Handout 10: Capital Markets

1 Introduction

The objective of this handout is to guide you through two key concepts in finance. The first is *diversification*. The first worked example shows you that there are large benefits to diversify your portfolio (or your choices) as long as the underlying assets (or lotteries) are not perfectly correlated. A portfolio manager can combine the two assets to maximize returns minimizing risk, reaching a portfolio that is strictly better than each asset by itself. A more general and developed version of this idea was awarded the Nobel Prize in economics (see here).

In the second worked example we will explore the problem of an agent that must choose to consume or save his wealth - i.e., the intertemporal choice problem. The intertemporal choice problem is an application of consumer choice (Handout 3) with two goods. The two goods are not different goods, but the same good in different moments in *time*. We show that savings depend on interest rates and the discount factor (how much the agent discounts the future). We vary this two parameters and show how consumption today/savings change with changes in them. For interest rates, in particular, we show that interest rates are related to the price of the future, and separate the of interest rates in income and substitution effects. This second worked example is an application of consumer theory that is the basis of most of what is done in macroeconomic today. We can do the same we did here for consumers but for firms deciding how much to invest - and thus start to understand the joint dynamics of consumption, savings, investment, inflation etc..

2 Portfolio Choice and Diversification

Consider an investor that has an endowment of \$10,000 that he wants to invest. He can either invest in

- a bond, which yields 1%,
- the stock market, which consist of two firms, firm A and firm B

Each firm's stock is costs \$P today, and will be worth \$4P in one year with probability $\frac{1}{2}$ or will drop to \$0 with probability $\frac{1}{2}$. Assume that the evolution of both stocks is independent: that is, the probability that stock of firm A rises in value does not vary or depend on what has happened to stock of firm B, and vice-versa. Finally, assume that the investor utility function is $U(w) = \sqrt{w}$, and there is no inflation.

1. Suppose that due to institutional regulations, the investor can invest only in bonds, or only in firm A. He cannot buy firm B stocks and he cannot buy both firm A stocks on bonds. What is the investor's utility of buying bonds? What if he invests only in Firm A stock? What does she prefer?

2. Now suppose there is a change in regulations. The investor can invest in either the stock market or in bonds, but not both. If the investor decides to invest in the stock market, he can choose how much he wants to invest in each company. Let x denote the fraction of wealth that the investor puts into firm A and $y = 1 - x$ the share on stock of firm B. If the investor maximizes her expected utility, what should be the value of x?

3. What is the expected value of the strategy in the previous item? Does the investor prefer to use this strategy or to invest in bonds only? What is the effect at work here?

4. Suppose again the investor can invest in either the stock market or in bonds, but not both. However, firm B changes their business line now and the stocks for firm A and firm B are negatively correlated, so that with probability 1/2 firm A goes bankrupt, firm B stocks are worth \$4P, and with probability 1/2 the opposite happens. What is the investor's expected utility with the portfolio you found item 2? Is the negative correlation hurting or helping?

5. Suppose Oliver works for firm A and he is investing for his retirement. Is the investment decision you found in last question still optimal? Why, or why not?

Solution.

1. If the investor chooses the bond, his expected utility is

$$EU_{bond} = \sqrt{10000 \times 1.01} \approx 100.5$$

since he makes 10000×1.01 for sure. Investing in company A

$$EU_A = \frac{1}{2}\sqrt{40000} + \frac{1}{2}\sqrt{0} = 100$$

Let's look at her payback in case of success above. If she invests all the money in company A, she will buy $10,000/P$ stocks. Each stock will then be valued at $4P$ and, thus, the payback in case of success is $\frac{10,000}{P}4P = 40,000$. Therefore, she prefers to buy the bond (given that is has a higher expected utility).

2. If she invests a share x of her portfolio on company A, the number of firm A stocks held by the investor is $\frac{10,000x}{P}$ (quantity of money over price of the stock) and the number of firm B stock is $\frac{10,000(1-x)}{P}$ (i.e., the rest of the money divided by the price of stock B)

Given that the outcomes of both stocks are independent, we have that the following outcomes are all equally likely, each with 25% probability:

(a) Both stocks quadruple in value

(b) Firm A quadruples , Firm B is worth 0

(c) Firm B quadruples , Firm A is worth 0

(d) Both stocks are worth 0.

The investors expected utility from investing a fraction x into firm A and $1-x$ into firm B is thus (each of the four terms is in the same order as in the list just above)

$$
\begin{aligned}
EU(x) = &\frac{1}{4}\sqrt{\left(\frac{10000x}{P}\right)4P + \left(\frac{10000(1-x)}{P}\right)4P} \\
&+ \frac{1}{4}\sqrt{\left(\frac{10000x}{P}\right)4P} + \frac{1}{4}\sqrt{\left(\frac{10000(1-x)}{P}\right)4P} + \frac{1}{4}\sqrt{0}
\end{aligned}
$$

Which we can simplify to

$$
\begin{aligned}
EU(x) = &\frac{1}{4}\sqrt{40000x + 40000(1-x)} \\
&+ \frac{1}{4}\sqrt{40000x} + \frac{1}{4}\sqrt{40000(1-x)}
\end{aligned}
$$

