Recitation 3 Practice Questions

1 Rational Choice

In the first period, $p_x = 2$ and $p_y = 1$ and Jon buys 11 units of x and 8 units of y. In the second period, $p_x = 1$ and $p_y = 2$ and Jon buys 10 units of x and 10 units of y. Prove that Jon's choices do not satisfy all 5 axioms of consumer theory in the specified ways. Be sure to cite specific axioms in your answer.

Hint: For some of your responses, you may find it useful to prove by contradiction. In order to prove the statement "A is true implies B is true" by contradiction, you assume that B is false and deduce that A must also be false. This relies on the fact that a logical statement and its contrapositive are equivalent. In the context of this question, a proof by contradiction would assume that Jon's choice ***do*** *satisfy all 5 axioms and conclude that he could not have made the choices that he did.*

1. (4 points) Prove using a graphical argument based on budget sets and/or indifference curves.

   Solution.

   The convexity of indifference curves implies that the indifference curve justifying the first (second) choice must lie above the budget set for point A (B) in Figure 1. This means the indifference curves must cross at some $x \in [10, 11]$. But crossing indifference curves violate non-satiation and transitivity.

2. Prove using an algebraic argument based on conditions from Jon's utility maximization problem.

   Solution. These are both interior solutions, so the FOCs must hold. In general, that is:

$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$

   Substituting the two price vectors:

$$\left.\frac{U_x}{U_y}\right|_A = 2$$
$$\left.\frac{U_x}{U_y}\right|_B = \frac{1}{2}$$

   The amount of x at point B is higher and the amount of y at point B is lower. Therefore, by the diminishing marginal rate of substitution, $MRS = \frac{U_x}{U_y}$ should be lower at point B compared to point A. But the relative price of x is higher at point B. Therefore the FOCs cannot hold at both points.

