# 5303 - Advanced Macroeconomics
# Assignment 5 Solutions

Alexandre Mendonça *

*Stockholm School of Economics*

December 2023

## Contents



*Department of Economics, Stockholm School of Economics. E-mail: alexandre.mendonca@phdstudent.hhs.se

## Exercise 1

Briefly evaluate the following statements (max 10 lines each):

(a) "Intergenerational trade never takes place in an overlapping generations model."

(b) "Given that population growth has almost stopped, it is now time to replace any pay-as-you-go pension system with a fully funded one (where the contributions of the young are invested and returned with interest when old)."

(c) "Studying Scandinavia leads one to conclude that the conclusions drawn in Prescott (2004) are false."

(d) "Linking households through altruistic bequests can fix a market economy that suffers from dynamic inefficiency."

### Answer:

(a) This is true in an overlapping generations model where people live for just two periods. Let's assume that some old person at time $t$ attempts to make a trade with a young person. This can take place in two ways: either the young person could give the old person some of the good for a promise of future repayment; or the old person could give the young person some good for a promise of future repayment. Since the old person will be dead in the next period, the young person cannot collect a debt and the now dead old person cannot enjoy a repayment. Therefore, no intergenerational trade can take place in this context. However, this is false if people live for more than two periods.

(b) It is true that the case for *introducing* a pay-as-you-go pension scheme is weaker the lower population growth is[1]. However, one cannot completely *abolish* a pay-as-you-go system from one day to another without imposing a 'grandfather clause' since this would obviously hurt the current old. Nevertheless, there may exist a Pareto improving reform under some circumstances, especially if labour supply is endogenous. However, constructing one is typically complicated.

[1] Check exercise 3.1d in Problem Set 1.

(c) Prescott (2004) argued that higher taxes imply lower hours worked. However, the data seems to show that Scandinavia has higher taxes, but also higher hours worked than continental Europe, which contradicts Prescott (2004)'s conclusions. According to Rogerson (2006), this pattern can be accounted by differences in the form of government spending. In fact, Rogerson (2006) argues that in Scandinavia a much larger share of government spending is devoted to family services such as child care and elderly care, which promote employment.

(d) This is false since a dynamic inefficient economy experiences overaccumulation of capital, i.e., high savings, whereas altruistic bequests, which correspond to a transfer from old to young without expectation of repayment, are a motive for saving more, not less.

## Exercise 2

Consider a two-period overlapping generations economy with production. Everyone has preferences represented by

$$u_t^h = \ln c_t^h(t) + \beta \ln c_t^h(t+1)$$

The population grows according to

$$N(t+1) = nN(t)$$

Everyone is endowed with one unit of labor when young and no labor when old. Production takes place according to

$$Y(t) = K(t)^{\theta} L(t)^{1-\theta}$$

wehre $K(t)$ is the aggregate savings of generation $t-1$ and $L(t)$ is the aggregate labor supply of those alive in period $t$. Markets are competitive.

(a) Suppose there is no government. Find an expression for the steady state capital per worker $k^*$ (the one where capital is not zero).

(b) Now suppose there is a government that taxes labor income at rate $\tau$ and transfers the revenue lump-sum immediately to the current old. Show that the steady state stock of capital per worker is decreasing in $\tau$.

(c) Now suppose the government taxes labor income at rate $\tau$ and uses the revenue to buy capital which is used in production. The tax receipts plus interest are then given back to the old in the next period. Show that raising $\tau$ a little bit above zero has no effect on the aggregate capital stock. Provide intuition.

## Answer:

(a) Let's start by writing down the budget constraints of the young and of the old:

$$\begin{cases} \text{Young: } \rightarrow c_t^h(t) \leq w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) \\ \text{Old: } \rightarrow c_t^h(t+1) \leq w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) \end{cases}$$

where $w(t)$ and $R(t)$ are the wage rate and the rental rate of capital at time $t$, respectively. The household's problem is thus given by:

$$\begin{aligned} \max_{\{c_t^h(t), c_t^h(t+1), l^h(t), k^h(t+1)\}} \quad & \ln c_t^h(t) + \beta \ln c_t^h(t+1) \\ \text{s.t. } & c_t^h(t) \leq w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) \\ & c_t^h(t+1) \leq w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) \end{aligned}$$

And our Lagrangian:

$$\begin{aligned} \mathcal{L} = & \ln c_t^h(t) + \beta \ln c_t^h(t+1) + \mu(t)\Big[w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) - c_t^h(t)\Big] \\ & + \mu(t+1)\Big[w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) - c_t^h(t+1)\Big] \end{aligned}$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the FOC:

$$\begin{aligned} \mathbf{c_t^h(t)}: &\ \tfrac{1}{c_t^h(t)} - \mu(t) = 0, \forall t \geq 1 \\ \mathbf{c_t^h(t+1)}: &\ \tfrac{\beta}{c_t^h(t+1)} - \mu(t+1) = 0, \forall t \geq 1 \\ \mathbf{l^h(t)}: &\ -\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1 \\ \mathbf{k^h(t+1)}: &\ -\mu(t) + \mu(t+1)R(t+1) = 0, \forall t \geq 1 \end{aligned}$$

