14.03/14.003: Microeconomic Theory and Public Policy 11/18/22

Recitation 9: Risk Aversion and Insurance

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Lotteries, Expected Utility, and Risk Aversion

Lotteries:

A **lottery** is a list L of probabilities and associated payoffs:

$$L = \begin{cases} \pi_1 & \text{with probability } p_1 \\ \pi_2 & \text{with probability } p_2 \\ ... & ... \\ \pi_n & \text{with probability } p \end{cases}$$

where p_i denotes a probability and π_i denotes a monetary payoff. $\sum_{i=1}^{n} p_i = 1$ (probabilities sum to 1).

Expected Utility Property:

The von Neuman Morgenstern **expected utility property** says that utility is linear in probabilities. For example, the expected utility of lottery L can be written as:

$$u(L) = p_1 u(\pi_1) + p_2 u(\pi_2) + ... + p_n u(\pi_n).$$

Alternatively, if we had two lotteries (L and L') and some $\beta \in [0, 1]$, then

$$u(\beta L + (1 - \beta)L') = \beta u(L) + (1 - \beta)u(L').$$

This is simply a generalization of the previous example (just think of each outcome in L as its own lottery that pays π_i with probability 1).

Example:

Suppose we have L and L', which are given by:

$$L = \begin{cases} 2 & \text{with probability } \frac{1}{2} \\ 4 & \text{with probability } \frac{1}{2} \end{cases}$$

$$L' = \begin{cases} 0 & \text{with probability } \frac{1}{4} \\ 6 & \text{with probability } \frac{3}{4} \end{cases}$$

What is the expected utility of L?

$$u(L) = \frac{1}{2}u(2) + \frac{1}{2}u(4).$$

What is the expected utility of $\frac{1}{3}L + \frac{2}{3}L'$?

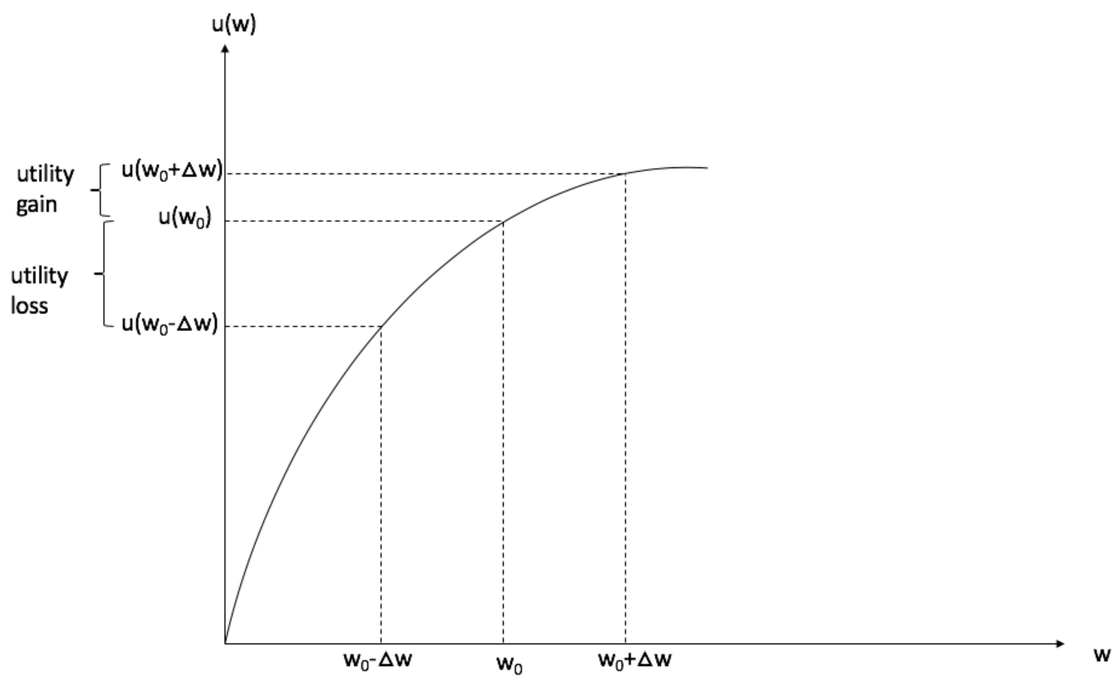
$$\begin{aligned} u\left(\frac{1}{3}L + \frac{2}{3}L'\right) &= \frac{1}{3}u(L) + \frac{2}{3}u(L') \\ &= \frac{1}{3}\left[\frac{1}{2}u(2) + \frac{1}{2}u(4)\right] + \frac{2}{3}\left[\frac{1}{4}u(0) + \frac{3}{4}u(6)\right] \\ &= \frac{1}{6}u(2) + \frac{1}{6}u(4) + \frac{1}{6}u(0) + \frac{1}{2}u(6). \end{aligned}$$

