Problem Set 2

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- Groups of up to four students may submit one set of solutions. Write each group member's name and student number clearly on the first page of your solutions. Group compositions are allowed to change from one home assignment to another.
- To obtain a good score, write clearly and logically, starting from the definitions and correctly deducing and motivating your answers.
- Please submit your answer via email or Canvas **by 13:15 on September 15th**. It's sufficient for one person from each group to make the submission.

Exercise 1 Consider the vectors $v_1 = (1, 0, 4), v_2 = (-4, 2, 1)$ in $\mathbb{R}^3$.

(1) Compute the distance between the two vectors for the following four metrics
   (a) $d_1(x, y) = \sum_{i=1}^{3} |x_i - y_i|$
   (b) $d_2(x, y) = \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2}$
   (c) d_∞ (see Example 6.1)
   (d) the discrete metrics (see Example 6.2)

(2) Compute the inner product of the two vectors. Are they orthogonal?

Exercise 2 For $x, y \in \mathbb{R}$, show, for each of the following, whether it is a metric or not.

(1) $d(x, y) = (x - y)^2$

(2) $d(x, y) = \sqrt{|x - y|}$

(3) $d(x, y) = |x^2 - y^2|$

Exercise 3 Draw a diagram of the subsets U of $(\mathbb{R}^2, d_2)$, are they open or closed? Determine their interior, closure, boundary, the set of accumulation points, the set of isolated points:

(a) $U = \{x \in \mathbb{R}^2 : x_2 > x_1^2\}$

(b) $U = \{x \in \mathbb{R}^2 : x_1 = x_2 = 1 - \frac{1}{k}, \text{ for some } k \in N\}$

Exercise 4 Consider the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \begin{cases} y = x^2, & x \neq 2 \\ y = 20, x = 2 \end{cases}$

(1) Draw its graph.

(2) Show it is not continuous by showing the $\epsilon - \delta$ definition is violated.

(3) Show it is not continuous by finding an open set V whose preimage is not open.

Exercise 5 Let f be a continuous real-valued function on a metric space (X, d_X). Let $Z = \{x \in X : f(x) = 0\}$. Show that Z is closed.