Final Examination: 5330 Advanced Microeconomic Theory

October 16, 2023, 14:00-18:00

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Instructions: Answer all questions.

1. (9 points) The n-good Cobb-Douglas utility function is

$$u(\mathbf{x}) = A\prod_{i=1}^{n} x_i^{\alpha_i}, \quad \text{where} \quad A > 0 \quad \text{and} \quad \sum_{i=1}^{n} \alpha_i = 1.$$

Compute the expenditure function $e(\mathbf{p}, u)$ using the general definition

$$e(\mathbf{p}, u) \equiv \min_{\mathbf{x}\in\mathbb{R}^n_+} \mathbf{p}\cdot\mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u$$

and also compute the Hicksian demand function $\mathbf{x}^h(\mathbf{p}, u)$.

2. (10 points) Suppose that a firm's production technology is described by a CES production function $y = f(x_1, x_2) \equiv (x_1^\rho + x_2^\rho)^{1/\rho}$ where $-\infty < \rho < 1$ and $\rho \neq 0$. Solve the firm's cost minimization problem

$$c(\mathbf{w}, y) \equiv \min_{\mathbf{x}\in\mathbb{R}^n_+} \mathbf{w}\cdot\mathbf{x} \quad \text{s.t.} \quad f(\mathbf{x}) \geq y$$

and show that the firm's cost function is

$$c(\mathbf{w}, y) = y\,(w_1^r + w_2^r)^{1/r}$$

where $r \equiv \rho/(\rho - 1)$.

3. (4 points) A per-unit tax $t > 0$ is levied on the output of a monopoly. The monopolist faces demand $q = p^{\epsilon}$ where $\epsilon > 1$ and has constant average costs. Show that the monopolist will increase price by more than the amount of the per-unit tax.

4. (14 points) When firms $j = 1, \ldots, J$ are active in a monopolistically competitive market, firm j faces the following demand function:

$$q^j = (p^j)^{-2} \left(\sum_{i=1, i \neq j}^{J} (p^i)^{-1/2} \right)^{-2}, \qquad j = 1, \ldots, J$$

Active or not, each of the many firms $j = 1, \ldots$ has identical costs $c(q) = cq + k$ where $c > 0$ and $k > 0$. Each firm chooses its price to maximize profits, given the prices chosen by the others.

(a) Show that each firm's demand is negatively sloped, with constant own-price elasticity, and that all goods are substitutes for each other.

(b) Show that if all firms raise their prices proportionately, the demand for any given good declines.

(c) Find the long-run Nash equilibrium number of firms.

5. (12 points) Consider an example of Cournot oligopoly in the market for some homogeneous good. Suppose there are $J = 2$ identical firms. Entry by additional firms is effectively blocked. Let each duopolist have constant average and marginal costs, but suppose that $0 < c^1 < c^2$, where c^j is the marginal cost of firm j. Firms sell output on a common market, so market price depends on the total output sold by all firms in the market. Let inverse market demand be the linear form,

$$p = a - b\left(\sum_{j=1}^{2} q^j\right),$$

where $a > 0$ and $b > 0$. We require $a > 2c^2$.

Show that firm 1 will have greater profits ($\pi^1 > \pi^2$) and produce a greater share of market output than firm 2 in the Nash equilibrium ($q^1 > q^2$).

6. (16 points) An exchange economy has three consumers and three goods. Consumers' utility functions and initial endowments are as follows:

$$u^1(x_1, x_2, x_3) = \min(x_1, x_2) \qquad \text{and} \qquad \mathbf{e}^1 = (1, 0, 0),$$

$$u^2(x_1, x_2, x_3) = \min(x_2, x_3) \qquad \text{and} \qquad \mathbf{e}^2 = (0, 1, 0),$$

$$u^3(x_1, x_2, x_3) = \min(x_1, x_3) \qquad \text{and} \qquad \mathbf{e}^3 = (0, 0, 1).$$

Find a Walrasian equilibrium and the associated Walrasian equilibrium allocation (WEA) for this economy.

