Advanced microeconomics problem set 4

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October 12, 2023

1 Exercise 3.35

1.1 Original question

Calculate the cost function and the conditional input demands for the linear production function $y = \sum_{i=1}^{n} \alpha_i x_i$.

1.2 Solution

Proof by contradiction First, we notice that all inputs are perfect substitutes. So we want to purchase the input that is most efficient in terms of money: $\frac{\alpha_i}{w_i}$ should be as high as possible. So we guess the conditional input demand is: $x_i > 0$ if good i is among the most efficient categories, and $x_i = 0$ if good i is not most efficient.

We then prove our guess using proof by contradiction. First, without lost of generality we assume good 1 is among the most efficient. Suppose there is an input plan where $x_i > 0$ if good i is not most efficient, then an improvement occurs where $x_i' = 0$ and x_1 is purchased $\frac{\alpha_i}{\alpha_1} x_i$ units more. The new output level is the same, while the change in cost is:

$$c_{new} - c_{old} = w_1 \frac{\alpha_i}{\alpha_1} x_i - w_i x_i = \frac{x_i}{a_1}(w_1 \alpha_i - w_i \alpha_1) \tag{1}$$

Since we assumed x_1 is among the most efficient categories, we have

$$\frac{\alpha_1}{w_1} > \frac{\alpha_i}{w_i} \rightarrow w_i \alpha_1 > w_1 \alpha_i \tag{2}$$

if i is not most efficient. So the change of cost: $c_{new} - c_{old} = \frac{x_i}{a_1}(w_1 \alpha_i - w_i \alpha_1) < 0$, which means the original plan is not cost-minimizing.
This improvement can be iterated, until we come to our original guess. So we formally come to the cost-minimizing input plan.
Remark: It is not sufficient to only provide guesses without proof, but we don't need to discuss where do our guesses come from.
Direct calculation Another way is to do direct calculation. The cost function is the solution to the cost-minimization problem

$$c(\mathbf{w}, y) = \min_{\mathbf{x} \in \mathbb{R}_+^n} \mathbf{w} \cdot \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) \geq y, \tag{3}$$

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which we can rewrite as follows

$$c(\mathbf{w}, y) = \min_{\mathbf{x} \in \mathbb{R}^n} \mathbf{w} \cdot \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) \geq y \;\; \& \;\; (x_i \geq 0 \quad \forall i \in \mathcal{I}) \tag{4}$$

Now turn this minimization problem into a maximization problem.

$$-c(\mathbf{w}, y) = \max_{\mathbf{x} \in \mathbb{R}^n_+} -\mathbf{w} \cdot \mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) \geq y \;\; \& \;\; (x_i \geq 0 \quad \forall i \in \mathcal{I}) \tag{5}$$

In order to bring the notation fully in line with the Kuhn-Tucker theorem, we can rewrite the constraints as follows

$$-c(\mathbf{w}, y) = \max_{\mathbf{x} \in \mathbb{R}^n_+} -\mathbf{w} \cdot \mathbf{x} \quad \text{s.t.} \quad y - f(\mathbf{x}) \leq \;\; \& \;\; (-x_i \leq 0 \quad \forall i \in \mathcal{I}) \tag{6}$$

Now let λ be a nonnegative multiplier associated with the production constraint. Let μ_i be a nonnegative multiplier associated with the i-th nonnegativity constraint. We form the Lagrangian.

$$\mathcal{L} = -\mathbf{w} \cdot \mathbf{x} - \lambda[y - f(\mathbf{x})] + \sum_{i=1}^{n} \mu_i x_i \tag{7}$$

An interior solution on $\mathbf{x} \in \mathbb{R}^n$ satisfies the Kuhn-Tucker conditions

$$\frac{\partial \mathcal{L}}{\partial x_j} = -w_i + \lambda f_i(\mathbf{x}) + \mu_i = 0 \quad \forall i \in \mathcal{I} \tag{FOC}$$

$$\lambda \geq 0 \tag{PC.1}$$

$$y - f(\mathbf{x}) \leq 0 \tag{PC.1}$$

$$\lambda(y - f(\mathbf{x})) = 0 \tag{CS.1}$$

$$\mu_i \geq 0 \tag{NN.1}$$

$$-x_i \leq 0 \tag{NN.2}$$

$$\mu_i x_i = 0 \tag{CS.2}$$

It is easy to verify that if $\mathbf{w} \gg 0$ the production constraint must bind in optimum, that is, $f(\mathbf{x}) = y$. Now let us impose the linear production technology. The first-order condition becomes

$$w_i = \lambda \alpha_i + \mu_i \quad \forall i \in \mathcal{I} \tag{8}$$

Since $\alpha_i > 0$, we can divide both sides by α_i and obtain

$$\frac{w_i}{\alpha_i} - \frac{\mu_i}{\alpha_i} = \lambda \quad \forall i \in \mathcal{I} \tag{9}$$

