Econometrics II (Spring 2024)
Suggested Solutions to Problem Set 3

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Question 1

See the do-file for the Stata implementation.

a.

Figure 1a presents the distribution of two $\hat{\beta}$s, one by the 2SLS and another by the OLS with $\tilde{x} = \beta_{FS} z$ as a regressor, together with their means. In my simulation, $\hat{\beta}$ when imposing the true first stage coefficient is close to the true value, 1, demonstrating the unbiasedness of the OLS estimator. In contrast, $\hat{\beta}$ estimated with the 2SLS exhibits the mean 1.906, which is nearly twice as large as the true value of 1. Besides the first moment, the distribution of $\hat{\beta}_{2SLS}$ exhibits a substantially large variance than that of the OLS estimates.

b.

The average absolute second stage bias is 0.32 when $\hat{\beta}_{FS} > \beta_{FS} = 1$, and 6.27 when $\hat{\beta}_{FS} < \beta_{FS}$. Hence, this bias is substantially smaller when $\hat{\beta}_{FS} > \beta_{FS}$.

Figure 2 presents the relationship between the first stage F-statistic and the absolute second stage bias. When we focus on the cases where the first stage F-statistic is greater than 10, which is a conventional rule of thumb value to judge if the instrument is weak, this figure shows that the absolute second stage bias turns out to *increase* as the first-stage F-statistic becomes larger.

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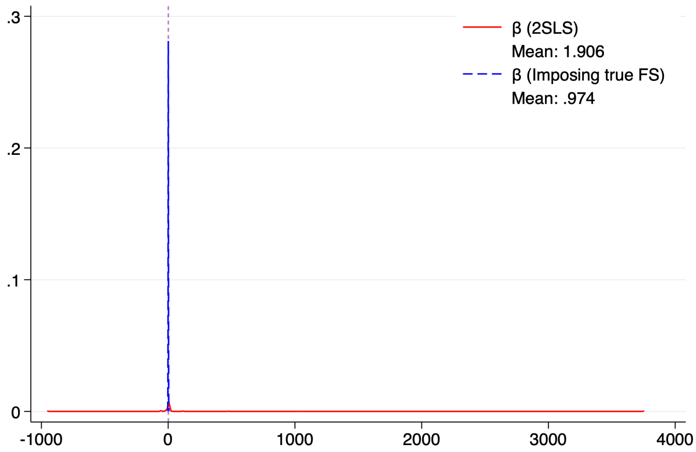
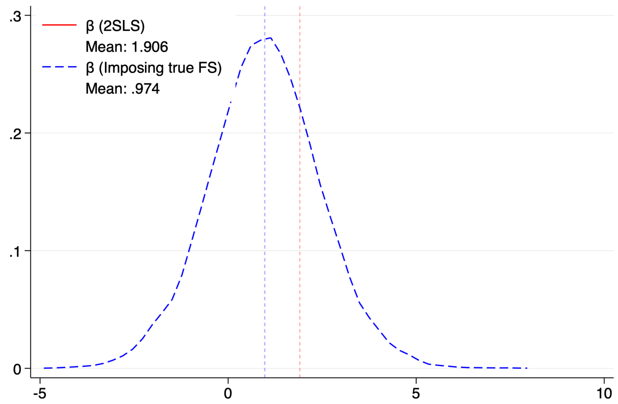
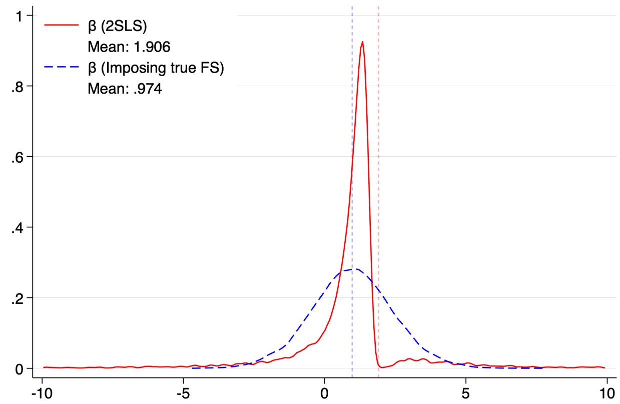
Figure 1: Q1.a. Kernel Density of $\hat{\beta}$