7. (2 points) Consider the following theorem about consumers:

THEOREM: If the binary relation $\succsim$ is complete, transitive, continuous and strictly monotonic, then there exists a continuous real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ which represents $\succsim$.

Explain how the utility function is defined given the binary relation $\succsim$ in the proof of this theorem. Illustrate using a graph how the utility function is defined, for the special case where there are just two goods, that is, $n = 2$.

8. (8 points) One of the most important results about exchange economies is the theorem about core and equilibria. Present a proof of the following theorem (using any results from earlier theorems that are needed):

THEOREM (Core and Equilibria in Competitive Economies): Consider an exchange economy $(u^i, \mathbf{e}^i)_{i \in \mathcal{I}}$. If each consumer's utility function u^i is strictly increasing on $\mathbb{R}^n_+$, then every Walrasian equilibrium allocation is in the core. That is,

$$W(\mathbf{e}) \subset C(\mathbf{e}).$$

Answers to the exam questions

2. (10 points) Suppose that a firm's production technology is described by a CES production function $y = f(x_1, x_2) \equiv (x_1^\rho + x_2^\rho)^{1/\rho}$ where $-\infty < \rho < 1$ and $\rho \neq 0$. Its cost minimization problem is then

$$-c(\mathbf{w}, y) \equiv \max_{x_1 \geq 0,\, x_2 \geq 0} -(w_1 x_1 + w_2 x_2) \quad \text{s.t.} \quad y - (x_1^\rho + x_2^\rho)^{1/\rho} \leq 0.$$

Assuming $y > 0$ and an interior solution, the Lagrangian function is

$$\mathcal{L}(x_1, x_2, \lambda) \equiv -(w_1 x_1 + w_2 x_2) - \lambda \left[y - (x_1^\rho + x_2^\rho)^{1/\rho} \right]$$

and the first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x_i} = -w_i + \lambda (1/\rho)(x_1^\rho + x_2^\rho)^{(1/\rho)-1} \rho x_i^{\rho-1} = 0, \quad i = 1, 2$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -y + (x_1^\rho + x_2^\rho)^{1/\rho} = 0$$

The first-order Lagrangian conditions reduce to 2 equations in 2 unknowns

$$\frac{w_1}{w_2} = \left(\frac{x_1}{x_2} \right)^{\rho-1}$$

$$y = (x_1^\rho + x_2^\rho)^{1/\rho}$$

Solving for x_1 yields

$$x_1 = x_2 \left(\frac{w_1}{w_2} \right)^{1/(\rho-1)}$$

$$y = \left(x_2^\rho \left(\frac{w_1}{w_2} \right)^{\rho/(\rho-1)} + x_2^\rho \right)^{1/\rho} \left[\left(\frac{w_2}{w_2} \right)^{\rho/(\rho-1)} \right]^{1/\rho}$$

$$y = \frac{x_2}{w_2^{1/(\rho-1)}} \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \right)^{1/\rho}$$

So, rearranging gives the conditional input demands

$$x_2 = y w_2^{1/(\rho-1)} \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \right)^{-1/\rho}$$

$$x_1 = x_2 \left(\frac{w_1}{w_2} \right)^{1/(\rho-1)} = y w_1^{1/(\rho-1)} \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \right)^{-1/\rho}$$

and the cost function

$$c(\mathbf{w}, y) = w_1 x_1(\mathbf{w}, y) + w_2 x_2(\mathbf{w}, y)$$

$$= \left(w_1 y w_1^{1/(\rho-1)} + w_2 y w_2^{1/(\rho-1)} \right) \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \right)^{-1/\rho}$$

$$= y \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \right) \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \right)^{-1/\rho}$$

Thus the firm's cost function is

$$c(\mathbf{w}, y) = y \left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)} \right)^{(\rho-1)/\rho}$$

