Normal-Form Games with Mixed Strategies

Martin Gregor
martin.gregor AT fsv.cuni.cz

JEB064 Game Theory and Applications

Sherlock vs. Moriarty

Arthur Conan Doyle: His Last Bow

- Sherlock Holmes wants to go from London to Dover by train and then proceed to the Continent, in order to escape from Professor Moriarty who pursues him. As the train pulls out of the station, Holmes sees Moriarty on the platform. Holmes takes it for granted that such a mighty opponent might secure a special train and overtake him.
- Holmes now faces the decision to continue to Dover, or to get off at Canterbury, the only intermediate station.
- Moriarty has the same options open to him. However, if he gets off at Canterbury, he cannot catch Holmes at Dover because a ship to France is leaving 15 minutes after arrival of the train.

Arthur Conan Doyle: His Last Bow

Outcomes

- Moriarty meets Holmes (Dover or Canterbury): Holmes is dead, Moriarty wins: payoffs $(0, 1)$
- Holmes escapes to the Continent: Holmes wins, Moriarty loses: payoffs $(1, 0)$
- Holmes escapes in Canterbury: Holmes survives for now but the future is unclear: payoffs (x, y), where $0 < x < 1$, $0 < y < 1$; $x + y \lesseqgtr 1$.

Normal-form game representation of the conflict:

Holmes/ Mortiary	Canterbury	Dover
Canterbury	$0, 1$	x, y
Dover	$1, 0$	$0, 1$

Arthur Conan Doyle: His Last Bow

Arthur Conan Doyle's solution

- Holmes gets off at Canterbury and watches Moriarty's train pass by.
- In the Doyle's story, Holmes is a higher level-k player than Moriarty.

  "There are limits, you see, to our friend's intelligence. It would have been a coup-de-maître had he deduced what I would deduce and acted accordingly."

But which level-k are Moriarty and Holmes?

- Level-0 Moriarty randomly stops. Level-1 Holmes compares $\frac{1}{2}0 + \frac{1}{2}x = \frac{x}{2}$ (Canterbury) with $0\frac{1}{2} + 1\frac{1}{2} = \frac{1}{2}$ (Dover) and gets off at Dover.
- Level-2 Moriarty expects Level-1 Holmes to get off at Dover and stops at Dover. Level-3 Holmes gets off at Canterbury.
- Level-4 Moriarty expects Level-3 Holmes to get off at Canterbury and stops at Canterbury. Level-5 Holmes gets off at Dover.

...

- Level-0 Holmes randomly stops. Level-1 Moriarty compares $\frac{1}{2}1 + \frac{1}{2}0 = \frac{1}{2}$ (Canterbury) with $y\frac{1}{2} + 1\frac{1}{2} = \frac{1+y}{2}$ (Dover) and gets off at Dover.
- Level-2 Holmes expects Level-1 Moriarty to get off at Dover and stops at Canterbury. Level-3 Moriarty gets off at Canterbury.
- Level-4 Holmes expects Level-3 Moriarty to get off at Canterbury and stops at Dover. Level-5 Moriarty gets off at Dover.

...

But which level-k are Moriarty and Holmes?

	0	1	2	3	4	5
Mortiary	random	Dover	Dover	Canterbury	Canterbury	Dover
Holmes	random	Dover	Canterbury	Canterbury	Dover	Dover

Assuming level-k players, the story is about

- level-1 Moriarty and level-2 Holmes, or
- level-2 Moriarty and level-3 Holmes, or
- level-5 Moriarty and level-6 Holmes, or
- ...

Notice: For best response of a level-k player, the key is her belief that the opponent is a level-$k-1$ player, and her ability to carry out reasoning up to level k. Therefore, even a player with a higher level reasoning (Sherlock Holmes) is a level-k player as long as he believes that the opponent is level-$k-1$ player.

Best responses in mixed strategies

- But we normally think of players who play mutual best responses.
- This is equivalent to infinite-k-level reasoning.
- In this game with a conflict, there are no mutual best responses (no pure-strategy Nash equilibrium).
- Now, what if the set of strategies (a set of prescriptions/manuals of actions) involves not only *deterministic* actions, but also *random* actions?

Example: Rock-Paper-Scissors

	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

- By experience, any deviation from randomizing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ over the three actions (Rock, Paper, Scissors) can be exploited by the opponent.
- For instance, playing Rock too often motivates the opponent to play Paper more often (and Scissors less often).
- Only randomizing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ over the three actions (Rock, Paper, Scissors) are mutual best responses.
- Well, of course, unless you can cheat: YouTube

Example: Matching Pennies (like Holmes and Moriarty)

- Players A and B have 1 coin each.
- Each chooses head or tail.
- If the same sides, A wins the coins.
- If different sides, B wins the coins.
- A mixed strategy is a *random mix* of actions head and tail.

		q	$1-q$
	A/B	head	tail
p	head	$1,-1$	$-1,1$
$1-p$	tail	$-1,1$	$1,-1$

- Player A expects q and:
  - if he plays head, his payoff is $q+(1-q)(-1)=2q-1$.
  - if he plays tail, his payoff is $q(-1)+(1-q)=1-2q$.
- Player B expects p and:
  - if she plays head, her payoff is $p(-1)+(1-p)=1-2p$.
  - if she plays tail, her payoff is $p+(-1)(1-p)=2p-1$.

