Introductory Statistics

2024 Lectures
Part 5 - Probability

Institute of Economic Studies
Faculty of Social Sciences
Charles University in Prague

Classical/logical definition of probability

- A mathematical definition of probability changed from its earliest formulation as a measure of belief to the modern approach of defining through Kolmogorov's axioms

Definition 12: (classical/logical definition of probability)
Let S be the sample space for an experiment having a finite number $N(S)$ of equally likely outcomes, and let $A \subset S$ be an event containing $N(A)$ elements. Then the probability of the event A, denoted by $P(A)$, is given by $P(A) = N(A)/N(S)$.

- Useful for simple experiments with finite sample space, such as tossing a fair die, fair coin etc.
- Two major limitations: sample space must be finite; and all outcomes of the experiment must be equally likely.

Frequency definition of probability

Example 17: Consider a biased coin. How would you find the probability of heads?

Definition 13: (frequency definition of probability)
Let n be the number of times that an experiment is repeated under identical conditions. Let A be an event in the sample space S, and define $n(A)$ to be the number of times in n repetitions of the experiment that the event A occurs. Then the probability of event A is equal to $P(A) = \lim_{n\to\infty} \frac{n(A)}{n}$.

- It can be applied for infinite sample space and outcomes need not be equally likely.
- Limitations: some experiments cannot be repeated under identical conditions (or at all). Also, it it may not be clear (but will be explained in Statistics JEB105), why is a convergence guaranteed for any such sequence of repetitions of an experiment and is the limit unique?

Subjective definition of probability

Example 18: How would you find probability of the event “the economy will be in recession next year”?

Definition 14: (subjective definition of probability)
A real number, $P(A)$, contained in $[0, 1]$ and chosen to express the degree of personal belief in the likelihood of occurrence or validity of event A, the number 1 being associated with certainty.

- It can be applied for infinite sample space, outcomes not equally likely and experiments which cannot be repeated
- Limitations: the subjective probability assigned to an event can vary depending on who is assigning the probabilities and the personal beliefs of the individual assigning the probabilities (even under identical information). Also, one needs to impose some rationality and coherence constraints on individuals.

Axiomatic/Kolmogorov definition of probability

Definition 15: (axiomatic definition of probability)
Suppose that $\mathcal{A}$ is a σ-algebra of subsets of S. Then probability P is defined as a set function on $\mathcal{A}$ satisfying

a) nonnegativity: For any event $A \in \mathcal{A}$:

$$P(A) \geq 0;$$

b) norming:

$$P(S) = 1;$$

c) countable additivity: For any sequence $A_1, A_2, \ldots$ of pairwise disjoint events $A_n \in \mathcal{A}, n \in \mathbb{N}$,

$$P(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n).$$

Definition 16: The triple $(S, \mathcal{A}, P)$ is called the probability space.

Example 19: A die is loaded such that the probability that the number i shows up is iK, $i = 1, \ldots, 6$, where K is a constant. Find the value of K.

Theorem 3: (consequences of the axioms)

a) $P(\emptyset) = 0$

b) $P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)$ for pairwise disjoint events A_i

c) if $A \subset B$ then $P(A) \leq P(B)$

d) if $A \subset B$ then $P(B - A) = P(B) - P(A)$

e) $P(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$

f) $P(A^c) = 1 - P(A)$

Warning! $P(A) = 0 \not\Rightarrow A = \emptyset$.

Example 20: (warning!)
Suppose that $P(A) = 0.4$, $P(B^c) = 0.6$, $P(A^c \cap B) = 0.1$. Find $P(A \cup B^c)$.

Theorem 4: (Inclusion and exclusion principle)
For any events $A_1, \ldots, A_n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i A_j) + \cdots + (-1)^{n-1} P(A_1 \ldots A_n).$$

Theorem 5: (Continuity of probability)
If $A_1, A_2, A_3, \ldots$ is either an increasing or decreasing sequence of events, then

$$\lim_{n \to \infty} P(A_n) = P(\lim_{n \to \infty} A_n).$$

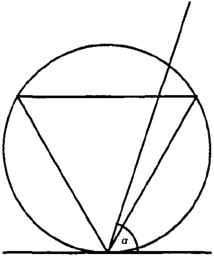
Special cases

- discrete probability:
  $S = \{s_1, \dots, s_n\}$
  Then $\mathcal{A}$ is a system of all subsets and for probability of any $A \in \mathcal{A}$ it suffices to identify $P(s_i) = p_i, \ \forall i = 1, \dots, n$. Then $P(A) = \sum_{i: s_i \in A} p_i$.
- geometric probability:
  $S \subset \mathbb{R}^n$ from Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ (generated by system of open subsets of $\mathbb{R}^n$) with finite and positive Lebesque measure (length, area, surface, ...) such that $\lambda(S) < \infty$.
  We can denote it also as $|S|$.
  Then probability of $A \in \mathcal{B}(S)$ is $P(A) = \frac{|A|}{|S|}$.

Example 21: Two ships using the same dock can arrive anytime during next 24 hours. One ship occupies dock for 1 hour, the other needs to spend 2 hours in docks. What is the probability that neither of the ships will wait to get to the docks?

Example 22: (Bertrand paradox)
We draw at random a chord onto a circle. What is the probability that it is longer than the side of the inscribed equilateral triangle?

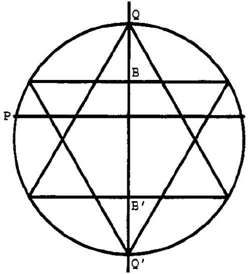
Solution 1: Choose point A as one of the ends of the chord. The chord is uniquely determined by the angle α chosen randomly from interval $(0, \pi)$. Favourable are angles from interval $[\frac{\pi}{3}, \frac{2\pi}{3}]$. Thus, the answer to the question is 1/3.



Bertrand paradox

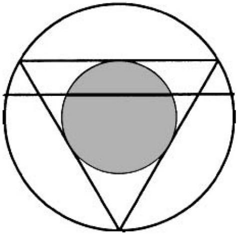
Example 22: (Bertrand paradox)
We draw at random a chord onto a circle. What is the probability that it is longer than the side of the inscribed equilateral triangle?

Solution 2: Let us draw a diameter QQ' perpendicular to the chord P. The chord is uniquely determined by a point on QQ'. Favourable are points on the line BB'. So, the answer is 1/2.

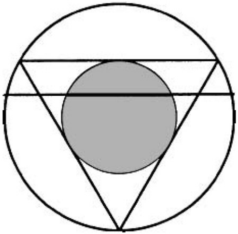


Example 22: (Bertrand paradox)
We draw at random a chord onto a circle. What is the probability that it is longer than the side of the inscribed equilateral triangle?

Solution 3: The location of the chord is uniquely determined by the location of its center (except when the center coincides with the center of the circle, which is an event with probability zero). Favourable are chords with centers inside the circle inscribed in the equilateral triangle. So, the answer is 1/4.



Example 22: (Bertrand paradox)
We draw at random a chord onto a circle. What is the probability that it is longer than the side of the inscribed equilateral triangle?



Solution 3: The location of the chord is uniquely determined by the location of its center (except when the center coincides with the center of the circle, which is an event with probability zero). Favourable are chords with centers inside the circle inscribed in the equilateral triangle. So, the answer is 1/4.

Three solutions, all valid, but yielding inconsistent results. Why?