Extensive-Form Games

Martin Gregor
martin.gregor AT fsv.cuni.cz

JEB064 Game Theory and Applications

Preliminaries

Static game of complete information

= an n-person decision problem, where once-and-for-all decisions are made simultaneously and independently, and information is complete

- *simultaneity* = each player makes a decision without information about decisions of the others
- *independently* = no ability to agree on decisions
- *complete information* = actions, outcomes, and payoffs are common knowledge

Dynamic game of complete information

= an n-person decision problem, where the decisions are made sequentially (in a given order) and independently, and information is complete

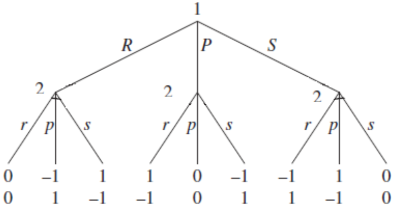
- in each move, it is described which previous moves are observed (information sets)

! not all previous moves must be observed

- *independently* = no ability to agree on decisions
- *complete information* = order of moves, players' information sets, actions, outcomes, and payoffs are common knowledge

Example: Sequential-move Rock-Paper-Scissors

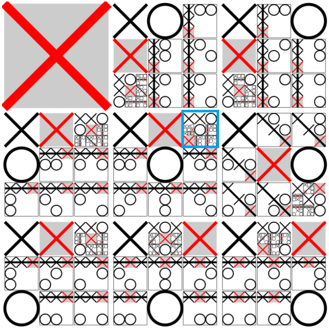
- Player 1 moves first.
- Player 2 observes Player 1's move and moves second.



- The order of moves (game tree) is a new (and extremely important) component.
- Specifically, this game features a second-mover advantage.
- Can you find an example of a game with a first-mover advantage? Gameplay (Last chair)

Example: Tic Tac Toe (3x3 grid)

- Is there a first-mover advantage in Tic Tac Toe (Naughts and Crossers)?



- In equilibrium (see later for a formal definition), the players draw/tie. (The equilibrium path is in the West box in the blue box.)

Example: The 1955 Federal Aid Highway Act

Sequential voting games are interesting extensive-form games.

- 3 outcomes
  - GB Albert Arnold 'Al' Gore, Sr., senator from Tennessee proposed to spend $18 billion on highways; the proposal included also Davis-Bacon Act requiring companies to pay union wages
  - A Amendment to delete the Davis-Bacon Act (only highways)
  - SQ Status quo (no highways, no union wages requirement)
- Preferences
  - Northern Democrats: $GB \succ_1 A \succ_1 SQ$
  - Southern Democrats: $A \succ_2 SQ \succ_2 GB$
  - Republicans: $SQ \succ_3 GB \succ_3 A$
  - $\rightarrow$ a pairwise voting cycle: $A \succ SQ \succ GB \succ A$
- Congress uses the 'amendment procedure':
  1. First, new proposals are compared (GB vs. A).
  2. Then, adopt or reject the winning proposal (winner vs. SQ).

The 1955 Federal Aid Highway Act

Recall $A \succ SQ \succ GB \succ A$.

- Sincere voting: GB wins in Vote 1, **SQ** wins in Vote 2.

- Strategic voting
  - In this game, everyone thinks of consequences of his or her vote. Here, players tend to strategically support a worse alternative if they expect a gain at a next round of voting.
  - Vote 2: All vote sincerely (if GB vs. SQ, then SQ; if A vs. SQ, then A).
  - Vote 1: If you vote for GB, you effectively vote for SQ. If you vote for A, you effectively vote for A. Hence, it is a vote over SQ vs. A, where **A** wins.

- Reality: Northern Democrats *understood* strategic incentives and supported Amendment in Stage 1.

The 1955 Federal Aid Highway Act

Unbundling

- Vote separately issue-by-issue, assuming that the preferences upon issues are *separable*.[1] Therefore, the order of voting is irrelevant.

	highways	no highways
union wages	GB	–
no union wages	A	SQ

- Majority preference on highways: $A \succ SQ$, i.e., highways supported.
- Majority preference on wages: $GB \succ A$, union wages (David-Bacon Act) supported.
- In total, **GB** wins.