Example: Matching Pennies

- Best response of Player A
  - $q < 1/2$: $P(q) = 0$
  - $q = 1/2$: $P(q) \in [0, 1]$
  - $q > 1/2$: $P(q) = 1$
- Best response of Player B
  - $p < 1/2$: $Q(p) = 1$
  - $p = 1/2$: $Q(p) \in [0, 1]$
  - $p > 1/2$: $Q(p) = 0$
- In a mixed-strategy equilibrium $(p^*, q^*) = (\frac{1}{2}, \frac{1}{2})$.

! Willingness to play *any* mixed strategy $0 < p < 1$ is equivalent to the *equality* between expected payoffs of each action (head, tail):

$$q = \frac{1}{2} : 2q - 1 = 1 - 2q$$

- **Payoff-equalizing property**: In the mixed-strategy equilibrium, an individual is indifferent between all actions that are mixed; each action provides an identical expected payoff. (If not, then it is strictly better to exclude the action with the lower expected payoff from the mix of actions.)

Matching Pennies in an experiment (Binmore 1987, 2007)

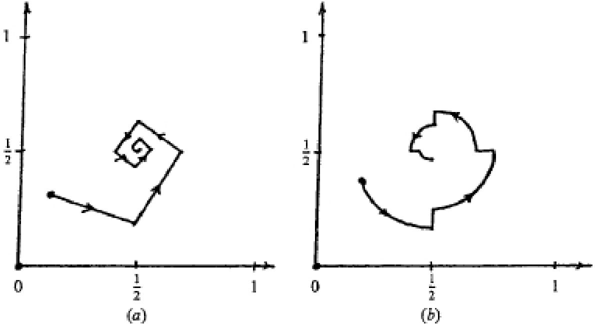


Figure 1.1
Approaching equilibrium in Matching Pennies

Mixed strategies

Finite strategy sets

- Let $S_i = \{s_{i1}, s_{i2}, \ldots, s_{im}\}$ be player i's finite set of pure strategies of size m.
- Let ΔS_i be the set of all probability distributions over S_i (the simplex of S_i of dimensionality $m-1$).
- A **mixed strategy** is an element $\sigma_i \in \Delta S_i$. That is, $\sigma_i = \{\sigma_i(s_{i1}), \sigma_i(s_{i2}), \ldots, \sigma_i(s_{im})\}$ is a probability distribution over S_i.

! Pure strategies also belong among mixed strategies.

- Recall that any $\sigma_i \in \Delta S_i$, where $\sigma_i = \{\sigma_i(s_{i1}), \sigma_i(s_{i2}), \ldots, \sigma_i(s_{im})\}$, satisfies: (i) $\sigma_i(s_{ik}) \geq 0$ for any $k = 1, \ldots, m$ and (ii) $\sum_k \sigma_i(s_{ik}) = 1$.
- Given a mixed strategy σ_i for player i, we will say that a pure strategy $s_i \in S_i$ is **in the support** of σ_i if and only if it occurs with positive probability, $\sigma_i(s_i) > 0$.

Mixed strategies

Beliefs about the opponents

- With mixed strategies, a **belief** for player i is a probability distribution $\pi_i \in \Delta S_{-i}$ over the pure (!) strategies of his opponents.
- The probability that player i assigns to his opponents playing $s_{-i} \in S_{-i}$ is denoted $\pi_i(s_{-i})$.
- If the belief is correct, $\pi_i = \sigma^*_{-i}$.

An extension to Continuous Strategy Sets:

- Let S_i be player i's pure-strategy set and assume that S_i is an **interval**.
- A mixed strategy for player i is a cumulative distribution function $F_i : S_i \to [0, 1]$, where $F_i(x) = \Pr\{s_i \leq x\}$.
- If $F_i(\cdot)$ is **differentiable** with density $f_i(\cdot)$, then we say that $s_i \in S_i$ is in the support of $F_i(\cdot)$ if $f_i(s_i) > 0$.

! Be careful when atoms exist!

- A belief is a cumulative probability distribution $\Pi_i : S_{-i} \to [0, 1]$, i.e., it is a belief over the pure (!) strategies of his opponents.

Mixed strategies

When the opponents of player i play a mixed strategy σ_{-i}, beliefs are correct, and ...

- player i plays a pure strategy s_i, the expected payoff of the player i is denoted

$$v_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) v_i(s_i, s_{-i});$$

- player i plays a mixed strategy σ_i, the expected payoff of the player i is denoted

$$v_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) v_i(s_i, \sigma_{-i}).$$

Nash equilibrium in mixed strategies

- A mixed-strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a **Nash equilibrium** (in mixed strategies) if for each player i, σ_i^* is a best response to σ_{-i}^*. That is, for each i,

$$v_i(\sigma_i^*, \sigma_{-i}^*) \geq v_i(\sigma_i, \sigma_{-i}^*) \quad \forall \sigma_i \in \Delta S_i.$$

- Payoff-equalizing property: If σ^* is a Nash equilibrium, and both s_i and s_i' are in the support of σ_i^*, then

$$v_i(s_i, \sigma_{-i}^*) = v_i(s_i', \sigma_{-i}^*) = v_i(\sigma_i^*, \sigma_{-i}^*).$$

Nash equilibrium in mixed strategies

Existence Theorem: Any n-player normal-form game with *finite* strategy sets S_i for all players has a Nash equilibrium in mixed strategies.

