5303 - Advanced Macroeconomics
Assignment 4 Solutions

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November 2023

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Exercise 1

Consider the model from MaCurdy (1981). Agents discount future utility at rate β and have the following period utility function over consumption and hours:

$$u(c_t, h_t) = \gamma_{1t} c_t^{\alpha_1} - \gamma_{2t} h_t^{\alpha_2}, 0 < \alpha_1 < 1, \alpha_2 > 1$$

The lifetime budget constraint is given by:

$$A_0 + \sum_{t=0}^{T} \frac{1}{(1+r)^t} w_t h_t = \sum_{t=0}^{T} \frac{1}{(1+r)^t} c_t$$

There is certainty about future wages and there is no human capital investment (i.e., the wage path is exogenous). Wages are measured with error.

(a) Write down the Lagrangian and first-order conditions for the agents' lifetime consumption/hours choice problem.

(b) Derive the labour supply function.

(c) What is the "marginal utility of wealth"?

(d) Derive the intertemporal elasticity of substitution of labour supply (*ies*). What does the *ies* mean?

(e) How would you consistently estimate the parameters of the labour supply function?

(f) What types of estimates of the *ies* do people typically get when they follow this procedure? Why?

Answer:

(a) The agent's problem is given by[1]:

$$\max_{\{c_t, h_t\}_{t=0}^{T}} \sum_{t=0}^{T} \frac{1}{(1+\rho)^t} (\gamma_{1t} c_t^{\alpha_1} - \gamma_{2t} h_t^{\alpha_2})$$

$$\text{s.t. } A_0 + \sum_{t=0}^{T} \frac{1}{(1+r)^t} w_t h_t = \sum_{t=0}^{T} \frac{1}{(1+r)^t} c_t$$

[1] $\beta = \frac{1}{(1+\rho)}$

where ρ is the agent's rate of time preference, $\alpha_1 \in]0,1[$ and $\alpha_2 \in]1,\infty[$ are time-invariant parameters common across workers and γ_{1t}, γ_{2t} are age-specific modifiers of "tastes", i.e. taste-shifters. The taste-shifters depend on all of the characteristics which plausibly affect an invidual's preferences at age t (such as number of children present at age t, education level, age, etc). The associated Lagrangian is given by:

$$\mathcal{L} = \sum_{t=0}^{T} \frac{1}{(1+\rho)^t}(\gamma_{1t}c_t^{\alpha_1} - \gamma_{2t}h_t^{\alpha_2}) + \lambda\Big[A_0 + \sum_{t=0}^{T} \frac{1}{(1+r)^t} w_t h_t - \sum_{t=0}^{T} \frac{1}{(1+r)^t} c_t\Big]$$

And the FOC are given by:

$\mathbf{c_t}$: $\frac{1}{(1+\rho)^t}\alpha_1\gamma_{1t}c_t^{\alpha_1-1} - \lambda\frac{1}{(1+r)^t} = 0, \forall t > 0$

$\mathbf{h_t}$: $-\frac{1}{(1+\rho)^t}\alpha_2\gamma_{2t}h_t^{\alpha_2-1} + \lambda\frac{1}{(1+r)^t}w_t = 0, \forall t > 0$

Updating the FOC on consumption:

$\mathbf{c_{t+1}}$: $\frac{1}{(1+\rho)^{t+1}}\alpha_1\gamma_{1t+1}c_{t+1}^{\alpha_1-1} - \lambda\frac{1}{(1+r)^{t+1}} = 0, \forall t > 0$

We can now divide both FOCs for consumption and obtain the following condition:

$$(1+\rho)\frac{\gamma_{1t}}{\gamma_{1t+1}}\Big(\frac{c_{t+1}}{c_t}\Big)^{1-\alpha_1} = 1 + r$$

which is our Euler equation.

(b) By the envelope theorem, we know that the Lagrangian multiplier λ is constant and equal to the marginal utility of wealth. Making use of the FOC for hours worked, we have that:

$$h_t = \Big[\frac{\lambda(1+\rho)^t w_t}{\alpha_2\gamma_{2t}(1+r)^t}\Big]^{\frac{1}{\alpha_2-1}}$$

which is our labour supply function. We can also write it in logarithmic form:

$$\ln h_t = \frac{1}{\alpha_2 - 1}\Big[\ln\lambda + t\ln(1+\rho) + \ln(w_t) - t\ln(1+r) - \ln\alpha_2 - \ln\gamma_{2t}\Big]$$

(c) As discussed in the previous question, we know that, by the envelope theorem, the "marginal utility of wealth" is equal to the Lagrangian multiplier λ of the lifetime budget constraint. In this particular setting, except for the value of the current wage rate, λ summarizes all information about lifetime wages and property income that a consumer requires to determine his/her optimal current consumption and labour supply. At any age, any path of wages or property income over a consumer's lifetime that keeps λ and the current wage constant implies the same optimal current consumption and labour-supply behaviour. To sum up, λ is the change in the value of the Lagrangian when the individual's lifetime budget constraint changes.