From which we can obtain the following equilibrium conditions:

$$\begin{aligned} R(t+1) &= r(t) \\ c_t^h(t+1) &= \beta r(t) c_t^h(t) \end{aligned}$$

Where the first condition is the no arbitrage condition between private borrowing and lending and holding capital. Substituting in the lifetime budget constraint:

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r(t)} - k^h(t+1)\Big[1 - \frac{R(t+1)}{r(t)}\Big]$$
$$c_t^h(t) = \Big(\frac{1}{1+\beta}\Big)\Big[w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r(t)}\Big]$$

And savings are given by:

$$s_t^h(t) = w(t)\Delta_t^h(t) - c_t^h(t) = \Big(\frac{\beta}{1+\beta}\Big)w(t)\Delta_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\frac{w(t+1)\Delta_t^h(t+1)}{r(t)}$$

Since $N(t+1) = nN(t)$, we know that aggregate savings are equal to $S_t(r(t)) = N(t)s_t^h(t)$.

Furthermore, the firm's problem is given by:

$$\max_{\{K(t),L(t)\}} K(t)^\theta L(t)^{1-\theta} - R(t)K(t) - w(t)L(t)$$

Taking the FOC:

$\mathbf{K(t)}$: $R(t) = \theta\Big(\frac{K(t)}{L(t)}\Big)^{\theta-1}, \forall t \geq 1$

$\mathbf{L(t)}$: $w(t) = (1-\theta)\Big(\frac{K(t)}{L(t)}\Big)^{\theta}, \forall t \geq 1$

Since labour endowments are fixed, total labour supply will be given by:

$$L(t) = \sum_{h=1}^{N(t)} \Delta_t^h(t) + \sum_{h=1}^{N(t-1)} \Delta_{t-1}^h(t) = N(t-1)\times 0 + N(t)\times 1 = N(t)$$

So we have that:

$$R(t) = \theta\Big(\frac{K(t)}{N(t)}\Big)^{\theta-1} \Rightarrow R(t+1) = \theta\Big(\frac{K(t+1)}{N(t+1)}\Big)^{\theta-1} = r(t)$$
$$w(t) = (1-\theta)\Big(\frac{K(t)}{N(t)}\Big)^{\theta}$$

And aggregate savings are thus given by:

$$
\begin{aligned}
S_t(r(t)) &= N(t)\Big(\frac{\beta}{1+\beta}\Big)w(t)\Delta_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\frac{w(t+1)\Delta_t^h(t+1)}{r(t)} \\
&= \Big(\frac{\beta}{1+\beta}\Big)(1-\theta)\Big(\frac{K(t)}{N(t)}\Big)^{\theta} \times 1 - \Big(\frac{1}{1+\beta}\Big)\frac{(1-\theta)\Big(\frac{K(t+1)}{N(t+1)}\Big)^{\theta} \times 0}{\theta\Big(\frac{K(t+1)}{N(t+1)}\Big)^{\theta-1}} \\
&= \Big(\frac{\beta}{1+\beta}\Big)(1-\theta)K(t)^{\theta}N(t)^{1-\theta}
\end{aligned}
$$

Recall that in a perfect foresight competitive equilbrium, all savings go to capital accumulation, such that:

$$
S_t(r(t)) = K(t+1) \Leftrightarrow \Big(\frac{\beta}{1+\beta}\Big)(1-\theta)K(t)^{\theta}N(t)^{1-\theta} = K(t+1)
$$

Capital per worker is given by $k(t) = \frac{K(t)}{N(t)}$. Rewriting our expression in per capita terms:

$$
\begin{aligned}
K(t+1) &= \Big(\frac{\beta}{1+\beta}\Big)(1-\theta)K(t)^{\theta}N(t)^{1-\theta} \\
N(t+1)k(t+1) &= \Big(\frac{\beta}{1+\beta}\Big)(1-\theta)k(t)^{\theta}N(t) \\
nN(t)k(t+1) &= \Big(\frac{\beta}{1+\beta}\Big)(1-\theta)k(t)^{\theta}N(t) \\
k(t+1) &= \Big(\frac{\beta}{1+\beta}\Big)\frac{1-\theta}{n}k(t)^{\theta}
\end{aligned}
$$

In a stationary equilibrium, capital per worker must be constant, such that $k(t+1) = k(t) = k$. It follows that:

$$
\begin{aligned}
k &= \Big(\frac{\beta}{1+\beta}\Big)\frac{1-\theta}{n}k^{\theta} \\
k &= \Big[\frac{\beta(1-\theta)}{(1+\beta)n}\Big]^{\frac{1}{1-\theta}}
\end{aligned}
$$