Since $y > 0$ and $f(\mathbf{0}) = 0$ by assumption, there must exist at least one good i which is used in production. Let us define as $S \subseteq \mathcal{I}$ the set of goods, which is used in production in optimum, i.e. $x_i > 0$. We define

$$S = \{i \in \mathcal{I} : x_i > 0\} \tag{10}$$

For the goods which are used in optimum, the non-negativity constraint does not bind. The complementary slackness condition in (CS.2) then implies

$$\mu_s = 0 \quad \forall s \in S \tag{11}$$

Hence, the first-order condition in (9) implies

$$\lambda = \frac{w_s}{\alpha_s} \quad \forall s \in S \tag{12}$$

Since λ is a constant, this equation tells us that for all goods used in production, they must have the same ratio of costs over efficiency (w_i/α_i). Note also that S can never be empty. There must always exist one element used in production, otherwise the firm could not satisfy the production constraint.
Now define $T = \mathcal{I} \setminus S$. This is the set of elements for which the condition in (10) does not hold, i.e. those goods that are not used in positive amounts in production. The nonnegativity constraint in (NN.2) implies that their quantities in optimum are equal to zero. We define

$$T = \{i \in \mathcal{I} : x_i = 0\} \tag{13}$$

The complementary slackness condition in (CS.2) does not really allow us to restrict the the multipliers further. We see that

$$\mu_t \geq 0 \quad \forall t \in T \tag{14}$$

Therefore, we can see that from (9) that for these goods the μ_t can drive a wedge between λ and the ratio of wages over efficiency.

$$\frac{w_t}{\alpha_t} \geq \lambda \quad \forall t \in T \tag{15}$$

Substituting for λ using (12) gives

$$\frac{w_t}{\alpha_t} \geq \frac{w_s}{\alpha_s} \quad \forall s \in S, \forall t \in T \tag{16}$$

Thus, any good m that is used in production must be in the set of goods with the lowest ratio between costs and marginal return. We define $\tilde{S}$ as

$$\tilde{S} = \{i \in \mathcal{I} : w_i/\alpha_i = \min(\{w_1/\alpha_1, ..., w_n/\alpha_n\})\} \tag{17}$$

This is the set of all goods with minimal cost to efficiency ratios. By construction, any of the goods in this set could potentially be used in optimum, as long as the production constraint is satisfied. Because $\mathbf{w} \gg 0$, we have that $\lambda > 0$, so the production constraint must be satisfied with equality. If the set S contains more than one element, a multitude of optimal input bundles are possible. The firm does not need to use each good in $\tilde{S}$, as shown in (16). Rather, condition (16) tells us that any good that is used in production must lie in $\tilde{S}$, i.e. it must possess a minimum cost to efficiency ratio.
As a final point, we can see that $\lambda > 0$ implies (via ()) that the production constraint must bind, i.e. $f(\mathbf{x}) = y$. We can characterize the solution to (3) as follows

$$x_i \geq 0 \quad \forall i \in \tilde{S} \tag{18}$$
$$x_i = 0 \quad \forall i \notin \tilde{S} \tag{19}$$
$$\sum_{i \in S} \alpha_i x_i = y \tag{20}$$

Effectively, equations (17) through (20) define the set of solutions of this problem as a function of the parameters $(\alpha, \mathbf{w}, y)$.

In order to arrive at a slightly simpler solution, we can assume a tie-breaking rule: the firm only uses the input in $\tilde{S}$ with the lowest index.[1] Let $m = \min(\tilde{S})$. Then the production constraint (20) implies

$$x_m = \frac{y}{\alpha_m} \tag{21}$$

In both cases, the cost function is then given by

$$c(\mathbf{w}, y) = \left(\frac{w_m}{\alpha_m}\right) y = \min(\{w_1/\alpha_1, ..., w_n/\alpha_n\})y \tag{22}$$

2 Exercise 3.42

2.1 Original question

We have seen that every Cobb-Douglas production function, $y = Ax_1^\alpha x_2^{1-\alpha}$ gives rise to a Cobb-Douglas cost function, $c(\mathbf{w}, y) = yAw_1^\alpha w_2^{1-\alpha}$, and every CES production function, $y = A(x_1^\rho + x_2^\rho)^{1\rho}$ gives rise to a CES cost function, $c(\mathbf{w}, y) = yA(w_1^r + w_2^r)^{1/r}$. For each pair of functions, show that the converse is also true. That is, starting with the respective cost functions, "work back-ward" to the underlying production function and show that it is of the indicated form. Justify your approach.

