5311 International Trade

Paul S. Segerstrom

Course Description

- This course provides an introduction to the literature on international trade.
- The first part of the course covers some essential mathematics for reading the literature on international trade. The topics covered are integration techniques, differential equations and optimal control theory.

Trade and Economic Growth

- We then use this mathematics to study models where investment choices are made by profit-maximising firms. In these models, the reward of future monopoly profits from innovating encourage firms to incur the costs of developing new products today. We use these models to study the implications of trade liberalization (North-North trade) and stronger intellectual property rights (North-South trade). Paul Romer won the 2018 Economic Nobel Prize for "integrating technological innovations into long-run macroeconomic analysis" and this part of the course explains Romer's contribution, in particular, Romer (1990, *Journal of Political Economy*).

Why Free Trade Increases Productivity Within Industries

- Next, we use the same mathematics to study a model of international trade with heterogeneous firms, called the Melitz model. In this model, firms differ in their productivities within an industry and firms make investment choices when they enter markets. The Melitz (2003, *Econometrica*) model provides an explanation for why free trade increases productivity within industries and it is currently the most influential model of international trade.

Why Some Countries are Rich and Other Countries are Poor

- The final part of the course is about economic history and builds on what we have learned from studying models of international trade. We discuss why free markets first appeared in the West and more generally, why the West became so economically prosperous compared to all other regions of the world.

Motivating Evidence

- Looking at 136 countries during the time period from 1950 to 2000, Wacziarg and Welsh (2008, *WBER*) find that for the typical country growing at an average annual rate of 1.1% before trade liberalization, its average annual growth rate jumps up to $1.1\% + 1.4\% = 2.5\%$ after trade liberalization.
- Why does opening up to trade have these effects?

Lecture 1: Integrals and Differential Equations

This lecture is based on

- Malcolm Pemberton and Nicholas Rau (2023), *Mathematics for Economists*, chapters 19 and 20.
- Malcolm Pemberton and Nicholas Rau (2023), *Mathematics for Economists*, chapter 23, 515-531, chapter 26, 583-597, 609-610, chapter 27 and chapter 28.

What is an integral?

- An **integral** is the area under the curve $y = f(x)$ and above the x-axis between $x = a$ and $x = b$ (where $b > a$). Let this area equal A.

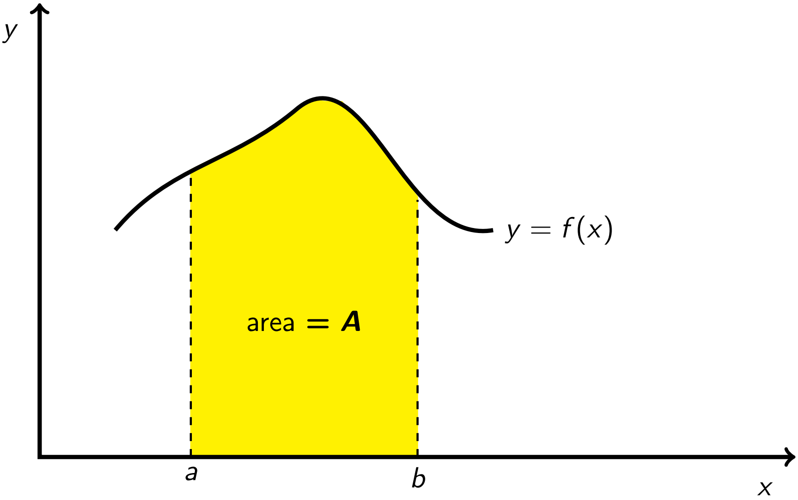

Figure: An integral.

- How do you calculate area A given the function $f(x)$? This is the fundamental question of integral calculus.
- For differential calculus, the fundamental question is, what is the slope of the tangent line to a function at a given point?
- So we have the contrast between slopes of curves versus area underneath curves.

- To find A, it is helpful to first define an area function $A(x)$ as the area under the curve $y = f(x)$ and above the x-axis between a and x, where $a < x < b$.

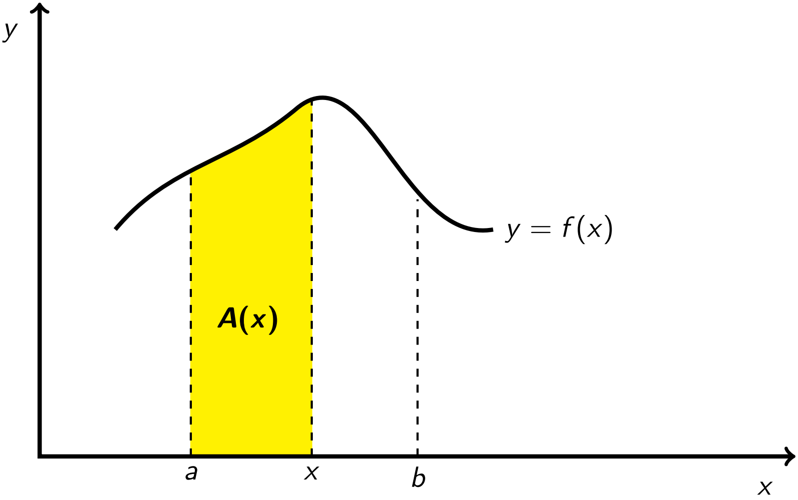

Figure: The area function $A(x)$. Obviously, $A(a) = 0$, $A(b) = A$.

- I want to look at how $A(x)$ changes as x increases. I will assume that $f(x)$ is locally increasing in x but the argument can be easily modified to handle other possibilities.

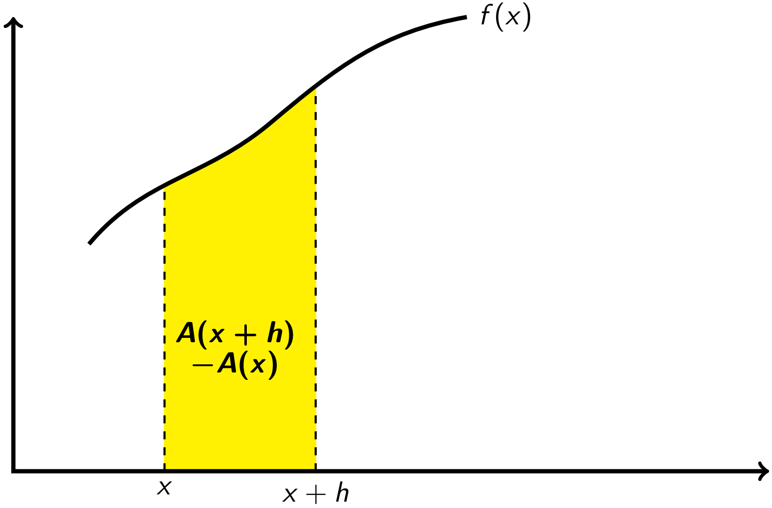

Figure: Looking at how $A(x)$ changes as x increases. Note that
$f(x) \cdot h \leq A(x+h) - A(x) \leq f(x+h) \cdot h$.

