5330 Advanced Microeconomic Theory

Lecture: The Consumer

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Introduction

This lecture is mainly based on

- Geoffrey Jehle and Philip Reny (2011), *Advanced Microeconomic Theory*, chapter 1, and any good mathematics textbook.

- Among the many good math textbooks out there, my personal favorite is Malcolm Pemberton and Nicholas Rau (2016), *Mathematics for Economists*. stronger at proofs of theorems
- Another very good math textbook is *Mathematics for Economics* by Michael Hoy, John Livernois, Chris McKenna, Ray Rees and Thanasis Stengos (2011). stronger at worked out examples

- There are four building blocks in any model of consumer choice. They are the consumption set, the feasible set, the preference relation and the behavioral assumption.

- The **consumption set**, X, represents the set of all alternatives that the consumer can choose from.
- We assume there is a finite, fixed but arbitrary number n of different goods. Let $\mathbf{x} \equiv (x_1, x_2, \ldots, x_n)$ be a vector containing different quantities of each of the n commodities and call $\mathbf{x}$ a **consumption bundle** or a **consumption plan**. A consumption bundle $\mathbf{x} \in X$ is thus represented by a point $\mathbf{x} \in \mathbb{R}^n_+$.
- We assume that the consumption set is the entire nonnegative orthant, $X = \mathbb{R}^n_+$.

- The **feasible set** $B \subset X$ represents all those alternative consumption bundles that are achievable given the economic realities the consumer faces.
- The **preference relation** specifies information about the consumer's tastes for the different objects of choice.
- Finally, the **behavioral assumption** expresses the guiding principle the consumer uses to make final choices. We assume that the consumer seeks to identify and select an available alternative that is most preferred in the light of his personal tastes.

Preference Relations

- We represent the consumer's preferences by a *binary relation*, $\succsim$, defined on the consumption set, X.
- If $\mathbf{x}^1 \succsim \mathbf{x}^2$, we say that "$\mathbf{x}^1$ is at least as good as $\mathbf{x}^2$".
- We require only that the consumer makes *binary* comparisons, that is, that she only examines two consumption bundles at a time and makes a decision regarding those two.

- **AXIOM 1**: Completeness. For all $\mathbf{x}^1$ and $\mathbf{x}^2$ in X, either $\mathbf{x}^1 \succsim \mathbf{x}^2$ or $\mathbf{x}^2 \succsim \mathbf{x}^1$.
- Axiom 1 formalizes the notion that the consumer *can* make comparisons.

- **AXIOM 2**: Transitivity. For any three elements $\mathbf{x}^1$, $\mathbf{x}^2$, and $\mathbf{x}^3$ in X, if $\mathbf{x}^1 \succsim \mathbf{x}^2$ and $\mathbf{x}^2 \succsim \mathbf{x}^3$, then $\mathbf{x}^1 \succsim \mathbf{x}^3$.
- Axiom 2 gives a very particular form to the requirement that the consumer's choices be *consistent*.

- Together, Axioms 1 and 2 constitute a formal definition of **rationality** as the term is used in economic theory. Rational economic agents are ones who have the ability to make choices, whose internal workings are at least minimally coherent, and whose choices display a logical consistency.
- **DEFINITION**: The binary relation $\succsim$ on the consumption set X is called a **preference relation** if it satisfies Axioms 1, and 2.

- **DEFINITION**: The binary relation $\succ$ on the consumption set X is defined as follows:

$$\mathbf{x}^1 \succ \mathbf{x}^2 \quad \text{if and only if} \quad \mathbf{x}^1 \succsim \mathbf{x}^2 \quad \text{and not} \quad \mathbf{x}^2 \succsim \mathbf{x}^1.$$

  The relation $\succ$ is called the **strict preference relation** induced by $\succsim$. The phrase $\mathbf{x}^1 \succ \mathbf{x}^2$ is read, "$\mathbf{x}^1$ is strictly preferred to $\mathbf{x}^2$."
- **DEFINITION**: The binary relation $\sim$ on the consumption set X is defined as follows:

$$\mathbf{x}^1 \sim \mathbf{x}^2 \quad \text{if and only if} \quad \mathbf{x}^1 \succsim \mathbf{x}^2 \quad \text{and} \quad \mathbf{x}^2 \succsim \mathbf{x}^1.$$

  The relation $\sim$ is called the **indifference relation** induced by $\succsim$. The phrase $\mathbf{x}^1 \sim \mathbf{x}^2$ is read, "$\mathbf{x}^1$ is indifferent to $\mathbf{x}^2$."
- For any pair $\mathbf{x}^1$ and $\mathbf{x}^2$, exactly one of three mutually exclusive possibilities exist: $\mathbf{x}^1 \succ \mathbf{x}^2$ or $\mathbf{x}^2 \succ \mathbf{x}^1$ or $\mathbf{x}^1 \sim \mathbf{x}^2$.

- **DEFINITION**: Let $\mathbf{x}^0$ be any point in the consumption set X. Relative to any such point, we can define the following subsets of X:

  $\succsim(\mathbf{x}^0) \equiv \{\mathbf{x} \,|\, \mathbf{x} \in X, \mathbf{x} \succsim \mathbf{x}^0\}$, called the "at least as good as" set.

  $\precsim(\mathbf{x}^0) \equiv \{\mathbf{x} \,|\, \mathbf{x} \in X, \mathbf{x} \precsim \mathbf{x}^0\}$, called the "no better than" set.

  $\prec(\mathbf{x}^0) \equiv \{\mathbf{x} \,|\, \mathbf{x} \in X, \mathbf{x} \prec \mathbf{x}^0\}$, called the "worse than" set.

  $\succ(\mathbf{x}^0) \equiv \{\mathbf{x} \,|\, \mathbf{x} \in X, \mathbf{x} \succ \mathbf{x}^0\}$, called the "preferred to" set.

  $\sim(\mathbf{x}^0) \equiv \{\mathbf{x} \,|\, \mathbf{x} \in X, \mathbf{x} \sim \mathbf{x}^0\}$, called the "indifference" set.

- For any bundle $\mathbf{x}^0$, the three sets $\prec(\mathbf{x}^0)$, $\succ(\mathbf{x}^0)$ and $\sim(\mathbf{x}^0)$ partition the consumption set.

- **AXIOM 3**: Continuity. For all $\mathbf{x} \in \mathbb{R}^n_+$, the "at least as good as" set $\succsim(\mathbf{x})$ and the "no better than" set $\precsim(\mathbf{x})$ are closed in $\mathbb{R}^n_+$.

- The continuity axiom guarantees that sudden preference reversals do not occur.
- Remember that a set D if closed if and only if every sequence $\{\mathbf{x}^k\}_{k=1}^{\infty}$ of points in D converging to some $\mathbf{x} \in \mathbb{R}^n_+$, it is also the case that $\mathbf{x} \in D$. (proof in Pemberton & Rao on page 727)
- Thus the continuity axiom can be equivalently expressed by saying that if each element $\mathbf{y}^n$ of a sequence of bundles is at least as good as $\mathbf{x}$, and $\mathbf{y}^n$ converges to $\mathbf{y}$, then $\mathbf{y}$ is at least as good as $\mathbf{x}$.

