5303 - Advanced Macroeconomics
Assignment 2 Solutions

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Introduction to Dynamic Macroeconomic Theory

Chapter 6 - Infinitely Lived Assets

Some definitions:

- $A \rightarrow$ Units of land
- $D(t) \rightarrow$ Total crop of the time t good in each period t
- $d(t) = \frac{D(t)}{A} \rightarrow$ Amount of crop per unit of land at time t
- $a^h(t) \rightarrow$ Quantity of land that individual h of generation t chooses to purchase when young at price $p(t)$ and expects to sell when old at the price that this individual expects will occur in the next period, $p^{h,e}(t+1)$

Proposition 1. [1] If there is unanimity of price expectations, namely,

$$p^{h,e}(t+1) = p^e(t+1)$$

for all h in generation t, then, in any equilibrium in which desired land purchases equal the total land supplied,

$$p(t) = \frac{d(t+1) + p^e(t+1)}{r(t)}$$

This proposition states that the present value of the expected value of land in the next period, including crops, equals today's price. Equivalently, the rate of return on land equals the rate of return on the private borrowing and lending.

Definition 1. [2] Given $p^e(t+1)$, a time t temporary equilibrium is a land price, $p(t)$, a gross interest rate, $r(t)$, land purchases, loans granted and received at t, and a pattern of consumption of time t good that:

1. maximize utility for all members of generation t subject to their budget constraints, and
2. clear the markets for land, loans, and the consumption good at time t

Alternatively, we can consider the following definition:

[1] Page 151 of the course textbook.
[2] Page 151 of the course textbook.

Definition 2. Given $\{u^h_t, \omega^h_t, A, D(t), D(t{+}1), p^e(t{+}1)\}$, a time t temporary equilibrium is a pair of prices $\{r(t), p(t)\}$ and a set of allocations $\{c^h_t, l^h(t), a^h(t)\}_{h=1}^{N(t)}$ such that the equilibrium conditions

1. $p(t) = \frac{d(t+1)+p^e(t+1)}{r(t)}$ and
2. $S_t(r(t)) = p(t)A(t)$

are fulfilled.

Notice that in the definition above, condition (1) is the market clearing condition for land, and condition (2) is the competitive equilibrium condition for an economy with land.

Definition 3. [3] A sequence of temporary equilibria is a sequence comprising a time t temporary equilibrium at each $t \geq 1$.

Definition 4. [4] A perfect foresight competitive equilibrium for an economy with land is a sequence of $p(t)$'s and $r(t)$'s and the other endogenous (dependent) variables such that the time t values are a time t temporary equilibrium for an economy with land where $p^e(t+1) = p(t+1)$.

Definition 5. [5] Given an economy with a stationary environment, in the sense that the environmental variables are the same in each period, a stationary equilibrium is a perfect foresight competitive equilibrium in which the consumptions are the same for every generation.

Exercise 6.4

Describe completely the sequence of temporary equilibria for each of the following economies.

(a) $N(t) = 100$, $u^h_t = c^h_t(t)c^h_t(t+1)$, $\omega^h_t = [2, 1]$, for all h and $t \geq 1$. $A = 100$, $d(t+1) = 1$, and $p^e(t+1) = 1$, for all $t \geq 1$.

(e) The same as a except that for all h:

$$\omega^h_t = \begin{cases} [2, 1], & \text{for } t \text{ odd} \\ [1, 1], & \text{for } t \text{ even} \end{cases}$$

[3] Page 156 of the course textbook.
[4] Page 160 of the course textbook.
[5] Page 169 of the course textbook.

Answer

(a) Since agents form expectations about prices, consumption when old is equal to planned consumption when old only when the expected price is realized. It is thus more accurate to refer to consumption when old as expected consumption when old, i.e. $c_t^{h,e}(t+1)$. In an economy with land, agents face the following budget constraints:

$$\begin{cases} \text{Young: } \rightarrow c_t^h(t) \leq \omega_t^h(t) - l^h(t) - p(t)a^h(t) \\ \text{Old: } \rightarrow c_t^{h,e}(t+1) \leq \omega_t^h(t+1) + r(t)l^h(t) + d(t+1)a^h(t) + p^e(t+1)a^h(t) \end{cases}$$

which can be combined into a single lifetime budget constraint:

$$c_t^h(t) + \frac{c_t^{h,e}(t+1)}{r(t)} \leq \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)} - a^h(t)\left[p(t) - \frac{d(t+1) + p^e(t+1)}{r(t)}\right]$$

The competitive choice problem is now given by:

$$\begin{aligned} \max_{\{c_t^h(t), c_t^{h,e}(t+1), l^h(t), a^h(t)\}} \quad & c_t^h(t)c_t^{h,e}(t+1) \\ \text{s.t. } \quad & c_t^h(t) \leq \omega_t^h(t) - l^h(t) - p(t)a^h(t) \\ & c_t^{h,e}(t+1) \leq \omega_t^h(t+1) + r(t)l^h(t) + d(t+1)a^h(t) + p^e(t+1)a^h(t) \end{aligned}$$

And our Lagrangian:

$$\begin{aligned} \mathcal{L} = {} & c_t^h(t)c_t^{h,e}(t+1) + \mu(t)\left[\omega_t^h(t) - l^h(t) - p(t)a^h(t) - c_t^h(t)\right] \\ & + \mu(t+1)\left[\omega_t^h(t+1) + r(t)l^h(t) + d(t+1)a^h(t) + p^e(t+1)a^h(t) - c_t^{h,e}(t+1)\right] \end{aligned}$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the FOC:

$$\begin{aligned} \mathbf{c_t^h(t)}\text{: } & c_t^{h,e}(t+1) - \mu(t) = 0, \forall t \geq 1 \\ \mathbf{c_t^{h,e}(t+1)}\text{: } & c_t^h(t) - \mu(t+1) = 0, \forall t \geq 1 \\ \mathbf{l^h(t)}\text{: } & -\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1 \\ \mathbf{a^h(t)}\text{: } & -\mu(t)p(t) + \mu(t+1)[d(t+1) + p^e(t+1)] = 0, \forall t \geq 1 \end{aligned}$$