(a) No Restriction on x-axis

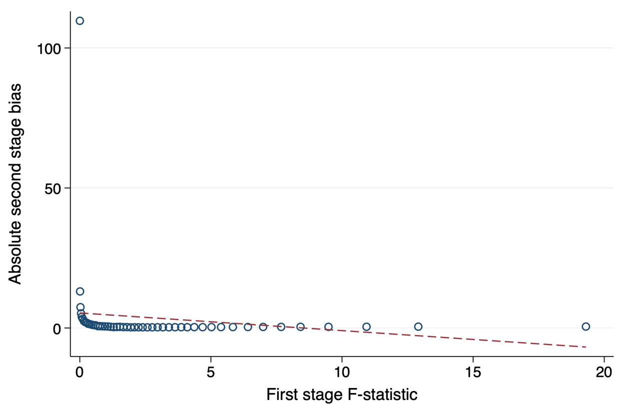


(b) X-axis = $[-10, 10]$

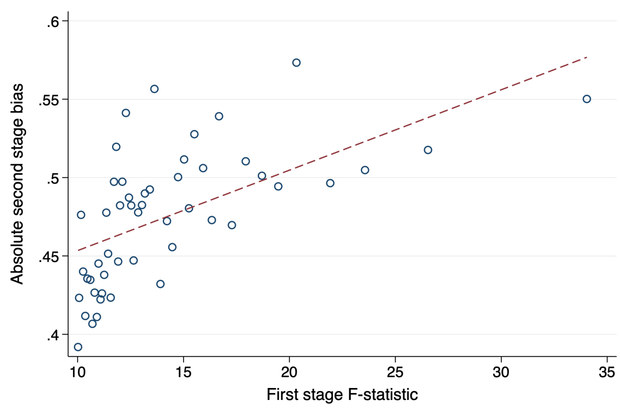


(c) Conditional on $\hat{\beta} \in [-10, 10]$

Figure 2: Q1.b. First stage F-statistic and Absolute second stage bias



(a) No Restriction on Fist stage F-stat.



(b) First stage F-stat. > 10

c.

Figure 3 presents the three bin-scatter plots over $\hat{\beta}_{FS}$: the residual sum of squares (RSS), the error sum of squares (ESS), and their ratio. The highest ratio of the RSS over the ESS across the simulations is 0.999, approximately 1, indicating that the RSS is always smaller the ESS.

