Advanced microeconomics problem set 7

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October 12, 2023

Exercise 5.32

Original question

Consider an economy with two consumers and one firm owned by consumer 1. There is a single consumption good and two time periods. The intertemporal utility functions are

$$u^i(x_1, x_2) = x_1 \cdot x_2,$$

where $e^1 = (19, 1)$ and $e^2 = (1, 9)$. The firm can store the good from period 1 to period 2 with technology $x_2(x_1) = -x_1$.

a) Suppose the consumers cannot trade. How much do they consume each period?

b) Suppose they can trade in a competitive economy. What are the Walrasian equilibrium prices and how much storage takes place?

c) interpret p_1 as spot price and p_2 as a futures price.

d) Suppose storage is costly. Show that the market makes both consumers strictly better off.

Solution

a) The firm maximizes profit, implying prices equal to

$$\pi = p_2 x_1 - p_1 x_1,$$

$$\frac{\partial \pi}{\partial x_1} = p_2 - p_1 = 0,$$

$$p_1 = p_2.$$

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The firm makes zero profit $\pi = 0$ with these prices. Normalizing $p_1 = 1$, we find the value of consumer 1's endowment is $p_1 x_1 + p_2 x_2 + \pi = 20$. The utility function represents Cobb-Douglas preferences, revealed by monotonic transformation $f(x) = x^{1/2}$, hence demand is

$$x_1^1 = \frac{y^1}{2p_1} = \frac{20}{2} = 10,$$

$$x_2^1 = \frac{y^1}{2p_2} = \frac{20}{2} = 10.$$

Consumer 1 is storing 9 units, the utility at this bundle is $u(x_1, x_2) = 100$. Consumer 2 cant trade and is therefore stuck with his/her endowment, it has utility level $u^2(e_1, e_2) = 9$.

b) If the economy opens up for trade and the firm is competitive, the firm again sets prices $p_1 = p_2$ and makes $\pi = 0$. Normalizing $p_1 = 1$, we find demand equal to

$$x_1^1 = \frac{y^1}{2} = 10,$$

$$x_2^1 = \frac{y^1}{2} = 10,$$

$$x_1^2 = \frac{y^2}{2} = 5,$$

$$x_2^2 = \frac{y^2}{2} = 5.$$

Consumer 1 stores 9 units and consumer 2 stores 4 units. The utlity for each individual is

$$u^1(x_1, x_2) = 10 \cdot 10 = 100,$$
$$u^2(x_1, x_2) = 5 \cdot 5 = 25.$$

c)

Exercise 5.39

In an economy with two types of consumer, each type has the respective utility function and endowments:

$$u^{1q}(x_1, x_2) = x_1 x_2 \qquad \text{and } e^1 = (8, 2),$$
$$u^{2q}(x_1, x_2) = x_1 x_2 \qquad \text{and } e^2 = (2, 8).$$

b) Characterise as precisely as possible the set of allocations that are in the core of this two-consumer economy.

c) Show that the allocation giving $x^{11} = (4,4)$ and $x^{21} = (6,6)$ is in the core.

d) Replicate this economy once so there are two consumers of each type, for a total of four consumers in the economy. Show that the double copy of the previous allocation, giving $x^{11} = x^{12} = (4,4)$ and $x^{21} = x^{22} = (6,6)$, is not in the core of the replicated economy.

Solution

b) DEFINITION 5.3: The core of an exchange economy with endowment e, denoted $C(e)$, is the set of all unblocked feasible allocations. In terms of the Edgeworth box, we require to be (1) 'in the box', (2) 'inside the lens', and (3) 'on the contract curve'.

(1) The allocations must exhaust all endowments: $x_1^1 + x_1^2 = 8 + 2$, $x_1^1 + x_1^2 = 2 + 8$.

(2) Each agent needs to be weakly better off than under autarky, i.e. $u^i(x^i) \geq u^i(e^i)$ for all $i \in \mathcal{I}$:

$$u^i(x^i) \geq u^1(e^1) = (8 * 2) = u^2(e^2) = (2 * 8).$$

(3) The consumers' indifference curves are tangent to each other:

$$\begin{aligned}
\frac{\frac{\partial u^{1q}(x_1,x_2)}{\partial x_1}}{\frac{\partial u^{1q}(x_1,x_2)}{\partial x_2}} &= \frac{\frac{\partial u^{2q}(x_1,x_2)}{\partial x_1}}{\frac{\partial u^{2q}(x_1,x_2)}{\partial x_2}} \\
\frac{x_2}{x_1} &= \frac{x_2}{x_1} = \frac{10 - x_2}{10 - x_1} \\
\frac{10 - x_1}{x_1} &= \frac{10 - x_2}{x_2} \\
\frac{10}{x_1} - 1 &= \frac{10}{x_2} - 1 \\
x_1 &= x_2
\end{aligned}$$

Conclude: The core is the set of allocations where $x_1 = x_2$ between $(x_1, x_2) = (4,4)$ for consumer 1 to $(x_1, x_2) = (6,6)$ for consumer 1.