From which we can obtain the following equilibrium conditions:

$$\begin{aligned} \frac{c_t^{h,e}(t+1)}{c_t^h(t)} &= r(t) \\ r(t) &= \frac{d(t+1) + p^e(t+1)}{p(t)} \end{aligned}$$

Substituting in the lifetime budget constraint:

$$c_t^h(t) + \frac{c_t^{h,e}(t+1)}{r(t)} = \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)} - a^h(t)\Big[p(t) - \frac{d(t+1)+p^e(t+1)}{r(t)}\Big]$$

$$c_t^h(t) + \frac{c_t^h(t)r(t)}{r(t)} = \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)} - a^h(t)\,\underbrace{\Big[p(t) - \frac{d(t+1)+p^e(t+1)}{r(t)}\Big]}_{=0}$$

$$c_t^h(t) = \frac{1}{2}\Big[\omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}\Big]$$

In a temporary equilibrium, the affordable consumption bundles depend only on the gross interest rate, $r(t)$, and on the endowments. Savings are given by:

$$s_t^h(t) = \omega_t^h(t) - c_t^h(t) = \frac{1}{2}\omega_t^h(t) - \frac{1}{2}\frac{\omega_t^h(t+1)}{r(t)}$$

Since the rates of return on loans and land are the same in equilibrium, the individual only cares about total savings, and is indifferent about their split between private lending and land purchases. We can now plug in the given values and get:

$$c_t^h(t) = 1 + \frac{1}{2r(t)}$$

$$s_t^h(t) = 1 - \frac{1}{2r(t)}$$

$$S_t(r(t)) = 100s_t^h(t) = 100 - \frac{50}{r(t)}$$

In equilibrium, we must have that:

$$S_t(r(t)) = p(t)A$$

$$S_t(r(t)) = \Big[\frac{d(t+1)+p^e(t+1)}{r(t)}\Big]A$$

$$r(t)S_t(r(t)) = [d(t+1)+p^e(t+1)]A$$

$$r(t)\Big[100 - \frac{50}{r(t)}\Big] = [1+1]\times 100$$

$$r(t) = \frac{5}{2}$$

From where it follows that:

$$\begin{aligned}
p(t) &= \frac{d(t+1) + p^e(t+1)}{r(t)} = \frac{1+1}{\frac{5}{2}} = \frac{4}{5} \\
c_t^h(t) &= 1 + \frac{1}{2 \times \frac{5}{2}} = \frac{6}{5} \\
c_t^{h,e}(t+1) &= r(t)c_t^h(t) = \frac{5}{2} \times \frac{6}{5} = 3 \\
s_t^h(t) &= 1 - \frac{1}{2 \times \frac{5}{2}} = \frac{4}{5}
\end{aligned}$$

Notice that the old choose their consumption based on their future price expectation, i.e $c_t^{h,e}(t+1)$ takes $p^e(t+1)$ in consideration. This means that actual consumption of the old may different from their expectation. Since we are not under perfect foresight, we need to check the feasibility condition at time t:

$$\begin{aligned}
\sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + \sum_{h=1}^{N(t)} c_t^h(t) &= Y(t) + D(t) \\
100c_{t-1}^h(t) + 100 \times \frac{6}{5} &= 300 + 1 \times 100 \\
c_{t-1}^h(t) &= 2.8 < c_{t-1}^{h,e}(t) = 3
\end{aligned}$$

Meaning that actual consumption of the old is lower than expected consumption.

(e) Making use of our previous results, we have that:

$$c_t^h(t) = \begin{cases} 1 + \frac{1}{2r(t)}, & \text{for } t \text{ odd} \\ \frac{1}{2} + \frac{1}{2r(t)}, & \text{for } t \text{ even} \end{cases}$$

$$s_t^h(t) = \begin{cases} 1 - \frac{1}{2r(t)}, & \text{for } t \text{ odd} \\ \frac{1}{2} - \frac{1}{2r(t)}, & \text{for } t \text{ even} \end{cases}$$

$$S_t(r(t)) = \begin{cases} 100 - \frac{50}{r(t)}, & \text{for } t \text{ odd} \\ 50 - \frac{50}{r(t)}, & \text{for } t \text{ even} \end{cases}$$

So at odd times, in equilibrium we must have that:

$$S_t(r(t)) = p(t)A \Leftrightarrow r(t)\Big[100 - \frac{50}{r(t)}\Big] = [1+1] \times 100 \Leftrightarrow r(t) = \frac{5}{2}$$

From where it follows that:

$$
\begin{aligned}
p(t) &= \frac{d(t+1)+p^e(t+1)}{r(t)} = \frac{1+1}{\frac{5}{2}} = \frac{4}{5} \\
c_t^h(t) &= 1 + \frac{1}{2\times\frac{5}{2}} = \frac{6}{5} \\
c_t^{h,e}(t+1) &= r(t)c_t^h(t) = \frac{5}{2}\times\frac{6}{5} = 3 \\
s_t^h(t) &= 1 - \frac{1}{2\times\frac{5}{2}} = \frac{4}{5}
\end{aligned}
$$

And at even times:

$$
S_t(r(t)) = p(t)A \Leftrightarrow r(t)\left[50 - \frac{50}{r(t)}\right] = [1+1]\times 100 \Leftrightarrow r(t) = 5
$$

From where it follows that:

$$
\begin{aligned}
p(t) &= \frac{d(t+1)+p^e(t+1)}{r(t)} = \frac{1+1}{5} = \frac{2}{5} \\
c_t^h(t) &= \frac{1}{2} + \frac{1}{2\times 5} = \frac{3}{5} \\
c_t^{h,e}(t+1) &= r(t)c_t^h(t) = 5\times\frac{3}{5} = 3 \\
s_t^h(t) &= \frac{1}{2} - \frac{1}{2\times 5} = \frac{2}{5}
\end{aligned}
$$

From a, we already know that at odd times the actual consumption of the old is lower than their expected consumption. For even times, it follows that:

$$
\begin{aligned}
\sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + \sum_{h=1}^{N(t)} c_t^h(t) &= Y(t) + D(t) \\
100c_{t-1}^h(t) + 100\times\frac{3}{5} &= 200 + 1\times 100 \\
c_{t-1}^h(t) &= 2.4 < c_{t-1}^{h,e}(t) = 3
\end{aligned}
$$

Meaning that at even times actual consumption of the old is also lower than expected consumption.

