Advanced microeconomics problem set 1

Zhaoqin Zhu *

September 12, 2023

1 Exercise 1.17

1.1 Original question

Suppose that preferences are convex but not strictly convex. Give a clear and convincing argument that a solution to the consumer's problem still exists, but that it need not be unique. Illustrate your argument with a two-good example.
(Note: for the "not unique" part you only need to find an example)

1.2 Solution

First, we need to argue that a utility function exists
We assume here that the preference relation is complete, transitive, continuous, strictly monotonic and convex. We can apply the following theorem (Theorem 1.1 on p.14): If the binary relation $\succsim$ is complete, transitive, continuous and strictly monotonic, then there exists a continuous real valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ which represents $\succsim$

Next, we argue that solution to consumer problem exists
Consumer's problem is:

$$\max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \quad \text{such that} \quad \mathbf{p}\cdot\mathbf{x} \leq y \tag{1}$$

We apply Weierstrass's Theorem (check in your math textbook or notes in course 5301):

Theorem 1. *Let $f : S \to \mathbb{R}$ be a continuous, real valued function where S is a non-empty compact subset of $\mathbb{R}^n$. Then there exists a vector $\mathbf{x}^* \in S$ such that $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ for all $\mathbf{x} \in S$*

Using the usual assumption that $\mathbf{p} \gg 0$ and $y \geq 0$, we know that the budget set B is non-empty and compact. Therefore the continuous, real-valued utility function restricted to the budget set will attain a maximum value at some vector $\mathbf{x}^*$.

Finally, we prove that it may not be unique
Since we do not assume the preference to be strictly convex, we can allow some "straight lines" in part of our indifference curve[1]. A trivial example may be that the consumer's

*zhaoqin.zhu@phdstudent.hhs.se

[1]It may be planes or hyperplanes in cases with many kinds of goods, but in a two-good example this is truly a straight line segment

utility function is $u = x_1 + x_2$, and the prices for both goods are the same, denoted as p. Then his problem becomes:

$$\max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) = x_1 + x_2 \quad \text{such that} \quad px_1 + px_2 \leq y \tag{2}$$

We see that any combination of x_1, x_2, as long as it exhausts the budget y, is optimal. So the solution is not unique.

2 Exercise 1.20

2.1 Original question

Suppose preferences are represented by the Cobb-Douglas utility function $u(\mathbf{x}) = Ax_1^{\alpha}x_2^{1-\alpha}$, where $0 < \alpha < 1$ and $A > 0$. Assuming an interior solution, solve for the Marshallian demand functions.

2.2 Solution

The Marshallian demand function $\mathbf{x}(\mathbf{p}, y)$ solves the consumer's problem for each combination of prices and income $(\mathbf{p}, y)$, that is,

$$\mathbf{x}(\mathbf{p}, y) = \arg\max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p}\cdot\mathbf{x} \leq y \tag{3}$$

For now, I will only assume that $u(\cdot)$ is strictly increasing and differentiable on $\mathbb{R}^n_+$. We will apply Lagrange/Kuhn-Tucker methods to solve this problem.[2] Define the index $\mathcal{I} = \{1, ..., n\}$. Let λ be a non-negative multiplier associated with the budget constraint. We form the Lagrangian

$$\mathcal{L} = u(\mathbf{x}) + \lambda(y - \mathbf{p}\cdot\mathbf{x}) \tag{4}$$

Lagrange Assuming an interior solution $\mathbf{x}^\star \gg 0$ exists, we know that since $u(\cdot)$ is strictly increasing on $\mathbf{x} \gg 0$, the budget constraint must always bind with equality. Therefore, we can invoke the Lagrange Theorem which tells us that there exists a $\lambda^\star$ such that an optimum satisfies the following first-order conditions

$$\frac{\partial\mathcal{L}}{\partial x_i} = u_i(\mathbf{x}^\star) - \lambda^\star p_i = 0 \quad \forall i \in \mathcal{I} \tag{5}$$

$$\frac{\partial\mathcal{L}}{\partial \lambda} = \mathbf{p}\cdot\mathbf{x}^\star - y = 0 \tag{6}$$