- We want to express the fundamental view that "wants" are essentially unlimited.
- **AXIOM 4'**: Local Nonsatiation. For all $\mathbf{x}^0 \in \mathbb{R}^n_+$, and for all $\epsilon > 0$, there exists some $\mathbf{x} \in B_\epsilon(\mathbf{x}^0) \cap \mathbb{R}^n_+$ such that $\mathbf{x} \succ \mathbf{x}^0$.
[Remember that the open ϵ-ball with center $\mathbf{x}^0$ and radius $\epsilon > 0$ is defined by $B_\epsilon(\mathbf{x}^0) \equiv \{\mathbf{x} \in \mathbb{R}^n \mid d(\mathbf{x}^0, \mathbf{x}) < \epsilon\}$.]
- Whereas local nonsatiation requires that a preferred alternative always exists, it does not rule out the possibility that the preferred alternative may involve less of some or even all commodities. To guarantee that more is always better than less, we need a stronger axiom:
- **AXIOM 4**: Strict Monotonicity. For all $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, if $\mathbf{x}^0 \geq \mathbf{x}^1$ then $\mathbf{x}^0 \succsim \mathbf{x}^1$, while if $\mathbf{x}^0 \gg \mathbf{x}^1$, then $\mathbf{x}^0 \succ \mathbf{x}^1$.

- **AXIOM 5'**: Convexity. If $\mathbf{x}^1 \succsim \mathbf{x}^0$, then $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succsim \mathbf{x}^0$ for all $t \in [0,1]$.

- **AXIOM 5**: Strict Convexity. If $\mathbf{x}^1 \neq \mathbf{x}^0$ and $\mathbf{x}^1 \succsim \mathbf{x}^0$, then $t\mathbf{x}^1 + (1-t)\mathbf{x}^0 \succ \mathbf{x}^0$ for all $t \in (0,1)$.

- Suppose we choose $\mathbf{x}^1 \sim \mathbf{x}^0$. Any convex combination of $\mathbf{x}^1$ and $\mathbf{x}^0$, such as $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1-t)\mathbf{x}^0$ will be a bundle containing a more "balanced" combination of x_1 and x_2 than does either "extreme" bundle $\mathbf{x}^1$ or $\mathbf{x}^0$. Axiom 5 requires that the consumer strictly prefers any such relatively balanced consumption bundles to both of the extremes between which she is indifferent.
- We require that tastes display some bias in favor of balance in consumption.

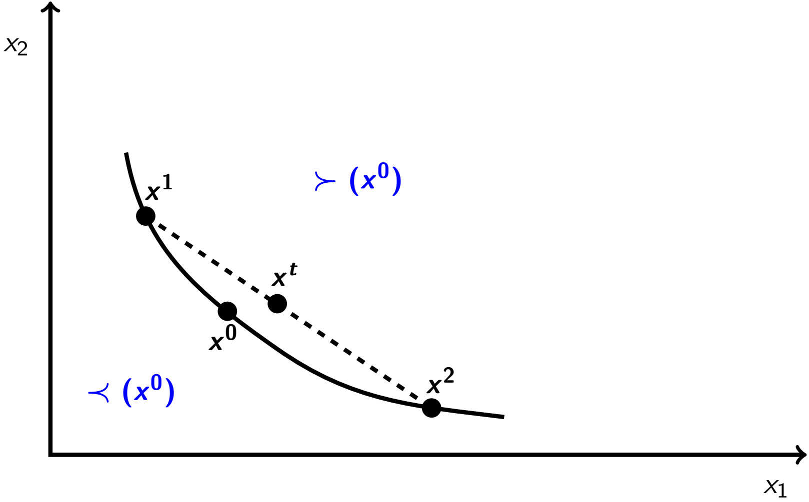

Figure: Preferences that satisfy strict convexity

The Utility Function

- **DEFINITION**: A real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called **a utility function representing the preference relation** $\succsim$ if, for all $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, $u(\mathbf{x}^0) \geq u(\mathbf{x}^1) \iff \mathbf{x}^0 \succsim \mathbf{x}^1$.

- In modern theory, a utility function is simply a convenient device for summarizing the same information about consumer preferences as the preference relation does – no more and no less.
- Sometimes (almost never) it is easier to work directly with the preference relation and its associated sets. Other times (almost always), especially when you would like to be able to use calculus methods, it is easier to work with preferences summarized by a utility function.

- **THEOREM**: If the binary relation $\succsim$ is complete, transitive, continuous and strictly monotonic, then there exists a continuous real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ which represents $\succsim$.

- **Proof**: Let the relation $\succsim$ be complete, transitive, continuous and strictly monotonic. Let $\mathbf{e} \equiv (1, 1, \ldots, 1) \in \mathbb{R}^n_+$ be a vector of ones, and consider the mapping $u : \mathbb{R}^n_+ \to \mathbb{R}$ defined so that the following condition is satisfied:

$$u(\mathbf{x})\mathbf{e} \sim \mathbf{x}$$

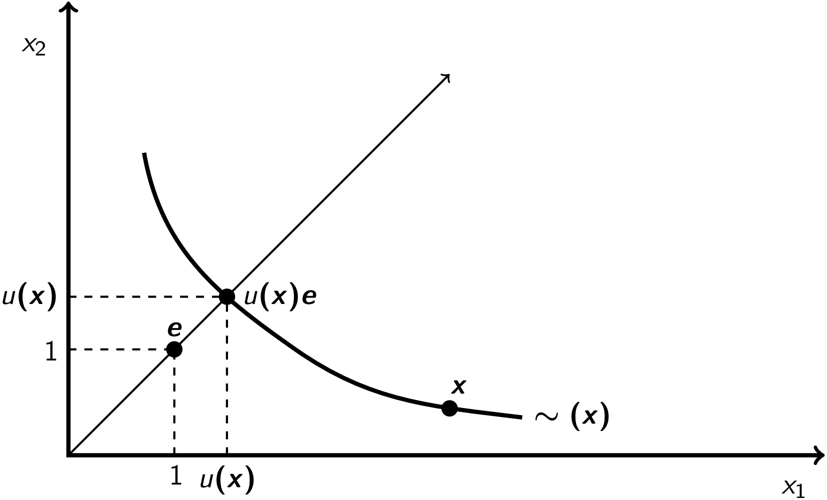

Figure: Constructing the utility function

- Does there always exist a number $u(\mathbf{x})$ satisfying equation $u(\mathbf{x})\mathbf{e} \sim \mathbf{x}$? Fix $\mathbf{x} \in X$ and consider the following two subsets of real numbers:

$$A \equiv \{t \geq 0 \,|\, t\mathbf{e} \succsim \mathbf{x}\}$$

$$B \equiv \{t \geq 0 \,|\, t\mathbf{e} \precsim \mathbf{x}\}.$$

  Note that if $t^* \in A \cap B$, then $t^*\mathbf{e} \sim \mathbf{x}$, so that setting $u(\mathbf{x}) = t^*$ would satisfy the equation $u(\mathbf{x})\mathbf{e} \sim \mathbf{x}$.

- First, we show that the continuity of $\succsim$ implies that both sets A and B are closed in $\mathbb{R}_+$. We use the following theorem (proof on page 727 in P&R):

  Set $D \subset \mathbb{R}^n$ is closed if and only if for every sequence $\{\mathbf{x}^k\}_{k=1}^{\infty}$ of points in D converging to some $\mathbf{x} \in \mathbb{R}^n$, it is also the case that $\mathbf{x} \in D$.