- Note that

$$f(x) \cdot h \leq A(x+h) - A(x) \leq f(x+h) \cdot h$$

implies that

$$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h).$$

- $$f(x) \leq \frac{A(x+h) - A(x)}{h} \leq f(x+h).$$
- Now look at the limit as h converges to zero and assume that the function is continuous:

$$f(x) \leq \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} \equiv A'(x) \leq \lim_{h \to 0} f(x+h) = f(x)$$

which implies that

$$f(x) = A'(x).$$

- This is one of the most remarkable and important results in all of mathematics: The derivative of the area function $A(x)$ is the curve's height function $f(x)$.
- It is known as the **Fundamental Theorem of Calculus**.

- I will now use this fundamental theorem of calculus to calculate area A.
- Definition: A function $P(x)$ is called a **primitive** of a function $f(x)$ if $P'(x) = f(x)$ for all x (for which the function is defined).
- Let $P(x)$ be **a** primitive of $f(x)$. [It does not have to be unique, in fact, it is never unique.]
- Then

$$
\begin{aligned}
P'(x) = A'(x) = f(x) \qquad & \text{for all } x \\
\Longrightarrow [P(x) - A(x)]' = P'(x) - A'(x) = 0 \qquad & \text{for all } x \\
\Longrightarrow P(x) - A(x) = c \qquad & \text{for some constant } c
\end{aligned}
$$

- Now $A(a) = 0$ implies that $P(a) = c$ and $A(b) = A$ implies that $P(b) - A = c$. Putting things together,

$$A = P(b) - P(a).$$

$$A = P(b) - P(a)$$

- We have solved for how to calculate the area underneath a function! This result justifies the following definition of an integral.
- Definition: The **integral** of a continuous function $f(x)$ from $x = a$ to $x = b$ is denoted by

$$\int_a^b f(x)\,dx$$

and equals

$$P(b) - P(a) \equiv P(x)\Big|_a^b$$

where $P(x)$ is any primitive of $f(x)$.

- Referring back to our earlier discussion,

$$A = \text{area} = \int_a^b f(x)\,dx = P(b) - P(a)$$

$$A(x) = \int_a^x f(t)\,dt = P(x) - P(a)$$

we can now state the fundamental theorem more formally.

- **Fundamental Theorem of Calculus** Let $f(\cdot)$ be a continous function and

$$A(x) = \int_a^x f(t)\,dt$$

for all x. Then $A'(x)$ exists and $A'(x) = f(x)$ for all x.

Significance

- Definition: A function $P(x)$ is called a **primitive** of a function $f(x)$ if $P'(x) = f(x)$ for all x.
- Note that $f(x)$ is a primitive of the function $f'(x)$ since $f'(x) = f'(x)$.
- Thus
$$\int_a^x f'(t)\,dt = f(x) - f(a)$$
$$f(x) = \int_a^x f'(t)\,dt + f(a)$$
- Knowing the function $f'(x)$ and the value of the f function at an arbitrary point $x = a$, we can calculate what the $f(x)$ function is at any point x.

- Starting with the function $f(x)$, we can calculate the function $f'(x)$ and $f(a)$ by differentiating the function $f(x)$.
- Starting with the function $f'(x)$ and $f(a)$, we can calculate the function $f(x)$ by integrating the function $f'(x)$ and adding $f(a)$.
- Differentiation and integration are inverse operations. Differentiation is like walking forward (finding the slope of a curve) and integration is like walking backwards (finding the area underneath a curve).
- This was first discovered by Isaac Barrow (1630-1677), a mathematics professor at Cambridge University in England. However, the full significance of the discovery was first recognized by Barrow's student, Isaac Newton (1643-1727). [Integral calculus was independently discovered by Gottfried Leibniz (1646-1716) and it is Leibniz's calculus that we use today].

Example

- Problem:

$$\int_0^1 x^n\,dx = ? \qquad \text{where } n \text{ is a positive integer.}$$

- Solution: First guess a primitive of x^n.

$$f(x) = x^n \Longrightarrow f'(x) = nx^{n-1}$$

$$P(x) = \frac{x^{n+1}}{n+1} + c \Longrightarrow P'(x) = x^n$$

$$\int_0^1 x^n\,dx = P(1) - P(0) = \left(\frac{1^{n+1}}{n+1} + c\right) - \left(\frac{0^{n+1}}{n+1} + c\right) = \frac{1}{n+1}$$

- Note that the answer does not depend on the choice of c, that is, does not depend on which primitive function is used.

Another example

- $$\int_1^3 (x^2 - 3x + 5)\,dx = \left(\frac{x^3}{3} - 3\frac{x^2}{2} + 5x\right)\Bigg|_1^3$$
  $$= \left(9 - \frac{27}{2} + 15\right) - \left(\frac{1}{3} - \frac{3}{2} + 5\right) = \frac{20}{3}$$

- In integral calculus, what result corresponds to the chain rule in differential calculus?
- $$[F(g(x))]' = F'(g(x)) \cdot g'(x) \qquad \text{by the chain rule.}$$
- Let $F'(u) \equiv f(u)$

$$\Longrightarrow \int_a^b F'(g(x))g'(x)\,dx = \int_a^b f(g(x))g'(x)\,dx = ?$$

- What is a primitive of $f(g(x))g'(x)$? Answer $F(g(x))$.

$$\Longrightarrow \qquad ? = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$$

- What function is F a primitive of? Answer $f(u)$ since $f(u) = F'(u)$.

- $$\implies \qquad ? = F(g(x))\Big|_a^b = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)\,du$$
- I have just proved **The Substitution Theorem for Integrals**: Assuming that $f(u)$ and $g'(x)$ are continuous functions,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example

- Problem:

$$\int_1^4 \left(x^2+10\right)^{50} 2x \, dx = ?$$

- Solution: Let $f(u) = u^{50}$, $g(x) = x^2 + 10$,

  $\Longrightarrow g'(x) = 2x$, $\Longrightarrow f(g(x))g'(x) = (x^2+10)^{50} 2x$.

- By the Substitution Theorem,

$$\int_1^4 \left(x^2+10\right)^{50} 2x \, dx = \int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du$$

$$= \int_{g(1)}^{g(4)} u^{50} \, du = \left. \frac{u^{51}}{51} \right|_{1+10}^{16+10} = \frac{26^{51}}{51} - \frac{11^{51}}{51}.$$

- The derivative of the product of two functions theorem tells us that

$$h(t) \equiv u(t) \cdot v(t) \qquad \Longrightarrow \qquad h'(t) = u'(t)v(t) + u(t)v'(t)$$

- What is the corresponding result for integrals (the inverse result)?

$$\int_c^x h'(t)\,dt = h(x) - h(c) = u(x)v(x) - u(c)v(c) = u(t)v(t)\Big|_c^x$$

$$= \int_c^x [u \cdot v]'\,dt = \int_c^x u'(t)v(t)\,dt + \int_c^x u(t)v'(t)\,dt$$

- I have just proved **The Integration By Parts Theorem**: Let u and v be continuously differentiable functions. Then

$$\int_c^x u(t)v'(t)\,dt = u(t)v(t)\Big|_c^x - \int_c^x u'(t)v(t)\,dt.$$

Example

- Problem:

$$\int_0^\infty te^{-rt}\,dt = ?$$

- This is a common type of calculation in dynamic models, encountered again and again.
- $r > 0$ is the relevant interest rate and the integral is the discounted value (e^{-rt}) of getting a payment t that grows linearly over time (from now to forever).

