Recitation 4: Linking Utility Maximization and Expenditure Minimization

Jon Cohen[1]

14.03 Fall 2022

Introduction

The goal of this recitation is to understand the connection between utility maximization and cost minimization. Throughout this handout, I will show results for a general utility function $u(x, y)$ and I will use a running example where $u(x, y) = \frac{3}{4}\ln x + \frac{1}{4}\ln y$.

Utility Maximization ("The Primal Problem")

Typical Setup

The consumer is trying to maximize her utility subject to a budget constraint:

$$\max_{x,y} u(x, y) \quad \text{subject to} \quad p_x x + p_y y \leq I.$$

We will assume the constraint binds (problem is well-behaved) and write the Lagrangian:[2]

$$\mathcal{L}(x, y, I) = u(x, y) - \lambda(p_x x + p_y y - I).$$

This gives us three FOCs:

$$\begin{aligned} \mathcal{L}_x &: \ u_x - \lambda p_x = 0 \\ \mathcal{L}_y &: \ u_y - \lambda p_y = 0 \\ \mathcal{L}_\lambda &: \ p_x x + p_y y - I = 0. \end{aligned}$$

Re-arranging, we see a general property of (interior) solutions:

$$\frac{u_x}{p_x} = \frac{u_y}{p_y} = \lambda.$$

In words, this says that marginal utility of the next dollar is the same across both goods. Solving for the optimal x and y in terms of p_x, p_y, and I yields the *uncompensated (Marshallian) demand functions*:

$$\begin{gathered} x^*(p_x, p_y, I) \\ y^*(p_x, p_y, I). \end{gathered}$$

Question: why are these called uncompensated demand functions?

[1] Notes adapted from a previous year's recitation by Carolyn Stein

[2] Note that in lecture, Professor Autor often writes $\mathcal{L}(x, y, I) = u(x, y) + \lambda(I - p_x x - p_y y)$. Can you see why these are the same?

Example:

Suppose utility is given by

$$u(x,y) = \frac{3}{4}\ln x + \frac{1}{4}\ln y \quad \text{subject to} \quad p_x x + p_y y \leq I.$$

The constraint binds, so the Lagrangian is

$$\mathcal{L}(x,y,I) = \frac{3}{4}\ln x + \frac{1}{4}\ln y - \lambda(p_x x + p_y y - I).$$

This gives us three FOCs:

$$\begin{aligned}
\mathcal{L}_x &: \ \frac{3}{4x} - \lambda p_x = 0 \\
\mathcal{L}_y &: \ \frac{1}{4y} - \lambda p_y = 0 \\
\mathcal{L}_\lambda &: \ p_x x + p_y y - I = 0.
\end{aligned}$$

Solving this system gives us the uncompensated (Marshallian) demand functions:

$$\begin{aligned}
x^*(p_x, p_y, I) &= \frac{3I}{4p_x} \\
y^*(p_x, p_y, I) &= \frac{I}{4p_y}.
\end{aligned}$$

The Value Function

Plugging $x^*(p_x, p_y, I)$ and $y^*(p_x, p_y, I)$ into $u(x,y)$ gives us the maximized utility function. We refer to this as the value function:

$$V(p_x, p_y, I) = u(x^*(p_x, p_y, I), y^*(p_x, p_y, I))$$

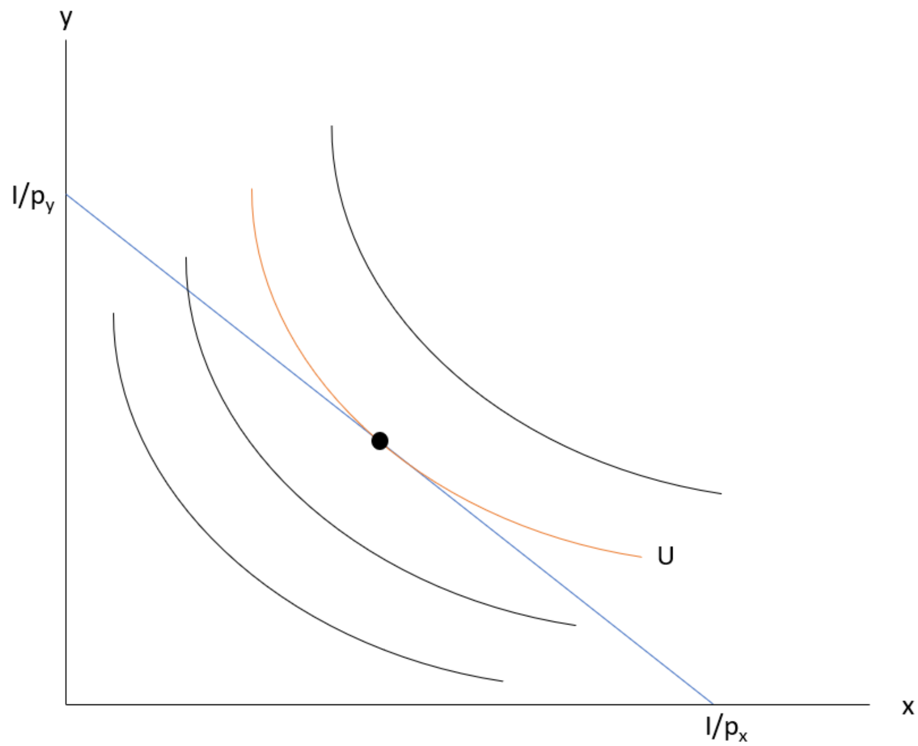
Example:

Plugging $x^*(p_x, p_y, I) = \frac{3I}{4p_x}$ and $y^*(p_x, p_y, I) = \frac{I}{4p_y}$ into $u(x,y) = \frac{3}{4}\ln x + \frac{1}{4}\ln y$ gives us the value function:

$$\begin{aligned}
V(p_x, p_y, I) &= u(x^*(p_x, p_y, I), y^*(p_x, p_y, I)) \\
&= \frac{3}{4}\ln\left(\frac{3I}{4p_x}\right) + \frac{1}{4}\ln\left(\frac{I}{4p_y}\right).
\end{aligned}$$