A mixed-strategy NE in Doyle's 'His Last Bow'

Holmes/ Mortiary	Canterbury q	Dover $1-q$
Canterbury p	$0, 1$	x, y
Dover $1-p$	$1, 0$	$0, 1$

- Holmes is indifferent if $(1-q)x = q$: $q^* = \frac{x}{1+x}$
- Mortiary is indifferent if $p = py + 1 - p$: $p^* = \frac{1}{2-y}$
- Equilibrium strategies

$$(p^*, q^*) = \left(\frac{1}{2-y}, \frac{x}{1+x}\right)$$

! Your mixing is driven by the opponent's values, not your values!

- Equilibrium (expected) payoffs

$$(v_H(p^*, q^*), v_M(p^*, q^*)) = (q^*, p^*)$$

A mixed-strategy NE in Doyle's 'His Last Bow'

- Holmes' payoff grows in x, Mortiary's payoff grows in y.

$$(v_H(p^*, q^*), v_M(p^*, q^*)) = (q^*, p^*) = \left(\frac{x}{1+x}, \frac{1}{2-y}\right).$$

- For example, think about the effect of a higher x:
  - Holmes successful escape in Canterbury is more attractive for Holmes.
  - Mortiary makes Holmes indifferent by stopping more likely in Canterbury.
  - $+$ Dover strategy: Holmes has a higher chance of successful escape in Dover.
  - $-/+$ Canterbury strategy: Holmes has (i) a lower chance but (ii) a higher value of successful escape in Canterbury. The overall effect is positive: $\frac{1}{1+x}x$ increases in x.
  - Intuitively, by Holmes' indifference, a higher payoff in Dover strategy must lead also to a higher payoff in Canterbury strategy.
  - Holmes preserves his mixed strategy. (Mortiary's payoffs don't change.)
- By analogy, think about the effect of a higher y.

Contests: Theory

Contests

Contest is a (competitive) game where players expend (irreversible!) resources to increase their probabilities of winning prizes (or, to increase their shares of the prizes).

- advertising competition (Friedman 1958)
- R&D contests (Fullerton and McAfee 1999)
- internal labor markets: relative performance schemes (Lazear and Rosen 1981; Rosen 1986)
- litigation
- sports (Hoehn and Szymanski 1999; Szymanski 2003)
- awarding Olympic games (Steward and Wu 1997)
- scientific and public awards
- status seeking (Congleton 1989; Konrad 1990, 1992)
- electoral campaigns
- competitive lobbying by special interests

The setup

competitors	$i \in \{1 \ldots n\}$
irreversible (sunk) payments	$x_i \geq 0$
value of the prize	R_i
contest-success function (CSF)	$p_i(x_1, \ldots, x_n)$

- CSF is interpreted in two ways:
  - deterministic outcome: p_i is a **share** of a **divisible** prize R_i, or
  - stochastic outcome: p_i is a **probability** of winning an **indivisible** prize R_i.
- This difference doesn't matter if competitors are risk-neutral (utility linear in payments).
  - deterministic: $u_i = p_i R_i - x_i$
  - stochastic, where $u(z) = z$:
    $U_i = p_i u_i(R_i - x_i) + (1 - p_i)u_i(-x_i) = p_i(R_i - x_i) + (1 - p_i)(-x_i) = p_i R_i - x_i$
- We assume risk-neutral players. Hence, the difference doesn't matter.
- Let $r_i = p_i R_i$ be the expected prize.

The setup

We will consider the following class of CSFs:

$$p_i(x_1, \ldots, x_n) = \begin{cases} \frac{1}{n} & \text{if} \quad x_1 = \cdots = x_n = 0 \\ \frac{x_i^v}{\sum_{j \in 1 \ldots n} x_j^v} & \text{otherwise} \end{cases}$$

$v \geq 0$ is a discrimination parameter.

- no discrimination ($v = 0$)

$$r_i = \frac{R_i}{n}$$

- TL: Tullock's lottery ($v = 1$)

$$r_i = \frac{x_i}{\sum_j x_j} R_i$$

- APA: all-pay-auction ($v \to \infty$)

$$r_i = 0 \quad \text{if} \quad x_i < \max_{j \in \{1 \ldots n\}} x_j$$

Our plan

For risk-neutral players, we will solve:

- Symmetric prizes $R_i = R$ (e.g., a resaleable prize)
    - Tullock's lottery, exogenous n ($n = 2, n > 2$)
    - Tullock's lottery, free entry (endogenous n)
    - All-pay-auction, exogenous n ($n = 2, n > 2$)
    - All-pay-auction, free entry (endogenous n)
- Asymmetric prizes
    - Tullock's lottery, $n = 2$

We will mainly analyze *intensity* of competition. We show that it depends on

- discrimination parameter v,
- the number of competitors n, and
- open vs. closed competition (entry costs).

Contests: Symmetry

Model 1: $v = 1$ (TL), $n = 2$

F.O.C. on $\pi_1 = r_1 - x_1 = R\frac{x_1}{x_1+x_2} - x_1$, using interior solution:

$$\frac{\partial \pi_1}{\partial x_1} = R\frac{x_2}{(x_1+x_2)^2} - 1 = 0$$

Best response of Player 1

$$\begin{aligned} x_2 &= 0 : x_1(x_2) > 0 \\ x_2 &< R : x_1(x_2) = \sqrt{x_2}(\sqrt{R} - \sqrt{x_2}) = \sqrt{x_2 R} - x_2 \\ x_2 &\geq R : x_1(x_2) = 0 \end{aligned}$$

Best response of Player 2

$$\begin{aligned} x_1 &= 0 : x_2(x_1) > 0 \\ x_1 &< R : x_2(x_1) = \sqrt{x_1}(\sqrt{R} - \sqrt{x_1}) = \sqrt{x_1 R} - x_1 \\ x_1 &\geq R : x_2(x_1) = 0 \end{aligned}$$

For positive outlays, we use $x_1(x_2) + x_2 = \sqrt{Rx_2} = \sqrt{Rx_1} = x_2(x_1) + x_1$ holds in equilibrium. Thus, we obtain symmetry, $x := x_1 = x_2$. Hence, $2x = \sqrt{Rx}$ and $x = x_1^* = x_2^* = \frac{R}{4}$. Total outlays are $x_1^* + x_2^* = \frac{R}{2}$.

