Inequalities, Household Behavior and the Macroeconomy
(Solving Consumption-Saving Models)

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Last lecture

- Discussed how to include borrowing limits in our model
- Zeldes(1989)
- Started to talk about solving models

Today

Solving models with a computer – in detail

- (Theory) Dynamic programming: Solve a series of simple problems instead of one complicated one
- (Practice) Understanding code to solve the consumption-saving problem
- I have to leave at 11:55

Full model

With uncertainty and borrowing limits:

$$V_0(z_0) = \max_{\{c_s(z_s), a_s(z_s)\}_{s=0}^{T}} \sum_{s=0}^{T} \sum_{z_s} \pi(z_s) \beta^s u(c_s(z_s))$$

$$\begin{aligned} s.t. \quad & a_s(z_s) = (1+r)a_{s-1}(z_{s-1}) + y_s(z_s) - c_s(z_s) \quad \forall s \in \{0, 1, \ldots, T\}, \ \forall z_s \\ & a_s(z_s) \geq -b \qquad \forall s \in \{0, 1, \ldots, T\}, \ \forall z_s \\ & a_T(z_T) \geq 0 \qquad \forall z_T \\ & a_{-1} \text{ given} \end{aligned}$$

Solving means finding optimal policies for all times and all histories.
Note that $a_{-1} \in z_0$.

Problem of agent of age t

$$V_t(z_t) = \max_{\{c_s(z_s), a_s(z_s)\}_{s=t}^T} \sum_{s=t}^{T} \sum_{z_s \subset z_t} \frac{\pi(z_s)}{\pi(z_t)} \beta^{s-t} u(c_s(z_s))$$

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s.t. \quad & a_s(z_s) = (1+r)a_{s-1}(z_{s-1}) + y_s(z_s) - c_s(z_s) \quad \forall s \in \{t, t+1, \ldots, T\}, \ \forall z_s \subset z_t \\
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& a_T(z_T) \geq 0 \qquad \forall z_T \\
& a_{t-1} \text{ given}
\end{aligned}$$

- Need to take into account only those future histories, which are continuations of the current (already realized) history. Notation: $z_s \subset z_t$
- If z_s is a continuation of z_t, then $\pi(z_s)/\pi(z_t)$ is the probability of z_s happening, conditional on z_t having happened. (Remember: π still means probabilities when looking from forward from period 0.)
- We can solve this for any reasonable history z_t. We are interested in histories that won't lead to $V_t = -\infty$.

Turn sequential problem into a sequence of problems

- Optimization problems of agents with different ages are not independent.
- Bellman's principle of optimality (1957):
  "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision"
- We have many decisions and constraints in the full problem. Separate them to simplify!
  - Decisions and constraints now
  - Decisions and constraints later

Problem of agent of age t

$$
\begin{aligned}
V_t(z_t) &= \max_{\{c_s(z_s), a_s(z_s)\}_{s=t}^{T}} \sum_{s=t}^{T} \sum_{z_s \subset z_t} \frac{\pi(z_s)}{\pi(z_t)} \beta^{s-t} u(c_s(z_s)) = \\
&= \max_{c_t(z_t), a_t(z_t)} \left\{ u(c_t(z_t)) + \max_{\{c_s(z_s), a_s(z_s)\}_{t+1}^{T}} \sum_{s=t+1}^{T} \sum_{z_s \subset z_t} \frac{\pi(z_s)}{\pi(z_t)} \beta^{s-t} u(c_s(z_s)) \right\}
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Bellman-equation

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Intuition:

- V_{t+1} incorporates everything about the future effects of your choices that you need to care about,
- so once you use this information through V_{t+1}, you don't need to care about future actions and constraints anymore to decide optimally today!

iid income shocks

- z_t denotes a whole history - very general framework, let's check something simpler!
- The simplest case is of iid income shocks. z_t is the series of past income shocks $z_t = \{y_0, y_1, \dots y_t\}$

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What does this mean?

- No need to keep track of whole histories
- Can condense information into the pair (a_{t-1}, y_t)
- (a_{t-1}, y_t) determines the "state": they contain all the knowledge we need to make optimal decisions.

