5303 - Advanced Macroeconomics
Assignment 1 Solutions

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Contents

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Chapter 1 - Describing the Environment

Some definitions

- $N(t) \to$ Population of generation t
- At any period t, there are $N(t-1)$ old people and $N(t)$ young people
- $Y(t) \to$ Economy's total endowment of the time t good
- $c_t^h(s) \to$ Consumption of the time **s** good by individual **h** of generation **t**
  - $c_t^h(t) \to$ Consumption of individual h of generation t when young
  - $c_t^h(t+1) \to$ Consumption of individual h of generation t when old
- $c_t^h = [c_t^h(t), c_t^h(t+1)] \to$ Ordered pair of consumption for individual **h** of generation **t**

Exercise 1.3

Let $N(t-1) = 2$ and $Y(t) = 2$ for all $t \geq 1$. Prove that the following allocation is feasible:

$$\begin{cases} c_0^h = \frac{1}{2} \quad \text{for} \quad h = 1,2 \\ c_1^1(1) = \frac{1}{4} \quad \text{and} \quad c_1^1(2) = \frac{3}{4} \\ c_1^2(1) = \frac{3}{4} \quad \text{and} \quad c_1^2(2) = \frac{1}{4} \\ c_t^h(t) = c_t^h(t+1) = \frac{1}{2} \quad \text{for} \quad h = 1,2 \quad \text{and all} \quad t \geq 2 \end{cases}$$

Answer:

Since $N(t-1) = 2, \forall t \geq 1$, we have that $N(t) = 2, \forall t$. Let's make use of the feasibility definition:

Definition 1. [1] A consumption allocation is **feasible** if the consumption path satisfies:

$$C(t) = \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) \leq Y(t), \quad \forall t \geq 1$$

[1] Page 10 of the course textbook.

Since feasibility must be satisfied for any period $t \geq 1$, we need to address all periods. For $t = 1$, we have:

$$\begin{aligned} C(1) &= \sum_{h=1}^{2} c_1^h(1) + \sum_{h=1}^{2} c_0^h(1) \\ &= c_1^1(1) + c_1^2(1) + c_0^1(1) + c_0^2(1) \\ &= \frac{1}{4} + \frac{3}{4} + \frac{1}{2} + \frac{1}{2} = 2 \leq Y(1) \end{aligned}$$

For $t = 2$, we have:

$$\begin{aligned} C(2) &= \sum_{h=1}^{2} c_2^h(2) + \sum_{h=1}^{2} c_1^h(2) \\ &= c_2^1(2) + c_2^2(2) + c_1^1(2) + c_1^2(2) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{3}{4} + \frac{1}{4} = 2 \leq Y(2) \end{aligned}$$

And for all periods $t \geq 3$, we have that:

$$\begin{aligned} C(t) &= \sum_{h=1}^{2} c_t^h(t) + \sum_{h=1}^{2} c_{t-1}^h(t) \\ &= c_t^1(t) + c_t^2(t) + c_{t-1}^1(t) + c_{t-1}^2(t) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2 \leq Y(t) \end{aligned}$$

Since $C(t) \leq Y(t), \forall t \geq 1$, we can conclude that the allocation is feasible.

Exercise 1.7

Prove the following: If an allocation is Pareto optimal, then it is efficient.

Answer:

Let's state a few definitions:

Definition 2. [2] A feasible consumption allocation is **efficient** if there is no alternative feasible allocation with more total consumption of some good and no less of any other good.

[2] Page 12 of the course textbook.

Definition 3. [3] Consumption allocation A is **Pareto superior** to consumption allocation B if:

1. no one strictly prefers B to A, and
2. at least one person strictly prefers A to B.

Definition 4. [4] A consumption allocation is **Pareto optimal** if it is feasible and if there does not exist a feasible consumption allocation that is Pareto superior to it.

Let's suppose that a given allocation $A = \left(\{c_t^h(t)\}_{h=1}^{N(t)}, \{c_{t-1}^h(t)\}_{h=1}^{N(t-1)}\right)$ is Pareto optimal but not efficient, i.e., that for some period t the resource constraint is not satisfied with equality: $C(t) < Y(t)$. This means that there are some unallocated resources in the economy at that period.

Given this, suppose that there is a second allocation B that assigns the unused resources in allocation A to at least one agent in the economy. Given that utility is increasing in consumption, this agent's utility would increase, while the utility of everybody else would remain the same, since their consumption is not being affected. This means that allocation B is Pareto superior to allocation A: at least one person is better off under allocation B and none of the other agents strictly prefers allocation A to B.

Therefore, since we found an allocation that is Pareto superior to allocation A, we can conclude that allocation A is not Pareto optimal. This it what is known as a Proof by Contrapositive.

Chapter 2 - Competitive Equilibrium

Some definitions:

- $\omega_t^h(s) \rightarrow$ Endowment of the time **s** good by individual **h** of generation **t**
- $\omega_t^h = [\omega_t^h(t), \omega_t^h(t+1)] \rightarrow$ Endowment of individual **h** of generation **t**
- Total time **t** endowment of the economy $\rightarrow Y(t) = \sum_{h=1}^{N(t)} \omega_t^h(t) + \sum_{h=1}^{N(t-1)} \omega_{t-1}^h(t)$

[3] Page 20 of the course textbook.
[4] Page 22 of the course textbook.

- $l^h(t) \rightarrow$ Lending of individual **h** of generation **t**, measured in terms of the time **t** good

- $r(t) \rightarrow$ Gross real interest rate at time **t**. Measured in terms of the time **t + 1** good per unit of the time **t** good

Exercise 2.3

Find the $S_t(r(t))$ function for the following cases:

(a) $N(t) = 100$; each h in generation t has the utility function

$$u_t^h = c_t^h(t)[c_t^h(t+1)]^\beta$$

with $\beta = 1$ and $[\omega_t^h(t), \omega_t^h(t+1)] = [2, 1]$.

(b) $N(t) = 100$; each h has the above utility function with $\beta = 1$ and

$$[\omega_t^h(t), \omega_t^h(t+1)] = \begin{cases} [2,1], h = 1, 2 \ldots, 50 \\ [1,1], h = 51, 52, \ldots, 100 \end{cases}$$

Answer:

(a) To find the aggregate savings function $S_t(r(t))$, we first need to solve the competitive choice problem for individual h of generation t, which consists in choosing an affordable consumption pair to maximize the utility function, constrained by the lifetime budget constraint, with $r(t)$ and the endowments viewed as given.

We know that individuals face the following budget constraints:

$$\begin{cases} \text{Young: } c_t^h(t) \leq \omega_t^h(t) - l^h(t) \\ \text{Old: } c_t^h(t+1) \leq \omega_t^h(t+1) + r(t)l^h(t) \end{cases}$$

Recall that no intergenerational trade takes place[5]. We can combine both budget constraints into a single lifetime budget constraint, in terms of the time t good:

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} \leq \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}$$

[5]For a detailed discussion on this, please refer to page 46 of the course textbook.

which basically states that the present discounted value of lifetime consumption must be smaller or equal than the present discounted value of endowments for all the agents in the economy and for all the generations.

