Inequalities, Household Behavior and the Macroeconomy
(Consumption - Basic Model)

Course Director: Zoltán Rácz

SSE, Department of Finance

April 2, 2024

Today

- Today we start our study of the standard consumption-saving problem, ...
- AKA permanent income hypothesis, lifecycle-model, or buffer-stock model. All these terms mean slightly different things, but they are related.
- This model is somewhat complex and will take a few lectures. Our strategy: introduce the full model, then try to understand it from several angles through various simplifications.
- These lectures will most likely look a bit dry relative to stuff to come later on.
- We will sprinkle in some empirics here and there and will discuss a couple of classic papers.

Why?

Three reasons why this model is important for talking about inequality:

- People often care about the welfare implications of income/wealth inequality. Hard to talk about welfare without modeling consumption.
- Consumption/saving decisions affect inequality.
- One might want to lower inequality by taxation. To understand the full effect of policy on welfare and inequality, we need a model of saving behavior.

General Model

- Decision maker: individual or household (interpretation varies across papers)
- Earns income, decides on consumption and saving
- Studying saving decisions is pointless without more time periods! $\Rightarrow$ need a dynamic model
- Income is uncertain by nature. $\Rightarrow$ need a stochastic model

Environment

- Finite time periods, labeled by $t = 0, 1, 2, \ldots, T$.
- Labor income is exogenous, stochastic and given by $y_t \geq 0$ in period t.

Environment

- Finite time periods, labeled by $t = 0, 1, 2, \ldots, T$.
- Labor income is exogenous, stochastic and given by $y_t \geq 0$ in period t.
- There is a constant (positive) interest rate r on savings a: the budget constraint is:

$$a_t = (1 + r)a_{t-1} + y_t - c_t \qquad \forall t \in \{0, 1, 2, \ldots, T\}$$

Saving today (a_t) equals:

- After-return savings from the previous period
- $+$ labor income
- $-$ consumption

Environment

- Finite time periods, labeled by $t = 0, 1, 2, \ldots, T$.
- Labor income is exogenous, stochastic and given by $y_t \geq 0$ in period t.
- There is a constant (positive) interest rate r on savings a: the budget constraint is:

$$a_t = (1 + r)a_{t-1} + y_t - c_t \qquad \forall t \in \{0, 1, 2, \ldots, T\}$$

  Saving today (a_t) equals:
  - After-return savings from the previous period
  - $+$ labor income
  - $-$ consumption
- Cannot end the final period with debt, so $a_T \geq 0$. Why?

Environment

- Finite time periods, labeled by $t = 0, 1, 2, \ldots, T$.
- Labor income is exogenous, stochastic and given by $y_t \geq 0$ in period t.
- There is a constant (positive) interest rate r on savings a: the budget constraint is:

$$a_t = (1 + r)a_{t-1} + y_t - c_t \qquad \forall t \in \{0, 1, 2, \ldots, T\}$$

  Saving today (a_t) equals:
  - After-return savings from the previous period
  - $+$ labor income
  - $-$ consumption
- Cannot end the final period with debt, so $a_T \geq 0$. Why?
- Sometimes we restrict borrowing in all time periods, i.e. savings is not allowed to go under a borrowing limit, so $a_t \geq -b_t \quad \forall t$.

Environment

- Finite time periods, labeled by $t = 0, 1, 2, \ldots, T$.
- Labor income is exogenous, stochastic and given by $y_t \geq 0$ in period t.
- There is a constant (positive) interest rate r on savings a: the budget constraint is:

$$a_t = (1+r)a_{t-1} + y_t - c_t \qquad \forall t \in \{0, 1, 2, \ldots, T\}$$

  Saving today (a_t) equals:
  - After-return savings from the previous period
  - $+$ labor income
  - $-$ consumption
- Cannot end the final period with debt, so $a_T \geq 0$. Why?
- Sometimes we restrict borrowing in all time periods, i.e. savings is not allowed to go under a borrowing limit, so $a_t \geq -b_t \quad \forall t$.
- Consumption has to be non-negative (usually). Most often we use preferences that ensure this without assuming it.

