JEB064 2022/2023 Sample solution to Homework 2

Competition through limited production capacities (5 points)

In the class, we analyzed a duopoly game where two identical firms initially commit to production capacities, denoted (q_1, q_2), and subsequently engage in price (Bertrand) competition subject to their production capacities. We made two observations: (i) If an equilibrium in price competition exists, it is an equilibrium in which both firms set the price that exactly uses the capacities (call it a clearing price). (ii) When the firms anticipate this equilibrium, they set Cournot-level production capacities.

Now let us focus more on point (ii). Take the economy with a linear demand and constant marginal cost that we had in the class, i.e., the market demand is $Q^m(p) = \frac{a-p}{b}$ and the marginal cost of production for any firm is $c \in (0, a)$. Suppose that Firm i expects that her competitor, Firm $-i$, sets her capacity at the Cournot level that we denote q^*.

1. Describe the residual market of Firm i as a function of parameters (a, b, c) and the price p_i. (That is, derive the demand function on this residual market.)

2. What is the optimal monopolist price $p^m_{res,i}$ on this residual market of Firm i?

3. Suppose Firm i, like Firm $-i$, sets her capacity at the Cournot level, $q_i = q^*$. In the subsequent price competition (where both firms have Cournot-level capacities, $q_1 = q_2 = q^*$), derive price p^* that clears the market, i.e., exactly uses all capacities. Is it indeed optimal for Firm i to set the price p^*?

4. Now, think about Firm i's decision in the capacity-setting stage. Suppose Firm i sets her capacity *below* the Cournot level, $q_i < q^*$. In the subsequent price competition (where Firm i has below-Cournot-level capacity and Firm $-i$ has Cournot-level capacity, $q_i < q_{-i} = q^*$), are there equilibrium prices or not? If so, what are the prices?

Sample solution

1. Describe the residual market of Firm i as a function of parameters (a, b, c) and the price p_i. (That is, derive the demand function on this residual market.)

   We know from the class that Cournot-level amount is $q^* = \frac{a-c}{3b}$. The residual demand function is
   $$Q_i(p_i) = Q^m(p_i) - q^* = \frac{a - p_i}{b} - \frac{a-c}{3b} = \frac{2a + c - 3p_i}{3b}.$$

2. What is the optimal monopolist price $p^m_{res,i}$ on this residual market of Firm i?

   The optimal monopolist price of Firm i is $p^m_{res,i} = \arg\max Q_i(p_i)(p_i - c)$. F.O.C. in this maximization problem writes $(2a + c - 3p_i - 3p_i + 3c)(p_i - c) = 0$. The optimal monopolist price is
   $$p^m_{res,i} = \frac{a + 2c}{3}.$$

Notice that this price is *identical* to the Cournot-level price!

3. Suppose Firm i, like Firm $-i$, sets her capacity at the Cournot level, $q_i = q^*$. In the subsequent price competition (where both firms have Cournot-level capacities, $q_1 = q_2 = q^*$), derive price p^* that clears the market, i.e., exactly uses all capacities. Is it indeed optimal for Firm i to set the price p^*?

   The demand price that clears the market at Cournot-level production is the Cournot-level price. We know from the class that the Cournot-level price is $p^* = \frac{a+2c}{3}$ and Cournot-level amount is $q^* = \frac{a-c}{3b}$. Firm i indeed sets $p_i = p^*$ because *it is a monopolist on the residual market, where the optimal monopolist price is equal to the clearing price,* $p^m_{res} = p^*$. Formally: (i) Setting $p^i < p^*$ doesn't affect demand (due to capacity constraint) and consequently only lowers profits. (i) Setting $p^i > p^m_{res} = p^*$ is unprofitable because price p^m_{res} is the optimal monopolist price on the respective residual market.

4. Now, think about Firm i's decision in the capacity-setting stage. Suppose Firm i sets her capacity *below* the Cournot level, $q_i < q^*$. In the subsequent price competition (where Firm i has below-Cournot-level capacity and Firm $-i$ has Cournot-level capacity, $q_i < q_{-i} = q^*$), are there equilibrium prices or not? If so, what are the prices?

   If an equilibrium in price competition exists, we know that capacities must be exactly used. This clearing price p *exceeds* the Cournot-level price, $p > p^*$.

   - Firm i: Her residual market is still the *same*. Its monopolist price is therefore also identical, $p^m_{res,i} = p^*$. This means that the monopolist price is *below* the clearing price, $p^m_{res,i} = p^* < p$. However, the firm doesn't decrease the price because if all capacities are used, this decrease has no effect on its demand (due to capacity being used). Consequently, $p_i = p$.
   - Firm $-i$: Her residual market is now higher. What is the effect on its monopolist price? To address this question, we write its residual demand function as $Q_{-i}(p_{-i}) = Q^m(p_{-i}) - q_i = \frac{a-p_{-i}}{b} - q_i$. The optimal monopolist price of Firm $-i$ is $p^m_{res,-i} = \arg\max Q_{-i}(p_{-i})(p_{-i} - c)$. F.O.C. in this maximization problem writes $a - p_{-i} - bq_i - p_{-i} + c = 0$. The optimal monopolist price is

     $$p^m_{res,-i} = \frac{a + c - bq_i}{2} > \frac{a + c - bq^*}{2}.$$

     Notice that the monopolist price is decreasing in the amount of the competitor. It means that Firm $-i$ now has a *larger* monopolist price than it used to have. Is this price above or below the clearing price? It is useful to write the clearing price:

     $$p = a - b(q_i + q_{-i}) = a - bq_i - bq^*$$

Firm $-i$ is willing to keep the price $p_{-i} = p$ if and only if her monopolist price is below the clearing price, $p^m_{res,-i} \leq p$ (otherwise she deviates by setting a higher price). This inequality is equivalent to $q_i \leq a - c - 2bq^*$. By inserting $q^* = \frac{a-c}{3b}$, the inequality is $q_i \leq q^*$, which is true.