(b) The budget constraints are now given by:

$$
\begin{cases}
\text{Young: } \rightarrow c_t^h(t) \leq (1-\tau)w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) \\
\text{Old: } \rightarrow c_t^h(t+1) \leq w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) + b(t+1)
\end{cases}
$$

where $b(t+1)$ is the pension benefit to each old person in period $t+1$. Solving the household's problem again, we obtain the same equilibrium conditions as in the case without government. Substituting in the lifetime budget constraint, we have that:

$$c_t^h(t) = \Big(\frac{1}{1+\beta}\Big)\Big[(1-\tau)w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1) + b(t+1)}{r(t)}\Big]$$

And savings:

$$\begin{aligned}
s_t^h(t) &= (1-\tau)w(t)\Delta_t^h(t) - c_t^h(t)\\
&= \Big(\frac{\beta}{1+\beta}\Big)(1-\tau)w(t)\Delta_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\frac{w(t+1)\Delta_t^h(t+1) + b(t+1)}{r(t)}
\end{aligned}$$

Recall that the budget constraint must be balanced in equilibrium $\forall t$, meaning that:

$$\begin{aligned}
N(t)b(t+1) &= \tau w(t+1)\Delta_{t+1}^h(t+1)N(t+1)\\
N(t)b(t+1) &= \tau(1-\theta)\Big(\frac{K(t+1)}{N(t+1)}\Big)^{\theta} \times 1 \times N(t+1)\\
b(t+1) &= \tau(1-\theta)K(t+1)^{\theta}N(t+1)^{1-\theta}\frac{1}{N(t)}
\end{aligned}$$

And aggregate savings are thus given by:

$$\begin{aligned}
S_t(r(t)) &= N(t)s_t^h(t)\\
&= N(t)\Big[\Big(\frac{\beta}{1+\beta}\Big)(1-\tau)w(t)\Delta_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\frac{w(t+1)\Delta_t^h(t+1) + b(t+1)}{r(t)}\Big]\\
&= \Big(\frac{\beta}{1+\beta}\Big)(1-\tau)(1-\theta)K(t)^{\theta}N(t)^{1-\theta} - \Big(\frac{1}{1+\beta}\Big)\frac{\tau(1-\theta)}{\theta}K(t+1)
\end{aligned}$$

And in equilibrium, we have that (recall that we are interested in capital per worker, $k$):

$$\begin{aligned}
K(t+1) &= S_t(r(t))\\
K(t+1) &= \Big(\frac{\beta}{1+\beta}\Big)(1-\tau)(1-\theta)K(t)^{\theta}N(t)^{1-\theta} - \Big(\frac{1}{1+\beta}\Big)\frac{\tau(1-\theta)}{\theta}K(t+1)\\
K(t+1) &= \frac{\beta(1-\tau)(1-\theta)\theta}{\theta(1+\beta)+\tau(1-\theta)}K(t)^{\theta}N(t)^{1-\theta}\\
N(t+1)k(t+1) &= \frac{\beta(1-\tau)(1-\theta)\theta}{\theta(1+\beta)+\tau(1-\theta)}k(t)^{\theta}N(t)\\
k(t+1) &= \frac{1}{n}\frac{\beta(1-\tau)(1-\theta)\theta}{\theta(1+\beta)+\tau(1-\theta)}k(t)^{\theta}
\end{aligned}$$

Notice that this formula corresponds to the one in the previous question when $\tau = 0$. In a stationary equilibrium, we have that:

$$k = \frac{1}{n}\frac{\beta(1-\tau)(1-\theta)\theta}{\theta(1+\beta)+\tau(1-\theta)}k^{\theta}$$
$$k(\tau) = \Big[\frac{1}{n}\frac{\beta(1-\tau)(1-\theta)\theta}{\theta(1+\beta)+\tau(1-\theta)}\Big]^{\frac{1}{1-\theta}}$$

Since we are only interested in the sign of the derivative, we can ignore the exponent and the factor $\frac{\beta\theta(1-\theta)}{n}$ and just focus on:

$$\frac{1-\tau}{\theta(1+\beta)+\tau(1-\theta)}$$

Taking the derivative:

$$\frac{\partial}{\partial\tau} = \frac{\theta-1-\theta(1+\beta)}{[\theta(1+\beta)+\tau(1-\theta)]^2} < 0$$

Meaning that an increase in the tax rate, $\tau$, decreases the steady state stock of capital.