Environment

- Finite time periods, labeled by $t = 0, 1, 2, \ldots, T$.
- Labor income is exogenous, stochastic and given by $y_t \geq 0$ in period t.
- There is a constant (positive) interest rate r on savings a: the budget constraint is:

$$a_t = (1 + r)a_{t-1} + y_t - c_t \qquad \forall t \in \{0, 1, 2, \ldots, T\}$$

  Saving today (a_t) equals:
  - After-return savings from the previous period
  - $+$ labor income
  - $-$ consumption
- Cannot end the final period with debt, so $a_T \geq 0$. Why?
- Sometimes we restrict borrowing in all time periods, i.e. savings is not allowed to go under a borrowing limit, so $a_t \geq -b_t \quad \forall t$.
- Consumption has to be non-negative (usually). Most often we use preferences that ensure this without assuming it.
- We denote $w_t = (1 + r)a_{t-1}$ (wealth is after-returns savings from the previous period). Starting wealth (w_0) is non-negative.

Preferences

The agent derives utility from the series of her consumption choices. So we can write utility as

$$U(c_0, c_1, c_2, \ldots, c_T)$$

This is obviously way too general to work with. Two steps of simplification (usually):

Preferences

The agent derives utility from the series of her consumption choices. So we can write utility as

$$U(c_0, c_1, c_2, \ldots, c_T)$$

This is obviously way too general to work with. Two steps of simplification (usually):

1. **Additive separability:**

$$U(c_0, c_1, c_2, \ldots, c_T) = u_0(c_0) + u_1(c_1) + u_2(c_2) + \cdots + u_T(c_T)$$

- Equivalent to $\partial U(c_0, c_1, c_2, \ldots, c_T)/\partial c_t$ being only a function of c_t. Is this sensible? Yes and no: Habits.

Preferences

The agent derives utility from the series of her consumption choices. So we can write utility as

$$U(c_0, c_1, c_2, \ldots, c_T)$$

This is obviously way too general to work with. Two steps of simplification (usually):

1. **Additive separability:**

$$U(c_0, c_1, c_2, \ldots, c_T) = u_0(c_0) + u_1(c_1) + u_2(c_2) + \cdots + u_T(c_T)$$

   - Equivalent to $\partial U(c_0, c_1, c_2, \ldots, c_T)/\partial c_t$ being only a function of c_t. Is this sensible? Yes and no: Habits.

2. **Exponential discounting:**

$$U(c_0, c_1, c_2, \ldots, c_T) = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots + \beta^T u(c_T)$$

where $\beta < 1$ is the discount factor and u is increasing and concave.

   - Only two objects (β and u) are sufficient to describe preferences for any number of periods.
   - Time preferences have a constant structure $\rightarrow$ helps when solving the model

Putting things together

The program of the agent can thus be written as:

$$\max_{\{c_t, a_t\}_{t=0}^{T}} E_0 \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\begin{aligned} s.t. \quad & a_t = (1+r)a_{t-1} + y_t - c_t \quad \forall t \in \{0, 1, \dots, T\} \\ & a_T \geq 0 \\ & a_{-1} \text{ given} \end{aligned}$$

Putting things together

The program of the agent can thus be written as:

$$\max_{\{c_t, a_t\}_{t=0}^{T}} E_0 \sum_{t=0}^{T} \beta^t u(c_t)$$

$$\begin{aligned} s.t. \quad a_t &= (1+r)a_{t-1} + y_t - c_t \quad \forall t \in \{0, 1, \ldots, T\} \\ a_T &\geq 0 \\ a_{-1} &\text{ given} \end{aligned}$$

- Two ways of solving the problem
  - Sequential approach: solve model at once. More straightforward intuition, we start with this
  - Dynamic programming: solve $T + 1$ one-period problems and combine their solutions into one. Easier to implement when programming.
- First we discuss the deterministic model, then stochastic

Deterministic problem - sequential approach

- Deterministic problem: the income sequence $\{y_t\}$ is known at time 0. We can drop the expectation operator.
- Note that in optimum s_T has to be 0.
- Maximize a multivariate function subject to equality constraints $\Rightarrow$ apply Lagrangian

Deterministic problem - sequential approach

- Deterministic problem: the income sequence $\{y_t\}$ is known at time 0. We can drop the expectation operator.
- Note that in optimum s_T has to be 0.
- Maximize a multivariate function subject to equality constraints $\Rightarrow$ apply Lagrangian

Original structure

- one constraint for each time period: $a_t = (1+r)a_{t-1} + y_t - c_t$
- optimize over c_t and a_t for all ts ($T+2$ constraints and $2(T+1)$ variables)

Consolidated budget constraint

- we can sum the $T+1$ budget contraints into one
- optimize over c_t for all ts (1 constraint and $T+1$ variables)

We do the first option, since easier to generalize to the case with borrowing constraints.