Figure 3: Q1.c. First Stage RSS, ESS, and Their Ratio over $\hat{\beta}_{FS}$

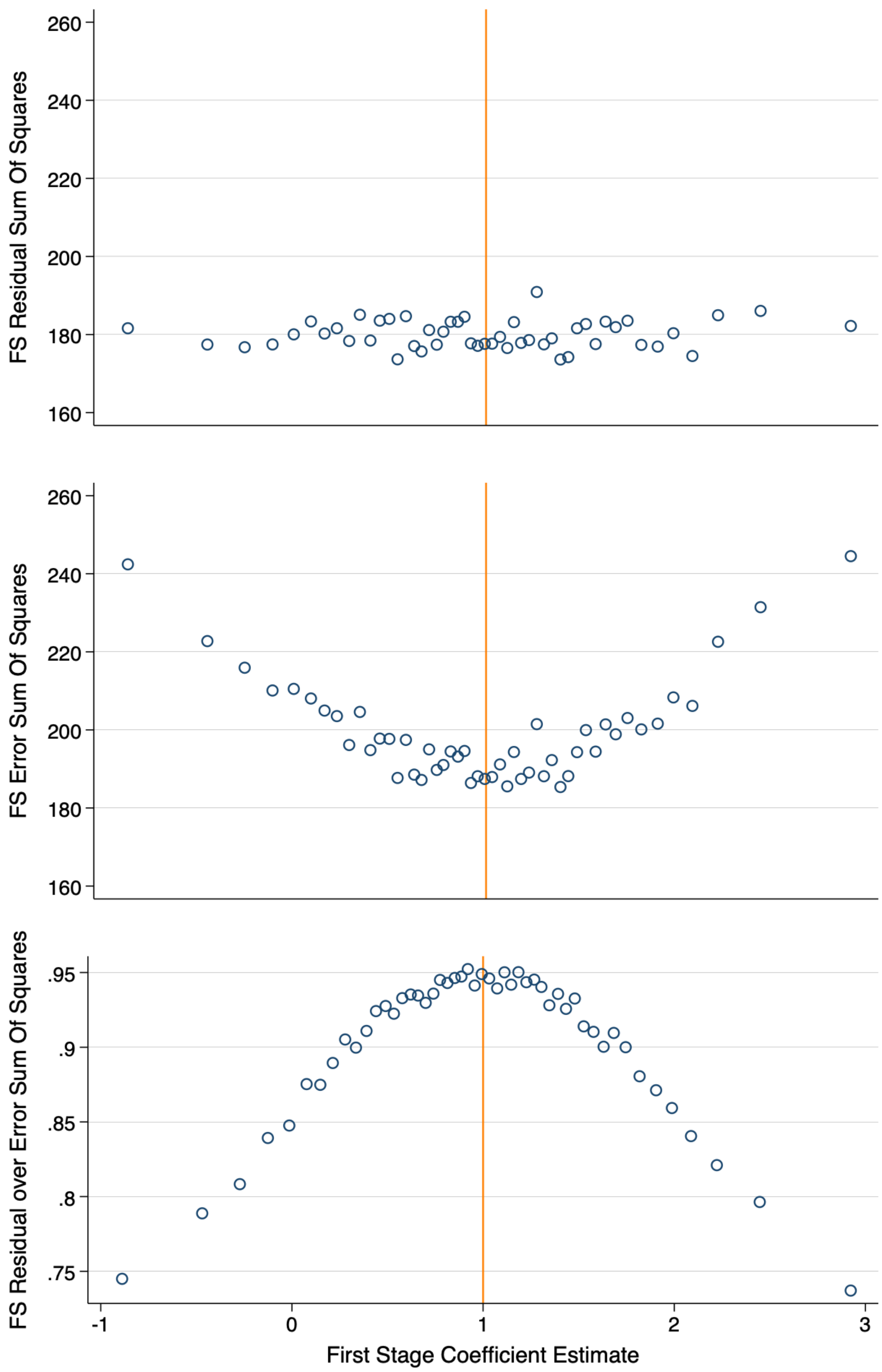
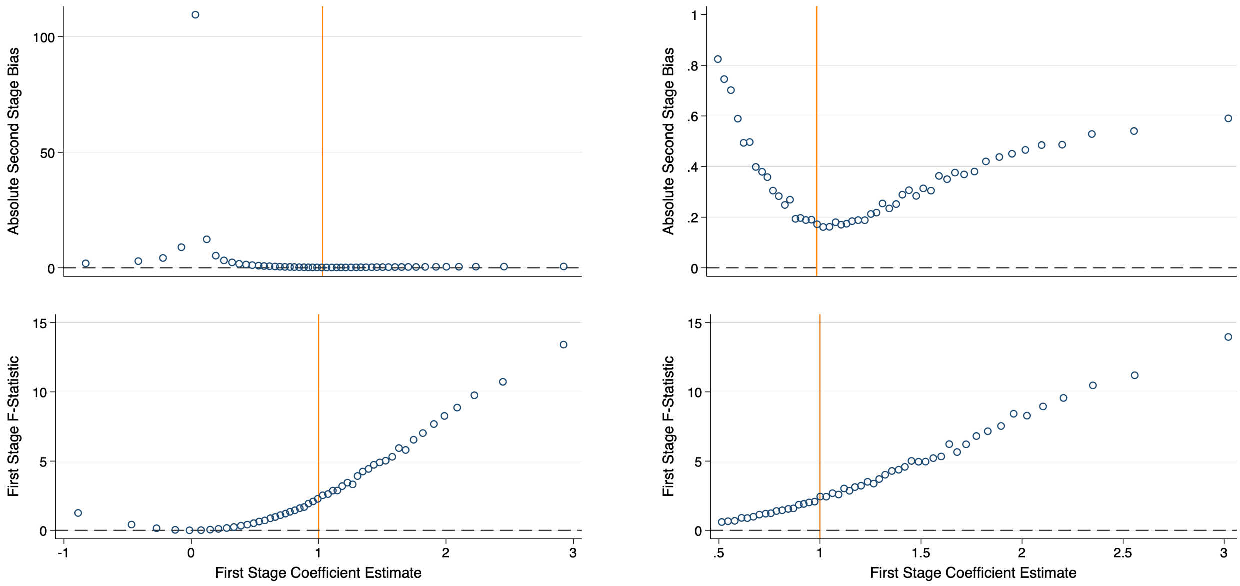