(c) The transfer to the old in period $t+1$ is now given by $b(t+1) = \tau w(t)\Delta_t^h(t)R(t+1)$. Furthermore, $K(t+1)$ now has two components: private and public savings. Private savings/investment are given by:

$$S_t(r(t)) = N(t)\Big[\Big(\frac{\beta}{1+\beta}\Big)(1-\tau)w(t)\Delta_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\frac{w(t+1)\Delta_t^h(t+1)+\tau w(t)\Delta_t^h(t)R(t+1)}{r(t)}\Big]$$
$$= N(t)w(t)\frac{\beta-\tau(1+\beta)}{1+\beta}$$

And public investment is given by $\tau w(t)N(t)$, meaning that in equilibrium we must have:

$$K(t+1) = N(t)w(t)\frac{\beta-\tau(1+\beta)}{1+\beta} + \tau w(t)N(t)$$
$$K(t+1) = \frac{\beta}{1+\beta}N(t)w(t)$$

which is the same results as in question (a). We can conclude that $\tau$ has no effect on the aggregate capital stock. This is because in the presence of such a scheme, the government is "saving" on behalf of the agents, much like in a fully-funded pension scheme. Here, a tax increase will reduce private savings, but will increase the return that the government pays back to agents when they are old. Since the interest received on the tax revenue is the same as the rental rate of capital, the aggregate stock of capital will remain unchanged. Moreover, However, the question refers to "small" values of $\tau$

since sufficiently large values of $\tau$ will make private savings negative, i.e. if $\tau > \frac{\beta}{1+\beta}$.

## Exercise 3

Consider a two-period overlapping generations economy with land. The number of people born in each period is $N(t) = 100$ for all $t$. Preferences are given by:

$$u_t^h = c_t^h(t).c_t^h(t+1)$$

Everyone has the endowment profile: $\omega_t^h = [30, 24]$. The southern part of the country has 100 units of land, each yielding a crop of $d^S(t) = 1$ in every period. The northern part of the country also has 100 units of land, but it is barren ($d^N(t) = 0$) and unpopulated. Ownership of southern land is in period 0 equally distributed among members of generation $-1$.

(a) What allocations are feasible in this economy?

(b) Solve for the stationary competitive equilibrium.

(c) Now suppose that people learn in period 0 that there will be global warming so that from period 2 on each unit of northern land will also yield a crop. In other words, $d^N(0) = 0$, $d^N(1) = 0$, but $d^N(t) = 1$ for $t = 2, 3, \dots$. In period 0, every old person gets one unit of northern land for free.

  (a) What is the new stationary competitive equilibrium (and thus the price of both types of land in period 1)?

  (b) What is the period 0 price of southern land $p^S(0)$. What is the period 0 price of northern land $p^N(0)$?

### Answer:

(a) Feasibility in an economy with land requires that:

$$C(t) = \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + \sum_{h=1}^{N(t)} c_t^h(t) \leq Y(t) + D(t), \forall t$$

where $D(t) = d(t)A$ and $Y(t) = \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t) + \sum_{h=1}^{N(t)} \omega_t^h(t)$. Substituing, we get that the set of feasible allocations in this economy is given by:

$$C(t) \leq 100 \times (30 + 24) + 100 \times 1 + 100 \times 0 = 5500, \forall t$$

(b) In an economy with land, agents face the following budget constraints[2]:

$$\begin{cases} \text{Young: } \rightarrow c_t^h(t) \leq \omega_t^h(t) - l^h(t) - p(t)a^h(t) \\ \text{Old: } \rightarrow c_t^h(t+1) \leq \omega_t^h(t+1) + r(t)l^h(t) + d(t+1)a^h(t) + p(t+1)a^h(t) \end{cases}$$

The competitive choice problem is given by:

$$\begin{aligned} \max_{\{c_t^h(t), c_t^h(t+1), l^h(t), a^h(t)\}} \quad & c_t^h(t)c_t^h(t+1) \\ \text{s.t. } & c_t^h(t) \leq \omega_t^h(t) - l^h(t) - p(t)a^h(t) \\ & c_t^h(t+1) \leq \omega_t^h(t+1) + r(t)l^h(t) + d(t+1)a^h(t) + p(t+1)a^h(t) \end{aligned}$$

And our Lagrangian:

$$\begin{aligned} \mathcal{L} = {} & c_t^h(t)c_t^h(t+1) + \mu(t)\Big[\omega_t^h(t) - l^h(t) - p(t)a^h(t) - c_t^h(t)\Big] \\ & + \mu(t+1)\Big[\omega_t^h(t+1) + r(t)l^h(t) + d(t+1)a^h(t) + p(t+1)a^h(t) - c_t^h(t+1)\Big] \end{aligned}$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the FOC:

$$\begin{aligned} \mathbf{c_t^h(t)}: &\ c_t^h(t+1) - \mu(t) = 0, \forall t \geq 1 \\ \mathbf{c_t^{h,e}(t+1)}: &\ c_t^h(t) - \mu(t+1) = 0, \forall t \geq 1 \\ \mathbf{l^h(t)}: &\ -\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1 \\ \mathbf{a^h(t)}: &\ -\mu(t)p(t) + \mu(t+1)[d(t+1) + p(t+1)] = 0, \forall t \geq 1 \end{aligned}$$