3. Prove using a revealed preference argument by comparing multiple bundles sequentially.

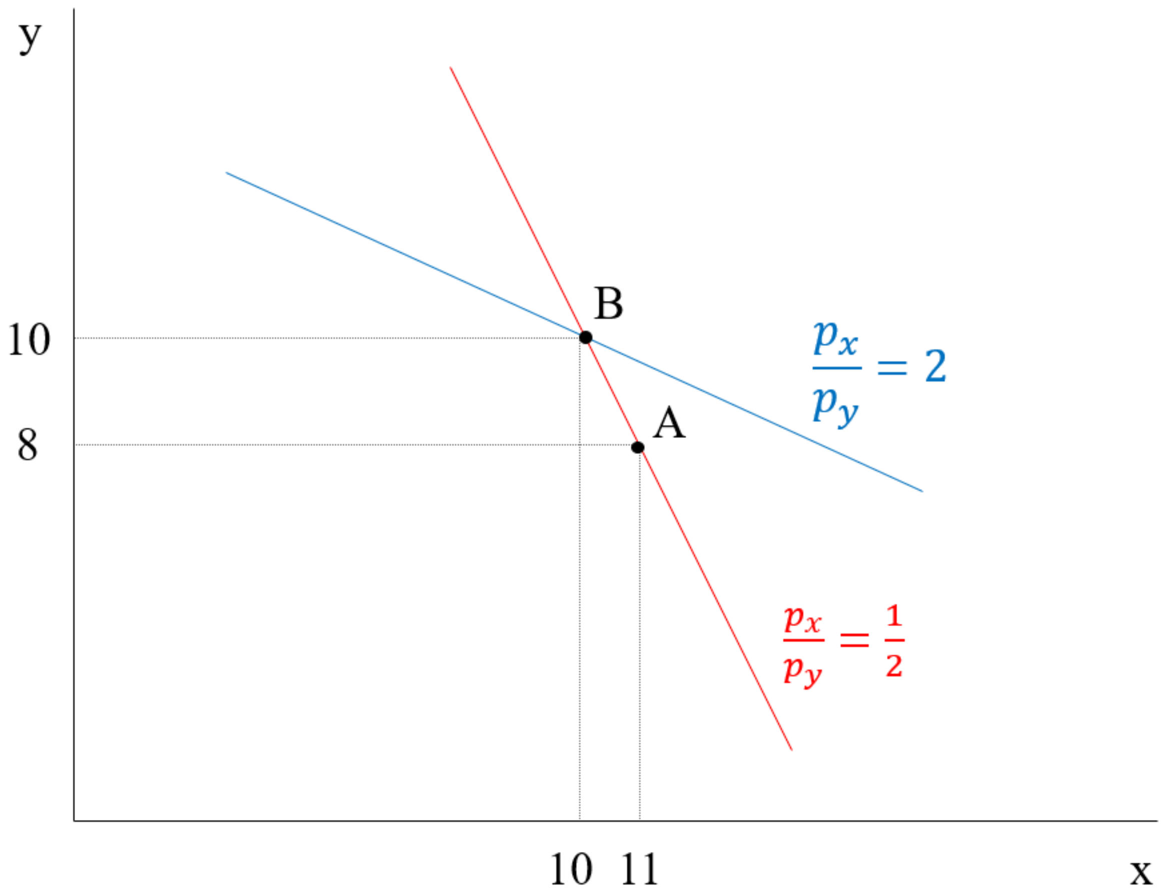

Figure 1: Budget Sets

Solution. By non-satiation, the consumer must strictly prefer C= $(11, 9.5)$ to A. But C is in the consumer's budget set in the second period, and the consumer chose B; so the consumer weakly prefers B to C. By transitivity, the consumer strictly prefers B to A. However, in the first period the consumer reveals that she weakly prefers A to B (since both bundles cost the same and she chose A). This is a contradiction, so the observed choices are inconsistent with revealed preference.

4. Consumer with preferences that satisfy all 5 axioms of consumer theory is indifferent between two bundles of X and Y: $(3, 2)$ and $(1, 5)$. Can we infer that she prefers the bundle $(2, 4)$ to either of the first two?

   Solution. Convex indifference curves imply that $0.5X + 0.5Y$, which is the bundle $(2, 3.5)$ is preferred to X and Y. By non-satiation $(2, 4)$ is preferred to $(2, 3.5)$ and by transitivity $(2, 4)$ is preferred to $(3, 2)$ and $(1, 5)$.

2 Regression Discontinuity

The basic idea of a regression discontinuity design (RDD) is to compare units just above a treatment cutoff and just below a treatment cutoff. David's lecture notes go through the math details, but the intuition is that these two types of units are virtually identical apart from their treatment exposure. In other words, the RDD is like an RCT for units just around the cutoff. The following questions will test your understanding of this.

1. High school students in the US take the PSAT only one time. Some scholarships have a minimum PSAT score cutoff. Suppose no students scoring below the threshold get a scholarship while all students scoring above the threshold get a scholarship? Do you think students just below the threshold are comparable to ones just above the threshold? Can you use this variation to evaluate the causal effect of the scholarship on student outcomes?

   Solution. It is likely that they are comparable. There is random chance in the exact score a student gets, so all students who score in that close region are likely comparable. Therefore comparing students above vs. below the threshold in that region isolates the effect of the scholarship.

2. What if students who scored above the threshold additionally needed to fill out a scholarship application form. Everyone who meets the threshold and fills out the application gets the scholarship, but only $x\%$ who scored high enough did so?

   Solution. Students scoring around the threshold are still likely comparable, so you can still use this variation. In this region around the cutoff, you can imagine there are two types of students: (1) those who will fill out the application if they score high enough and (2) those who will not. $x\%$ are type 1 (denoted as C for compliers) and $1 - x\%$ are type 2 (denoted as N for never-takers). Therefore only $x\%$ of people are affected by whether they're above or below the cutoff. If the effect of treatment for everyone were a constant β, the average change in outcomes across the discontinuity would be $x\% \cdot \beta$. Therefore to recover β, you divide the average change in outcomes across the discontinuity by the average change in treatment across the discontinuity.

Note that the effect you're recovering is specific to the people who comply with the cutoff. For example, suppose that the effect of the scholarship on outcomes is β_1 for type 1 students and β_2 for type 2 students. Because only type 1 students scoring above the threshold get the scholarship, the average change in outcomes across the discontinuity would be $x\% \cdot \beta_1$

3. What if students could retake the test multiple times and use their highest score for the scholarship?

   Solution. You would likely be concerned that students just below the cutoff are no longer comparable to those just above the cutoff. On the first test try, students are a similar mix of motivated and lazy ones both just above and just below the cutoff. The motivated students would retake the test while the lazy ones wouldn't. Once you take the maximum of all scores, students just below the cutoff are disproportionately lazy while those just above the cutoff are disproportionately motivated. Therefore comparing students across the cutoff is confounded by student motivation.