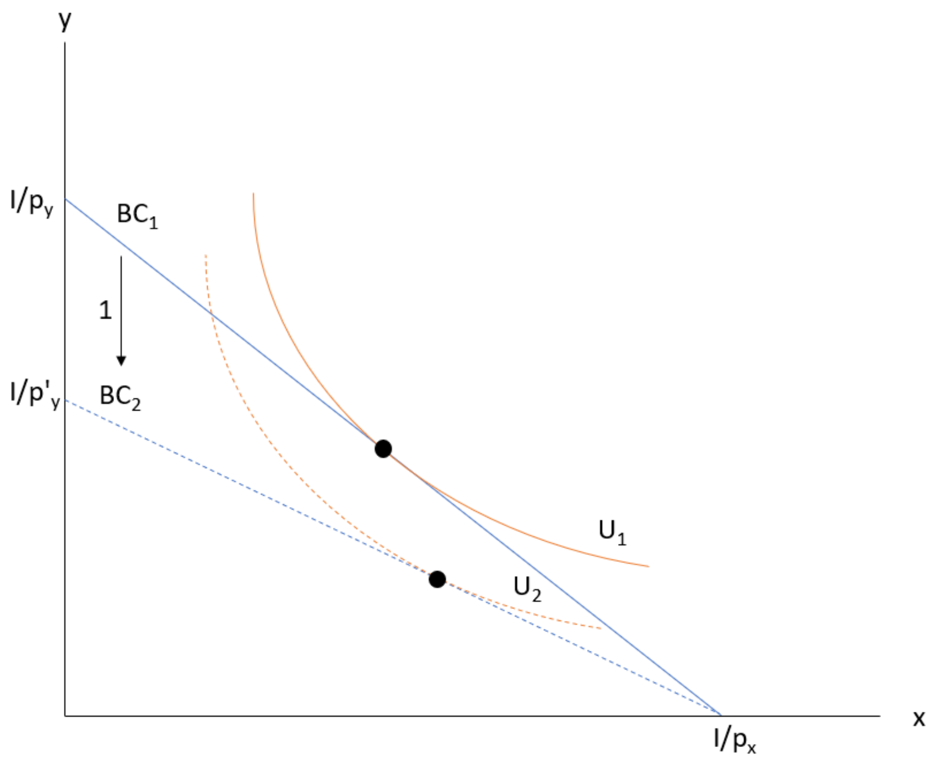
Graphical Explanation

In this problem, the budget is fixed at I, and we searching for the highest indifference curve that lies on the budget line.



Uncompensated (Marshallian) Demand Functions

What happens when one price (say p_y) goes up? First, the slope of the budget line changes, because now y is more expensive relative to x. We call this the "substitution effect." But simultaneously, the budget line has moved closer to the origin - the consumer now effectively has less wealth. We call this the "income effect." Graphically, both of these changes happen as the budget line rotates from BC_1 to BC_2. The consumer can no longer afford the same level of utility.



Expenditure Minimization ("The Dual Problem")

Typical Setup

This time, the consumer is trying to minimize her expenditure subject to a utility constraint:

$$\min_{x,y} p_x x + p_y y \quad \text{subject to} \quad u(x,y) \geq U.$$

We will assume the constraint binds (problem is well-behaved) and write the Lagrangian:

$$\mathcal{L}(x,y,I) = p_x x + p_y y - \mu(u(x,y) - U).$$

This gives us three FOCs:

$$\begin{aligned}
\mathcal{L}_x &: \; p_x - \mu u_x = 0 \\
\mathcal{L}_y &: \; p_y - \mu u_y = 0 \\
\mathcal{L}_\mu &: \; u(x,y) - U = 0.
\end{aligned}$$

Re-arranging, we see a general property of (interior) solutions:

$$\frac{p_x}{u_x} = \frac{p_y}{u_y} = \mu.$$

In words, this says that marginal cost of the next util is the same across both goods. Solving for the optimal x and y in terms of p_x, p_y, and U yields the *compensated (Hicksian) demand functions*:

$$x^h(p_x, p_y, U)$$
$$y^h(p_x, p_y, U).$$

Question: why are these called compensated demand functions?

Example:

In our example, this problem becomes

$$\min_{x,y} p_x x + p_y y \quad \text{subject to} \quad \frac{3}{4}\ln x + \frac{1}{4}\ln y \geq U.$$

The constraint binds, so the Lagrangian is

$$\mathcal{L}(x, y, V) = p_x x + p_y y - \mu\left(\frac{3}{4}\ln x + \frac{1}{4}\ln y - U\right).$$

This gives us three FOCs:

$$\begin{aligned}
\mathcal{L}_x &: \ p_x - \mu\frac{3}{4x} = 0 \\
\mathcal{L}_y &: \ p_y - \mu\frac{1}{4y} = 0 \\
\mathcal{L}_\mu &: \ \frac{3}{4}\ln x + \frac{1}{4}\ln y - U = 0.
\end{aligned}$$

Solving this system gives us the compensated (Hicksian) demand functions:

$$\begin{aligned}
x^h(p_x, p_y, U) &= e^U\left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}} \\
y^h(p_x, p_y, U) &= e^U\left(\frac{p_x}{3p_y}\right)^{\frac{3}{4}}.
\end{aligned}$$