Chapter 8 - A Storage Technology

Some definitions:

- A storage technology is a linear, constant returns to scale, intertemporal technology that transforms time t goods into time $t+1$ goods.

- For every unit of good stored, λ units are returned in the next period, where $\lambda \geq 0$
- $K(t+1) \rightarrow$ Total amount of the time t good that is stored until time $t+1$
- $k^h(t+1) \rightarrow$ Amount of time t good that is invested (stored) by individual h at date t

Definition 6. [6] A path of total consumption $C(1), C(2), C(3), \ldots, C(t)$, is feasible for an economy with storage if, given $K(1)$, there exists a nonnegative $K(t)$ sequence for $t > 1$ that satisfies:

$$C(t) + K(t+1) \leq Y(t) + \lambda K(t), \forall t \geq 1$$

(where $Y(t)$ includes the crop from land)

Definition 7. [7] A perfect foresight competitive equilibrium for an economy with storage and land is a nonnegative sequence of land prices and interest rates and of total storage amounts, $\{p(t), r(t), K(t+1)\}_{t=1}^{\infty}$, that for all $t \geq 1$ satisfy:

(i) $S_t(r(t)) = p(t)A + K(t+1)$

(ii) $p(t) = \frac{p(t+1)+d(t+1)}{r(t)}$

(iii) $r(t) \geq \lambda$, and

(iv) $K(t+1)\left[1 - \frac{\lambda}{r(t)}\right] = 0$

given initial conditions $\{a^h(1), k^h(1), \omega_0^h(1)\}_{h=1}^{N(0)}$ and sequences $\{D(t), \{\omega_t^h\}_{h=1}^{N(t)}\}_{t=1}^{\infty}$.

Exercise 8.1

Suppose $N(t) = 1$, $Y(t) = 1$ for all $t \geq 0$, $\lambda = 2$, $K(1) = 0$, and the utility function is $u_t^h = c_t^h(t)c_t^h(t+1)$, for all h and $t \geq 1$. Show that the following consumption allocation is feasible but not Pareto optimal:

$$c_t^h = \left[\frac{1}{2}, \frac{1}{2}\right] \text{ for all } h \text{ and } t \geq 1, \text{ and } c_0^h(1) = \frac{1}{2}$$

[6] Page 210 of the course textbook.
[7] Page 215 of the course textbook.

Answer

Let's start with the **Feasibility** considerations. Recall our definition of feasibility 6 above, which requires that:

$$C(t) + K(t+1) \leq Y(t) + \lambda K(t), \forall t \geq 1$$

Therefore, at $t = 1$, we must have that:

$$C(1) + K(2) \leq Y(1) + \lambda K(1) \Leftrightarrow 1 \times \left(\frac{1}{2} + \frac{1}{2}\right) + K(2) \leq 1 + 2 \times 0$$

which holds as long as $K(2) = 0$. At $t = 2$, we must have that:

$$C(2) + K(3) \leq Y(2) + \lambda K(2) \Leftrightarrow 1 \times \left(\frac{1}{2} + \frac{1}{2}\right) + K(3) \leq 1 + 2 \times 0$$

which holds as long as $K(3) = 0$. By iterating forward, we can see that at each time t, if $K(t+1) = 0$, the allocation $c_t^h = \left[\frac{1}{2}, \frac{1}{2}\right]$ is feasible.

Now for **Pareto Optimality** considerations. Let's consider an alternative allocation where at all times each individual h invests $x \in]0, 0.5[$ in storage when young, such that $K(t+1) = x, \forall t$. The new allocation is thus given by:

$$c_t^h = \left[\frac{1}{2} - x, \frac{1}{2} + \lambda x\right]$$

For it to be Pareto superior to the original allocation, it must be feasible and utility improving. This allocation is utility improving if:

$$\begin{aligned}
u_t^h\left(\frac{1}{2} - x, \frac{1}{2} + \lambda x\right) &> u_t^h\left(\frac{1}{2}, \frac{1}{2}\right) \\
\left(\frac{1}{2} - x\right)\left(\frac{1}{2} + \lambda x\right) &> \frac{1}{4} \\
\frac{1}{2}x - 2x^2 &> 0 \\
x &\in \left]0, \frac{1}{4}\right[
\end{aligned}$$

Moreover, this allocation is feasible if:

$$\begin{aligned}
C(t) + K(t+1) &\leq Y(t) + \lambda K(t) \\
\left(\frac{1}{2} - x + \frac{1}{2} + \lambda x\right) + x &\leq 1 + \lambda x \\
1 - x + \lambda x + x &\leq 1 + \lambda x \\
1 &\leq 1
\end{aligned}$$

which is true for any $x \in \left]0, \frac{1}{4}\right[$. We can thus conclude that there is an infinity of alternative Pareto superior allocations, meaning that our original allocation is not Pareto optimal.

Exercise 8.4

Describe an economy without land with a stationary equilibrium that satisfies $r > \lambda$.

Answer

Making use of definitions 7 and 5 above, we know that a stationary equilibrium for an economy with storage must satisfy the following conditions:

(i) $S(r) = K$

(ii) $r \geq \lambda$, and

(iii) $K\left[1 - \frac{\lambda}{r}\right] = 0$

Since we have that $r > \lambda$, this means that the return on private borrowing and lending is greater than the return on storage. This means that agents will not want to invest in storage, i.e. $K = 0$. Therefore, aggregate savings will also be 0, i.e. $S(r) = K = 0$ and the interest rate is computed in the standard way. The stationary equilibrium is thus given by:

$$\{r(t), K(t)\} = \{r, 0\}, \forall t \geq 1$$

Exercise 8.5

Describe an economy without land with a stationary equilibrium that satisfies $r = \lambda$.