It is convenient now to define a new parameter $r \equiv \rho/(\rho - 1)$. Then

$$c(\mathbf{w}, y) = y \left(w_1^r + w_2^r \right)^{1/r}.$$

7. (2 points) Let $\mathbf{e} \equiv (1, 1, \ldots, 1) \in \mathbb{R}^n_+$ be a vector of ones, and consider the mapping $u : \mathbb{R}^n_+ \to \mathbb{R}$ defined so that the following condition is satisfied:

$$u(\mathbf{x})\mathbf{e} \sim \mathbf{x}$$

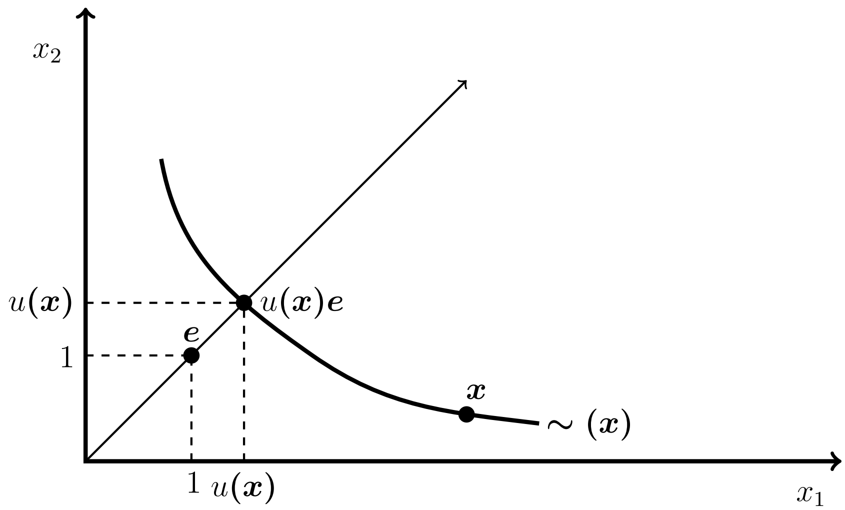

Figure 1: Constructing the utility function

8. (8 points) **Proof**: The theorem claims that if $\mathbf{x}(\mathbf{p}^*)$ is a WEA for equilibrium prices $\mathbf{p}^*$, then $\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$. To prove it, suppose $\mathbf{x}(\mathbf{p}^*)$ is a WEA and assume that $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$.

Because $\mathbf{x}(\mathbf{p}^*)$ is a WEA, we know from the earlier Lemma that $\mathbf{x}(\mathbf{p}^*) \in F(\mathbf{e})$, so $\mathbf{x}(\mathbf{p}^*)$ is feasible.

However, because $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$, we can find a coalition S and another allocation $\mathbf{y}$ such that

$$\sum_{i\in S} \mathbf{y}^i = \sum_{i\in S} \mathbf{e}^i$$

and

$$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)) \quad \text{for all } i \in S,$$

with at least one inequality strict.

Multiplying both sides of the summation by $\mathbf{p}^*$ yields

$$\mathbf{p}^* \cdot \sum_{i\in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i\in S} \mathbf{e}^i$$

From

$$u^i(\mathbf{y}^i) \geq u^i(\mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i)) \quad \text{for all } i \in S$$

and the last Lemma, we know that

$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) = \mathbf{p}^* \cdot \mathbf{e}^i,$$

with at least one inequality strict.

Given

$$\mathbf{p}^* \cdot \mathbf{y}^i \geq \mathbf{p}^* \cdot \mathbf{x}^i(\mathbf{p}^*, \mathbf{p}^* \cdot \mathbf{e}^i) = \mathbf{p}^* \cdot \mathbf{e}^i,$$

summing over all consumers in S, we obtain

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i > \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i,$$

which contradicts the earlier equation

$$\mathbf{p}^* \cdot \sum_{i \in S} \mathbf{y}^i = \mathbf{p}^* \cdot \sum_{i \in S} \mathbf{e}^i.$$

Since we assumed $\mathbf{x}(\mathbf{p}^*) \notin C(\mathbf{e})$ and obtained a contradiction, it must be that $\mathbf{x}(\mathbf{p}^*) \in C(\mathbf{e})$.

$\square$