Figure 4: Q1.d. Absolute Second Stage Bias over $\hat{\beta}_{FS}$



(a) No Restriction on $\hat{\beta}_{FS}$

(b) $\hat{\beta}_{FS} > 0.5$

d.

Figure 4 presents the bin-scatter plots of the absolute second stage bias and the first stage F-statistic, respectively, over $\hat{\beta}_{FS}$. The absolute bias is largest when $\hat{\beta}_{FS} = -.00017$, approximately 0. Focusing on the case with $\hat{\beta}_{FS} > 0.5$, the absolute bias is smallest when $\hat{\beta}_{FS} = 0.785$, somewhat close to the true β_{FS}. In this case, the first stage F-statistic does not appear to be a useful diagnostic to detect bias. While the absolute bias decreases as $\hat{\beta}_{FS}$ becomes closer to the true β_{FS}, this is not the case for the first stage F-statistic: it is monotonically increasing in $\hat{\beta}_{FS}$.

e.

This question as a whole illustrates sources of the bias in the 2SLS and how we can or cannot use particular measures to detect it, in the situation where the instrument is modeled to be valid but the sample is small.

First of all, the bias in the 2SLS estimator is due to the fact that we estimate the first stage. As Figure 1a shows, the 2SLS estimates are, on average, biased and its distribution exhibits significant variability in our simulation, while $\hat{\beta}$ is centered around the true β_x.

Although the first-stage F-statistic is the most widely used diagnostic for the 2SLS bias, the sample first-stage F-statistic we compute does not appear to be a good measure of the bias in the 2SLS in general. Figure 2 shows that while the overall relationship between the (sample) first-stage F-statistic and the absolute second-stage bias seems negative, the bias is actually larger for the larger first-stage F-statistic once we focus on large enough values of the F-statistic. A small first-stage F-statistic implies serious 2SLS bias. Yet, it is not necessarily the case for larger values of the first-stage F-statistic.

Note that it does NOT contradict the results in the lecture and other references that the 2SLS bias decreases with the F-statistic. The F-statistic in that context refers to the *population* F-statistic, fixed in the data-generating process we are simulating.[1] The F-statistic we have for each simulation is the sample F-statistic (the estimate of the population F-statistic). Hence, our finite sample simulation results are all conditional on the same population F-statistic.