Deterministic problem - sequential approach

We should take a look at the FOCs (first order conditions...)

To do so, we first need to set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \{u(c_t) + \lambda_t [a_{t-1}(1+r) + y_t - c_t - a_t]\}$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \implies u'(c_t) = \lambda_t$$

$$\frac{\partial \mathcal{L}}{\partial a_t} = 0 \implies \lambda_{t+1}\beta(1+r) = \lambda_t$$

Deterministic problem - sequential approach

We should take a look at the FOCs (first order conditions...)

To do so, we first need to set up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t \{u(c_t) + \lambda_t [a_{t-1}(1+r) + y_t - c_t - a_t]\}$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \implies u'(c_t) = \lambda_t$$
$$\frac{\partial \mathcal{L}}{\partial a_t} = 0 \implies \lambda_{t+1}\beta(1+r) = \lambda_t$$

Combining the two equations above we get the Euler equation:

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

Euler-equation

Optimal decisions are characterized by the Euler equation:

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

After rearranging

$$\frac{1}{1+r} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

Euler-equation

Optimal decisions are characterized by the Euler equation:

$$u'(c_t) = \beta(1+r)u'(c_{t+1})$$

After rearranging

$$\frac{1}{1+r} = \frac{\beta u'(c_{t+1})}{u'(c_t)}$$

- LHS: price of period $t+1$ consumption in terms of period t consumption
- RHS: marginal rate of substitution of consumption in periods $t+1$ vs t

Euler-equation

Optimal decisions are characterized by the Euler equation:

$$u'(c_t) = \beta(1 + r)u'(c_{t+1})$$

- The decision maker wants to smooth marginal utilities over time, ...
- ... taking into account impatiance (β) and the price of moving consumption across periods (r).
- r is a small positive number, which is more or less constant across time and households
- a reasonable β is probably slightly below 1.

An equivalent condition can be also derived in stochastic settings. Let's see how!

Stochastic problem - sequential approach

- We again want to solve a constrained optimization problem where our decision variables are c_t and a_t for all periods $t \in \{0, 1, \ldots, T\}$
- But our decision variables might be stochastic: for example, the optimal c_2 might need to depend on income received up to period 2.

Stochastic problem - sequential approach

- We again want to solve a constrained optimization problem where our decision variables are c_t and a_t for all periods $t \in \{0, 1, \ldots, T\}$
- But our decision variables might be stochastic: for example, the optimal c_2 might need to depend on income received up to period 2.
- Optimizing with respect to random variables sounds confusing. How to go around it?
  - We assume there are only finitely many possible histories throughout the time span of the model,
  - and optimize with respect to decisions conditional on every possible history!

Stochastic problem - sequential approach

Notation:

- $z^t = [z_1, z_2, ..., z_t]$ represents a history of all relevant variables up to period t. E.g. z_t might directly affect labor income y_t, but it could also bring information about future income $y_{t+1}, y_{t+2}....$
- Period t consumption and savings depend on z^t: we have to pick an optimal c and a conditional on each possible history up to that point!
- $\pi(z^t)$ is the measure related to z^t ($=$ the probability that history z^t is realized)

Stochastic problem - sequential approach

Notation:

- $z^t = [z_1, z_2, ..., z_t]$ represents a history of all relevant variables up to period t. E.g. z_t might directly affect labor income y_t, but it could also bring information about future income $y_{t+1}, y_{t+2}....$
- Period t consumption and savings depend on z^t: we have to pick an optimal c and a conditional on each possible history up to that point!
- $\pi(z^t)$ is the measure related to z^t ($=$ the probability that history z^t is realized)

This notation gives us two things:

1. we can write expected utility as a sum:

$$\mathbb{E}[u(c_t)] = \sum_{z^t} \pi(z^t) u(c_t(z^t))$$

2. we have a formal way to say that the budget constraints have to hold in every **history**:

$$a_t(z^t) = a_{t-1}(z^{t-1})(1+r) + y_t(z^t) - c_t(z^t) \quad \forall t, z^t$$

Stochastic problem - sequential approach

The Lagrangian could be written as

$$\mathcal{L} = \sum_{t=0}^{T} \{\beta^t \sum_{z^t} \pi(z^t)[u(c_t(z^t)) + \lambda_t(z^t)[a_{t-1}(z^{t-1})(1+r) + y_t(z^t) - c_t(z^t) - a_t(z^t)]]\}$$

The FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_t(z_t)} = 0 \implies u'(c_t(z_t)) = \lambda_t(z_t)$$
$$\frac{\partial \mathcal{L}}{\partial a_t(z_t)} = 0 \implies \sum_{z^{t+1}=(z^t, z_{t+1})} \pi(z^{t+1})\lambda_{t+1}(z^{t+1})\beta(1+r) = \pi(z^t)\lambda_t(z_t)$$

- Note: In the second FOC we are summing only over those histories z^{t+1} which are continuations of history z^t!
- If z^{t+1} is a continuation of history z^t, then $\pi(z^{t+1})/\pi(z^t)$ equals the conditional expectation that z^{t+1} will happen, given that z^t took place.

Stochastic Euler-equation

The two FOCs imply

$$u'(c_t(z_t)) = \beta(1+r) \sum_{z^{t+1}=(z^t, z_{t+1})} \frac{\pi(z^{t+1})}{\pi(z^t)} u'(c_{t+1}(z_{t+1}))$$

so we end up with the stochastic version of the Euler equation:

$$u'(c_t) = \beta(1+r)\mathbb{E}_t\left[u'(c_{t+1})\right]$$

- Same intuition as deterministic case,
- except that we have the conditional expectation of marginal utility on the RHS.

Simplifications

$$u'(c_t) = \beta(1+r)\mathbb{E}_t\left[u'(c_{t+1})\right]$$

Unfortunately, it is hard to work with expectations and non-linear functions at once. Two approaches:

Simplifications

$$u'(c_t) = \beta(1+r)\mathbb{E}_t\left[u'(c_{t+1})\right]$$

Unfortunately, it is hard to work with expectations and non-linear functions at once. Two approaches:

Learn to live with it:

- numerical solutions
- learn some more about how risk affects consumption decisions
- we do this most of the course

Simplifications

$$u'(c_t) = \beta(1+r)\mathbb{E}_t\left[u'(c_{t+1})\right]$$

Unfortunately, it is hard to work with expectations and non-linear functions at once. Two approaches:

Learn to live with it:

- numerical solutions
- learn some more about how risk affects consumption decisions
- we do this most of the course

Simplify using one of the approaches below:

1. Forget about uncertainty and go back to the deterministic model
2. Assume u' is a linear function of c
3. Much richer asset structure $\Rightarrow$ complete markets

Simplifications

$$u'(c_t) = \beta(1+r)\mathbb{E}_t\left[u'(c_{t+1})\right]$$

Unfortunately, it is hard to work with expectations and non-linear functions at once. Two approaches:

Learn to live with it:

- numerical solutions
- learn some more about how risk affects consumption decisions
- we do this from next class

Simplify using one of the approaches below:

1. Forget about uncertainty and go back to the deterministic model
2. Assume u' is a linear function of c
3. Much richer asset structure $\Rightarrow$ complete markets

Solution of Deterministic Model

We saw that the Euler-equation

$$u'(c_t) = \beta(1 + r)u'(c_{t+1})$$

holds for all periods.

- $\Rightarrow$ c_0 determines the entire consumption path up to period T
- higher c_0 implies a uniformly higher consumption profile and vice versa. This means:
  - if c_0 is too high, the implied consumption path will violate some of the constraints.
  - if c_0 is too low, the implied consumption path is not optimal, since a_T will be positive.

Solution of Deterministic Model

We saw that the Euler-equation

$$u'(c_t) = \beta(1 + r)u'(c_{t+1})$$

holds for all periods.

- $\Rightarrow \quad c_0$ determines the entire consumption path up to period T
- higher c_0 implies a uniformly higher consumption profile and vice versa. This means:
  - if c_0 is too high, the implied consumption path will violate some of the constraints.
  - if c_0 is too low, the implied consumption path is not optimal, since a_T will be positive.

