5303 - Advanced Macroeconomics
Practice Exam Solutions

Alexandre Mendonça *

Stockholm School of Economics

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*Department of Economics, Stockholm School of Economics. E-mail: alexandre.mendonca@phdstudent.hhs.se

Exercise 1

Evaluate the following statements (max 10 lines each).

(a) Differences in human capital account for only a small share of the differences in per capita incomes between rich and poor countries.

(b) Taxes on savings are not distortive in a two-period OLG economy, because savings have to be zero in any equilibrium, with or without taxes.

(c) Going from an allocation that is not Pareto optimal to one that is is a desirable because it constitutes a Pareto improvement.

(d) Ricardian equivalence holds in any competitive equilibrium. This means that government deficits do not matter.

Answer:

(a) True. Classic development accounting exercises find that differences in TFP account for the majority of the cross-country differences in GDP per capita, while differences in physical and human capital play only small roles. However, it is not so easy to capture differences in human capital. Hall and Jones (1999) use average years of schooling. Lagakos et al. (2018) point out that quality of education, not quantity, matters. In particular, they study US immigrants and find a bigger role for human capital.

(b) This is true in an economy without storage and capital. However, it is not true in an economy with storage or capital. In such an economy, savings do not need to be zero and a tax on savings would change the return on capital or storage, thus affecting the consumption decisions

(c) This is false. If someone has all the resources in the economy, this is typically Pareto optimal since any different resource allocation would make that person worse-off. But shifting from an allocation where all agents have something, but all could have more resources, to a situation where a single agent owns all the resources in the economy and all the other agents have nothing is clearly not a Pareto improvement.

(d) Recall the definition of the Ricardian Equivalence:

Proposition 1. *Given an initial equilibrium under some pattern of lump-sum taxation and government borrowing, alternative (intertemporal) patterns* *of lump-sum taxation that keep the present value (at the initial equilibrium's interest rates) of each individual's total tax liability equal to that in the initial equilibrium are equivalent in the following sense. Corresponding to each alternative taxation pattern is a pattern of government borrowing such that the initial equilibrium's consumption allocation, including consumption of the government, and the initial equilibrium's gross interest rates are an equilibrium under the alternative taxation pattern.*

The statement is false since the Ricardian equivalence does not hold in any competitive equilibrium. In fact, it only holds in a rather narrow set of alternative fiscal policies, all of which have to impose the same present value (given interest rates common to all the competitive equilibria in the set) of taxes for each member of each generation. However, provided that the Ricardian Equivalence does hold, it is indeed true that deficits do not matter for consumption or welfare.

Exercise 2

Consider a model in which there are two periods. In period 2 there are two states, denoted s_G and s_B. The state is s_G with probability p and s_B with probability $1-p$. Each agent receives income e_1 in period 1 and $e_2(s)$ in state s of period 2, where $e_2(s_G) \geq e_2(s_B)$. All agents have the same preference over consumption:

$$U = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta\mathbb{E}\Big[\frac{c_2(s)^{1-\gamma}}{1-\gamma}\Big]$$

where $\mathbb{E}$ denotes the expectations operator with respect to the state s. All markets are competitive. Agents can buy a bond, b, at price 1 in period 1 which pays an (endogenous) interest rate $1+r$ in period 2 and is in zero supply. Agents start with initial assets of zero, that is, they have no bond holding. There is no capital and no firms in this economy.

(a) Derive the optimality condition for the agent's bond holdings and state the bond market clearing condition. What are the implications of the bond market clearing condition for the equilibrium trading of consumption across time?

(b) Assume that $e_1 = 2$ in period 1, $e_2(s_G) = 3$ and $e_2(s_B) = 1$ in period 2, $\beta = \frac{1}{2}$, $\gamma = 1$ and $p = \frac{1}{2}$. What is the interest rate $1+r$ in equilibrium?

(c) Now assume instead that $e_2(s_G) = 2$ and $e_2(s_B) = 2$ in period 2. What is the interest rate $1 + r$ in equilibrium? Compare to your answer in part (b) and provide intuition.