Therefore, there is an equilibrium in the price-setting stage with the unique clearing price $p_1 = p_2 = p = a - bq_i - bq^* > p^*$.

Battlefields (5 points)

Like in the class, suppose Attacker and Defender fight over Battlefields $i = 1, 2$. Attacker's strategy is (a_1, a_2), where $a_1 \in \mathbb{N}, a_2 \in \mathbb{N}, a_1 + a_2 = A$. Defender's strategy is (d_1, d_2), where $d_1 \in \mathbb{N}, d_2 \in \mathbb{N}, d_1 + d_2 = D$. (We disregard strategies with unused resources. Zero is defined to belong among natural numbers.)

In Battlefield i, Attacker wins if $a_i > d_i$, Defender wins if $d_i > a_i$, and there is a tie if $a_i = d_i$. Win implies payoff 1, tie implies payoff $\frac{1}{2}$, and loss implies payoff 0. The total payoff is the sum of payoffs across the two battlefields.

For simplicity, suppose $A \geq D \geq 2$ and A is odd.

1. *Tie, Tie*: Under which (A, D) there exists a pure-strategy Nash equilibrium with Ties in both battlefields?
2. *Win, Loss*: Under which (A, D) there exists a pure-strategy Nash equilibrium with Attacker's Win in Battlefield 1 and Attacker's Loss in Battlefield 2?
3. *Win, Tie*: Under which (A, D) there exists a pure-strategy Nash equilibrium with Attacker's Win in Battlefield 1 and Tie in Battlefield 2?
4. *Win, Win*: Under which (A, D) there exists a pure-strategy Nash equilibrium with Attacker's Wins in both battlefields?
5. Under which (A, D) there exist multiple pure-strategy Nash equilibria? How do these equilibria differ?

Sample solution For a profile to be a pure-strategy NE, we need that none player profits from any deviation.

1. *Tie, Tie*: A double tie, $a_1 - d_1 = 0 = a_2 - d_2$, implies $A = D$. For each player, it is then always possible to deviate to (Win, Loss), but this deviation doesn't change the total payoff. Therefore, (Tie, Tie) exists if $A = D$.
2. *Win, Loss*: In this outcome, Attacker can always implement a double tie by replicating Defender, which implies an identical payoff. But then she has extra resources $A - D$

that can be used to implement (Win, Tie). Therefore, Attacker doesn't deviate to (Win, Tie) only if these extra resources are zero, $A = D$. Therefore, (Win, Loss) exists only if $A = D$.

3. *Win, Tie*: Defender cannot implement a single win, which means $d_1 = 0$ (otherwise, Defender transfers resources from Battlefield 1 to Battlefield 2 to win Battlefield 2) and $D < a_1$ (otherwise Defender can win Battlefield 1). Attacker cannot implement a double win, which means no extra resources in Battlefield 1 exist, $a_1 = d_1 + 1$, and consequently $a_1 = 1$. But then $D < a_1 = 1$ is impossible by assumption.

4. *Win, Win*: Defender cannot implement any tie (not even win). That is, $D < \min\{a_1, a_2\}$. A necessary condition is $D < \frac{A-1}{2}$ (notice A is odd).

5. There are two environments with multiple pure-strategy NE:

   - If $A = D$, *any strategy profile is a Nash equilibrium.* There exist multiple Nash equilibria with double ties (when $a_1 = d_1$), multiple Nash equilibria with (Win, Loss) (when $a_1 > d_1$) and multiple Nash equilibria with (Loss, Win) (when $a_1 < d_1$).
   - If $D < \frac{A-1}{2}$, there exist *at least two Nash equilibria with double win.* One is $(a_1, a_2, d_1, d_2) = (D + 1, A - D - 1, D, 0)$ and the other is $(a_1, a_2, d_1, d_2) = (A - D - 1, D + 1, 0, D)$. The higher is the difference $A - 2D$, the higher is the number of additional NE.

Auction items for resale (2 points)

Suppose two bidders bid for an item in an auction, and both bidders then want to resell the item on the secondary market. The price on the secondary market is $v \geq 3$, and the price is common knowledge. Bidders can submit only integers.

1. First-price auction: Find all *symmetric* Nash equilibria in pure strategies.

2. Second-price auction: Find all *symmetric* Nash equilibria in pure strategies.

Sample solution In this auction, bidder's values are identical, $v \equiv v_1 = v_2$. In a symmetric equilibrium, there will be a tie when both submit a common bid b.

1. First-price auction: Like in the class, we check necessary equilibrium conditions.

   - If $b > v$, each bidder is better off with a loss, $0 > v - b$.
   - If $b = v$, each bidder is indifferent over tie and loss, $0 = 0$. This is a NE.
   - If $b = v - 1$, each bidder strictly prefers tie over win, as $\frac{1}{2} > 0$. This is a NE.

- If $b = v - 2$, each bidder is indifferent over tie and win, $\frac{2}{2} = 1$. This is a NE.
- If $b < v - 2$, each bidder strictly prefers win over tie, as $\frac{1}{2} > 0$.

Therefore, a symmetric NE (tie) exists when $b \in \{v - 2, v - 1, v\}$.

2. Second-price auction: Recall best responses from the class. If $b_{-i} < v_i$, the best response is $b_i > b_{-i}$ (win). If $b_{-i} = v_i$, the best response is any $b_i \geq 0$ (loss, tie and win are equivalent). If $b_{-i} > v_i$, the best response is $b_i < b_{-i}$ (loss).

   Therefore, a unique symmetric NE (tie) exists when $b = v$.