From which we can obtain the following equilibrium conditions:

$$\begin{aligned} c_t^h(t+1) &= r(t)c_t^h(t) \\ r(t) &= \frac{d(t+1) + p(t+1)}{p(t)} \end{aligned}$$

Substituting in the lifetime budget constraint:

$$\begin{aligned} c_t^h(t) + \frac{c_t^{h,e}(t+1)}{r(t)} &= \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)} - a^h(t)\Big[p(t) - \frac{d(t+1) + p^e(t+1)}{r(t)}\Big] \\ c_t^h(t) + \frac{c_t^h(t)r(t)}{r(t)} &= \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)} - a^h(t)\underbrace{\Big[p(t) - \frac{d(t+1) + p^e(t+1)}{r(t)}\Big]}_{=0} \\ c_t^h(t) &= \frac{1}{2}\Big[\omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}\Big] \end{aligned}$$

[2]Remember that a stationary equilibrium is, by definition, also a perfect foresight equilibrium, hence the lack of the expectation term.

And savings are given by:

$$s_t^h(t) = \omega_t^h(t) - c_t^h(t) = \frac{1}{2}\omega_t^h(t) - \frac{1}{2}\frac{\omega_t^h(t+1)}{r(t)}$$

And aggregate savings are given by $S_t(r(t)) = N(t)s_t^h(t)$. Recall that a competitive equilibrium in an economy with land requires the following conditions to hold:

$$\begin{aligned} S_t(r(t)) &= p(t)A \\ r(t) &= \frac{d(t+1)+p(t+1)}{p(t)} \end{aligned}$$

Furthermore, a stationary equilibrium is such that $p(t) = p(t+1) = p$, $r(t) = r$ and $d(t+1) = d$, from where it follows that:

$$r = \frac{d+p}{p} \Leftrightarrow p = \frac{d}{r-1}$$

And in equilibrium, we have that:

$$\begin{aligned} S(r) &= pA \\ 100 \times \left(\frac{1}{2} \times 30 - \frac{1}{2} \times \frac{24}{r}\right) &= \left(\frac{d}{r-1}\right)A \\ 100 \times \left(15 - \frac{12}{r}\right) &= \left(\frac{1}{r-1}\right) \times 100 \\ 15r^2 - 28r + 12 &= 0 \end{aligned}$$

which has two roots, i.e. $r = \left(\frac{2}{3}, \frac{6}{5}\right)$, with corresponding prices:

$$\begin{cases} r = \frac{2}{3} \Rightarrow p = \frac{d}{r-1} = \frac{1}{\frac{2}{3}-1} = -3 \\ r = \frac{6}{5} \Rightarrow p = \frac{d}{r-1} = \frac{1}{\frac{6}{5}-1} = 5 \end{cases}$$

Since the price of land must be nonnegative, we must have $r = \frac{6}{5}$ in equilibrium. Consumption and savings are thus given by, $\forall t \geq 1$:

$$\begin{aligned} c_t^h(t) &= \frac{1}{2}\left[30 + \frac{24}{\frac{6}{5}}\right] = 25 \\ c_t^h(t+1) &= r(t)c_t^h(t+1) = \frac{6}{5} \times 25 = 30 \\ s_t^h(t) &= \frac{1}{2}\left[30 - \frac{24}{\frac{6}{5}}\right] = 5 \end{aligned}$$

Furthermore, this consumption allocation clears the market since:

$$C(t) = \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + \sum_{h=1}^{N(t)} c_t^h(t) = 100 \times (30 + 25) = 5500 \leq Y(t), \forall t$$

(c) (i) Making use of the previous results, our equilibrium condition is now given by:

$$\begin{aligned} S(r) &= pA \\ 100 \times \left(\frac{1}{2} \times 30 - \frac{1}{2} \times \frac{24}{r}\right) &= \left(\frac{d}{r-1}\right) A \\ 100 \times \left(15 - \frac{12}{r}\right) &= \left(\frac{1}{r-1}\right) \times 200 \\ 15r^2 - 29r + 12 = 0 \end{aligned}$$

which has two roots, i.e. $r = \left(\frac{3}{5}, \frac{4}{3}\right)$, with corresponding prices:

$$\begin{cases} r = \frac{3}{5} \Rightarrow p = \frac{d}{r-1} = \frac{1}{\frac{3}{5}-1} = -\frac{5}{2} \\ r = \frac{4}{3} \Rightarrow p = \frac{d}{r-1} = \frac{1}{\frac{4}{3}-1} = 3 \end{cases}$$

Since the price of land must be nonnegative, we must have $p^N(t) = p^S(t) = p = 3, \forall t \geq 1$, and $r = \frac{4}{3}$ in equilibrium. Consumption and savings are thus given by, $\forall t \geq 2$:

$$\begin{aligned} c_t^h(t) &= \frac{1}{2}\left[30 + \frac{24}{\frac{4}{3}}\right] = 24 \\ c_t^h(t+1) &= r(t) c_t^h(t+1) = \frac{4}{3} \times 24 = 32 \\ s_t^h(t) &= \frac{1}{2}\left[30 - \frac{24}{\frac{4}{3}}\right] = 6 \end{aligned}$$