- So suppose that $\succsim$ is continuous and $A \equiv \{t \geq 0 \,|\, t\mathbf{e} \succsim \mathbf{x}\}$ is not closed in $\mathbb{R}_+$. Then there exists some sequence $\{t^k\}_{k=1}^{\infty}$ of points in A that converges to a point t^0 that is not in A.
- It follows that there exists some sequence $\{t^k\mathbf{e}\}_{k=1}^{\infty}$ of points in $\succsim(\mathbf{x})$ that converges to a point $t^0\mathbf{e}$ that is not in $\succsim(\mathbf{x})$. Therefore the set $\succsim(\mathbf{x})$ is not closed.
- But we have assumed that $\succsim$ is continuous, which implies that $\succsim(\mathbf{x})$ is closed. Contradiction. Therefore, A must be closed (and a similar argument establishes that B must be closed).

- So the continuity of $\succsim$ implies that both sets $A \equiv \{t \geq 0 \,|\, t\mathbf{e} \succsim \mathbf{x}\}$ and $B \equiv \{t \geq 0 \,|\, t\mathbf{e} \precsim \mathbf{x}\}$ are closed in $\mathbb{R}_+$.
- Also, by strict monotonicity, $t \in A$ implies that $t' \in A$ for all $t' \geq t$. Consequently, A must be a closed interval of the form $[\underline{t}, \infty)$.
- Similarly, strict monotonicity and closedness of B in $\mathbb{R}_+$ implies that B must be a closed interval of the form $[0, \bar{t}]$.
- Now for any $t \geq 0$, completeness of $\succsim$ implies that either $t\mathbf{e} \succsim \mathbf{x}$ or $t\mathbf{e} \precsim \mathbf{x}$, that is, $t \in A \cup B$. But this means that $\mathbb{R}_+ = A \cup B = [0, \bar{t}] \cup [\underline{t}, \infty)$. We conclude that $\underline{t} \leq \bar{t}$ so that $A \cap B \neq \emptyset$.

- Next, we show that there is only one number $t \geq 0$ such that $t\mathbf{e} \sim \mathbf{x}$.
- But this follows easily because if $t_1\mathbf{e} \sim \mathbf{x}$ and $t_2\mathbf{e} \sim \mathbf{x}$, then by the transitivity of $\sim$, $t_1\mathbf{e} \sim t_2\mathbf{e}$.
- So, by strict monotonicity, it must be the case that $t_1 = t_2$.
- We conclude that for every $\mathbf{x} \in X$, there is exactly one number $u(\mathbf{x})$ such that $u(\mathbf{x})\mathbf{e} \sim \mathbf{x}$ is satisfied. Having constructed a utility function assigning each bundle in X a number, we show next that this utility function represents the preferences $\succsim$.

- Consider two bundles x^1 and x^2, and their associated utility numbers $u(\mathsf{x}^1)$ and $u(\mathsf{x}^2)$, which by definition satisfy $u(\mathsf{x}^1)\mathbf{e} \sim \mathsf{x}^1$ and $u(\mathsf{x}^2)\mathbf{e} \sim \mathsf{x}^2$. Then we have the following:

$$
\begin{aligned}
\mathsf{x}^1 \succsim \mathsf{x}^2 \quad &\Longleftrightarrow \quad u(\mathsf{x}^1)\mathbf{e} \sim \mathsf{x}^1 \succsim \mathsf{x}^2 \sim u(\mathsf{x}^2)\mathbf{e} \\
&\Longleftrightarrow \quad u(\mathsf{x}^1)\mathbf{e} \succsim u(\mathsf{x}^2)\mathbf{e} \\
&\Longleftrightarrow \quad u(\mathsf{x}^1) \geq u(\mathsf{x}^2)
\end{aligned}
$$

- It remains only to show that the utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ representing $\succsim$ is continuous.
- It suffices to show that the inverse image under u of every open ball in $\mathbb{R}$ is open in $\mathbb{R}^n_+$. Because open balls in $\mathbb{R}$ are merely open intervals, this is equivalent to showing that $u^{-1}((a, b))$ is open in $\mathbb{R}^n_+$ for every $a < b$. We use the following theorem (proof on page 731 in P&R):

Let D be any subset of $\mathbb{R}^m$. The following conditions are equivalent:

1. $f : D \to \mathbb{R}^n$ is continuous.
2. For every open ball B in $\mathbb{R}^n$, $f^{-1}(B)$ is open in D.
3. For every open set S in $\mathbb{R}^n$, $f^{-1}(S)$ is open in D.

- Now,
$$\begin{aligned} u^{-1}((a,b)) &= \{\mathbf{x} \in \mathbb{R}^n_+ \mid a < u(\mathbf{x}) < b\} \\ &= \{\mathbf{x} \in \mathbb{R}^n_+ \mid a\,\mathbf{e} \prec u(\mathbf{x})\mathbf{e} \prec b\,\mathbf{e}\} \\ &= \succ (a\,\mathbf{e}) \bigcap \prec (b\,\mathbf{e}) \end{aligned}$$
- By the continuity of $\succsim$, the sets $\precsim (a\,\mathbf{e})$ and $\succsim (b\,\mathbf{e})$ are closed in $X = \mathbb{R}^n_+$. Therefore the complements of these sets, $\succ (a\,\mathbf{e})$ and $\prec (b\,\mathbf{e})$, are open in $\mathbb{R}^n_+$.

- Next, we use the following theorem (proof on pages 742-3 in P&R):

  The intersection of any finite number of open sets in $\mathbb{R}^n$ is an open set in $\mathbb{R}^n$, (also true for any subset of a metric space, for example, $\mathbb{R}^n_+$)

  and the following definition:

  Let D (for example, $\mathbb{R}^n_+$) be a subset of $\mathbb{R}^n$. Then a subset S of D is open in D if for every $\mathbf{x} \in S$ there is an $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \cap D \subset S$.

- It follows that $u^{-1}((a,b))$, being the intersection of two open sets in $\mathbb{R}^n_+$, is itself open in $\mathbb{R}^n_+$. Therefore u is continuous.

$\square$

- All we require of the consumer via the preference relation is the ability to rank consumption bundles from best to worse.
- All that we require of the utility function representing those preferences is that it reflect that same ranking on a simple numerical scale by assigning larger numbers to more preferred bundles, smaller numbers to less preferred bundles, and the same number to indifferent bundles.
- But if some function u is able to do that, will it not be the case that the function $v = u + 5$, or the function $v = u^3$, or the function $v = \log u$, also assigns a higher number to a more preferred bundle, a lower number to a less preferred bundle, and the same number to indifferent bundles?

- **THEOREM**: Let $\succsim$ be a preference relation on $\mathbb{R}^n_+$ and suppose $u(\mathbf{x})$ is a utility function that represents it. Then $v(\mathbf{x})$ also represents $\succsim$ if and only if $v(\mathbf{x}) = f(u(\mathbf{x}))$ for every $\mathbf{x}$, where $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing on the set of values taken on by u.
  $f(u) \equiv u + 5$ implies that $f'(u) = 1 > 0$

- **THEOREM**: Let $\succsim$ be represented by $u : \mathbb{R}^n_+ \to \mathbb{R}$. Then:

  1. $u(\mathbf{x})$ is strictly increasing if and only if $\succsim$ is strictly monotonic.
  2. $u(\mathbf{x})$ is quasiconcave if and only if $\succsim$ is convex.
  3. $u(\mathbf{x})$ is strictly quasiconcave if and only if $\succsim$ is strictly convex.