Generalization to $n > 2$ players

Denote $x_{-i} := \sum_{j\in 1\ldots n, j\neq i} x_j$. F.O.C. on $\pi_i = r_i - x_i = R\frac{x_i}{x_i+x_{-i}} - x_i$:

$$\frac{\partial \pi_i}{\partial x_i} = R\frac{x_{-i}}{(x_i + x_{-i})^2} - 1 = 0$$

By symmetry, $x := x_i = \frac{x_{-i}}{n-1}$

$$R\frac{(n-1)x}{(nx)^2} - 1 = \frac{R}{x}\frac{n-1}{n^2} - 1 = 0 \qquad x = \frac{n-1}{n^2}R$$

- Individual expenditure x decreases in n down to zero

$$\frac{dx}{dn} = \left[-\frac{1}{n^2} + \frac{2}{n^3}\right]R = \frac{2-n}{n^3}R \leq 0$$

- Total expenditures grow in n up to R

$$nx = \frac{n-1}{n}R = (1 - \tfrac{1}{n})R$$

- Expected payoff $r_i - x_i$ falls in n down to zero

$$r_i - x_i = \frac{R}{n} - \frac{n-1}{n^2}R = \frac{R}{n^2}$$

Generalization to $v \in [1,2]$, $n = 2$

Consider Player 1. F.O.C. on $\pi_1 = r_1 - x_1 = R\frac{x_1^v}{x_1^v + x_2^v} - x_1$, using interior solution:

$$\frac{\partial \pi_1}{\partial x_1} = R\frac{vx_1^{v-1}x_2^v}{(x_1^v + x_2^v)^2} - 1 = 0$$

By symmetry, $x := x_1 = x_2$:

$$R\frac{vx^{2v-1}}{4x^{2v}} = 1$$

$$x = v\frac{R}{4}$$

At $v = 2$, the prize is completely dissipated, $2x = R$.

Generalization to $v \in [1, \frac{n}{n-1}]$, $n > 2$

Consider Player 1. F.O.C. on $\pi_1 = r_1 - x_1 = R\frac{x_1^v}{x_1^v + X} - x_1$, where $X = x_2^v + \cdots + x_n^v$, using interior solution:

$$\frac{\partial \pi_1}{\partial x_1} = R\frac{v x_1^{v-1} X}{(x_1^v + X)^2} - 1 = 0$$

By symmetry, $x := x_1$ and $X = (n-1)x^v$:

$$R\frac{v x^{2v-1}(n-1)}{n^2 x^{2v}} = 1$$

$$x = v\frac{(n-1)R}{n^2}$$

At $v = \frac{n}{n-1}$, the prize is completely dissipated, $nx = R$.

Model 2: $v = 1$, free entry

Firm i considers entry:

$$x_{-i} < R : x_i(x_{-i}) = \sqrt{x_{-i}}(\sqrt{R} - \sqrt{x_{-i}})$$
$$x_{-i} \geq R : x_i(x_{-i}) = 0$$

This gives that for $x_{-i} < R$, positive profits exist for an entrant.

- Entry when additional profits exist ($x_{-i} < R$)
- No entry when no more profits ($x_{-i} \geq R$)
- Entry stops once no more profits, i.e. total outlays equal prize, $\sum_j x_j = x_{-i} = R$
- Profits of an investing firm are zero (complete prize-dissipation),

$$\pi_i = \frac{x_i}{x_i + x_{-i}} R - x_i = \frac{x_i}{R} R - x_i = 0.$$

Application: Executive compensation[1]

- Idea: To increase company value, the shareholders give financial incentives to CEOs (variable executive compensation = prize R).
- Empirics: Here, the prize is measured by CEO Pay Gap (total CEO compensation minus median compensation of top 5 vice presidents).
- Each CEO competes by making costly effort x_i. (i) The larger is variable compensation/prize R_i, the higher is optimal effort x_i^*. (ii) The larger is effort, the higher is productivity of capital (a higher future company value).
- Investors: Institutional investors understand the role of incentives and buy if the company increases variable executive compensation. Retail investors don't understand the role of incentives. Noise traders help to clear the market.
- Empirics: The share of institutional ownership is larger in companies with higher variable executive compensation because within this mechanism, higher compensation is a signal of a higher future company value.

! A high variable executive compensation may also be a signal of managerial entrenchment (e.g., evidence on CEO's compensation packages set by co-opted non-executive directors), i.e., a lower company value.

[1] Source: Cheong, C. S., Yu, C. F. J., Zurbruegg, R., Brockman, P. (2021). Tournament incentives and institutional ownership. *International Review of Economics & Finance*, 74, 418–433.

What are the best responses in a perfectly-discriminating contest?