[1] A stronger assumption is independence of values.

Definitions

Tree

- A *game tree* is a set of *nodes* $x \in X$ with a precedence relation $x > x'$, which means x precedes x''.
- The precedence relation is transitive, asymmetric ($x > x' \Rightarrow \neg x' > x$) and incomplete.
- Every node in a game tree has only one predecessor (i.e., history characterizes each node), except for the *root* of the tree, x_0, which is not preceded by any other $x \in X$.

Tree

- A *game tree* is a set of *nodes* $x \in X$ with a precedence relation $x > x'$, which means x precedes x''.
- The precedence relation is transitive, asymmetric ($x > x' \Rightarrow \neg x' > x$) and incomplete.
- Every node in a game tree has only one predecessor (i.e., history characterizes each node), except for the *root* of the tree, x_0, which is not preceded by any other $x \in X$.
- Nodes that do not precede other nodes are *terminal nodes*, $Z \subset X$.
- The terminal nodes the *outcomes* of the game.
- Each non-terminal node is assigned to a player ("a player moves in this node").

Definitions

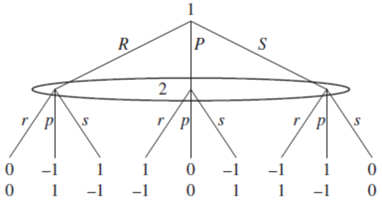
Every player i has a collection of *information sets* $h_i \in H_i$ that partition the nodes of the game at which i moves such that:

- If h_i is a singleton (includes only a single node x) then player i who moves at x knows that he is at x and nowhere else.
- If $x \in h_i$ and $x' \in h_i, x \neq x'$, then player i who moves at x doesn't know whether he is at x or x'.

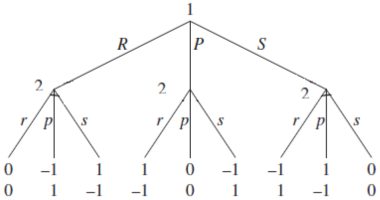
Pure strategy (a generalization of a complete 'plan of actions')

- A pure strategy for player i is a complete plan of play that describes which *pure action* player i will choose *at each of his information sets*. It is a mapping $s_i : H_i \to A_i$ that assigns an action $s_i(h_i) \in A_i(h_i)$ for every information set $h_i \in H_i$.

Example: Rock-Paper-Scissors



Simultaneous-move version



Sequential-move version

- When several players move simultaneously, multiple trees characterize the game equivalently. (Here, either Player 1 moves first and her move is unobserved by Player 2, or Player 2 moves first and her move is unobserved by Player 1).

(Im)perfect information

How do we introduce exogenous uncertainty?

- *Nature* is a non-strategic player that plays an exogenous *mixed* strategy.

Dynamic games of complete information

1. A game of *perfect information*: The game is of complete information and every information set is a singleton.
   - Every player observes all previous moves, including random moves of Nature; e.g., chess.
2. A game of *imperfect information*: The game is of complete information but some information sets contain several nodes.
   - A player doesn't observe previous moves of the opponents or moves of Nature.

Sequential-move Rock-Paper-Scissors

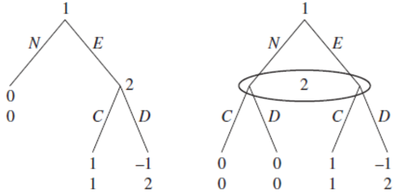
- **Every** extensive form has a **unique** normal-form representation.
- A normal-form representation of an extensive-form game is useful to quickly find all Nash equilibria.

Player 2/Player 1	R	P	S
(r, r, r)	$0, 0$	$-1, 1$	$1, -1$
(r, r, p)	$0, 0$	$-1, 1$	$-1, 1$
...	...	...	...
(p, s, r)	$1, -1$	$1, -1$	$1, -1$
...	...	...	...
(s, s, s)	$0, 0$	$-1, 1$	$1, -1$

- Player 2's strategy (p, s, r) weakly dominates all other strategies.
- Player 2 plays (p, s, r) in all Nash equilibria.