3 Indirect utility function and expenditure function

Let $U = [\alpha x^r + (1-\alpha)y^r]^{\frac{1}{r}}$ be the utility function (it is called constant elasticity of substitution (CES) utility function), where x and y are two goods, and $\alpha \in (0,1)$ and $r > 0$ are parameters. Denote p_x and p_y as respectively the prices of the two goods x and y, and I as the income of the consumer. This question has challenging algebra, so be careful!

1. Derive the Marshallian demand functions $d_x(p_x, p_y, I)$, $d_y(p_x, p_y, I)$, and the indirect utility function $V(p_x, p_y, I)$.

   Solution.

   The primal problem is:

$$\begin{aligned} \max\ U &= [\alpha x^r + (1-\alpha)y^r]^{\frac{1}{r}} \\ s.t.\ p_x x + p_y y &\leq I \end{aligned}$$

   The Lagrangian for this problem is:

$$= [\alpha x^r + (1-\alpha)y^r]^{\frac{1}{r}} - \lambda\left(p_x x + p_y y - I\right)$$

First order conditions are:

$$\begin{aligned}
\frac{\partial}{\partial x} &= \alpha x^{r-1}[\alpha x^r + (1-\alpha)y^r]^{\frac{1}{r}-1} - \lambda p_x = 0 \\
\frac{\partial}{\partial y} &= (1-\alpha)y^{r-1}[\alpha x^r + (1-\alpha)y^r]^{\frac{1}{r}-1} - \lambda p_y = 0 \\
\frac{\partial}{\partial \lambda} &= I - p_x x - p_y y = 0.
\end{aligned}$$

From first two equations we have:

$$\frac{\alpha}{1-\alpha}\left(\frac{x}{y}\right)^{r-1} = \frac{p_x}{p_y}.$$

Substituting it back into the budget constraint we get the Marshallian (uncompensated) demand functions:

$$\begin{aligned}
d_x\left(p_x, p_y, I\right) &= I\left(\frac{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{1}{r-1}}}{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}}\right) \\
d_y\left(p_x, p_y, I\right) &= I\left(\frac{\alpha^{\frac{1}{r-1}} p_y^{\frac{1}{r-1}}}{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}}\right)
\end{aligned}$$

Substituting Marshallian demand functions into the utility function we get indirect utility function:

$$\begin{aligned}
V\left(p_x, p_y, I\right) &= I\left[\alpha\left(\frac{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{1}{r-1}}}{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}}\right)^r\right. \\
&\quad \left. +(1-\alpha)\left(\frac{\alpha^{\frac{1}{r-1}} p_y^{\frac{1}{r-1}}}{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}}\right)^r\right]^{\frac{1}{r}} \\
&= I\left(\alpha(1-\alpha)\right)^{\frac{1}{r}}\left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1-r}{r}}
\end{aligned} \tag{1}$$

2. Apply Roy's Identity to find Marshallian demand functions.

Solution. Roy's identity:

$$
\begin{aligned}
d_x(p_x, p_y, I) &= -\frac{\frac{\partial V}{\partial p_x}}{\frac{\partial V}{\partial I}} \\
&= -\frac{I\left(\alpha(1-\alpha)\right)^{\frac{1}{r}} \frac{1-r}{r}\frac{r}{r-1}(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{1}{r-1}} \left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1-r}{r}-1}}{\left(\alpha(1-\alpha)\right)^{\frac{1}{r}} \left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1-r}{r}}} \\
&= I\left(\frac{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{1}{r-1}}}{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
d_y\left(p_x, p_y, I\right) &= -\frac{\frac{\partial V}{\partial p_y}}{\frac{\partial V}{\partial I}} = \\
&= -\frac{I\left(\alpha(1-\alpha)\right)^{\frac{1}{r}} \frac{1-r}{r}\frac{r}{r-1}\alpha^{\frac{1}{r-1}} p_y^{\frac{1}{r-1}} \left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1-r}{r}-1}}{\left(\alpha(1-\alpha)\right)^{\frac{1}{r}} \left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1-r}{r}}} \\
&= I\left(\frac{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{1}{r-1}}}{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}}\right)
\end{aligned}
$$

3. Derive the Hicksian demand functions $h_x(p_x, p_y, U_0)$, $h_y(p_x, p_y, U_0)$, and the expenditure function $E\left(p_x, p_y, U_0\right)$.