(d) From MaCurdy (1981), we know that the intertemporal elasticity of substitution determines the response of hours of work at age t to a shift in the age t wage rate holding λ or the marginal utility of wealth constant. In the literature on consumer demand, this particular elasticity is known as the specific substitution effect. This elasticity can be used to predict the response of labour supply/hours worked to evolutionary wage changes (differences in wages across time for the same consumer), i.e., it provides the information one needs to describe a consumer's dynamic behaviour. It is given by:

$$ies = \frac{\partial \ln h_t}{\partial \ln w_t} = \frac{1}{\alpha_2 - 1}$$

Since it is positive, we would expect an increase in hours worked following an increase in the wage rate.

(e) Remember that the labour supply function in logarithmic form is given by:

$$\ln h_t = \frac{1}{\alpha_2 - 1}\Big[\ln \lambda + t\ln(1+\rho) + \ln(w_t) - t\ln(1+r) - \ln \alpha_2 - \ln \gamma_{2t}\Big]$$

In order to estimate these parameters, we can first make use of the first differencing method:

$$\Delta \ln h_t = \frac{1}{\alpha_2 - 1}\Big[\ln(1+\rho) + \Delta \ln(w_t) - \ln(1+r) - \Delta \ln \gamma_{2t}\Big]$$

Notice that we were able to remove the temporal component t from the RHS and the unknown parameter λ, as well as α_2. Essentially, we were able to eliminate individual effects, thus avoiding the introduction of incidental parameters[2].

Approximating[3], we have that:

$$\Delta \ln h_t = \frac{1}{\alpha_2 - 1}\Big[\Delta \ln(w_t) + (\rho - r) - \Delta \ln \gamma_{2t}\Big]$$

By looking at the equation above, we can see that there are three main reasons why hours worked could change over the lifecycle, namely a change in wages, w_t, a change in preferences for work, γ_{2t}, and/or having $\rho \neq r$. Since wages are measured with error, we could now perform an IV or

[2] Incidental parameters are parameters which, from a theoretical point of view, are of secondary importance.

[3] $\ln(1+x) \approx x$ works well for small changes

2SLS regression to obtain the parameter estimates. Following MaCurdy (1981)'s approach, you could use family background variables (such as father and mother's education), education, age, age squared, interaction variables between education and age and dummy variables for each year of the sample as the set of instruments to run the regression. By doing so, you would impose a typical life cycle profile for wages.

(f) When making use of the procedure described above, the *ies* estimates lie in the range $0.1 - 0.23$, meaning that they are not that high. The reason for this is that the wage profile is exogenous. MaCurdy (1981) calculates the co-variation of the typical hours profile and the typical wage profile. Given that the latter has three times the slope of the former, one can only get a small elasticity.
Furthermore, further research has showed that measures of elasticities can be downward biased either because borrowing constraints are omitted (Domeij and Floden, 2006), human capital accumulation is omitted (Imai and Keane, 2004; Wallenius, 2011), or when the extensive margin of labor supply (fraction of life worked) is ignored (Rogerson and Wallenius, 2009).

Exercise 2

Consider the following two-period OLG model with human capital. The number of young agents at any date t is fixed at N.

Agent preferences are described by $u_t^h = \ln c_t^h(t) + \ln c_t^h(t+1)$

The production function is given by $Y_t = S_t^\gamma L_t^{1-\gamma}$

where L_t is the labour of young (unskilled) workers and S_t is the labour of the old (skilled) workers.

When young, agents spend a fraction m_t of their time working and the remaining time $1 - m_t$ getting an education (acquiring human capital).

Their human capital accumulation is described by $h_{t+1} = h_t + (1 - m_t)\theta h_t$

where h_t is human capital of the current old, h_{t+1} is human capital acquired by the current young, and $\theta > 1$ is a parameter.

The wage per unit of time worked when young is w_t. When young, agents cannot use the human capital they have acquired in that period.