- But **not every** normal form has a **unique** extensive-form representation!

	C	D
N	$0, 0$	$0, 0$
E	$1, 1$	$-1, 2$



! A game in an extensive-form gives us 'more information' about the game (about the order of moves). So the extensive form should be preferred over the normal form.

- Representation is irrelevant for NE but relevant for those solution concepts that use the order of moves.

Randomizing

There are 2 ways of randomizing:

- *mixed strategy*: a player randomizes in the *ex ante* stage

= ex ante, the player randomly selects a manual; then, at each node (on each page) she deterministically follows the instructions

- *behavioral strategy*: a player randomizes in the *interim* stage

= ex ante, the player deterministically selects a manual; then, at each node (on each page), she randomizes over alternative instructions on the page

Behavioral strategy

- A *behavioral strategy* specifies for each information set $h_i \in H_i$ an independent probability distribution over $A_i(h_i)$ and is denoted by $\sigma_i : H_i \to \Delta A_i(h_i)$, where $\sigma_i(a_i(h_i))$ is the probability that player i plays action $a_i(h_i) \in A_i(h_i)$ in information set h_i.
- With perfect recall (unlimited memory of the past), any randomization can be represented by either mixed or behavioral strategies. There is **no difference** between solving the game with the sets of mixed strategies or behavioral strategies (but behavioral strategies are easier to deal with).

Equilibrium path

Let $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ be a Nash equilibrium profile of behavioral strategies in an extensive-form game. We say that an information set is **on the equilibrium path** if given σ^* it is reached with positive probability. We say that an information set is **off the equilibrium path** if given σ^* it is never reached.

First difference:

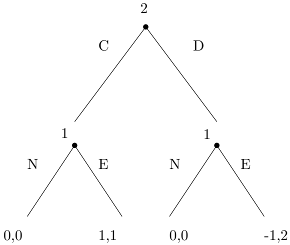
- On the equilibrium path, if a player changes his/her action, the outcome changes.
- Off the equilibrium path, if a player changes his/her action, the outcome does *not* change.

Second difference:

- Players form beliefs (about the opponents' actions) on the equilibrium path and off the equilibrium path.
- Players observe the actions of the opponents on the equilibrium path (beliefs are *verified*).
- Players don't observe the actions of the opponents off the equilibrium path (beliefs are *not* verified).

Nash equilibrium doesn't care about these differences. Only requires correct beliefs.

Credible and non-credible threats



	C	D
N, N	$0, 0$	$0, 0$
N, E	$0, 0$	$-1, 2$
E, N	$1, 1$	$0, 0$
E, E	$1, 1$	$-1, 2$

Nash equilibria

- $(E, N; C)$: Eq. path is $C \to E$. $a_1(D) = N$ is not played, observed, or verified.
- $(N, N; D)$: Eq. path is $D \to N$. $a_1(C) = N$ is not played, observed, or verified.

Sequential rationality in games with perfect information

What if rational players play in the off equilibrium information sets **as if** their off-equilibrium actions matter?

- Given strategies $\sigma_{-i} \in \Delta S_{-i}$ of i's opponents, we say that a player i is a playing a best response in her information set h_i if she maximizes her expected utility given her beliefs at the information set (and given that she is at the information set).
- If player i is playing a best response to σ_{-i} in each of her information sets, the player is *sequential rational*.

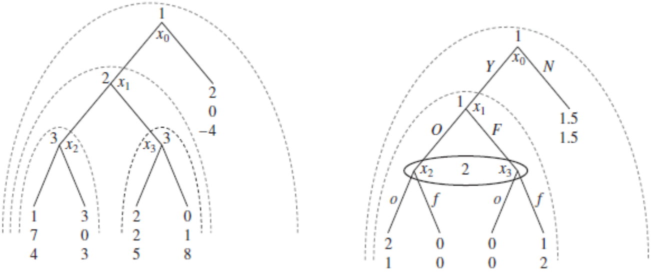
How do we check sequential rationality for *perfect information games* (e.g., chess)?

