Delta Method

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Applied Econometrics II

Bootstrap and Delta Method

- We know how to construct confidence intervals for parameter estimates: $\hat{\theta}_k \pm 1.96 SE(\hat{\theta}_k)$
- Often we are asked to construct standard errors or confidence intervals around model outputs that are not just parameter estimates: ie: $h(x_i, \hat{\theta})$.
- Sometimes we can't even write $g(x_i, \theta)$ as an explicit function of θ ie: $\Psi(h(x_i, \theta), \theta) = 0$.
- Two options:
  1. Delta Method
  2. Bootstrap

Delta Method

Delta method works by considering a Taylor Expansion of $g(x_i, \theta)$.

$$h(z) \approx h(z_0) + h'(z_0)(z - z_0) + o(||z - z_0||)$$

Assume that θ_n is asymptotically normally distributed so that:

$$\sqrt{n}(\theta_n - \theta_0) \sim N(0, \Sigma)$$

(How do we get this: OLS? GMM? MLE?). Then we have that

$$\sqrt{n}(h(\theta_n) - h(\theta_0)) \sim N(0, D(\theta)'\Sigma D(\theta))$$

Where $D(\theta) = \frac{\partial h(x_i, \theta)}{\partial \theta}$ is the Jacobian of g with respect to theta evaluated at θ.
We need g to be continuously differentiable around the center of our expansion θ.

Delta Method: Examples

Start with something simple: $g(\theta) = \overline{X}_1 \cdot \overline{X}_2$ with $(X_{1i}, X_{2i}) \sim IID$. We know the CLT applies so that:

$$\sqrt{n}\begin{pmatrix} \overline{X}_1 - \mu_1 \\ \overline{X}_2 - \mu_2 \end{pmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma\right]$$

The Jacobian is just $D(\theta) = \begin{pmatrix} \frac{\partial g(\theta)}{\partial \theta_1} \\ \frac{\partial g(\theta)}{\partial \theta_2} \end{pmatrix} = \begin{pmatrix} s_2 \\ s_1 \end{pmatrix}$

So,

$$V(Y) = D(\theta)'\Sigma D(\theta) = \begin{pmatrix} \mu_2 & \mu_1 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \mu_2 \\ \mu_1 \end{pmatrix}$$

$$\sqrt{n}(\overline{X}_1\overline{X}_2 - \mu_1\mu_2) \sim N(0, \mu_2^2\sigma_{11}^2 + 2\mu_1\mu_2\sigma_{12} + \mu_1^2\sigma_{22}^2)$$

Delta Method: Examples

Think about a simple logit:

$$\mathbb{P}(Y_i = 1|X_i) = \frac{\exp^{\beta_0+\beta_1 X_i}}{1+\exp^{\beta_0+\beta_1 X_i}} \quad \mathbb{P}(Y_i = 0|X_i) = \frac{1}{1+\exp^{\beta_0+\beta_1 X_i}}$$

Remember the “trick” to use GLM (log-odds):

$$\log \mathbb{P}(Y_i = 1|X_i) - \log \mathbb{P}(Y_i = 0|X_i) = \beta_0 + \beta_1 X_i$$

- Suppose that we have estimated $\hat{\beta_0}, \hat{\beta_1}$ via GLM/MLE but we want to know the confidence interval for the probability: $P(Y_i = 1|X_i, \hat{\theta})$
- The derivatives are a little bit tricky, but the idea is the same.
- This is what STATA should be doing when you type: `mfx, compute`

Delta Method: Other Examples

Often we have a regression like:

$$logY_i = \beta_0 + \beta_1 X_i + \gamma Income_i + \epsilon_i$$

And we are interested in β_1/γ so that we have β_i in units of "dollars". Again Delta Method Works fine here.

Delta Method: Some Failures

But we need to be careful. Suppose that $\theta \approx 0$ and

- $h(x) = |X|$
- $h(x) = 1/X$
- $h(x) = \sqrt{X}$

These situations can arise in practice when we have weak instruments or other problems.