These two first-order conditions imply that an interior optimum satisfies the following to conditions

$$\frac{u_i(\mathbf{x}^\star)}{u_j(\mathbf{x}^\star)} = \frac{p_i}{p_j} \tag{7}$$

$$\mathbf{p}\cdot\mathbf{x}^\star = y \tag{8}$$

[2]Before moving on, I will make a few theoretical statements about this problem. It is fine if you are not familiar with this yet, but you will encounter this in the Maths course and I think it is good if you have seen it a couple of times. The objective function u is continuous. The budget set $B(\mathbf{p}, y) \subseteq \mathbb{R}^n_+$ is non-empty. It is closed and bounded and therefore compact by the Heine-Borel Theorem (see Definition A1.8 on p. 515 of the Mathematical Appendix. The Extreme Value Theorem A1.10 then guarantees the existence of a maximum. Furthermore, if u is quasiconcave, the solution is unique. Given the restrictions put on α, it can be shown that the Cobb-Douglas utility function is quasiconcave (by a similar argument as you will encounter in my solution to the fourth problem) and the solution is unique.

So both methods arrive at the same result, when $u(\cdot)$ is strictly increasing on its domain. Let us now move from the general case to the particular setting given in the problem. We have two goods and so $n = 2$ and the index set is $\mathcal{I} = \{1, 2\}$. Therefore, a solution satisfies the two conditions

$$\frac{u_1(\mathbf{x}^\star)}{u_2(\mathbf{x}^\star)} = \frac{p_1}{p_2} \tag{9}$$

$$p_1 x_1 + p_2 x_2 = y \tag{10}$$

We can now impose the specification $u(\mathbf{x}) = A x_1^\alpha x_2^{1-\alpha}$ of the utility function. The utility function is differentiable and the marginal utility functions are given by

$$u_1(\mathbf{x}^\star) = \alpha A \left(\frac{x_2}{x_1}\right)^{1-\alpha} \qquad u_2(\mathbf{x}^\star) = (1-\alpha) A \left(\frac{x_1}{x_2}\right)^{\alpha} \tag{11}$$

We see that for $\alpha \in (0, 1)$ and $\mathbf{x} \gg 0$, the partial derivatives are strictly positive, so the utility function is strictly increasing on $\mathbb{R}^n_+$. Therefore, we can apply all the results derived earlier to our given setting. Plugging the marginal utilities into (9) gives

$$\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} = \frac{p_1}{p_2} \tag{12}$$

By rearranging (12) one obtains

$$p_2 x_2 = \left(\frac{1-\alpha}{\alpha}\right) p_1 x_1 \tag{13}$$

Using (13) to substitute for $p_2 x_2$ in the budget constraint in (10) gives

$$p_1 x_1 \left(\frac{1-\alpha}{\alpha} + 1\right) = y$$

which implies

$$x_1 = \frac{\alpha y}{p_1} \tag{14}$$

We could now substitute for x_1 in (9) and solve for the optimal x_2. This yields

$$x_2 = \frac{(1-\alpha) y}{p_2} \tag{15}$$

We are ready to assemble the Marshallian demand function, which is given by

$$\mathbf{x}(\mathbf{p}, y) = \begin{bmatrix} \frac{\alpha y}{p_1} \\ \frac{(1-\alpha) y}{p_2} \end{bmatrix} \tag{16}$$

Notice that the Marshallian demand function expresses the endogenous variables of the model (the choice of $\mathbf{x}$) as a function of the model's **exogenous** parameters alone. This is what it usually means to solve the economic model.

3 Exercise 1.23

3.1 Original question

Prove Theorem 1.3, which is:
Let $\succsim$ be represented by $u : \mathbb{R}^n_+ \mapsto \mathbb{R}$. Then:

1. $u(\mathbf{x})$ is strictly increasing if and only if $\succsim$ is strictly monotonic.
2. $u(\mathbf{x})$ is quasiconcave if and only if $\succsim$ is convex.
3. $u(\mathbf{x})$ is strictly quasiconcave if and only if $\succsim$ is strictly convex.