- 3 types of strategies: loss (zero), tie, win (overbid)
  - tie is clearly dominated by win because $\frac{R}{2} > \epsilon > 0$
  - $x_{-i} \in [0, R)$: win is the best response, $x_i > x_{-i}; x_i < R$, so $\pi = R - x_i > 0$
  - $x_{-i} \geq R$: loss is the best response, $x_i = 0$, and $\pi = 0$
- There is no intersection of best responses in $x_1 \times x_2$.
- No pure-strategy Nash equilibrium.
- Nevertheless, there is a mixed-strategy Nash equilibrium.

Model 3: Equilibrium in mixed strategies

- Mixed strategies over full support:
  - Probability density functions $f_1(x_1)$, and $f_2(x_2)$ over $[0, R]$
  - Cumulative probability distributions $F_1(x_1), F_2(x_2)$ over $[0, R]$
- Conditional on x_1, probability of beating player 2 is $\Pr(x_2 \leq x_1 \mid x_1) = F_2(x_1)$.
- By payoff-equalizing property and assuming $F(0) = 0$:

$$RF_2(x_1) - x_1 = RF_2(0) - 0 = 0$$

$$F_2(x) = \frac{x}{R}, f_2(x) = \frac{dF_2(x)}{dx} = \frac{1}{R}$$

- By symmetry, $f(x) := f_1(x) = f_2(x) = \frac{1}{R}$. Each player uniformly randomizes.
- Total expected outlays are $2E(x) = 2\frac{R}{2} = R$.

$$E(x) = \int_0^R xf(x)dx = \int_0^R \frac{x}{R}dx = \left[\frac{x^2}{2R}\right]_0^R = \frac{R}{2}$$

Generalization to $n > 2$ players

- Mixing for n players
  - probability of beating 1 player: $F(x)$
  - probability of beating $n-1$ players: $F(x)^{n-1}$
  - again symmetry
  - by equalizing property

$$RF(x)^{n-1} - x = 0 \qquad F(x) = \left(\frac{x}{R}\right)^{\frac{1}{n-1}}$$

- Total *expected* outlays:
  - by equalizing property, each action gives expected payoff 0
  - thus, expected payoff of any mixed strategy is also 0
  - by symmetry, each wins with probability $\frac{1}{n}$

$$\frac{R}{n} - E(x) = 0$$

  - full dissipation: $nE(x) = R$
- With endogenous entry, both participants and non-participants get zero. There is no difference to total expected outlays if free entry is allowed to APA.

Contests: Asymmetry

Model 4: Asymmetric prizes, $v = 1$ (TL), $n = 2$

- Interior best responses yield $x_1 + x_2 = \sqrt{x_2 R_1} = \sqrt{x_1 R_2}$, because:

$$\begin{aligned} 0 < x_2 < R_1 &: x_1(x_2) = \sqrt{x_2}(\sqrt{R_1} - \sqrt{x_2}) \\ x_2 \geq R_1 &: x_1(x_2) = 0 \\ 0 < x_1 < R_2 &: x_2(x_1) = \sqrt{x_1}(\sqrt{R_2} - \sqrt{x_1}) \\ x_1 \geq R_2 &: x_2(x_1) = 0 \end{aligned}$$

- Using the best responses, $x_1 R_2 = x_2 R_1$. Entering into the F.O.C.:

$$\frac{d\pi_1}{dx_1} = R_1 \frac{x_2}{(x_1 + x_2)^2} - 1 = \frac{x_1 R_2}{(x_1 + x_1 \frac{R_2}{R_1})^2} - 1 = \frac{R_1^2 R_2}{x_1 (R_1 + R_2)^2} - 1 = 0$$

$$(x_1^*, x_2^*) = \left(R_1 \frac{R_1 R_2}{(R_1 + R_2)^2}, R_2 \frac{R_1 R_2}{(R_1 + R_2)^2} \right)$$

- Total effort is $x_1 + x_2 = \frac{R_1 R_2}{R_1 + R_2} = \sigma(1 - \sigma)(R_1 + R_2)$, where $\sigma := \frac{R_1}{R_1 + R_2}$.
- More symmetry ($\sigma \to \frac{1}{2}$) makes the contest more intensive.

Generalization to $v \in [1, 2]$, $n = 2$

Can we get the win probabilities without the explicit equilibrium efforts?

- From the first-order conditions (best responses) of both players:

$$R_1 \frac{v x_1^{v-1} x_2^v}{(x_1^v + x_2^v)^2} = R_2 \frac{v x_2^{v-1} x_1^v}{(x_1^v + x_2^v)^2}$$

- Hence, the ratio of efforts is the ratio of values of prizes:

$$\frac{x_1}{x_2} = \frac{R_1}{R_2} = \frac{\frac{R_1}{R_1+R_2}}{\frac{R_2}{R_1+R_2}} = \frac{\sigma}{1-\sigma}$$

- Therefore, the win probability p_i can be expressed through σ:

$$p_1 = \frac{x_1^v}{x_1^v + x_2^v} = \frac{(\frac{x_1}{x_2})^v}{(\frac{x_1}{x_2})^v + 1} = \frac{(\frac{\sigma}{1-\sigma})^v}{(\frac{\sigma}{1-\sigma})^v + 1} = \frac{\sigma^v}{\sigma^v + (1-\sigma)^v}$$

- Specifically for $v = 1$, $(p_1, p_2) = (\sigma, 1-\sigma)$.

Model 5: Asymmetric costs, $v = 1$, $n = 2$

- Suppose Firm 1 is a more productive company with a higher return on its investments than Firm 2.

! Firm 1 has a higher opportunity cost of its lobbying expenditures.