Then, further exploration tells us that the absolute second-stage bias seems to stem from two sources. First, the overfit in the first stage leads to the bias in $\hat{\beta}_{2SLS}$. As Figure 4 shows, the bias tends to increase as $\hat{\beta}_{FS}$ goes away from the true first stage. While the first-stage F-statistic tends to be monotonically increasing as $\hat{\beta}_{FS}$ goes away from 0, the second-stage bias is minimized around the true first stage and increases in $\hat{\beta}_{FS} > 1$ even though the first F-statistic is also increasing there. This reflects the overfitting in Figure 3. The RSS is always smaller than the true ESS, and the ratio of the RSS to the ESS becomes smaller as as $\hat{\beta}_{FS}$ moves away from the true first stage. The first stage thus attempts to fit well regardless of the true errors, by adjusting $\hat{\beta}_{FS}$, fitting even to a part of the variation coming from ϵ_i. In other words, in a small sample, the correlation between z_i and ϵ_i may not be exactly zero, and $\hat{\beta}_{FS}$ may capture such finite-sample correlation. This then results in the finite sample correlation between the first-stage fitted value and the error term in the second stage, causing the bias in the second stage.

Why is the sample F-statistic indicate not a good indicator of the 2SLS bias? In this just-identified case of our simulations, the sample F-statistic is the square of the t-statistic on the coefficent $\hat{\beta}_{FS}$ under $H_0 : \beta_{FS} = 0$: $\hat{F} = \hat{t}^2 = \left(\hat{\beta}_{FS}/SE(\hat{\beta}_{FS})\right)^2$, which picks up how far $\hat{\beta}_{FS}$ is from 0. This involves the RSS in the denominator when computing the standard error. Given that the first stage RSS does not vary with $\hat{\beta}_{FS}$, the sample F-statistic is simply increasing in $|\hat{\beta}_{FS}|$. This monotonicity is observed in Figure 4. On the other hand, as in this figure, the larger $\hat{\beta}_{FS}$ away from the true first stage is associated with the larger second stage bias. This is why the larger values of the sample F-statistic does not indicate the smaller bias, as we might presume.

In addition to the overfitting in the first stage, what also matters is the *weak* first stage, measured by whether the coefficient is close to 0 or not. This is evident from Figure 4, which shows that the 2SLS bias becomes largest around $\hat{\beta}_{FS} \approx 0$ and becomes smaller as $\hat{\beta}_{2SLS}$ moves away from 0. This is due to the fact that $\hat{\beta}_{2SLS}$ is the ratio of the reduced-form estimate and the first stage estimate; thus, a $\hat{\beta}_{FS}$ closer to 0 means that you are dividing the reduced-form estimate by a smaller number, inflating $\hat{\beta}_{2SLS}$ in its absolute value.

Overall, the results imply that a small F-statistic can be a caution sign for the 2SLS bias, and the first-stage coefficient can be more informative in this regard. In some contexts, if we may expect a certain first-stage relationship, this can guide us in inferring the potential 2SLS bias from the estimated first-stage result. Note that these results are based on a small sample, and a larger sample size tends to reduce the bias here, as long as the instrument is valid.

[1] We may have multiple distinct values of the population F-statistic in cases where, for instance, multiple instrumental variables (with different levels of the first-stage strength) are available and each combination of the instruments has a corresponding F-statistic in the population.

Question 2

1.

This scenario is 'purely hypothetical', because in general we can never observe the fraction of always- and never-takers, as well as compliers, and defiers in our sample. To know the composition of our sample, we need to see the take-up patterns of each unit in response to getting assigned to the treatment and not being assigned. In the next question, we assess if the randomisation is successful or not, assuming that we know the fraction of each type in the sample. This is not possible in practice: typically we assume that the randomisation is successful and then estimate the fraction of each type. Besides, we know that there are no defiers, which we also need to assume in general.

2.

We will show that the randomisation is NOT successful. To do so, let $Z_i = \mathbb{1}\{\text{assigned to treatment}\}_i$, $D_i = \mathbb{1}\{\text{took up treatment}\}_i$, define A_i, N_i, C_i as the event that unit i is an always-taker, never-taker, and complier, respectively. With a slight abuse of notation, we denote the fraction of the units satisfying a certain condition Π_i by $\Pr(\Pi_i)$.