Another way to think about the expected utility property: you can pull probabilities outside of the utility function.

Risk Aversion:

Agents are risk-averse if their utility functions are concave (they display diminishing marginal returns). Consider an agent with a concave utility function who has initial wealth w_0. He is considering a bet that pays $+\Delta w$ with probability $\frac{1}{2}$ and $-\Delta w$ with probability $\frac{1}{2}$. Notice two things: first, the expected value of the bet is 0. Second, the utility loss from $-\Delta w$ is greater (in absolute value) than the utility gain from $+\Delta w$. Therefore, the expected *utility* of the bet is negative. This comes from the concavity of the utility function.

Figure 1: 50/50 Bet for a Risk-Averse Agent



More generally, there are two concepts here:

1. **The utility of expected wealth**. This is the utility if you gave the agent the expected value of the bet *for sure*. Let w be the agent's wealth after the bet. Then the utility of expected wealth is

$$u\left(E[w]\right) = u\left(p_1 w_1 + p_2 w_2 + ... + p_n w_n\right).$$

2. **The expected utility of wealth**. This is the expected utility if the agent takes the bet. Again, let w be the agent's wealth after the bet. Then the expected utility of wealth is

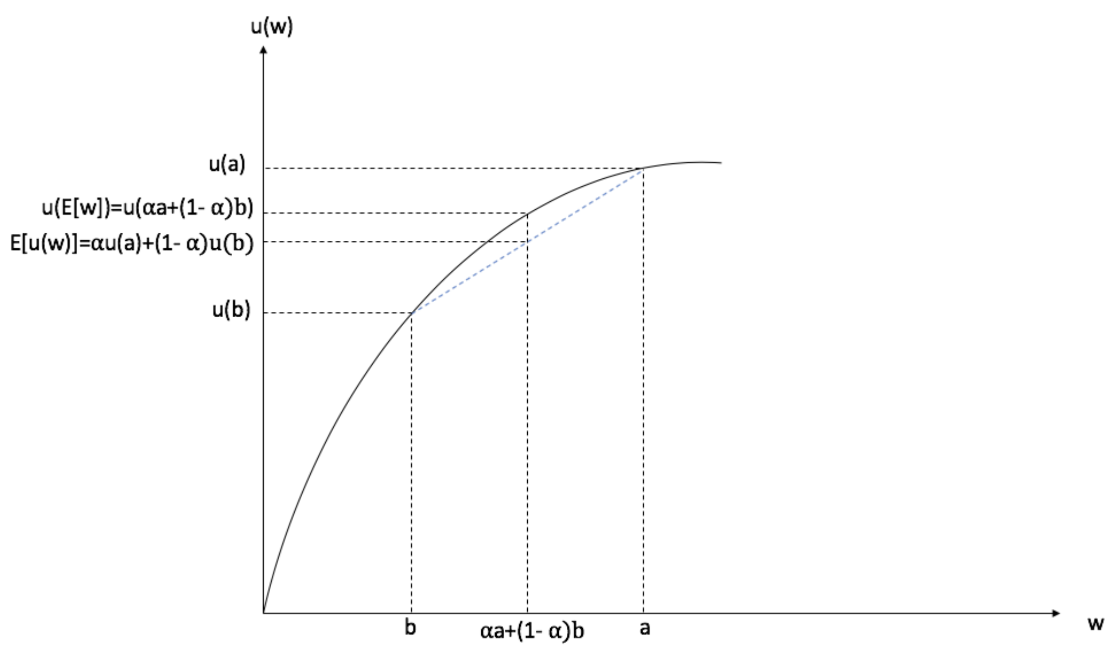
$$E\left[u(w)\right] = p_1 u(w_1) + p_2 u(w_2) + ... + p_n u(w_n).$$

