JEB064 2022/2023 Sample solution to Homework 3

Winning with Rock rocks! (4 points)

Two players play Rock-Paper-Scissors. In contrast to the standard game, the value of victory depends on the winning move. Precisely:

- Winning by Rock is treated as most valuable (payoff 3), winning by Paper as moderately valuable (payoff 2) and winning by Scissors is least valuable (payoff 1).
- Losing gives zero payoff.
- Tie is a fair lottery. (After a tie, each player wins with probability $\frac{1}{2}$.) Therefore, there is either a direct victory (e.g., Scissors against Paper) or an indirect victory (e.g., Scissors against Scissors, as long as the draw from the fair lottery is successful.)

We will be comparing the equilibrium in this game with the equilibrium in the standard Rock-Paper-Scissors where the value of victory is independent on the winning move.

1. Find a symmetric Nash equilibrium in mixed strategies.
2. What is the probability that a player ties? Is the probability lower or higher than in the standard game?
3. What is the probability that a player wins by Rock directly, respectively indirectly? Is the probability lower or higher than in the standard game?
4. What are the probabilities that a player wins by Paper directly, respectively indirectly? Are the probabilities lower or higher than in the standard game?

Sample solution We look for a symmetric equilibrium, where each player's mixed strategy is identical. Let $r \in [0, 1]$ be the probability of Rock, $p \in [0, 1]$ be the probability of Paper, and $s \in [0, 1]$ be the probability of Scissors, where $r + p + s = 1$. Also, let V be the expected payoff of each player.

If all pure strategies are played with a positive probability, then their expected payoffs are identical and equal to the expected payoff,

$$\frac{3}{2}r + 0 + 3s = V$$
$$2r + p + 0 = V$$
$$0 + p + \frac{1}{2}s = V$$

From the second and third equation, $s = 4r$. Entering into the first equation,

$$(r, s) = \left(\frac{2}{27}V, \frac{8}{27}V \right).$$

Entering into the second equation,

$$p = \frac{23}{27}V.$$

Using $r + p + s = 1$, the expected value is $V = \frac{27}{33} = \frac{9}{11}$.

Nash equilibrium. Using V,

$$(r, p, s) = \left(\tfrac{2}{33}, \tfrac{23}{33}, \tfrac{8}{33}\right).$$

Tie. The probability of a tie is

$$rr + pp + ss = \frac{597}{1089} \doteq 54.8\%.$$

To compare, in the standard game, $(r, p, s) = (\frac{11}{33}, \frac{11}{33}, \frac{11}{33})$. The probability of a tie is $3rr = \frac{363}{1089} \doteq 33.3\%$. Tie is much more likely than in the standard game.

Rock wins. The probability of winning by Rock directly (Rock against Scissors) is $rs = \frac{16}{1089}$, and the probability of winning indirectly is $\frac{rr}{2} = \frac{2}{1089}$. The total probability is

$$rs + \frac{rr}{2} = \frac{18}{1089} = 1.7\%.$$

To compare, in the standard game, the total probability is $\frac{3}{2}rr = \frac{1}{2}\frac{363}{1089} \doteq 16.7\%$. Win by Rock is much **less likely** than in the standard game.

Paper wins. The probability of winning by Paper directly (Paper against Rock) is $pr = \frac{46}{1089} \doteq 4.2\%$, and the probability of winning indirectly is $\frac{pp}{2} = \frac{1}{2}\frac{529}{1089} \doteq 24.2\%$. To compare, in the standard game, the direct-win probability is $rr = \frac{121}{1089} \doteq 11.1\%$ and the indirect-win probability is $\frac{pp}{2} = \frac{1}{2}\frac{121}{1089} \doteq 5.5\%$. Winning by Paper directly is **less likely** than in the standard game, but winning by Paper indirectly is much **more likely** than in the standard game.

More contestants, more money to win (2 points)

There is a grant agency that allocates money to scientific fields depending on the number of grant applications in each field. For n applications in a field, the total amount of money is $R(n) = n^k$, where $k \geq 0$. In each field, the money is distributed by a grant competition with a structure of Tullock lottery.

All participants in grant competition are identical. Suppose you are a grant applicant and the number of applications in your field increases because new researchers have switched from another field to your field.

1. Suppose the grant agency allocates all available money to fields proportionally to the number of applications in fields. Are you better off or worse off when new researchers switch from another field to your field?

2. For which k are you worse off and for which k are you better off when the number of applications in your field increases?