Answer:

(a) The budget constraints for each period in this economy are given by, for any state $s \in \{s_G, s_B\}$:

$$c_1 = e_1 - b$$
$$c_2(s) = e_2(s) + (1 + r)b$$

And the consumer's problem is thus given by:

$$\max_{\{c_1, c_2(s), b\}} \frac{c_1^{1-\gamma}}{1-\gamma} + \beta\mathbb{E}\Big[\frac{c_2(s)^{1-\gamma}}{1-\gamma}\Big]$$
$$\text{s.t. } c_1 = e_1 - b$$
$$c_2(s) = e_2(s) + (1 + r)b$$

And our Lagrangian:

$$\mathcal{L} = \frac{c_1^{1-\gamma}}{1-\gamma} + \beta\mathbb{E}\Big[\frac{c_2(s)^{1-\gamma}}{1-\gamma}\Big] + \mu(1)[e_1 - b - c_1] + \mu(2)[e_2(s) + (1+r)b - c_2(s)]$$

Taking the first order conditions:

$$\mathbf{c_1}\text{: } c_1^{-\gamma} - \mu(1) = 0$$
$$\mathbf{c_2(s)}\text{: } \beta\mathbb{E}[c_2(s)^{-\gamma}] - \mu(2) = 0$$
$$\mathbf{b}\text{: } -\mu(1) + (1 + r)\mu(2) = 0$$

From which we obtain the following equilibrium condition:

$$\mathbb{E}[c_2(s)^{-\gamma}] = \frac{1}{\beta(1 + r)} c_1^{-\gamma}$$

For a constant population size, N, since the bond is in zero supply, market clearing in the bond market implies that:

$$Nb = 0 \Rightarrow b = 0$$

Therefore, at equilibrium we have that:

$$c_1 = e_1$$
$$c_2(s) = e_2(s)$$

(b) Let's first find the expected consumption in period 2:

$$\mathbb{E}[c_2(s)^{-\gamma}] = \mathbb{E}[e_2(s)^{-\gamma}] = p \times e_2(s_G)^{-\gamma} + (1-p) \times e_2(s_B)^{-\gamma} = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times 1 = \frac{2}{3}$$

Making use of our equilibrium condition from question (a), it follows that:

$$\mathbb{E}[c_2(s)^{-\gamma}] = \frac{1}{\beta(1+r)} c_1^{-\gamma} \Leftrightarrow \frac{2}{3} = \frac{1}{\frac{1}{2} \times (1+r)} \times \frac{1}{2} \Leftrightarrow 1 + r = \frac{3}{2}$$

(c) Once again, let's first find the expected consumption in period 2:

$$\mathbb{E}[c_2(s)^{-\gamma}] = \mathbb{E}[e_2(s)^{-\gamma}] = p \times e_2(s_G)^{-\gamma} + (1-p) \times e_2(s_B)^{-\gamma} = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

Making use of our equilibrium condition from question (a), it follows that:

$$\mathbb{E}[c_2(s)^{-\gamma}] = \frac{1}{\beta(1+r)} c_1^{-\gamma} \Leftrightarrow \frac{1}{2} = \frac{1}{\frac{1}{2} \times (1+r)} \times \frac{1}{2} \Leftrightarrow 1 + r = 2 > \frac{3}{2}$$

In the economy with uncertainty (question (b)), the average income in period 2 is equal to the average income in period 2 in the economy without uncertainty (question (c)). However, due to the uncertainty factor in question (b), the interest rate is lower than in the case without uncertainty, and it decreases in the amount of risk. Since the marginal utility of the stated preferences is strictly convex, there is a precautionary savings motive for the households, which is reflected by the fact that the interest rate is lower in the economy with uncertainty. In fact, the interest rate has to fall to accommodate the precautionary savings motive and to bring the bond market down to the zero-bond-demand equilibrium.

Exercise 3

Assume there are families. Agents live for two periods. They are children in the first and adults in the second period.

As adults, agents earn income depending on their skill level, $e_t(t+1)$, and how much time they spend working. If they are skilled, $e_t(t+1) = 1$, they work in the manufacturing sector at wage rate w^s. If they are unskilled, $e_t(t+1) = 0$, they work in the agricultural sector at wage rate w^u.

