5330 - Advanced Microeconomic Theory
Answers to Problem Set 2

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September 7, 2023

1 Exercise 1.39

Theorem 1.9 states that: Under assumption 1.2 we have the following relations between the Hicksian and the Marshallian demand functions for $\mathbf{p} \gg 0$, $y \geq 0$, $u \in U$ and $i = 1, .., n$:

1. $x_i(\mathbf{p}, y) = x_i^h(\mathbf{p}, v(\mathbf{p}, y)$
2. $x_i^h(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u)$

We will prove the second relation in theorem 1.9. Under assumption 1.2, we know that $u(\cdot)$ is continuous and strictly quasiconcave. Therefore, we know that a solution exists and is unique. Consequently, the solutions to both constrained optimization problems exist and are unique. Hence, the Marshallian $x_i(\mathbf{p}, y)$ and Hicksian demand functions $\mathbf{x}^h(\mathbf{p}, u)$ are well-defined.

To prove the second relation, we let: $\mathbf{x}^0 = \mathbf{x}(\mathbf{p}^0, y^0)$ and let $u^0 = u(\mathbf{x}^0)$, Since we know that $u(\cdot)$ is strictly increasing. This tells us that at the optimum solution to the expenditure minimization problem must bind at the constraint on utility.

We have $e(\mathbf{p}^0, u^0) = \mathbf{p}^0 \cdot \mathbf{x}^0$ by definition of $e(\cdot)$. Thus, when solving the expenditure minimization problem for the parameters: $(\mathbf{p}^0, u^0)$ we can represent expenditure through the Hicksian demand $\mathbf{x}^h(\mathbf{p}^0, u^0)$:

$$e(\mathbf{p}^0, u^0) = \mathbf{p}^0 \cdot \mathbf{x}^h(\mathbf{p}^0, u^0) \tag{1}$$

Also, when solving the maximization utility problem for $(\mathbf{p}^0, y^0)$ we get: $y^0 = \mathbf{p}^0 \cdot \mathbf{x}^0$. Thus, the constraint of income is binding. This implies that income equals expenditure at the Marshallian solution: $e(\mathbf{p}^0, u^0) = y^0$.

By Theorem 1.8 we can write Marshallian demand as: $\mathbf{x}(\mathbf{p}^0, e(\mathbf{p}^0, u^0)) = \mathbf{x}^0$. Since we know that at the solution income level equals expenditure it must also be true that:

$$e(\mathbf{p}^0, u^0) = y^0 \Rightarrow \mathbf{p}^0 \cdot \mathbf{x}^h(\mathbf{p}^0, u^0) = \mathbf{p}^0 \cdot \mathbf{x}^0$$

$$\Rightarrow \mathbf{p}^0 \cdot \mathbf{x}(\mathbf{p}^0, e(\mathbf{p}^0, u^0)) = \mathbf{p}^0 \cdot \mathbf{x}^h(\mathbf{p}^0, u^0)$$

Therefore, we have proved that: $x^h(\mathbf{p}^0, u^0) = \mathbf{x}(\mathbf{p}^0, e(\mathbf{p}^0, u^0)$
Thus, we have proved the second relation in Theorem 1.9.

2 Exercise 1.51

(a) The utility function is $U(x_1, x_2) = (x_1)^{\frac{1}{2}} + (x_2)^{\frac{1}{2}}$. We need to find the demand functions that maximize utility subject to budget constraint, as shown below:

$$\text{Max } \; U(x_1, x_2) \text{ s.t } \; p_i x_i \leq y$$

To do so, we first need to set up Lagrangian function:

$$L = (x_1)^{\frac{1}{2}} + (x_2)^{\frac{1}{2}} + \lambda(y - p_1x_1 - p_2x_2)$$

Write down first order conditions:

$$\frac{\partial L}{\partial x_1} = \frac{1}{2}(x_1)^{-1/2} - \lambda p_1 = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{1}{2}(x_2)^{-1/2} - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = y - p_1x_1 - p_2x_2 = 0$$

Divide first and second order conditions:

$$\frac{0.5(x_1)^{-1/2}}{0.5(x_2)^{-1/2}} = \frac{p_1\lambda}{p_2\lambda} = (\frac{x_1}{x_2})^{-1/2} \text{ »> } (\frac{x_2}{x_1})^{1/2} = \frac{p_1}{p_2} = \frac{x_2}{x_1} = \frac{(p_1)^2}{(p_2)^2}$$

Express x_2 :

$$\frac{x_2}{x_1} = \frac{(p_1)^2}{(p_2)^2} \text{ »> } x_2 = \frac{p_1^2 x_1}{p_2^2}$$

Plug expression for x_2 in the budget constraint and derive x_1:

$$p_1x_1 + p_2x_2 = p_1x_1 + P_2\frac{(p_1)^2x_1}{p_2^2} = y \text{ »> } x_1(p_1 + \frac{(p_1)^2}{p_2}) = y \text{ »> } x_1^* = \frac{y}{p_1 + \frac{p_1^2}{p_2}} = \frac{yp_2}{p_1p_2 + p_1^2}$$