When old, agents just work and earn a wage of v_t per unit of human capital they acquired when young. All markets are perfectly competitive.

(a) Describe the key feature(s) of the human capital accumulation equation.

(b) Solve for the agent's optimal consumption, savings and human capital investment profile.

(c) What is the growth rate for output in this economy? (Hint: $S_t = Nh_t$ and $L_t = Nm_t$)

(d) Define a competitive equilibrium for this economy.

(e) Suppose the government initiates a mandatory PAYG social security program, taxing the young a lump-sum amount T_1 and returning the old a lump-sum amount T_2. Will countries that run such a program invest more in education than before? Show all steps. Provide a clear intutition for your answer.

Answer:

(a) The capital accumulation equation tells us that each young agent at time t acquires the human capital provided by the old agents at time t plus a fraction of the same old agents' human capital, which depends on the amount of time the individual spends on getting an education. The higher the investment in education, they higher the human capital level of the current young (the more they learn from old agents). Notice that the agents can only use the human capital they acquired when young when they are old.

(b) The agent solves the following problem [4]:

$$\max_{\{c_t(t), c_t(t+1), m_t\}} \ln c_t(t) + \ln c_t(t+1)$$
$$\text{s.t. } c_t(t) = w_t m_t$$
$$c_t(t+1) = v_{t+1} h_{t+1} = v_{t+1}\big[h_t + (1 - m_t)\theta h_t\big]$$

And our Lagrangian:

$$\mathcal{L} = \ln c_t(t) + \ln c_t(t+1) + \mu(t)\Big[w_t m_t - c_t(t)\Big] + \mu(t+1)\Big[v_{t+1}\big[h_t + (1 - m_t)\theta h_t\big] - c_t(t+1)\Big]$$

[4] Since all individuals are identical, we drop the h superscript. Furthermore, we know that there will be no borrowing nor lending in this economy, meaning that there are no savings.

Taking the FOC:

$$\mathbf{c_t(t)}: \frac{1}{c_t(t)} - \mu(t) = 0, \forall t > 0$$
$$\mathbf{c_t(t+1)}: \frac{1}{c_t(t+1)} - \mu(t+1) = 0, \forall t > 0$$
$$\mathbf{m_t}: \mu(t)w_t - \mu(t+1)v_{t+1}\theta h_t = 0, \forall t > 0$$

From which we obtain the following equilibrium condition:

$$\frac{c_t(t+1)}{c_t(t)} = \frac{v_{t+1}}{w_t}\theta h_t$$

Making use of our budget constraints, we have that:

$$\begin{aligned}
\frac{c_t(t+1)}{c_t(t)} &= \frac{v_{t+1}}{w_t}\theta h_t \\
\frac{v_{t+1}\big[h_t + (1-m_t)\theta h_t\big]}{w_t m_t} &= \frac{v_{t+1}}{w_t}\theta h_t \\
m_t &= \frac{1+\theta}{2\theta}
\end{aligned}$$

meaning that the fraction of time spent working is constant over time. Therefore, it follows that:

$$\begin{aligned}
h_{t+1} &= \frac{1+\theta}{2}h_t \\
c_t(t) &= \frac{1+\theta}{2\theta}w_t \\
c_t(t+1) &= \frac{1+\theta}{2}v_{t+1}h_t
\end{aligned}$$

(c) The aggregate production function in this economy is given by:

$$Y_t = S_t^{\gamma} L_t^{1-\gamma}$$

and each generation consists of N individuals. Making use of the hint above, the gross output growth rate is given by:

$$\begin{aligned}
\frac{Y_{t+1}}{Y_t} &= \Big(\frac{S_{t+1}}{S_t}\Big)^{\gamma}\Big(\frac{L_{t+1}}{L_t}\Big)^{1-\gamma} \\
&= \Big(\frac{Nh_{t+1}}{Nh_t}\Big)^{\gamma}\Big(\frac{Nm_{t+1}}{Nm_t}\Big)^{1-\gamma} \\
&= \Big(\frac{h_{t+1}}{h_t}\Big)^{\gamma}\underbrace{\Big(\frac{m_{t+1}}{m_t}\Big)^{1-\gamma}}_{=1} \\
&= \Big(\frac{1+\theta}{2}\Big)^{\gamma}
\end{aligned}$$

We can thus conclude that the growth rate of output in this economy is constant over time.