The competitive choice problem is thus given by:

$$\max_{\{c_t^h(t), c_t^h(t+1)\}} c_t^h(t)[c_t^h(t+1)]^\beta$$
$$\text{s.t. } c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} \leq \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}$$

Since we assume that more is always better, the lifetime budget constraint will hold with equality for an utility maximizing individual. We can thus rewrite it in terms of $c_t^h(t+1)$:

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}$$
$$c_t^h(t+1) = \omega_t^h(t+1) + r(t)\big[\omega_t^h(t) - c_t^h(t)\big]$$

And we can also rewrite the competitive choice problem:

$$\max_{\{c_t^h(t)\}} c_t^h(t)\big[\omega_t^h(t+1) + r(t)[\omega_t^h(t) - c_t^h(t)]\big]^\beta$$

Taking the first-order-condition:

$c_t^h(t)$:

$$\big[\omega_t^h(t+1) + r(t)[\omega_t^h(t) - c_t^h(t)]\big]^\beta - \beta r(t) c_t^h(t)\big[\omega_t^h(t+1) + r(t)[\omega_t^h(t) - c_t^h(t)]\big]^{\beta-1} = 0$$
$$[c_t^h(t+1)]^\beta = \beta r(t) c_t^h(t)[c_t^h(t+1)]^{\beta-1}$$
$$c_t^h(t+1) = \beta r(t) c_t^h(t)$$

Notice that the result above corresponds to $MRS(t, t+1) = r(t)$. Substituting in the lifetime budget constraint:

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} = \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}$$
$$c_t^h(t) + \frac{\beta r(t) c_t^h(t)}{r(t)} = \omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}$$
$$c_t^h(t) = \Big(\frac{1}{1+\beta}\Big)\big[\omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}\big]$$

Which is our demand function for consumption when young, i.e.:

$$c_t^h(t) = \chi_t^h(r(t), \omega_t^h(t), \omega_t^h(t+1)) = \Big(\frac{1}{1+\beta}\Big)\big[\omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}\big]$$

Substituting $\beta = 1$ and $\omega_t^h = (2, 1)$, we have that:

$$c_t^h(t) = \frac{1}{2} \times \left[2 + \frac{1}{r(t)}\right] = 1 + \frac{1}{2r(t)}$$

And the savings function is given by:

$$\begin{aligned}
s_t^h(t) &= \omega_t^h(t) - \chi_t^h(r(t), \omega_t^h(t), \omega_t^h(t+1)) \\
&= \left(\frac{\beta}{1+\beta}\right)\omega_t^h(t) - \left(\frac{1}{1+\beta}\right)\left[\frac{\omega_t^h(t+1)}{r(t)}\right] \\
\textit{substituting:} \quad &= 2 - \left[1 + \frac{1}{2r(t)}\right] = 1 - \frac{1}{2r(t)}
\end{aligned}$$

And the Aggregate Savings Function, $S_t(r(t))$, is thus given by:

$$\begin{aligned}
S_t(r(t)) &= \sum_{h=1}^{N(t)} (\omega_t^h(t) - c_t^h(t)) = \sum_{h=1}^{N(t)} s_t^h(r(t)) \\
&= 100 \times \left[1 - \frac{1}{2r(t)}\right] \\
&= 100 - \frac{50}{r(t)}
\end{aligned}$$

(b) Since the utility function is the same as before, we can make use of our previous results, accounting for the fact that half of each generation t receives twice the amount of the time t good when young, compared to the other half. The demand functions for consumption are thus given by:

$$\begin{aligned}
c_t^{(1-50)}(t) &= \frac{1}{2} \times \left[2 + \frac{1}{r(t)}\right] = 1 + \frac{1}{2r(t)} \\
c_t^{(51-100)}(t) &= \frac{1}{2} \times \left[1 + \frac{1}{r(t)}\right] = \frac{1}{2} + \frac{1}{2r(t)}
\end{aligned}$$

And the savings functions:

$$\begin{aligned}
s_t^{(1-50)}(t) &= 2 - \left[1 + \frac{1}{2r(t)}\right] = 1 - \frac{1}{2r(t)} \\
s_t^{(51-100)}(t) &= 1 - \left[\frac{1}{2} + \frac{1}{2r(t)}\right] = \frac{1}{2} - \frac{1}{2r(t)}
\end{aligned}$$

Meaning that the Aggregate Savings Function is given by:

$$\begin{aligned} S_t(r(t)) &= \sum_{h=1}^{N(t)} s_t^h(r(t)) \\ &= 50 \times \left[1 - \frac{1}{2r(t)}\right] + 50 \times \left[\frac{1}{2} - \frac{1}{2r(t)}\right] \\ &= 75 - \frac{50}{r(t)} \end{aligned}$$

Exercise 2.4

Describe completely the competitive equilibria for the following economies:

(a) $N(t) = 100, \forall t \geq 0$. Each member of generation t, $t \geq 0$, has the Exercise 2.3 utility function with $\beta = 1$ and $[\omega_t^h(t), \omega_t^h(t+1)] = [2, 1]$ for all h and $t \geq 0$.

(e) Same as a, except that $\forall t \geq 0$:

$$[\omega_t^h(t), \omega_t^h(t+1)] = \begin{cases} [2,1], h = 1, 2, \ldots, 60 \\ [1,1], h = 61, 62, \ldots, 100 \end{cases}$$

(f) Same as a, except that $\forall t \geq 0$:

$$[\omega_t^h(t), \omega_t^h(t+1)] = \begin{cases} [1,1], t = 1, 3, 5, \ldots \\ [2,1], t = 2, 4, 6, \ldots \end{cases}$$

Answer:

Let's state a few definitions/propositions:

Definition 5. [6] A **competitive equilibrium** is a set of prices and quantities that satisfy the following two conditions:

1. The quantities that are relevant for a particular person maximize that person's utility in the set of all quantities that are affordable given the prices and the person's endowment.

2. The quantities clear all markets at date t

[6] Page 45 of the course textbook.

Proposition 1. [7] If the quantities and the sequence $\{r(t)\}$ are a competitive equilibrium, then $\{r(t)\}$ satisfies

$$S_t(r(t)) = 0$$

for every t.

Proposition 2. [8] If $\{r(t)\}$ satisfies the condition above for every t, then there exist quantities such that they and the $r(t)$ sequence are an equilibrium.

(a) Recall our results from Exercise 2.3:

$$c_t^h(t) = \chi_t^h(r(t), \omega_t^h(t), \omega_t^h(t+1)) = \Big(\frac{1}{1+\beta}\Big)\big[\omega_t^h(t) + \frac{\omega_t^h(t+1)}{r(t)}\big]$$
$$s_t^h(t) = \Big(\frac{\beta}{1+\beta}\Big)\omega_t^h(t) - \Big(\frac{1}{1+\beta}\Big)\Big[\frac{\omega_t^h(t+1)}{r(t)}\Big]$$

Substituting, we have that:

$$c_t^h(t) = 1 + \frac{1}{2r(t)}$$
$$s_t^h(t) = 1 - \frac{1}{2r(t)}$$
$$S_t(r(t)) = 100 - \frac{50}{r(t)}$$

A competitive equilibrium is such that the goods market clears and the market for borrowing and lending clears each period, i.e.:

$$S_t(r(t)) = 0, \forall t \geq 1 \Leftrightarrow 100 - \frac{50}{r(t)} = 0 \Leftrightarrow r(t) = \frac{1}{2}$$

So individual consumption and savings are thus given by:

$$c_t^h(t) = 1 + \frac{1}{2 \times \frac{1}{2}} = 2, \forall t \geq 1$$
$$c_t^h(t+1) = \beta r(t) c_t^h(t) = \frac{1}{2} \times 2 = 1, \forall t \geq 1$$
$$c_0^h(1) = \omega_0^h(1) = 1$$
$$s_t^h(t) = 1 - \frac{1}{2 \times \frac{1}{2}} = 0, \forall t \geq 1$$

We can thus conclude that there are no savings in this economy and that all the individuals in all generations consume their income/endowment.