- *Backward induction*: Begin with nodes that directly precede the terminal nodes and then inductively move backwards through the game.

! In our example, $a_1(C) = N$ is not sequentially rational.

Sequential rationality in games with imperfect information

- A *proper subgame* G of an extensive-form game Γ consists of only a single node and all its successors in Γ with the property that if $x \in G$ and $x' \in h(x)$, then $x' \in G$. The subgame G is itself a game tree with its information sets and payoffs inherited from Γ.



- Let Γ be an n-player extensive-form game. A behavioral strategy profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*)$ is a *subgame-perfect Nash equilibrium* if for every proper subgame G of Γ the restriction of σ^* to G is a Nash equilibrium in G.

Application: Bundling and entry

Bundling (Belleflamme and Peitz 2015, Ch. 16.3.2)

Model with *endogenous* entry and bundling

- Bundling = selling multiple products in a single package
- Famous Microsoft antitrust cases 2004-07: bundling PC operating systems and workgroup operating systems; bundling Windows OS and Windows Media Player

Assumptions

- Firm 1 is operating in Market A and is protected (e.g., by licence).
- Firm 2 is operating in Market B and is not protected.
- Firm 1 considers entry into Market B.
- Consumers have independent valuations (WTP) (α_a, α_b)
- $\alpha_a \sim [0, 1]$, $\alpha_b \sim [0, 1]$
- Product A: zero variable cost, fixed cost $f > 0$
- Product B: zero variable cost, fixed cost $f > 0$
- Status quo: Firm 1 is monopolist in A-market, Firm 2 is monopolist in B-market.
- Entry: If Firm 1 enters Market B, there is price (Bertrand) competition.

Setup

Tree

- Node 1: Firm 1 enters (pays fixed cost) or not.
- Node 2a: If no entry, status quo.
- Node 2b: If entry, Firm 1 announces a separate price or a 1-1 bundling price.
- Pricing (simultaneous-move) subgames: Firm 2 sets a price p_2 and Firm 1 sets a price. Specifically, Firm 1 can set a separate price p_1 or a single price p_{ab} for a 1-1 bundle of goods.

Recall that in the status quo, both firms are monopolists on their markets.

- For any monopolist, the profit maximizing price is

$$p^m = \arg\max p(1-p) = \frac{1}{2}.$$

- The profits are $\pi^{SQ} = p^m(1-p^m) - f = \frac{1}{4} - f$.

Entry with a separate price

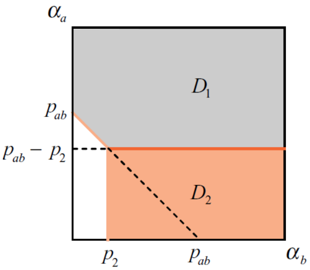
- Pricing subgame: Equilibrium is $p_1 = p_2 = 0$.

Entry with bundling (p_{ab}, p_2)

Pricing subgame: Firm 2's best response to p_{ab}

- demand is $(1 - p_2)(p_{ab} - p_2)$ (D_1 consume both goods, D_2 only good B)
- profits are $p_2(1 - p_2)(p_{ab} - p_2) = p_2(1 - p_2)p_{ab} - p_2^2(1 - p_2)$
- by F.O.C., $3p_2^2 - 2(p_{ab} + 1)p_2 + p_{ab} = 0$; we take the lower root:

$$P_2(p_{ab}) = \frac{2(p_{ab}+1) - \sqrt{4(p_{ab}+1) - 12p_{ab}}}{6} = \frac{p_{ab} + 1 - \sqrt{1 - p_{ab} + p_{ab}^2}}{3}$$



Entry with bundling (p_{ab}, p_2)