We assume that children do not make any decisions.

Parents decide how many children, n^h_{t+1}, to have and whether or not to educate their children. Having children involves a time cost, $f = \frac{1}{2}$ units of time per child. Children are born unskilled, $e_{t+1}(t+1) = 0$. Parents can choose to educate their children so that they become skilled ($e_{t+1}(t+2) = 1$), but parents can also choose not to educate them so that they remain unskilled ($e_{t+1}(t+2) = 0$). Educating children involves a time-cost $g = \frac{1}{3}$ units of time per child, for the unskilled parents, but is costless for the skilled parents (schools are located in urban areas).

All parents care about their own consumption and also about their children's quality (income level). The utility function of an adult, (born at t but adults at $t+1$), is given by:

$$u^h_t = c^h_t(t+1) + a(n^h_{t+1})n^h_{t+1}\frac{\left[e_{t+1}(t+2)w^s + [1 - e_{t+1}(t+2)]w^u\right]^{\frac{1}{2}}}{\frac{1}{2}}$$

where $h = s$ (skilled) or $h = u$ (unskilled), and where the degree of altruism per child is given by:

$$a(n^h_{t+1}) = \frac{[n^h_{t+1}]^{-\frac{1}{2}}}{\frac{1}{2}}$$

(a) How much time can a skilled and unskilled parent devote to working, if total time is T?

(b) Write down the budget constraint for the skilled and unskilled parent.

(c) In 1800, the average woman had 7 children, 94% of the population (the parents) lived in rural areas and were unskilled, and the remaining 6% of the population (the parents) lived in urban areas and were skilled. The manufacturing wage was somewhat higher than the wage in agriculture. Let $w^s = 4$ and $w^u = \frac{8}{3}$. You will now show that in 1800, the skilled parents chose to have few but educates children and that the unskilled parents chose to have many but uneducated children. By 1940 this had changed. Technological progress had caused real wages to rise in both sectors, the share of population living in rural areas had declined and the average woman had 2 children. You will use this model to show that by 1940, everyone in this model had few but educated children.

i. Solve the skilled parents' problem in 1800 (number of children and whether or not to educate their children).

ii. Solve the unskilled parents' problem in 1800 (number of children and whether or not to educate their children). Hint: begin by solving for number of children, then compare whether parents prefer to educate.

iii. Solve the skilled and the unskilled parents' problem in 1940 assuming that wages have increased: $w^s = 8$ and $w^u = 4$.

Answer:

(a) Given that the total time is T, each skilled parent devotes the following amount of time to working:

$$l^s_{t+1} = T - fn^s_{t+1}$$

whereas each unskilled parent devotes the following amount of time to working:

$$l^u_{t+1} = T - fn^u_{t+1} - gn^u_{t+1}e_{t+1}(t+2)$$

(b) Given salaries w^s and w^u for skilled and unskilled parents, respectively, consumption for an adult born at time t is given by:

$$c^h_t(t+1) = w^h l^h_{t+1}, h \in \{s, u\}$$

Therefore, the budget constraints are given by:

$$\begin{aligned}
c^s_t(t+1) &= w^s l^s_{t+1} = w^s[T - fn^s_{t+1}] \\
c^u_t(t+1) &= w^u l^u_{t+1} = w^u[T - fn^u_{t+1} - gn^u_{t+1}e_{t+1}(t+2)]
\end{aligned}$$

(c) Prior to solving the problem, we will rewrite the utility function in the following way:

$$\begin{aligned}
u^h_t &= c^h_t(t+1) + a(n^h_{t+1})n^h_{t+1}\frac{\left[e_{t+1}(t+2)w^s + [1 - e_{t+1}(t+2)]w^u\right]^{\frac{1}{2}}}{\frac{1}{2}} \\
u^h_t &= c^h_t(t+1) + \frac{[n^h_{t+1}]^{-\frac{1}{2}}}{\frac{1}{2}}n^h_{t+1}\frac{\left[e_{t+1}(t+2)w^s + [1 - e_{t+1}(t+2)]w^u\right]^{\frac{1}{2}}}{\frac{1}{2}} \\
u^h_t &= c^h_t(t+1) + 4[n^h_{t+1}]^{\frac{1}{2}}\left[e_{t+1}(t+2)w^s + [1 - e_{t+1}(t+2)]w^u\right]^{\frac{1}{2}}
\end{aligned}$$