Furthermore, this consumption allocation clears the market since:

$$\begin{aligned} C(t) = \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + \sum_{h=1}^{N(t)} c_t^h(t) &= 100 \times (32 + 24) = 5600 \\ &\leq \\ Y(t) = 100 \times (30 + 24) + 100 \times 1 + 100 \times 1 &= 5600, \forall t \geq 2 \end{aligned}$$

(ii) Making use of the no-arbitrage conditions, we know that the rate of return on each type of land must be the same:

$$r(0) = \frac{d^N(0) + p^N(1)}{p^N(0)} = \frac{p^N(1)}{p^N(0)} = \frac{3}{p^N(0)}$$
$$r(0) = \frac{d^S(0) + p^S(1)}{p^S(0)} = \frac{1 + p^S(1)}{p^S(0)} = \frac{4}{p^S(0)}$$

From where it follows that:

$$\frac{3}{p^N(0)} = \frac{4}{p^S(0)} \Leftrightarrow p^N(0) = \frac{3}{4}p^S(0)$$

Substituting in the market clearing condition, we have that:

$$\begin{aligned}
S_0(r(0)) &= p(0)A \\
100 \times \left(\frac{1}{2} \times 30 - \frac{1}{2} \times \frac{24}{r(0)}\right) &= 100 \times \left(p^S(0) + p^N(0)\right) \\
15 - \frac{12}{\frac{4}{p^S(0)}} &= \frac{3}{4}p^S(0) + p^S(0) \\
p^S(0) &= \frac{60}{19} \Rightarrow p^N(0) = \frac{3}{4}p^S(0) = \frac{45}{19}
\end{aligned}$$

# Exercise 4

Consider a two-period overlapping generations economy with production. The population is given by $N(t) = 1$. Preferences are represented by

$$u_t^h = \ln c_t^h(t) + \beta \ln c_t^h(t+1)$$

People are endowed with labor: $\Delta_t^h = \left[\frac{1}{2}, \frac{1}{2}\right]$.
Suppose that firms have access to the following production function:

$$Y(t) = K(t)^\theta L(t)^{1-\theta}$$

where $K(t)$ is the period $t$ capital stock and $L(t)$ is the period $t$ supply of labor. The aggregate resource constraint is: $C(t) + K(t+1) = Y(t)$.

(a) Find explicit expressions for the steady state equilibrium capital stock and interest rate. What are the numerical values when $\beta = 1$ and $\theta = \frac{1}{3}$?

(b) Discuss why the interest rate is so high in the economy.

(c) Find the golden rule level of capital. Is the steady state equilibrium dynamically efficient?

## Answer:

(a) Let's start by solving the household's problem. In a production economy, the budget constraints of the young and of the old are given by:

$$\begin{cases} \text{Young: } \rightarrow c_t^h(t) \leq w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) \\ \text{Old: } \rightarrow c_t^h(t+1) \leq w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) \end{cases}$$

where $w(t)$ and $R(t)$ are the wage rate and the rental rate of capital at time $t$, respectively. The household's problem is thus given by:

$$\begin{aligned} \max_{\{c_t^h(t), c_t^h(t+1), l^h(t), k^h(t+1)\}} \quad & \ln c_t^h(t) + \beta \ln c_t^h(t+1) \\ \text{s.t. } \quad & c_t^h(t) \leq w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) \\ & c_t^h(t+1) \leq w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) \end{aligned}$$

And our Lagrangian:

$$\begin{aligned} \mathcal{L} = & \ln c_t^h(t) + \beta \ln c_t^h(t+1) + \mu(t)\Big[w(t)\Delta_t^h(t) - l^h(t) - k^h(t+1) - c_t^h(t)\Big] \\ & + \mu(t+1)\Big[w(t+1)\Delta_t^h(t+1) + r(t)l^h(t) + R(t+1)k^h(t+1) - c_t^h(t+1)\Big] \end{aligned}$$

where $\mu(t)$ and $\mu(t+1)$ are nonnegative Lagrangian multipliers associated with the budget constraints of the young and of the old, respectively. Taking the FOC:

$$\begin{aligned} \mathbf{c_t^h(t)}: \quad & \tfrac{1}{c_t^h(t)} - \mu(t) = 0, \forall t \geq 1 \\ \mathbf{c_t^h(t+1)}: \quad & \tfrac{\beta}{c_t^h(t+1)} - \mu(t+1) = 0, \forall t \geq 1 \\ \mathbf{l^h(t)}: \quad & -\mu(t) + \mu(t+1)r(t) = 0, \forall t \geq 1 \\ \mathbf{k^h(t+1)}: \quad & -\mu(t) + \mu(t+1)R(t+1) = 0, \forall t \geq 1 \end{aligned}$$