[7] Page 48 of the course textbook.
[8] Page 48 of the course textbook.

Moreover, it is straightforward to show that this allocation is feasible and efficient:

$$C(t) = 100 \times 2 + 100 \times 1 = 300 = Y(t), \forall t \geq 1$$

(e) This economy is identical to the previous one, except for the fact that the population is divided into two groups, the first consisting of 60 "high-income" households, with endowment stream $(2, 1)$, and the second consisting of 40 "low-income" households, with endowment stream $(1, 1)$. We can once again make use of our results from Exercise 2.3, this time differentiating by household type:

$$\begin{aligned}
c_t^{(1-60)}(t) &= 1 + \frac{1}{2r(t)} \\
c_t^{(61-100)}(t) &= \frac{1}{2}\left[1 + \frac{1}{r(t)}\right] = \frac{1}{2} + \frac{1}{2r(t)} \\
s_t^{(1-60)}(t) &= 1 - \frac{1}{2r(t)} \\
s_t^{(61-100)}(t) &= 1 - \left[\frac{1}{2} + \frac{1}{2r(t)}\right] = \frac{1}{2} - \frac{1}{2r(t)}
\end{aligned}$$

And the Aggregate Savings are thus given by:

$$\begin{aligned}
S_t(r(t)) &= 60 \times s_t^{(1-60)}(t) + 40 \times s_t^{(61-100)}(t) \\
&= 60 \times \left[1 - \frac{1}{2r(t)}\right] + 40 \times \left[\frac{1}{2} - \frac{1}{2r(t)}\right] \\
&= 80 - \frac{50}{r(t)}
\end{aligned}$$

A competitive equilibrium is such that:

$$S_t(r(t)) = 0, \forall t \geq 1 \Leftrightarrow 80 - \frac{50}{r(t)} = 0 \Leftrightarrow r(t) = \frac{5}{8}$$

So individual consumption and savings of the 60 "high-income" individuals are thus given by:

$$\begin{aligned}
c_t^{(1-60)}(t) &= 1 + \frac{1}{2 \times \frac{5}{8}} = \frac{9}{5} = 1.8 \\
c_t^{(1-60)}(t+1) &= \beta r(t) c_t^{(1-60)}(t) = \frac{5}{8} \times \frac{9}{5} = \frac{9}{8} = 1.125 \\
s_t^{(1-60)}(t) &= 1 - \frac{1}{2 \times \frac{5}{8}} = \frac{1}{5} = 0.2
\end{aligned}$$

And for the 40 "low-income" individuals:

$$c_t^{(61-100)}(t) = \frac{1}{2} + \frac{1}{2 \times \frac{5}{8}} = \frac{13}{10} = 1.3$$
$$c_t^{(61-100)}(t+1) = \beta r(t) c_t^{(1-60)}(t) = \frac{5}{8} \times \frac{13}{10} = \frac{13}{16} = 0.8125$$
$$s_t^{(61-100)}(t) = \frac{1}{2} - \frac{1}{2 \times \frac{5}{8}} = -\frac{3}{10} = -0.3$$

Each of the 60 "high-income" individuals lends $\frac{1}{5}$ units of the time t good to the 40 "low-income" indidividuals who then each pay back $\frac{5}{8} \times \frac{3}{10} = \frac{3}{16}$ units of the good when old. Furthermore, as before, we have $c_0^h(1) = 1$. Also, it is straightforward to show that this allocation is feasible and efficient:

$$C(t) = 60 \times \frac{9}{5} + 40 \times \frac{13}{10} + 60 \times \frac{9}{8} + 40 \times \frac{13}{16} = 260$$

$$Y(t) = 60 \times 2 + 40 \times 1 + 60 \times 1 + 40 \times 1 = 260$$

(f) Once again, we have two groups of individuals with different endowments, the first group consisting of people born at even times and the second group consisting of people born at odd times. For illustrative purposes, let's set $t = even$ if a person is born/lives at even times and $t = odd$ if a person is born/lives at odd times. The endowment streams are given by:

$$\text{Even Generations:} \rightarrow [\omega^h_{even}(even), \omega^h_{even}(odd)] = [2, 1]$$
$$\text{Odd Generations:} \rightarrow [\omega^h_{odd}(odd), \omega^h_{odd}(even)] = [1, 1]$$

Making use of our previous results:

$$c^h_{even}(even) = 1 + \frac{1}{2r(even)}$$
$$c^h_{odd}(odd) = \frac{1}{2} + \frac{1}{2r(odd)}$$
$$s^h_{even}(even) = 1 - \frac{1}{2r(even)}$$
$$s^h_{odd}(odd) = \frac{1}{2} - \frac{1}{2r(odd)}$$

It is clear that aggregate endowment changes depending on whether we are at an odd or an even time:

$$Y(even) = 100\omega^h_{even}(even) + 100\omega^h_{odd}(even) = 300$$
$$Y(odd) = 100\omega^h_{odd}(odd) + 100\omega^h_{even}(odd) = 200$$

So Aggregate Savings change as well:

$$S_{even}(r(even)) = 100 \times \left[1 - \frac{1}{2r(even)}\right] = 100 - \frac{50}{r(even)}$$
$$S_{odd}(r(odd)) = 100 \times \left[\frac{1}{2} - \frac{1}{2r(odd)}\right] = 50 - \frac{50}{r(odd)}$$

And in a competitive equilibrium, we have that:

$$S_{even}(r(even)) = 0 \Leftrightarrow r(even) = \frac{1}{2}$$
$$S_{odd}(r(odd)) = 0 \Leftrightarrow r(odd) = 1$$

With these interest rates, no individual will save at equilibrium, meaning that individuals consume their whole endowment at each time, i.e.:

$$s^h_{even}(even) = 1 - \frac{1}{2 \times \frac{1}{2}} = 0$$
$$s^h_{odd}(odd) = \frac{1}{2} - \frac{1}{2 \times 1} = 0$$
$$c^h_{even} = (2, 1)$$
$$c^h_{odd} = (1, 1)$$
$$c^h_0(1) = 1$$

Moreover, it is straightforward to show that this allocation is feasible and efficient:

$$C(even) = 100 \times 2 + 100 \times 1 = 300 = Y(even)$$

$$C(odd) = 100 \times 1 + 100 \times 1 = 200 = Y(odd)$$

Exercise 2.5

Prove that the competitive equilibrium of each of the economies in question 2.4 is not Pareto optimal.

Answer:

Note: Please refer to Definitions 1, 3 and 4 above.

(a) Recall that the optimal consumption allocation is given by $c^h_t = (2, 1), \forall t \geq 1, \forall h$, whereas the initial old have $c^h_0(1) = 1$. This allocation gives the following utility:

$$u_t^h(2,1) = 2 \times 1^1 = 2$$

Consider the following alternative allocation: $c_t^h = (1,2), \forall t, \forall h$. We now have that:

$$\begin{aligned} u_t^h(1,2) &= 1 \times 2^1 = 2 \\ c_0^h(1) &= 2 \end{aligned}$$

Now, all members of all generations $t \geq 1$ have the same utility level as before but the old of period 1 are better off. Moreover, this allocation is also feasible:

$$C(t) = 100 \times 1 + 100 \times 2 = 300 = Y(t)$$

Since we found a Pareto superior allocation, the competitive equilibrium allocation is not Pareto optimal.

