JEB064 2021/2022 Sample solution to Homework 1

Let's Make a Deal (4 points)

Suppose you are a staunch follower of Let's Make a Deal show and you realize that the car is surprisingly often located behind curtain A. More specifically, you observe the relative frequencies of the car location $p_A = \frac{1}{2}$ for curtain A, $p_B = \frac{1}{3}$ for curtain B and $p_C = \frac{1}{6}$ for curtain C.

You are invited to the show. As a smart student, you take the frequencies as the probabilities of the location of the car in your play. You know that the showmaster never reveals a curtain with a car and expect that she is randomizing curtains when your initial choice hits location of the car.

In this asymmetric environment, is it always optimal to switch from the initially chosen curtain or not? More specifically:

1. There are 6 combinations of an initially chosen curtain (by you) and subsequently revealed curtain (by the showmaster). Describe all these combinations (situations).

2. For each combination (situation), derive how your initial beliefs over location of the car, (p_A, p_B, p_C), change into posterior beliefs, (q_A, q_B, q_C). To clarify, in a situation when you initially choose A and the showmaster reveals B, the posterior belief q_A means the probability that the car is located in A,

$$q_A = \Pr(\text{Car in A}|\text{A initially chosen, B then revealed}).$$

3. For each possible combination (situation), find your optimal strategy (keep your initial choice or switch).

Sample solution We borrow the table from the lecture but change the priors to be non-uniform:

You/Car	A ($\frac{1}{2}$)	B ($\frac{1}{3}$)	C ($\frac{1}{6}$)
A	B ($\frac{1}{2}$), C ($\frac{1}{2}$)	C	B
B	C	A ($\frac{1}{2}$), C ($\frac{1}{2}$)	A
C	B	A	A ($\frac{1}{2}$), B ($\frac{1}{2}$)

A bit more formally, we list triples of conditional probabilities of revelation of A, B and C:

You/Car	A ($\frac{1}{2}$)	B ($\frac{1}{3}$)	C ($\frac{1}{6}$)
A	$(0, \frac{1}{2}, \frac{1}{2})$	$(0, 0, 1)$	$(0, 1, 0)$
B	$(0, 0, 1)$	$(\frac{1}{2}, 0, \frac{1}{2})$	$(1, 0, 0)$
C	$(0, 1, 0)$	$(1, 0, 0)$	$(\frac{1}{2}, \frac{1}{2}, 0)$

The posterior for the revealed curtain is zero, and the sum of the two other posteriors is 1. Therefore, to evaluate the optimal choice, it is sufficient to check if the posterior exceeds $\frac{1}{2}$ or not. Take each of the 6 situations:

- A is initially chosen, B is revealed: It is optimal to *keep* initial curtain A, because $q_A > 1 - q_A = q_C$.
$$q_A = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{6}} = \frac{3}{5}$$
- A is initially chosen, C is revealed: It is optimal to switch to curtain C, because $q_A < 1 - q_A = q_C$.
$$q_A = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{3}} = \frac{3}{7}$$
- B is initially chosen, A is revealed: Both choices are optimal, because $q_B = 1 - q_B = q_C$.
$$q_B = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{6}} = \frac{1}{2}$$
- B is initially chosen, C is revealed: It is optimal to switch to curtain A, because $q_B < 1 - q_B = q_A$.
$$q_B = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{2}} = \frac{1}{4}$$
- C is initially chosen, A is revealed: It is optimal to switch to curtain B, because $q_C < 1 - q_C = q_B$.
$$q_C = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{3}} = \frac{1}{5}$$
- C is initially chosen, B is revealed: It is optimal to switch to curtain A, because $q_C < 1 - q_C = q_A$.
$$q_C = \frac{\frac{1}{12}}{\frac{1}{12} + \frac{1}{2}} = \frac{1}{7}$$

Retrieving CEO's preferences (4 points)

As a consultant, you are expected to provide advice to a company on risky projects. By conducting an interview with the CEO, you learn the following:

- There are 4 certain outcomes, and the CEO's preferences are $x \succ y \succ w \succ z$.
- The CEO is indifferent between getting the outcome y with certainty and a lottery $(\alpha, 1-\alpha)$ over outcomes (x, w), where $\alpha \in (0, 1)$. Formally, the CEO's preferences over lotteries satisfy $(\alpha, 0, 1-\alpha, 0) \sim (0, 1, 0, 0)$.
- The CEO is indifferent between getting the outcome w with certainty and a lottery $(\beta, 1-\beta)$ over outcomes (y, z), where $\beta \in (0, 1)$. Formally, the CEO's preferences over lotteries satisfy $(0, \beta, 0, 1-\beta) \sim (0, 0, 1, 0)$.
- Find Bernoulli utilities (u_x, u_y, u_w, u_z) that characterize the CEO's preferences over lotteries. (Hint: Without loss of generality, impose $u_y = 1$ and $u_w = 0$.)
- Find $\gamma \in (0, 1)$ under which the CEO is indifferent between the outcome y with certainty and a lottery $(\gamma, 1-\gamma)$ over outcomes (x, z).
- Find $\delta \in (0, 1)$ under which the CEO is indifferent between the outcome w with certainty and a lottery $(\delta, 1-\delta)$ over outcomes (x, z).
- Calculate ratio $\frac{\delta}{\gamma}$.

Sample solution First, we insert $u_y = 1$ and $u_w = 0$ into the first manager's indifference, $\alpha u_x + (1-\alpha)u_w = u_y$. This implies $\alpha u_x = 1$, and thus

$$u_x = \frac{1}{\alpha}.$$

Second, we insert $u_y = 1$ and $u_w = 0$ into the second manager's indifference, $\beta u_y + (1-\beta)u_z = u_w$. This implies $\beta + (1-\beta)u_z = 0$, and thus

$$u_z = -\frac{\beta}{1-\beta}.$$

Third, we express the third manager's indifference as $\gamma u_x + (1-\gamma)u_z = u_y$. By entering Bernoulli's utilities,

$$\gamma = \frac{\alpha}{1-\beta+\alpha\beta}.$$

Fourth, we express the fourth manager's indifference as $\delta u_x + (1-\delta)u_z = u_w$. By entering Bernoulli's utilities,

$$\delta = \frac{\beta\alpha}{1 - \beta + \alpha\beta}.$$

Finally, the ratio is $\frac{\delta}{\gamma} = \beta$.

Double draw with a revision of the initial bet (4 points)

Consider the choice of a red vs. white bet when the decision-maker expects two draws with the option to revise her initial bet.

1. Suppose White bag contains 1 red ball and 3 white balls and Black bag contains 1 red ball and 3 black balls. White bag occurs with probability $\frac{1}{3}$ and Black bag with probability $\frac{2}{3}$. For a risk-averse decision-maker, is the initial red bet better than the initial white bet or not?

2. Take any $n \in N$ such that $n \geqslant 2$. Suppose White bag contains 1 red ball and n white balls and Black bag contains 1 red ball and n black balls. White bag occurs with probability $\frac{1}{n}$ and Black bag with probability $\frac{n-1}{n}$. For which n is the initial white bet better than the initial red bet if the decision-maker is risk-averse?

Sample solution

1. For the first draw alone, the decision-maker is indifferent over the bets, since

$$\Pr(\text{red}) = \frac{1}{4} = \frac{1}{3}\frac{3}{4} = \Pr(\text{white}).$$

The optimal bet for the second draw depends on her updated beliefs about the bags.

- Red ball: Beliefs about the bags (White, Black) don't change, $(\frac{1}{3}, \frac{2}{3})$. Red bet and white bet are identical. Their choice doesn't affect the lottery over outcomes.
- Black ball: Beliefs about the bags are $(0, 1)$. Red bet is optimal.
- White ball: Beliefs about the bags are $(1, 0)$. White bet is optimal.

Like in the class, we calculate the lotteries, L'_r and L'_w.