From which we can obtain the following equilibrium conditions:

$$\begin{aligned} R(t+1) &= r(t) \\ c_t^h(t+1) &= \beta r(t) c_t^h(t) \end{aligned}$$

Where the first condition is the no arbitrage condition between private borrowing and lending and holding capital. Substituting in the lifetime budget constraint:

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r(t)} - k^h(t+1)\Big[1 - \frac{R(t+1)}{r(t)}\Big]$$
$$c_t^h(t) = \Big(\frac{1}{1+\beta}\Big)\Big[w(t)\Delta_t^h(t) + \frac{w(t+1)\Delta_t^h(t+1)}{r(t)}\Big]$$

And savings are given by:

$$s_t^h(t) = w(t)\Delta_t^h(t) - c_t^h(t) = \Big(\frac{\beta}{1+\beta}\Big)w(t)\Delta_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\frac{w(t+1)\Delta_t^h(t+1)}{r(t)}$$

Since $N(t) = 1, \forall t$, we know that aggregate savings are equal to individuals savings, i.e. $S_t(r(t)) = s_t^h(t)$.

Furthermore, the firm's problem is given by:

$$\max_{\{K(t),L(t)\}} K(t)^\theta L(t)^{1-\theta} - R(t)K(t) - w(t)L(t)$$

Taking the FOC:

$\mathbf{K(t)}$: $R(t) = \theta\Big(\frac{K(t)}{L(t)}\Big)^{\theta-1}, \forall t \geq 1$

$\mathbf{L(t)}$: $w(t) = (1-\theta)\Big(\frac{K(t)}{L(t)}\Big)^{\theta}, \forall t \geq 1$

Note that labour endowments are fixed, meaning that the total labour supply is also fixed for every period $t$ and given by:

$$L(t) = \sum_{h=1}^{N(t)} \Delta_t^h(t) + \sum_{h=1}^{N(t-1)} \Delta_{t-1}^h(t) = \frac{1}{2} \times 1 + \frac{1}{2} \times 1 = 1$$

So we have that:

$$R(t) = \theta K(t)^{\theta-1} \Rightarrow R(t+1) = \theta K(t+1)^{\theta-1} = r(t)$$
$$w(t) = (1-\theta)K(t)^\theta$$

And aggregate savings are thus given by:

$$\begin{aligned}
S_t(r(t)) &= \Big(\frac{\beta}{1+\beta}\Big)w(t)\Delta_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\frac{w(t+1)\Delta_t^h(t+1)}{r(t)} \\
&= \Big(\frac{\beta}{1+\beta}\Big)\Delta_t^h(t)(1-\theta)K(t)^\theta - \Big(\frac{1}{1+\beta}\Big)\frac{\Delta_t^h(t+1)(1-\theta)K(t+1)^\theta}{\theta K(t+1)^{\theta-1}} \\
&= \Big(\frac{\beta}{1+\beta}\Big)\Delta_t^h(t)(1-\theta)K(t)^\theta - \Big(\frac{1}{1+\beta}\Big)\Big(\frac{1-\theta}{\theta}\Big)\Delta_t^h(t+1)K(t+1)
\end{aligned}$$

**Definition 1.** [3] A perfect foresight competitive equilibrium for an economy with labour endowments and a production function of $\gamma(t)F(K(t), L(t))$ is a sequence of $K(t)$, $r(t)$, $w(t)$ and $R(t)$ for $t \geq 1$ such that, given an initial $K(1) > 0$,

$$\begin{aligned} S_t(r(t)) &= K(t+1) \\ r(t) &= R(t+1) \\ w(t) &= \frac{\partial[\gamma(t)F(K(t), L(t))]}{\partial L(t)} \\ R(t) &= \frac{\partial[\gamma(t)F(K(t), L(t))]}{\partial K(t)} \end{aligned}$$

hold for all $t \geq 1$.

Making use of definition 1 above, we know that in a perfect foresight competitive equilbrium, all savings go to capital accumulation, such that:

$$S_t(r(t)) = K(t+1)$$
$$\Big(\frac{\beta}{1+\beta}\Big)\Delta_t^h(t)(1-\theta)K(t)^\theta - \Big(\frac{1}{1+\beta}\Big)\Big(\frac{1-\theta}{\theta}\Big)\Delta_t^h(t+1)K(t+1) = K(t+1)$$
$$K(t+1) = \Big(\frac{\theta(1-\theta)\beta\Delta_t^h(t)}{(1+\beta)\theta + (1-\theta)\Delta_t^h(t+1)}\Big)K(t)^\theta$$