- Problem:
$$\int_0^\infty te^{-rt}\,dt = ?$$
- Solution: Use the integration by parts theorem.
- Let $u(t) = t$ and $v'(t) = e^{-rt}$.
- Then $u'(t) = 1$ and $v(t) = e^{-rt}/(-r)$.
-
$$\int_0^\infty te^{-rt}\,dt = \int_c^x u(t)v'(t)\,dt = u(t)v(t)\Big|_c^x - \int_c^x u'(t)v(t)\,dt$$
$$= t\frac{e^{-rt}}{-r}\Big|_0^\infty - \int_0^\infty 1\frac{e^{-rt}}{-r}\,dt$$

- Now, by L'Hopital's Rule,

$$\lim_{t\to\infty} te^{-rt} = \lim_{t\to\infty} \frac{t}{e^{rt}} = \lim_{t\to\infty} \frac{1}{re^{rt}} = 0$$

so

$$\int_0^\infty te^{-rt}\,dt = t\frac{e^{-rt}}{-r}\bigg|_0^\infty - \int_0^\infty 1\frac{e^{-rt}}{-r}\,dt = \frac{1}{r}\int_0^\infty e^{-rt}\,dt$$

$$= \frac{1}{r}\frac{e^{-rt}}{-r}\bigg|_0^\infty = \frac{1}{r^2}$$

Second Example

- Problem:
$$\int_0^\infty t^2 e^{-rt}\,dt = ?$$
- This is another common type of calculation in dynamic models. $r > 0$ is the relevant interest rate and the integral is the discounted value (e^{-rt}) of getting a payment t^2 that grows at a faster than linear rate over time.
- Use the integration by parts theorem.
- Let $u(t) = t^2$ and $v'(t) = e^{-rt}$.
- Then $u'(t) = 2t$ and $v(t) = e^{-rt}/(-r)$.

- Let $u(t) = t^2$ and $v'(t) = e^{-rt}$.
- Then $u'(t) = 2t$ and $v(t) = e^{-rt}/(-r)$.
- $$\int_0^\infty t^2 e^{-rt}\, dt = \int_c^x u(t)v'(t)\, dt = u(t)v(t)\Big|_c^x - \int_c^x u'(t)v(t)\, dt$$
  $$= t^2 \frac{e^{-rt}}{-r}\Big|_0^\infty - \int_0^\infty 2t \frac{e^{-rt}}{-r}\, dt$$
- Now, by L'Hopital's Rule,

$$\lim_{t\to\infty} t^2 e^{-rt} = \lim_{t\to\infty} \frac{t^2}{e^{rt}} = \lim_{t\to\infty} \frac{2t}{re^{rt}} = \lim_{t\to\infty} \frac{2}{r^2 e^{rt}} = 0$$

$$\int_0^\infty t^2 e^{-rt}\, dt = \frac{2}{r}\int_0^\infty t e^{-rt}\, dt = \frac{2}{r}\frac{1}{r^2} = \frac{2}{r^3}.$$

The Logarithm Function

- Definition: If x is a positive real number, we define **the natural logarithm** of x to be
$$\ln x \equiv \int_1^x \frac{1}{t}\, dt.$$
- This is a function defined in terms of an integral. The idea of the natural logarithm is due to John Napier (1550-1617).
- It is hard to believe that this is one of the most used functions in economics but life has many surprises!

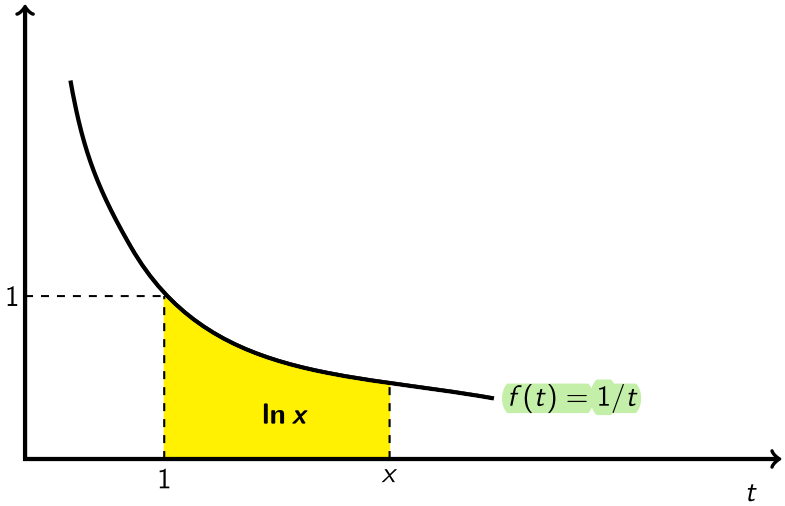

Figure: The Logarithm Function.

- Why is the natural logarithm such an important function? Answer: because it has several special properties.
- (1) $\ln 1 = 0$

$$\text{since } \ln 1 \equiv \int_1^1 \frac{1}{t}\, dt = 0$$

- (2) $(\ln x)' = 1/x$ for all $x > 0$
  by the fundamental theorem of calculus (remember $A'(x) = f(x)$).

- (3) $\ln ab = \ln a + \ln b$ for all $a, b > 0$

Proof of (3) $\ln ab = \ln a + \ln b$ for all $a, b > 0$

- $$\ln ab = \int_1^{ab} \frac{1}{t}\, dt = \int_1^{a} \frac{1}{t}\, dt + \int_a^{ab} \frac{1}{t}\, dt = \ln a + \int_a^{ab} \frac{1}{t}\, dt$$
- To make further progress, we use the substitution theorem for integrals!
- Let $g(t) = t/a$ and let $f(g(t)) = a/t = 1/g(t)$. Then $g'(t) = 1/a$ and $f(u) = 1/u$. Let $\hat{b} \equiv ab$.

$$\int_a^{ab} \frac{1}{t}\, dt = \int_a^{ab} \frac{a}{t}\frac{1}{a}\, dt = \int_a^{\hat{b}} f(g(t))g'(t)\, dt = \int_{g(a)}^{g(\hat{b})} f(u)\, du$$

$$= \int_a^{ab} f(g(t))g'(t)\, dt = \int_{g(a)}^{g(ab)} f(u)\, du$$

- Let $g(t) = t/a$ and let $f(g(t)) = a/t = 1/g(t)$. Then $g'(t) = 1/a$ and $f(u) = 1/u$. Let $\hat{b} \equiv ab$.
- $$\int_a^{ab} \frac{1}{t}\,dt = \int_{g(a)}^{g(ab)} f(u)\,du = \int_1^b \frac{1}{u}\,du = \ln b$$
- So
$$\ln ab = \ln a + \int_a^{ab} \frac{1}{t}\,dt = \ln a + \ln b$$
- The logarithm function converts a multiplication into an addition.