From the description of the question, we know the following:

$$
\begin{aligned}
&\Pr(A_i) = 0.3,\ \Pr(N_i) = 0.3,\ \Pr(C_i) = 0.4, \\
&\Pr(Z_i = 1) = \Pr(Z_i = 0) = 0.5, \\
&\Pr(Z_i = 1, D_i = 0) = 0.2,\ \Pr(Z_i = 1, D_i = 1) = 0.3, \\
&\Pr(Z_i = 0, D_i = 1) = 0.2,\ \Pr(Z_i = 0, D_i = 0) = \Pr(D_i = 0) - \Pr(Z_i = 0, D_i = 1) = 0.3.
\end{aligned}
$$

Then, to show the imbalance of the fraction of a certain type, it suffices to show that the fraction of the always-takers is not balanced across the groups with $Z_i = 0$ and $Z_i = 1$:

Always-taker

$$
\begin{aligned}
&\Pr(A_i|Z_i = 0) = \frac{\Pr(Z_i = 0, D_i = 1)}{\Pr(Z_i = 0)} = \frac{0.2}{0.5} = 0.4, \\
&\Pr(A_i, Z_i = 1) = \Pr(A_i) - \Pr(Z_i = 0, D_i = 1) = 0.3 - 0.2 = 0.1, \\
&\Rightarrow \Pr(A_i|Z_i = 1) = \frac{\Pr(A_i, Z_i = 1)}{\Pr(Z_i = 1)} = \frac{0.1}{0.5} = 0.2 \neq \Pr(A_i|Z_i = 0) = 0.4,
\end{aligned}
$$

indicating that the fraction of the always-takers is not the same amongst those with $Z_i = 1$ and with $Z_i = 0$ and hence the randomisation is not successful.

Analogously, we can show the imbalance of the never-takers and the balance of the compliers:

Never-takers

$$
\begin{aligned}
&\Pr(N_i|Z_i=1) = \frac{\Pr(Z_i=1,D_i=0)}{\Pr(Z_i=1)} = \frac{0.2}{0.5} = 0.4,\\
&\Pr(N_i,Z_i=0) = \Pr(N_i) - \Pr(Z_i=1,D_i=0) = 0.3-0.2 = 0.1,\\
&\Rightarrow\ \Pr(N_i|Z_i=0) = \frac{\Pr(N_i,Z_i=0)}{\Pr(Z_i=0)} = \frac{0.1}{0.5} = 0.2 \neq \Pr(N_i|Z_i=1) = 0.4;
\end{aligned}
$$

Compliers

$$
\begin{aligned}
&\Pr(C_i,Z_i=1) = \Pr(Z_i=1,D_i=1) - \Pr(A_i,Z_i=1) = 0.3-0.1 = 0.2,\\
&\Pr(C_i,Z_i=0) = \Pr(Z_i=0,D_i=0) - \Pr(N_i,Z_i=0) = 0.3-0.1 = 0.2,\\
&\Rightarrow \Pr(C_i|Z_i=1) = \frac{\Pr(C_i,Z_i=1)}{\Pr(Z_i=1)} = \frac{0.2}{0.5} = 0.4,\\
&\quad\ \Pr(C_i|Z_i=0) = \frac{\Pr(C_i,Z_i=0)}{\Pr(Z_i=0)} = 0.4,\\
&\Rightarrow \Pr(C_i|Z_i=1) = \Pr(C_i|Z_i=0) = 0.4.
\end{aligned}
$$

3.

We continue to use the notation from Question 2.2. Since the first-stage regression is equivalent to the difference-in-means estimator, the coefficient estimate of the first stage regression is given by

$$
\begin{aligned}
&\frac{1}{\sum_i Z_i}\sum_i D_i Z_i - \frac{1}{\sum_i(1-Z_i)}\sum_i D_i(1-Z_i)\\
=&\Pr(D_i=1|Z_i=1) - \Pr(D_i=1|Z_i=0)\\
=&\frac{\Pr(D_i=1,Z_i=1)}{\Pr(Z_i=1)} - \frac{\Pr(D_i=1,Z_i=0)}{\Pr(Z_i=0)}\\
=&\frac{0.3}{0.5} - \frac{0.2}{0.5} = 0.2,
\end{aligned}
$$

which is NOT identical to the true fraction of compliers in the sample, 0.4.