We are looking for a c_0, such that with the implied consumption path:

1. the budget constraint is satisfied in all periods
2. $a_T = 0$

To find this initial consumption level (and hence the entire optimal path), we first derive the consolidated budget constraint.

Consolidated budget constraint

We can rearrange the budget constraint for any period t as

$$\frac{a_t}{1+r} + \frac{c_t - y_t}{1+r} = a_{t-1} \tag{1}$$

Start from budget constraint of period 0:

$$a_0 + c_0 - y_0 = w_0$$

Substituting out a_0 by using (1) for $t = 1$ we get

$$\frac{a_1}{1+r} + \frac{c_1 - y_1}{1+r} + c_0 - y_0 = w_0$$

Iterating this procedure for all t up to T we end up with

$$\frac{a_T}{(1+r)^T} + \sum_{t=0}^{T} \frac{c_t - y_t}{(1+r)^t} = w_0$$

Consolidated budget constraint

we saw how to get:

$$\frac{a_T}{(1+r)^T} + \sum_{t=0}^{T} \frac{c_t - y_t}{(1+r)^t} = w_0$$

After imposing $a_T = 0$ and rearranging we obtain the consolidated budget constraint:

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = w_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

Consolidated budget constraint

we saw how to get:

$$\frac{a_T}{(1+r)^T} + \sum_{t=0}^{T} \frac{c_t - y_t}{(1+r)^t} = w_0$$

After imposing $a_T = 0$ and rearranging we obtain the consolidated budget constraint:

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = w_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

- The present value of total consumption equals the sum of the present value of total income and initial wealth.

Consolidated budget constraint

we saw how to get:

$$\frac{a_T}{(1+r)^T} + \sum_{t=0}^{T} \frac{c_t - y_t}{(1+r)^t} = w_0$$

After imposing $a_T = 0$ and rearranging we obtain the consolidated budget constraint:

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = w_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

- The present value of total consumption equals the sum of the present value of total income and initial wealth.
- Technically we showed: the sequence of budget constraints implies the consolidated budget constraint. Is the opposite statement true?
  Yes! But only when:
  - no uncertainty and
  - no budget constraints

Consolidated budget constraint

We can view the **deterministic** saving problem **without borrowing constraints** as a simple optimal consumption problem:

$$\max_{\{c_t\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to:

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = w_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

- the consumer allocates her wealth over $T+1$ 'goods', which correspond to consumption in different time periods;
- relative price of c_t and c_0 is $1/(1+r)^t$;
- total wealth is initial wealth plus discounted labor income over the horizon (i.e. human capital)

Therefore in this case the dynamic structure of the problem is irrelevant.

Solution of Deterministic Model

If we assume

- $\beta(1+r) = 1$ (no relative impatience)

the Euler-equation becomes

$$u'(c_t) = u'(c_{t+1})$$

implying

$$c_t = c_{t+1}$$

Consumption is constant over the life cycle!

- Without assuming $\beta(1+r) = 1$ we could still get a closed-form solution for some utility functions (e.g. quadratic, CRRA, CARA)

Solution of Deterministic Model

So for all period t, we have

$$c_t = c_0$$

Recall the consolidated budget constraint:

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} = w_0 + \sum_{t=0}^{T} \frac{y_t}{(1+r)^t}$$

The two conditions provide us with the solution:

$$c_t = \tilde{R}\left[w_0 + \sum_{s=0}^{T} \frac{y_s}{(1+r)^s}\right] \quad \forall t \in \{0, 1, 2, \ldots, T\}$$

where

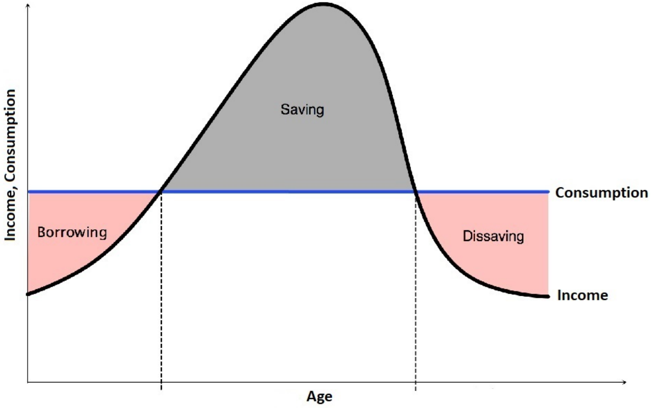
$$\frac{1}{\tilde{R}} = \sum_{t=0}^{T} \frac{1}{(1+r)^t} = \frac{1 - \left(\frac{1}{1+r}\right)^{T+1}}{1 - \frac{1}{1+r}}$$

The agent spreads initial wealth + human capital evenly across all periods, consumes its annuity value. 'Permanent Income Hypothesis' (PIH)

Deterministic model: constant consumption

What does the constant consumption condition imply for savings?