- **ASSUMPTION**: In the following definitions, whenever $f : D \to \mathbb{R}$ is a real-valued function, we will assume $D \subset \mathbb{R}^n$ is a convex set. When we take $\mathbf{x}^1 \in D$ and $\mathbf{x}^2 \in D$, we will let $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1-t)\mathbf{x}^2$, for $t \in [0,1]$, denote the convex combination of $\mathbf{x}^1$ and $\mathbf{x}^2$. Because D is convex, we know that $\mathbf{x}^t \in D$.
- **DEFINITION**: $f : D \to \mathbb{R}$ is **quasiconcave** if, for all $\mathbf{x}^1$ and $\mathbf{x}^2$ in D,

$$f(\mathbf{x}^t) \geq \min\left[f(\mathbf{x}^1), f(\mathbf{x}^2)\right] \text{ for all } t \in [0,1].$$

- **DEFINITION**: $f : D \to \mathbb{R}$ is **strictly quasiconcave** if, for all $\mathbf{x}^1 \neq \mathbf{x}^2$ in D,

$$f(\mathbf{x}^t) > \min\left[f(\mathbf{x}^1), f(\mathbf{x}^2)\right] \text{ for all } t \in (0,1).$$

The Consumer's Problem

- We view the consumer as having a consumption set $X = \mathbb{R}^n_+$ containing all conceivable alternatives in consumption.
- Her inclinations and attitudes toward them are described by the preference relation $\succsim$ defined on $\mathbb{R}^n_+$.
- The consumer's circumstances limit the alternatives she is actually able to achieve, and we collect these all together into a feasible set $B \subset \mathbb{R}^n_+$.
- Finally, we assume the consumer is motivated to choose the most preferred feasible alternative according to her preference relation. The consumer seeks

$$\mathbf{x}^* \in B \text{ such that } \mathbf{x}^* \succsim \mathbf{x} \text{ for all } \mathbf{x} \in B.$$

- We make the following assumptions that will be maintained unless stated otherwise.
- **ASSUMPTION**: The consumer's preference relation $\succsim$ is complete, transitive, continuous, strictly monotonic and strictly convex on $\mathbb{R}^n_+$. Therefore by the previous theorems it can be represented by a real-valued utility function u that is continuous, strictly increasing and strictly quasiconcave on $\mathbb{R}^n_+$.

- Our concern is with an individual consumer operating within a **market economy**. By a market economy, we mean an economic system in which transactions between agents are mediated by markets. There is a market for each good in the consumption set and in these markets, a price p_i prevails for each commodity.
- We suppose that prices are strictly positive, so $p_i > 0, i = 1, \ldots, n$.
- Moreover, we assume the individual consumer is an insignificant force on every market. By this we mean, specifically, that the size of each market relative to the potential purchases of the individual consumer is so large that no matter how much or how little the consumer might purchase, there will be no preceptible effect on any market price.
- We take the vector of market prices $\mathbf{p} \gg \mathbf{0}$ as *fixed* from the consumer's point of view.

- The consumer is endowed with a fixed money income $y > 0$ and all current purchases must be paid for out of current resources.
- Because the purchase of x_i units of commodity i at price p_i per unit requires an expenditure of $p_i x_i$ dollars, the requirement that expenditure not exceed income can be stated as $\sum_{i=1}^{n} p_i x_i \leq y$ or, more compactly, as $\mathbf{p} \cdot \mathbf{x} \leq y$. (In general, we will take prices to be row vectors and quantities to be column vectors.)
- Thus the feasible set (also called the **budget set**) is

$$B = \{\mathbf{x} \,|\, \mathbf{x} \in \mathbb{R}^n_+, \mathbf{p} \cdot \mathbf{x} \leq y\}.$$

- The consumer's problem can thus be cast equivalently as the problem of maximizing the utility function subject to the budget constraint:

$$\max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} \leq y.$$

- Note that if $\mathbf{x}^*$ solves this **utility-maximization problem**, then $u(\mathbf{x}^*) \geq u(\mathbf{x})$ for all $\mathbf{x} \in B$, which means that $\mathbf{x}^* \succsim \mathbf{x}$ for all $\mathbf{x} \in B$.

- Under the assumptions on preferences, the utility function $u(\mathbf{x})$ is real-valued and continuous.
- The budget set B is a nonempty (it contains $\mathbf{0} \in \mathbb{R}^n_+$ since $y > 0$), closed, bounded (because all prices are strictly positive) and thus compact subset of $\mathbb{R}^n$. Here we are using the definition:

  A set S in $\mathbb{R}^n$ is called **compact** if it is closed and bounded.

- By the Weierstrass theorem, we are therefore assured that a maximum of $u(\mathbf{x})$ over B exists. (proof on pages 731-2 in P&R)

  THEOREM (Weierstrass Existence of Extreme Values):
  Let $f : S \to \mathbb{R}$ be a continuous real-valued function where S is a nonempty compact subset of $\mathbb{R}^n$. Then there exists a vector $\mathbf{x}^* \in S$ and a vector $\tilde{\mathbf{x}} \in S$ such that

  $$f(\tilde{\mathbf{x}}) \leq f(\mathbf{x}) \leq f(\mathbf{x}^*) \text{ for all } \mathbf{x} \in S.$$

- Moreover, because B is convex and the objective function is strictly quasiconcave, the maximizer of $u(\mathbf{x})$ over B is *unique*.

  DEFINITION: A function $f : D \to \mathbb{R}$ is strictly quasiconcave if, for all $\mathbf{x}^1 \neq \mathbf{x}^2$ in the convex set $D \subset \mathbb{R}^n$ and $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1-t)\mathbf{x}^2$, $f(\mathbf{x}^t) > \min[f(\mathbf{x}^1), f(\mathbf{x}^2)]$ for all $t \in (0,1)$.

- Proof: Suppose both $\mathbf{x}^1$ and $\mathbf{x}^2$ are maximizers of $u(\mathbf{x})$ over B. Then any convex combination $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1-t)\mathbf{x}^2$ is also in B and $u(\mathbf{x}^t) > \min[u(\mathbf{x}^1), u(\mathbf{x}^2)] = u(\mathbf{x}^1) = u(\mathbf{x}^2)$. Contradiction.

$\square$

- Because preferences are strictly monotonic, the solution $\mathbf{x}^*$ will satisfy the budget constraint with equality, lying on, rather than inside, the boundary of the budget set.
- Thus, because $y > 0$ and $\mathbf{x}^* \geq \mathbf{0}$ but $\mathbf{x}^* \neq \mathbf{0}$, we know that $x_i^* > 0$ for at least one good i.
- Clearly the solution vector $\mathbf{x}^*$ is a function of the set of prices and income, so we write $x_i^* = x_i(\mathbf{p}, y)$, $i = 1, \ldots, n$, or in vector notation, $\mathbf{x}^* = \mathbf{x}(\mathbf{p}, y)$. These are called the ordinary or **Marshallian demand functions**.

- How do we solve the consumer's problem

$$\max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p}\cdot\mathbf{x} \leq y$$

in a concrete example where the function u is specified? First, we explain the general principles for how the problem is solved and then we do an example.

- First, consider the simpler problem with just two variables and an equality constraint:

$$\max_{x_1,x_2} f(x_1,x_2) \quad \text{subject to} \quad g(x_1,x_2) = 0$$

Note that the equality constraint $p_1x_1 + p_2x_2 = y$ can be conveniently rewritten as $g(x_1,x_2) \equiv p_1x_1 + p_2x_2 - y = 0$.