2.2 Solution

The cost function is the solution to the cost-minimization problem

$$c(\mathbf{w}, y) = \min_{\mathbf{x}\in\mathbb{R}^n_+} \mathbf{w}\cdot\mathbf{x} \quad \text{s.t} \quad f(\mathbf{x}) \geq y \tag{23}$$

We can rewrite

$$-c(\mathbf{w}, y) = \max_{\mathbf{x}\in\mathbb{R}^n_+} -\mathbf{w}\cdot\mathbf{x} \quad \text{s.t} \quad f(\mathbf{x}) \geq y \tag{24}$$

By definition, the cost function satisfies

$$c(\mathbf{w}, y) = \mathbf{w}\cdot\mathbf{x}(\mathbf{w}, y) \tag{25}$$

Let λ be a nonnegative multiplier associated with the production constraint. We form the Lagrangian.

$$\mathcal{L} = -\mathbf{w}\cdot\mathbf{x} - \lambda(y - f(\mathbf{x})) \tag{26}$$

An interior solution satisfies the Kuhn-Tucker conditions

$$\frac{\partial\mathcal{L}}{x_i} = -w_i + \lambda f_i(\mathbf{x}) = 0 \quad \forall i \in \mathcal{I} \tag{27}$$

$$\lambda \geq 0 \tag{28}$$

$$f(\mathbf{x}) \geq y \tag{29}$$

$$\lambda(y - f(\mathbf{x})) = 0 \tag{30}$$

If $\mathbf{w} \gg 0$ and $f_i(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}_{++}$, the multiplier must be strictly positive, $\lambda > 0$, and therefore the production constraint must bind, $f(\mathbf{x}) = y$. Therefore an interior optimum satisfies

$$\frac{f_i(\mathbf{x})}{f_j(\mathbf{x})} = \frac{w_i}{w_j} \quad \forall i, j \in \mathcal{I} \tag{31}$$

$$y = f(\mathbf{x}) \tag{32}$$

[1]The cost function is not affected by this assumption, as you can easily verify for yourself.

Cobb-Douglas Now assume $n = 2$ and impose our Cobb-Douglas specification for f. It is easy to verify that this function is strictly increasing for all $\mathbf{x} \in \mathbb{R}_+ +$. In order to map the properties of the cost function into conditional factor demands, we can apply Shephard's Lemma (given that the solution will be unique), which gives us that

$$x_1(\mathbf{w}, y) = \alpha y \left(\frac{w_2}{w_1}\right)^{1-\alpha} \qquad x_2(\mathbf{w}, y) = (1-\alpha) y \left(\frac{w_1}{w_2}\right)^{\alpha} \tag{33}$$

Dividing $x_1(\mathbf{w}, y)$ by $x_2(\mathbf{w}, y)$ we obtain

$$\frac{x_1(\mathbf{w}, y)}{x_2(\mathbf{w}, y)} = \frac{\alpha}{1-\alpha}\left(\frac{w_1}{w_2}\right) \tag{34}$$

This expression tells us that the price ratio is proportional to the ratio of goods, something which will prove to be useful soon. Recall that the cost function satisfies

$$c(\mathbf{w}, y) = Ayw_2\left(\frac{w_1}{w_2}\right)^{\alpha} \tag{35}$$

Substituting for $c(\mathbf{w}, y)$ using (25) and dividing both sides by w_2 gives

$$\left(\frac{w_1}{w_2}\right) x_1 + x_2 = A\left(\frac{w_1}{w_2}\right)^{\alpha} y \tag{36}$$

Substituting for w_1/w_2 using (34) and rearranging gives

$$\frac{x_2}{1-\alpha} = A\left(\frac{x_2}{x_1}\right)^{\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{\alpha} y$$

Since $y = f(\mathbf{x})$ in optimum, we retrieve the production function

$$f(\mathbf{x}) = \frac{x_1^{\alpha} x_2^{1-\alpha}}{A\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \tag{37}$$

