Inequalities, Household Behavior and the Macroeconomy
(Income uncertainty and Inequality)

Course Director: Zoltán Rácz

SSE, Department of Finance

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Last time

- Found that uncertainty increases savings in model economy; unless preferences are quadratic
- Tried to understand why: - Risk aversion and prudence
- In data, people with higher income uncertainty seem to hold more wealth.

Today

- (Theory) Precautionary premium, reminder
- (Theory of coding) Solving infinite horizon models
- (Coding) How does wealth inequality arise in our model economy due to income uncertainty?

Precautionary premium - Intuition

- $$U'(C_t) = \beta(1+r)\mathbb{E}\Big[U'(C_{t+1})\Big]$$
- $$\mathbb{E}\Big[U'(C_{t+1})\Big] > U'\Big(\mathbb{E}[C_{t+1}]\Big)$$

  if $U''' > 0$. This means that consuming the average value for sure leads to too low marginal utility (i.e. C_{t+1} is too high, assuming C_t is unchanged). So without uncertainty, you should optimally consume less in the next period, i.e. save less and consume more now.
- By how much? PP tells you. Losing how much future consumption (relative to expected value) is equivalent to having uncertainty in consumption, in terms of ending up with the same expected marginal utility

  $$\mathbb{E}\Big[U'(C_{t+1})\Big] = U'\Big(\mathbb{E}[C_{t+1}] - PP\Big)$$
- We are interested in keeping expected marginal utility constant, since that implies constant saving today (due to the Euler-equation).

Wealth inequality from income uncertainty and precautionary savings

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- Will work an infinite horizon model. Why?
  - so that you see how they work
  - we can ignore age and its relation to wealth inequality (for today at least)

The problem

$$\max_{\{c_t, a_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\begin{aligned} s.t. \quad & a_t = (1+r)a_{t-1} + y_t - c_t \quad \forall t \in \{0, 1, \dots\} \\ & a_t \geq -b \quad \forall t \in \{0, 1, \dots\} \\ & a_{-1} \text{ given} \end{aligned}$$

Same as before, but we sum up until infinity!

Next, we discuss how to solve this model on the computer.

So what changes?

Two kinds of changes in today's code (31_*infinite_horizon.jl*), relative to the previous one.

1. Changes due to working an infinite horizon
   - Conceptual stuff: Natural borrowing constraint depended on age until now. This will become one number.
   - Iteration: Happened backwards, over ages. One iteration per possible age. Still solve backwards. We just iterate until the solution becomes approximately constant, i.e. doesn't change as you iterate further.
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   - We used to have one consumption and saving policy and one value function for each age. We will have just one of each!
2. Switching to a faster solution method
   - So far we iterated using the Bellman-equation. Will rely on the Euler-equation from now on
   - So far we had a fixed grid for cash-on-hand. Will use a fixed grid for **savings**.
   - Much faster and more precise! (comes handy for your assignment)
   - A bit tricky.

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- In infinite horizon:

$$x_t = c_t + a_t \geq a_t \geq -\sum_{s=t+1}^{\infty} \frac{\min\{y_s\}}{(1+r)^{s-t}} = -\frac{\min\{y\}}{r}$$

assuming that y is iid.

So how do we solve models? – Bigger picture

We have two useful equations:

- The Bellman-equation

$$V_t(x_t) = \max_{a_t \geq -b} \left\{ u(x_t - a_t) + \beta \mathbb{E}_t \left[V_{t+1}(a_t(1+r) + y_{t+1}) \right] \right\}$$

- The Euler-equation

$$u'(c_t) = \beta(1+r)\mathbb{E}_t \left[u'(c_{t+1}) \right]$$

In both equations, we have

- Stuff related to time t on the LHS
- The **expectation** of something related to time $t+1$ on the LHS

This is why we always iterate backwards!