And since a stationary equilibrium is such that $K(t+1) = K(t) = K$, we have that:

$$K = \Big(\frac{\theta(1-\theta)\beta\Delta_t^h(t)}{(1+\beta)\theta + (1-\theta)\Delta_t^h(t+1)}\Big)K^\theta \Leftrightarrow K = \Big(\frac{\theta(1-\theta)\beta\Delta_t^h(t)}{(1+\beta)\theta + (1-\theta)\Delta_t^h(t+1)}\Big)^{\frac{1}{1-\theta}}$$
$$r = \theta K^{\theta-1} = \Big(\frac{(1+\beta)\theta + (1-\theta)\Delta_t^h(t+1)}{(1-\theta)\beta\Delta_t^h(t)}\Big)$$
$$w = (1-\theta)K^\theta = (1-\theta)\Big(\frac{\theta(1-\theta)\beta\Delta_t^h(t)}{(1+\beta)\theta + (1-\theta)\Delta_t^h(t+1)}\Big)^{\frac{\theta}{1-\theta}}$$

Substituting, we have:

$$\begin{aligned} K &= \Big(\frac{\frac{1}{3} \times (1-\frac{1}{3}) \times 1 \times \frac{1}{2}}{(1+1) \times \frac{1}{3} + (1-\frac{1}{3}) \times \frac{1}{2}}\Big)^{\frac{1}{1-\frac{1}{3}}} = \frac{1}{27} \\ r &= \Big(\frac{(1+1) \times \frac{1}{3} + (1-\frac{1}{3}) \times \frac{1}{2}}{(1-\frac{1}{3}) \times 1 \times \frac{1}{2}}\Big) = 3 \\ w &= (1-\frac{1}{3}) \times \Big(\frac{\frac{1}{3} \times (1-\frac{1}{3}) \times 1 \times \frac{1}{2}}{(1+1) \times \frac{1}{3} + (1-\frac{1}{3}) \times \frac{1}{2}}\Big)^{\frac{\frac{1}{3}}{1-\frac{1}{3}}} = \frac{2}{9} \end{aligned}$$

[3] Page 238 of the course textbook.

While the exercise did not ask for this, we can also find consumption and savings:

$$
\begin{aligned}
c_t^h(t) &= \left(\frac{1}{1+1}\right) \times \left[\frac{2}{9} \times \frac{1}{2} + \frac{\frac{2}{9} \times \frac{1}{2}}{3}\right] = \frac{2}{27} \\
c_t^h(t+1) &= \beta r(t) c_t^h(t) = 1 \times 3 \times \frac{2}{27} = \frac{2}{9} \\
s_t^h(t) &= \left(\frac{1}{1+1}\right) \times \left[\frac{2}{9} \times \frac{1}{2}\right] - \left(\frac{1}{1+1}\right) \times \left[\frac{\frac{2}{9} \times \frac{1}{2}}{3}\right] = \frac{1}{27}
\end{aligned}
$$

Notice that savings are equal to aggregate capital, as we imposed in the equilibrium condition. We can also check whether this allocation is feasible:

**Definition 2.** [4] An allocation is <u>feasible</u> if there is a sequence $K(t)$ such that $K(t) \geq 0$ and

$$Y(t) \geq C(t) + K(t+1)$$

Making use of definition 2 above, we have that:

$$\left(\frac{1}{27}\right)^{\frac{1}{3}} \geq \frac{2}{27} + \frac{2}{9} + \frac{1}{27} \Leftrightarrow \frac{1}{3} \geq \frac{1}{3}$$

Meaning that this allocation is feasible.

(b) Since the intertemporal discount factor, $\beta$, is quite high, we know that agents are quite patient, i.e., they put a lot of value in future consumption. However, given that the labor endowment profile is completely smooth over the life cycle, the incentives to save are quite weak. The lack of savings leads to capital scarcity, which translates into a high marginal productivity of capital (interest rate). Furthermore, if the interest rate was equal to 1, people would not want to save, meaning that the aggregate stock of capital would be zero, which corresponds to an infinite marginal productivity of capital, a contradiction!

(c) Let's refer to the following definition:

**Definition 3.** The <u>golden rule level of capital</u> is the steady state level of capital that maximizes consumption.

Let's make use of the resource constraint (remember that $L(t) = 1$):

$$C + K = Y = F(K, 1) \Leftrightarrow C = F(K, 1) - K$$

[4] Page 236 of the course textbook.

Taking the FOC with respect to capital:

$$F'(K,1) = 1 \Leftrightarrow \theta K^{\theta-1} = 1 \Leftrightarrow K^{gr} = \left(\frac{1}{\theta}\right)^{\frac{1}{\theta-1}} = \left(\frac{1}{\frac{1}{3}}\right)^{\frac{1}{\frac{1}{3}-1}} = \left(\frac{1}{3}\right)^{\frac{3}{2}} = \frac{\sqrt{3}}{9} \approx 0,1925$$

which is our golden rule level of capital. Remember that the stationary equilibrium level of capital is given by $K^* = \frac{1}{27} \approx 0,0370$. Since the golden rule level of capital its higher than the stationary equilibrium level of capital, the economy is dynamically efficient.