CES Now assume $n = 2$ and impose our CES specification for f. From Shephard's Lemma we get that

$$x_1(\mathbf{w}, y) = Aw_1^{\rho-1}\left(w_1^{\rho} + w_2^{\rho}\right)^{1/\rho-1} \qquad x_2(\mathbf{w}, y) = Aw_2^{\rho-1}\left(w_1^{\rho} + w_2^{\rho}\right)^{1/\rho-1} \tag{38}$$

Dividing $x_1(\mathbf{w}, y)$ by $x_2(\mathbf{w}, y)$ we obtain

$$\frac{x_1(\mathbf{w}, y)}{x_2(\mathbf{w}, y)} = \left(\frac{w_1}{w_2}\right)^{r-1} \tag{39}$$

Note that we can rewrite the CES cost function as follows

$$c(\mathbf{w}, y) = yAw_2\left(\left(\frac{w_1}{w_2}\right)^{r} + 1\right)^{1/r} \tag{40}$$

Proceeding in the same manner as for the Cobb-Douglas we

$$\left(\frac{w_1}{w_2}\right) x_1 + x_2 = yA\left(\left(\frac{w_1}{w_2}\right)^{r} + 1\right)^{1/r} \tag{41}$$

Substituting for w_1/w_2 using (40) and rearranging we obtain

$$x_2\left(\left(\frac{x_1}{x_2}\right)^{\frac{r}{r-1}}+1\right)=yA\left(\left(\frac{x_1}{x_2}\right)^{\frac{r}{r-1}}+1\right)^{1/r} \tag{42}$$

We can then impose $y=f(\mathbf{x})$ and retrieve the production function

$$f(\mathbf{x})=A^{-1}x_2\left(\left(\frac{x_1}{x_2}\right)^{\frac{r}{r-1}}+1\right)^{\frac{r-1}{r}}=A^{-1}\left(x_1^{\frac{r}{r-1}}+x_2^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}} \tag{43}$$

3 Exercise 3.44

3.1 Original question

Derive the profit function for a firm with the Cobb-Douglas technology, $y=x_1^{\alpha}x_2^{\beta}$. What restrictions on α and β are required to ensure that the profit function is well-defined? Explain.

3.2 Solution

Maybe it is useful to first think about what we are after. Wikipedia tells us that

> *In mathematics, a well-defined expression or unambiguous expression is an expression whose definition assigns it a unique interpretation or value.*

I take this to say that we need to find parameters such that a function for the firm's profit (if quantities are optimally chosen) that solves the optimization problem of the firm and maps prices $(\mathbf{w},y)$ into a level of profits.
To begin, I assume that $p>0$ and $\mathbf{w}>\mathbf{0}$. Furthermore, I assume that $\alpha>0$ and $\beta>0$, so that the production function is a strictly increasing function of both inputs. The profit maximization problem for the firm is as follows-

$$\max_{\mathbf{x}\in\mathbb{R}^2_+} px_1^{\alpha}x_2^{\beta}-w_1x_1-w_2x_2$$

The first order conditions for this problem are-

$$p\alpha x_1^{\alpha-1}x_2^{\beta}-w_1=0 \tag{44}$$

$$p\beta x_1^{\alpha}x_2^{\beta-1}-w_2=0 \tag{45}$$

Use the second FOC to write: $x_1=[\frac{p\beta x_2^{\beta-1}}{w_2}]^{-\frac{1}{\alpha}}$. Substitute into the first FOC to get:

$$x_2=p^{\frac{1}{1-\alpha-\beta}}\cdot(\frac{\alpha}{w_1})^{\frac{\alpha}{1-\alpha-\beta}}\cdot(\frac{\beta}{w_2})^{\frac{1-\alpha}{1-\alpha-\beta}} \tag{46}$$

This expression for x_2 gives us one restriction: $1 \neq \alpha + \beta$. We can also conclude-

$$x_1 = p^{\frac{1}{1-\alpha-\beta}} \cdot (\frac{\alpha}{w_1})^{\frac{1-\beta}{1-\alpha-\beta}} \cdot (\frac{\beta}{w_2})^{\frac{\beta}{1-\alpha-\beta}} \tag{47}$$

