Handout 5: Production Functions and Cost Minimization

1 Introduction

In this handout we are going to focus on the problem of a firm. First, we will discuss the production function and its properties. The production function shows how inputs - capital and labor, for instance - are transformed in output. We will focus on the concepts of returns to scale, marginal returns and marginal product. Second, we discuss the relation of the production function with average and marginal costs. Third, we discuss the cost minimization problem of the firm, which is intimately related with the consumer choice problem (Handout 3). We will mathematically explore the following idea: given that a firm must produce a given quantity, what is the cost minimizing way to combine inputs? Within this example, we highlight the differences between short and long-run cost functions.

From a big picture perspective, we want to write the cost function of a firm - which comes from the production function - to be able to solve the profit maximization problem of the firm, from which we will be able to derive an equation of the supply.

2 Production Function

For each of the following statements, answer whether they are True/False. Explain your choice.

1. A production function $F(K, L)$ can have constant returns to scale *and* diminishing marginal returns to both K and L.

2. A production function $F(K, L)$ can have *increasing* returns to scale and a diminishing marginal returns to both K and L.

3. Suppose for a production function $F(K, L)$, the Marginal Product of Labor is increasing in Labor at some level of L. Then the Average Product of Labor is definitely increasing in Labor at L.

Solution.

1. *True.* A production function has constant returns to scale (CRS) if, for any $t > 0$:

$$F(tK, tL) = tF(K, L)$$

that is, doubling both inputs doubles the output. For instance the Cobb-Douglas production function $F(K,L) = K^\alpha L^\beta$ with $\alpha \in (0,1)$ and $\beta = 1-\alpha$ has *constant* returns to scale

$$F(tK,tL) = (tK)^\alpha (tL)^{1-\alpha} = t^{\alpha+1-\alpha} K^\alpha L^{1-\alpha} = tF(K,L)$$

and a diminishing marginal returns to both K and L. For labor, for instance, the marginal return is given by:

$$\frac{\partial F(K,L)}{\partial L} = (1-\alpha)\left(\frac{K}{L}\right)^\alpha$$

The marginal return of labor is thus decreasing in labor as long as $\alpha < 1$ (which implies that $1-\alpha > 0$, since L appears in the denominator. More generally, any property relative to returns to scale are defined by changing all inputs *jointly*. On the other hand, marginal returns are related with changing each input *separately*.

2. *True*. for instance the Cobb-Douglas production function $F(K,L) = K^\alpha L^\beta$ with $\alpha = \frac{2}{3}$ and $\beta = \frac{2}{3}$ has *increasing* returns to scale:

$$F(tK,tL) = (tK)^{2/3} (tL)^{2/3} = t^{4/3} K^{2/3} L^{2/3} = t^{4/3} F(K,L)$$

For $t > 1$, $t^{\frac{4}{3}} > t$. Therefore:

$$F(tK,tL) = t^{4/3} F(K,L) > tF(K,L)$$

On the other hand, we have diminishing marginal returns can still be decreasing. For labor, for instance, the marginal return is given by:

$$\frac{\partial F(K,L)}{\partial L} = \frac{2}{3}\frac{K^{2/3}}{L^{1/3}}$$

which is decreasing in L. As in the case of Item 1, returns to scale are defined by changing all inputs *jointly*, while marginal returns are related with changing each input *separately*.

3. *False*. The marginal product of labor (MPL) is defined as:

$$MPL \equiv \frac{\partial F(K,L)}{\partial L}$$

The average product of labor is defined as:

$$APL = \frac{F(K,L)}{L}$$

Let's first think about how these two concepts are related. The marginal product of labor represents how much the extra unit of labor increases output at the margin. The average product of labor represents the output per worker. If the marginal worker is more productive than the average, that is, $MPL > APL$, we expect the average productivity to increase. On the other hand, if the marginal worker is less productive than the average, we expect the average productivity to decrease. Therefore, $MPL > APL$ means that APL is increasing, while $MPL < APL$ implies that APL is decreasing. Note that this does *not* depend if MPL is decreasing or increasing, just on which is larger. Mathematically, we can see this argument as follows:

$$\frac{\partial APL}{\partial L} = \frac{\frac{\partial F(K,L)}{\partial L}L - F(K,L)}{L^2} = \frac{MPL - APL}{L}$$

Therefore, APL is increasing in L if and only if $MPL > APL$ - which can happen with MPL being either increasing or decreasing.