- To solve the problem

$$\max_{x_1, x_2} f(x_1, x_2) \quad \text{subject to} \quad g(x_1, x_2) = 0,$$

first we mechanically construct a new function, called the **Lagrangian**, where λ is a new variable:

$$\mathcal{L}(x_1, x_2, \lambda) \equiv f(x_1, x_2) - \lambda\, g(x_1, x_2).$$

- Then we solve for critical points of $\mathcal{L}$:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -g(x_1, x_2) = 0$$

- These are three equations in the three unknowns x_1, x_2 and λ. Lagrange's method asserts that if we can find values x_1^* , x_2^* and λ^* that solve these three equations simultaneously, then we will have a critical point of $f(x_1, x_2)$ along the constraint $g(x_1, x_2) = 0$.
- Lagrange's method "works" for functions with any finite number of variables and in problems with any finite number of constraints, as long as the number of constraints is less than the number of variables being chosen.
- Suppose we have a function of n variables and we face m constraints, where $m < n$:

$$\max_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \quad \text{subject to} \quad g^j(x_1, \ldots, x_n) = 0, \; j = 1, \ldots, m.$$

- First we mechanically construct the **Lagrangian** function:

$$\mathcal{L}(\mathbf{x}, \lambda) \equiv f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j \, g^j(\mathbf{x})$$

  where $\mathbf{x} \equiv (x_1, \ldots, x_n)$ and $\lambda \equiv (\lambda_1, \ldots, \lambda_m)$.
- Then we solve for critical points of $\mathcal{L}$:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \, \frac{\partial g^j}{\partial x_i} = 0 \quad i = 1, \ldots, n$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = -g^j(\mathbf{x}) = 0 \quad j = 1, \ldots, m.$$

- In principle, these $n + m$ equations can be solved for the $n + m$ values $\mathbf{x}^*$ and λ^*. All solution vectors $\mathbf{x}^*$ will then be candidates for the solution to the constrained optimization problem.

- **THEOREM (Lagrange)**: Let $f(\mathbf{x})$ and $g^j(\mathbf{x})$, $j = 1, \ldots, m$ be continuously differentiable real-valued functions over some domain $D \subset \mathbb{R}^n$. Let $\mathbf{x}^*$ be an interior point of D and suppose that $\mathbf{x}^*$ is an optimum (maximum or minimum) of f subject to the constraints $g^j(\mathbf{x}^*) = 0$, $j = 1, \ldots, m$. If the gradient vectors $\nabla g^j(\mathbf{x}^*)$, $j = 1, \ldots, m$, are linearly independent, then there exist m unique numbers λ_j^*, $j = 1, \ldots, m$, such that

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_i} = \frac{\partial f(\mathbf{x}^*)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j^* \frac{\partial g^j(\mathbf{x}^*)}{\partial x_i} = 0 \quad i = 1, \ldots, n.$$

- **THEOREM (Kuhn-Tucker)**: Let $f(\mathbf{x})$ and $g^j(\mathbf{x})$, $j = 1, \ldots, m$ be continuous real-valued functions defined over some domain $D \subset \mathbb{R}^n$. Let $\mathbf{x}^*$ be an interior point of D and suppose that $\mathbf{x}^*$ maximizes $f(\mathbf{x})$ on D subject to the constraints $g^j(\mathbf{x}) \leq 0$, $j = 1, \ldots, m$, and that f and each g^j are continuously differentiable on an open set containing $\mathbf{x}^*$. If the gradient vectors $\nabla g^j(\mathbf{x}^*)$ associated with constraints j that bind at $\mathbf{x}^*$ are linearly independent, then there exists a unique vector $\lambda^* \equiv (\lambda_1^*, \ldots, \lambda_m^*) \in \mathbb{R}^m$ such that $(\mathbf{x}^*, \lambda^*)$ satisfies the Kuhn-Tucker conditions:

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_i} = \frac{\partial f(\mathbf{x}^*)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j^* \frac{\partial g^j(\mathbf{x}^*)}{\partial x_i} = 0 \quad i = 1, \ldots, n.$$

$$\lambda_j^* \geq 0 \qquad g^j(\mathbf{x}^*) \leq 0 \qquad \lambda_j^* g^j(\mathbf{x}^*) = 0 \qquad j = 1, \ldots, m.$$

- Now we return to the consumer's problem

$$\max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} \leq y.$$

- As we have noted, a solution $\mathbf{x}^*$ exists and is unique. If we rewrite the constraint as $\mathbf{p} \cdot \mathbf{x} - y \leq 0$ and then form the Lagrangian, we obtain

$$\mathcal{L}(\mathbf{x}, \lambda) \equiv u(\mathbf{x}) - \lambda\,[\mathbf{p} \cdot \mathbf{x} - y].$$

- Assuming that the solution $\mathbf{x}^*$ is strictly positive, we can apply Kuhn-Tucker methods to characterize it. If $\mathbf{x}^* \gg \mathbf{0}$ solves the consumer's problem, then by the Kuhn-Tucker Theorem, there exists a $\lambda^* \geq 0$ such that $(\mathbf{x}^*, \lambda^*)$ satisfy the following Kuhn-Tucker conditions:

- $$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i = 0 \quad i = 1, \ldots, n.$$
$$\lambda^* \geq 0 \qquad \mathbf{p} \cdot \mathbf{x}^* - y \leq 0 \qquad \lambda^* \left[\mathbf{p} \cdot \mathbf{x}^* - y\right] = 0.$$
- Now, by strict monotonicity, $\mathbf{p} \cdot \mathbf{x}^* - y \leq 0$ must be satisfied with equality, so $\lambda^*[\mathbf{p} \cdot \mathbf{x}^* - y] = 0$ becomes redundant. Consequently, these conditions reduce to
$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i = 0 \quad i = 1, \ldots, n.$$
$$\lambda^* \geq 0 \qquad \mathbf{p} \cdot \mathbf{x}^* - y = 0.$$
- There are two possibilities. Either $\nabla u(\mathbf{x}^*) = \mathbf{0}$ or $\nabla u(\mathbf{x}^*) \neq \mathbf{0}$. Under strict monotonicity, the first case is mathematically possible but never occurs in economic practice. We shall simply assume therefore that $\nabla u(\mathbf{x}^*) \neq \mathbf{0}$.

- Example: $y = f(x) \equiv x^3$

  $f'(x) = 3x^2 > 0$ if $x > 0$

  $f'(x) = 3x^2 > 0$ if $x < 0$

  $f'(x) = 3x^2 = 0$ if $x = 0$

- Thus, by strict monotonicity, $\partial u(\mathbf{x}^*)/\partial x_i > 0$ for some $i = 1, \ldots, n$.
- Because $p_i > 0$ for all i, it is clear from

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i = 0 \quad i = 1, \ldots, n$$

  that the Lagrange multiplier will be strictly positive at the solution, because $\lambda^* = u_i(\mathbf{x}^*)/p_i > 0$.
- Consequently, for all j, $\partial u(\mathbf{x}^*)/\partial x_j = \lambda^* p_j > 0$ and for any two goods j and k,

$$\frac{\partial u(\mathbf{x}^*)/\partial x_j}{\partial u(\mathbf{x}^*)/\partial x_k} = \frac{p_j}{p_k}.$$

  This says that at the optimum, the marginal rate of substitution between any two goods must be equal to the ratio of the goods' prices.

- The previous first-order conditions are merely necessary conditions for a local optimum. However, for the particular problem at hand, these necessary first-order conditions are in fact *sufficient* for a global optimum.