$\Rightarrow$ a cost disadvantage in the contest

- Let the cost functions be $c_1(x_1) = \gamma x_1$ and $c_2(x_2) = x_2$, where $\gamma > 1$.
- From the best responses in a lobbying contest for a *symmetric* prize:

$$\frac{x_2}{(x_1 + x_2)^2} \frac{R}{\gamma} = 1, \frac{x_1}{(x_1 + x_2)^2} R = 1$$

- Therefore, $\frac{x_2}{x_1} = \gamma > 1$.
- Firm 1 invests relatively less, $x_1 < \gamma x_1 = x_2$, and therefore $r_1 < r_2$.
- Contest theory predicts that *new and growing firms get lower support* from the government than old and declining firms not only due to missing networks but also because of a higher opportunity cost of competitive lobbying.

Contests: Applications

Application: Course evaluations

'Golden Course' at the Faculty of Social Sciences

- 1 indivisible prize R
- in a symmetric NE, average investment is $E(x) = \frac{R}{n}$
- low investments frequent, high investments rare
- ex post: 1 winner, others are losers

Course evaluation at the Institute of Economic Studies (IES)

- 1 divisible prize R
- the prize is divided into small prizes where the value of a small prize is proportional to the course rank $K = 0, \dots, n-1$ (zero is the worst rank)
- in a symmetric NE, the course rank K is the number of successes in $n-1$ independent pairwise comparisons with the other courses ($n-1$ Bernoulli trials)
- for an investment x, probability of success in a single trial is $F(x)$
- probability of a course rank $K = k$:

$$\Pr(K = k \mid x) = \binom{n-1}{k} F(x)^k (1 - F(x))^{n-k-1}$$

Application: Course evaluation at the IES

- the prize for rank $K = k$ is kr_{n-1}, where r_{n-1} is a 'prize per unit of rank' when the number of courses is n
- all small prizes yield in total the large prize R (w.l.o.g., we use n is even)

$$\sum_{k=0}^{n-1} kr_{n-1} = \frac{n}{2}(n-1)r_{n-1} = R : r_{n-1} = \frac{2R}{n(n-1)}$$

- the expected course rank is $E(K \mid x) = (n-1)F(x)$
- thus, for an investment x, the expected prize is

$$E(Kr_{n-1} \mid x) = E(K \mid x)r_{n-1} = (n-1)F(x)\frac{2R}{n(n-1)} = F(x)\frac{2R}{n}$$

- expected payoff from an investment x is $\pi(x) = F(x)\frac{2R}{n} - x$
- by equalizing property, $\frac{d\pi(x)}{dx} = f(x)\frac{2R}{n} - 1 = 0$:

$$f(x) = \frac{n}{2R}$$

- each player *uniformly randomizes* over $x \in [0, \frac{2R}{n}]$ even if $n > 2$!
- average investment is again $E(x) = \frac{R}{n}$

Application: British litigation

American litigation is a standard contest (all players pays their costs). In contrast, in British litigation, the loser pays *all* costs. What is the difference?

- 2 risk-neutral contestants, investments x_1 and x_2
- Tullock's lottery
- Expected cost for Player 1 is *invariant* to her level of investments, x_1:

$$\frac{x_1}{x_1 + x_2} 0 + \frac{x_2}{x_1 + x_2}(x_1 + x_2) = x_2$$

- $\pi_1(x) = p_1(x_1, x_2)R - x_2 = \frac{x_1}{x_1+x_2} R - x_2$
- Both risk-neutral players aggressively compete up to their budget limits, because

$$\frac{dp_1(x_1, x_2)}{dx_1} = \frac{x_2}{(x_1 + x_2)^2} > 0.$$

Does the campaign finance cap decrease total campaign expenditures?

- The cap constrains spending of high spenders.
- The drop of their expenditures encourages extra expenditures of low spenders.
- The overall effect is ambiguous.

We will analyze an extremely discriminative case of the electoral campaign competition (APA).

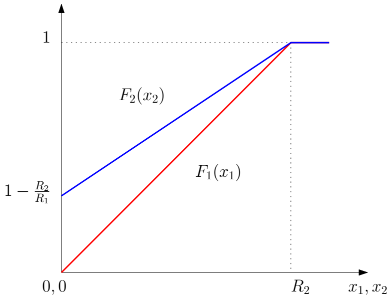
Campaign finance cap for $v = \infty$ (Che and Gale, 1998)

Suppose $R_2 < R_1$. The equilibrium differs depending on the cap:

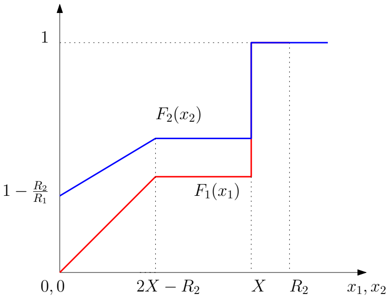
- **Ineffective** cap, $X \geq R_2$: mass point at zero, mixing on $[0, R_2]$ as if no cap.
- **Moderate** cap, $\frac{R_2}{2} \leq X < R_2$: mix on $[0, 2X - R_2]$, mass points at zero and X
- **Restrictive** cap, $X < \frac{R_2}{2}$: mass points at X

For ineffective cap, notice $\pi_1(x_1) = R_1 - R_2 = 0$ and $\pi_2(x_2) = 0$.