i. Skilled parents do not pay the education cost, meaning that they will trivially choose to educate their children. In particular, if the skilled parent chooses not to educate his/her children, then $e_{t+1}(t+2) = 0$ and the parent gets $2(w^u)^{\frac{1}{2}}$ in the utility function, *ceteris paribus.* Should the parent choose to educate his/her children, then $e_{t+1}(t+2) = 1$ and the parent gets $2(w^s)^{\frac{1}{2}}$ in the utility function, *ceteris paribus.* Since $w^s > w^u$ by assumption, the parent will always choose to educate his/her children, i.e. $e_{t+1}(t+2) = 1$. The utility maximization problem is thus given by:

$$\max_{\{c_t^s(t+1), n_{t+1}^s\}} c_t^s(t+1) + 4[n_{t+1}^s]^{\frac{1}{2}}\left[w^s\right]^{\frac{1}{2}}$$
$$\text{s.t. } c_t^s(t+1) = w^s[T - fn_{t+1}^s]$$

Our Lagrangian is thus given by:

$$\mathcal{L} = c_t^s(t+1) + 4[n_{t+1}^s]^{\frac{1}{2}}\left[w^s\right]^{\frac{1}{2}} + \mu(t+1)\left[w^s[T - fn_{t+1}^s] - c_t^s(t+1)\right]$$

Taking the FOC:

$\mathbf{c_t^s(t+1)}$: $1 - \mu(t+1) = 0, \forall t$

$\mathbf{n_{t+1}^s}$: $2[n_{t+1}^s]^{-\frac{1}{2}}\left[w^s\right]^{\frac{1}{2}} - \mu(t+1)w^s f = 0, \forall t$

From which we get the following equilibrium condition:

$$n_{t+1}^s = \left[\frac{2}{f(w^s)^{\frac{1}{2}}}\right]^2$$

Substituting:

$$n_{t+1}^s = \left[\frac{2}{\frac{1}{2}(4)^{\frac{1}{2}}}\right]^2 = 4$$

And consumption and utility of the skilled parent are given by:

$$c_t^s(t+1) = 4[T - \frac{1}{2} \times 4] = 4T - 8$$
$$u_t^s = 4T - 8 + 4 \times [4]^{\frac{1}{2}} \times \left[4\right]^{\frac{1}{2}} = 4T + 8$$

ii. For the unskilled parent, the choice of education matters. On the one hand, educating children increases the 'altruistic' part of utility but, on the other hand, it reduces consumption and, therefore, the utility from consumption. The choice of whether to educate children or not depends on the relative size of these utility changes. We need to address the parents' utility for both possible values of $e_{t+1}(t+2)$. The utility maximization problem of the unskilled parent is given by:

$$\max_{\{c_t^u(t+1),n_{t+1}^u\}} c_t^u(t+1) + 4[n_{t+1}^u]^{\frac{1}{2}}\big[e_{t+1}(t+2)w^s + [1-e_{t+1}(t+2)]w^u\big]^{\frac{1}{2}}$$
$$\text{s.t. } c_t^u(t+1) = w^u[T - fn_{t+1}^u - gn_{t+1}^u e_{t+1}(t+2)]$$

Our Lagrangian is thus given by:

$$\begin{aligned}\mathcal{L} &= c_t^u(t+1) + 4[n_{t+1}^u]^{\frac{1}{2}}\big[e_{t+1}(t+2)w^s + [1-e_{t+1}(t+2)]w^u\big]^{\frac{1}{2}}\\ &+ \mu(t+1)\big[w^u[T - fn_{t+1}^u - gn_{t+1}^u e_{t+1}(t+2)] - c_t^u(t+1)\big]\end{aligned}$$

Taking the FOC:

$\mathbf{c_t^u(t+1)}$: $1 - \mu(t+1) = 0, \forall t$

$\mathbf{n_{t+1}^u}$: $2[n_{t+1}^u]^{-\frac{1}{2}}\big[e_{t+1}(t+2)w^s + [1-e_{t+1}(t+2)]w^u\big]^{\frac{1}{2}} - \mu(t+1)w^u[f + ge_{t+1}(t+2)] = 0, \forall t$

From which we get the following equilibrium condition:

$$n_{t+1}^u = \left[\frac{2\big[e_{t+1}(t+2)w^s + [1-e_{t+1}(t+2)]w^u\big]^{\frac{1}{2}}}{w^u[f + ge_{t+1}(t+2)]}\right]^2$$

We need to address two cases:

Case I: Parents educate children $\rightarrow e_{t+1}(t+2) = 1$
Substituting:

$$n_{t+1}^u = \left[\frac{2(w^s)^{\frac{1}{2}}}{w^u[f + ge_{t+1}(t+2)]}\right]^2 = \left[\frac{2 \times (4)^{\frac{1}{2}}}{\frac{8}{3}[\frac{1}{2} + \frac{1}{3}]}\right]^2 = \frac{81}{25} = 3.24$$

And consumption and utility are given by:

$$\begin{aligned}c_t^u(t+1) &= \frac{8}{3}[T - \frac{1}{2} \times \frac{81}{25} - \frac{1}{3} \times \frac{81}{25}] = \frac{8}{3}T - \frac{36}{5}\\ u_t^u &= \frac{8}{3}T - \frac{36}{5} + 4 \times \left(\frac{81}{25}\right)^{\frac{1}{2}} \times 4^{\frac{1}{2}} = \frac{8}{3}T + \frac{36}{5}\end{aligned}$$

Case II: Parents do not educate children $\rightarrow e_{t+1}(t+2) = 0$
Substituting:

$$n_{t+1}^u = \left[\frac{2(w^u)^{\frac{1}{2}}}{w^u f}\right]^2 = \left[\frac{2 \times \left(\frac{8}{3}\right)^{\frac{1}{2}}}{\frac{8}{3} \times \frac{1}{2}}\right]^2 = 6$$

And consumption and utility are given by:

$$c_t^u(t+1) = \frac{8}{3}[T - \frac{1}{2} \times 6] = \frac{8}{3}T - 8$$
$$u_t^u = \frac{8}{3}T - 8 + 4 \times 6^{\frac{1}{2}} \times \left(\frac{8}{3}\right)^{\frac{1}{2}} = \frac{8}{3}T + 8$$

Comparing both cases, we can see that unskilled parents get a higher utility from not educating their children (Case II). This means that unskilled parents choose to have many children and not educate them. This is in line with the setting of the exercise.

iii. Starting with the skilled parents, we can substitute and get that:

$$n_{t+1}^s = \left[\frac{2}{\frac{1}{2}(8)^{\frac{1}{2}}}\right]^2 = 2$$
$$c_t^s(t+1) = 8[T - \frac{1}{2} \times 2] = 8T - 8$$
$$u_t^s = 8T - 8 + 4 \times [2]^{\frac{1}{2}} \times [8]^{\frac{1}{2}} = 8T + 8$$

For the unskilled parents, we must once again address both cases:

Case I: Parents educate children $\rightarrow e_{t+1}(t+2) = 1$
Substituting:

$$n_{t+1}^u = \left[\frac{2 \times (8)^{\frac{1}{2}}}{4[\frac{1}{2} + \frac{1}{3}]}\right]^2 = \frac{72}{25} = 2.88$$
$$c_t^u(t+1) = 4[T - \frac{1}{2} \times \frac{72}{25} - \frac{1}{3} \times \frac{72}{25}] = 4T - \frac{48}{5}$$
$$u_t^u = 4T - \frac{48}{5} + 4 \times \left(\frac{72}{25}\right)^{\frac{1}{2}} \times 8^{\frac{1}{2}} = 4T + \frac{48}{5}$$

Case II: Parents do not educate children $\rightarrow e_{t+1}(t+2) = 0$
Substituting:

$$n_{t+1}^u = \left[\frac{2 \times 4^{\frac{1}{2}}}{4 \times \frac{1}{2}}\right]^2 = 4$$
$$c_t^u(t+1) = 4[T - \frac{1}{2} \times 4] = 4T - 8$$
$$u_t^u = 4T - 8 + 4 \times 4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4T + 8$$

Under the 1940 parameters, unskilled parents get a higher utility from educating their children (Case I). This means that unskilled parents choose to have fewer children than in 1800 and to educate them.