   Solution. There are two ways to solve the problem. First, we can solve the dual problem. Second, we know that $V\left(p_x, p_y, E\left(p_x, p_y, U_0\right)\right) = U_0$, thus solving the equation $V\left(p_x, p_y, I\right) = U_0$ for I, we can find the expenditure function $E\left(p_x, p_y, U_0\right)$. Then knowing the expenditure function $E\left(p_x, p_y, U_0\right)$, we can find Hicksian demand functions using the fact that $h_x(p_x, p_y, U_0) = d_x(p_x, p_y, E\left(p_x, p_y, U_0\right))$ and $h_y(p_x, p_y, U_0) = d_y(p_x, p_y, E\left(p_x, p_y, U_0\right))$. I will solve the problem using both methods.

   (i) The dual problem is:

$$
\begin{aligned}
\min\ I &= p_x x + p_y y \\
s.t.\ \left[\alpha x^r + (1-\alpha) y^r\right]^{\frac{1}{r}} &\geq U_0
\end{aligned}
$$

The Lagrangian for this problem is:

$$
^{D} = p_x x + p_y y - \lambda^D \left(\left[\alpha x^r + (1-\alpha) y^r\right]^{\frac{1}{r}} - U_0\right)
$$

First order conditions:

$$
\begin{aligned}
\frac{\partial^D}{\partial x} &= p_x - \lambda^D \alpha x^{r-1}[\alpha x^r + (1-\alpha)y^r]^{\frac{1-r}{r}} = 0 \\
\frac{\partial^D}{\partial y} &= p_y - \lambda^D (1-\alpha) y^{r-1}[\alpha x^r + (1-\alpha)y^r]^{\frac{1-r}{r}} = 0 \\
\frac{\partial^D}{\partial \lambda} &= U_0 - [\alpha x^r + (1-\alpha)y^r]^{\frac{1-r}{r}} = 0 \qquad (2)
\end{aligned}
$$

From first two equations we have:

$$
\frac{\alpha}{1-\alpha}\left(\frac{x}{y}\right)^{r-1} = \frac{p_x}{p_y}.
$$

Substituting it back into the dual constraint ($[\alpha x^r + (1-\alpha)y^r]^{\frac{1}{r}} = U_0$) we get Hicksian (compensated) demand functions:

$$
\begin{aligned}
h_x\left(p_x, p_y, U_0\right) &= \frac{U_0(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{1}{r-1}}}{(\alpha(1-\alpha))^{\frac{1}{r}}\left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1}{t}}} \\
h_y\left(p_x, p_y, M\right) &= \frac{U_0 \alpha^{\frac{1}{r-1}} p_y^{\frac{1}{r-1}}}{(\alpha(1-\alpha))^{\frac{1}{r}}\left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1}{r}}}
\end{aligned}
$$

Substituting Hicksian demand functions into the dual objective function we find the expenditure function:

$$
\begin{aligned}
E\left(p_x, p_y, U_0\right) &= p_x h_x(p_x, p_y, U_0) + p_y h_y\left(p_x, p_y, M\right) \qquad (3) \\
&= U_0\left(\alpha(1-\alpha)\right)^{-\frac{1}{r}}\left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{r-1}{r}} \qquad (4)
\end{aligned}
$$

(ii) Let us solve (1) for I when $V\left(p_x, p_y, I\right) = U_0$. We immediately get:

$$
E\left(p_x, p_y, U_0\right) = U_0\left(\alpha(1-\alpha)\right)^{-\frac{1}{r}}\left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{r-1}{r}}.
$$

Thus Hicksian demand functions are given by

$$\begin{aligned} h_x\left(p_x, p_y, U_0\right) &= d_x\left(p_x, p_y, E\left(p_x, p_y, U_0\right)\right) \\ &= \frac{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{1}{r-1}} E\left(p_x, p_y, U_0\right)}{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}} \\ &= \frac{U_0 (1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{1}{r-1}}}{(\alpha(1-\alpha))^{\frac{1}{r}} \left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1}{t}}}, \end{aligned}$$

$$\begin{aligned} h_x\left(p_x, p_y, U_0\right) &= d_y\left(p_x, p_y, E\left(p_x, p_y, U_0\right)\right) \\ &= \frac{\alpha^{\frac{1}{r-1}} p_y^{\frac{1}{r-1}} E\left(p_x, p_y, U_0\right)}{(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}} \\ &= \frac{U_0 \alpha^{\frac{1}{r-1}} p_y^{\frac{1}{r-1}}}{(\alpha(1-\alpha))^{\frac{1}{r}} \left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1}{r}}}. \end{aligned}$$

