_k = \frac{k^2-k+2}{5k^2+4k-9}$

(3) $(x_k)_{k\in\mathbb{N}}$ with $x_k = \sqrt{k+1} - \sqrt{k}$

(4) $(x_k)_{k\in\mathbb{N}}$ with $x_k = (-1)^k$

Exercise 2 Assume we are working in $\mathbb{R}$ with the Euclidean distance.

(1) Show that if $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, then $(x_k^2)_{k\in\mathbb{N}}$ is also a Cauchy sequence.

(2) Give an example of a Cauchy sequence $(x_k^2)_{k\in\mathbb{N}}$ such that $(x_k)_{k\in\mathbb{N}}$ is not a Cauchy sequence.

(3) Take the sequence $x_1 = 1$ and $x_{k+1} = x_k + (-1)^k k^3$. Is it a Cauchy sequence?

Exercise 3 Let (X_1, p) and (X_2, q) be two complete metric spaces. Let $X = X_1 \times X_2$. A metric on X is defined as follows:

$$d(a, b) = (p(a_1, b_1)^2 + q(a_2, b_2)^2)^{\frac{1}{2}}$$

where $a, b \in X$. Show that (X, d) is complete.

Exercise 4 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 2x + 2$.

(a) Show that $f : \mathbb{R} \to \mathbb{R}$ is not a contraction. Identify the possibly empty set of fixed points.

(b) Let $\alpha \in (0, \frac{1}{2})$ and define $X_\alpha = [1-\alpha, 1+\alpha]$. Show that $f(X_\alpha) \subset X_\alpha$ and that $f : X_\alpha \to X_\alpha$ is a contraction.

(c) Is X_α a complete set? What does the Contraction Mapping Theorem imply for X_α and $f(x)$?

Exercise 5

(1) Consider metric space (X, d) where $X = \mathbb{R}$ and d is the usual distance. Is the set $[0, 1]$ compact? What about $\mathbb{R}$? In any of these sets fails to be compact, give an example of a covering that set that does not have a finite subcovering.

(2) Let (X, d) be a metric space. Let $A \subseteq X$ be compact and $B \subseteq X$ be closed. Show that $C = A \cap B$ is compact.

(3) Let (X, d) be a set with the discrete metric. Show that (X, d) is compact if and only if X is finite.