Campaign finance cap for $v = \infty$ (Che and Gale, 1998)



(a) no cap

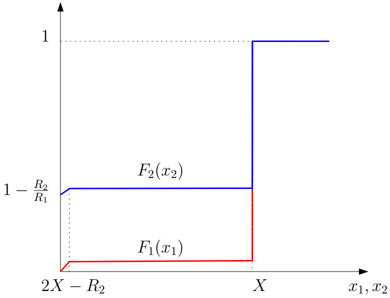


(b) moderate cap X

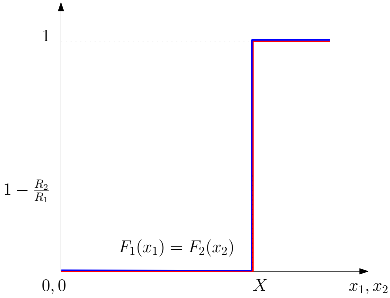
In the equilibrium with a moderate cap $X < R_2$, competition is absent on $x \in (2X - R_2, X)$ and all this probability mass shifts to $x = X$.

Campaign finance cap for $v = \infty$ (Che and Gale, 1998)

There is a 'structural change' when a moderate cap changes into a restrictive cap:



(a) just moderate cap, $X = \frac{R_2}{2} + \epsilon$



(b) just restrictive cap, $X = \frac{R_2}{2} - \epsilon$

What if the cap X increases (i.e., the cap is relaxed)?

- **Restrictive** cap: Total spending grows up to $2\frac{R_2}{2} = R_2$.
- At $X = \frac{R_2}{2}$, total spending *step-wise drops* down because Player 2 begins to abstain with probability $1 - \frac{R_2}{R_1} > 0$:

$$\frac{R_2}{2} + \frac{R_2}{R_1}\frac{R_2}{2} < 2\frac{R_2}{2} = R_2$$

- **Moderate** cap: Total spending is constant.
- **Ineffective** cap: No effect.

Application: Labor unions vs. trade unions

Endogenous prizes in an APA

What if special interests can choose what they lobby for?

- 2 special interest groups (e.g., labor unions vs. trade unions) compete over a policy (e.g., employee protection) by lobbying in a *policy contest*.
- Policy contest is a lottery over policies *proposed* by the contestants.
- Consider a policy $x \in R$.
- Lobby A's utility is $u_A(x) = -(0 - x)^2 = -x^2$ (concave, bliss point at 0).
- Lobby B's utility is $u_B(x) = -(2 - x)^2$ (concave, bliss point at 2).

Timeline

- In Stage 1, Lobby A announces $a \in [0, 1]$ and Lobby B announces $b \in [1, 2]$.
- In Stage 2, the two lobbies are engaged in an *all-pay auction*.
- In APA, prizes are $R_A(a, b) = u_A(a) - u_A(b)$ and $R_B(a, b) = u_B(b) - u_B(a)$.

What are the expected utilities of subgame in Stage 2?

- In APA with asymmetric valuations (say $R_A < R_B$), the weaker player gains nothing, $\pi_A = 0$.
- The stronger player, by equalizer property, earns $R_B F_A(x_B) - x_B$ for any $x_B \in [0, R_A]$.
- We use that the weaker player abstains with probability $F_A(0) = 1 - \frac{R_A}{R_B}$, hence the expected payoff of the stronger player is

$$\pi_B = \pi_B(0) = R_B F_A(0) - 0 = R_B - R_A.$$

Therefore, subgame in Stage 2 implies the following expected utilities:

$$U_A(a,b) = \begin{cases} u_A(b) + R_A - R_B & \text{if } R_A \geq R_B, \\ u_A(b) & \text{if } R_A \leq R_B. \end{cases}$$

Endogenous prizes

- In Stage 1, the proposals determine prizes in the subsequent APA:

$$R_A(a,b) = u_A(a) - u_A(b) = -a^2 - (-b^2) = b^2 - a^2$$
$$R_B(a,b) = u_B(b) - u_B(a) = -(2-b)^2 - [-(2-a)^2] = (2-a)^2 - (2-b)^2$$

- Suppose A decides to converge to B's proposal. What is the marginal effect on the levels of prizes?

$$\frac{\partial R_A(a,b)}{\partial a} = -2a < 0$$
$$\frac{\partial R_B(a,b)}{\partial a} = -4 + 2a < 0$$

! Convergence reduces both prizes, but asymmetrically (due to concavity).

Endogenous prizes

If A decides to converge on $a \in [0, 1]$, what is her marginal benefit?

- Weaker player ($R_A < R_B$): The marginal benefit is zero, because for the weaker contestant, $U_A(a, b) = u_A(b)$.
- Stronger player ($R_A \geq R_B$): The marginal benefit is positive,

$$\frac{\partial U_A(a, b)}{\partial a} = \frac{\partial R_A(a, b)}{\partial a} - \frac{\partial R_B(a, b)}{\partial a} = -2a + 4 - 2a = 4(1 - a) \geq 0.$$

! Convergence makes you relatively stronger (a higher difference in stakes, $R_A - R_B$, if $a < 1$).

! You are destroying part of your prize in order to destroy a larger part of the opponent's prize.

- $U_A(a, b)$ is flat for $a \in [0, 2 - b]$ and then it is increasing for $a \in [2 - b, 1]$.
- The best response is at $a(b) = 1$. By symmetry, $b(a) = 1$.

In the equilibrium, the proposals fully *converge*, $a^* = b^* = 1$.