Leibniz Rule

- **Leibniz Rule**: If $g_y \equiv \partial g / \partial y$ is continuous and h_1 and h_2 are differentiable, then

$$\frac{d}{dy} \int_{h_1(y)}^{h_2(y)} g(x, y)\, dx = \int_{h_1(y)}^{h_2(y)} \frac{\partial g(x, y)}{\partial y}\, dx$$

$$+ h_2'(y) g(h_2(y), y) - h_1'(y) g(h_1(y), y)$$

- First note that Leibniz Rule is a generalization of the Fundamental Theorem of Calculus:

$$\frac{d}{dy}\int_{h_1(y)}^{h_2(y)} g(x,y)\,dx = \int_{h_1(y)}^{h_2(y)} \frac{\partial g(x,y)}{\partial y}\,dx$$

$$+h_2'(y)g(h_2(y),y) - h_1'(y)g(h_1(y),y)$$

$$\frac{d}{dy}\int_{a}^{y} g(x)\,dx = \int_{a}^{y} 0\,dx + 1\cdot g(y) - 0\cdot g(a) = g(y)$$

$$\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x)$$

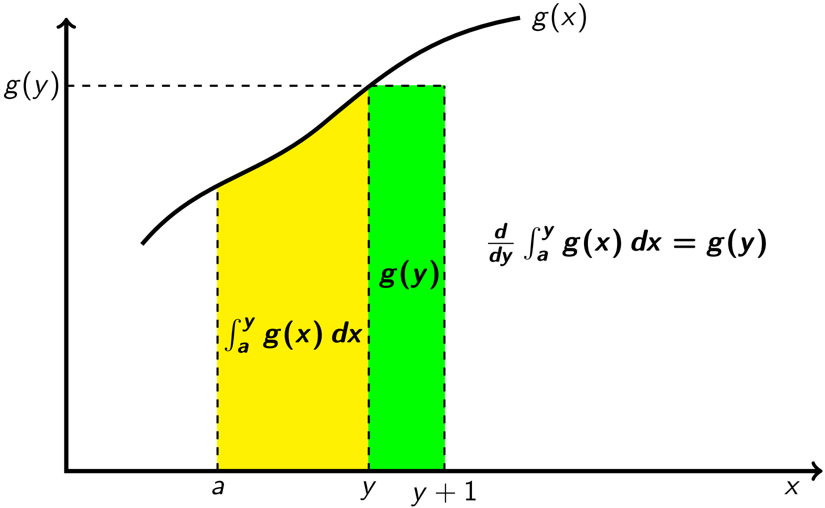

Figure: Intuition for the Fundamental Theorem of Calculus

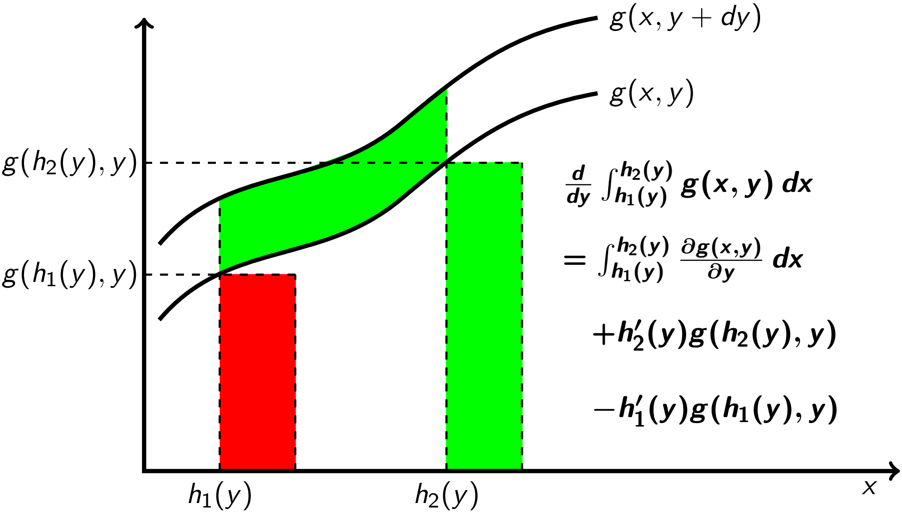

Figure: Intuition for Leibniz Rule

Example

- Suppose a small business earns profits $\pi(t)$ over time t and the interest rate r is constant over time.
- Then the discounted profits from time s to time T is

$$V(s, r) \equiv \int_s^T \pi(t) e^{-r(t-s)} dt.$$

- How do discounted profits change over time? What is $\partial V(s, r)/\partial s$?

- $$\frac{d}{dy}\int_{h_1(y)}^{h_2(y)} g(x,y)\,dx = \int_{h_1(y)}^{h_2(y)} \frac{\partial g(x,y)}{\partial y}\,dx$$
$$+h_2'(y)g(h_2(y),y) - h_1'(y)g(h_1(y),y)$$

- $$\frac{\partial V(s,r)}{\partial s} = \frac{d}{ds}\int_s^T \pi(t)e^{-r(t-s)}dt = \int_s^T \pi(t)e^{-rt}re^{rs}\,dt$$
$$+0 - 1\pi(s)e^{-r(s-s)}$$

- $$\frac{\partial V(s,r)}{\partial s} = r\int_s^T \pi(t)e^{-r(t-s)}\,dt - \pi(s) = rV(s,r) - \pi(s)$$

- $$\frac{\partial V(s,r)}{\partial s} = rV(s,r) - \pi(s)$$
$$rV(s,r) = \frac{\partial V(s,r)}{\partial s} + \pi(s)$$
$$r = \frac{\partial V(s,r)/\partial s}{V(s,r)} + \frac{\pi(s)}{V(s,r)}$$
- The no-arbitrage condition: the risk-free interest rate equals the capital gain rate plus the dividend rate.
- If r was greater than capital gain plus dividend, no one would want to own the firm. If r was less than capital gain plus dividend, no one would want to own bonds paying the interest rate r.
- r must adjust to clear the market.

Linear First Order Differential Equations

- We begin by solving the following type of differential equation:

$$\dot{y}(t) + a \cdot y(t) = b.$$

This differential equation is

- **linear** because $\dot{y}$ and y are not raised to any power other than one,
- **first order** because only $\dot{y}$ and y show up, not $\ddot{y}$, etc,
- **autonomous** because a and b do not depend on t.