The Expenditure Function

Plugging $x^h(p_x, p_y, U)$ and $y^h(p_x, p_y, U)$ into $p_x x + p_y y$ gives us the minimized expenditure function. We refer to this as just the expenditure function:

$$E(p_x, p_y, U) = p_x x^h + p_y y^h.$$

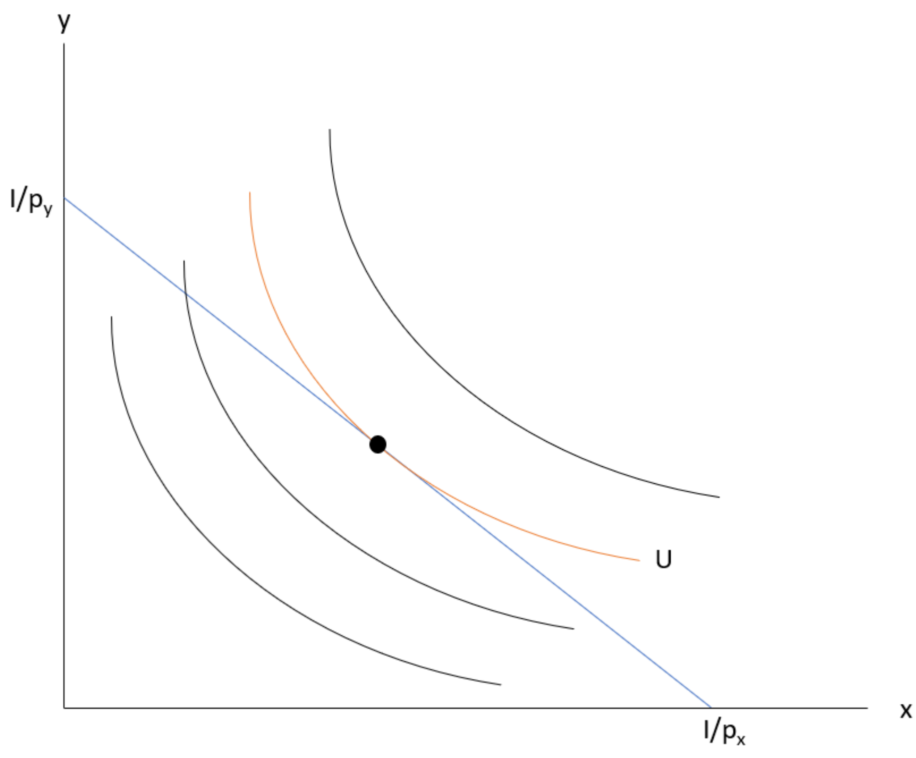
Example:

Plugging $x^h(p_x, p_y, U) = e^U \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}}$ and $y^h(p_x, p_y, U) = e^U \left(\frac{p_x}{3p_y}\right)^{\frac{3}{4}}$ into $p_x x + p_y y$ gives us the value function:

$$\begin{aligned}
E(p_x, p_y, U) &= p_x x^h + p_y y^h \\
&= p_x e^U \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}} + p_y e^U \left(\frac{p_x}{3p_y}\right)^{\frac{3}{4}} \\
&= e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}} \left(3^{\frac{1}{4}} + 3^{-\frac{3}{4}}\right) \\
&= \frac{e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}} \left(3^1 + 3^0\right)}{3^{\frac{3}{4}}} \\
&= \frac{4e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}}}
\end{aligned}$$