- **THEOREM (Sufficiency of Consumer's First-Order Conditions)**: Suppose that $u(\mathbf{x})$ is continuous and quasiconcave on $\mathbb{R}^n_+$, and that $(\mathbf{p}, y) \gg \mathbf{0}$. If u is differentiable at $\mathbf{x}^*$ and $(\mathbf{x}^*, \lambda^*) \gg \mathbf{0}$ solves the first-order necessary conditions, then $\mathbf{x}^*$ solves the consumer's maximization problem at prices $\mathbf{p}$ and income y.

- **Proof**: We shall employ the following fact that we state without proof: For all x, $\mathsf{x}^1 \geq \mathbf{0}$, because u is quasiconcave, $\nabla u(\mathsf{x})(\mathsf{x}^1 - \mathsf{x}) \geq 0$ whenever $u(\mathsf{x}^1) \geq u(\mathsf{x})$ and u is differentiable at x.

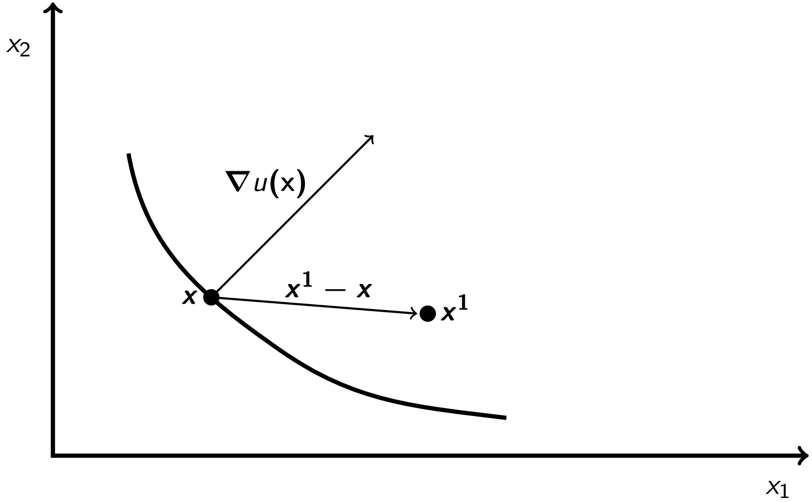

Figure: Graphical intuition for the fact stated without proof. Note that as drawn, $\nabla u(\mathbf{x}) \cdot (\mathbf{x}^1 - \mathbf{x}) = \|\nabla u(\mathbf{x})\| \, \|\mathbf{x}^1 - \mathbf{x}\| \cos\theta > 0$

- We are using the following theorems and definition to interpret the graph:
- **THEOREM**: Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^n$. Let θ be the angle between them. Then

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta.$$

- **DEFINTION**: Two vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if and only $\mathbf{x} \cdot \mathbf{y} = 0$.
- **THEOREM**: Let $u : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. At any point $\mathbf{x}$ in the domain of u at which $\nabla u(\mathbf{x}) \neq \mathbf{0}$, the gradient vector $\nabla u(\mathbf{x})$ points at $\mathbf{x}$ into the direction in which u increases most rapidly.

- Now, suppose that $\nabla u(\mathbf{x}^*)$ exists and $(\mathbf{x}^*, \lambda^*) \gg \mathbf{0}$ solves the first-order necessary conditions. Then

$$\nabla u(\mathbf{x}^*) = \lambda^* \mathbf{p} \quad \text{and} \quad \mathbf{p} \cdot \mathbf{x}^* = y.$$

If $\mathbf{x}^*$ is not utility-maximizing, then there must be some $\mathbf{x}^0 \geq \mathbf{0}$ such that

$$u(\mathbf{x}^0) > u(\mathbf{x}^*) \quad \text{and} \quad \mathbf{p} \cdot \mathbf{x}^0 \leq y.$$

Because u is continuous and $y > 0$, the preceding inequalities imply that

$$u(t\mathbf{x}^0) > u(\mathbf{x}^*) \quad \text{and} \quad \mathbf{p} \cdot t\mathbf{x}^0 < y$$

for some $t < 1$ that is close enough to one. Letting $\mathbf{x}^1 \equiv t\mathbf{x}^0$, we then have

$$\nabla u(\mathbf{x}^*)(\mathbf{x}^1 - \mathbf{x}^*) = (\lambda^* \mathbf{p}) \cdot (\mathbf{x}^1 - \mathbf{x}^*) = \lambda^* (\mathbf{p} \cdot \mathbf{x}^1 - \mathbf{p} \cdot \mathbf{x}^*) < 0$$

- However, because $u(t\mathbf{x}^0) \equiv u(\mathbf{x}^1) > u(\mathbf{x}^*)$ and u is quasiconcave, it should be that $\nabla u(\mathbf{x}^*)(\mathbf{x}^1 - \mathbf{x}^*) \geq 0$, contradicting the previously derived equation $\nabla u(\mathbf{x}^*)(\mathbf{x}^1 - \mathbf{x}^*) < 0$. Thus $\mathbf{x}^*$ must be utility-maximizing.

$\square$

An Example: CES Utility

- The function $u(x_1, x_2) \equiv (x_1^\rho + x_2^\rho)^{1/\rho}$ where $-\infty < \rho < 1$ and $\rho \neq 0$, is known as a **CES utility function**. You can easily verify that this utility function represents preferences that are strictly monotonic and strictly convex.
- In the currently most influential model of international trade due to Melitz (2003, *Econometrica*), consumers are assumed to have CES preferences, so this example is actually quite important.
- The consumer's problem is to find a nonnegative consumption bundle solving

$$\max_{x_1, x_2} (x_1^\rho + x_2^\rho)^{1/\rho} \quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 - y \leq 0.$$

- To solve this problem, we first form the associated Lagrangian function

$$\mathcal{L}(x_1, x_2, \lambda) \equiv f(x_1, x_2) - \lambda\, g(x_1, x_2)$$

$$\mathcal{L}(x_1, x_2, \lambda) \equiv (x_1^\rho + x_2^\rho)^{1/\rho} - \lambda\, (p_1 x_1 + p_2 x_2 - y).$$

- Because preferences are strictly monotonic, the budget constraint will hold with equality at the solution. Assuming an interior solution, the Kuhn-Tucker conditions coincide with the ordinary first-order Lagrangian conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{1/\rho - 1} \rho x_1^{\rho - 1} - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{1/\rho - 1} \rho x_2^{\rho - 1} - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -g(x_1, x_2) = y - p_1 x_1 - p_2 x_2 = 0$$

- $$\frac{\frac{1}{\rho}(x_1^\rho + x_2^\rho)^{1/\rho-1}\rho x_1^{\rho-1}}{\frac{1}{\rho}(x_1^\rho + x_2^\rho)^{1/\rho-1}\rho x_2^{\rho-1}} = \frac{\lambda p_1}{\lambda p_2} \quad \text{simplifies to}$$
$$\left(\frac{x_1}{x_2}\right)^{\rho-1} = \frac{p_1}{p_2}$$
- This equation corresponds to the earlier
$$\frac{\partial u(\mathbf{x}^*)/\partial x_j}{\partial u(\mathbf{x}^*)/\partial x_k} = \frac{p_j}{p_k}$$
and says that at the optimum, the marginal rate of substitution between the two goods must be equal to the ratio of the goods' prices.

