Introductory Statistics

2024 Lectures
Part 6 - Counting

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Counting in classical probability

- In the classical interpretation of probability all outcomes of the experiment are equally likely, and the probability of an event is obtained as the relative frequency of outcomes that favor this event
- Techniques for counting elements of certain sets – essential in cases when simple enumeration is no longer feasible...part of combinatorics, or combinatorial analysis

a) Sampling with replacement, ordered objects

Definition 17: If A and B are two sets, their Cartesian product $A \times B$ is defined as the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

Example 23: If $A = \{x, y\}$ and $B = \{1, 2, 3\}$, then

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}.$$

Counting in classical probability

Theorem 6: (multiplication rule)
If $A_1, \ldots, A_n$, are finite sets, with A_i consisting of k_i elements ($i = 1, \ldots, n$), then the Cartesian product $A_1 \times \cdots \times A_n$ contains $\prod_{i=1}^{n} k_i$ elements.

- It is useful in composite experiments where we count number of ordered sequences selected with replacement from some set.

Example 24: The total number of possible initials consisting of two letters of Czech alphabet (first name, family name) is $42^2 = 1764$. Each two-letter initial is an element of the set $A \times A$, where A is the alphabet, so $k_1 = k_2 = 42$.

Permutations

b) Sampling without replacement, ordered objects

Definition 18: (permutation)
An ordered sequence of k elements selected without replacement from a set of n distinct elements ($n \geq k$) is called a permutation of k out of n elements.

Theorem 7: The number of permutations of k out of n, denoted P_n^k, equals

$$P_n^k = n(n-1)\dots(n-k+1).$$

- P_n^n is called n factorial denoted by $n! = n(n-1)\dots 2 \cdot 1$, with a convention $0! = 1$
- Using factorials,

$$P_n^k = \frac{n!}{(n-k)!}.$$

Example 25: (Birthday problem)
If r randomly chosen persons attend a party, what is the probability p_r that none of them will have a birthday on the same day?

Assume:

- Each year has 365 days
- Each day is equally likely to be a birthday of a person
- No twins (triplets etc) attend the party

Combinations

c) Sampling without replacement, objects not ordered

Definition 19: (combination)
A subset of size k from a set of size n ($n \geq k$), regardless of the order, is called a combination of k out of n.

Theorem 8: The number of combinations of k out of n, denoted C_n^k, equals

$$C_n^k = \frac{P_n^k}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

- $\binom{n}{k}$ is called a binomial coefficient. Read "n choose k".

Example 26: In a tank there are 10 fishes, 3 yellow and 7 black. We randomly select 3 of them. What is the probability that exactly 1 yellow fish gets selected?

d) Sampling with replacement, objects not ordered

- consider n boxes represented by $n-1$ walls between them
- placing k indistinguishable objects in the boxes can be represented as a sequence of length $n-1+k$ elements
- the number of such samples is thus equal to the number of possibilities to choose k elements out of $n-1+k$ without replacement,

$$\binom{n+k-1}{k}$$

Example 27: Suppose a recipe for a cake called for 5 pinches of spice, out of 9 different spices. How many different cakes based on this recipe can you make?

Basic properties of binomial coefficient

Theorem 9: For $n, k \in \mathbb{N} \cup \{0\}$, such that $k \leq n$, we have

a) $\binom{n}{n} = \binom{n}{0} = 1$;

b) $\binom{n}{k} = \binom{n}{n-k}$;

c) $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$;

d) (Newton's binomial formula) for any $n > 0$ and real x and y

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k};$$

Basic properties of binomial coefficient

e) $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n;$

f) $\binom{n}{0} - \binom{n}{1} + \cdots + (-1)^n \binom{n}{n} = 0;$

g) $\binom{n}{0}\binom{n}{k} + \binom{n}{1}\binom{n}{k-1} + \cdots + \binom{n}{k}\binom{n}{0} = \binom{2n}{k}.$