$\square$

3 Productivity and Costs

Consider the following production function:

$$F(K,L) = L + 2\sqrt{K}L^2 - L^3$$

1. Plot $F(K,L)$ for $\overline{K} = 1$ for values of L between 0 and 1.2 using any software. Interpret the shape of the production function.

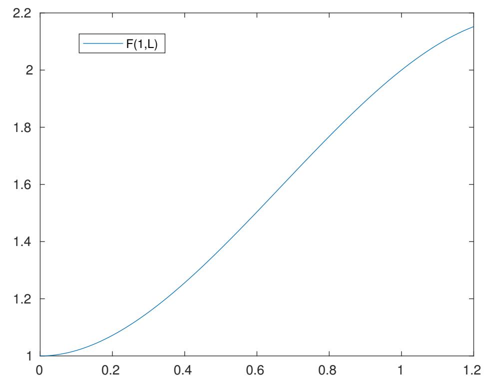
2. Compute the Average Product of Labor (APL), the Marginal Product of Labor (MPL) and find the labor such that $APL = MPL$. What is the economic interpretation of the point where $APL = MPL$?

3. (Hard) Compute the Marginal Cost (MC) of producing Q units if K is fixed at $\overline{K}$. For $w = r = 1$,

plot the MC and Average Total Cost (ATC) using a software. What is the relation between these two curves? Can they cross more than once?

Solution.

1. For $\overline{K} = 1$, the production function looks like



   Labor is very productive with low levels of L. Each worker has a lot of capital to work with. Eventually, however, the production plant gets crowded and output starts to increase slowly as we change labor. One example is a restaurant's kitchen. Imagine that there are two people using the kitchen: they have access to all of the equipment, don't have to wait for anyone else to finish their tasks etc.. The third chef will also be very productive, since now they can specialize in tasks without interfering with each others work. As the number of workers increases in the kitchen, however, their marginal effect is reduced: they have to share equipment etc., and thus production increases slowly with labor. Eventually, the kitchen can get crowded enough that the marginal worker is essentially useless.

2. The Average Product of Labor (APL) can be computed as

$$\frac{F(K, L)}{L} = 1 + 2\sqrt{K}L - L^2$$

   The Marginal Product of Labor (MPL):

$$\frac{\partial F(K, L)}{\partial L} = 1 + 4\sqrt{K}L - 3L^2$$

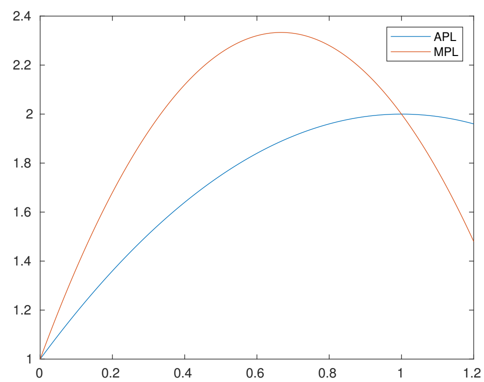
Equating the two curves:

$$1 + 2\sqrt{K}L - L^2 = 1 + 4\sqrt{K}L - 3L^2$$

$$2\sqrt{K}L - L^2 = 4\sqrt{K}L - 3L^2$$

$$2L\left(\sqrt{K} - L\right) = 0$$

Thus: $L = \sqrt{K}$ at the point where $MPL = APL = 1 + K$. Note that this point depends on capital - the other input in production. For $K = 1$, for instance:



The point where the $MPL = APL$ is the point where APL is at either its maximum or minimum. For a detailed explanation, see the answer of Question 1, Item 2. If the MPL crosses the APL from above, as in this case, this means that APL is increasing before they cross and decreasing afterwards - which implies that they cross at a maximum of the APL.