Pricing subgame: Firm 1's best response to p_2

- demand is $1 - p_{ab} + p_2 - \frac{p_2^2}{2}$
- profits are $p_{ab}(1 - p_{ab} + p_2) - p_{ab}\frac{p_2^2}{2}$; by F.O.C.:

$$1 - 2p_{ab} + p_2 - \frac{p_2^2}{2} = 0$$

$$P_{ab}(p_2) = \frac{1 + p_2}{2} - \frac{p_2^2}{4}$$

Solving for NE:

$$(p_{ab}^*, p_2^*) = (P_{ab}(p_2), P_2(p_{ab})) \approx (0.61, 0.24)$$

Profits in NE:

$$(\pi_1^*, \pi_2^*) \approx (0.369 - 2f, 0.067 - f)$$

The subgame-perfect Nash equilibrium

Node 2b: Enter with a separate or bundled price?

- Firm 1 prefers bundled price to avoid price war, $0.369 - 2f > 0.25 - 2f$.

Node 1: To enter (with a bundled price) or not to enter?

- Firm 1 enters if and only if (approximately)

$$0.369 - 2f > 0.25 - f$$

$$f < 0.119$$

- Firm 2 obviously prefers to be a monopolist, $0.25 - f > 0.067 - f$.

A sufficiently large fixed cost deters entry.

Application: Agenda-setting power

How strong is agenda-setting power in committees?

Agenda-setters are more 'powerful' than the other members.

- Example: Irish National Lottery Sports Capital Grant Allocations[2]
- 26 Irish counties are ranked by received (per-capita) grants
- Minister of Arts, Sports and Tourism: 1997–2002: Jim McDaid (North Donegal), 2002–2007: John O'Donoghue (South Kerry)

County	1999–2002	2003–207
North Donegal	1st	23rd
South Kerry	10th	1st

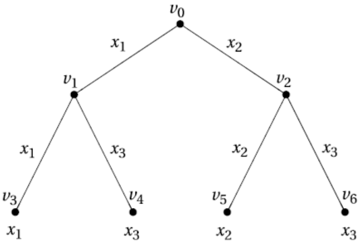
- Minister of Finance: 1999–2004: Charlie McCreevy (North Kildare), 2005–2007: Brian Cowen (Offaly)

County	1999–2004	2005–207
North Kildare	2nd	26th
Offaly	20th	6th

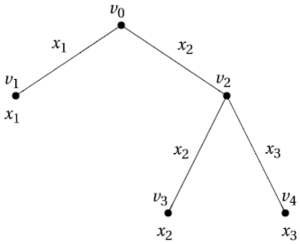
[2] Source: Considine, J., et al. (2008) Irish National Lottery Sports Capital Grant Allocations, 1999–2007: Natural Experiments on Political Influence. *Economic Affairs*, 28 (3), 38–44.

We will analyze only the agenda-setting power (control of the *order* of proposals).

- Suppose committee members are non-cooperative.
- The committee uses sequential pairwise (simple majority) voting.
- The committee chairman sets the agenda (= the order of proposals).



Amendment procedure
US, Canada, UK, Switzerland, Sweden, Finland
Anglo-Saxon



Successive procedure
EU, Norway, European Parliament
Euro-Latin

Agenda-setting power

- If a Condorcet winner (an alternative that wins in each pairwise vote) exists, then the agenda is irrelevant (for both amendment and successive procedure).
- Consider a 'Condorcet triplet' $A \succ B \succ C \succ A$. Suppose Chairman wants to implement A (and wants to eliminate C).
- Amendment procedure: Chairman sets the agenda $\{A, C, B\}$.
  - Node 2a: A vs B, where A wins.
  - Node 2b: C vs B, where B wins. In this node, C won Round 1 but is outvoted.
  - Node 1 is *effectively* A vs B (nominally A vs C), where A wins.
- Successive procedure: Chairman sets the agenda $\{A, B, C\}$.
  - Node 2: B vs C, where B wins. In this node, C is proposed but is outvoted.
  - Node 1: A vs B, where A wins.
  - Intuitively, C is outvoted in the last non-terminal node.
- For the amendment procedure, the threat is optimally eliminated early.
- For the successive procedure, the threat is optimally avoided.

