Problem Set 4

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Exercise 1 Let $m \in R$ with $m \geq 2$. Suppose $A_1, A_2, \ldots, A_m$ are convex sets in $\mathbb{R}^n$, show if the following sets are convex or not. If it is, prove; if it is not, show one counter example.

Solution 1 $A_1, A_2, \ldots, A_m$ are convex: $\forall x_i, y_i \in A_i, \lambda x_i + (1-\lambda)y_i \in A_i, \forall \lambda \in [0,1]$.

(1) $\cup_{i=1}^m A_i$

False. Consider the following counterexample: Let $R = [-2,-1]$ and $P = [1,2]$. Both sets are convex. Their union $U = [-2,-1] \cup [1,2]$, however, is not convex. Let $x = -1 \in U$ and $y = 1 \in U$. Take the convex combination $z = 1/2x + 1/2y = 0 \notin U$. So, the union of convex sets is not necessarily convex.

(2) $\cap_{i=1}^m A_i$

True. Take two arbitrary points x and y in $\cap_{i=1}^m A_i$. Since they belong to $\cap_{i=1}^m A_i$, they also belong to each of the convex sets $A_1, A_2, \ldots, A_m$. It follows that $\lambda x + (1-\lambda)y$ is in the intersection of all the A_i sets for any $\lambda \in [0,1]$. Thus $\cap_{i=1}^m A_i$ is convex.

(3) $\times_{i=1}^m A_i$

True. Consider two arbitrary vectors $(x_i, \ldots, x_m), (y_1, \ldots, y_m) \in \times_{i=1}^m A_i$, then their convex combination will be $\lambda(x_i, \ldots, x_m) + (1-\lambda)(y_1, \ldots, y_m) = (\lambda x_1, \ldots, \lambda x_m) + ((1-\lambda)y_1, \ldots, (1-\lambda)y_m) = (\lambda x_1 + (1-\lambda)y_1, \ldots, \lambda x_m + (1-\lambda)y_m)$ for an arbitrary $\lambda \in [0,1]$. Since $\lambda x_i + (1-\lambda)y_i \in A_i$, $i = 1, \ldots, m$ it follows directly that $(x_1 + (1-\lambda)y_1, \ldots, \lambda x_m + (1-\lambda)y_m) \in \times_{i=1}^m A_i$. Thus $\times_{i=1}^m A_i$ is convex.

(4) $\sum_{i=1}^m A_i$

True. The convex combination of two arbitrary vectors $(x_i + \cdots + x_m), (y_1 + \cdots + y_m) \in \sum_{i=1}^m A_i$, is $\lambda(x_i + \cdots + x_m) + (1-\lambda)(y_1 + \cdots + y_m) = (\lambda x_1 + \cdots + \lambda x_m) + ((1-\lambda)y_1 + \cdots + (1-\lambda)y_m) = (\lambda x_1 + (1-\lambda)y_1 + \cdots + \lambda x_m + (1-\lambda)y_m)$ for an arbitrary $\lambda \in [0,1]$. Since $\lambda x_i + (1-\lambda)y_i \in A_i$, $i = 1, \ldots, m$ it follows directly that $(x_1 + (1-\lambda)y_1, \ldots, \lambda x_m + (1-\lambda)y_m) \in \sum_{i=1}^m A_i$. Thus $\sum_{i=1}^m A_i$ is convex.

Exercise 2 Apply Fourier-Motzkin elimination method to solve the following system of linear inequalities:

$$-x_1 - x_2 - x_3 \leq -1 \wedge 3x_1 - x_2 - x_3 \leq 1 \wedge -x_1 + 3x_2 - x_3 \leq -2 \wedge -x_1 - x_2 + 3x_3 \leq 3$$

Solution 2 We first eliminate x_1. Normalize the coefficient of x_1 to 1 in each inequality, we have:

$$\begin{aligned}
&x_1 \geq 1 - x_2 - x_3 \\
&x_1 \leq 1/3 + 1/3x_2 + 1/3x_3 \\
&x_1 \geq 2 + 3x_2 - x_3 \\
&x_1 \geq -3 - x_2 + 3x_3
\end{aligned}$$

which can be summarized as

$$\max\{1 - x_2 - x_3, 2 + 3x_2 - x_3, -3 - x_2 + 3x_3\} \leq x_1 \leq 1/3 + 1/3x_2 + 1/3x_3 \tag{1}$$

If the system of inequalities has a solution, the lower bound of x_1 must be no greater than its upper bound. Thus,

$$\begin{aligned}
1 - x_2 - x_3 &\leq 1/3 + 1/3x_2 + 1/3x_3 \\
2 + 3x_2 - x_3 &\leq 1/3 + 1/3x_2 + 1/3x_3 \\
-3 - x_2 + 3x_3 &\leq 1/3 + 1/3x_2 + 1/3x_3
\end{aligned}$$

We then eliminate x_2. Normalize the coefficient of x_2 to 1 in each inequality, we have:

$$x_2 \leq -5/8 + 1/2x_3 \wedge x_2 \geq 1/2 - x_3 \wedge x_2 \geq -5/2 + 2x_3$$

which can be summarized as

$$\max\{1/2 - x_3, -5/2 + 2x_3\} \leq x_2 \leq -5/8 + 1/2x_3 \tag{2}$$

If the system of inequalities has a solution, the lower bound of x_2 must be no greater than its upper bound. Thus,

$$1/2 - x_3 \leq -5/8 + 1/2x_3 \wedge -5/2 + 2x_3 \geq -5/8 + 1/2x_3$$

This further implies

$$3/4 \leq x_3 \leq 5/4 \tag{3}$$

To summarize, the set of solutions to our system of linear inequalities consists of all $x \in \mathbb{R}^3$ such that inequalities 3, 2 and 1 hold.