Answer

Since we now have that $r = \lambda$, this means that individuals are indifferent between investing in storage and investing in the loan market. Therefore, we may have a positive stationary value of K (all individuals can still choose to invest in the loan market), and the equilibrium condition is now given by $S(r) = K$. The stationary equilbrium is thus given by:

$$\{r(t), K(t)\} = \{\lambda, K\}, \forall t \geq 1$$

Note that the actual division between private loans and storage is irrelevant for a specific individual since all that matters is the net position, i.e. whether he/she is a borrower or a lender.

Other exercises

Exercise 1

Consider the following economy:

Preferences are given by $u_t^h = \log c_t^h(t) + \beta \log c_t^h(t+1), \beta = \frac{2}{3}, N = 100$

There are no financial assets but private borrowing and lending

Endowments for type-1 households: $\omega_t^h = [3, 1]$ for $h = 1, \ldots, 50$

Endowments for type-2 households: $\omega_t^h = [1, 3]$ for $h = 51, \ldots, 100$

(a) Solve for $r(t)$ and s_t^h in the competitive equilibrium.

(b) Now suppose that you are not allowed to borrow, so that $0 \leq s_t^h$. What happens to $r(t)$? Why?

Answer

(a) Let's solve the competitive choice problem:

$$\max_{\{c_t^h(t), c_t^h(t+1), l^h(t)\}} \log c_t^h(t) + \beta \log c_t^h(t+1)$$
$$\text{s.t. } c_t^h(t) \leq \omega_t^h(t) - l^h(t)$$
$$c_t^h(t+1) \leq \omega_t^h(t+1) + r(t) l^h(t)$$

And our Lagrangian:

$$\mathcal{L} = \log c_t^h(t) + \beta \log c_t^h(t+1) + \mu(t) \Big[\omega_t^h(t) - l^h(t) - c_t^h(t)\Big]$$
$$+ \mu(t+1) \Big[\omega_t^h(t+1) + r(t) l^h(t) - c_t^h(t+1)\Big]$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the FOC:

$$\begin{aligned}
\mathbf{c_t^h(t)}&: \ \tfrac{1}{c_t^h(t)} - \mu(t) = 0, \forall t \geq 1\\
\mathbf{c_t^h(t+1)}&: \ \tfrac{\beta}{c_t^h(t+1)} - \mu(t+1) = 0, \forall t \geq 1\\
\mathbf{l^h(t)}&: \ -\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1
\end{aligned}$$

From which we can obtain the following equilibrium condition:

$$c_t^h(t+1) = \beta r(t) c_t^h(t)$$

which is the same condition of the classical Cobb-Douglas case. This should not come as surprising since we are now working with a log-transformation of it. Our time t demand function for consumption when young and savings are thus still given by (you only need to plug the equilibrium condition back in the lifetime budget constraint):

$$\begin{aligned}
c_t^h(t) &= \Big(\frac{1}{1+\beta}\Big)\Big[\omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}\Big]\\
s_t^h(t) &= \Big(\frac{\beta}{1+\beta}\Big)\omega_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\Big[\frac{\omega_t^h(t+1)}{r(t)}\Big]
\end{aligned}$$

Substituting:

$$c_t^h(t) = \begin{cases} \frac{9}{5} + \frac{3}{5r(t)}, & \text{for } h = 1, \dots, 50 \\ \frac{3}{5} + \frac{9}{5r(t)}, & \text{for } h = 51, \dots, 100 \end{cases}$$

$$s_t^h(t) = \begin{cases} \frac{6}{5} - \frac{3}{5r(t)}, & \text{for } h = 1, \dots, 50 \\ \frac{2}{5} - \frac{9}{5r(t)}, & \text{for } h = 51, \dots, 100 \end{cases}$$

And Aggregate Savings are thus given by:

$$S_t(r(t)) = 50 \times \Big[\frac{6}{5} - \frac{3}{5r(t)}\Big] + 50 \times \Big[\frac{2}{5} - \frac{9}{5r(t)}\Big] = 80 - \frac{120}{r(t)}$$

In a competitive equilibrium, we must have that:

$$S_t(r(t)) = 0 \Leftrightarrow r(t) = \frac{3}{2}$$

And individual savings are given by:

$$s_t^h(t) = \begin{cases} \frac{6}{5} - \frac{3}{5\times\frac{3}{2}} = \frac{4}{5}, \text{ for } h = 1, \ldots, 50 \\ \frac{2}{5} - \frac{9}{5\times\frac{3}{2}} = -\frac{4}{5}, \text{ for } h = 51, \ldots, 100 \end{cases}$$

(b) We are now imposing a borrowing constraint, i.e. $s_t^h(t) \geq 0, \forall t$. The intuition behind this is as follows: in the private loan market, there are individuals who supply funds (lenders) and individuals who demand them (borrowers). When demand for funds falls, since supply must match demand at equilibrium, the equilibrium interest rate must fall as well. For individuals $h = 1, \ldots, 50$, their savings are nonnegative when:

$$\frac{6}{5} - \frac{3}{5r(t)} \geq 0 \Leftrightarrow r(t) \geq \frac{1}{2}$$

And for individuals $h = 51, \ldots, 100$, nonnegative savings require:

$$\frac{2}{5} - \frac{9}{5r(t)} \geq 0 \Leftrightarrow r(t) \geq \frac{9}{2}$$

Since we have a borrowing constraint, i.e. $s_t^h(t) \geq 0$ and we know that a competitive equilibrium requires $S_t(r(t)) = 0$, this means that individual savings must be 0 at all times t. This is achieved when $r(t) = \frac{1}{2}$. Individuals $h = 51, \ldots, 100$ are already prevented from borrowing by the borrowing constraint and, as long as $r(t) \leq \frac{9}{2}$, they will not want to lend. Individuals $h = 1, \ldots, 50$ will want to lend as long as $r(t) \geq \frac{1}{2}$. Therefore, to have $s_t^h(t) = 0, \forall t, \forall h$, we must have $r(t) = \frac{1}{2}$.

Exercise 2

Consider an overlapping generations environment with land where population is given by $N(t) = 1$, the quantity of land is $A = 1$, each unit of which delivers a crop $d(t) = 1$ in every period. Land ownership in period 0 is equally distributed among members of generation -1. Endowments are $\omega_t^h = [7, 10]$, and preferences are represented by $u_t^h = c_t^h(t)c_t^h(t+1)$ for all h and all t. Purchasing land in period t entitles the buyer to the next period's crop $d(t+1)$ and the revenue from selling it in period $t+1$. Initially, there is no government.