(e) The competitive equilibrium allocation is given by, $\forall t \geq 1$:

$$\begin{aligned} c_t^{(1-60)} &= \left(\frac{9}{5}, \frac{9}{8}\right) = (1.8, 1.125) \\ c_t^{(61-100)} &= \left(\frac{13}{10}, \frac{13}{16}\right) = (1.3, 0.8125) \\ c_0^h(1) &= 1, \forall h \end{aligned}$$

With the following corresponding utility levels:

$$\begin{aligned} u_t^{(1-60)}\left(\frac{9}{5}, \frac{9}{8}\right) &= \frac{9}{5} \times \left(\frac{9}{8}\right)^1 = \frac{81}{40} = 2.025 \\ u_t^{(61-100)}\left(\frac{13}{10}, \frac{13}{16}\right) &= \frac{13}{10} \times \left(\frac{13}{16}\right)^1 = \frac{169}{160} = 1.05625 \end{aligned}$$

Consider the alternative allocation $c_t^{(1-60)} = (1.5, 1.4)$ and $c_t^{(61-100)} = (1.15, 1)$. The utility levels are now:

$$\begin{aligned} u_t^{(1-60)}(1.5, 1.4) &= 1.5 \times 1.4^1 = 2.1 \\ u_t^{(61-100)}(1.15, 1) &= 1.15 \times 1^1 = 1.15 \end{aligned}$$

All members of all generations $t \geq 1$ have a higher utility level than before and the rich old of period 1 are better off, whereas the poor old of period 1 remain the same. Moreover, this allocation is also feasible:

$$C(t) = 60 \times 1.5 + 40 \times 1.15 + 60 \times 1.4 + 40 \times 1 = 60(2+1) + 40(1+1) = Y(t)$$

Since we found a Pareto superior allocation, the competitive equilibrium allocation is not Pareto optimal.

(f) The competitive equilibrium allocation is given by, $\forall t \geq 1$:

$$c^h_{even} = (2, 1), \text{ for } t = even$$
$$c^h_{odd} = (1, 1), \text{ for } t = odd$$

With the following corresponding utility levels:

$$u^h_{even} = 2 \times 1^1 = 2, \text{ for } t = even$$
$$u^h_{odd} = 1 \times 1^1 = 1, \text{ for } t = odd$$

Consider the alternative allocation:

$$c^h_{even} = \left(\frac{7}{4}, \frac{6}{5}\right), \text{ for } t = even$$
$$c^h_{odd} = \left(\frac{4}{5}, \frac{5}{4}\right), \text{ for } t = odd$$

The utility levels are now:

$$u^h_{even} = \frac{7}{4} \times \frac{6}{5} = \frac{21}{10} > 2, \text{ for } t = even$$
$$u^h_{odd} = \frac{4}{5} \times \frac{5}{4} = 1, \text{ for } t = odd$$

Under this allocation, individuals born at even times are better off, as well as the initial old, who now consume $c^h_0(1) = \frac{6}{5} > 1$. Individuals born at odd times remain unchanged. Moreover this allocation is also feasible since:

$$C(even) = 100 \times \frac{7}{4} + 100 \times \frac{5}{4} = 300 = Y(even)$$
$$C(odd) = 100 \times \frac{4}{5} + 100 \times \frac{6}{5} = 200 = Y(odd)$$

Since we found a Pareto superior allocation, the competitive equilibrium allocation is not Pareto optimal.

Chapter 3 - Introducing a Government

Some definitions:

- $t^h_t = [t^h_t(t), t^h_t(t+1)] \rightarrow$ Set of taxes and transfers faced by individual **h** of generation **t**

- $t^h_t(s) \rightarrow$ Tax in the form of time s equals t or $t+1$ good payable by person h of generation t
- Pretax endowment $\rightarrow [\omega^h_t(t), \omega^h_t(t+1)]$
- Posttax and transfer endowment $\rightarrow [\omega^h_t(t) - t^h_t(t), \omega^h_t(t+1) - t^h_t(t+1)]$
- $p(t) \rightarrow$ Price of one government bond at time t
- $b^h(t) \rightarrow$ Quantity of government bonds purchased by individual **h** of generation **t**

Exercise 3.1

(Taxes on young, transfers to old; or "social security" schemes) Here we consider only balanced budget schemes, those for which the taxes collected from the young at any date t equal the transfers at that date to the old. Assume throughout that people when young know and take into account the transfers they will receive when old.

(a) Consider the Exercise 2.4a economy and consider a scheme in which each person when young is taxed 1 unit and each person when old receives 1 unit as a transfer. Describe the competitive equilibrium and show that it is Pareto superior to that obtained in the absence of the scheme

(c) Consider the Exercise 2.4a economy but with $N(0) = 100$ and $N(t) = 2N(t-1)$ for all $t \geq 1$. If each person when young is taxed 1 unit, then what is the maximum that can be given to every person when old? Compare the competitive equilibria with and without this scheme.

(d) Consider the Exercise 2.4a economy but with $N(0) = 100$, $N(t) = 2N(t-1)$ for $t = 1, 2, \ldots, 10$ and $N(t) = N(10)$ for $t \geq 10$. Describe some alternative social security schemes for this economy. Relate the "difficulties" that arise to those that currently plague the U.S. social security system.

Answer:

(a) Consider the following vector of taxes: $t^h_t = (1, -1)$, i.e., the young are taxed 1 unit and the old receive a 1 unit subsidy. Now individuals face the following budget constraints:

$$\begin{cases} \text{Young: } c_t^h(t) \leq \omega_t^h(t) - t_t^h(t) - l^h(t) \\ \text{Old: } c_t^h(t+1) \leq \omega_t^h(t+1) - t_t^h(t+1) + r(t)l^h(t) \end{cases}$$

Once again, we can combine both into a single lifetime budget constraint, in terms of the time t good:

$$c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} \leq \omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)}$$

Since the lifetime budget constraint must hold with equality for an individual-maximizing individual, we can rewrite it in terms of $c_t^h(t+1)$:

$$c_t^h(t+1) = r(t)[\omega_t^h(t) - t_t^h(t) - c_t^h(t)] + \omega_t^h(t+1) - t_t^h(t+1)$$

And the competitive choice problem is thus given by:

$$\max_{\{c_t^h(t)\}} c_t^h(t)\Big[r(t)[\omega_t^h(t) - t_t^h(t) - c_t^h(t)] + \omega_t^h(t+1) - t_t^h(t+1)\Big]^\beta$$

Taking the first-order condition:

$c_t^h(t)$:

$$\begin{aligned} &\Big[r(t)[\omega_t^h(t) - t_t^h(t) - c_t^h(t)] + \omega_t^h(t+1) - t_t^h(t+1)\Big]^\beta \\ &\qquad - \beta r(t)c_t^h(t)\Big[r(t)[\omega_t^h(t) - t_t^h(t) - c_t^h(t)] + \omega_t^h(t+1) - t_t^h(t+1)\Big]^{\beta-1} = 0 \\ &[c_t^h(t+1)]^\beta = \beta r(t)c_t^h(t)[c_t^h(t+1)]^{\beta-1} \\ &\quad c_t^h(t+1) = \beta r(t)c_t^h(t) \end{aligned}$$