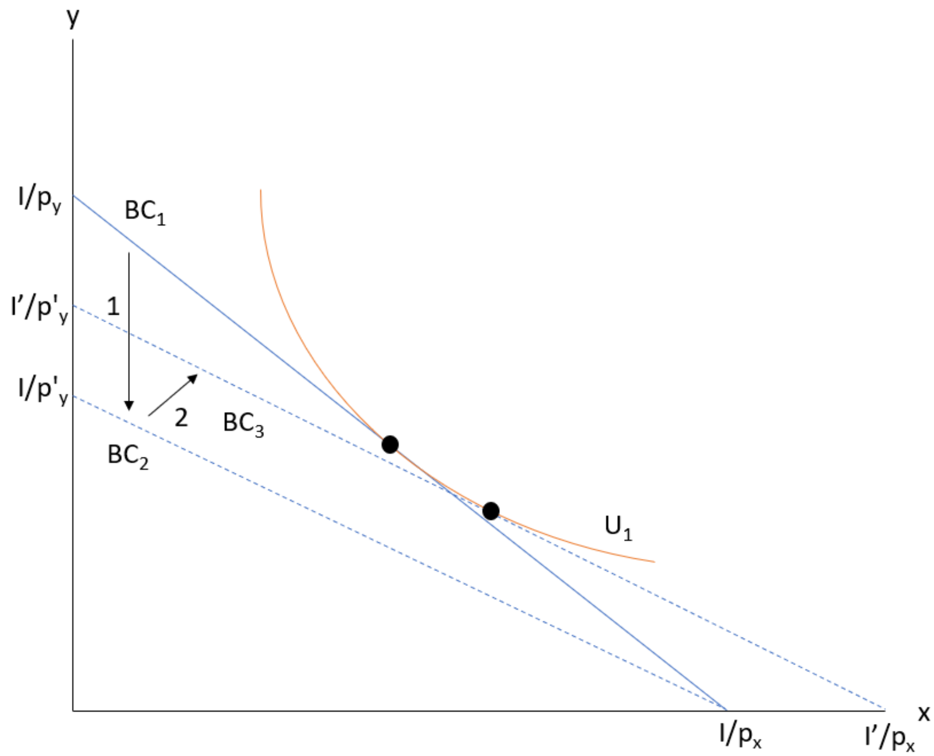
Graphical Explanation

In this problem, the indifference curve is fixed at U, and we are searching for the lowest budget line that lies on the indifference curve.



Compensated (Hicksian) Demand Functions

What happens when one price (say p_y) goes up? The slope of the budget line changes, because now y is more expensive relative to x. We call this the "substitution effect." However, the consumer stays at the same level of utility so the consumer does *not* become effectively poorer. Therefore there is *no* "income effect." Graphically, this change happens in two steps. First, p_y changes and the budget constraint rotates from BC_1 to BC_2. Then, to restore the consumer to her previous utility (i.e. to "undo" the income effect), the budget constraint shifts out to BC_3. The consumer can afford the same level of utility, but it required her income to increase from I to I'.



The Link Between Utility Maximization and Expenditure Minimization

The Indirect Utility Function and the Expenditure Function are Inverses

Suppose you maximize utility subject to I, where I is the solution to the expenditure minimization problem, given by $E(p_x, p_y, U)$. Then maximized utility will be U, the utility constraint from the expenditure minimization problem. Mathematically, this says

$$V(p_x, p_y, \underbrace{E(p_x, p_y, U)}_{I}) = U.$$

Now, suppose you minimize expenditure subject to U, where U is the solution to the utility maximization problem, given by $V(p_x, p_y, I)$. Then minimized expenditure will be I, the income constraint from the maximization problem. Mathematically, this says

$$E(p_x, p_y, \underbrace{V(p_x, p_y, I)}_{U}) = I.$$

Example:

Let's calculate $V(p_x, p_y, E(p_x, p_y, U))$:

$$\begin{aligned}
V(p_x, p_y, E(p_x, p_y, U)) &= V\left(p_x, p_y, \underbrace{\frac{4e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}}}}_{E(p_x,p_y,U)}\right) \\
&= \frac{3}{4}\ln\left(\frac{3\frac{4e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}}}}{4p_x}\right) + \frac{1}{4}\ln\left(\frac{\frac{4e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}}}}{4p_y}\right) \\
&= \frac{3}{4}\ln\left(\frac{3e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}} p_x}\right) + \frac{1}{4}\ln\left(\frac{e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}} p_y}\right) \\
&= \ln\left(\left(\frac{3e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}} p_x}\right)^{\frac{3}{4}} \left(\frac{e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}} p_y}\right)^{\frac{1}{4}}\right) \\
&= \ln\left(\frac{e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}}} \left(\frac{3}{p_x}\right)^{\frac{3}{4}} \left(\frac{1}{p_y}\right)^{\frac{1}{4}}\right) \\
&= \ln\left(e^U\right) \\
&= U.
\end{aligned}$$