Plug x_1^* into the expression for x_2:

$$x_2^* = \frac{(p_1)^2x_1}{p_2^2} = \frac{p_1^2}{p_2^2}\frac{yp_2}{p_1p_2 + p_1^2} = \frac{yp_1}{p_1p_2 + p_2^2}$$

Marshallian demand functions are thus the following:

$$x_1^*(p_1, p_2, y) = \frac{yp_2}{p_1p_2 + p_1^2}$$

$$x_2^*(p_1, p_2, y) = \frac{yp_1}{p_1p_2 + p_2^2}$$

(b) The Slutsky equation showing the effects on x_1 when p_2 changes is the following:

$$\frac{\partial x_1(p, y)}{\partial p_2} = \frac{\partial x_1^h(p, u^*)}{\partial p_2} - x_2(p, y)\frac{\partial x_1(p, y)}{\partial y}$$

Here the change in x_1 when p_2 changes is the total effect (TE) and this term equals substitution term (SE) (the first entry on the right hand side of the equation) minus income effect (IE) (second entry on the right hand side of the equation). The above equation can therefore be expressed as:

$$TE = SE - IE$$

Thus

$$SE = TE + IE$$

Given the information above, in order to compute substitution term, we first need to calculate total effect and income effect.
First lets calculate income effect (IE) which equals ordinary demand for x_2 multiplied by partial derivative of x_1 with respect to income:

$$x_2(p, y) = \frac{yp_1}{p_1 p_2 + p_2^2}$$

$$\frac{\partial x_1(p, y)}{\partial y} = \frac{p_2}{p_1 p_2 + p_1^2}$$

By multiplying these two terms we get the income effect:

$$\frac{yp_1}{p_1 p_2 + p_2^2} \frac{p_2}{p_1 p_2 + p_1^2} = \frac{yp_1 p_2}{p_1^2 p_2^2 + p_1^3 p_2 + p_2^3 p_1 + p_1^2 p_2^2} = \frac{yp_1 p_2}{p_1 p_2 (p_1 p_2 + p_1^2 + p_2^2 + p_1 p_2)} = \frac{y}{(p_1 + p_2)^2}$$

Now lets calculate total effect (TE):

$$\frac{\partial x_1(p, y)}{\partial p_2} = \frac{\partial y p_2 (p_1 p_2 + p_1^2)^{-1}}{\partial p_2} =$$

$$y(p_1 p_2 + p_1^2)^{-1} - yp_2 (p_1 p_2 + p_1^2)^{-2} p_1 =$$

$$\frac{y}{p_1 p_2 + p_1^2} - \frac{yp_1 p_2}{(p_1 p_2 + p_1^2)^2} =$$

$$\frac{y(p_1 p_2 + p_1^2) - yp_1 p_2}{(p_1 p_2 + P_1^2)^2} =$$

$$\frac{yp_1^2}{p_1^2 (p_1 + p_2)^2} = \frac{y}{(p_1 + p_2)^2}$$

Substitution term is therefore the following:

$$\frac{\partial x_1^h(p, u^*)}{\partial p_2} = SE = TE + IE = \frac{y}{(p_1 + p_2)^2} + \frac{y}{(p_1 + p_2)^2} = \frac{2y}{(p_1 + p_2)^2}$$

Because substitution term is clearly positive, we can state that increase in p_2 leads to increase in x_1^h.
(c) In order to clasify goods as either gross substitutes or complements, we need to look at what happens to the demand of one good when the price of another changes. If the demand for x_1 increases when p_2 increases, goods are substitutes. In the previous part of the exercise we already calculated demand for x_1 with respect to changes in p_2:

$$\frac{\partial x_1(p, y)}{\partial p_2} = \frac{y}{(p_1 + p_2)^2} > 0$$

This term is clearly positive. Because of the symmetry, the effect on x_2 when p_1 changes is also positive. The two goods are therefore gross substitutes as the demand increases when the price of other good increases.

3 Exercise 1.54

The n-good Cobb-Douglas utility function is $u(x) = A\prod_{i=1}^{n} x_i^{\alpha_i}$, $A > 0$, $\sum_{i=1}^{n} \alpha_i = 1$.

(a) If we maximize utility subject to a budget constraint we obtain

$$L = A\prod_{i=1}^{n} x_i^{\alpha_i} - \lambda[\sum_{j=1}^{n} p_j x_j - y]$$

The FOCs are:

$$\frac{\partial L}{\partial x_i} = A\frac{\alpha_i \prod_{i=1}^{n} x_i^{\alpha_i}}{x_i} - \lambda p_i = 0,$$
$$\frac{\partial L}{\partial \lambda} = -\sum_{j=1}^{n} p_j x_j + y = 0.$$

Taking the ratio of any of the first $n$conditions to obtain

$$\frac{p_i}{p_j} = \frac{A\frac{\alpha_i \prod_{k=1}^{n} x_k^{\alpha_k}}{x_i}}{A\frac{\alpha_j \prod_{k=1}^{n} x_k^{\alpha_k}}{x_j}} = \frac{\alpha_i x_j}{\alpha_j x_i}$$