We can see that the equilibrium condition is the same as before, meaning that taxes do not affect the intertemporal MRS of the consumer. Plugging in the lifetime budget constraint:

$$\begin{aligned} c_t^h(t) + \frac{c_t^h(t+1)}{r(t)} &= \omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)} \\ c_t^h(t) + \frac{\beta r(t)c_t^h(t)}{r(t)} &= \omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)} \\ c_t^h(t) &= \Big(\frac{1}{1+\beta}\Big)[\omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)}] \end{aligned}$$

Which is our demand function for consumption when young, i.e.:

$$\begin{aligned} c_t^h(t) &= \chi\big(r(t), \omega_t^h(t) - t_t^h(t), \omega_t^h(t+1) - t_t^h(t+1)\big) \\ &= \Big(\frac{1}{1+\beta}\Big)\big[\omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)}\big] \end{aligned}$$

Substituting $\beta = 1$ and $\omega_t^h = (2, 1)$, we have that:

$$c_t^h(t) = \frac{1}{2}\big[2 - 1 + \frac{1-(-1)}{r(t)}\big] = \frac{1}{2} + \frac{1}{r(t)}$$

And the savings function is given by

$$\begin{aligned} s_t^h(t) &= \omega_t^h(t) - t_t^h(t) - \chi\big(r(t), \omega_t^h(t) - t_t^h(t), \omega_t^h(t+1) - t_t^h(t+1)\big) \\ &= \frac{1}{2} - \frac{1}{r(t)} \end{aligned}$$

And the Aggregate Savings Function is given by:

$$S_t(r(t)) = 100 \times \Big[\frac{1}{2} - \frac{1}{r(t)}\Big] = 50 - \frac{100}{r(t)}$$

A competitive equilibrium is such that, $\forall t \geq 1$:

$$S_t(r(t)) = 0 \Leftrightarrow 50 - \frac{100}{r(t)} = 0 \Leftrightarrow r(t) = 2$$

So individual consumption and savings are given by:

$$\begin{aligned} c_t^h(t) &= \frac{1}{2} + \frac{1}{2} = 1 \\ c_t^h(t+1) &= \beta r(t) c_t^h(t) = 2 \times 1 = 2 \\ c_0^h(1) &= \omega_0^h(1) - t_0^h(1) = 1 - (-1) = 2 \\ s_t^h(t) &= \frac{1}{2} - \frac{1}{2} = 0 \end{aligned}$$

This allocation is feasible since:

$$C(t) = 100 \times 1 + 100 \times 2 = 300 = Y(t)$$

And the government budget constraint holds, $\forall t \geq 0$:

$$\sum_{h=1}^{N(t)} t_t^h(t) + \sum_{h=1}^{N(t-1)} t_{t-1}^h(t) = 100 \times 1 + 100 \times (-1) = 0$$

Now we need to check whether the equilibrium with this tax scheme is

Pareto superior to the equilibrium without any tax scheme. Recall that in the case with no taxes, we have $c_t^h = (2, 1), \forall t$ for a corresponding utility level of $u_t^h(2, 1) = 2, \forall t$ and $c_0^h(1) = 1$.

With the tax scheme, we have $c_t^h = (1, 2), \forall t$ for a corresponding utility level of $u_t^h(1, 2) = 2, \forall t$ and $c_0^h(1) = 2$. Since the initial old are better off and all the individuals of all generations $t \geq 1$ have the same utility level, we can conclude that the allocation with the tax scheme is Pareto superior to the allocation with no tax scheme.

(c) The question we are now interested in is what is the maximum subsidy $t_{t-1}^h(t)$ that we can give to old people, given that $t_t^h(t) = 1$? The tax vector we are now facing is $t_t^h = (1, x)$. Let's make use of our previous results:

$$c_t^h(t) = \Big(\frac{1}{1+\beta}\Big)\big[\omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)}\big] = \frac{1}{2} + \frac{1-x}{2r(t)}$$

$$s_t^h(t) = \omega_t^h(t) - t_t^h(t) - c_t^h(t) = \frac{1}{2} - \frac{1-x}{2r(t)}$$

And the Aggregate Savings Function is now given by:

$$S_t(r(t)) = \sum_{h=1}^{N(t)} s_t^h(t) = N(t)\Big[\frac{1}{2} - \frac{1-x}{2r(t)}\Big]$$

In a competitive equilibrium, we must have that, $\forall t$:

$$S_t(r(t)) = 0 \Leftrightarrow N(t)\Big[\frac{1}{2} - \frac{1-x}{2r(t)}\Big] = 0 \Leftrightarrow r(t) = 1 - x$$

For this allocation to be feasible, the aggregate subsidy should not exceed the resources collected through taxation, meaning that the government budget constraint must be such that:

$$N(t)t_t^h(t) + N(t-1)t_{t-1}^h(t) \geq 0$$

Since $t_{t-1}^h(t)$ is maximized when the government budget constraint holds with equality, i.e., all tax revenues are allocated to old individuals, it follows that:

$$\begin{aligned} N(t)t_t^h(t) + N(t-1)t_{t-1}^h(t) &= 0 \\ 2N(t-1) \times 1 + N(t-1) \times x &= 0 \\ x &= -2 \end{aligned}$$

Meaning that $r(t) = 1 - x = 3$. Individual consumption and savings are thus given by:

$$\begin{aligned}
c_t^h(t) &= \frac{1}{2} + \frac{1-(-2)}{2\times 3} = 1 \\
c_t^h(t+1) &= \beta r(t) c_t^h(t) = 3\times 1 = 3 \\
c_0^h(1) &= 3 \\
s_t^h(t) &= \frac{1}{2} - \frac{1-(-2)}{2\times 3} = 0
\end{aligned}$$

And the utility is given by:

$$u_t^h(1,3) = 3 > u_t^h(2,1) = 2$$

Since all individuals of generation t have a higher utility level and the initial old are better off under this tax scheme, we can conclude that this allocation is Pareto superior to the allocation without the tax scheme.

(d) We can make use of our results in a and c and design the following social security scheme:

$$\begin{cases} t \in [0,9] \rightarrow t_t^h = (1,-2) \text{ and } c_t^h = (1,3) \\ t \geq 10 \rightarrow t_t^h = (1,-1) \text{ and } c_t^h = (1,2) \end{cases}$$

This allocation is clearly Pareto superior to the one without any scheme since all the agents now have an utility level at least equal to the one without any scheme.

An interesting point to make here is that the old of generations $t < 10$ are better off than the old of generations $t > 10$. This happens because, without any population growth for $t \geq 10$, keeping the same utility level for everyone would require increasing taxes on the young. Without this tax increase, the government budget scheme would no longer be balanced, as one can see below for $t = 11$:

$$N(t)t_t^h(t) + N(t-1)t_{t-1}^h(t) = N(11)\times 1 + N(10)\times(-2) = -N(10) < 0$$

Therefore, the entire pension scheme must be adjusted to account for the lack of population growth.