Why is this the case? Note that it holds that

$$
\begin{aligned}
\Pr(D_i=1|Z_i=1) &= \Pr(A_i|Z_i=1) + \Pr(C_i|Z_i=1),\\
\Pr(D_i=1|Z_i=0) &= \Pr(A_i|Z_i=0),
\end{aligned}
$$

where $\Pr(A_i|Z_i=1) = 0.2 \neq \Pr(A_i|Z_i=0) = 0.4$ from Question 2.2. Therefore, taking the difference of these two terms, we have $\Pr(C_i|Z_i=1) = 0.4$ and $\Pr(A_i|Z_i=1) - \Pr(A_i|Z_i=0) = -0.2$, and we do not get the true fraction of compliers because of this second term: the imbalance of the fraction of the always-takers in the groups with $Z_i=1$ and $Z_i=0$. If, on the other hand, the randomisation is successful in the sense that $\Pr(A_i|Z_i=1) = \Pr(A_i|Z_i=0)$, then the first-stage regression indeed gives the true fraction of compliers.

4.

With another abuse of notation, denote the sample average by $\mathbb{E}$. Let $\{Y_i(d)\}_{d\in\{0,1\}}$ be the potential outcome, Y_i be the observed outcome, where $d \in \{0,1\}$ is the treatment status, and

$$Y_i = \begin{cases} 1 & \text{if } N_i, \\ 5 & \text{if } A_i, \\ 4 + D_i \cdot 2 = 4 + Z_i \cdot 2 & \text{if } C_i. \end{cases}$$

Since the instrument is binary, the second stage coefficient on D_i is given by the ratio of the reduced-form estimate to the first-stage estimate, i.e.,

$$\frac{\mathbb{E}[Y_i|Z_i = 1] - \mathbb{E}[Y_i|Z_i = 0]}{\mathbb{E}[D_i|Z_i = 1] - \mathbb{E}[D_i|Z_i = 0]}.$$

From Question 2.3, we know the denominator is equal to 0.2.

Then, consider the numerator (the reduced form): for each $z \in \{0,1\}$, with type $\in \{A_i, N_i, C_i\}$,

$$\begin{aligned} \mathbb{E}[Y_i|Z_i = z] &= \mathbb{E}[\mathbb{E}[Y_i|\text{ type, } Z_i = z]|Z_i = z] \\ &= \mathbb{E}[Y_i|A_i, Z_i = z] \cdot \Pr(A_i|Z_i = z) \qquad \text{(always-takers)} \\ &+ \mathbb{E}[Y_i|N_i, Z_i = z] \cdot \Pr(N_i|Z_i = z) \qquad \text{(never-takers)} \\ &+ \mathbb{E}[Y_i|C_i, Z_i = z] \cdot \Pr(C_i|Z_i = z) \qquad \text{(compliers).} \end{aligned}$$

Hence, we have

$$\begin{aligned} \mathbb{E}[Y_i|Z_i = 1] &= 5 \cdot 0.2 + 1 \cdot 0.4 + 6 \cdot 0.4 = 3.8, \\ \mathbb{E}[Y_i|Z_i = 0] &= 5 \cdot 0.4 + 1 \cdot 0.2 + 4 \cdot 0.4 = 3.8, \\ \Rightarrow \ \mathbb{E}[Y_i|Z_i = 1] &- \mathbb{E}[Y_i|Z_i = 0] = 0, \end{aligned}$$

which means that the second stage coefficent on D_i is 0, which is different from the true 'local average treatment effect' (LATE), 2. It indicates the IV estimate is downward biased. This is due to the imbalance of the always- and never-takers across the groups with $Z_i = 1$ and $Z_i = 0$: the always-takers have the larger outcome than the never-takers but more of them are in the $Z_i = 0$ group and the never-takers are more in the $Z_i = 1$ group; thus, the differences in the outcome and in their distribution bias the true LATE downward. The failure of the randomisation, therefore, leads to the imbalanced distribution of different types and consequently induces the *selection bias* in the LATE estimation.