Similarly, if we do the math, we find $E(p_x, p_y, V(p_x, p_y, I)) = I$. This also means that if just given the indirect utility function, we could calculate the expenditure function by inverting (solving for I):

$$\begin{aligned}
\underbrace{V(p_x, p_y, I)}_{U} &= \frac{3}{4}\ln\left(\frac{3I}{4p_x}\right) + \frac{1}{4}\ln\left(\frac{I}{4p_y}\right) \\
U &= \ln\left(\left(\frac{3I}{4p_x}\right)^{\frac{3}{4}} \left(\frac{I}{4p_y}\right)^{\frac{1}{4}}\right) \\
e^U &= \frac{I}{4}\left(\frac{3}{p_x}\right)^{\frac{3}{4}} \left(\frac{1}{p_y}\right)^{\frac{1}{4}} \\
\underbrace{I}_{E(p_x,p_u,U)} &= \frac{4e^U p_x^{\frac{3}{4}} p_y^{\frac{1}{4}}}{3^{\frac{3}{4}}}.
\end{aligned}$$

Similarly, we could solve for the indirect utility function by inverting the expenditure function.

Utility Maximization and Expenditure Minimization Yield the Same Optimal "Basket"

Suppose we solve

$$\max_{x,y} u(x,y) \quad \text{subject to} \quad p_x x + p_y y \leq I$$

to find $x^*(p_x, p_y, I)$ and $y^*(p_x, p_y, I)$. Then we solve

$$\min_{x,y} p_x x + p_y y \quad \text{subject to} \quad u(x,y) \geq U = V(p_x, p_y, I)$$

to find $x^h(p_x, p_y, U)$ and $y^h(p_x, p_y, U)$. Then $x^* = x^h$ and $y^* = y^h$.

Example:

Recall that

$$\begin{aligned} x^*(p_x, p_y, I) &= \frac{3I}{4p_x} \\ y^*(p_x, p_y, I) &= \frac{I}{4p_y} \end{aligned}$$

and

$$\begin{aligned} x^h(p_x, p_y, U) &= e^U \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}} \\ y^h(p_x, p_y, V) &= e^U \left(\frac{p_x}{3p_y}\right)^{\frac{3}{4}}. \end{aligned}$$

If we plug in $U = V(p_x, p_y, I) = \frac{3}{4}\ln\left(\frac{3I}{4p_x}\right) + \frac{1}{4}\ln\left(\frac{I}{4p_y}\right)$ into the compensated (Hicksian) demands, we get

$$\begin{aligned} x^h(p_x, p_y, U) &= e^U \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}} \\ &= e^{\frac{3}{4}\ln\left(\frac{3I}{4p_x}\right) + \frac{1}{4}\ln\left(\frac{I}{4p_y}\right)} \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}} \\ &= e^{\ln\left(\left(\frac{3I}{4p_x}\right)^{\frac{3}{4}}\left(\frac{I}{4p_y}\right)^{\frac{1}{4}}\right)} \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}} \\ &= \left(\frac{3I}{4p_x}\right)^{\frac{3}{4}} \left(\frac{I}{4p_y}\right)^{\frac{1}{4}} \left(\frac{3p_y}{p_x}\right)^{\frac{1}{4}} \\ &= \frac{3I}{4p_x} \\ &= x^*(p_x, p_y, I). \end{aligned}$$

Similarly, (omitting some math), we find

$$\begin{aligned} y^h(p_x, p_y, V) &= e^U \left(\frac{p_x}{3p_y}\right)^{\frac{3}{4}} \\ &= \frac{I}{4p_y} \\ &= y^*(p_x, p_y, I). \end{aligned}$$