Substitute x_1 and x_2 in the objective function to write-

$$px_1^{\alpha}x_2^{\beta} - w_1x_1 - w_2x_2 = p^{\frac{1}{1-\alpha-\beta}} \cdot (\frac{\alpha}{w_1})^{\frac{\alpha}{1-\alpha-\beta}} \cdot (\frac{\beta}{w_2})^{\frac{\beta}{1-\alpha-\beta}} - [\frac{p\alpha^{1-\beta}\beta^{\beta}}{w_2^{\beta}w_1^{\alpha}}]^{\frac{1}{1-\alpha-\beta}} - [\frac{p\alpha^{\alpha}\beta^{1-\alpha}}{w_2^{\beta}w_1^{\alpha}}]^{\frac{1}{1-\alpha-\beta}}$$

Notice that[2]

$$\begin{aligned}
w_1x_1 &= w_1p^{-\frac{1}{\alpha+\beta-1}}[(\frac{w_1}{\alpha})^{\frac{\alpha}{\alpha+\beta}}(\frac{w_2}{\beta})^{\frac{\beta}{\alpha+\beta}}]^{\frac{1}{\alpha+\beta-1}}(\frac{w_1}{\alpha})^{-\frac{\beta}{\alpha+\beta}}(\frac{w_2}{\beta})^{\frac{\beta}{\alpha+\beta}} && (48)\\
&= p^{-\frac{1}{\alpha+\beta-1}}(\frac{w_1}{\alpha})^{\frac{\alpha}{(\alpha+\beta)(\alpha+\beta-1)}}(\frac{w_2}{\beta})^{\frac{\beta}{(\alpha+\beta)(\alpha+\beta-1)}}w_1^{\frac{\alpha}{\alpha+\beta}}w_2^{\frac{\beta}{\alpha+\beta}}(\frac{1}{\alpha})^{-\frac{\beta}{\alpha+\beta}}(\frac{1}{\beta})^{\frac{\beta}{\alpha+\beta}} && (49)\\
&= p^{-\frac{1}{\alpha+\beta-1}}w_1^{\frac{\alpha}{\alpha+\beta-1}}(\frac{w_2}{\beta})^{\frac{\beta}{\alpha+\beta-1}}(\frac{1}{\alpha})^{\frac{\alpha+\beta(\alpha+\beta-1)}{(\alpha+\beta)(\alpha+\beta-1)}} && (50)\\
&= p^{-\frac{1}{\alpha+\beta-1}}w_1^{\frac{\alpha}{\alpha+\beta-1}}(\frac{w_2}{\beta})^{\frac{\beta}{\alpha+\beta-1}}(\frac{1}{\alpha})^{\frac{\alpha}{\alpha+\beta-1}}(\frac{1}{\alpha})^{\frac{1-\alpha-\beta}{\alpha+\beta-1}} && (51)\\
&= p^{-\frac{1}{\alpha+\beta-1}}(\frac{w_1}{\alpha})^{\frac{\alpha}{\alpha+\beta-1}}(\frac{w_2}{\beta})^{\frac{\beta}{\alpha+\beta-1}}\alpha && (52)
\end{aligned}$$

Going through the same step for w_2x_2 we have

$$w_2x_2 = p^{-\frac{1}{\alpha+\beta-1}}(\frac{w_1}{\alpha})^{\frac{\alpha}{\alpha+\beta-1}}(\frac{w_2}{\beta})^{\frac{\beta}{\alpha+\beta-1}}\beta \tag{53}$$

So, the above expression simplifies to-

$$\pi(p, \mathbf{w}) = [p \cdot (\frac{\alpha}{w_1})^{\alpha} \cdot (\frac{\beta}{w_2})^{\beta}]^{\frac{1}{1-\alpha-\beta}} \cdot [1 - \alpha - \beta]$$

A competitive firm will not operate at a point where they get negative profits. Therefore, it follows that $1 - \alpha - \beta > 0 \implies \alpha + \beta < 1$. To summarize, the requirement is:

$$\alpha > 0, \beta > 0, \alpha + \beta < 1 \tag{54}$$

Some students have asked me why we cannot have $\alpha + \beta = 1$. Take the following example, in which we have a firm producing with one good and a linear production technology $f(\mathbf{x}) = Ax^{\alpha}$ with $\alpha = 1$.

$$f(\mathbf{x}) = Ax \tag{55}$$

[2] credit to one of the groups who wrote this out super clear (if you want your name here please contact me :D)