- When $X = \mathbb{R}^2_+$, the (absolute value of the) slope of an indifference curve is called the **marginal rate of substitution**. This slope measures, at any point, the rate at which the consumer is willing to give up x_2 in exchange for x_1, such that he remains indifferent after the exchange.
- Let $x_2 = f(x_1)$ be the function describing an indifference curve. It follows that for all x_1, $u(x_1, f(x_1)) = \text{constant}$. Its derivative with respect to x_1 must equal zero:

$$\frac{\partial u(x_1, x_2)}{\partial x_1} + \frac{\partial u(x_1, x_2)}{\partial x_2} f'(x_1) = 0.$$

Thus, the marginal rate of substitution is

$$MRS_{12}(\mathbf{x}) \equiv -f'(x_1) = \frac{\partial u(\mathbf{x})/\partial x_1}{\partial u(\mathbf{x})/\partial x_2}.$$

- Returning to our calculations, $(x_1/x_2)^{\rho-1} = p_1/p_2$ implies that

$$\frac{x_1}{x_2} = \left(\frac{p_1}{p_2}\right)^{1/(\rho-1)} \text{ and } y = p_1 x_1 + p_2 x_2$$

  We have reduced the three equations in three unknowns to only two equations in the two unknowns of particular interest, x_1 and x_2.
- Substituting yields

$$y = p_1 x_2 \left(\frac{p_1}{p_2}\right)^{1/(\rho-1)} + p_2 x_2 = x_2 \left(p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\right) p_2^{-1/(\rho-1)}$$

- It is convenient now to define a new parameter $r \equiv \rho/(\rho - 1)$. Then $r - 1 = (\rho - (\rho - 1))/(\rho - 1) = 1/(\rho - 1)$ and

$$y = x_2 \left(p_1^r + p_2^r\right) p_2^{1-r}.$$

- Rearranging yields the ordinary or **Marshallian demand functions**:

$$x_2(\mathbf{p}, y) \equiv \frac{p_2^{r-1}\, y}{p_1^r + p_2^r}$$

$$x_1(\mathbf{p}, y) \equiv \frac{p_1^{r-1}\, y}{p_1^r + p_2^r}$$

where the latter equation comes from

$$x_1 = x_2 \left(\frac{p_1}{p_2}\right)^{1/(\rho-1)} = \frac{p_2^{r-1}\, y}{p_1^r + p_2^r} \left(\frac{p_1}{p_2}\right)^{r-1} = \frac{p_1^{r-1}\, y}{p_1^r + p_2^r}.$$

The Indirect Utility Function

- The ordinary utility function $u(\mathbf{x})$ is defined over the consumption set X and represents the consumer's preferences directly. It is therefore referred to as the **direct utility function**.
- Given prices $\mathbf{p}$ and income y, the consumer chooses a utility-maximizing bundle $\mathbf{x}(\mathbf{p}, y)$. The level of utility achieved when $\mathbf{x}(\mathbf{p}, y)$ is chosen thus will be the highest level permitted by the consumer's budget constraint facing prices $\mathbf{p}$ and income y.
- The relationship among prices, income and the maximized value of utility can be summarized by a real-valued function $v : \mathbb{R}^{n+1}_{++} \to \mathbb{R}$ defined as follows:

$$v(\mathbf{p}, y) \equiv u(\mathbf{x}(\mathbf{p}, y)) = \max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} \leq y$$

and is called the **indirect utility function**.

- **THEOREM (Properties of the Indirect Utility Function)**: If $u(\mathbf{x})$ is continuous and strictly increasing on $\mathbb{R}^n_+$, then $v(\mathbf{p}, y)$ is

  1. Continuous on $\mathbb{R}^n_{++} \times \mathbb{R}_+$,

  2. Homogeneous of degree zero in $(\mathbf{p}, y)$,

  3. Strictly increasing in y,

  4. Decreasing in $\mathbf{p}$,

  5. Quasiconvex in $(\mathbf{p}, y)$,

  6. Roy's Identity: If $v(\mathbf{p}, y)$ is differentiable at $(\mathbf{p}^0, y^0)$ and $\partial v(\mathbf{p}^0, y^0)/\partial y \neq 0$, then

$$x_i(\mathbf{p}^0, y^0) = -\frac{\partial v(\mathbf{p}^0, y^0)/\partial p_i}{\partial v(\mathbf{p}^0, y^0)/\partial y}, \quad i = 1, \ldots, n.$$

- **Proof**:

$$v(\mathbf{p}, y) \equiv u(\mathbf{x}(\mathbf{p}, y)) = \max_{\mathbf{x} \in \mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} \leq y$$

- Property 1 follows from the **Theorem of the Maximum**, which tells us (roughly) that if the objective function u and the constraint are continuous in the parameters $(\mathbf{p}, y)$, and if the domain (feasible set) is a compact set, then the maximum value function v is a continuous function of the parameters. We now state this theorem precisely.

- Consider the maximization problem

$$\max_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x},\mathbf{a}) \quad \text{s.t.} \quad g^j(\mathbf{x},\mathbf{a}) \leq 0, \quad j = 1,\ldots,m,$$

  where $\mathbf{x}$ is a vector of choice variables and $\mathbf{a}$ is a vector of parameters that may enter the objective function, the constraints or both.
- The set of parameters is a subset A of $\mathbb{R}^l$. For each $\mathbf{a} \in A$, we assume that there is at least one $\mathbf{x}$ that satisfies all the constraints.
- Suppose for a moment that for each $\mathbf{a} \in A$ there is at least one solution $\mathbf{x}(\mathbf{a})$ to the constrained maximization problem. Then the value function is well defined and given by

$$V(\mathbf{a}) \equiv \max_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x},\mathbf{a}) \quad \text{s.t.} \quad g^j(\mathbf{x},\mathbf{a}) \leq 0, \quad j = 1,\ldots,m.$$

- **DEFINITION (Constraint-Continuity)**: Say that constraint-continuity is satisfied if each $g^j : \mathbb{R}^n \times A \longrightarrow \mathbb{R}$ is continuous, and for every $(\mathbf{x}^0, \mathbf{a}^0) \in \mathbb{R}^n \times A$ satisfying the m constraints, and for every sequence $\mathbf{a}^k$ in A converging to $\mathbf{a}^0$, there is a sequence $\mathbf{x}^k$ in $\mathbb{R}^n$ converging to $\mathbf{x}^0$ such that $(\mathbf{x}^k, \mathbf{a}^k)$ satisfies the constraints for every k.

- The constraints

$$\mathbf{0} \leq \mathbf{x} \qquad \mathbf{p} \cdot \mathbf{x} \leq y$$

obviously satisfy constraint-continuity.

- **THEOREM (The Theorem of the Maximum)**: Suppose that S [the set of $(\mathbf{x}, \mathbf{a}) \in \mathbb{R}^n \times A$ that satisfy all the constraints] is compact, that $f : D \longrightarrow \mathbb{R}$ is continuous, and that constraint-continuity is satisfied. Then

  (i) A solution to the constrained maximization problem exists for every $\mathbf{a} \in A$ and therefore the value function $V(\mathbf{a})$ is defined for all of A.

  (ii) The value function $V : A \longrightarrow \mathbb{R}$ is continuous.

  (iii) If for every $\mathbf{a} \in A$, the solution to the constrained maximization problem is unique and given by the function $\mathbf{x}(\mathbf{a})$, then $\mathbf{x} : A \longrightarrow \mathbb{R}^n$ is continuous.

