5330 Assignment 3

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Exercise 3.11

The production function is of the form $f(x_1, x_2) = Ax_1^{\alpha}x_2^{\beta}, A > 0, \alpha > 0, \beta > 0$. First we calculate the marginal rate of technical substitution:

$$MRTS_{12}(x) = \frac{\partial f(x)/\partial x_1}{\partial f(x)/\partial x_2} = \frac{A\alpha x_1^{\alpha-1}x_2^{\beta}}{A\beta x_1^{\alpha}x_2^{\beta-1}} = \frac{\alpha x_2}{\beta x_1}$$

Taking logs of both sides and then totally differentiating yields

$$lnMRTS_{12}(x) = ln\left(\frac{\alpha}{\beta}\right) + ln\left(\frac{x_2}{x_1}\right)$$

$$d\,MRTS_{12}(x) = d\,ln\left(\frac{x_2}{x_1}\right)$$

Thus the elasticity of substitution is

$$\boldsymbol{\sigma}_{12} = \frac{d\,ln(x_2/x_1)}{d\,ln(f_1(x)/f_2(x))} = \mathbf{1}$$

Exercise 3.32

Define the cost function as $c(y) \equiv atc(y)y$, where $atc(y)$ is the average cost. Taking the derivative with respect to y gives the following expression for the marginal cost $mc(y)$:

$$mc(y) = \frac{\partial c(y)}{\partial y} = \frac{\partial atc(y)}{\partial y}y + atc(y)$$

Assuming $y > 0$, it follows that:

- when average cost is declining $\frac{\partial atc(y)}{\partial y} < \mathbf{0}$, marginal cost must be less than average cost: $mc(y) = -negative * positive + atc(y) = negative + atc(y) \rightarrow \mathbf{mc(y) < atc(y)}$;
- when average cost is constant $\frac{\partial atc(y)}{\partial y} = \mathbf{0}$, marginal cost must equal average cost: $mc(y) = -0 * positive + atc(y) = 0 + atc(y) \rightarrow \mathbf{mc(y) = atc(y)}$;
- and when average cost is increasing $\frac{\partial atc(y)}{\partial y} > \mathbf{0}$, marginal cost must be greater than average cost: $mc(y) = positive * positive + atc(y) = positive + atc(y) \rightarrow \mathbf{mc(y) > atc(y)}$.

Exercise 8.17

For the CES production function $y = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho}$ $\sum_{i=1}^{n} \alpha_i = 1$ $0 \neq \rho < 1$

prove:

(a) $\lim_{\rho \to 0} y = \prod_{i=1}^{n} x_i^{\alpha_i}$ $\longrightarrow$ We want to show how this form of CES reduces to the linear homogeneous Cobb-Douglas form when ρ goes to zero

First step is to take the natural log of the CES production function, which becomes:

$$\lim_{\rho \to 0} \ln y = \frac{\ln \sum_{i=1}^{n} \alpha_i x_i^{\rho}}{\rho}$$

the problem here is that when ρ goes to zero there is gonna be zero in the numerator and the denomenator, will be indeterminate. We will therefore use L'Hôpital's rule as both numerator and denominator tend to zero.

L'Hôpital's rule: $\lim_{x \to a} \frac{m(x)}{n(x)} = \lim_{x \to a} \frac{m'(x)}{n'(x)}$

Using L'Hôpital's rule:

$$\lim_{\rho \to 0} \ln y = \frac{\ln \sum_{i=1}^{n} \alpha_i x_i^{\rho}}{\rho} = \frac{d\left(\ln \sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)}{d\rho} \Longleftrightarrow \frac{d\left(\ln \sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)}{1} \Longleftrightarrow d\left(\ln \sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)$$

$$\Longleftrightarrow \frac{1}{\sum_{i=1}^{n} \alpha_i x_i^{\rho}} \times \sum \alpha_i x_i^{\rho} \ln x_i \Longleftrightarrow \frac{\sum_{i=1}^{n} \alpha_i x_i^{\rho} \ln x_i}{\sum_{i=1}^{n} \alpha_i x_i^{\rho}}$$

Now put in 0 for ρ as for the limit $\rho \to 0$:

$$\frac{\sum_{i=1}^{n} \alpha_i x_i^{0} \ln x_i}{\sum_{i=1}^{n} \alpha_i x_i^{0}} \Longleftrightarrow \frac{\sum_{i=1}^{n} \alpha_i \ln x_i}{\sum_{i=1}^{n} \alpha_i} \Longleftrightarrow \frac{\sum_{i=1}^{n} \ln x_i^{\alpha_i}}{\sum_{i=1}^{n} \alpha_i} \Longleftrightarrow \frac{\sum_{i=1}^{n} \ln x_i^{\alpha_i}}{1} \Longleftrightarrow \sum_{i=1}^{n} \ln x_i^{\alpha_i}$$

by $\sum \alpha_i = 1$

$\Longleftrightarrow \ln \prod_{i=1}^{n} x_i^{\alpha_i}$, now converting this expression to an expression for $\lim_{\rho \to 0} y$ rather than $\lim_{\rho \to 0} \ln y$ using e:

$$\boxed{\ln \prod_{k=a}^{b} f(k) = \sum_{k=a}^{b} \ln f(k)}$$

$$e^{\ln \prod_{i=1}^{n} x_i^{\alpha_i}} \Longleftrightarrow \prod_{i=1}^{n} x_i^{\alpha_i}, \text{ hence, } \lim_{\rho \to 0} y = \underline{\prod_{i=1}^{n} x_i^{\alpha_i}} \quad \text{Q.E.D}$$

which is what we set out to prove

(b) $\lim_{\rho \to -\infty} y = \min\{x_1, \dots, x_n\}$

So we want to prove that the limit when ρ tends to minus infinity for the CES production function $y = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho}$ becomes $\min\{x_1, \dots, x_n\}$.
To do this, we use the:

Thm 4 (Hardy, Littlewood & Pólya, 1934)

$$\lim_{r \to \infty} \mathfrak{M}_r(a) = \text{Max } a, \quad \lim_{r \to -\infty} \mathfrak{M}_r(a) = \text{Min } a$$

If a_k is the largest a, or one of the largest, and $r > 0$, we have

$$q_k^{1/r} a_k \leq \mathfrak{M}_r(a) \leq a_k$$

$\mathfrak{M}_r(a) = \mathfrak{M}(a, p) = \left(\frac{\sum p a^r}{\sum p}\right)^{\frac{1}{r}}$ by (2.2.2) in (Hardy et al, 1934)

Now using this for the CES production function as it can adopt the similar form as $\mathfrak{M}_r(a)$

$$y = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho} \iff \left(\frac{\left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)}{\underbrace{\sum_{i=1}^{n} \alpha_i}_{\sum_{i=1}^{n} \alpha_i = 1 \text{ given by the question}}}\right)^{\frac{1}{\rho}}$$

where
$\mathfrak{M}_r(a) = y$
$p_i = \alpha_i$
$a_i = x_i$
$r = \rho$

by thm 4 (Hard et al. 1934) this is $\min\{x_1, \dots, x_n\}$ when $\rho \to -\infty$. Hence, following the thm directly we can write:

$$\lim_{\rho \to -\infty} y = \left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)^{1/\rho} = \left(\frac{\left(\sum_{i=1}^{n} \alpha_i x_i^{\rho}\right)}{\sum_{i=1}^{n} \alpha_i}\right)^{\frac{1}{\rho}} = \underbrace{\min\{x_1, \dots, x_n\}}_{= \min(a)}$$

which is what we set out to prove, Q.E.D

Exercise 3.26

Calculate the cost function and conditional input demands for the Leontief production function in exercise 3.10

3.10 Leontief production function: $y = \min\{\alpha x_1, \beta x_2\}$ $\alpha > 0$ and $\beta > 0$

Central assumption of Leontief production function: fixed proportions of inputs required for production. (both x_1 and x_2 are required)
$\Rightarrow$ no substitutability between factors
$\Rightarrow$ x_1 and x_2 are perfect complements
$\Rightarrow$ $x_1 \neq 0$, $x_2 \neq 0$

1.) Conditional demand for inputs:
Production occurs at: $y = \alpha x_1 = \beta x_2$, hence: $y = \alpha x_1 \Leftrightarrow x_1 = \frac{y}{\alpha}$

$y = \beta x_2 \Leftrightarrow x_2 = \frac{y}{\beta}$

2.) Cost function:
Total cost $= C = w_1 x_1 + w_2 x_2$

w_1 = unit cost of x_1
w_2 = unit cost of x_2

$\hookrightarrow$ Stemming from def 3.5 where all input prices $\vec{w} \gg 0$ and $y \in f(\mathbb{R}^n_+)$ where the cost function can be constructed as
$c(\vec{w}, y) = \vec{w} \cdot \vec{x}(\vec{w}, y)$
In our two-input case $\vec{w} = (w_1, w_2)$ and $\vec{x} = (x_1, x_2)$

Plug in conditional demand for inputs x_1 and x_2 in c:

$$c = w_1\left(\frac{y}{\alpha}\right) + w_2\left(\frac{y}{\beta}\right) \Leftrightarrow c = y\left(\frac{w_1}{\alpha} + \frac{w_2}{\beta}\right)$$

Exercise 3.29

(a) A wage rate w_1^0, firm B uses more of input 1.
According to Shepard's lemma

$$x_1(w_1, w_2, y) = \frac{\partial C(w_1, w_2, y)}{\partial w_1}$$

The use of input 1 is equal to the slope of C in the figure.

At w_1^0, $\dfrac{\partial C_A(w_1, w_2, y)}{\partial w_1} < \dfrac{\partial C_B(w_1, w_2, y)}{\partial w_1}$

(C^B is "steeper" at w_1^0) which means firm B uses more of input 1 at w_1^0.

Similarly, compare $x_1(w_1, w_2, y)$ at point w_1^1

At w_1^1 $\dfrac{\partial C_A(w_1, w_2, y)}{\partial w_1} > \dfrac{\partial C_B(w_1, w_2, y)}{\partial w_1}$

(C^A is "steeper" at w_1^1) which means that firm A uses more of input 1 at w_1^1

(b) Firm B's production function has the higher elasticity of substitution. It is shown in fig. 3.8 that the use of input 1 is constant for firm A. In other words, in order to produce a certain amount of output y, a certain amount of input 1 must be used, regardless of how pricey it is. The usage of input 1 cannot be replaced by the usage of other inputs for firm A and thus, the elasticity of substitution can be seen as zero.

For firm B, as the price of input 1 increases, the usage of input 1 decreases, holding the output level y unchanged. This implies that the usage of input 1 can be substituted by another input on a certain level, and thus, the elasticity of substitution for firm B is larger than zero.