3.2 Solution

We begin with a lemma with proof: $u(\mathbf{x}^0) > u(\mathbf{x}^1) \iff \mathbf{x}^0 \succ \mathbf{x}^1$
Proof for this lemma: Pick any $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$ such that $u(\mathbf{x}^0) > u(\mathbf{x}^1)$. This statement tells us that $u(\mathbf{x}^0) \geq u(\mathbf{x}^1)$, but not $u(\mathbf{x}^1) \geq u(\mathbf{x}^0)$. Otherwise, if both statements were to hold, we would get $u(\mathbf{x}^0) = u(\mathbf{x}^1)$. Thus,

$$u(\mathbf{x}^0) > u(\mathbf{x}^1) \iff \Big(u(\mathbf{x}^0) \geq u(\mathbf{x}^1)\Big) \;\&\; \neg \Big(u(\mathbf{x}^1) \geq u(\mathbf{x}^0)\Big)$$

where $\neg$ means that the statement in brackets is not true. If $u(\cdot)$ represents $\succsim$, we get that

$$u(\mathbf{x}^0) > u(\mathbf{x}^1) \iff \Big(\mathbf{x}^0 \succsim \mathbf{x}^1\Big) \;\&\; \neg \Big(\mathbf{x}^1 \succsim \mathbf{x}^0\Big)$$

The expression on the right-hand side is just what we usually denote by $\mathbf{x}^0 \succ \mathbf{x}^1$. Hence,

$$u(\mathbf{x}^0) > u(\mathbf{x}^1) \iff \mathbf{x}^0 \succ \mathbf{x}^1$$

Remark: Although the lemma is simple, we still need to prove it before we can use it. With this lemma it is legitimate to follow our common sense and continue.

(1) Start by listing all the relevant definitions.

Strict monotonicity: Preferences satisfy *strict monotonicity* if they satisfy Axiom 4 (p. 10), that is, for all $\mathbf{x}^1, \mathbf{x}^0 \in \mathbb{R}^n_+$, if $\mathbf{x}^0 \geq \mathbf{x}^1$ then $\mathbf{x}^0 \succsim \mathbf{x}^1$, while if $\mathbf{x}^0 \gg \mathbf{x}^1$, then $\mathbf{x}^0 \succ \mathbf{x}^1$.

Strictly increasing function: A function $f : X \mapsto \mathbb{R}$ where $X \subseteq \mathbb{R}^n$ is said to be *strictly increasing* if $f(\mathbf{x}^0) \geq f(\mathbf{x}^1)$ whenever $\mathbf{x}^0 \geq \mathbf{x}^1$ and if in addition, $f(\mathbf{x}^0) > f(\mathbf{x}^1)$ whenever $\mathbf{x}^0 \gg \mathbf{x}^1$ (see A 1.17 on p.529).

Proof: Suppose then that u is strictly increasing. Pick any $\mathbf{x}^0 \geq \mathbf{x}^1$. Then since u is strictly increasing, it follows that $u(\mathbf{x}^0) \geq u(\mathbf{x}^1)$. Since u represents $\succsim$, it follows that $\mathbf{x}^0 \succsim \mathbf{x}^1$. Likewise, if $\mathbf{x}^0 \gg \mathbf{x}^1$, then $u(\mathbf{x}^0) > u(\mathbf{x}^1)$ and representation now implies $\mathbf{x}^0 \succ \mathbf{x}^1$. Thus, the preference relation $\succsim$ represented by u is strictly monotonic.

Now suppose that $\succsim$ is strictly monotonic. If $\mathbf{x}^0 \geq \mathbf{x}^1$, then $\mathbf{x}^0 \succsim \mathbf{x}^1$, which implies $u(\mathbf{x}^0) \geq u(\mathbf{x}^1)$, since u represents the preference relation $\succsim$. If $\mathbf{x}^0 \gg \mathbf{x}^1$, then $\mathbf{x}^0 \succ \mathbf{x}^1$, which implies $u(\mathbf{x}^0) \gg u(\mathbf{x}^1)$. Hence, u is strictly increasing. This proves the first claim.

(2) Start by listing all the relevant definitions.

Quasiconcave (Function): A function $f : X \mapsto \mathbb{R}$ where $X \subseteq \mathbb{R}^n$ is quasiconcave if and only if for all $\mathbf{x}^0$ and $\mathbf{x}^1$ in X

$$f(\mathbf{x}^t) \geq \min[f(\mathbf{x}^0), f(\mathbf{x}^1)] \quad \forall t \in [0,1] \tag{17}$$

where $\mathbf{x}^t = t\mathbf{x}^0 + (1-t)\mathbf{x}^1$.