Discussion about commitment

- In this model, lobbies publicly *commit to proposals* but cannot commit to expenses.
- By announcing a proposal but not expenses, a lobby affects the subsequent opponent's expenses.
- What if the lobbies cannot commit?
- Then, Lobby A sets $a = 0$ and Lobby B sets $b = 2$. (The winner sets her first-best policy.)
- If the lobbies commit to expenses but not proposal, the model is equivalent to the model without any commitment. (The winner sets her first-best policy.)
- If the lobbies commit to *both proposals and expenses*, the model is equivalent to the model without any commitment. (Proposal and expenses are set *simultaneously*, hence changing the proposal doesn't affect the competitor's expenses.)

Application: Grading competition

Multiple battlefields

Colonel Blotto Game

- 2 players compete over n multiple battlefields
- each player has a fixed amount of (use-or-lose) resources, e.g., 1
- a pure strategy is then a distribution $\mathrm{x} \in \Delta^{n-1}$

= multiple APAs with a joint 'use-or-lose' budget constraint

- applications: military strategy, network defense (hackers!), electoral competition over multiple districts

Continuous Lotto Game

- K players
- continuous: an infinite number of battlefields
- lotto: total resources limited only *in expectation*
- a pure strategy is a c.d.f. $F(x)$, where $x \geq 0$ and $E(x)$ is constant

Grading competition (Gregor, 2021)

Setup

- K lecturers (each 1 course), K electives
- a continuum of ex ante symmetric students
- each student has to take $1 \leq M < K$ electives
- a lecturer's strategy: a random course grade x (continuous) distributed by $F(x)$
- each lecturer maximizes the number/share of students in his class, σ
- grading inflation constraint: the average grade for all students (or, for a representative sample) cannot exceed the fixed level (here, suppose $E(x) \leq 1$)

Timeline

- Grading schemes: Each lecturer i sets $F_i(x)$.
- Electives shopping: Each student observes 'grade offers' $(x_1, \ldots, x_K)$ and in the 'electives shopping period' selects M courses with the highest grades.

Solution for $M = 1$, $K = 2$

$$F(x) \equiv F_1(x) = F_2(x) = \frac{x}{2}, x \in [0, 2]$$

- Are these offers best responses? Consider Lecturer 2 who considers a change in $F_2(x)$:

$$\sigma_2 = \int \underbrace{F_1(x)}_{\Pr(x_1 \le x)} \underbrace{f_2(x)}_{\Pr(x_2 = x)} dx = \int \frac{x}{2} f_2(x) dx$$

- By constraint on offers, $\int x f_2(x) dx = 1$ holds for any $f_2(x)$. Hence, σ_2 is **constant** for any admissible $f_2(x)$,

$$\sigma_2 = \frac{1}{2} \int x f_2(x) dx = \frac{1}{2}.$$

- Therefore, Lecturer 2 has no strict incentive to deviate from $F_2(x) = \frac{x}{2}$.

Solution for $M = 1$, $K \geq 2$

$$F(x) \equiv F_1(x) = F_2(x) = \ldots = F_K(x) = \left(\frac{x}{K}\right)^{\frac{1}{K-1}}, x \in [0, K]$$

- Again, we can easily check best responses.
- How to quickly verify this solution? Consider Lecturer K. She expects $1/K$ of students:

$$\sigma_K = \int_0^\infty \underbrace{F^{K-1}(x)}_{\Pr(x_1 \leq x)(\Pr x_2 \leq x)\ldots(\Pr x_{K-1} \leq x)} \underbrace{f_K(x)}_{\Pr(x_K = x)} \, dx = \frac{1}{K}$$

- This is transformed into the constraint $\int x \, f(x) \, dx = 1$ by setting $F^{K-1}(x) = \frac{x}{K}$.
- Then, we know that Lecturer K has no strict incentive to deviate.
- Inequality (more L-shaped density) grows in the number of courses.
- Recall APA of K players competing over a single prize.

Solution for $M = K - 1,\ K \geq 2$

- $K - 1$ courses are selected, and only the least attractive course is not taken.
- From symmetry, each lecturer expects $\frac{K-1}{K}$ of students.
- Lecturer K's objective:

$$\sigma_K = \int_0^\infty \left[1 - \underbrace{(1 - G(x))^{K-1}}_{\Pr(x_1 \geq x)\Pr(x_2 \geq x)\ldots\Pr(x_{K-1} \geq x)} \right] g(x)\ dx = \frac{K-1}{K}$$

- We can transform it into the constraint $\int x\ g(x)\ dx = 1$, or equivalently $\int x \frac{K-1}{K}\ g(x)\ dx = \frac{K-1}{K}$, by setting

$$1 - (1 - G(x))^{K-1} = x \frac{K-1}{K}.$$

- By rearranging,

$$G(x) = 1 - \left(1 - \frac{K-1}{K} x\right)^{\frac{1}{K-1}}.$$

Comparison $(K = 2, 3)$

	$M = 1$	$M = K - 1$
$K = 2$	$F(x) = \frac{x}{2},\ x \in [0, 2]$	$G(x) = \frac{x}{2},\ x \in [0, 2]$
$K = 3$	$F(x) = \sqrt{\frac{x}{3}},\ x \in [0, 3]$	$G(x) = 1 - \sqrt{\frac{3-2x}{3}},\ x \in [0, \frac{3}{2}]$

For $K = 3$, compare $f(x) = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{4x}}$ and $g(x) = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3-2x}}$.

- $f(x) \geq g(x)$ for $x \in [0, \frac{1}{2}]$
- $f(x) \leq g(x)$ for $x \in [\frac{1}{2}, \frac{3}{2}]$
- $f(x) \geq g(x)$ for $x \in [\frac{3}{2}, 3]$

Comparison ($K = 3$)