Duality is the equivalence between constrained maximization problems and an associated constrained minimization problem. The main recitation notes and the problem set give examples to show this equivalency. This brief note gives additional intuition based on the first-order conditions.

1 Primal Problem

Primal:

$$\begin{aligned} \max_{x,y} z &= f(x,y) \\ s.t.\ g(x,y) &= \bar{k} \end{aligned}$$

Primal Lagrangian:

$$\mathcal{L}_P = f(x,y) + \lambda_P(\bar{k} - g(x,y))$$

We'll take the first-order conditions with respect to our actual choice variables x and y along with our Lagrange multiplier λ_P.

Primal FOC:

$$\frac{\partial \mathcal{L}_P}{\partial x} = f_x(x,y) - \lambda_P g_x(x,y) = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}_P}{\partial y} = f_y(x,y) - \lambda_P g_y(x,y) = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}_P}{\partial \lambda_P} = \bar{k} - g(x,y) = 0 \tag{3}$$

Intuitively, $f_x(x,y)$ and $f_y(x,y)$ is the direct value of changing x and y on the objective we value, while $g_x(x,y)$ and $g_y(x,y)$ represent the indirect value of changing x and y based on the effect on the constraint. This indirect effect is scaled by the Lagrange multiplier λ_P, which measures how valuable it is to relax the constraint.

2 Preparing for duality equivalency

To prepare for showing the duality equivalency, let's divide both sides of (1) and (2) by -λ_P:

$$\frac{\partial \mathcal{L}_P}{\partial x} = g_x(x,y) - \frac{1}{\lambda_P} f_x(x,y) = 0 \tag{4}$$

$$\frac{\partial \mathcal{L}_P}{\partial y} = g_y(x,y) - \frac{1}{\lambda_P} f_y(x,y) = 0 \tag{5}$$

$$\frac{\partial \mathcal{L}_P}{\partial \lambda_P} = \bar{k} - g(x,y) = 0 \tag{6}$$

The above three equations implicitly define $z^* = f(x^*, y^*)$, where $g(x^*, y^*) = \overline{k}$ by (3).

To see where the dual problem is coming from, think about the optimization problem that would lead to equations (4)-(5). It looks like the objective would be $g(x, y)$, the constraint would be $f(x, y)$, and the lagrange multiplier would be $\frac{1}{\lambda_P}$. The constraint is set up in terms of $g(x, y) = \overline{k}$, but we know at the solution that an equivalent constraint that must hold is $f(x^*, y^*) = z^*$.

Do you think that "new" optimization problem should try to maximize or minimize $g(x, y)$? The formal mathematical way to prove this is with an object called the "Bordered Hessian", but let's simplify things by assuming that both $f(x, y)$ and $g(x, y)$ are strictly increasing in both x and y. In this maximization problem, we want to keep increasing x and y for the objective $f(x, y)$ but the constraint $g(x, y)$ is stopping us. $f(x, y)$ is the thing we like, but $g(x, y)$ is the thing we don't like. So in the "new" optimization problem where $g(x, y)$ is the objective and $f(x, y)$ is the constraint, it's natural to think that it will be a minimization problem.

Now let's explicitly set up that "new" minimization problem. This is referred to as the dual problem.

3 Dual Problem

Dual:

$$\begin{aligned} \min_{x,y} k &= g(x, y) \\ s.t.\ f(x, y) &= z^* \end{aligned}$$

Dual Lagrangian:

$$\mathcal{L}_D = g(x, y) + \lambda_D(z^* - f(x, y))$$

Dual FOC:

$$\frac{\partial \mathcal{L}_D}{\partial x} = g_x(x, y) - \lambda_D f_x(x, y) = 0 \tag{7}$$

$$\frac{\partial \mathcal{L}_D}{\partial y} = g_y(x, y) - \lambda_D f_y(x, y) = 0 \tag{8}$$

$$\frac{\partial \mathcal{L}_D}{\partial \lambda_D} = z^* - f(x, y) = 0 \tag{9}$$

Let's first compare the Lagrange multiplier FOCs of (6) vs. (9). We know that (6) is satisfied, by definition, by $x = x^*$ and $y = y^*$. These values of x and y produce $z^* = f(x^{ast}, y^*)$. Therefore the FOC in (9) will hold at those values!

Now let's compare the other FOCs between (4) and (5) vs. (7) and (8). If $\frac{1}{\lambda_D} = \lambda_P$, then these would be identical. But recall that x and y are the "actual" choices while the λ's are "artificial" choice variables. So if we use the primal solutions to the actual choices x^* and y^* and set the Lagrange multiplier $\lambda_D = \frac{1}{\lambda_P^*}$, then the FOCs (7)-(9) will all be satisfied. Duality!

4 Intuition Takeaway

Both maximization problems trade off benefits vs. costs. At the optimum, marginal benefits equal marginal costs for all choices. For simplicity, again suppose that both $f(x, y)$ and $g(x, y)$ are both increasing in x and y. The primal problem treats maximizing the objective $f(x, y)$ as the benefit, which comes at a cost of increasing the constraint $g(x, y)$. In contrast, the dual problem flips the script by treating keeping the constraint $g(x, y)$ as the benefit, which comes at a cost of decreasing the objective $f(x, y)$.