The firm simply chooses inputs in order to meet the production constraint:

$$x = A^{-1}y \tag{56}$$

Then output y would be chosen in order to maximize total profits

$$\max_{y \in \mathbb{R}_+} py - A^{-1}y \tag{57}$$

If $A^{-1} < p$, the function is strictly increasing on $\mathbb{R}_+$, so it is unbounded and no maximum exists. Likewise if marginal costs are equal to the price (this would be the case in competitive markets), the firm makes zero profit independent of the level of output y, so any $y \in \mathbb{R}_+$ would constitute a solution. Lastly, if $A^{-1} < p$, then the firm makes a loss on any positive amount of output produced, and therefore it chooses to not produce at all $y = 0$. So is the profit function well-defined? I would say, that in order to guarantee for any level of parameters $(\mathbf{w}, p)$ that the profit function exists, we need to eliminate the possibility of infinite profits, which we can only do if we rule out constant returns (and hence impose that $\alpha + \beta < 1$)[3].

4 Exercise 4.6

4.1 Original question

A firm j in a competitive industry has total cost function $c^j(q) = aq + b_j q^2$, where $a > 0$, q is firm output, and b_j is different for each firm.

1. If $b_j > 0$ for all firms, what governs the amount produced by each of them? Will they produce equal amounts of output? Explain.

2. What happens if $b_j < 0$ for all firms?

4.2 Solution

The important thing to note here is that the industry is *competitive*, meaning that each firm takes the price as given. We have seen this concept in the consumer and firm's problems studied earlier. Goods are sold and bought at a fixed price. Underlying this concept is the assumption that no actor is big enough that its actions have an impact on the price.
A firm's profits are given by

$$\pi_j(q_j) = pq_j - c_j(q_j) \tag{58}$$

Production is determined optimally by the firm, in order to maximize profits

$$q^\star = \arg\max_{x \in \mathbb{R}_+} \pi_j(q_j) \tag{59}$$

The first-order condition for an interior optimum is given by

$$\frac{\partial \pi_j}{\partial q_j} = p - \frac{\partial c_j(q_j^\star)}{\partial q_j} = 0 \tag{60}$$

[3] This seemingly annoying feature of constant return to scale is actually quite useful: given an appropriate price, the output and input can be any level, which makes market equilibrium easier

The second-order sufficient condition for a global maximum is given by

$$\frac{\partial^2 \pi_j}{\partial q_j^2} = -\frac{\partial^2 c_j(q_j^\star)}{\partial q_j^2} < 0 \tag{61}$$

Plug in the given cost function, the first-order condition becomes

$$\frac{\partial \pi_j}{\partial q_j} = p - (a + 2b_j q_j^\star) = 0 \tag{62}$$

which we can solve for the optimal quantity

$$q_j^\star = \frac{p-a}{2b_j} > 0 \tag{63}$$

which is strictly greater than zero if $b_j > 0$ and $p - a > 0$. If we compute the second-order partial derivative, we can see that the profit function is concave

$$\frac{\partial^2 \pi_j}{\partial q_j^2} = -2b_j < 0 \tag{64}$$

Hence, $q_j^\star$ constitutes a local maximum. At this maximum, profits are given by

$$\pi_j(q_j^\star) = \frac{(p-a)^2}{4b_j} > 0 \tag{65}$$

As long as $p > a$, any makes a strictly positive profit by producing a positive quantity. Therefore a competitive market per se is not associated with zero profits.[4]The quantity produced depends on the firm-specific cost parameter b_j. A higher b_j, decreases both output and profits.

b) If $b_j < 0$, we can repeat the exercise above, and obtain the second-order condition

$$\frac{\partial^2 \pi_j}{\partial q_j^2} = -2b_j > 0 \tag{66}$$

which implies that profits are globally convex in output. The profit function turns out to be strictly positive for all q_j as long as $p - a > 0$. For quantities above a certain threshold, $q > (2b_j)^{-1}a$, costs are actually decreasing in output. As a result, the firm will maximize profits by letting output go to infinity, $q_j \to \infty$. For example, with $p > 0$ and $c(q) = 10q - 5q^2$ for some firm, then the firm can set $q = 100$ to achieve zero cost, and if it sets $q = 200$ the profit will increase even more.

[4]Competitive market means firms are price takers, so marginal costs are equal to the price. The reason why profit is always positive is that the price is not affected by quantity supplied. As a side note, it is common to assume constant returns for the representative firm, and assume that prices are such that a zero-profit condition is satisfied.

5 Exercise 4.7

5.1 Original question

Technology for producing q gives rise to the cost function $c(q) = aq + bq^2$. The market demand for q is $p = \alpha - \beta q$.