Assume that y_t is hump-shaped over the life cycle (this is the case in the data...)



Consumption over the life-cyle: what does the data say?

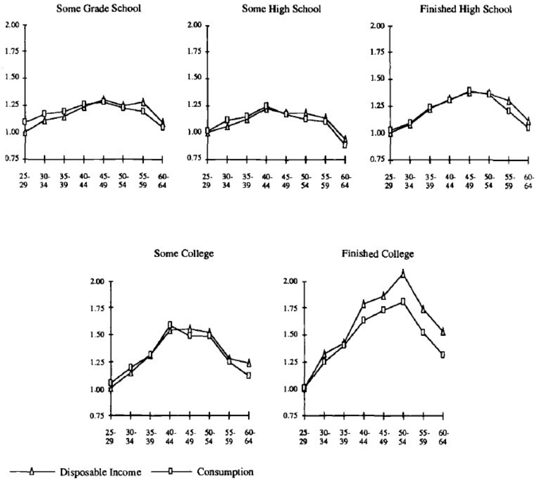
Does the PIH match the facts?

No!

In particular:

1. Consumption tracks labor income over the life cycle... - **Consumption is Hump Shaped**
2. Consumption Drops at retirement - **Consumption Retirement Puzzle**

Average income and consumption over age; 1960-1961 CES



Deterministic model: taking stock

- Our model implied something that wildly contradicts data. What should we do?

Deterministic model: taking stock

- Our model implied something that wildly contradicts data. What should we do?
- Let's revisit our assumptions!

Assumptions:

1. $\beta(1+r) = 1$
2. No uncertainty
3. No borrowing constraints

Deterministic model: taking stock

- Our model implied something that wildly contradicts data. What should we do?
- Let's revisit our assumptions!

Assumptions:

1. $\beta(1 + r) = 1$
   - This actually doesn't affect smoothing that much with realistic parameter values
2. No uncertainty
3. No borrowing constraints

Deterministic model: taking stock

- Our model implied something that wildly contradicts data. What should we do?
- Let's revisit our assumptions!

Assumptions:

1. $\beta(1+r) = 1$
   - This actually doesn't affect smoothing that much with realistic parameter values
2. No uncertainty
   - We check this next - in the simplest way possible
3. No borrowing constraints

Deterministic model: taking stock

- Our model implied something that wildly contradicts data. What should we do?
- Let's revisit our assumptions!

Assumptions:

1. $\beta(1 + r) = 1$
   - This actually doesn't affect smoothing that much with realistic parameter values
2. No uncertainty
   - We check this next - in the simplest way possible
3. No borrowing constraints
   - Next lecture

Simplifications

$$u'(c_t) = \beta(1+r)\mathbb{E}_t\left[u'(c_{t+1})\right]$$

Unfortunately, it is hard to work with expectations and non-linear functions at once. Two approaches:

Learn to live with it:

- numerical solutions
- learn some more about how risk affects consumption decisions
- we do this from next class

Simplify using one of the approaches below:

1. Forget about uncertainty and go back to the deterministic model
2. Assume u' is a linear function of c
3. Much richer asset structure $\Rightarrow$ complete markets

Stochastic model with quadratic utility

- Allow income to be stochastic. How to keep the model tractable?
- We need all these:
  - assume a quadratic utility function:

$$u(c) = -(c - \overline{c})^2$$

  - assume the bliss point $\overline{c}$ is never reached.
  - allow negative consumption
- Only for convenience, we also assume $\beta(1 + r) = 1$.

Stochastic model with quadratic utility

- Allow income to be stochastic. How to keep the model tractable?
- We need all these:
  - assume a quadratic utility function:

$$u(c) = -(c - \overline{c})^2$$

  - assume the bliss point $\overline{c}$ is never reached.
  - allow negative consumption
- Only for convenience, we also assume $\beta(1+r) = 1$.