Convex (Preferences): A preference relation $\succsim$ defined on $X \subseteq \mathbb{R}^n_+$ is said to be convex if for all $\mathbf{x}^1$ and $\mathbf{x}^0$ in X satisfying $\mathbf{x}^0 \succsim \mathbf{x}^1$, it holds that $t\mathbf{x}^0 + (1-t)\mathbf{x}^0 \succsim \mathbf{x}^0$ for all $t \in [0,1]$.

Proof Suppose that u is quasiconcave. Now pick any $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$ such that $\mathbf{x}^0 \succsim \mathbf{x}^1$. Furthermore, pick any $t \in [0,1]$ and define $\mathbf{x}^t = t\mathbf{x}^0 + (1-t)\mathbf{x}^1$. Since u represents $\succsim$, $\mathbf{x}^0 \succsim \mathbf{x}^1$ implies that $u(\mathbf{x}^0) \geq u(\mathbf{x}^1)$. Hence, $u(\mathbf{x}^0) \geq \min[u(\mathbf{x}^0), u(\mathbf{x}^1)]$. Quasiconcavity of u implies that $u(\mathbf{x}^t) \geq u(\mathbf{x}^1)$. Together, these inequalities imply $u(\mathbf{x}^t) \geq u(\mathbf{x}^1)$. The representativeness of u then implies that $\mathbf{x}^t \succsim \mathbf{x}^1$, which proves that the preference relation $\succsim$ represented by u is convex.

The proof in the other direction proceeds analogously. Suppose that $\succsim$ is convex. Now, pick any $\mathbf{x}^0, \mathbf{x}^1$ in X satisfying $\mathbf{x}^0 \succsim \mathbf{x}^1$. Since $u(\cdot)$ represents the preference relation, it follows that $u(\mathbf{x}^0) \geq u(\mathbf{x}^1)$. Hence, the minimum operator satisfies $\min[u(\mathbf{x}^0), u(\mathbf{x}^1)] = u(\mathbf{x}^1)$. Furthermore, convexity implies that $\mathbf{x}^t \succsim \mathbf{x}^1$ and thus $u(\mathbf{x}^t) \geq u(\mathbf{x}^1)$ for any $t \in [0,1]$. It follows that $u(\mathbf{x}^t) \geq \min[u(\mathbf{x}^0), u(\mathbf{x}^1)]$ for all $t \in [0,1]$, so u is quasiconcave.

(3) Start by listing all the relevant definitions.

Strict Quasiconcavity (Function): A function $f : X \mapsto \mathbb{R}$ is strictly quasiconcave if and only if for all $\mathbf{x}^0, \mathbf{x}^1$ in X satisfying $\mathbf{x}^0 \neq \mathbf{x}^1$

$$f(\mathbf{x}^t) > \min[f(\mathbf{x}^0), f(\mathbf{x}^1)] \quad \forall t \in (0,1) \tag{18}$$

where $\mathbf{x}^t = t\mathbf{x}^0 + (1-t)\mathbf{x}^1$.

Strict Convexity (Preferences): We say that a preference relation $\succsim$ is strictly convex if for all $\mathbf{x}^0, \mathbf{x}^1$ in X satisfying $\mathbf{x}^0 \neq \mathbf{x}^1$ and $\mathbf{x}^0 \succsim \mathbf{x}^1$, it holds that $\mathbf{x}^t \succ \mathbf{x}^1$ for all $t \in (0,1)$, where $\mathbf{x}^t = t\mathbf{x}^0 + (1-t)\mathbf{x}^1$.

Proof Pick any $\mathbf{x}^0, \mathbf{x}^1$ in X satisfying $\mathbf{x}^0 \neq \mathbf{x}^1$ and $\mathbf{x}^0 \succsim \mathbf{x}^1$ and assume that u is strictly quasiconcave. Furthermore, pick any scalar $t \in [0,1]$ and define $\mathbf{x}^t = t\mathbf{x}^0 + (1-t)\mathbf{x}^1$. Strict quasiconcavity of u implies that $u(\mathbf{x}^t) > \min[u(\mathbf{x}^0), u(\mathbf{x}^1)]$. Since $\mathbf{x}^0 \succsim \mathbf{x}^1$, it must be that $u(\mathbf{x}^0) > u(\mathbf{x}^1)$ and therefore $\min[u(\mathbf{x}^0), u(\mathbf{x}^1)] = u(\mathbf{x}^1)$. It follows that $u(\mathbf{x}^t) > u(\mathbf{x}^1)$. Hence, $\mathbf{x}^t \succ \mathbf{x}^1$. We conclude that the preference relation $\succsim$ represented by u must be strictly convex.