1. If $a > 0$, if $b < 0$, and if there are J firms in the industry, what is the short-run equilibrium market price and the output of a representative firm?
2. If $a > 0$ and $b < 0$, what is the long-run equilibrium market price and number of firms? Explain.
3. If $a > 0$ and $b > 0$, what is the long-run equilibrium market price and number of firms? Explain.

5.2 Solution

I take this task to mean that we are in a setting with Cournot competition, since there is no mentioning of a competitive industry. Hence, I will rewrite aggregate demand $p = \alpha - \beta Q$ where aggregate quantity $Q = \sum_{i \in \mathcal{I}} q_i$ is simply the sum of individual firm's quantities. In the Cournot competition, each firm maximizes profits in its own output q_i taking the output placed on the market by the other firms, which we can stack into a vector $\mathbf{q}_{-j}$ as given.[5]. Hence, the inverse demand function can be written as

$$p(q_j, \mathbf{q}_{-j}) = \alpha - \beta \left(q_j + \sum_{i \neq j} q_i \right) \tag{67}$$

A firm's profits given the strategies of its competitors are given by

$$\pi_j(q_j, \mathbf{q}_{-j}) = p(q_j, \mathbf{q}_{-j})q_j - c(q_j) \tag{68}$$

Production is determined optimally by the firm, in order to maximize profits

$$q^\star = \arg \max_{q_j \in \mathbb{R}_+} \pi_j(q_j, \mathbf{q}_{-j}) \tag{69}$$

The first-order condition for an interior optimum is given by

$$\frac{\partial \pi_j}{\partial q_j} = p(q_j, \mathbf{q}_{-j}) + \frac{\partial p(q_j, \mathbf{q}_{-j})}{\partial q_j} q_j - \frac{\partial c(q_j)}{\partial q_j} = 0 \tag{70}$$

The second-order necessary condition for a local maximum is given by

$$\frac{\partial^2 \pi_j}{\partial (q_j)^2} = 2\frac{\partial p(q_j, \mathbf{q}_{-j})}{\partial q_j} + \frac{\partial^2 p(q_j, \mathbf{q}_{-j})}{\partial (q_j)^2} q_j - \frac{\partial^2 c(q_j)}{\partial (q_j)^2} < 0 \tag{71}$$

We now plug in the given functional form for the cost function, and market demand. The relevant partial derivatives are

$$\frac{\partial p(q_j, \mathbf{q}_{-j})}{\partial q_j} = -\beta \qquad \frac{\partial^2 p(q_j, \mathbf{q}_{-j})}{\partial (q_j)^2} = 0 \tag{72}$$

[5]That is, $\mathbf{q}_{-j} = \{q_1, q_2, ..., q_{j-1}, q_{j+1}, ..., q_n\}$

$$\frac{\partial c(q_j)}{\partial q_j} = a + bq_j \qquad \frac{\partial^2 c(q_j)}{\partial (q_j)^2} = 2b \tag{73}$$

Thus, the first-order condition in (78) specializes to

$$\frac{\partial \pi_j}{\partial q_j} = \alpha - a - \beta \sum_{i \neq j}^{J} q_i - 2(\beta + b)q_j = 0 \tag{74}$$

Moreover, the second condition is given by

$$\frac{\partial^2 \pi_j}{\partial q_j^2} = -2(b + \beta) < 0 \tag{75}$$

In order for an interior optimum to be a maximum, we need the second condition to be strictly negative, i.e. $\partial^2 \pi_j / \partial q_j^2 < 0$. This requires $b + \beta > 0$. Since β regulates the decline of demand in quantity, and b regulates the convexity of the cost function, i.e., the speed at which marginal costs increase/decrease, the condition tells us that in order for there t be an interior maximum, prices must fall more rapidly than marginal costs fall if quantities are increase. Note that I am not imposing any assumptions on prices here, and similarly for marginal costs. From the view of economic intuition, it would be sensible to impose $p \geq 0$. I am abstaining here from any such assumptions.
In order for a positive quantity to be produced, it needs to be that $\alpha - a > 0$. Otherwise, demand is "too small" for there to be production at the given cost structure (you can also verify $\alpha - a > 0$ in equation (74)). We can then write (74) as:

$$q_j(\mathbf{q}_{-j}) = \frac{\alpha - a - \beta \sum_{i \neq j}^{J} q_j}{2(\beta + b)} \tag{76}$$