- Step 1: We will first find the general solution to the homogeneous form of the differential equation (where $b = 0$):

$$\dot{y}(t) + a \cdot y(t) = 0$$

- Rearranging terms, we obtain

$$\dot{y}(t) = -a \cdot y(t)$$

$$\frac{\dot{y}(t)}{y(t)} = -a$$

- Integrating both sides, we obtain

$$\int \frac{\dot{y}(t)}{y(t)} dt = \int -a \, dt$$

$$\ln y + c_1 = -at + c_2$$

- The last step used the chain rule for taking the derivative of a composition of two functions.
- One function maps t to $y(t)$ and the second function maps $y(t)$ to $\ln y(t)$.
- The chain rule as usually stated is:

$$[F(g(x)]' = F'(g(x)) \cdot g'(x).$$

- The chain rule applied yields

$$\frac{d}{dt}[\ln y + c_1] = (\ln y)' \cdot y'(t) = \frac{1}{y} \cdot y'(t) = \frac{y'(t)}{y(t)} = \frac{\dot{y}(t)}{y(t)}.$$

- $$\ln y + c_1 = -at + c_2$$
$$\ln y = -at + c_3$$
- Taking the exponential of both sides,
$$e^{\ln y} = e^{-at+c_3}$$
- Thus, the general solution to the homogenous form is
$$y_h \equiv C \cdot e^{-at}$$
where C is an arbitrary constant of integration.
- Check that the solution really does satisfy the differential equation:
$$\dot{y} + a \cdot y = (-a)Ce^{-at} + aCe^{-at} = 0 \qquad \text{for all } t$$

- Step 2: Find a particular solution to

$$\dot{y}(t) + a \cdot y(t) = b.$$

- Try

$$y(t) = c.$$

$$\Longrightarrow \dot{y}(t) = 0$$

$$\dot{y}(t) + a \cdot y(t) = b \qquad \Longrightarrow 0 + a \cdot c = b \qquad \Longrightarrow c = \frac{b}{a}$$

- Thus, a particular solution is the steady-state value of y:

$$y_p(t) = \frac{b}{a} \qquad \text{when } a \neq 0.$$

- If $a = 0$, then the general solution to

$$\dot{y}(t) = b$$

is obviously

$$y(t) = C + bt.$$

- Step 3: Add the general solution of the homogeneous form and a particular solution of the complete differential equation to obtain the general solution

$$y = y_h + y_p.$$

- Claim:

$$y(t) = Ce^{-at} + \frac{b}{a}$$

is the general solution to

$$\dot{y}(t) + a \cdot y(t) = b.$$

- Proof: Let $y_1(t)$ be any solution to the differential equation. Let $z(t) \equiv y_1(t) - y_p(t)$ where y_p is the particular solution b/a.

$$\dot{z} = \dot{y}_1 - \dot{y}_p = (b - ay_1) - (b - ay_p) = -a(y_1 - y_p) = -az$$

$$\dot{z} + az = 0 \qquad \text{homogeneous form}$$

$$z(t) = Ce^{-at} \qquad \text{general solution}$$

- $$z(t) = Ce^{-at} \qquad \text{general solution}$$
$$z(t) \equiv y_1(t) - y_p(t)$$
$$y_1(t) = z(t) + y_p(t) = Ce^{-at} + y_p(t)$$
- But $y_1(t)$ is any solution to the differential equation. Thus the general solution to the differential equation is
$$y = y_h + y_p = Ce^{-at} + \frac{b}{a}.$$

Example

- Solve for $y(t)$ where

$$\dot{y} + 2y = 8 \qquad \text{and} \qquad y(0) = 6.$$

- The homogeneous form $\dot{y} + 2y = 0$ has general solution $y_h = Ce^{-2t}$.
- A particular steady-state solution is $y_p = 4$. (Check: $0 + 2 \cdot 4 = 8$)
- Thus the general solution is

$$y(t) = y_h(t) + y_p(t) = Ce^{-2t} + 4.$$

- Also want to satisfy the initial condition $y(0) = 6$, which implies that $6 = C + 4$ or $C = 2$.

$$\implies y(t) = 2e^{-2t} + 4.$$

The non-autonomous (or general) case

- Next, we solve the non-autonomous linear first order differential equation
$$\dot{y}(t) + a(t) \cdot y(t) = b(t)$$
where $a(t)$ and $b(t)$ are known functions of t.
- We will solve this differential equation using a trick, called the **integrating factor** [discovered by Leonhard Euler in 1739].
- The idea is as follows: We want to solve by integrating both sides. By first multiplying both sides by a particular function (the integrating factor), and then integrating, the integration becomes easier.

- Given the differential equation

$$\dot{y}(t) + a(t) \cdot y(t) = b(t),$$

let

$$A(t) \equiv \int_{t_0}^{t} a(x)\, dx$$

where t_0 is an arbitrary point in time. Then the integrating factor used to solve the differential equation is

$$e^{A(t)}.$$

- First note that

$$\frac{d}{dt}\left[e^{A(t)}y(t)\right] = e^{A(t)}\dot{y}(t) + \dot{A}(t)e^{A(t)}y(t)$$

$$= e^{A(t)}\left[\dot{y}(t) + a(t)y(t)\right] = e^{A(t)}b(t).$$

- Since we want to integrate using t as the upper limit of integration, it is convenient to change variables now:

$$\frac{d}{dx}\left[e^{A(x)}y(x)\right] = e^{A(x)}\left[\dot{y}(x) + a(x)y(x)\right] = e^{A(x)}b(x) \qquad \text{for all } x.$$

- Now integrate

$$\int_{t_0}^{t} e^{A(x)}\left[\dot{y}(x) + a(x)y(x)\right]dx = \int_{t_0}^{t} e^{A(x)}b(x)\,dx.$$

- $$\int_{t_0}^{t} e^{A(x)} \left[\dot{y}(x) + a(x)y(x)\right] dx = \int_{t_0}^{t} e^{A(x)} b(x)\, dx$$
$$= \int_{t_0}^{t} \frac{d}{dx}\left[e^{A(x)} y(x)\right] dx = e^{A(x)} y(x)\Big|_{t_0}^{t}$$
$$= e^{A(t)}y(t) - e^{A(t_0)}y(t_0) = e^{A(t)}y(t) - y(t_0)$$
where we have used the fact that
$$A(t) \equiv \int_{t_0}^{t} a(x)\, dx \quad \text{implies that} \quad A(t_0) \equiv \int_{t_0}^{t_0} a(x)\, dx = 0.$$
- $$e^{-A(t)}\left[\int_{t_0}^{t} e^{A(x)} b(x)\, dx\right] = e^{-A(t)}\left[e^{A(t)}y(t) - y(t_0)\right]$$

- Solving for the differential equation

$$\dot{y}(t) + a(t) \cdot y(t) = b(t)$$

for $y(t)$ yields

$$y(t) = e^{-A(t)} \left[y(t_0) + \int_{t_0}^{t} e^{A(x)} b(x)\, dx \right]$$

where

$$A(t) \equiv \int_{t_0}^{t} a(x)\, dx.$$

- This solution to the differential equation is hard to remember, so it is better to rederive from first principles with each application.