Exercise 3.2

(Consumption versus income taxes) To get started, we write versions of the budget constraints of the young and the old that include a flat-rate consumption tax, z_c, and a flat-rate income tax, z_y:

1. $c_t^h(t) \leq (1 - z_y)\omega_t^h(t) - l^h(t) - z_c c_t^h(t)$
2. $c_t^h(t+1) \leq (1 - z_y)[\omega_t^h(t+1) + r(t)l^h(t)] - z_c c_t^h(t+1)$

Show, without making any particular assumptions about preferences or endowments, that all nonnegative pairs (z_c, z_y) satisfying $\frac{1-z_y}{1+z_c} = g$ for a given, constant g satisfying $0 < g < 1$ are equivalent. As part of your answer, define *equivalent*. (Hint: What is the constraint an individual faces when making consumption decisions? How does $r(t)$ enter into that constraint?)

Answer:

Definition 6. Nonnegative tax pairs (z_y, z_c) are **equivalent** when they do not affect individual consumption and savings decisions.

We need to show that pairs that satisfy

$$\frac{1 - z_y}{1 + z_c} = g \in]0, 1[$$

are equivalent. Making use of the given budget constraints, we can combine them into a single lifetime utility constraint:

$$c_t^h(t) + \frac{c_t^h(t+1)}{(1 - z_y)r(t)} \leq \Big(\frac{1 - z_y}{1 + z_c}\Big)\Big(\omega_t^h(t) + \frac{\omega_t^h(t+1)}{(1 - z_y)r(t)}\Big)$$

which basically states that the present value of consumption must be the equal to the after-tax present value of endowments. Since g is constant by assumption, the only way taxes affect the lifetime budget constraint is via the interest rate, i.e. $(1 - z_y)r(t)$. This means that the consumption and savings decisions are the same under two alternative tax plans if the after-tax interest rate is the same in both plans. Without making any assumptions for the functional form of the utility function, we know that the equilibrium condition for this economy is given by:

$$(1 - z_y)r(t) = MRS = \frac{\frac{\partial u_t^h}{\partial c_t^h(t)}}{\frac{\partial u_t^h}{\partial c_t^h(t+1)}}$$

By knowing the utility function, we would be able to retrieve equilibrium consumption. However, we are nonetheless able to infer that consumption at time t is a function of the after-tax interest rate $(1 - z_y)r(t)$. The same can be said of the aggregate savings function, S_t. Moreover, under a competitive equilibrium, we must have that:

$$S_t((1 - z_y)r(t)) = 0$$

Notice that, even though we cannot make any particular assumptions about preferences, the setting we are working with requires us to assume that preferences have "nice" properties (the utility function is strictly increasing in each of its arguments, is differentiable and is convex). Let's consider two alternative tax plans and denote them by $\left(z_y^1, z_c^1\right)$ and (z_y^2, z_c^2), such that:

$$\frac{1 - z_y^1}{1 + z_c^1} = \frac{1 - z_y^2}{1 + z_c^2} = g \in]0, 1[$$

For them to be consistent with a competitive equilibrium allocation, we must have that:

$$S_t((1 - z_y^1)r^1(t)) = 0$$
$$S_t((1 - z_y^2)r^2(t)) = 0$$

From where it follows that:

$$S_t((1 - z_y^1)r^1(t)) = S_t((1 - z_y^2)r^2(t))$$

Which implies that:

$$(1 - z_y^1)r^1(t) = (1 - z_y^2)r^2(t)$$

Meaning that both tax plans are equivalent.

Exercise 3.3

(Endowment taxes versus government borrowing) Consider the economy of Exercise 3.1a. Assume there is a government that will raise 25 units of time 1 good and transfer these units to members of generation 0. Compare the following ways of financing this scheme.

(a) The government collects $\frac{1}{4}$ unit of time 1 good from each member of generation 1.

(b) The government at $t = 1$ sells securities that are titles to time 2 good (one-period bonds). At $t = 1$ it also announces that the members of generation 1 will be taxed equally at $t = 2$ (when they are old) to pay off the bonds. (Hint: Show that the equilibrium under scheme a is also an equilibrium under this scheme.)

(c) The government at $t = 1$ sells securities that are titles to time 2 good (one-period bonds). It sells enough to get 25 units of time 1 good. It will tax the members of generation 2 equally at $t = 2$ to pay off the securities it sells at $t = 1$.

Answer:

Recall our previous results:

$$\begin{aligned} c_t^h(t) &= \chi\big(r(t), \omega_t^h(t) - t_t^h(t), \omega_t^h(t+1) - t_t^h(t+1)\big) \\ &= \Big(\frac{1}{1+\beta}\Big)\Big[\omega_t^h(t) - t_t^h(t) + \frac{\omega_t^h(t+1) - t_t^h(t+1)}{r(t)}\Big] \\ c_0^h(1) &= \omega_0^h(1) - t_0^h(1) \end{aligned}$$

(a) In this setting, the government wants to raise 25 units of the time 1 good that can be evenly distributed among the initial old. To do so, the government decides to tax individuals born in generation 1, so the tax vector is given by:

$$t_t^h = \begin{cases} -\frac{1}{4}, & \text{for } t = 0 \\ \left(\frac{1}{4}, 0\right), & \text{for } t = 1 \\ (0,0), & \forall t > 1 \end{cases}$$

The demand function for consumption and the savings when young are given by, for the generation born at time $t = 1$:

$$\begin{aligned} c_1^h(1) &= \frac{1}{2} \times \Big(2 - \frac{1}{4} + \frac{1-0}{r(1)}\Big) = \frac{7}{8} + \frac{1}{2r(1)} \\ s_1^h(1) &= 2 - \frac{1}{4} - \Big[\frac{7}{8} + \frac{1}{2r(1)}\Big] = \frac{7}{8} - \frac{1}{2r(1)} \end{aligned}$$

And for all generations born at time $t > 1$:

$$\begin{aligned} c_t^h(t) &= \frac{1}{2} \times \Big(2 - 0 + \frac{1-0}{r(t)}\Big) = 1 + \frac{1}{2r(t)} \\ s_t^h(t) &= 2 - 0 - \Big[1 + \frac{1}{2r(t)}\Big] = 1 - \frac{1}{2r(t)} \end{aligned}$$

And Aggregate Savings are given by:

$$S_1(r(1)) = 100 \times \left[\frac{7}{8} - \frac{1}{2r(1)}\right]$$
$$S_t(r(t)) = 100 \times \left[1 - \frac{1}{2r(t)}\right], \forall t > 1$$

Under a competitive equilibrium, we have that

$$S_1(r(1)) = 0 \Leftrightarrow r(1) = \frac{4}{7}$$
$$S_t(r(t)) = 0 \Leftrightarrow r(t) = \frac{1}{2}, \forall t > 1$$

Therefore, optimal consumption allocations are given by:

$$c_0^h(1) = 1 + \frac{1}{4} = \frac{5}{4}$$
$$c_1^h(1) = \frac{7}{8} + \frac{1}{2 \times \frac{4}{7}} = \frac{7}{4}$$
$$c_1^h(2) = \beta r(1) c_1^h(1) = \frac{4}{7} \times \frac{7}{4} = 1$$
$$c_t^h(t) = (2, 1), \forall t > 1$$