4. Apply Shephard's lemma to find Hicksian demand functions.

   Solution. Shephard's lemma:

$$\begin{aligned} h_x\left(p_x, p_y, U_0\right) &= \frac{\partial E}{\partial p_x} = \frac{U_0 (1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{1}{r-1}}}{(\alpha(1-\alpha))^{\frac{1}{r}} \left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1}{t}}}. \\ h_y\left(p_x, p_y, U_0\right) &= \frac{\partial E}{\partial p_y} = \frac{U_0 \alpha^{\frac{1}{r-1}} p_y^{\frac{1}{r-1}}}{(\alpha(1-\alpha))^{\frac{1}{r}} \left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1}{r}}}. \end{aligned}$$

5. Find $V\left(p_x, p_y, E\left(p_x, p_y, U_0\right)\right)$ and $E\left(p_x, p_y, V\left(p_x, p_y, I\right)\right)$ and explain.

   Solution.

$$\begin{aligned} V\left(p_x, p_y, E\left(p_x, p_y, U_0\right)\right) &= U_0 \\ E\left(p_x, p_y, V\left(p_x, p_y, I\right)\right) &= I \end{aligned}$$

   This is the general property which follows from the fact that primal and dual problems are equivalent.

6. Suppose now that the utility function is given by $\widehat{U} = g([\alpha x^r + (1-\alpha) y^r]^{\frac{1}{r}})$, where $g(.)$ is a strictly increasing function. Find indirect utility and expenditure functions $\widehat{V}\left(p_x, p_y, I\right)$,

$\widehat{E}\left(p_x, p_y, \widehat{U}_0\right)$. How are they related to $V(p_x, p_y, I)$ and $E(p_x, p_y, U_0)$ found in 2? Why?

Solution.

Since $g(.)$ is a strictly increasing function, $\widehat{U}$ is just a monotone transformation of U and thus consumer's preferences over consumption bundles are identical in both utility functions, which implies that the consumer makes the same choices $d_x(p_x, p_y, I)$ and $d_y(p_x, p_y, I)$ when she has preferences U and $\widehat{U}$. Thus

$$\begin{aligned}\widehat{V}(p_x, p_y, I) &= g(U(d_x(p_x, p_y, I), d_y(p_x, p_y, I))) = g(V(p_x, p_y, I)) \\ &= g\left(I\left(\alpha(1-\alpha)\right)^{\frac{1}{r}}\left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{1-r}{r}}\right).\end{aligned}$$

To derive $\widehat{E}\left(p_x, p_y, \widehat{U}_0\right)$ i will use the fact that $\widehat{E}\left(p_x, p_y, \widehat{V}\left(p_x, p_y, \widehat{I}\right)\right) = \widehat{I}$ and solve $\widehat{V}\left(p_x, p_y, \widehat{I}\right) = \widehat{U}_0$ for $\widehat{I}$ to find $\widehat{E}\left(p_x, p_y, \widehat{U}_0\right)$:

$$\begin{aligned}g(V(p_x, p_y, I)) &= \widehat{U}_0 \\ V(p_x, p_y, I) &= g^{-1}\left(\widehat{U}_0\right) \\ \widehat{E}\left(p_x, p_y, \widehat{U}_0\right) &= E\left(p_x, p_y, g^{-1}\left(\widehat{U}_0\right)\right) \\ &= g^{-1}\left(\widehat{U}_0\right)\left(\alpha(1-\alpha)\right)^{-\frac{1}{r}}\left[(1-\alpha)^{\frac{1}{r-1}} p_x^{\frac{r}{r-1}} + \alpha^{\frac{1}{r-1}} p_y^{\frac{r}{r-1}}\right]^{\frac{r-1}{r}},\end{aligned}$$

where $g^{-1}(.)$ is the inverse function of $g(.)$.