So we see that the solution to the firm's problem takes the form of a function over the space of its competitors quantities. In the language of game theory, we would say that (76) defines a best-reponse of firm j to the strategy of its competitors.
If we wrote down the best-response for all firms $j \in \mathcal{I}$, we would get a system of J equations in J unknowns that could be solved in order to obtain the equilibrium quantities $\mathbf{q}$. If the cost structure is the same for all firms, we can see that there is no real difference among firms. Then all will produce the same output, denoted as $q_j = q$, and aggregate output can be denoted as $Q = \sum q_j = Jq$.

$$q = \frac{\alpha - a - \beta(J-1)q}{(J+1)\beta + 2b} \tag{77}$$

We can then solve for q

$$q = \frac{\alpha - a}{\beta(J-1) + 2(\beta + b)} = \frac{\alpha - a}{\beta(J+1) + 2b} \tag{78}$$

This is the short-run equilibrium output for the representative firm. The short-run equilibrium price is given by

$$p = \alpha - \frac{\beta J(\alpha - a)}{\beta(J+1) + 2b} \tag{79}$$

Let us briefly recap what we have done. We have first solved a general optimization problem of an individual firm. That solution takes the form of a function taking as given the quantities supplied by its competitors. After that we have imposed a simplifying assumption, namely that all firms are identical, so we only have to solve for the quantity of one representative firm. We have then solved for an equilibrium, a quantity supplied by the representative firm, which if adopted by all other firms in the market, will render that quantity optimal. We have determined an equilibrium in the Cournot oligopoly.

b) The critical requirement for an equilibrium is that $\beta + b > 0$. Prices must fall at a greater rate than marginal costs if quantities increase. From equation (74) we see that if $\beta + b < 0$, there exists a threshold q above which the profit function is increasing in quantity. That implies that the firm can generate infinite profits by producing infinite amounts, even if prices are negative. Therefore, it seems reasonable to assume that $\beta + b > 0$.

For a market with J identical firms, the representative firm generates a profit π_j that satisfies

$$\Pi(J) = [\alpha - \beta Q - a - bq]\, q \tag{80}$$

$$\begin{aligned}
\Pi(J) &= \left(\alpha - \beta J \frac{\alpha - a}{\beta(J+1) + 2b} - a - b\frac{\alpha - a}{\beta(J+1)+2b}\right) q & (81)\\
&= (\alpha - a)\left[1 - \frac{\beta J + b}{\beta(J+1)+2b}\right] q & (82)\\
&= (\alpha - a)\left[\frac{\beta + b}{\beta(J+1)+2b}\right] q & (83)\\
&= \left[\frac{\beta + b}{\beta(J+1)+2b}\right] \frac{\alpha - a}{\beta(J+1)+2b} & (84)\\
&= (\beta + b)\left(\frac{(\alpha - a)}{\beta(J+1)+2b}\right)^2 & (85)
\end{aligned}$$

which is positive if $\beta + b > 0$. Note that the inequality does not depend in any way on the number of firms J. Regardless of J, the representative firm will always generate a strictly positive profit. The existence of profits generates entry of new firms into the market. So we conclude that the long-run equilibrium involves an infinite number of firms $J = +\infty$. Now consider what happens to output in the long-run.

$$\lim_{J\to\infty} q(J) = \lim_{J\to\infty} \frac{\alpha - a}{\beta(J+1)+2b} = 0 \tag{86}$$

Since $\beta > 0$ by assumption, the representative firm's output converges to zero as J tends to infinity. In the limit profits are zero, as quantities tend to zero

$$\lim_{J\to\infty} \Pi(J) = 0 \tag{87}$$

Observe now that we can write the aggregate quantity Q as

$$Q(J) = J\frac{\alpha - a}{\beta(J+1)+2b} = \frac{\alpha - a}{\beta\left(1 + \frac{1}{J}\right) + \frac{2b}{J}} \tag{88}$$

From this we can infer that as $J \to \infty$, the aggregate demand converges to a positive quantity

$$\lim_{J\to\infty} Q(J) = \frac{\alpha - a}{\beta} > 0 \tag{89}$$

The long-run market price is given by

$$\lim_{J \to \infty} p(J) = \alpha - \beta \left(\frac{\alpha - a}{\beta} \right) = a > 0 \tag{90}$$

This is the marginal cost at $q = 0$, so essentially the long-run equilibrium in this setting "undoes" all the scale effects.

c) For $\alpha > 0$ and $b > 0$, the condition $b + \beta > 0$ must hold, so the analysis is the same as above.