Example

- Solve
$$\dot{y}(t) - 2 \cdot y(t) = e^{2t} \qquad \text{given } y(0) = 1.$$
- Find the integrating factor:
$$A(t) \equiv \int_{t_0}^{t} a(x)\,dx = \int_0^t -2\,dx = -2x\Big|_0^t = -2t$$
- Next use the integrating factor $e^{A(t)} = e^{-2t}$

$$\int_0^t e^{A(x)}\,[\dot{y}(x) - 2y(x)]\,dx = \int_0^t \frac{d}{dx}\left[e^{A(x)}y(x)\right]dx = \int_0^t e^{A(x)}e^{2x}\,dx$$

- $$\int_0^t e^{A(x)} [\dot{y}(x) - 2y(x)]\, dx = \int_0^t \frac{d}{dx}\left[e^{A(x)} y(x)\right] dx = \int_0^t e^{A(x)} e^{2x}\, dx$$
$$= e^{A(x)} y(x)\Big|_0^t = e^{A(t)} y(t) - y(0) = e^{-2t} y(t) - 1$$
$$= \int_0^t e^{A(x)} e^{2x}\, dx = \int_0^t e^{-2x} e^{2x}\, dx = x\Big|_0^t = t$$
- Therefore, we have
$$e^{-2t} y(t) - 1 = t \qquad \text{or} \qquad y(t) = e^{2t}(t+1).$$

- We can check that $y(t) = e^{2t}(t+1)$ is the solution to the differential equation

$$\dot{y}(t) - 2 \cdot y(t) = e^{2t} \qquad \text{given } y(0) = 1.$$

- $$y(0) = e^{2 \cdot 0}(0+1) = 1$$

$$\dot{y}(t) = 2e^{2t}(t+1) + e^{2t} \cdot 1 = e^{2t}(2t+3)$$

$$\dot{y}(t) - 2 \cdot y(t) = e^{2t}(2t+3) - 2 \cdot e^{2t}(t+1) = e^{2t}.$$

Eigenvalues, Eigenvectors and Differential Equations

- Eigenvalues and eigenvectors play important roles in a variety of branches of mathematics.
- It turns out that the eigenvalues of a $n \times n$ matrix are numbers that summarise important properties of that matrix.
- In dynamic economic models where matrix equations describe how the economy evolves over time, it turns out that the solution path of the economy can be expressed in terms of the eigenvalues of the transition matrix and its corresponding eigenvectors.
- In solving a system of differential equations, the bulk of the work is solving for eigenvalues and eigenvectors.

- We begin our study of this important topic by defining what eigenvalues and eigenvectors are:

- Definition: For a n x n matrix $\mathbf{A}$ ($a_{ij} \in \Re$ for all i, j), an **eigenvalue** of $\mathbf{A}$ is a number r (real or complex) satisfying

$$|\mathbf{A} - r \cdot \mathbf{I}| = 0$$

and corresponding to r, a non-zero vector $\mathbf{v}$ such that

$$(\mathbf{A} - r \cdot \mathbf{I})\mathbf{v} = \mathbf{0}$$

is called an **eigenvector** of $\mathbf{A}$ corresponding to the eigenvalue r.

- Comment 1:

$$(\mathbf{A} - r \cdot \mathbf{I})\,\mathbf{v} = \mathbf{0} \qquad \Longleftrightarrow \qquad \mathbf{A}\mathbf{v} = r\mathbf{v}$$

The transformation defined by the matrix $\mathbf{A}$ maps the eigenvector to a multiple (the eigenvalue r) of itself.

- Comment 2: Let $\mathbf{B} \equiv \mathbf{A} - r \cdot \mathbf{I}$. Then

$$\mathbf{B}\mathbf{v} = \mathbf{0} \text{ for non-zero } \mathbf{v} \iff |\mathbf{B}| = 0 \iff |\mathbf{A} - r \cdot \mathbf{I}| = 0.$$

$$\mathbf{B}\mathbf{v} = \mathbf{0} \text{ always has the trivial solution } \mathbf{v} = \mathbf{0}.$$

Thus, corresponding to each eigenvalue, there is a non-zero eigenvector.

- Comment 3:

$$|\mathbf{A} - r \cdot \mathbf{I}|$$

is a polynomial equation of degree n in the unknown number r. By the **fundamental theorem of algebra**,

$$\begin{aligned}|\mathbf{A} - r \cdot \mathbf{I}| &= b_n r^n + b_{n-1} r^{n-1} + \cdots + b_1 r + b_0 \\ &= (r - r_1)(r - r_2)(r - r_3) \cdots (r - r_n).\end{aligned}$$

Thus the **characteristic equation** of the matrix **A**

$$|\mathbf{A} - r \cdot \mathbf{I}| = 0$$

has n solutions called **characteristic roots** or **eigenvalues**, namely, $r_1, r_2, \ldots, r_n$.
[with the understanding that some of the r_i's could be complex numbers and $r_i = r_j$ for some $i \neq j$ is possible (repeated roots)]

- In the rest of this lecture, I will focus on the $n = 2$ case. However all of the results generalize in a natural way for arbitrary n.
- For $n = 2$,

$$|\mathbf{A} - r \cdot \mathbf{I}| = 0$$

$$\begin{vmatrix} a_{11} - r & a_{12} \\ a_{21} & a_{22} - r \end{vmatrix} = 0$$

$$(a_{11} - r)(a_{22} - r) - a_{12}a_{21} = 0$$

$$r^2 - (a_{11} + a_{22})r + (a_{11}a_{22} - a_{12}a_{21}) = 0 \quad \text{polynomial of degree 2}$$

$$r^2 - \mathrm{tr}(\mathbf{A})r + |\mathbf{A}| = 0$$

- We can solve the characteristic equation $r^2 - \text{tr}(\mathbf{A})r + |\mathbf{A}| = 0$ using the quadratic formula

$$ar^2 + br + c = 0 \quad \Longrightarrow \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to obtain

$$r = \frac{\text{tr}(\mathbf{A}) \pm \sqrt{[\text{tr}(\mathbf{A})]^2 - 4\,|\mathbf{A}|}}{2},$$

which yields 2 eigenvalues or characteristic roots. Thus we obtain

- real and distinct roots if $[\text{tr}(\mathbf{A})]^2 > 4\,|\mathbf{A}|$,
- real and repeated roots if $[\text{tr}(\mathbf{A})]^2 = 4\,|\mathbf{A}|$,
- complex and distinct roots if $[\text{tr}(\mathbf{A})]^2 < 4\,|\mathbf{A}|$.

Example

- $$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -4 & 4 \end{bmatrix}$$

- $$\operatorname{tr}(\mathbf{A}) = 4 + 4 = 8 \qquad |\mathbf{A}| = 4 \cdot 4 - (-1)(-4) = 16 - 4 = 12$$

- We obtain real and distinct roots since $[\operatorname{tr}(\mathbf{A})]^2 = 8^2 = 64 > 4\,|\mathbf{A}| = 4 \cdot 12 = 48.$

$$r = \frac{\operatorname{tr}(\mathbf{A}) \pm \sqrt{[\operatorname{tr}(\mathbf{A})]^2 - 4\,|\mathbf{A}|}}{2} = \frac{8 \pm \sqrt{64 - 48}}{2} = \frac{8 \pm \sqrt{16}}{2}$$

$$r = \frac{8 \pm 4}{2} = 6 \text{ or } 2$$

- For the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -1 \\ -4 & 4 \end{bmatrix}$$

  there are two distinct real eigenvalues, $r_1 = 2$ and $r_2 = 6$.