c) $x^{11} = (4,4)$ and $x^{21} = (6,6)$ is in the core since

(1) $4 + 6 = 8 + 2$, $4 + 6 = 2 + 8$,

(2) $4 * 4 \geq 8 * 2$, $6 * 6 \geq 2 * 8$,

(3) $4 = 4$, $6 = 6$,

d) Consider the coalition $S = \{11, 12, 21\}$. Let

$$x_S^{11} = 1/2(e^1 + x^{11}), x_S^{12} = 1/2(e^1 + x^{12}), x_S^{21} = x^{21}.$$

The proposed allocation is feasible for the coalition:

$$1/2[(8,2)+(4,4)]+1/2[(8,2)+(4,4)]+(6,6)=(18,12)$$
$$2e^1+e^2=2(8,2)+(2,8)=(18,12)$$

The consumers in S prefer the proposed assignment of goods:

$$u^{11}(6,3)=18>u^{11}(4,4)=16$$
$$u^{12}(6,3)=18>u^{12}(4,4)=16$$
$$u^{21}(6,6)=36>u^{21}(6,6)=36$$

Conclude: The allocation $x^{11}=x^{12}=(4,4)$ and $x^{21}=x^{22}=(6,6)$, is not in the core of the replicated economy since it is blocked by S.

1 Exercise 5.40

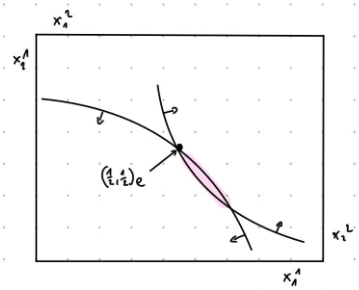
5.40 In a pure exchange economy, consumer i *envies* consumer j if $\mathbf{x}^j \succ^i \mathbf{x}^i$. (Thus, i envies j if i likes j's bundle better than his own.) An allocation $\mathbf{x}$ is therefore *envy free* if $\mathbf{x}^i \succsim^i \mathbf{x}^j$ for all i and j. We know that envy-free allocations will always exist, because the equal-division allocation, $\bar{\mathbf{x}} = (1/I)\mathbf{e}$, must be envy free. An allocation is called **fair** if it is both envy free *and* Pareto efficient.

(a) In an Edgeworth box, demonstrate that envy-free allocations need not be fair.

(b) Under Assumption 5.1 on utilities, prove that every exchange economy having a strictly positive aggregate endowment vector possesses at least one fair allocation.

(a) As stated above, the equal division allocation is always envy free.

However, it is rarely the case that it is Pareto efficient as well.



The shaded lens is a Pareto improvement for both consumers.

So the allocation is envy-free, but not fair.

(b) Note: The preference relation $x \succsim^i y$ is represented by: $u^i(x) \geq u^i(y)$.

Let I be the number of consumers.

Assumption 5.1: u^i is continuous, strongly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.

Under these assumptions we know that the utility maximization problem

$$v^i(p, y) = \max u^i(x^i) \quad \text{s.t.} \quad p \cdot x^i \leq p \cdot e^i \quad , \quad i \in I$$

has a unique solution and that the Marshallian demand is $x^i(p, p \cdot e^i)$,

according to theorem 5.1

THEOREM 5.1 **Basic Properties of Demand**

If u^i satisfies Assumption 5.1 then for each $\mathbf{p} \gg \mathbf{0}$, the consumer's problem (5.2) has a unique solution, $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$. In addition, $\mathbf{x}^i(\mathbf{p}, \mathbf{p} \cdot \mathbf{e}^i)$ is continuous in $\mathbf{p}$ on $\mathbb{R}^n_{++}$.

Additionally, all assumptions for theorem 5.5 are fulfilled:

THEOREM 5.5 **Existence of Walrasian Equilibrium**

If each consumer's utility function satisfies Assumption 5.1, and $\sum_{i=1}^{I} \mathbf{e}^i \gg \mathbf{0}$, then there exists at least one price vector, $\mathbf{p}^* \gg \mathbf{0}$, such that $\mathbf{z}(\mathbf{p}^*) = \mathbf{0}$.

Thus, we know that a Walrasian Equilibrium exists for p^*.

The WEA is accordingly: $x(p^*) = \left(x^1(p^*, p^* \cdot e^1), \dots, x^I(p^*, p^* e^I) \right)$

This allocation is Pareto efficient because of the first Welfare theorem.

- **THEOREM (First Welfare Theorem)**:
  Consider an exchange economy $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$. If each consumer's utility function u^i is strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation is Pareto efficient.

However, we still need to show that the allocation is envy-free in order to show fairness. We need to show that $u^i(x^i) \geq u^i(x^j) \quad \forall i \neq j \in I$

Plug in the allocation obtained from the WE:

$$u^i\left(x^i(p^*, p^* \cdot e^i)\right) \geq u^i\left(x^j(p^*, p^* \cdot e^i)\right)$$

The left side is exactly the indirect utility function $v^i(p^*, p^* \cdot e^i)$ and by definition the indirect utility function denotes the highest possible utility for consumer i, given the budget restrictions.

Accordingly, $v^i(p^*, p^* \cdot e^i) = u^i\left(x^i(p^*, p^* \cdot e^i)\right) \geq u^i\left(x^j(p^*, p^* \cdot e^i)\right)$

holds and consumer i is envy-free.

We conclude, that $x(p^*)$ is not only PE but also envy-free.

Thus, it is fair. $\square$