Then the Euler-equation $u'(c_t) = \beta(1+r)\mathbb{E}_t\big[u'(c_{t+1})\big]$ simplifies to

$$-(c_t - \overline{c}) = \mathbb{E}_t\big[-(c_{t+1} - \overline{c})\big]$$

implying consumption is a random walk:

$$c_t = \mathbb{E}_t\big[c_{t+1}\big]$$

Stochastic model with quadratic utility

If consumption is a random walk, then

- consumption growth equals a forecast error:

$$c_{t+1} - c_t = c_{t+1} - \mathbb{E}_t\left[c_{t+1}\right]$$

- ... which should be uncorrelated to everything known at time t or before.
- This is a testable implication!
- So people have tried and

Stochastic model with quadratic utility

If consumption is a random walk, then

- consumption growth equals a forecast error:

$$c_{t+1} - c_t = c_{t+1} - \mathbb{E}_t\left[c_{t+1}\right]$$

- ... which should be uncorrelated to everything known at time t or before.
- This is a testable implication!
- So people have tried and **failed**.

So what to do if consumption is not a random walk?

(before throwing our new assumptions out the window...)

- We can assume our model holds only for a subset of the population! Why is this interesting?
  - We learn how relevant economically the deviation is between reality and a PIH model
  - We can assess whether the implications on the rest of the population (i.e. not PIHG consumers) look reasonable
- Campbell and Mankiw (1989) do this!

Campbell and Mankiw (1989)

A famous test of the PIH: assume there are two types of agents in the economy:

1. λ share of all consumers always consume their current income ('hand-to-mouth' consumers): $\Delta c_{1,t} = \Delta y_t$
2. $1-\lambda$ PIH consumers $\Delta c_{2,t} = \epsilon_t$ (ϵ_t is due to unexpected changes in income)

Aggregate consumption growth in this economy is:

$$\lambda \Delta c_{1,t} + (1-\lambda)\Delta c_{2,t} = \lambda \Delta y_t + (1-\lambda)\epsilon_t$$

We can test $\lambda = 0$ by regressing consumption growth on income growth...

Campbell and Mankiw (1989)

A famous test of the PIH: assume there are two types of agents in the economy:

1. λ share of all consumers always consume their current income ('hand-to-mouth' consumers): $\Delta c_{1,t} = \Delta y_t$
2. $1 - \lambda$ PIH consumers $\Delta c_{2,t} = \epsilon_t$ (ϵ_t is due to unexpected changes in income)

Aggregate consumption growth in this economy is:

$$\lambda \Delta c_{1,t} + (1 - \lambda)\Delta c_{2,t} = \lambda \Delta y_t + (1 - \lambda)\epsilon_t$$

We can test $\lambda = 0$ by regressing consumption growth on income growth...

...OLS would be biased, though! Can you say why?

Campbell and Mankiw (1989) circumvent this problem with an IV regression, instrumenting current income growth with lagged income and consumption growth.

PIH and the data

Estimated λ ranges from 0.35 to 0.7

A substantial fraction of hand-to-mouth consumers

How to explain hand-to-mouth consumers?

Two possibilities:

1. Keynesian agents, for whom $c_t = \alpha + \beta y_t$
2. A theory of borrowing constrained agents

Table 1 UNITED STATES 1953–1986
$\Delta c_y = \mu + \lambda \Delta y_t$

Row	Instruments	First-stage regressions: Δc equation	First-stage regressions: Δy equation	λ estimate (s.e.)	Test of restrictions
1	None (OLS)	—	—	0.316 (0.040)	—
2	$\Delta y_{t-2}, \ldots, \Delta y_{t-4}$	−0.005 (0.500)	0.009 (0.239)	0.417 (0.235)	−0.022 (0.944)
3	$\Delta y_{t-2}, \ldots, \Delta y_{t-6}$	0.017 (0.209)	0.026 (0.137)	0.506 (0.176)	−0.034 (0.961)
4	$\Delta c_{t-2}, \ldots, \Delta c_{t-4}$	0.024 (0.101)	0.045 (0.028)	0.419 (0.161)	−0.009 (0.409)
5	$\Delta c_{t-2}, \ldots, \Delta c_{t-6}$	0.081 (0.007)	0.079 (0.007)	0.523 (0.131)	−0.016 (0.572)
6	$\Delta i_{t-2}, \ldots, \Delta i_{t-4}$	0.061 (0.010)	0.028 (0.082)	0.698 (0.235)	−0.016 (0.660)
7	$\Delta i_{t-2}, \ldots, \Delta i_{t-6}$	0.102 (0.002)	0.082 (0.006)	0.584 (0.137)	−0.025 (0.781)
8	$\Delta y_{t-2}, \ldots, \Delta y_{t-4}$, $\Delta c_{t-2}, \ldots, \Delta c_{t-4}$, $c_{t-2} - y_{t-2}$	0.007 (0.341)	0.068 (0.024)	0.351 (0.119)	−0.033 (0.840)
9	$\Delta y_{t-2}, \ldots, \Delta y_{t-4}$, $\Delta c_{t-2}, \ldots, \Delta c_{t-4}$, $\Delta i_{t-2}, \ldots, \Delta i_{t-4}$, $c_{t-2} - y_{t-2}$	0.078 (0.026)	0.093 (0.013)	0.469 (0.106)	−0.029 (0.705)

Note: The columns labeled "First-stage regressions" report the adjusted R^2 for the OLS regressions of the two variables on the instruments; in parentheses is the p-value for the null that all the coefficients except the constant are zero. The column labeled "λ estimate" reports the IV estimate of λ and, in parentheses, its standard error. The column labeled "Test of restrictions" reports the adjusted R^2 of the OLS regression of the residual on the instruments; in parenthesis is the p-value for the null that all the coefficients are zero.

Solving the Stochastic model with quadratic utility

To gain more intuition, we can look a bit more in detail at the solution. We already saw:

$$c_t = \mathbb{E}_t\left[c_{t+1}\right]$$

Some derivations:

- By the law of iterated expectations (AKA 'tower property')

$$c_0 = \mathbb{E}_0\left[c_{t+1}\right]$$

 follows for all t.

- if the borrowing limit holds in all histories, then they also hold in expectation:

$$\mathbb{E}_0\left[\frac{a_t}{1+r}\right] + \mathbb{E}_0\left[\frac{c_t - y_t}{1+r}\right] = \mathbb{E}_0\left[a_{t-1}\right]$$

- From this we can derive the analog of the consolidated budget constraint:

$$\mathbb{E}_0\left[\sum_{t=0}^{T} \frac{c_t}{(1+r)^t}\right] = w_0 + \mathbb{E}_0\left[\sum_{t=0}^{T} \frac{y_t}{(1+r)^t}\right]$$

Solving the Stochastic model with quadratic utility

This implies a similar solution to the deterministic model:

$$c_0 = \tilde{R}_T \Big[w_0 + \mathbb{E}_0 \Big[\sum_{s=0}^{T} \frac{y_s}{(1+r)^s} \Big] \Big]$$

where

$$\frac{1}{\tilde{R}_T} = \sum_{t=0}^{T} \frac{1}{(1+r)^t} = \frac{1 - \left(\frac{1}{1+r}\right)^{T+1}}{1 - \frac{1}{1+r}}$$

Define human capital as

$$h_0 = \mathbb{E}_0 \Big[\sum_{s=0}^{T} \frac{y_s}{(1+r)^s} \Big]$$

so that we can write

$$c_0 = \tilde{R}_T \left[w_0 + h_0 \right]$$

Solving the Stochastic model with quadratic utility

Of course, we could also solve for optimal consumption at time 1! That would be

$$c_1 = \tilde{R}_{T-1}\left[w_1 + h_1\right]$$

This is related to c_0, since

- $w_1 = (1+r)[w_0 - c_0 + y_0]$
- $h_0 = y_0 + \frac{\mathbb{E}_0\left[h_1\right]}{1+r}$
- $(1 - \tilde{R}_T)(1+r) = \tilde{R}_{T-1}$

Putting all these together implies

$$c_1 - c_0 = \tilde{R}_{T-1}\left(h_1 - \mathbb{E}_0\left[h_1\right]\right)$$

or written it out

$$c_1 - c_0 = \tilde{R}_{T-1}\left(\sum_{s=1}^{T} \frac{y_s - \mathbb{E}_0\left[y_s\right]}{(1+r)^s}\right)$$

Therefore consumption changes due to surprises in current and expected income.