The following inequalities are true:

- If the agent is risk-averse, then $u\left(E[w]\right) > E\left[u(w)\right]$
- If the agent is risk-loving, then $u\left(E[w]\right) < E\left[u(w)\right]$
- If the agent is risk-neutral, then $u\left(E[w]\right) = E\left[u(w)\right]$

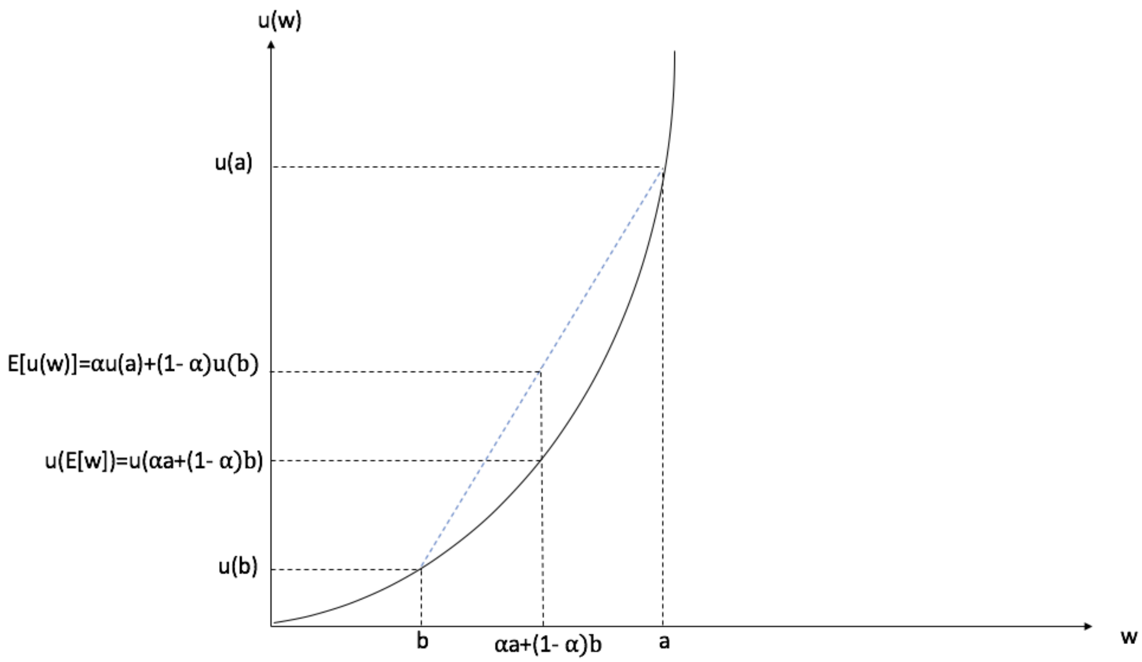
For those of you who have taken a probability class, this is Jensen's inequality. To see why these inequalities are true graphically, consider again a risk-averse person with initial wealth of zero. They are considering a bet where they win a with probability α and win b with probability $(1-\alpha)$. The expected value of this bet is $\alpha a + (1-\alpha)b$, and it will lie somewhere on the x-axis between a and b. The utility of getting the expected value of this bet for sure is given by $u\left(E[w]\right) = u(\alpha a + (1-\alpha)b)$, and this lies on the utility curve. However, the expected utility of the bet is given by $E\left[u(w)\right] = \alpha u(a) + (1-\alpha)u(b)$. This lies somewhere on the dotted blue line connecting $u(a)$ and $u(b)$. Since the blue line is everywhere below the utility curve, we can see that no matter what α is, $u\left(E[w]\right) > E\left[u(w)\right]$.

Figure 2: Risk Aversion



Now consider a risk-loving person with an initial wealth of zero. Now we see the blue dotted line is above the utility curve, and so $u\left(E[w]\right) < E\left[u(w)\right]$.