(a) What allocations are feasible in this environment?

(b) Find the stationary competitive equilibrium.

(c) A government is suddenly created in period 0. It taxes the crop; $\frac{2}{5}$ of the crop is collected in each period and distributed equally among the old. Find the stationary competitive equilibrium. Verify that the initial old lose but that everyone else gains from this policy.

Answer

(a) Feasibility in an economy with land requires that:

$$\sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + \sum_{h=1}^{N(t)} c_t^h(t) \leq Y(t) + D(t), \forall t$$

where $D(t) = d(t)A$ and $Y(t) = \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t) + \sum_{h=1}^{N(t)} \omega_t^h(t)$. Substituing, we get that the set of feasible allocations in this economy is given by:

$$c_{t-1}^h(t) + c_t^h(t) \leq (7+10) + 1 \times 1 = 18, \forall t$$

(b) Making use of our results from Exercise 6.4, we have that:

$$\begin{aligned}
c_t^h(t) &= \left(\frac{1}{2}\right)\left[\omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}\right] \\
s_t^h(t) &= \left(\frac{1}{2}\right)\omega_t^h(t) - \left(\frac{1}{2}\right)\left[\frac{\omega_t^h(t+1)}{r(t)}\right] \\
S_t(r(t)) &= \sum_{h=1}^{N(t)} s_t^h(t)
\end{aligned}$$

Substituting:

$$\begin{aligned}
c_t^h(t) &= \frac{7}{2} + \frac{5}{r(t)} \\
s_t^h(t) &= \frac{7}{2} - \frac{5}{r(t)} \\
S_t(r(t)) &= \frac{7}{2} - \frac{5}{r(t)}
\end{aligned}$$

And we know that a competitive equilibrium in an economy with land requires the following conditions to hold:

$$\begin{aligned}
S_t(r(t)) &= p(t)A \\
p(t) &= \frac{d(t+1) + p(t+1)}{r(t)}
\end{aligned}$$

And making use of definition 5 above, we know that a stationary equilibrium is such that $p(t) = p(t+1) = p$, $r(t) = r$ and $d(t+1) = d$, from where it follows that:

$$p = \frac{d+p}{r} \Leftrightarrow p = \frac{d}{r-1}$$

$$S(r) = pA \Leftrightarrow \frac{7}{2} - \frac{5}{r} = \Big(\frac{d}{r-1}\Big)A$$

Substituing, we reach the following equation:

$$7r^2 - 19r + 10 = 0$$

which has two roots, i.e. $r = \left(2, \frac{5}{7}\right)$, with corresponding prices:

$$\begin{cases} r = 2 \Rightarrow p = \frac{d}{r-1} = \frac{1}{2-1} = 1 \\ r = \frac{5}{7} \Rightarrow p = \frac{d}{r-1} = \frac{1}{\frac{5}{7}-1} = -\frac{7}{2} \end{cases}$$

Since the price of land must be nonnegative, we must have $r = 2$ at equilibrium. Consumption and savings are thus given by, $\forall t \geq 1$:

$$\begin{aligned} c_t^h(t) &= \frac{7}{2} + \frac{5}{2} = 6 \\ c_t^h(t+1) &= r(t)c_t^h(t) = 2 \times 6 = 12 \\ s_t^h(t) &= \frac{7}{2} - \frac{5}{2} = 1 \end{aligned}$$

And for the initial old:

$$\begin{aligned} c_0^h(1) &= \omega_0^h(1) + d(1)A + p(1)A \\ &= 10 + 1 \times 1 + 1 \times 1 \\ &= 12 \end{aligned}$$

Furthermore, this consumption allocation clears the market since:

$$c_{t-1}^h(t) + c_t^h(t) = 12 + 6 = 18, \forall t$$

(c) The government now decides to impose a tax τ_d on dividends d (the crop), which will be paid by the individual when old since that is when he/she collects the crop. At the same time it taxes the old, the government issues a lump-sum transfer ϕ to the the old, which is equal to the tax that was imposed. Since there is only 1 individual in each generation, this exercise might seem a bit counterintuitive. The old person's budget constraint is now given by:

$$c_t^h(t+1) \leq \omega_t^h(t+1) + r(t)l^h(t) + a^h(t)\Big[(1-\tau_d)d(t+1) + p^e(t+1)\Big] - \phi$$

We are now able to solve the competitive choice problem:

$$\begin{aligned}
\max_{\{c_t^h(t), c_t^h(t+1), l^h(t), a^h(t)\}} \quad & c_t^h(t)c_t^h(t+1) \\
\text{s.t. } & c_t^h(t) \leq \omega_t^h(t) - l^h(t) - p(t)a^h(t) \\
& c_t^h(t+1) \leq \omega_t^h(t+1) + r(t)l^h(t) + a^h(t)\Big[(1-\tau_d)d(t+1) + p^e(t+1)\Big] - \phi
\end{aligned}$$

And our Lagrangian:

$$\begin{aligned}
\mathcal{L} = & \, c_t^h(t)c_t^h(t+1) + \mu(t)\Big[\omega_t^h(t) - l^h(t) - p(t)a^h(t) - c_t^h(t)\Big] \\
& + \mu(t+1)\Big[\omega_t^h(t+1) + r(t)l^h(t) + a^h(t)\Big[(1-\tau_d)d(t+1) + p^e(t+1)\Big] - \phi - c_t^h(t+1)\Big]
\end{aligned}$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the FOC:

$$\begin{aligned}
\mathbf{c_t^h(t)}: \; & c_t^h(t+1) - \mu(t) = 0, \forall t \geq 1 \\
\mathbf{c_t^h(t+1)}: \; & c_t^h(t) - \mu(t+1) = 0, \forall t \geq 1 \\
\mathbf{l^h(t)}: \; & -\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1 \\
\mathbf{a^h(t)}: \; & -\mu(t)p(t) + \mu(t+1)[(1-\tau_d)d(t+1) + p^e(t+1)] = 0, \forall t \geq 1
\end{aligned}$$