3. To compute the marginal cost, we must compute the conditional labor demand. Given a capital level of $\overline{K}$, labor demand to produce Q units must solve the following Equation:

$$Q = L(Q) + 2\sqrt{\overline{K}}L(Q)^2 - L(Q)^3$$

We can compute the marginal cost only using the equation above - that is, without solving explicitly for $L(Q)$. The marginal cost is given by:

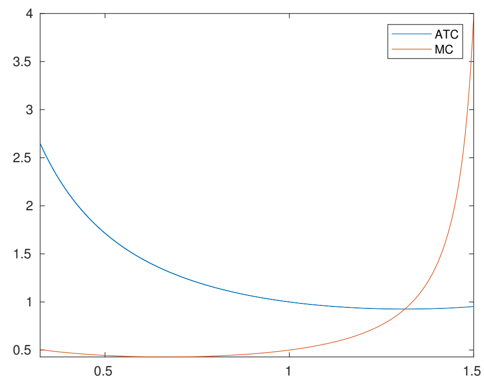
$$MC(Q) = wL^{'}(Q)$$

Note that we can use $Q = L + 2\sqrt{K}L^2 - L^3$ and derive both sides with respect to Q to get:

$$1 = \left(1 + 4\sqrt{K}L - 3L^2\right) L^{'}(Q) \Rightarrow L^{'}(Q) = \left[1 + 4\sqrt{K}L - 3L^2\right]^{-1}$$

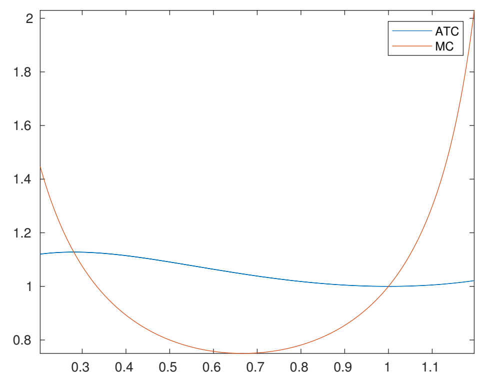
$$MC(Q) = \frac{w}{1 + 4\sqrt{K}L - 3L^2}$$

For $w = r = K = 1$, the ATC and MC are such that:



The MC and ATC have a relation very close to the one between MPL and APL. The marginal cost represents the cost of producing one extra unit. If the extra unit costs less than the average, the average cost will be reduced, that is: $MC < ATC$ implies that ATC is decreasing. Equivalently, $MC > ATC$ implies that ATC is increasing. For most production functions, the MC will cross ATC from below. This means that $MC < ATC$ before they cross - ATC decreasing - and $MC > ATC$ after they cross - ATC increasing. In this case, the MC crosses the ATC at the minimum (as in the figure above).

For other production functions, MC and ATC can cross more than once. We worked so far with $F(K, L) = L + 2\sqrt{K}L^2 - L^3$. If we consider this other production function $G(K, L) = 1 + 2\sqrt{K}L^2 - L^3$, we have that ATC and MC for $\overline{K} = 1, w = r = 1$ are given by:



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4 Short and Long-Run Cost Functions

Consider a firm with the Cobb-Douglas production function

$$F(L, K) = L^{1/3}K^{1/3}$$

and assume that factor prices are $w = 1$ and $r = 1$.

1. Write down the cost minimization problem and compute the long run cost function of the firm (when the firm can choose both K and L).
2. Write down the cost minimization problem and compute the Short Run cost function of the firm if capital is fixed at $\overline{K}$.
3. Using any software, plot the long and short run cost curves for $\overline{K} = 2, 4$ and 10. What is the relation between them?

Solution.