Here, both procedures give the chairman identical agenda-setting power. But in general, the chairman can implement less outcomes under the amendment procedure than under the successive procedure (Banks, 1985; Barbera and Gerber, 2017).

Application: Monetary policy

Monetary policy (Riboni and Ruge-Murcia, 2010)

Boards in central banks adopt formal and informal procedures:

- *individualistic committee*: simple majority voting
- *consensual/collegial committee*: supermajority voting (e.g., $\frac{2}{3}$ to beat the status quo/SQ)
- *chairman-driven committee*: a simple majority vote over the chairman's proposal

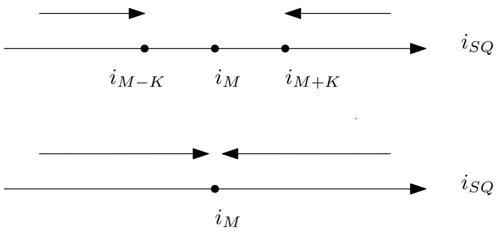
The analysis is manageable, because:

- Monetary policy (interest rate i) is single-dimensional.
- Preferences satisfy single-peakedness.

Riboni and Ruge-Murcia (2010) test which voting procedure fits the data best.

Consensual/supermajority committee

- First, the committee votes (by simple majority) whether to increase or decrease i_{SQ} (current/status-quo interest rate). Second, the interest rate i is sequentially increased (or decreased) by a vote of a *supermajority*.
- If i_{SQ} is close to the median rate i_M, there is no supermajority, and $i = i_{SQ}$.
- If i_{SQ} is far from i_M, then i approaches i_M up to the point where the supermajority ceases to exist, which means incomplete convergence to i_M.

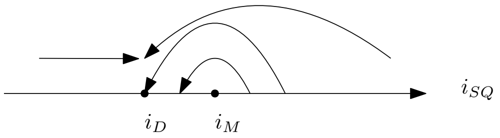


Individualistic/frictionless committee

- Like above; only simple majority is used in the second step; $i = i_M$.

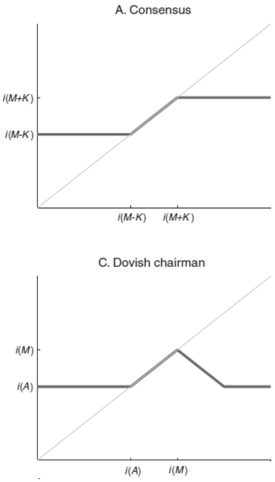
Dovish chairman

- First, find the mirror policy to i_{SQ} (ties with i_{SQ}). Implementable policies are between i_{SQ} and the mirror policy.
- Dovish chairman selects the closest implementable policy (closest to his bliss point).
- If i_{SQ} is far from i_M, then chairman implements his rate i_D.
- If i_{SQ} is close to the median rate i_M, then there is a dovish rate, but not as dovish as the dovish chairman's rate i_D.

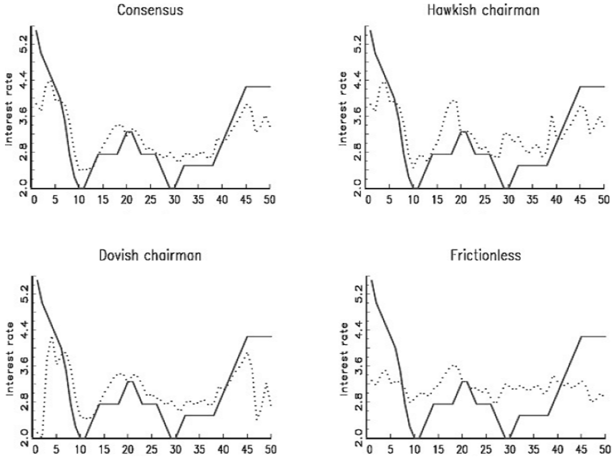


Hawkish chairman is by analogy.

Monetary policy function



Consensus fits best all 5 central banks: Bank of Canada, the Bank of England, the ECB, the Swedish Riksbank, and the U.S. Federal Reserve. See for Bank of Canada:



Application: Coordinated budgeting

What if total spending is set first?