(d) Please refer to the definition below:

Definition 1. A competitive equilibrium is a sequence of allocations $\{c_t(t), c_t(t+1), m_t, h_t\}_{t=1}^{\infty}$ and wages $\{w_t, v_t\}_{t=1}^{\infty}$ such that:

(i) Given wages and the human capital accumulation equation, the allocation solves the individual's utility maximization problem.

(ii) Given wages, the firm's profit function is maximized.

(iii) All markets clear.

(e) The budget constraints are now given by:

$$\begin{cases} \text{Young } \rightarrow c_t(t) = w_t m_t - T_1 \\ \text{Old } \rightarrow c_t(t+1) = v_{t+1}\big[h_t + (1-m_t)\theta h_t\big] + T_2 \end{cases}$$

Since there is no population growth and the government budget constraint must be balanced, we know that in equilibirum we must have $T_1 = T_2 = T$. The equilibrium condition will be the same as in question (b) so we can once again make use of our budget constraints:

$$\begin{aligned} \frac{c_t(t+1)}{c_t(t)} &= \frac{v_{t+1}}{w_t}\theta h_t \\ \frac{v_{t+1}\big[h_t + (1-m_t)\theta h_t\big] + T_2}{w_t m_t - T_1} &= \frac{v_{t+1}}{w_t}\theta h_t \\ m_t &= \frac{1+\theta}{2\theta} + T\left[\frac{w_t + v_{t+1}\theta h_t}{2v_{t+1}\theta h_t w_t}\right] \end{aligned}$$

Which is greater than the previous fraction of time spent working[5], i.e. countries that run such a program will invest less in education than before. This happens because the lump-sum tax on the young decreases their consumption possibilities, meaning that they now choose to work more in order to retain their consumption patterns.

Exercise 3

Consider the following environment. Each generation consists of 50 type H individuals with a high earnings profile, $\omega_t^H = [3, 1]$, and 50 type L individuals

[5] Since $T, \theta > 0$ by assumption and w_t, v_{t+1} and h_t must be positive in equilibrium.

with a low earnings profile, $\omega_t^L = [\frac{9}{4}, 1]$. Each individual has preferences that can be captured by the utility function

$$u_t^h = \log c_t^h(t) + \beta c_t^h(t+1)$$

with $\beta = \frac{2}{3}$. When young, each individual can invest in accumulating human capital using a linear technology where 1 unit of endowments yields 1 unit of human capital, k_t^h. When old, each individual can use his/her human capital to produce the consumption good according to the following production function:

$$y_t^{\text{type i}}(t+1) = k_t^{\text{type i}} k_t^{\text{type j}}$$

There is no private borrowing/lending in this economy.

(a) Solve for the amount of human capital accumulated by each type of individual. Is there a coordination problem?

(b) Suppose the government wants to increase the level of schooling and introduces a subsidy at rate $\theta = 0.25$ on human capital accumulation. Assume the subsidy is financed by a proportional tax, τ, on the endowments of the old. Did the government succeed in its goal of increasing the level of schooling?

(c) Would a mandatory schooling system (a minimum level of human capital accumulation) have a different effect?

Answer:

We will solve the general problem, including the subsidy, θ, and the proportional tax, τ, and then go through each of the subquestions. The household problem is given by:

$$\max_{\{c_t^h(t), c_t^h(t+1), k_t^h\}} \ln c_t^h(t) + \beta c_t^h(t+1)$$
$$\text{s.t. } c_t^h(t) = \omega_t^h(t) - (1-\theta)k_t^h$$
$$c_t^h(t+1) = (1-\tau)\omega_t^h(t+1) + k_t^h k_t^j$$

for every $h = \{H, L\}$ and $j \neq h$. Our Lagrangian is given by:

$$\mathcal{L} = \ln c_t^h(t) + \beta c_t^h(t+1) + \mu(t)\Big[\omega_t^h(t) - (1-\theta)k_t^h - c_t^h(t)\Big] + \mu(t+1)\Big[(1-\tau)\omega_t^h(t+1) + k_t^h k_t^j - c_t^h(t+1)\Big]$$

Taking the FOC:

$\mathbf{c_t^h(t)}$: $\frac{1}{c_t^h(t)} - \mu(t) = 0, \forall t > 0$

$\mathbf{c_t^h(t+1)}$: $\beta - \mu(t+1) = 0, \forall t > 0$

$\mathbf{k_t^h}$: $-(1-\theta)\mu(t) + \mu(t+1)k_t^j = 0, \forall t > 0$

From which we get the following equilibrium condition (we substitute $c_t^h(t)$ for the budget constraint of the young agents):

$$\frac{1-\theta}{\omega_t^h(t) - (1-\theta)k_t^h} = \beta k_t^j, \forall t, \forall h, h \neq j$$

which is our Euler equation. Making use of the equilibrium conditions for both types of agents:

$$\frac{1-\theta}{\omega_t^H(t) - (1-\theta)k_t^H} = \beta k_t^L$$
$$\frac{1-\theta}{\omega_t^L(t) - (1-\theta)k_t^L} = \beta k_t^H$$

we can combine them to get relationship between the human capital investment of both types of agents:

$$k_t^L \left[\omega_t^H(t) - (1-\theta)k_t^H\right] = k_t^H \left[\omega_t^L(t) - (1-\theta)k_t^L\right]$$
$$k_t^L = \frac{\omega_t^L(t)}{\omega_t^H(t)} k_t^H$$

Substituting back in the Euler equation:

$$\frac{1-\theta}{\omega_t^H(t) - (1-\theta)k_t^H} = \beta k_t^L$$
$$\frac{1-\theta}{\omega_t^H(t) - (1-\theta)k_t^H} = \beta\Big(\frac{\omega_t^L(t)}{\omega_t^H(t)} k_t^H\Big)$$
$$\left(k_t^H\right)^2 - \frac{\omega_t^H(t)}{1-\theta} k_t^H + \frac{\omega_t^H(t)}{\beta\omega_t^L(t)} = 0$$

We now have a 2^{nd} degree equation from which we can obtain individual investment in human capital. Making use of the numerical values for the endomwents and β:

$$k_t^L = \frac{3}{4} k_t^H$$
$$\left(k_t^H\right)^2 - \frac{3}{1-\theta} k_t^H + 2 = 0$$

Let's now address each subquestion:

(a) In the baseline case, there is not tax nor subsidy, i.e. $\theta = \tau = 0$. Substituting:

$$\left(k_t^H\right)^2 - 3k_t^H + 2 = 0$$

which has the following solutions:

$$\begin{cases} k_t^H = 1 \Rightarrow k_t^L = \frac{3}{4} \\ k_t^H = 2 \Rightarrow k_t^L = \frac{3}{2} \end{cases}$$

We are in the presence of two solutions, one where both types of agents choose a high level of human capital investment and another where both types of agents choose a low level of human capital investment. There is a coordination problem since both types of agents would be better off in the scenario where both are in the equilibrium with high human capital investment.

(b) We now have a subsidy on human capital accumulation, $\theta = \frac{1}{4}$, and a proportional tax τ on the endowments of the old. Since the government budget constraint must be balanced in equilibrium, we have that:

$$\begin{aligned} \theta(k_t^H + k_t^L) &= \tau(\omega_{t-1}^H(t) + \omega_{t-1}^L(t)) \\ \theta(k_t^H + \frac{3}{4}k_t^H) &= \tau(1+1) \\ \tau &= \frac{7}{8}\theta k_t^H \end{aligned}$$

And in order to find the human capital investment, we must now solve:

$$\left(k_t^H\right)^2 - \frac{3}{1-\frac{1}{4}}k_t^H + 2 = 0$$

which has the following solutions:

$$\begin{cases} k_t^H = 2 - \sqrt{2} < 1 \Rightarrow k_t^L = \frac{3}{2} - \frac{3\sqrt{2}}{4} = \frac{6-3\sqrt{2}}{4} < \frac{3}{4} \\ k_t^H = 2 + \sqrt{2} > 2 \Rightarrow k_t^L = \frac{3}{2} + \frac{3\sqrt{2}}{4} = \frac{6+3\sqrt{2}}{4} > \frac{3}{2} \end{cases}$$

By comparing both cases, we cannot say with certainty whether the government succeeded in its goal of increasing the level of schooling, since it would depend on the amount of human capital accumulated by each type of individual (i.e. the solution under consideration). However, it is worth noting that the case where both types of agents choose a high level of schooling yields a higher utility than in the case without government intervention whereas the case where both types of agents choose a low level of schooling yields a lower utility than in the case without government intervention

(c) If the government were to set the mandatory level of human capital above the minimum level from the case with no taxes and subsidies (i.e. $\frac{3}{4}$), there would be an increase in human capital and, consequently, in the production of the consumption good. Therefore, a mandatory schooling system would have a different effect than the previous scheme since all agents would be forced to have a minimum level of human capital.