A little more simplifying:

$$
EU(x) = \frac{1}{4}\sqrt{40000}\left[1 + \sqrt{x} + \sqrt{1-x}\right]
$$

Therefore, the problem of choosing x to maximize expected utility is given by

$$
\max_x \quad \sqrt{x} + \sqrt{1-x}
$$

Note that constants (terms that do not depend on x) in the expected utility either multiplying (as long as positive) or summing do not have to be taken into account for the optimization, since they are not affected by the choice. The FOC is given by

$$\frac{1}{x^{1/2}} = \frac{1}{(1-x)^{1/2}} \Rightarrow x = 1/2$$

The optimal portfolio of stocks is to invest 1/2 of the wealth in each.

3. The expected value is still \$20,000 (as it was investing in A only). To see that:

$$EV_A = \frac{1}{2}40,000 + \frac{1}{2}0 = 20,000$$

$$EV_{A,B} = \frac{1}{4}40,000 + \frac{1}{4}20,000 + \frac{1}{4}20,000 + \frac{1}{4}0 = 20,000$$

However, the expected utility at the optimum portfolio now is:

$$EU_{A,B} = \frac{1}{4}\sqrt{40,000} + \frac{1}{4}\sqrt{20,000} + \frac{1}{4}\sqrt{20,000} + \frac{1}{4}\sqrt{0} \approx 120.71$$

which is larger than the utility of investing in bonds. The investor is better off by *diversifying* his investments, because by doing so she reduced his risk without reducing his expected payoffs. Since she is risk-averse, that makes her strictly better off.

4. Note that the expected value of any combination of Firm A and Firm B is the same, namely \$20000. However, the investor's utility is maximized when he fully diversifies and buys 1/2 of A and 1/2 of B, in which case his utility is $\sqrt{20000} = 141$. The negative correlation helps, because she is now able to obtain \$20000 with probability 1, and because she is risk-averse he values certainty.

5. If the investor works for A, then the investment we found is not optimal, because

there is a new source of correlation: namely when A goes bankrupt, the investor also loses his job. Because the investor is risk-averse, he would like to hedge against this event, by investing more than half of her savings in B, so that when A goes bankrupt the investor's returns to investment are still relatively high.

□

3 Intertemporal Choice

Suppose that households live for two periods $t = 1, 2$ and die at the end of period 2. They have wealth in the first period $W > 0$ but no wealth in the second period. Their utility over consumption in the first and second period is given by

$$U(c_1, c_2) = \sqrt{c_1} + \beta\sqrt{c_2}$$

where c_1 is consumption in period 1, c_2 is consumption in period 2 and $\beta \in (0, 1]$ is a preference parameter. Buying 1 unit of consumption costs \$1 in both periods. The households can save by investing in a safe asset at the market interest rate r. Assume that the household gets no utility from leaving any money behind after death.

1. Write down the household budget constraints for periods 1, 2. Then, write an expression for the household's intertemporal budget constraint in terms of today's dollars.

2. How much of its income will the household consume in each of the two periods and how much will it save given the interest rate r?

3. Does increasing β increase or decrease savings? What is the intuition behind this result?

4. Does a higher interest rate increase or decrease savings? Provide the intuition for this result.

Solution.

1. The budget constraint in period 1 is:

$$c_1 + s = W$$

The budget constraint in period 2 is

$$c_2 = (1 + r)s$$

Combining the two constraints by substituting s, we have

$$c_1 + \frac{c_2}{1 + r} = W$$

2. From the tangency condition, we have

$$\frac{2\sqrt{c_2}}{2\beta\sqrt{c_1}} = 1 + r \iff c_2 = \beta^2(1 + r)^2 c_1$$

Substituting for c_2 in the budget constraint yields:

$$c_1 + \frac{\beta^2(1 + r)^2 c_1}{1 + r} = W \iff c_1 = \frac{W}{1 + \beta^2(1 + r)}$$

and this implies that

$$c_2 = \frac{W(1 + r)^2\beta^2}{1 + (1 + r)\beta^2}$$

Savings are equal to

$$s = W - c_1 = W - \frac{W}{1 + \beta^2(1+r)} = W\frac{\beta^2(1+r)}{1 + \beta^2(1+r)}$$

3. Taking the partial derivative yields:

$$\frac{\partial s}{\partial \beta} = \frac{2\beta W(1+r)}{(1 + \beta^2(1+r))^2} > 0$$

A higher β implies higher savings. The intuition behind this result is that β is a preference parameter capturing the relative preference for second period consumption compared to first period. A higher β implies relatively higher utility from second period consumption and so the household wants to save more in order to consume more in the second period.

4. Taking the partial derivative yields:

$$\frac{\partial s}{\partial r} = \frac{\beta^2 W}{(1 + \beta^2(1+r))^2} > 0$$

Increasing the interest rate increases savings and decreases consumption in period 1. The intuition behind this result is that the substitution effect is larger than the income effect (in absolute value), so savings increase.

□