Figure 3: Risk Loving

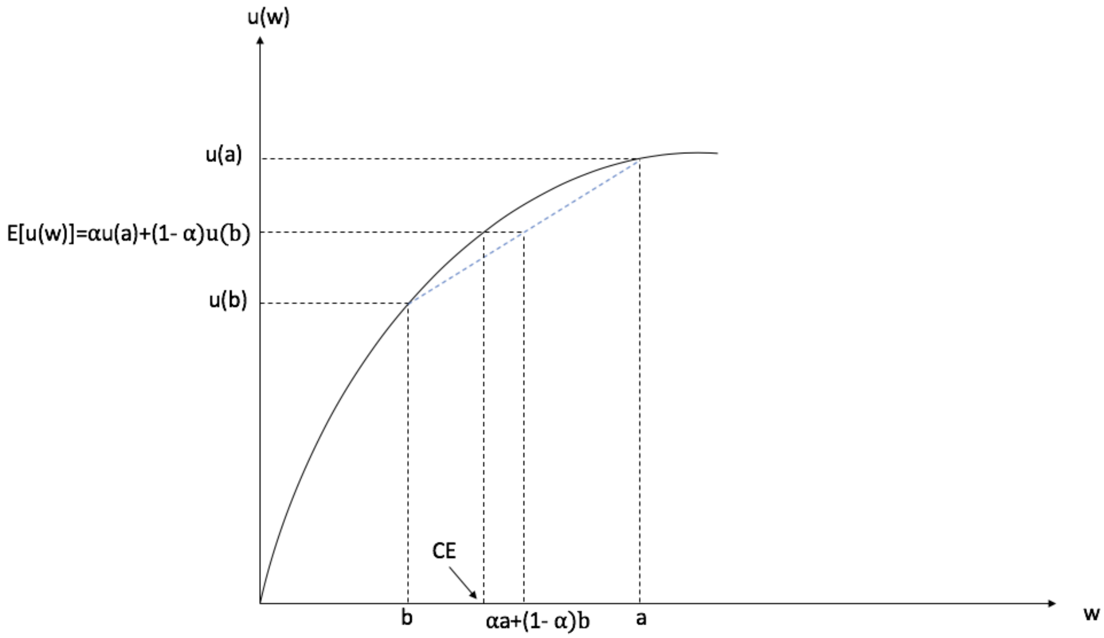


Question: what would the picture look like for a risk-neutral agent?

Certainty Equivalent:

The certainty equivalent is the amount of money you would accept for sure instead of taking the bet. It is labelled on Figure 4 below.

Figure 4: Certainty Equivalent



How do we solve for the certainty equivalent? Set the expected utility of the lottery equal to the utility of the CE for sure:

$$\underbrace{\alpha u(a) + (1-\alpha)u(b)}_{E[u(w)]} = u(CE)$$

$$u^{-1}\left(E[u(w)]\right) = CE.$$

For a risk-averse agent, the certainty equivalent is less than the expected value of the bet, i.e. $CE < E[w]$. **Question:** what about for a risk-loving agent? Why? Can you draw the picture?

Insurance

Insurance Demand:

Consider a person with initial wealth w_0, probability of accident p, and loss if they have an accident of L. If this person does not have insurance, her expected utility is:

$$(1-p)u(w_0) + pu(w_0 - L)$$

where $u(\cdot)$ is the utility function. Now, suppose that insurance companies charge θ per dollar of insurance. So for example, if you buy \$2 of insurance, you pay a premium of 2θ. If you have an accident, you will receive \$2 in compensation. If this person buys A units of insurance, her expected utility is:

$$(1-p)u(w_0 - \theta A) + pu(w_0 - L - \theta A + A).$$

How much insurance will this person demand? She will solve:

$$\max_A \{(1-p)u(w_0 - \theta A) + pu(w_0 - L - \theta A + A)\}$$

which yields the FOC:

$$\begin{aligned} -\theta(1-p)u'(w_0 - \theta A^*) + (1-\theta)pu'(w_0 - L - \theta A^* + A^*) = 0 \\ (1-\theta)pu'(w_0 - L - \theta A^* + A^*) = \theta(1-p)u'(w_0 - \theta A^*). \end{aligned}$$