1. The firm cost minimization problem is given by:

$$\min_{L,K} \quad wL + rK \quad s.t. \quad F(L, K) = Q$$

which states that the firm is minimizing the total cost of producing Q units by choosing the optimal combination of K and L. At the optimum, it must be that the marginal rate of transformation between capital and labor is equal to the absolute value of the ratio of wages and rental rate of capital. Intuitively, this means that the ratio at which the firm is willing to trade capital for labor ($MRTS$) is equal to the ratio the market is willing to trade one for the other (ratio of the prices of each input). This is the analogous of the firm for the condition that the marginal rate of substitution between two goods is equal to the ratio of prices for the consumer. Mathematically:

$$MRTS = -\frac{\frac{1}{3}L^{-2/3}K^{1/3}}{\frac{1}{3}L^{1/3}K^{-2/3}} = -1 = -\frac{w}{r}$$
$$K = L$$

We can then substitute this into our quantity constraint $F(L, K) = Q$

$$K^{1/3}K^{1/3} = Q$$
$$K^2 = Q^3$$
$$K^*(Q) = Q^{3/2}$$
$$L^*(Q) = Q^{3/2}$$

We denote $K^*(Q)$ and $L^*(Q)$ as conditional factor demands. They are the demand of the firm for inputs *conditional* on producing a quantity Q. Substituting these factor demands in the cost function gives the long run cost function:

$$C_{LR}(Q) = wL^*(Q) + rK^*(Q) = 2Q^{3/2}$$

2. The firm cost minimization problem is given by:

$$\min_{L} \quad wL + r\overline{K} \quad s.t. \quad F(L, \overline{K}) = Q$$

The firm chooses labor to minimize costs and guarantee a production of Q. Note that this optimization problem has only one feasible point, i.e., only one level of labor that satisfies the constraint. Therefore, this level is the solution of the minimization problem. Mathematically, in the short run, if the firm cannot adjust the level of capital and must produce using $\overline{K}$, we have that for a given level of labor L, output Q will be given by

$$Q = L^{1/3}\overline{K}^{1/3}$$

Therefore, to produce Q units, the the short run labor demand $L_{SR}(Q)$ is (inverting the equation above:

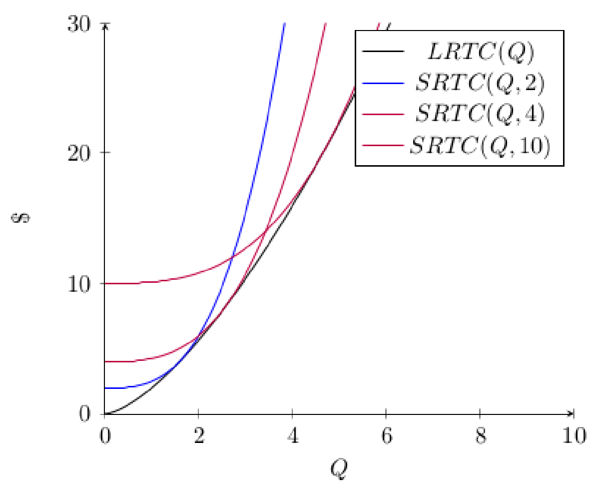
$$L_{SR}(Q, \bar{K}) = \frac{Q^3}{\bar{K}}$$

Therefore, the total cost in the short run will be given by:

$$\begin{aligned} C_{SR}(Q, \bar{K}) &= wL_{SR}(\cdot) + r\bar{K} \\ &= \frac{Q^3}{\bar{K}} + \bar{K} \end{aligned}$$

3. We plot $C_{LR}(Q)$, $C_{SR}(Q, 2)$, $SRTC(Q, 4)$, and $C_{SR}(Q, 10)$, that is, the long and short run cost curves varying Q. For the short run curves, we plot the cost function for $\overline{K} = 2, 4$ and 10. You can do this in any software (e.g.: Excel). Your graph should look as the following figure.

Figure 1: Short Run vs Long Run Cost



There are two important aspects of the long vs short run cost curves:

(a) The long-run cost curve is always below the short run cost curve: in the long run, the firm has a larger choice set to minimize cost, so costs must be lower! If capital is expensive, for instance, the firm can substitute capital with labor. If capital is relatively cheap, the firm can do the opposite and fire workers and increase the amount of capital used in production.

(b) The long and short run cost curves are tangent exactly at one point: this point is such that given the quantity, the short run fixed amount of capital is exactly the same as the one the firm would choose to keep in the long run. For this reason, the long-run cost curve is generally called a *lower envelope* of short run cost curves.

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