- If you already know something, you can compute its expectation.
- But not the other way round!
- Good direction: from RHS to LHS

Solving finite vs infinite horizon problems

In both setups, we iterate backwards! But different interpretation

Finite horizon:

- There is a final period. Typically obvious policy and value functions
- Then we compute policies of all ages going backwards. Applying the Bellman or Euler-equations for a particular t each time. Stuff on RHS refers to agents one year older than stuff on LHS.
- Done when we got to the first period.

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Infinite horizon:

- We are looking for a solution such that stuff on LHS is identical to stuff RHS. Same value function, same policies.
- Why? Nothing should depend on the time period, only on cash-on-hand! In Infinite horizon, all time periods look the same!
- We guess some policies and a value function to get started.
- We use iterations using the Bellman or Euler-equation to update our guesses. RHS is old guess, LHS is new one.
- We are done when the new objects obtained from the LHS are close enough to the current old guesses.

Value function iteration vs endogenous grid point method

So far we solved our model with value function iteration.

Value function iteration:

- Iterate using the Bellman-equation
- Use a fixed grid for cash-on-hand
- Slow (maximization for each grid point), but reliable
- Works for many kinds of models

Endogenous grid point method:

- Iterate using the Euler-equation
- Use a fixed grid for **savings**
- Very fast
- Perfect for consumption/saving decisions

Endogenous grid point method

1. Start with a given level of savings a_j (jth grid point)
2. Figure out: what could have been my optimal consumption if at the same I decided to save a_j? Solve:
$$u'(c_j) = \beta(1+r)\mathbb{E}_t\Big[u'(c_{t+1})\Big]$$
In our case with CRRA:
$$c_j = \Big(\beta(1+r)\mathbb{E}\Big[u'(c(a_j(1+r)+y))\Big]\Big)^{-1/\gamma}$$
3. What was your cash on hand, if you consumed c_j and saved a_j?
$$c_j + a_j$$
4. To obtain the new consumption policy, we can interpolate the vector of c_js on the vector of $c_j + a_j$s. Obtain new consumption policy by interpolating the vector of a_js on the vector of $c_j + a_j$s.

Endogenous grid point method

We don't have to keep track of the value function when performing EGM. If you are interested in it, you can:

5. New value function be computed as

$$V_{newguess}(c_j + a_j) = u(c_j) + \beta \mathbb{E}_t \Big[V_{oldguess}(a_j(1+r) + y_{t+1}) \Big]$$

   for relevant grid points.

6. Then we can interpolate this too.

Some takeaways from simulations

Two economic forces in balance:

- Impatience:
  A declining consumption (and eventual 0 wealth) would be optimal (without uncertainty)
- Precautionary savings:
  Want a wealth cushion to protect smoothness of consumption from income fluctuations

Which one is stronger? Depends on cash-on-hand!

- Why? Precautionary saving motive is stronger for the poor since prudence is a decreasing function of consumption (and hence of wealth).
- For the poor:
$$\text{Precautionary saving motive} > \text{Impatience motive}$$
- For the rich:
$$\text{Precautionary saving motive} < \text{Impatience motive}$$
- The poor choose savings to have an increasing coh path between now and the next period, in expectation. The rich do the opposite. $\Rightarrow$ everyone strives toward an equilibrium wealth level w^* in the long term.

Some takeaways from simulations

Wealth distribution:

- Everyone strives toward an equilibrium wealth level w^*
- Income shocks throw people around $\Rightarrow$ non-trivial distribution around w^*.
- The distribution is stable over time (after 'burn-in' period)
- But individuals still move around all the time! Position in the wealth distribution depends on past income shocks, especially the more recent ones.

What happens if we turn off one of the forces?

- Turn off impatience: choose $\beta > \frac{1}{1+r}$
  $\Rightarrow$ no steady-state distribution, everybody's wealth grows towards ∞.
- Turn off uncertainty in income
  $\Rightarrow$ boring steady-state distribution, everybody's wealth is 0.