Wealth and income as states

$$V_t(a_{t-1}, y_t) = \max_{c_t(a_{t-1},y_t), a_t(a_{t-1},y_t)} \left\{ u(c_t(a_{t-1}, y_t)) + \beta \mathbb{E}_t \Big[V_{t+1}(a_t(a_{t-1}, y_t), y_{t+1}) \Big] \right\}$$

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We can substitute out consumption using the constraint! $\Rightarrow$ We only have one decision variable

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To simplify notation, we often write

$$V_t(a_{t-1}, y_t) = \max_{a_t \geq -b} \left\{ u((1+r)a_{t-1} + y_t - a_t) + \beta \mathbb{E}_t \left[V_{t+1}(a_t, y_{t+1}) \right] \right\}$$

It is still true that the current policy a_t is a function of the current state (a_{t-1}, y_t)!

Cash-on-hand as state

Actually, we can do better!

- We don't care where our current resources come from, only matters, how much!
- Cash-on-hand

$$x_t = (1 + r)a_{t-1} + y_t$$

  is the perfect state variable (when income is iid).

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Again we can substitute out!

$$V_t(x_t) = \max_{a_t \geq -b} \left\{ u(x_t - a_t) + \beta \mathbb{E}_t \Big[V_{t+1}(a_t(1 + r) + y_{t+1}) \Big] \right\}$$

What is a 'solution' of a model?

- Closed-form solution:

$$\text{Interesting variable} = \text{Formula involving parameters \& other given quantites}$$

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In our model,

- given a set of parameters,
- for any age and
- for any cash-on-hand level;
- we want to be able to compute optimal consumption and saving, and the value function.

Solving our consumption-saving model

Last period is trivial:

- Save nothing, consume all cash-on-hand: $a_T(x_T) = 0 \quad c_T(x_T) = x_T$
- Value coincides with utility from eating everything: $V_T(x_T) = u(x_T)$

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What to do next? We continue with $t = T - 1$ based on

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- We can compute value and policies for a given $x = x_{T-1}$.
- But we want to solve for the whole functions (we want to evaluate the solutions potentially anywhere). How to do that on the computer?
- Interpolation! We store approximations of the exact solution functions.
- We compute the exact values at certain points for x (grid points) and then we perform interpolation over these exact values.
- Then we can get a(n approximated) solution when evaluating c_{T-1}, a_{T-1} or V_{T-1} for any x without going back to the Bellman-equation all the time.

Solving our consumption-saving model

What to do with the previous periods? We iterate backwards based on

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So (knowing V_T) we can compute everything in $T-1$, then we can do $T-2$, and so on until $t=0$.

In the end we have

- an interpolated value function
- an interpolated consumption function
- an interpolated saving function

for all ages $t \in \{0, 1, 2, \ldots, T\}$. Then we are done!

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  - coh should be high enough to satisfy the borrowing limit at the end of the current period without negative consumption:

$$x_t = c_t + a_t \geq a_t \geq -b$$

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  - Ignore wealth levels that lead to $-\infty$ value. A debt that cannot be surely repaid results in $-\infty$ value (as it forces negative consumption with positive probability). Natural borrowing limit

$$x_t = c_t + a_t \geq a_t \geq -\sum_{s=t+1}^{T} \frac{\min\{y_s\}}{(1+r)^{s-t}}$$

is the discounted value of future income **in the worst case**. This is the largest debt that can surely be repaid (under 0 future consumption).

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Taking the maximum of the above two limits works well.

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- How to space points over the grid?
  - Even is simple
  - Usually better to have relatively more points at the bottom (low wealth), as functions are more curved there (more curvature $\Rightarrow$ more errors)

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- How to space points over the grid?
  - Even is simple
  - Usually better to have relatively more points at the bottom (low wealth), as functions are more curved there (more curvature $\Rightarrow$ more errors)
- How many grid points?
  - A lot. Converge to true solution as the grid is getting more dense.
  - If further increasing the number of points has little effect on solutions, you probably have enough.

Infinite horizon

Infinite horizon means maximizing something like this

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Does it make sense to study infinite horizon models? What do you think?

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- Not realistic
- Cannot study lifecycle patterns
- Difficult decisions when calibrating. For example, mean income should be
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Good:

- One state variable fewer (age) $\Rightarrow$ easier to solve, simpler to understand solution
- Captures dynastic behavior in a compact way

Infinite horizon - how to solve them

- Can derive a similar Bellman equation to finite horizon problems (deeper math due to infinity)
- But there is no age t, so same value function in each period:

$$V(x) = \max_{a \geq -b} \left\{ u(x - a) + \beta \mathbb{E}\Big[V(a(1 + r) + y') \Big] \right\}$$

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How to solve this functional equation numerically?

- Start with any policy guess (e.g. eating everything, like people of age T in the finite horizon model). Implies a guess for the value function.
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- This procedure is guaranteed to converge to the unique true solution under quite broad assumptions. (Banach fixed-point theorem)