- U.S. prior 1974: Each spending bill voted independently; aggregate level of spending determined as residually.
- CBA74 (Congressional Budget Act): Budget Committee proposes total spending (budget resolution); Appropriations Committee proposes a budget
- It was expected that redistributive efforts partly shift from Budget Committee into Appropriations Committee, where the game is zero-sum, not common-pool.
- Unexpected outcome: Total spending has increased!

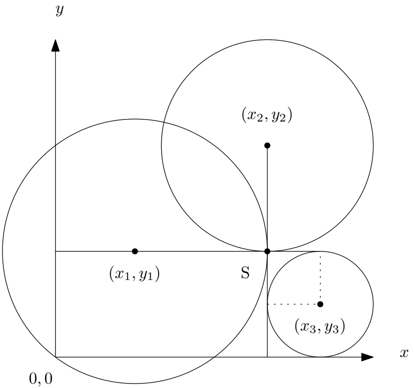
Budget processes

budget	(x, y)
parties	$i \in \{1, 2, 3\}$
optimal allocation of Party i	$(x_i, , y_i)$
Euclidean preferences	$u_i = -(x - x_i)^2 - (y - y_i)^2$
total budget	$X := x + y$
sequential budgeting (A)	x first, y second
coordinated budgeting (B)	X first, x second

What are the differences between coordinated spending and spending caps?

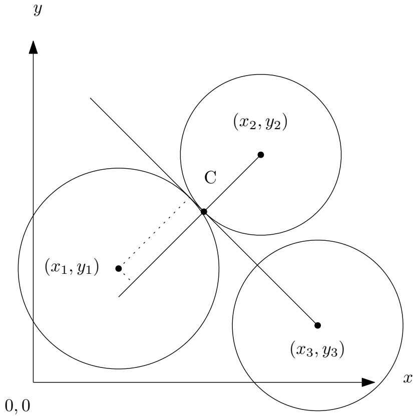
- Coordination imposes an *exact* level of total spending, not only a cap.
- Budget items are set by simple majority voting, not by ministers/parties alone.
- Moreover, with these preferences, budget items bring wide (universal) benefits, not narrow (group) benefits.

Sequential budgeting (S)



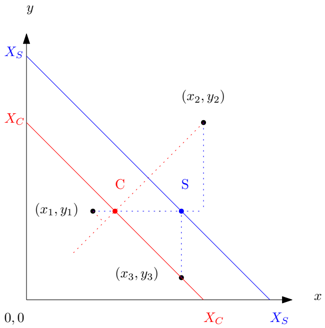
Step 1: x selected
Step 2: y selected (X is given residually)

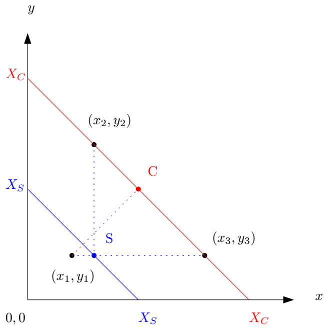
Coordinated budgeting (C)



Step 1: X selected
Step 2: x selected (y is given residually)

$X_S > X_C$ or $X_S < X_C$?


y
X_S
X_C
(x_2, y_2)
C
S
(x_1, y_1)
(x_3, y_3)
x
0, 0
X_C
X_S


y
X_C
(x_2, y_2)
C
X_S
S
(x_3, y_3)
(x_1, y_1)
x
0, 0
X_S
X_C

Scenarios

Restrictive coordination

- Extreme parties prefer *smaller* budgets than the centrist party.
- Sequential budgeting brings ad hoc agreements, in which the centrist party participates = high spending.
- Coordination initiates agreement of low spenders in the first stage, in which the centrist party doesn't participate = low spending.

Expansionary coordination

- Extreme parties prefer *larger* budgets than the centrist party.
- Sequential budgeting brings ad hoc restrictive agreements, in which the centrist party participates = low spending.
- Coordination initiates agreement of high spenders in the first stage, in which the centrist party doesn't participate = high spending.