From which we can obtain the following equilibrium conditions:

$$\begin{aligned}
\frac{c_t^h(t+1)}{c_t^h(t)} &= r(t) \\
p(t) &= \frac{(1-\tau_d)d(t+1) + p^e(t+1)}{r(t)}
\end{aligned}$$

Let's pause here and reflect about what we just did: when an individual makes investment decisions, he/she takes into account the effective rate of return that he/she may get from every available alternative. When choosing between investing in private loans or in land, agents will consider the after-tax return (i.e. the effective gain) they will get from both. Since there is no tax on private loans, only the RHS of the second equilibrium condition will change. In equilibrium, the price of land, $p(t)$, must be equal to the present value effective return on land, to make the individuals indifferent between both asset types. Plugging our equilibrium conditions in the lifetime budget constraint, it follows that:

$$c_t^h(t) = \Big(\frac{1}{1+\beta}\Big)\Big[\omega_t^h(t) + \frac{\omega_t^h(t+1) - \phi}{r(t)}\Big]$$
$$s_t^h(t) = \Big(\frac{\beta}{1+\beta}\Big)\omega_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\Big[\frac{\omega_t^h(t+1) - \phi}{r(t)}\Big]$$

Substituting (note that $D(t+1) = d(t+1)$):

$$c_t^h(t) = \frac{7}{2} + \frac{10-\phi}{2r(t)}$$
$$s_t^h(t) = \frac{7}{2} - \frac{10-\phi}{2r(t)}$$

Since we know that the government budget constraint must be balanced, i.e. the total amount of subsidies paid has to be equal to the amount of taxes collected, the subsidy is given by:

$$\sum_{h=1}^{N(t-1)} t_{t-1}^h(t) + \sum_{h=1}^{N(t)} t_t^h(t) = 0 \Leftrightarrow \frac{2}{5} + \phi = 0 \Leftrightarrow \phi = -\frac{2}{5}$$

Once again, making use of definition 5 above, we know that a stationary equilibrium is such that $p(t) = p(t+1) = p$, $r(t) = r$ and $d(t+1) = d$ (and we have $d(t) = 1$), so the market clearing condition is now given by:

$$S(r) = pA \Leftrightarrow \frac{7}{2} - \frac{10+\frac{2}{5}}{2r} = \frac{1-\frac{2}{5}}{r-1}A$$

since $p = \frac{(1-\tau_d)d}{r-1}$. We now have the following equation:

$$35r^2 - 93r + 52 = 0$$

which has two roots, i.e. $r = \Big(\frac{4}{5}, \frac{13}{7}\Big)$, with corresponding prices:

$$\begin{cases} r = \frac{4}{5} \Rightarrow p = \frac{(1-\tau_d)d}{r-1} = \frac{1-\frac{2}{5}}{\frac{4}{5}-1} = -3 \\ r = \frac{13}{7} \Rightarrow p = \frac{(1-\tau_d)d}{r-1} = \frac{1-\frac{2}{5}}{\frac{13}{7}-1} = \frac{7}{10} \end{cases}$$

Since the price of land must be nonnegative, we must have $r = \frac{13}{7}$ at equilibrium. Consumption and savings are thus given by, $\forall t \geq 1$:

$$\begin{aligned}
c_t^h(t) &= \frac{7}{2} + \frac{10 + \frac{2}{5}}{2 \times \frac{13}{7}} = \frac{63}{10} > 6 \\
c_t^h(t+1) &= r(t)c_t^h(t) = \frac{13}{7} \times \frac{63}{10} = \frac{117}{10} < 12 \\
s_t^h(t) &= \frac{7}{2} - \frac{10 + \frac{2}{5}}{2 \times \frac{13}{7}} = \frac{7}{10} < 1
\end{aligned}$$

And for the initial old:

$$\begin{aligned}
c_0^h(1) &= \omega_0^h(1) + (1 - \tau_d)d(1)A + p(1)A - \phi \\
&= 10 + \frac{3}{5} \times 1 + \frac{7}{10} \times 1 + \frac{2}{5} \\
&= \frac{117}{10} < 12
\end{aligned}$$

All individuals from generations $t \geq 1$ gain from this policy since:

$$u_t^h\left(\frac{63}{10}, \frac{117}{10}\right) = \frac{7371}{100} > u_t^h(6, 12) = 72$$

But the initial old are worse-off since their consumption is now inferior to that of the case with no tax.

Exercise 3

Consider the following OLG environment where everyone lives for two periods and $N(t) = 1$. There is no storage or capital. Half the population is "poor" with the endowment profile $\omega_t^{poor} = [5, 3]$ and the other half is "rich" with the endowment profile $\omega_t^{rich} = [5, 4]$.

Preferences are described by $u_t = \ln c_t^h(t) + \ln c_t^h(t+1)$

Wasteful government purchases are given by $G(t) = \frac{3}{2}$ (and need to be funded with taxes and/or government debt).

Suppose all the poor consume the same and likewise with the rich.

(a) Suppose the tax rate on the endowments of the young is 30 percent and that the old are not taxed. Find the competitive equilibrium. (There is no need for government debt in this scenario. Why?)

(b) Suppose the tax/debt policy is changed in period 0 from that described in (a). In period 0, no taxes are levied. Consider two alternative scenarios.

(i) From period 1 onwards, a proportional tax rate of $\frac{3}{7}$ (approx. 43 percent) is levied on the old. Find the competitive equilibrium, including the quantity of government debt issued in each period. Does it feature the same consumption allocation as in (a)? Why or why not?

(ii) From period 1 onwards, all old people (rich and poor alike) have to pay a lump-sum of $\frac{3}{2}$ to the government. Find the competitive equilibrium, including the quantity of debt issued each period. Does it feature the same consumption allocation as in (a)? Why or why not?