- $$(\mathbf{A} - r_1 \mathbf{I})\,\mathbf{v}_1 = \mathbf{0} \qquad \Longleftrightarrow \qquad \begin{bmatrix} 4-2 & -1 \\ -4 & 4-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  Let $v_1 = 1$. Then $2 \cdot 1 - v_2 = 0$ implies that $v_2 = 2$, so an eigenvector corresponding to $r_1 = 2$ is

$$\mathbf{v}_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

  Notice that $-4v_1 + (4-2)v_2 = -4 + 2 \cdot 2 = 0$

- For the matrix

$$\mathbf{A} = \begin{bmatrix} 4 & -1 \\ -4 & 4 \end{bmatrix}$$

there are two distinct real eigenvalues, $r_1 = 2$ and $r_2 = 6$.

- $$(\mathbf{A} - r_2\mathbf{I})\,\mathbf{v}_2 = \mathbf{0} \qquad \Longleftrightarrow \qquad \begin{bmatrix} 4-6 & -1 \\ -4 & 4-6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $v_1 = 1$. Then $-2 \cdot 1 - v_2 = 0$ implies that $v_2 = -2$, so an eigenvector corresponding to $r_2 = 6$ is

$$\mathbf{v}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Notice that $-4v_1 + (4-6)v_2 = -4 + -2 \cdot -2 = 0$

Properties of Eigenvalues and Eigenvectors

- We will now establish some important properties of eigenvalues and their corresponding eigenvectors.
- Consider a 2 x 2 matrix $\mathbf{A}$ with 2 distinct eigenvalues (r_1 and r_2) and corresponding eigenvectors ($\mathbf{v}_1$ and $\mathbf{v}_2$).
- Then $(\mathbf{A} - r_i\mathbf{I})\,\mathbf{v}_i = \mathbf{0} \qquad \Longrightarrow \mathbf{A}\mathbf{v}_i = r_i\mathbf{v}_i \qquad$ for $i = 1, 2$.

- Property 1: $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.
- Proof: By definition, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \qquad \text{only holds when } c_1 = c_2 = 0.$$

- Proof: By definition, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0} \qquad \text{only holds when } c_1 = c_2 = 0.$$

Assuming that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$, it follows that

$$\mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = \mathbf{A}\mathbf{0}$$

$$c_1\mathbf{A}\mathbf{v}_1 + c_2\mathbf{A}\mathbf{v}_2 = \mathbf{0}$$

$$c_1 r_1\mathbf{v}_1 + c_2 r_2\mathbf{v}_2 = \mathbf{0}$$

Now subtracting r_1 times the first equation from the last equation yields

$$r_1(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) - (c_1 r_1\mathbf{v}_1 + c_2 r_2\mathbf{v}_2) = r_1\mathbf{0} - \mathbf{0}$$

$$c_2(r_1 - r_2)\mathbf{v}_2 = \mathbf{0}$$

- $$c_2(r_1 - r_2)\mathbf{v}_2 = \mathbf{0}$$

  implies that $c_2 = 0$ (since $r_1 \neq r_2$ and $\mathbf{v}_2 \neq \mathbf{0}$). Then

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$$

  implies that $c_1 = 0$ as well. Thus $\mathbf{v}_1$ and $\mathbf{v}_2$ must be linearly independent. Q.E.D.

Second Property of Eigenvalues and Eigenvectors

- These two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ can be used to construct a 2 x 2 matrix $\mathbf{P} = [\mathbf{v}_1 \ \mathbf{v}_2]$ by letting the first column of $\mathbf{P}$ be the vector $\mathbf{v}_1$ and the second column of $\mathbf{P}$ be the vector $\mathbf{v}_2$.
- Then the linear independence of $\mathbf{v}_1$ and $\mathbf{v}_2$ implies that the matrix $\mathbf{P}$ has a well-defined inverse matrix $\mathbf{P}^{-1}$.
- Consider now the effect of multiplying $\mathbf{A}$ by the matrix $\mathbf{P}$ of its eigenvectors.

$$\mathbf{AP} = \mathbf{A}[\mathbf{v}_1 \ \mathbf{v}_2] = [\mathbf{A}\mathbf{v}_1 \ \mathbf{A}\mathbf{v}_2] = [r_1\mathbf{v}_1 \ r_2\mathbf{v}_2]$$

$$= \left[\begin{array}{cc} \mathbf{v}_1 & \mathbf{v}_2 \end{array}\right] \left[\begin{array}{cc} r_1 & 0 \\ 0 & r_2 \end{array}\right] = \mathbf{P} \left[\begin{array}{cc} r_1 & 0 \\ 0 & r_2 \end{array}\right]$$

- From before,

$$\mathbf{AP} = \mathbf{P}\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

- Since $\mathbf{P}$ has an inverse, it follows furthermore that

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{P}^{-1}\mathbf{P}\begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$$

  This is the second important property that we wanted to establish.

- Using the matrix $\mathbf{P}$ of eigenvectors, the matrix $\mathbf{A}$ can be transformed [left multiplication by $\mathbf{P}^{-1}$, right multiplication by $\mathbf{P}$] into a matrix that is very easy to work with, a diagonal matrix with distinct eigenvalues along the diagonal and zero entries off the diagonal.

Systems of linear differential equations

- To illustrate the usefulness of this second property, we will now show that it can be used to find the general solution to a system of two linear differential equations:

$$\begin{aligned} \dot{y}_1(t) &= a_{11}y_1(t) + a_{12}y_2(t) \\ \dot{y}_2(t) &= a_{21}y_1(t) + a_{22}y_2(t) \end{aligned}$$

- In matrix form, this system is

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \qquad \text{or} \qquad \dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$$

- Assume that the eigenvalues of the 2×2 matrix $\mathbf{A}$ are real and distinct (r_1 and r_2) and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be corresponding eigenvectors.
- To solve this system of differential equations, we first construct a new "transformed" vector

$$\mathbf{z} \equiv \mathbf{P}^{-1}\mathbf{y} \qquad \text{where} \qquad \mathbf{P} = [\mathbf{v}_1\ \mathbf{v}_2].$$

- Then $\mathbf{Pz} = \mathbf{y}$ and

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\dot{\mathbf{y}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{y} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \mathbf{z},$$

which implies that $\dot{z}_1(t) = r_1 \cdot z_1(t)$ and $\dot{z}_2(t) = r_2 \cdot z_2(t)$.

- $\dot{z}_1(t) = r_1 \cdot z_1(t)$ and $\dot{z}_2(t) = r_2 \cdot z_2(t)$ are simple differential equations with solutions $z_1(t) = c_1 e^{r_1 t}$ and $z_2(t) = c_2 e^{r_2 t}$.
- Then $\mathbf{y} = \mathbf{P}\mathbf{z}$ implies that

$$\mathbf{y} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} c_1 e^{r_1 t} \\ c_2 e^{r_2 t} \end{bmatrix} = c_1 \mathbf{v}_1 e^{r_1 t} + c_2 \mathbf{v}_2 e^{r_2 t}$$

- This is the general solution to the system of two linear differential equations.
- What is remarkable is that one can write down the general solution just by knowing the two eigenvalues and corresponding eigenvectors of the "transition" matrix $\mathbf{A}$. Thus eigenvalues and eigenvectors play central roles in the analysis of dynamic economic models!