Solving for x_j, substituting to the lambda-FOC, and using $\sum_{j=1}^{n} \alpha_j = 1$ gives

$$y = \sum_{j=1}^{n} p_j \frac{\alpha_j x_i p_i}{\alpha_i p_j} = \frac{x_i p_i}{\alpha_i} \sum_{j=1}^{n} \alpha_j = \frac{x_i p_i}{\alpha_i}.$$

The Marshallian demands are therefore $x_i^*(p, y) = (\alpha_i / p_i)y$

(b) Substituting the optimal demands to the utility function, and using $\sum_{i=1}^{n} \alpha_i = 1$, to express the indirect utility function

$$v(p, y) = u(x^*(p, y)) = A\prod_{i=1}^{n} x_i^{\alpha_i} = A\prod_{i=1}^{n} [y(\alpha_i / p_i)]^{\alpha_i} = Ay\prod_{i=1}^{n} (\alpha_i / p_i)^{\alpha_i}.$$

(c) Since $e(p, u) = y$ when $u = v(p, y)$ it follows that $u = Ae(p, u)\prod_{i=1}^{n} (\alpha_i / p_i)^{\alpha_i}$. The expenditure function is therefore

$$e(p, u) = \frac{u}{A}\prod_{i=1}^{n} (p_i / \alpha_i)^{\alpha_i}.$$

(d) The Hicksian demands $x_i^h(p, u)$ are equal to the Marshallian demands $x_i^*(p, y)$ when $y = e(p, u)$, thus

$$x_i^h(p, u) = x_i^*(p, e(p, u)) = (\alpha_i / p_i)\frac{u}{A}\prod_{i=1}^{n} (p_i / \alpha_i)^{\alpha_i}.$$

4 Exercise 1.57

(a) The Stone-Geary utility function is $u(x) = \prod_{j=1}^{n}(x_j - a_j)^{b_j}$ where $\sum_{j=1}^{n} b_j = 1$. We want the expenditure function so we minimize expenditure subject to the utility level being at least u. The Lagrangian is

$$L = \sum_{i=1}^{n} p_i x_i - \lambda[\prod_{j=1}^{n}(x_j - a_j)^{b_j} - u].$$

The first order conditions are

$$\frac{\partial L}{\partial x_i} = p_i - \lambda b_i (x_i - a_i)^{-1} \prod_{j=1}^{n}(x_j - a_j)^{b_j} = 0$$

for all $i = 1, 2, \ldots, n$ and

$$\frac{\partial L}{\partial \lambda} = \prod_{j=1}^{n}(x_j - a_j)^{b_j} = u.$$

The first order conditions give

$$p_i x_i - p_i a_i = \lambda b_i u$$

for all $i = 1, 2, \ldots, n$. So

$$\lambda u = (p_i/b_i)(x_i - a_i)$$

for all $i = 1, 2, \ldots, n$. Hence

$$x_j - a_j = (p_i/b_i)(x_i - a_i)(b_j/p_j)$$

for all $i, j = 1, 2, \ldots, n$ as λu is independent from i. Substitute this in the lambda FOC. So

$$\begin{aligned} u &= \prod_{j=1}^{n}[(p_i/b_i)(x_i - a_i)(b_j/p_j)]^{b_j} \\ &= \prod_{j=1}^{n}[(p_i/b_i)(x_i - a_i)]^{b_j} \prod_{j=1}^{n}(b_j/p_j)]^{b_j} \\ &= (p_i/b_i)(x_i - a_i) \prod_{j=1}^{n}(b_j/p_j)^{b_j} \end{aligned}$$

using $\sum_{j=1}^{n} b_j = 1$. It follows that

$$x_i^h(p, u) = a_i + (b_i/p_i) u \prod_{j=1}^{n}(p_j/b_j)^{b_j}$$

Summing over i and using $\sum_{i=1}^{n} b_i = 1$, gives the expenditure function

$$e(p, u) = \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i a_i + u \prod_{j=1}^{n}(p_j/b_j)^{b_j}.$$

Since $e(p, u) = y$, it follows that the indirect utility function is

$$v(p, y) = (y - \sum_{i=1}^{n} p_i a_i) \prod_{j=1}^{n}(b_j/p_j)^{b_j}.$$

(b) The Marshallian demands, expressed by using the equality between $x_i^h(p,u)$ and $x_i^*(p,y)$ when $u = v(p,y)$

$$x_i^*(p,y) = x_i^h(p, v(p,y)) = a_i + (b_i/p_i)(y - \sum_{i=1}^{n} p_i a_i) \prod_{j=1}^{n}(b_j/p_j)^{b_j} \prod_{j=1}^{n}(p_j/b_j)^{b_j}$$
$$= a_i + (b_i/p_i)(y - \sum_{i-1}^{n} a_i)$$

show that b_i measures the share of the 'discretionary income' $y - \sum_{i-1}^{n} a_i$ that will be spent on 'discretionary' purchases of good x_i in excess of the subsistence level a_i.