Structural Form and Reduced Form VAR

The structural model (or system) ($p = 1$, $n = 2$; see also equations 5.17 and 5.18 p. 218 in the textbook)

$$\begin{aligned} y_t &= b_{10} - b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{yt} \\ z_t &= b_{20} - b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{zt} \end{aligned} \tag{1}$$

where ε_{yt} is a white-noise disturbance term (with $E\varepsilon_{yt} = 0$ and $E\varepsilon_{yt}^2 = \sigma_y^2$), ε_{zt} is another white-noise disturbance term (with $E\varepsilon_{zt} = 0$ and $E\varepsilon_{zt}^2 = \sigma_z^2$), and ε_{ys} and ε_{zt} are uncorrelated for all s and t.

The structural system in (1) can be written on reduced form noticing:

$$\begin{aligned} y_t + b_{12} z_t &= b_{10} + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{yt} \\ z_t + b_{21} y_t &= b_{20} + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{zt} \end{aligned}$$

$$\Longleftrightarrow$$

$$\begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix}_{2\times 2} \begin{bmatrix} y_t \\ z_t \end{bmatrix}_{2\times 1} = \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix}_{2\times 1} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}_{2\times 2} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix}_{2\times 1} + \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}_{2\times 1}$$

$$\Longrightarrow$$

$$BX_t = \Gamma_0 + \Gamma_1 X_{t-1} + \varepsilon_t$$

$$\Longrightarrow$$

$$B^{-1} B X_t = B^{-1}\Gamma_0 + B^{-1}\Gamma_1 X_{t-1} + B^{-1}\varepsilon_t$$

$$\Longrightarrow$$

$$X_t = A_0 + A_1 X_{t-1} + e_t \tag{2}$$

where

$$B = \begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix}_{2\times 2} \quad \Gamma_0 = \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix}_{2\times 1} \quad \Gamma_1 = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}_{2\times 2}$$

$$X_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}_{2\times 1} \quad X_{t-1} = \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix}_{2\times 1} \quad \varepsilon_t = \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}_{2\times 1}$$

and

$$
\begin{aligned}
A_0 &= \begin{bmatrix} a_{10} \\ a_{20} \end{bmatrix}_{2\times 1} = \begin{bmatrix} b_{10} - b_{20}b_{12} \\ b_{20} - b_{10}b_{21} \end{bmatrix} \times \frac{1}{\Delta} \\
A_1 &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} = \begin{bmatrix} \gamma_{11} - \gamma_{21}b_{12} & \gamma_{12} - \gamma_{22}b_{12} \\ \gamma_{21} - \gamma_{11}b_{21} & \gamma_{22} - \gamma_{12}b_{21} \end{bmatrix} \times \frac{1}{\Delta} \\
e_t &= \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}_{2\times 1} = \begin{bmatrix} \varepsilon_{yt} - b_{12}\varepsilon_{zt} \\ \varepsilon_{zt} - b_{21}\varepsilon_{yt} \end{bmatrix} \times \frac{1}{\Delta} \\
\Delta &= \det B = 1 - b_{12}b_{21}
\end{aligned}
$$

Furthermore, (2) is on matrix form but can of course be expressed equation wise as

$$
\begin{aligned}
y_t &= a_{10} + a_{11}y_{t-1} + a_{12}z_{t-1} + e_{1t} \\
z_t &= a_{20} + a_{21}y_{t-1} + a_{22}z_{t-1} + e_{2t}
\end{aligned}
\tag{3}
$$

Note that (2) or (3) is called the reduced form of the system in (1), and are the versions that will be used in, e.g., an estimation context.

The properties of the new disturbance terms e_{1t} and e_{2t} can be summarized as follows

$$
\begin{aligned}
E(e_t) &= E\begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}_{2\times 1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}_{2\times 1} = 0_{2\times 1} \\
E(e_t e_t') &= E\begin{bmatrix} e_{1t}^2 & e_{1t}e_{2t} \\ e_{2t}e_{1t} & e_{2t}^2 \end{bmatrix}_{2\times 2} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}_{2\times 2} \overset{def}{=} \Sigma_e
\end{aligned}
$$

where

$$
\sigma_1^2 = \left(\sigma_y^2 + b_{12}^2\sigma_z^2\right)/\Delta^2 \tag{i}
$$

$$
\sigma_2^2 = \left(\sigma_z^2 + b_{21}^2\sigma_y^2\right)/\Delta^2 \tag{ii}
$$

$$
\sigma_{12} = \sigma_{21} = -\left(b_{21}\sigma_y^2 + b_{12}\sigma_z^2\right)/\Delta^2 \tag{iii}
$$

and recall that

$$
E(\varepsilon_t\varepsilon_t') = E\begin{bmatrix} \varepsilon_{1t}^2 & \varepsilon_{1t\varepsilon 2t} \\ \varepsilon_{2t\varepsilon 1t} & \varepsilon_{2t}^2 \end{bmatrix}_{2\times 2} = \begin{bmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_z^2 \end{bmatrix}_{2\times 2} \overset{def}{=} \Sigma_\varepsilon
$$