Exercise 3 Let C be the convex cone spanned by the vectors $(1,2,2)^T, (3,2,5)^T$ and $(1,1,0)^T$. For each value of t determine if the point $(t,t,t)^T$ lies in C or find a hyperplane that separates $(t,t,t)^T$ from C.
Solution 3
We use Farka's Lemma: $Ax = b, x \geq 0$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & t \\ 2 & 2 & 1 & t \\ 2 & 5 & 0 & t \end{array}\right] -R_2 + 2R_1, -2R_3 + R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & t \\ 0 & 4 & 1 & t \\ 0 & 1 & 2 & t \end{array}\right] 4R_3 - R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 1 & t \\ 0 & 4 & 1 & t \\ 0 & 0 & 7 & 3t \end{array}\right]$$

$$\implies x_3 = \tfrac{3t}{7}, x_2 = \tfrac{1t}{7}, x_1 = \tfrac{1t}{7} \implies x = t\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

Thus, for $t \geq 0, x_1, x_2, x_3$ are greater or equal to zero as well. Therefore, for $t \geq 0$ the point lies in the cone. For $t < 0$ we want to find a hyperplane that, according to Farka's Lemma, satisfies the following conditions: $y^T A \geq 0^T$, $y^T b < 0$. We choose $y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and

$b = \begin{pmatrix} t \\ t \\ t \end{pmatrix}$ with $t < 0$: $y^T A = (1,3,1) \geq (0,0,0)$ and $(1,0,0)(t,t,t)^T = t < 0$

Thus, we know that a separating hyperplane is given by the set:

$$\{y \in \mathbb{R}^3 : y_1 = 0\}$$

We show that this in fact a separating hyperplane. Let $z \in C$, there exists $x \geq 0$ such that $Ax = z$ (Note that $Ax = (a_1, a_2, a_3)x$). This point lies in the positive halfspace since

$$y^T z = y^T(a_1 x_1 + a_2 x_2 + a_3 x_3) = x_1 y^T a_1 + x_2 y^T a_2 + x_3 y^T a_3 \geq 0.$$

However, for $t < 0$ we know $y^T t < 0$. So it is in the negative halfspace. This establishes that $y^T x = 0$ is a separating hyperplane.

Exercise 4 For each of the following functions determine whether it is convex or concave. Are they quasi-concave? Motivate your answer.
Solution 4

(1) $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x) = (x_1^2 + x_2^2)^2$

We can decompose f into $f(x) = h_1(h_2(x))$ with $h_1(t) = t^2$, $t \geq 0$, and $h_2(x) = x_1^2 + x_2^2$. h_1 is convex by Example 17.1 and h_2 is convex by by Example 17.1 and Theorem 17.6 (b). Moreover, h_1 is nondecreasing since $h_1'(x) = 2x \geq 0$ for all $x \geq 0$ and defined over the convex interval $(0, \infty)$. By Theorem 17.6 (d), $f(x)$ is convex. By Theorem 17.7 (a)

f is quasiconcave if and only if for each $r \in R$, the set $\{x \in C : f(x) \geq r\}$ is a convex set. The upper contour set of f consists of points with distance from the origin in the x_1, x_2 plane greater than or equal to $r^{1/4}$. It is not convex, since, for example, the linepiece between such two points that have coordinates of the opposite sign includes points that are closer to the origin than $c^{1/4}$. f is not quasiconcave.

(2) $f : \mathbb{R}^2_+ \to \mathbb{R}$ with $f(x) = \sqrt{x_1 + x_2}$

Claim: linear functions are convex and concave. Using definition 17.1 f is convex if $\forall \lambda \in [0,1] : f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x2)$. Functions $f = x$ as well as $-f = -x$ are therefore convex since $\forall \lambda \in [0,1] : f(\lambda x_1 + (1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda) f(x_2)$. And since f is concave if and only if $-f$ is convex, the claim follows.

We can decompose f into $f(x) = h_1(h_2(x))$ with $h_1(t) = \sqrt{t}$, $t \geq 0$, and $h_2(x) = x_1 + x_2$. h_1 is concave by Theorem 17.4 as $-h_1''(x) = 1/4x^{-3/2} \geq 0$ for all $x \geq 0$. h_2 is concave by claim above and Theorem 17.6 (b). Moreover, h_1 is nondecreasing since $h_1'(x) = 1/2x \geq 0$ for all $x \geq 0$ and defined over the convex interval $(0, \infty)$. By Theorem 17.6 (d), $f(x)$ is concave. By Theorem 17.7 (a) $f(x)$ is quasiconcave.