And the utility levels:

$$u_1^h = \frac{7}{4} \times 1^1 = \frac{7}{4}$$
$$u_t^h = 2, \forall t > 1$$

(b) The economy is the same as before but the government now issues treasury bonds at time 1 to finance the transfer to the initial old. To pay off this debt, the government imposes an unspecified tax $t_1^h(2)$ on the people from generation 1 when they are old, i.e. at time $t = 2$. The following generations will face no taxes. The tax vector is now given by:

$$t_t^h = \begin{cases} -\frac{1}{4}, & \text{for } t = 0 \\ (0, x), & \text{for } t = 1 \\ (0, 0), & \forall t > 1 \end{cases}$$

while the government borrows $p(1)B(1) = 25$. Making use of our previous results, we have that the demand function for consumption and savings when young at time $t = 1$ are given by:

$$c_1^h(1) = \frac{1}{2} \times \left(2 + \frac{1-x}{r(1)}\right) = 1 + \frac{1-x}{2r(1)}$$
$$s_1^h(1) = 2 - \left[1 + \frac{1-x}{2r(1)}\right] = 1 - \frac{1-x}{2r(1)}$$

And Aggregate Savings are given by:

$$S_1(r(1)) = 100 \times \left[1 - \frac{1-x}{2r(1)}\right] = 100 - \frac{50(1-x)}{r(1)}$$

Recall that the equilibrium condition on aggregate savings is given by:

$$S_t(r(t)) = p(t)B(t) = \frac{B(t)}{r(t)}, \forall t$$

where we make use of the *Present Value Condition*, i.e. the no-arbitrage condition on the price of bonds: $r(t) = \frac{1}{p(t)}$. This equilibrium condition captures the requirement that total private savings at time t are equal to the time t value of outstanding government debt at time t, both measured in units of the time t good, and the requirement that government debt and private debt earn the same return. Making use of this condition at time 1, we have that:

$$S_1(r(1)) = p(1)B(1) \Leftrightarrow 100 - \frac{50(1-x)}{r(1)} = 25 \Leftrightarrow r(1) = \frac{2(1-x)}{3}$$

Moreover, we know that the *time t government budget constraint* must be balanced at all times. For $t = 1$, we have:

$$\begin{aligned}\sum_h t_1^h(1) + \sum_h t_0^h(1) - B(0) + p(1)B(1) &= 0 \\ 0 - \frac{1}{4} \times 100 - 0 + 25 &= 0 \\ 0 &= 0\end{aligned}$$

So it is balanced. For $t = 2$, we have that (recall that $p(1)B(1) = 25 \Rightarrow B(1) = \frac{25}{p(1)} = 25r(1)$):

$$\begin{aligned}\sum_h t_2^h(2) + \sum_h t_1^h(2) - B(1) + p(2)B(2) &= 0 \\ 0 + 100x - 25r(1) + 0 &= 0 \\ 100x - 25r(1) &= 0\end{aligned}$$

So at equilibrium, we must have that:

$$100x - 25r(1) = 0 \Leftrightarrow 100x - 25 \times \frac{2(1-x)}{3} = 0 \Leftrightarrow x = \frac{1}{7} \Rightarrow r(1) = \frac{4}{7}$$

So optimal consumption allocations are given by:

$$\begin{aligned} c_0^h(1) &= 1 + \frac{1}{4} = \frac{5}{4} \\ c_1^h(1) &= 1 + \frac{1 - \frac{1}{7}}{2 \times \frac{4}{7}} = \frac{7}{4} \\ c_1^h(2) &= \beta r(1) c_1^h(1) = \frac{4}{7} \times \frac{7}{4} = 1 \\ c_t^h(t) &= (2, 1), \forall t > 1 \end{aligned}$$

which are the same consumption alllocation as the previous exercise. And the utility levels:

$$\begin{aligned} u_1^h &= \frac{7}{4} \times 1^1 = \frac{7}{4} \\ u_t^h &= 2, \forall t > 1 \end{aligned}$$

which are also the same as before. This is no coincidence, it is actually a particular instance of a more general result called the **Ricardian Equivalence**:

Proposition 3. [9] *Given an initial equilibrium under some pattern of lump-sum taxation and government borrowing, alternative (intertemporal) patterns of lump-sum taxation that keep the present value (at the initial equilibrium's interest rates) of each individual's total tax liability equal to that in the initial equilibrium are equivalent in the following sense. Corresponding to each alternative taxation pattern is a pattern of government borrowing such that the initial equilibrium's consumption allocation, including consumption of the government, and the initial equilibrium's gross interest rates are an equilibrium under the alternative taxation pattern.*

The definition above seems a bit cumbersome. The main idea here is that when the government needs to finance itself (in this case to transfer 25 units of the good to the initial old), it can do so by either taxing agents now (at time $t = 1$, as in question a) or by borrowing now and taxing later (by issuing bonds at $t = 1$ and taxing members of generation 1 at time $t = 2$, as in question b). Under these policies, the consumption pattern and the prices won't be affected, as long as the present value of income is not affected. This

[9] Page 71 of the course textbook.

is true above so we can conclude that the Ricardian Equivalence holds.

(c) The setting is the same of question b but this time the debt is paid off by imposing an unspecified tax on generation 2 at time $t = 2$. Our tax vector is now given by:

$$t_t^h = \begin{cases} -\frac{1}{4}, & \text{for } t = 0 \\ (0,0), & \text{for } t = 1 \\ (x,0), & \text{for } t = 2 \\ (0,0), & \forall t > 2 \end{cases}$$

while the government borrows $p(1)B(1) = 25$. Making use of our previous results, we have that, for time $t = 1$:

$$\begin{aligned} c_1^h(1) &= 1 + \frac{1}{2r(1)} \\ s_1^h(1) &= 2 - \left[1 + \frac{1}{2r(1)}\right] = 1 - \frac{1}{2r(1)} \end{aligned}$$

And Aggregate Savings are given by:

$$S_1(r(1)) = 100 \times \left[1 - \frac{1}{2r(1)}\right] = 100 - \frac{50}{r(1)}$$

In equilibrium, we must have that:

$$S_1(r(1)) = p(1)B(1) \Leftrightarrow r(1) = \frac{2}{3}$$

As before, the *time t government budget constraint* must be balanced at all times. For time $t = 1$, it follows that:

$$\begin{aligned} \sum_h t_1^h(1) + \sum_h t_0^h(1) - B(0) + p(1)B(1) &= 0 \\ 0 - \frac{1}{4} \times 100 - 0 + 25 &= 0 \\ 0 &= 0 \end{aligned}$$

So it is balanced. For time $t = 2$, we have:

$$\begin{aligned} \sum_h t_2^h(2) + \sum_h t_1^h(2) - B(1) + p(2)B(2) &= 0 \\ 100x - 0 - 25r(1) + 0 &= 0 \\ 100x - 25r(1) &= 0 \end{aligned}$$

So at equilibrium, we must have that:

$$100x - 25r(1) = 0 \Leftrightarrow 100x - 25 \times \frac{2}{3} = 0 \Leftrightarrow x = \frac{1}{6}$$

Meaning that the demand function for consumption and savings of the young at time $t = 2$ are such that:

$$\begin{aligned} c_2^h(2) &= \frac{1}{2}\Big(2 - \frac{1}{6} + \frac{1-0}{r(2)}\Big) = \frac{11}{12} + \frac{1}{2r(2)} \\ s_2^h(2) &= 2 - \frac{1}{6} - \Big[\frac{11}{12} + \frac{1}{2r(2)}\Big] = \frac{11}{12} - \frac{1}{2r(2)} \end{aligned}$$