Pick any $\mathbf{x}^0, \mathbf{x}^1$ in X satisfying $\mathbf{x}^0 \neq \mathbf{x}^1$ and $\mathbf{x}^0 \succsim \mathbf{x}^1$ and suppose that $\succsim$ is strictly convex. Furthermore, pick any scalar $t \in (0,1)$ and define $\mathbf{x}^t = t\mathbf{x}^0 + (1-t)\mathbf{x}^1$. Strict convexity of $\succsim$ implies that $\mathbf{x}^t \succ \mathbf{x}^1$. Then the utility function satisfies $u(\mathbf{x}^0) \geq u(\mathbf{x}^1)$ and $u(\mathbf{x}^t) > u(\mathbf{x}^1)$. It follows from these two inequalities and the definition of the minimum operator that $u(\mathbf{x}^t) > \min[u(\mathbf{x}^0), u(\mathbf{x}^1)]$. Therefore, u is strictly quasiconcave.

4 Exercise 1.24

4.1 Original question

Let $u(x)$ represent some consumer's monotonic preferences over $x \in \mathbb{R}^n_+$. For each of the functions $f(x)$ that follow, state whether or not f also represents the preferences of this consumer. In each case, be sure to justify your answer with either an argument or a counterexample.

(a) $f(x) = u(x) + (u(x))^3$

(b) $f(x) = u(x) - (u(x))^2$

(c) $f(x) = u(x) + \sum_{i=1}^n x_i$

4.2 Solution

(a) **True**: Define $v : \mathbb{R} \to \mathbb{R}$ as follow: $v(x) = x + x^3$. This function is strictly increasing in $\mathbb{R}$ since $v'(x) = 1 + 2x^2$ is positive for all $x \in \mathbb{R}$. So $f(x) = v(u(x))$ is a positive monotone transformation of u, which means f represents the same preference as u.

(b) **False**: Suppose $u(x_1) = 1, u(x_2) = 2$. Then by definition of utility function we have $x_2 \succsim x_1$. However, $f(x_1) = 0 > f(x_2) = -2$, which is not in accordance to $x_2 \succsim x_1$, so f does not represent the same preference.

(c) **False**: Consider the Leontief preference[3] over bundles $(x_1, x_2) \in \mathbb{R}^2_+$:

$$u(x_1, x_2) = \min\{x_1, x_2\} \tag{19}$$

So we pick $\mathbf{x}^1 = (1, 2)$ and $\mathbf{x}^2 = (1, 3)$. Then we have $u(\mathbf{x}^1) = \min\{1, 2\} = 1$ and $u(\mathbf{x}^2) = \min\{1, 3\} = 1$, which means $\mathbf{x}^1 \sim \mathbf{x}^2$. However, according to $f(u(x))$, $f(\mathbf{x}^1) = \min\{1, 2\} + 1 + 2 = 4 < f(\mathbf{x}^2) = 5$, which implies $x^2 \succ x^1$. So f and u are different preferences.

5 Exercise 1.29 cake-eating problem

5.1 Original question

An infinitely lived agent owns 1 unit of a commodity that he consumes over his lifetime. The commodity is perfectly storable and he will receive no more than he has now. Consumption of the commodity in period t is denoted x_t, and his lifetime utility function is given by

$$u(\mathbf{x}) = \sum_{t=0}^{\infty} \beta^t \ln(x_t) \quad \text{where } \ 0 < \beta < 1 \tag{20}$$

Calculate his optimal level of consumption in each period.