Answer

(a) The government wants to fund wasteful government purchases of $\frac{3}{2}$ each period by taxing the endowments of the young by 30%. Essentially, the government wants to make purchases now and taxes now. The tax vector is given by $t_t^h = (0.3\omega_t^h(t), 0)$. Let's start by solving the competitive choice problem:

$$\max_{\{c_t^h(t), c_t^h(t+1), l^h(t)\}} \ln c_t^h(t) + \ln c_t^h(t+1)$$
$$\text{s.t. } c_t^h(t) \leq \omega_t^h(t) - t_t^h(t) - l^h(t)$$
$$c_t^h(t+1) \leq \omega_t^h(t+1) - t_t^h(t+1) + r(t)l^h(t)$$

And our Lagrangian:

$$\mathcal{L} = \ln c_t^h(t) + \ln c_t^h(t+1) + \mu(t)\Big[\omega_t^h(t) - t_t^h(t) - l^h(t) - c_t^h(t)\Big]$$
$$+ \mu(t+1)\Big[\omega_t^h(t+1) - t_t^h(t+1) + r(t)l^h(t) - c_t^h(t+1)\Big]$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the FOC:

$\mathbf{c_t^h(t)}$: $\frac{1}{c_t^h(t)} - \mu(t) = 0, \forall t \geq 1$

$\mathbf{c_t^h(t+1)}$: $\frac{1}{c_t^h(t+1)} - \mu(t+1) = 0, \forall t \geq 1$

$\mathbf{l^h(t)}$: $-\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1$

From which we can obtain the standard equilibrium condition:

$$\frac{c_t^h(t+1)}{c_t^h(t)} = r(t)$$

Plugging in the lifetime budget constraint:

$$c_t^h(t) = \frac{1}{2}\Big[\omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)}\Big]$$
$$s_t^h(t) = \omega_t^h(t) - t_t^h(t) - c_t^h(t) = \frac{1}{2}\Big[\omega_t^h(t) - t_t^h(t)\Big] - \frac{1}{2}\Big[\frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)}\Big]$$

Substituting, we have:

$$c_t^{rich}(t) = \frac{1}{2}\Big[5 \times (1-0.3) + \frac{4-0}{r(t)}\Big] = \frac{7}{4} + \frac{2}{r(t)}$$
$$s_t^{rich}(t) = \frac{1}{2}\Big[5 \times (1-0.3)\Big] - \frac{1}{2}\Big[\frac{4-0}{r(t)}\Big] = \frac{7}{4} - \frac{2}{r(t)}$$
$$c_t^{poor}(t) = \frac{1}{2}\Big[5 \times (1-0.3) + \frac{3-0}{r(t)}\Big] = \frac{7}{4} + \frac{3}{2r(t)}$$
$$s_t^{poor}(t) = \frac{1}{2}\Big[5 \times (1-0.3)\Big] - \frac{1}{2}\Big[\frac{3-0}{r(t)}\Big] = \frac{7}{4} - \frac{3}{2r(t)}$$

And Aggregate Savings are thus given by:

$$S_t(r(t)) = \frac{1}{2}s_t^{poor}(t) + \frac{1}{2}s_t^{rich}(t) = \frac{7}{4} - \frac{7}{4r(t)}$$

Since the government needs to fund its wasteful purchases, we need to check whether the government budget constraint holds at all times t:

$$\frac{1}{2}t_t^{poor}(t) + \frac{1}{2}t_t^{rich}(t) = G(t), \forall t$$
$$\frac{1}{2} \times \frac{3}{10} \times 5 + \frac{1}{2} \times \frac{3}{10} \times 5 = \frac{3}{2}, \forall t$$
$$\frac{3}{2} = \frac{3}{2}, \forall t$$

Given that the government budget is balanced at all times t, it can fully fund its purchases each period, meaning that it does not need to issue debt. Therefore, in a competitive equilibrium, the interest rate should be that aggregate savings are zero, meaning that:

$$S_t(r(t)) = 0 \Leftrightarrow r(t) = 1$$

From where it follows that:

$$
\begin{aligned}
c_t^{rich}(t) &= \frac{7}{4} + \frac{2}{1} = \frac{15}{4} \\
c_t^{rich}(t+1) &= r(t)c_t^{rich}(t) = \frac{15}{4} \\
s_t^{rich}(t) &= \frac{7}{4} - \frac{2}{1} = -\frac{1}{4} \\
c_t^{poor}(t) &= \frac{7}{4} + \frac{3}{2} = \frac{13}{4} \\
c_t^{poor}(t+1) &= r(t)c_t^{poor}(t) = \frac{13}{4} \\
s_t^{poor}(t) &= \frac{7}{4} - \frac{3}{2} = \frac{1}{4}
\end{aligned}
$$

(b) (i) Now the government wants to fund wasteful government purchases of $\frac{3}{2}$ each period by taxing the endowments of the old from period 1 onwards by $\frac{4}{7}$. Since the initial old were not taxed, the government needs to issue debt at period $t = 1$ to finance its expenditures, such that $P(1)B(1) = G(1) = \frac{3}{2}$. This is because only agents born at $t \geq 1$ will be taxed when old. Essentially, the government wants to make purchases now and decides to borrow now and tax later. The tax vector is now given by $t_t^h = (0, \frac{3}{7}\omega_t^h(t+1))$. Making use of our previous results:

$$
\begin{aligned}
c_t^{rich}(t) &= \frac{1}{2}\Big[5 - 0 + \frac{4 \times (1 - \frac{3}{7})}{r(t)}\Big] = \frac{5}{2} + \frac{8}{7r(t)} \\
s_t^{rich}(t) &= \frac{1}{2}\Big[5 - 0\Big] - \frac{1}{2}\Big[\frac{4 \times (1 - \frac{3}{7})}{r(t)}\Big] = \frac{5}{2} - \frac{8}{7r(t)} \\
c_t^{poor}(t) &= \frac{1}{2}\Big[5 - 0 + \frac{3 \times (1 - \frac{3}{7})}{r(t)}\Big] = \frac{5}{2} + \frac{6}{7r(t)} \\
s_t^{poor}(t) &= \frac{1}{2}\Big[5 - 0\Big] - \frac{1}{2}\Big[\frac{3 \times (1 - \frac{3}{7})}{r(t)}\Big] = \frac{5}{2} - \frac{6}{7r(t)}
\end{aligned}
$$