Sample solution In Tullock lottery with n symmetric contestants, the profit is $\pi(n) = \frac{R(n)}{n^2}$. With our specification, $\pi(n) = n^{k-2}$. When the money is divided proportionally, $R = n$ (as if $k = 1$), and the profit is decreasing in n: $\pi = \frac{1}{n}$. In general, the profit is decreasing when $k < 2$ (you are worse off if the field is larger), is constant when $k = 2$, and increases if $k > 2$ (you are better off if the field is larger).

Multiple prizes (3 points)

Two identical prizes (e.g., 2 mobile phones) have to be allocated among four contestants. Their valuations are $R_1 > R_2 > R_3 > R_4 > 0$.

Suppose that the contestants compete in pairs, and each pair competes in an separate all-pay auction over a single prize. For example, a pair $(1, 2)$ competes over one prize, and a pair $(3, 4)$ competes over another prize.

How do you form the pairs if you are interested in *minimizing the expected profits* (payoffs) of the contestants?

Sample solution For an APA of a pair (A, B) with valuations $R_A \geq R_B$, we know that the equilibrium expected payoffs (profits) are $\pi_A = R_A - R_B$ and $\pi_B = 0$. We have three ways how to divide four contestants into pairs:

- Pairs $(1, 2)$ and $(3, 4)$: The total profits are $(R_1 + R_3) - (R_2 + R_4)$.
- Pairs $(1, 3)$ and $(2, 4)$: The total profits are $(R_1 + R_2) - (R_3 + R_4)$.
- Pairs $(1, 4)$ and $(2, 3)$: Again, the total profits again $(R_1 + R_2) - (R_3 + R_4)$.

It is easy to see that the total profits are minimized in the first case, because

$$(R_1 + R_3) - (R_2 + R_4) < (R_1 + R_2) - (R_3 + R_4)$$

is equivalent to

$$R_3 < R_2,$$

which is true by assumption. Intuitively, to minimize profits, it is better to create two close contests rather than a pair of contests, where one contest is very close but the other contest is very asymmetric.

Elimination contest (4 points)

In elimination contests, participants with the worst performance are eliminated sequentially. Consider such an elimination contest for 4 players and Tullock lottery.

- In Round 1, four players exert efforts (a_1, a_2, a_3, a_4). Based on Tullock contest-success function, three players pass into Round 2.
- In Round 2, the three players exert efforts (b_1, b_2, b_3). Based on Tullock contest-success function, two players pass into Round 3.
- In Round 3, the two players exert efforts (c_1, c_2). Based on Tullock contest-success function, a winner is selected and receives a prize of value R.

To calculate the probability that multiple players pass into the next round, use the same method that we have established in Tullock contest of pairs.

1. Using that the setting is symmetric, compute the equilibrium amounts (a^*, b^*, c^*).
2. What is the expected profit for each of the players before the competition starts (i.e., ex ante expected profit)?

Sample solution We express the optimization from the perspective of Player 1.

- In Round 3, the prize is R. The contest-success function (probability to win the prize) is
$$p_1 = \frac{c_1}{c_1 + c_2}.$$
We know that the equilibrium is $c^* = \frac{R}{4}$. The expected profit is $\frac{1}{2}R - c^* = \frac{R}{4}$. This constitutes the prize in the previous Round 2.
- In Round 2, the prize is $\frac{R}{4}$. The contest-success function (probability to win the prize which is access to Round 3) is
$$p_1 = \frac{b_1 + b_2 + b_1 + b_3}{b_1 + b_2 + b_1 + b_3 + b_2 + b_3} = \frac{2b_1 + b_2 + b_3}{2(b_1 + b_2 + b_3)}.$$
We calculate F.O.C. on the profit to obtain $b^* = \frac{R}{36}$. The expected profit is $\frac{2}{3}\frac{R}{4} - \frac{R}{36} = \frac{5R}{36}$.
- In Round 1, the prize is $\frac{5R}{36}$. The contest-success function (probability to win the prize which is access to Round 2) is
$$p_1 = \frac{a_1 + a_2 + a_3 + a_1 + a_2 + a_4 + a_1 + a_3 + a_4}{3(a_1 + a_2 + a_3 + a_4)} = \frac{3a_1 + 2a_2 + 2a_3 + 2a_4}{3(a_1 + a_2 + a_3 + a_4)}.$$
We calculate F.O.C. on the profit to obtain $a^* = \frac{5R}{576}$. The expected profit is $\frac{3}{4}\frac{5R}{36} - \frac{5R}{576} = \frac{55R}{576} \doteq 9.5\%R$. This is also the ex ante expected profit for each player. (The ex post profit of course depends on the round into which the player gets, where we can interpret victory as Round 4 with only 1 player.)