[3] This is a classic preference, please remember this name

5.2 Solution

This problem is known as the cake-eating problem. It is a model of intertemporal choice. The agent is gifted one cake at the beginning of his life and has to choose at which point to eat how much. We note a few things in the beginning. The agent discounts future utility with a factor less than unity, $\beta < 1$. Ceteris paribus, the agent prefers to incur utility from consumption as early as possible. The earliest point in time being the present, $t = 0$. Now note that the period utility function $\ln(x)$ is strictly concave on its entire domain. Effectively, this means that the agent prefers an equal allocation of consumption across all periods. Given the limited availability of resources, the concavity of period utility and the impatience encapsulated in β introduce a trade-off between consumption today and in the future. Thirdly, note that unlike in previous problems, the domain of $u(x)$ is an infinite-dimensional commodity space.
Formally, the consumer's problem is

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(x_t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} x_t \leq 1 \tag{21}$$

where $0 < \beta < 1$. Strictly speaking, Lagrange's Theorem as represented in Theorem A2.16 only applies if the domain is a subset of an Euclidean space (with finite dimension), but let us just proceed in the usual manner. Let λ be a non-negative multiplier associated with the resource constraint. We form the Lagrangian

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \ln(x_t) + \lambda[1 - \sum_{t=0}^{\infty} x_t] \tag{22}$$

The period utility $\ln(x_t)$ is strictly increasing on $\mathbb{R}^n_{++}$. Hence, the weighted sum of these functions, which is $u(\mathbf{x})$, is also strictly increasing. Since the objective function is strictly increasing, we know that in optimum, the budget constraint must bind. Then, an interior solution $\mathbf{x}^\star \gg 0$ satisfies the first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_t} = \frac{\beta^t}{x_t} - \lambda = 0 \quad \forall t \geq 0 \tag{23}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{t=0}^{\infty} x_t - 1 = 0 \tag{24}$$

for all t. Since equation (23) holds for all dates t, we can use it to relate marginal utility in periods 0 and t as follows

$$\frac{\beta^t}{x_t} = \frac{\beta^0}{x_0} \tag{25}$$

which implies

$$x_t = x_0 \beta^t \tag{26}$$

for all t. We note from the first-order condition (23) that at an interior solution the multiplier is strictly positive, $\lambda > 0$, confirming that the resource constraint holds with equality. Substituting for x_t in the resource constraint (24) using (26) gives

$$\sum_{t=0}^{\infty} x_0 \beta^t = 1 \tag{27}$$

Since it is a summation of stuff multiplying with x_0, we can extract the x_0 out, and get:

$$x_0 \sum_{t=0}^{\infty} \beta^t = 1 \tag{28}$$

where the $\sum_{t=0}^{\infty} \beta^t$ is called geometric series. The sum of geometric series follows, notice that we break it into two halfs, first we calculate a general solution of summing to any finite number n, then we calculate the limit of this general solution[4] :

$$\sum_{t=0}^{\infty} \beta^t = \lim_{n \longrightarrow \infty} \sum_{t=0}^{n} \beta^t \tag{29}$$
$$= \lim_{n \longrightarrow \infty} \frac{\beta^0(1-\beta^n)}{1-\beta} \tag{30}$$
$$= \frac{\beta^0(1-\beta^\infty)}{1-\beta} \tag{31}$$
$$= \frac{1}{1-\beta} \tag{32}$$

In summary, we can write the **geometric series** on the left-hand side in closed-form as

$$x_0(1-\beta)^{-1} = 1$$

which in turn implies

$$x_0 = (1-\beta) \tag{33}$$

From (33) and (26) one obtains a consumption at any date t

$$x_t = \beta^t(1-\beta) \tag{34}$$

We see that the agent's consumption is strictly decreasing in time and that the speed of this decrease is regulated by his level of patience β.

6 Exercise 1.28, not in the homework, only for reference

6.1 Original question

In the proof of Theorem 1.4 we use the fact that if $u(\cdot)$ is quasiconcave and differentiable at $\mathbf{x}$ and $u(\mathbf{y}) \geq u(\mathbf{x})$, then $\nabla u(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0$. Prove this fact in the following two steps.

a) Prove that if $u(\mathbf{y}) \geq u(\mathbf{x})$, the quasiconcavity of $u(\cdot)$ and its differentiability at $\mathbf{x}$ imply that the derivative of $u((1-t)\mathbf{x} + t\mathbf{y})$ with respect to t must be non-negative at $t = 0$.

b) Compute the derivative of $u((1-t)\mathbf{x} + t\mathbf{y})$ with respect to t evaluated at $t = 0$ and show that it is $\nabla u(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})$.