And Aggregate Savings are thus given by:

$$
S_t(r(t)) = \frac{1}{2}s_t^{poor}(t) + \frac{1}{2}s_t^{rich}(t) = \frac{5}{2} - \frac{1}{r(t)}
$$

Since the government needs to finance its wasteful purchases, it will do so by issuing debt. Starting in period 1, the government needs to borrow $P(1)B(1) = \frac{3}{2}$ to fund its spending. Under a competitive equilibrium, it follows that:

$$S_1(r(1)) = P(1)B(1) \Leftrightarrow \frac{5}{2} - \frac{1}{r(1)} = \frac{3}{2} \Leftrightarrow r(1) = 1 \Rightarrow B(1) = \frac{3}{2}$$

And the consumption allocations are given by:

$$\begin{aligned}
c_1^{rich}(1) &= \frac{5}{2} + \frac{8}{7 \times 1} = \frac{51}{14} \\
c_1^{rich}(2) &= r(1) \times c_1^{rich}(1) = \frac{51}{14} \\
c_1^{poor}(1) &= \frac{5}{2} + \frac{6}{7 \times 1} = \frac{47}{14} \\
c_1^{poor}(2) &= r(1) \times c_1^{poor}(1) = \frac{47}{14}
\end{aligned}$$

Notice that even though the interest rate is the same as before, the consumption allocation is different. This is expected since, under the new policy, rich and poor agents pay different amounts, which was not the case before. Therefore the present value of tax payments is not preserved for each person, meaning that the Ricardian equivalence does not hold.

In period $t = 2$, the government uses the tax revenues to pay off the debt from period $t = 1$[8] and then issues debt again to the wasteful government purchases, meaning that we have $P(2)B(2) = G(2) = \frac{3}{2}$. Under a competitive equilibrium, it follows that:

$$S_2(r(2)) = P(2)B(2) \Leftrightarrow r(2)S_2(r(2)) = B(2) \Leftrightarrow r(2) = 1 \Rightarrow B(2) = \frac{3}{2}$$

As you can probably already perceive, the economy is in a knife-edge equilibrium, meaning that $B(t) = \frac{3}{2}, \forall t$, $r(t) = 1, \forall t$ and the allocations are given by:

$$\begin{aligned}
c_t^{rich} &= \left[\frac{51}{14}, \frac{51}{14}\right] \\
c_t^{poor} &= \left[\frac{47}{14}, \frac{47}{14}\right]
\end{aligned}$$

(ii) In order to fund the wasteful government purchases of $\frac{3}{2}$ each period, the government now decides to impose a lump-sum tax on the old of $\frac{3}{2}$ from period 1 onwards. The idea is the same as the previous question, in the sense that the government wants to finance current expenditures by borrowing now and taxing later. We can once again make use of

[8] Notice that tax revenues are given by $\frac{1}{2} \times \frac{3}{7} \times 4 + \frac{1}{2} \times \frac{3}{7} \times 3 = \frac{3}{2}$, which is exactly the amount of outstading government debt from the previous period, $B(1) = \frac{3}{2}$.

our previous results:

$$
\begin{aligned}
c_t^{rich}(t) &= \frac{1}{2}\left[5 - 0 + \frac{4 - \frac{3}{2}}{r(t)}\right] = \frac{5}{2} + \frac{5}{4r(t)} \\
s_t^{rich}(t) &= \frac{1}{2}\Big[5 - 0\Big] - \frac{1}{2}\left[\frac{4 - \frac{3}{2}}{r(t)}\right] = \frac{5}{2} - \frac{5}{4r(t)} \\
c_t^{poor}(t) &= \frac{1}{2}\left[5 - 0 + \frac{3 - \frac{3}{2}}{r(t)}\right] = \frac{5}{2} + \frac{3}{4r(t)} \\
s_t^{poor}(t) &= \frac{1}{2}\Big[5 - 0\Big] - \frac{1}{2}\left[\frac{3 - \frac{3}{2}}{r(t)}\right] = \frac{5}{2} - \frac{3}{4r(t)}
\end{aligned}
$$

And Aggregate Savings are thus given by:

$$
S_t(r(t)) = \frac{1}{2}s_t^{poor}(t) + \frac{1}{2}s_t^{rich}(t) = \frac{5}{2} - \frac{1}{r(t)}
$$

As before, the government needs to issue debt in order to finance its expenditure. Starting in period 1, the government needs to borrow $P(1)B(1) = \frac{3}{2}$ to fund its spending. Under a competitive equilibrium, it follows that:

$$
S_1(r(1)) = P(1)B(1) \Leftrightarrow \frac{5}{2} - \frac{1}{r(1)} = \frac{3}{2} \Leftrightarrow r(1) = 1 \Rightarrow B(1) = \frac{3}{2}
$$

And the consumption allocations are given by:

$$
\begin{aligned}
c_1^{rich}(1) &= \frac{5}{2} + \frac{5}{4 \times 1} = \frac{15}{4} \\
c_1^{rich}(2) &= r(1) \times c_1^{rich}(1) = \frac{15}{4} \\
c_1^{poor}(1) &= \frac{5}{2} + \frac{3}{4 \times 1} = \frac{13}{4} \\
c_1^{poor}(2) &= r(1) \times c_1^{poor}(1) = \frac{13}{4}
\end{aligned}
$$

Unlike the previous exercise, the Ricardian equivalence holds since everyone pays the same amount when old now as they were paying when young before. This is true since $r(t) = 1$, meaning that the present value of the tax burden under this scenario is identical to that of question (a).

As before, the government uses the tax revenues in period $t = 2$ to pay off its debt from the previous period. In order to finance its new wasteful purchases, the government once again issues debt, such that $P(2)B(2) = G(2) = \frac{3}{2}$. As you can probably perceive, the economy is once again in a knife-edge equilibrium with $B(t) = \frac{3}{2}, \forall t$, $r(t) = 1, \forall t$

and the allocations are given by:

$$c_t^{rich} = \left[\frac{15}{4}, \frac{15}{4}\right]$$
$$c_t^{poor} = \left[\frac{13}{4}, \frac{13}{4}\right]$$