- Initial red bet, $L'_r = (\frac{7}{16}, \frac{8}{16}, \frac{1}{16})$ over $(0, 1, 2)$

$$\Pr(2 \text{ wins}) = \Pr(\text{red, red}) = \tfrac{1}{4}\tfrac{1}{4} = \tfrac{1}{16}$$

$$\Pr(1 \text{ win}) = \Pr(\text{red, white}) + \Pr(\text{red, black}) + \Pr(\text{white, white}) + \Pr(\text{black, red}) =$$

$$= \tfrac{1}{4}\left(\tfrac{1}{4} + \tfrac{1}{2} + \tfrac{3}{4}\right) + \tfrac{1}{2}\tfrac{1}{4} = \tfrac{8}{16}$$

- Initial white bet, $L'_w = (\frac{9}{16}, \frac{4}{16}, \frac{3}{16})$ over $(0, 1, 2)$

$$\Pr(2 \text{ wins}) = \Pr(\text{white, white}) = \tfrac{1}{4}\tfrac{3}{4} = \tfrac{3}{16}$$

$$\Pr(1 \text{ win}) = \Pr(\text{white, red}) + \Pr(\text{red, white}) + \Pr(\text{black, red}) =$$

$$= \tfrac{1}{4}\left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \tfrac{1}{2}\tfrac{1}{4} = \tfrac{4}{16}$$

Answer: Both lotteries have an identical expected payoff $\frac{10}{16}$ but L'_r is less risky than L'_w.

2. In the class, we had a special case of $n = 2$. In the previous example, we had a special case of $n = 3$. Now, we generalize to any $n \geqslant 2$.

   To begin with, for the first draw, the decision-maker is again indifferent over the bets, since $\Pr(\text{red}) = \frac{1}{n+1} = \frac{1}{n}\frac{n}{n+1} = \Pr(\text{white})$.

   The optimal bet for the second draw depends on her updated beliefs about the bags.

   - Red ball: Beliefs about the bags (White, Black) don't change, $(\frac{1}{n}, \frac{n-1}{n})$. Red bet and white bet are identical. Their choice doesn't affect the lottery over outcomes.
   - Black ball: Beliefs about the bags are $(0, 1)$. Red bet is optimal.
   - White ball: Beliefs about the bags are $(1, 0)$. White bet is optimal.

   Like in the class, we calculate the lotteries, L'_r and L'_w.

   - Initial red bet, $L'_r = \left(\frac{n^2-n+1}{(n+1)^2}, \frac{3n-1}{(n+1)^2}, \frac{1}{(n+1)^2}\right)$ over $(0, 1, 2)$

   $$\Pr(2 \text{ wins}) = \Pr(\text{red, red}) = \tfrac{1}{n+1}\tfrac{1}{n+1} = \tfrac{1}{(n+1)^2}$$

   $$\Pr(1 \text{ win}) = \Pr(\text{red, white}) + \Pr(\text{red, black}) + \Pr(\text{white, white}) + \Pr(\text{black, red}) =$$

   $$= \tfrac{1}{n+1}\left(\tfrac{1}{n+1} + \tfrac{n-1}{n+1} + \tfrac{n}{n+1}\right) + \tfrac{n-1}{n+1}\tfrac{1}{n+1} = \tfrac{3n-1}{(n+1)^2}$$

   - Initial white bet, $L'_w = \left(\frac{n^2}{(n+1)^2}, \frac{n+1}{(n+1)^2}, \frac{n}{(n+1)^2}\right)$ over $(0, 1, 2)$

   $$\Pr(2 \text{ wins}) = \Pr(\text{white, white}) = \tfrac{1}{n+1}\tfrac{n}{n+1} = \tfrac{n}{(n+1)^2}$$

   $$\Pr(1 \text{ win}) = \Pr(\text{white, red}) + \Pr(\text{red, white}) + \Pr(\text{black, red}) =$$

   $$= \tfrac{1}{n+1}\left(\tfrac{1}{n+1} + \tfrac{1}{n+1}\right) + \tfrac{n-1}{n+1}\tfrac{1}{n+1} = \tfrac{n+1}{(n+1)^2}$$

   Answer: For any $n \geqslant 2$, both lotteries have an identical expected payoff $\frac{3n+1}{(n+1)^2}$, but L'_r is less risky than L'_w.