(3) $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x) = (x_1 + x_2)^3$

We can decompose f into $f(x) = h_1(h_2(x))$ with $h_1(t) = t^3$, $t \in \mathbb{R}$, and $h_2(x) = x_1 + x_2$. h_1 is neither concave or convex by Theorem 17.4 as $h_1''(x) = 6x < 0$ for $x < 0$ and $-h_1''(x) = -6x < 0$ for $x > 0$. It follows that f is neither concave or convex

The upper contour set of f is a halfspace, namely the set of solutions to a single linear inequality $x_1 + x_2 \geq r^{1/3}$, $r \in R$. It is convex by Example 14.1. By Theorem 17.7 (b) $f(x)$ is quasiconcave.

Exercise 5 Solve the problem:

$$\max\ f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2} \text{ with } 4x_1 + x_2 \leq 100,\ x_1 + x_2 \leq 60,\ x_1, x_2 \geq 0$$

Solution 5 The goal function is given by the addition of two continuous functions. Hence, by Theorem 8.4 (b), it is continuous. The feasible set is the intersection of 3 closed sets and, by Theorem 7.3 (b), it is closed. We can show that it is bounded as follows: The bounds for x_1 can be calculated by setting $x_2 = 0$ in $4x_1 + x_2 \leq 100$ and solving: $0 \leq x_1 \leq 25$. For x_2, we know that $x_2 \geq 0$ and $x_2 \leq 60 - x_1$, $x_2 \leq 100 - 4x_1$ which implies that $x_2 \leq 60$. Therefore, the feasible set is bounded. By Heine-Borel we conclude that it is compact. It is also nonempty, as it contains point $(x_1, x_2) = (0, 0)$. By the Extreme Value Theorem, we conclude that this constrained maximization problem has a solution.

Rewrite the problem in the standard form: maximize $f(x) = \sqrt{x_1} + \sqrt{x_2}$ subject to $h_1(x) = -x_1 \leq 0, h_2(x) = -x_2 \leq 0, h_3(x) = 4x_1 + x_2 - 100 \leq 0, h_4(x) = x_1 + x_2 - 60 \leq 0$.

The constraints h_1, h_2, h_3, h_4 are affine functions, so we are in Case 3 of Theorem 19.4: a maximum must satisfy the KKT conditions. The Lagrangian is

$$\mathcal{L}(x, \mu) = \sqrt{x_1} + \sqrt{x_2} - \mu_1(-x_1) - \mu_2(-x_2) - \mu_3(4x_1 + x_2 - 100) - \mu_4(x_1 + x_2 - 60) - .$$

In a local maximum x, the following KKT-conditions must hold:

$$1/2x_1^{-1/2} + \mu_1 - 4\mu_3 - \mu_4 = 0 \tag{4}$$
$$1/2x_2^{-1/2} + \mu_2 - \mu_3 - \mu_4 = 0 \tag{5}$$
$$x_1 \geq 0 \tag{6}$$
$$x_2 \geq 0 \tag{7}$$
$$4x_1 + x_2 \leq 100 \tag{8}$$
$$x_1 + x_2 \leq 60 \tag{9}$$
$$\mu_1, \mu_2, \mu_3, \mu_4 \geq 0 \tag{10}$$
$$\mu_1 x_1 = 0 \tag{11}$$
$$\mu_2 x_2 = 0 \tag{12}$$
$$\mu_3(4x_1 + x_2 - 100) = 0 \tag{13}$$
$$\mu_4(x_1 + x_2 - 60) = 0 \tag{14}$$

Apparently $(x_1, x_2) = (0, 0)$ cannot be optimal in our utility maximization. Further, since the the square root and its first derivative are, respectively, increasing and decreasing functions, we know that the value of $\sqrt{a} + \sqrt{b} > \sqrt{a+b}$. It follows that in optimum, $x_1 > 0, x_2 > 0, \mu_1 = \mu_2 = 0$. Now distinguish four cases, depending on whether the nonnegativity constraints h_3, h_4 are binding:

1. $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$: By (4) and (5): $x_1 = x_2 = 0$.
2. $\mu_1 = \mu_2 = \mu_3 = 0, \mu_4 > 0$: By (4), (5) and (14): $x_1 = x_2 = 30$, contradicting (13).
3. $\mu_1 = \mu_2 = \mu_4 = 0, \mu_3 > 0$: By (4), (5) and (13): $x_1 = 5, x_2 = 80$, contradicting (14).
4. $\mu_1 = \mu_2 = 0, \mu_3 > 0, \mu_4 > 0$: By (13) and (14): $x_1 = 40/3, x_2 = 140/3$.

We showed that there is a maximum. We argued that $(x_1, x_2) = (0, 0)$ cannot be optimal and found only one additional candidate. Conclude: the function is maximized if

$$x^* = \left(\frac{40}{3}, \frac{140}{3}\right)$$

and the maximal value is

$$f(x^*) = 2\sqrt{\frac{5}{3}(9 + 2\sqrt{14})}.$$