6.2 Solution

a) Recall that the definition of a partial derivative. Suppose that $f : D \mapsto \mathbb{R}$ where $D \subseteq \mathbb{R}^n$. Then if $\mathbf{x}$ is an interior point of D, the partial derivative of f with respect to x_i at $\mathbf{x}$ is defined as

$$\frac{\partial f(x)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ..., x_n) - f(x_1, ..., x_i, ..., x_n)}{h} \tag{35}$$

[4]Here is β to the power of n, usually the superscript is reserved for power calculation, unless specified. You may also want to google geometric sequences

We are asked to show that the partial derivative of the convex combination of two bundles $(1-t)\mathbf{x}+t\mathbf{y}$ with regards to its modulus t, evaluated at the point $t=0$ is non-negative. Formally, we write

$$\frac{\partial}{\partial t}u((1-t)\mathbf{x}+t\mathbf{y})\Big|_{t=0} \geq 0 \tag{36}$$

Generally for any $t \in [0,1]$ our object of interest can be expressed as follows, using the definition of the partial derivative in (35),

$$\frac{\partial}{\partial t}u((1-t)\mathbf{x}+t\mathbf{y}) = \lim_{h\to 0}\frac{u((1-(t+h))\mathbf{x}+(t+h)\mathbf{y})-u((1-t)\mathbf{x}+t\mathbf{y})}{h} \tag{37}$$

Hence, if we evaluate the partial derivative at $t=0$, the expression becomes

$$\frac{\partial}{\partial t}u((1-t)\mathbf{x}+t\mathbf{y})\Big|_{t=0} = \lim_{h\to 0}\frac{u((1-h)\mathbf{x}+h\mathbf{y})-u(\mathbf{x})}{h} \tag{38}$$

Now note that we can write $(1-t)\mathbf{x}+t\mathbf{y}$ as $\mathbf{x}+h(\mathbf{y}-\mathbf{x})$, so the right-hand side of (38) is the *directional derivative* of u at $\mathbf{x}$ in the direction of $\mathbf{y}-\mathbf{x}$. Since u is differentiable at $\mathbf{x}$ by assumption, the directional derivative exists in all directions and this particular limit is well-defined. The quasiconcavity of u and the assumption that $u(\mathbf{y}) \geq u(\mathbf{x})$ imply that for all $h \in [0,1]$

$$\begin{aligned}
u((1-h)\mathbf{x}+h\mathbf{y})-u(\mathbf{x}) &\geq \min[u(\mathbf{y}),u(\mathbf{x})]-u(\mathbf{x})\\
&\geq u(\mathbf{x})-u(\mathbf{x})\\
&\geq 0
\end{aligned}$$

This last inequality and equation (38) then imply (36).

b) From page 276 in Pemberton & Rau (2016), or page 555 in Jehle & Reny (2011), we know that the following holds

$$\begin{aligned}
\frac{du((1-t)\mathbf{x}+t\mathbf{y})}{dt} &= \frac{\partial u((1-t)\mathbf{x}+t\mathbf{y})}{\partial x_1}(y_1-x_1)\\
&+\frac{\partial u((1-t)\mathbf{x}+t\mathbf{y})}{\partial x_2}+(y_2-x_2)\\
&...+\frac{\partial u((1-t)\mathbf{x}+t\mathbf{y})}{\partial x_n}+(y_n-x_n)\\
&=\sum_{i=1}^{n}\frac{\partial u((1-t)\mathbf{x}+t\mathbf{y})}{\partial x_i}(y_i-x_i)
\end{aligned}$$

Evaluating this expression at $t=0$, we get that

$$\begin{aligned}
\frac{du((1-t)\mathbf{x}+t\mathbf{y})}{dt}\Big|_{t=0} &= \sum_{i=1}^{n}\frac{\partial u(\mathbf{x})}{\partial x_i}(y_i-x_i)\\
&= \nabla u(\mathbf{x})\cdot(\mathbf{y}-\mathbf{x})
\end{aligned}$$

where $\nabla u(\mathbf{x})$ is the gradient of u at $\mathbf{x}$, i.e. a column vector of length n containing all the partial derivatives of u at $\mathbf{x}$, $\nabla u(\mathbf{x}) = \{\partial u(\mathbf{x})/\partial x_1, ..., \partial u(\mathbf{x})/\partial x_n\}$.[5]

[5] This "proof" is a bit heuristic, admittedly.