And Aggregate Savings at time $t = 2$:

$$S_2(r(2)) = 100 \times \Big[\frac{11}{12} - \frac{1}{2r(2)}\Big]$$

At equilibrium, it follows that:

$$S_2(r(2)) = 0 \Leftrightarrow r(2) = \frac{6}{11}$$

Therefore, optimal consumption allocations are given by:

$$\begin{aligned} c_0^h(1) &= 1 + \frac{1}{4} = \frac{5}{4} \\ c_1^h(1) &= 1 + \frac{1}{2 \times \frac{2}{3}} = \frac{7}{4} \\ c_1^h(2) &= \beta r(1) c_1^h(1) = \frac{2}{3} \times \frac{7}{4} = \frac{7}{6} \\ c_2^h(2) &= \frac{11}{12} + \frac{1}{2 \times \frac{6}{11}} = \frac{11}{6} \\ c_2^h(3) &= \beta r(2) c_2^h(2) = \frac{6}{11} \times \frac{11}{6} = 1 \\ c_t^h &= (2, 1), \forall t > 2 \end{aligned}$$

And the utility levels:

$$\begin{aligned} u_1^h &= \frac{7}{4} \times \Big(\frac{7}{6}\Big)^1 = \frac{49}{24} > \frac{7}{4} \\ u_2^h &= \frac{11}{6} \times 1^1 = \frac{11}{6} < 2 \\ u_t^h &= 2, \forall t > 2 \end{aligned}$$

Individuals born at time $t = 0$ and $t \geq 3$ remain the same, whereas individuals born at time $t = 1$ are better off, compared to the previous cases, and individuals born at time $t = 2$ are worse off. We can thus conclude that the

Ricardian Equivalence does not hold.

Exercise 3.4

(Borrowing forever) As in Exercise 3.3, the government sells enough securities at $t = 1$ to get 25 units of time 1 good, which it transfers to members of generation 0. However, instead of taxing the members of generation 2 at $t = 2$, it sells enough new securities (new one-period bonds) to pay off the securitites it issued at $t = 1$. At every subsequent date, it keeps doing this; it sells enough new securities to pay off the securitites that come due. As completely as you can, describe the competitive equilibrium under this scheme for the Exercise 2.4a economy. Is the equilibrium under this scheme Pareto superior to the competitive equilibrium in the absence of any policy? Is it Pareto optimal?

Answer:

Like Exercise 3.3, the government raises 25 units of the time 1 consumption good, by issuing treasury bonds. This time, instead of raising taxes, the government pays off its debt by issuing new treasury bonds at subsequent times to pay off the previous period's debt. Our tax vector is thus given by:

$$t_t^h = \begin{cases} -\frac{1}{4}, & \text{for } t = 0 \\ (0, 0), & \forall t \geq 1 \end{cases}$$

Making use of our previous results, we have that, $\forall t \geq 1$:

$$\begin{aligned} c_t^h(t) &= 1 + \frac{1}{2r(t)} \\ c_t^h(t+1) &= \beta r(t) c_t^h(t) \\ s_t^h(t) &= 2 - t_t^h(t) - c_t^h(t) = 1 - \frac{1}{2r(t)} \\ S_t(r(t)) &= \sum_{h=1}^{N(t)} s_t^h(t) = 100 - \frac{50}{r(t)} \end{aligned}$$

and $c_0^h(1) = \frac{5}{4}$. At time $t = 1$, the government borrows $p(1)B(1) = 25$, meaning that in equilibrium we must have:

$$S_1(r(1)) = p(1)B(1) \Leftrightarrow r(1) = \frac{2}{3} \Rightarrow B(1) = 25 \times r(1) = \frac{50}{3}$$

And consumption is given by:

$$
\begin{aligned}
c_1^h(1) &= 1 + \frac{1}{2 \times \frac{2}{3}} = \frac{7}{4} = 2 - \frac{1}{4} \\
c_1^h(2) &= \frac{2}{3} \times \frac{7}{4} = \frac{7}{6} = 1 + \frac{1}{6}
\end{aligned}
$$

Notice that consumption of the young born at time $t = 1$ is given by the endowment minus the investment in government bonds whereas their consumption when old is given by the endowment plus their return on the government bonds.

This means that at period $t = 2$ the government borrows $p(2)B(2) = B(1) = \frac{50}{3}$ to repay the previous debt. At equilibrium, we must have that:

$$
S_2(r(2)) = p(2)B(2) \Leftrightarrow r(2) = \frac{3}{5} \Rightarrow B(2) = \frac{50}{3} \times r(2) = 10
$$

And consumption is given by:

$$
\begin{aligned}
c_2^h(2) &= 1 + \frac{1}{2 \times \frac{3}{5}} = \frac{11}{6} = 2 - \frac{1}{6} \\
c_2^h(3) &= \frac{3}{5} \times \frac{11}{6} = \frac{11}{10} = 1 + \frac{1}{10}
\end{aligned}
$$

where, once again, consumption of the young born at time $t = 2$ is given by the endowment minus the investment in government bonds whereas their consumption when old is given by the endowmment plus their return on the government bonds.

At period $t = 3$, the government borrows $p(3)B(3) = B(2) = 10$ to repay the previous debt. Solving for equilibrium, we have that:

$$
\begin{aligned}
S_3(r(3)) = p(3)B(3) \Leftrightarrow r(3) = \frac{5}{9} \Rightarrow B(3) &= 10 \times \frac{5}{9} = \frac{50}{9} \\
c_3^h(3) &= 1 + \frac{1}{2 \times \frac{5}{9}} = \frac{19}{10} = 2 - \frac{1}{10} \\
c_3^h(4) &= \frac{5}{9} \times \frac{19}{10} = \frac{19}{18} = 1 + \frac{1}{18}
\end{aligned}
$$

where, once again, consumption of the young born at time $t = 3$ is given by the endowment minus the investment in government bonds whereas their consumption when old is given by the endowmment plus their return on the government bonds.

If we keep rolling over the debt until $t \to \infty$, we have that $B(t) \to 0$, $c_t^h \to (2, 1)$ and $r(t) \to \frac{1}{2}$, i.e. the economy converges to the equilibrium without government.

The utility levels are given by:

$$\begin{aligned}
c_0^h(1) &= \frac{5}{4} > 1 \\
u_1^h &= \frac{7}{4} \times \frac{7}{6} = \frac{49}{24} > 2 \\
u_2^h &= \frac{11}{6} \times \frac{11}{10} = \frac{121}{60} > 2 \\
u_3^h &= \frac{19}{10} \times \frac{19}{18} = \frac{361}{180} > 2 \\
&\dots \\
u_\infty^h &\to 2
\end{aligned}$$

Since everyone is better off that in the case without government, we can conclude that this equilibrium is Pareto superior to the one without any policy. However, this equilibrium is not Pareto optimal because imposing a policy $t_t^h = (0.5, -0.5)$ with $c_t^h = (1.5, 1.5)$ makes everyone better off:

$$\begin{aligned}
c_0^h(1) &= \frac{3}{2} > \frac{5}{4} > 1 \\
u_1^h &= \frac{9}{4} > \frac{49}{24} > 2 \\
u_2^h &= \frac{9}{4} > \frac{121}{60} > 2 \\
u_3^h &= \frac{9}{4} > \frac{361}{180} > 2
\end{aligned}$$

and so on. This allocation can also be achieved in a context where the government raises 50 units of the time 1 good and rolls the debt over forever.