Exercise 4

Assume that each generation in this economy consists of 100 individuals with the earnings profile: $\omega_t^h = [3, 1]$, and that there exists a storage technology with storage parameter $\lambda = 1$.

Assume that each individual has preferences that can be captured by the following utility function:

$$u_t^h = \log c_t^h(t) + \beta c_t^h(t+1)$$

with $\beta = \frac{2}{3}$. Assume that the government needs to finance government expenditure equal to $G = 10$ and decides to use a capital income tax (i.e. tax on storage) to do so, and that the government announces in period t what the capital tax will be in period $t+1$. Solve for the capital tax rate. Is it unique?

Answer:

Let's denote the tax on storage by τ. Given the return on storage, λ, the government budget constraint at time t is given by:

$$G(t) \leq \tau\lambda K(t+1)$$

where $G(t)$ it the government expenditure. Therefore, at equilibrium:

$$K(t+1) = \frac{G(t)}{\tau\lambda}$$

The individuals' problem is given by:

$$\begin{aligned} \max_{\{c_t^h(t), c_t^h(t+1), k^h(t+1)\}} \quad & \log c_t^h(t) + \beta c_t^h(t+1) \\ \text{s.t.} \quad & c_t^h(t) = \omega_t^h(t) - k^h(t+1) \\ & c_t^h(t+1) = \omega_t^h(t+1) + (1-\tau)\lambda k^h(t+1) \end{aligned}$$

Our Lagrangian is given by:

$$\begin{aligned} \mathcal{L} = \log c_t^h(t) + \beta c_t^h(t+1) + \mu(t)[\omega_t^h(t) - k^h(t+1) - c_t^h(t)] \\ + \mu(t+1)[\omega_t^h(t+1) + (1-\tau)\lambda k^h(t+1) - c_t^h(t+1)] \end{aligned}$$

Taking the FOC:

$\mathbf{c_t^h(t)}$: $\frac{1}{c_t^h(t)} - \mu(t) = 0, \forall t \geq 1$

$\mathbf{c_t^h(t+1)}$: $\beta - \mu(t+1) = 0, \forall t \geq 1$

$\mathbf{k^h(t+1)}$: $-\mu(t) + \mu(t+1)(1-\tau)\lambda = 0, \forall t \geq 1$

From which we get the following equilibrium condition:

$$c_t^h(t) = \frac{1}{\beta(1-\tau)\lambda}$$

Therefore, individual saving are given by:

$$s_t^h(t) = \omega_t^h(t) - c_t^h(t) = \omega_t^h(t) - \frac{1}{\beta(1-\tau)\lambda}$$

Since there is no population growth, aggregate savings are given by:

$$S_t(t) = N s_t^h(t) = N\Big[\omega_t^h(t) - \frac{1}{\beta(1-\tau)\lambda}\Big]$$

In equilibrium, we must have that:

$$S_t(t) = K(t+1)$$

Substituting:

$$\begin{aligned} N\Big[\omega_t^h(t) - \frac{1}{\beta(1-\tau)\lambda}\Big] &= \frac{G(t)}{\tau\lambda} \\ 100\Big[3 - \frac{1}{\frac{2}{3} \times (1-\tau) \times 1}\Big] &= \frac{10}{\tau \times 1} \\ -30\tau^2 + 16\tau - 1 &= 0 \end{aligned}$$

And we get the following solutions:

$$\tau \approx 0.07 \vee \tau \approx 0.46$$

We can thus conclude that the solution is not unique. However, individuals are better off under the low tax case (you can check this yourself).