Example

- Solve
$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$$
where
$$\mathbf{A} = \begin{bmatrix} 4 & -1 \\ -4 & 4 \end{bmatrix}$$
and the initial conditions are $y_1(0) = 1$ and $y_2(0) = 0$.
- For the **A** matrix, we have already shown that there are two distinct real eigenvalues, $r_1 = 2$ and $r_2 = 6$, with corresponding eigenvectors
$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

- Thus the general solution is

$$\mathbf{y} = c_1 \mathbf{v}_1 e^{r_1 t} + c_2 \mathbf{v}_2 e^{r_2 t}$$

$$\left[\begin{array}{c} y_1(t) \\ y_2(t) \end{array}\right] = c_1 \left[\begin{array}{c} 1 \\ 2 \end{array}\right] e^{2t} + c_2 \left[\begin{array}{c} 1 \\ -2 \end{array}\right] e^{6t}$$

- The initial conditions $y_1(0) = 1$ and $y_2(0) = 0$ imply that

$$1 = c_1 + c_2$$

$$0 = c_1 2 - 2c_2$$

$$\Longrightarrow c_1 = c_2 \qquad \Longrightarrow c_1 = .5 \text{ and } c_2 = .5$$

- Thus the solution is

$$y_1(t) = .5e^{2t} + .5e^{6t} \qquad y_2(t) = e^{2t} - e^{6t}.$$

How do we solve $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$?

- Let $\mathbf{y}_h$ be the general solution to the homogeneous system $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$.
- Let $\mathbf{y}_p$ be a particular solution to $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$.
- Then the general solution to $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$ is $\mathbf{y}_h + \mathbf{y}_p$.

- How do we find a particular solution?
  Set $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b} = \mathbf{0}$ and solve for the steady-state equilibrium.

$$\mathbf{A}\mathbf{y}_p = -\mathbf{b}$$

$$\mathbf{y}_p = -\mathbf{A}^{-1}\mathbf{b} \qquad \text{if } |\mathbf{A}| \neq 0$$

- For $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{b}$, when is the steady-state equilibrium $\mathbf{y}_p = -\mathbf{A}^{-1}\mathbf{b}$ stable?
- The general solution is $\mathbf{y}_h + \mathbf{y}_p$.

$$\lim_{t\to\infty} \mathbf{y}_h(t) + \mathbf{y}_p = \mathbf{y}_p \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \mathbf{y}_h(t) = \mathbf{0}$$

- Case 1: Real and distinct roots

$$\mathbf{y}_h(t) = c_1\mathbf{v}_1 e^{r_1 t} + c_2\mathbf{v}_2 e^{r_2 t} \qquad \Longrightarrow \text{need } r_1 < 0 \text{ and } r_2 < 0$$

- Case 2: Real and repeated roots
- Case 3: Complex roots

- General Theorem: The steady-state equilibrium is stable if and only if the eigenvalues all have negative real parts.

Can we determine stability from the matrix $\mathbf{A}$?

- $$r_1, r_2 = \frac{\mathrm{tr}(\mathbf{A}) \pm \sqrt{[\mathrm{tr}(\mathbf{A})]^2 - 4\,|\mathbf{A}|}}{2}$$
- Assume that $|\mathbf{A}| \neq 0$ since otherwise there is no steady-state to converge to.
- If $|\mathbf{A}| < 0$, then $\sqrt{[\mathrm{tr}(\mathbf{A})]^2 - 4\,|\mathbf{A}|} > \sqrt{[\mathrm{tr}(\mathbf{A})]^2} = |\mathrm{tr}(\mathbf{A})|$, which implies that the eigenvalues are real and of opposite sign, so the steady-state is unstable (saddle-point equilibrium).
- If $|\mathbf{A}| > 0$ and $\mathrm{tr}(\mathbf{A}) < 0$, then the real part of both eigenvalues is negative, so the steady-state is stable.
- If $|\mathbf{A}| > 0$ and $\mathrm{tr}(\mathbf{A}) > 0$, then the real part of both eigenvalues is positive, so the steady-state is unstable.
- Thus the check for stability is $|\mathbf{A}| > 0$ and $\mathrm{tr}(\mathbf{A}) < 0$.

Non-linear differential equation systems

- Consider the non-linear differential equation system

$$\dot{y}_1(t) = F(y_1(t), y_2(t))$$

$$\dot{y}_2(t) = G(y_1(t), y_2(t))$$

  where the functions $F(\cdot)$ and $G(\cdot)$ are continuously differentiable but not linear.
- Furthermore, suppose that a steady-state exists, denoted by $(\bar{y}_1, \bar{y}_2)$, that is, $F(\bar{y}_1, \bar{y}_2) = G(\bar{y}_1, \bar{y}_2) = 0$.
- I will now construct the first degree Taylor polynomials of F and G around $(\bar{y}_1, \bar{y}_2)$.

- $$\dot{y}_1(t) \approx F(\bar{y}_1, \bar{y}_2) + \frac{\partial F(\bar{y}_1, \bar{y}_2)}{\partial y_1}(y_1 - \bar{y}_1) + \frac{\partial F(\bar{y}_1, \bar{y}_2)}{\partial y_2}(y_2 - \bar{y}_2)$$

  $$\dot{y}_2(t) \approx G(\bar{y}_1, \bar{y}_2) + \frac{\partial G(\bar{y}_1, \bar{y}_2)}{\partial y_1}(y_1 - \bar{y}_1) + \frac{\partial G(\bar{y}_1, \bar{y}_2)}{\partial y_2}(y_2 - \bar{y}_2)$$

- Since $F(\bar{y}_1, \bar{y}_2) = G(\bar{y}_1, \bar{y}_2) = 0$, this system can be expressed in matrix form as

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \approx \begin{bmatrix} \frac{\partial F(\bar{y}_1,\bar{y}_2)}{\partial y_1} & \frac{\partial F(\bar{y}_1,\bar{y}_2)}{\partial y_2} \\ \frac{\partial G(\bar{y}_1,\bar{y}_2)}{\partial y_1} & \frac{\partial G(\bar{y}_1,\bar{y}_2)}{\partial y_2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} - \begin{bmatrix} \bar{y}_1 \frac{\partial F(\bar{y}_1,\bar{y}_2)}{\partial y_1} + \bar{y}_2 \frac{\partial F(\bar{y}_1,\bar{y}_2)}{\partial y_2} \\ \bar{y}_1 \frac{\partial G(\bar{y}_1,\bar{y}_2)}{\partial y_1} + \bar{y}_2 \frac{\partial G(\bar{y}_1,\bar{y}_2)}{\partial y_2} \end{bmatrix}$$

$\dot{\mathbf{y}} \approx \mathbf{A}\mathbf{y} + \mathbf{b}$ linear approximation differential equation system

- Theorem: If $|\mathbf{A}| \neq 0$ and the eigenvalues of $\mathbf{A}$ do not have zero real parts, then in a neighborhood of it's steady state $(\bar{y}_1, \bar{y}_2)$, trajectories of the non-linear

$$\dot{y}_1(t) = F(y_1(t), y_2(t))$$

$$\dot{y}_2(t) = G(y_1(t), y_2(t))$$

and linear approximation

$$\dot{\mathbf{y}} \approx \mathbf{A}\mathbf{y} + \mathbf{b}$$

differential equation systems have the same qualitative properties.