Question 3

We want to show that we can obtain the average covariate value of the complier subpopulation, i.e., $\mathbb{E}[X|D(1)=1, D(0)=0]$, by regressing DX on D, instrumeted by Z. Following the hint, our goal is to show the following:

$$\frac{\mathbb{E}[XD|Z=1]-\mathbb{E}[XD|Z=0]}{\mathbb{E}[D|Z=1]-\mathbb{E}[D|Z=0]} = \mathbb{E}[X|D(1)=1, D(0)=0].$$

We start with the denominator of the population Wald estimator $\mathbb{E}[D|Z=1]-\mathbb{E}[D|Z=0]$. Since Z is fully randomised, we have $(D(1), D(0)), X \perp\!\!\!\perp Z$. Then, observe that

$$\begin{aligned}
\mathbb{E}[D|Z=1]-\mathbb{E}[D|Z=0] &= \mathbb{E}[D(1)|Z=1]-\mathbb{E}[D(0)|Z=0] \\
&= \mathbb{E}[D(1)]-\mathbb{E}[D(0)] \qquad (\because \text{ indepenence of } Z) \\
&= \mathbb{E}[D(1)-D(0)] \\
&= \quad 1\cdot \Pr(D(1)-D(0)=1) \quad (\text{compliers}) \\
&\quad + \underbrace{0\cdot \Pr(D(1)-D(0)=0)}_{=0} \quad (\text{always- and never-takers}) \\
&\quad + (-1)\cdot \underbrace{\Pr(D(1)-D(0)=-1)}_{=0 \text{ by no-defier assumption}} \quad (\text{defiers}) \\
&= \Pr(D(1)-D(0)=1).
\end{aligned}$$

This means that the *first-stage*, the regression of D on Z, measures the share of the compliers.

Next, we consider the denominator of the Wald estimator, $\mathbb{E}[XD|Z=1]-\mathbb{E}[XD|Z=0]$. Because the covariate X is independent of Z, we have

$$\begin{aligned}
&\mathbb{E}[XD|Z=1]-\mathbb{E}[XD|Z=0] \\
&= \mathbb{E}[XD(1)|Z=1]-\mathbb{E}[XD(0)|Z=0] \\
&= \mathbb{E}[XD(1)]-\mathbb{E}[XD(0)] \qquad (\because \text{ independence of } Z) \\
&= \mathbb{E}\left[X\left(D(1)-D(0)\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[X|D(1)-D(0)\right]\left(D(1)-D(0)\right)\right] \\
&= \quad \mathbb{E}[X|\underbrace{D(1)-D(0)=1}_{=D(1)=1, D(0)=0}]\cdot 1\cdot \Pr(D(1)-D(0)=1) \quad (\text{compliers}) \\
&\quad + \underbrace{\mathbb{E}\left[X|D(1)-D(0)=0\right]\cdot 0\cdot \Pr(D(1)-D(0)=0)}_{=0} \quad (\text{always- and never-takers}) \\
&\quad + \mathbb{E}\left[X|D(1)-D(0)=-1\right]\cdot(-1)\cdot \underbrace{\Pr(D(1)-D(0)=-1)}_{=0 \text{ by no-defier assumption}} \quad (\text{defiers}) \\
&= \mathbb{E}\left[X|D(1)=1, D(0)=0\right]\cdot \Pr(D(1)-D(0)=1).
\end{aligned}$$

Combining these two results gives

$$\begin{aligned}
\frac{\mathbb{E}[XD|Z=1]-\mathbb{E}[XD|Z=0]}{\mathbb{E}[D|Z=1]-\mathbb{E}[D|Z=0]} &= \frac{\mathbb{E}\left[X|D(1)=1, D(0)=0\right]\cdot \Pr(D(1)-D(0)=1)}{\Pr(D(1)-D(0)=1)} \\
&= \mathbb{E}[X|D(1)=1, D(0)=0].
\end{aligned}$$