Problem Set 4

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- Groups of up to four students may submit one set of solutions. Write each group member's name and student number clearly on the first page of your solutions. Group compositions are allowed to change from one home assignment to another.
- To obtain a good score, write clearly and logically, starting from the definitions and correctly deducing and motivating your answers.
- Please submit your answer **before 13:15 on September 29th**.

Exercise 1 Let $m \in \mathbb{R}$ with $m \geq 2$. Suppose $A_1, A_2, \cdots, A_m$ are convex sets in $\mathbb{R}^n$, show if the following sets are convex or not. If it is, prove; if it is not, show one counter example:

(1) $\cup_{i=1}^m A_i$

(2) $\cap_{i=1}^m A_i$

(3) $\times_{i=1}^m A_i = \{(x_1, x_2, \cdots, x_m) : \ x_i \in A_i, \ i = 1, 2, \cdots, m\}$

(4) $\sum_{i=1}^m A_i = \{x_1 + x_2 + \cdots + x_m : \ x_i \in A_i, \ i = 1, 2, \cdots, m\}$

Exercise 2 Apply Fourier-Motzkin elimination method to solve the following system of linear inequalities

$$\begin{cases} -x_1 - x_2 - x_3 \leq -1 \\ 3x_1 - x_2 - x_3 \leq 1 \\ -x_1 + 3x_2 - x_3 \leq -2 \\ -x_1 - x_2 + 3x_3 \leq 3 \end{cases}$$

Exercise 3 Let C be the convex cone generated by the vectors $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. For each value of t determine if the point $\begin{pmatrix} t \\ t \\ t \end{pmatrix}$ lies in C or find a hyperplane that separates $\begin{pmatrix} t \\ t \\ t \end{pmatrix}$ from C.

Exercise 4 For each of the following functions determine whether it is convex or concave. Are they quasi-concave? Motivate your answer.

(1) $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x) = (x_1^2 + x_2^2)^2$

(2) $f : \mathbb{R}_+^2 \to \mathbb{R}$ with $f(x) = \sqrt{x_1 + x_2}$

(3) $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x) = (x_1 + x_2)^3$

Exercise 5 Solve the problem:

$$\max\ f(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2} \quad \text{with} \quad 4x_1 + x_2 \leq 100 \quad , \quad x_1 + x_2 \leq 60 \quad , \quad x_1, x_2 \geq 0$$