This implicitly defines A^*. Now, suppose that the price of insurance is actuarily fair. In other words, the insurance company charges the expected cost of the insurance. This would imply that $\theta = p$. This allows us to simplify the FOC more:

$$\begin{aligned} (1-p)pu'(w_0 - L - \theta A^* + A^*) &= p(1-p)u'(w_0 - \theta A^*). \\ u'(w_0 - L - \theta A^* + A^*) &= u'(w_0 - \theta A^*) \\ w_0 - L - \theta A^* + A^* &= w_0 - \theta A^* \\ -L + A^* &= 0 \\ A^* &= L. \end{aligned}$$

Is this a minimum or a maximum? The SOC is:

$$p(1-p)\left[(1-\theta)u''(w_0 - L - \theta A^* + A^*) + \theta u''(w_0 - \theta A^*)\right].$$

The sign of this expression depends on the sign of $u''(\cdot)$. If $u''(\cdot) < 0$ (the person is risk-averse), then $A^* = L$ is the maximum. The agent wants to buy full insurance. Why? Because a risk-averse agent wants to distribute wealth evenly across states of the world. Note that if $u''(\cdot) > 0$ (the person is risk-loving), then $A^* = L$ is the minimum. Here, the logic is the reverse: a risk-loving agent wants to concentrate all her wealth in one state of the world.

Two Ways to Quote the Price of Insurance

- Price per dollar: if insurance costs θ per dollar, then for every θ you pay in premium, the insurance will pay out \$1 in the bad state.
- Total price: if your insurance policy pays out A in the bad state, then the total cost is $x = \theta A$.

Willingness to Pay for Insurance:

We have established that risk-averse agents will want to purchase full insurance at actuarily fair prices. But would they be willing to pay more? Yes. How much more? Until they are indifferent between purchasing the insurance and not purchasing the insurance. Let x be the agent's willingness to pay for full insurance. It is defined by the indifference condition:

$$\begin{aligned} \underbrace{pu(w_0) + (1-p)u(w_0 - L)}_{\text{expected utility w/o insurance}} &= \underbrace{pu(w_0 - x) + (1-p)u(w_0 - x - L + L)}_{\text{expected utility w/ insurance}} \\ pu(w_0) + (1-p)u(w_0 - L) &= pu(w_0 - x) + (1-p)u(w_0 - x) \\ pu(w_0) + (1-p)u(w_0 - L) &= u(w_0 - x) \\ x &= w_0 - u^{-1}\left(pu(w_0) + (1-p)u(w_0 - L)\right) \end{aligned}$$

Example:

Suppose you start with $w_0 = 100$, $p = 0.1$, and $L = 90$. Your utility is given by

$$u(w) = \ln(w).$$

The actuarily fair price of full insurance is \$0.1 per dollar insured. For full insurance, this comes out to $0.1 \times 90 = 9$. Your willingness to pay for full insurance is the value of x that solves

$$\begin{aligned}
0.9\ln(100) + 0.1\ln(10) &= 0.9\ln(100 - x) + 0.1\ln(100 - x - 90 + 90)\\
0.9\ln(100) + 0.1\ln(10) &= \ln(100 - x)\\
e^{0.9\ln(100)+0.1\ln(10)} &= 100 - x\\
x &= 100 - e^{0.9\ln(100)+0.1\ln(10)} \approx 20.57.
\end{aligned}$$

If the insurance company can charge anywhere between these two values (\$9 and \$20.57 for full insurance, or \$0.1 and \$0.23 per dollar of insurance) and you will choose to purchase full insurance. What if the insurance company instead charged \$0.30 per dollar of insurance. Would you choose full insurance, partial insurance, or no insurance? We can set up the maximization:

$$\max_A \{0.9\ln(100 - 0.3A) + 0.1\ln(100 - 90 - 0.3A + A)\} = \max_A \{0.9\ln(100 - 0.3A) + 0.1\ln(10 + 0.7A)\}$$

which yields the FOC:

$$\begin{aligned}
\frac{0.9(0.3)}{100 - 0.3A^*} &= \frac{0.1(0.7)}{10 + 0.7A^*}\\
0.27(10 + 0.7A^*) &= 0.07(100 - 0.3A^*)\\
2.7 + (0.27)(0.7)A^* &= 7 - (0.07)(0.3)A^*\\
A^*\left((0.27)(0.7) + (0.07)(0.3)\right) &= 7 - 2.7\\
A^* &= \frac{7 - 2.7}{(0.27)(0.7) + (0.07)(0.3)} \approx 20.48
\end{aligned}$$