- To prove property 2 (homogeneous of degree zero in $(\mathbf{p}, y)$), we must show that $v(\mathbf{p}, y) = v(t\mathbf{p}, ty)$ for all $t > 0$. But $v(t\mathbf{p}, ty) = [\max u(\mathbf{x}) \text{ s.t. } t\mathbf{p} \cdot \mathbf{x} \leq ty]$, which is clearly equivalent to $v(\mathbf{p}, y) = [\max u(\mathbf{x}) \text{ s.t. } \mathbf{p} \cdot \mathbf{x} \leq y]$ because we may divide both sides of the constraint by $t > 0$ without affecting the set of bundles satisfying it.

- To prove 3, we shall make some additional assumptions although property 3 can be shown to hold without them. We assume that the solution $\mathbf{x}(\mathbf{p}, y)$ to the constrained maximization problem is strictly positive and differentiable, where $(\mathbf{p}, y) \gg \mathbf{0}$ and that u is differentiable with $\partial u(\mathbf{x})/\partial x_i > 0$ for all $\mathbf{x} \gg \mathbf{0}$.
- Then the first-order conditions are

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_i} = \frac{\partial u(\mathbf{x}^*)}{\partial x_i} - \lambda^* p_i = 0 \quad i = 1, \ldots, n.$$

Note that because both p_i and $\partial u(\mathbf{x})/\partial x_i$ are strictly positive, so too is λ^*. Our additional differentiability assumptions allow us to now apply the Envelope Theorem to establish that $v(\mathbf{p}, y)$ is strictly increasing in y.

- **THEOREM (The Envelope Theorem)**: Consider the constrained maximization problem where there is just one constraint, and suppose the objective function f and the constraint function g are continuously differentiable in $(\mathbf{x}, \mathbf{a})$ on an open subset $W \times U$ of $\mathbb{R}^n \times A$. For each $\mathbf{a} \in U$, suppose that $\mathbf{x}(\mathbf{a}) \in W$ uniquely solves the problem, is continuously differentiable in $\mathbf{a}$ on U, and that the constraint $g(\mathbf{x}, \mathbf{a}) \leq 0$ is binding for every $\mathbf{a} \in U$. Let $\mathcal{L}(\mathbf{x}, \mathbf{a}, \lambda)$ be the problem's associated Lagrangian function and let $(\mathbf{x}(\mathbf{a}), \lambda(\mathbf{a}))$ solve the Kuhn-Tucker conditions. Finally, let $V(\mathbf{a})$ be the problem's associated value function. Then, the Envelope Theorem states that for every $\mathbf{a} \in U$,
$$\frac{\partial V(\mathbf{a})}{\partial a_j} = \left.\frac{\partial \mathcal{L}}{\partial a_j}\right|_{\mathbf{x}(\mathbf{a}),\, \lambda(\mathbf{a})} \qquad j = 1, \ldots, m.$$

- Returning to our proof of property 3, the Lagrangian function is

$$\mathcal{L}(\mathbf{x}, \lambda) \equiv u(\mathbf{x}) - \lambda\,[\mathbf{p} \cdot \mathbf{x} - y],$$

  so applying the Envelope Theorem yields

$$\frac{\partial V(\mathbf{a})}{\partial a_j} = \frac{\partial v(\mathbf{p}, y)}{\partial y} = \left.\frac{\partial \mathcal{L}}{\partial a_j}\right|_{\mathbf{x}(\mathbf{a}),\, \lambda(\mathbf{a})} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial y} = \lambda^* > 0.$$

  Thus, $v(\mathbf{p}, y)$ is strictly increasing on $y > 0$. So, because v is continuous, it is then strictly increasing on $y \geq 0$.
- We have reached a surprising conclusion: in the consumer's constrained maximization problem, the Lagrangian multiplier λ has an economic interpretation. $\lambda^* > 0$ is the marginal utility of income.

- To prove property 4 (v decreasing in $\mathbf{p}$), we can also use the Envelope Theorem. The Lagrangian function is

$$\mathcal{L}(\mathbf{x}, \lambda) \equiv u(\mathbf{x}) - \lambda\,[\mathbf{p} \cdot \mathbf{x} - y],$$

so

$$\frac{\partial V(\mathbf{a})}{\partial a_j} = \frac{\partial v(\mathbf{p}, y)}{\partial p_j} = \left.\frac{\partial \mathcal{L}}{\partial a_j}\right|_{\mathbf{x}(\mathbf{a}),\,\lambda(\mathbf{a})} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial p_j} = -\lambda^* x_j^* \leq 0.$$

- Given what we have just derived, property 6 (Roy's Identity) follows immediately:

$$-\frac{\partial v(\mathbf{p}^0, y^0)/\partial p_i}{\partial v(\mathbf{p}^0, y^0)/\partial y} = -\frac{-\lambda^* x_i^*}{\lambda^*} = x_i^* = x_i(\mathbf{p}^0, y^0).$$

  Roy's identity says that the consumer's Marshallian demand for good i is simply the ratio of the partial derivatives of indirect utility with respect to p_i and y after a sign change.

- All that remains is to prove property 5 (v is quasiconvex in $(\mathbf{p}, y)$).

- **ASSUMPTION**: In the following definitions, whenever $f : D \to \mathbb{R}$ is a real-valued function, we will assume $D \subset \mathbb{R}^n$ is a convex set. When we take $\mathbf{x}^1 \in D$ and $\mathbf{x}^2 \in D$, we will let $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1-t)\mathbf{x}^2$, for $t \in [0, 1]$, denote the convex combination of $\mathbf{x}^1$ and $\mathbf{x}^2$. Because D is convex, we know that $\mathbf{x}^t \in D$.
- **DEFINITION**: $f : D \to \mathbb{R}$ is **quasiconvex** if, for all $\mathbf{x}^1$ and $\mathbf{x}^2$ in D,

$$f(\mathbf{x}^t) \leq \max\left[f(\mathbf{x}^1), f(\mathbf{x}^2)\right] \text{ for all } t \in [0, 1].$$

- **DEFINITION**: $f : D \to \mathbb{R}$ is **strictly quasiconvex** if, for all $\mathbf{x}^1 \neq \mathbf{x}^2$ in D,

$$f(\mathbf{x}^t) < \max\left[f(\mathbf{x}^1), f(\mathbf{x}^2)\right] \text{ for all } t \in (0, 1).$$

- Property 5 says that a consumer would prefer one of any two extreme budget sets to any average of the two. The key to the proof is to concentrate on the budget sets.
- Let B^1, B^2 and B^t be the budget sets available when prices and income are $(\mathbf{p}^1, y^1)$, $(\mathbf{p}^2, y^2)$ and $(\mathbf{p}^t, y^t)$, respectively, where $\mathbf{p}^t \equiv t\mathbf{p}^1 + (1-t)\mathbf{p}^2$ and $y^t \equiv ty^1 + (1-t)y^2$. Then,

$$\begin{array}{rcl} B^1 & = & \{\mathbf{x} \mid \mathbf{p}^1 \cdot \mathbf{x} \leq y^1\} \\ B^2 & = & \{\mathbf{x} \mid \mathbf{p}^2 \cdot \mathbf{x} \leq y^2\} \\ B^t & = & \{\mathbf{x} \mid \mathbf{p}^t \cdot \mathbf{x} \leq y^t\} \end{array}$$

- Suppose we could show that every choice the consumer can possibly make when she faces budget B^t is a choice that could have been made when she faced either budget B^1 or budget B^2.
- It then would be the case that every level of utility she can achieve facing B^t is a level she could have achieved either when facing B^1 or when facing B^2.
- Then, of course, the maximum level of utility that she can achieve over B^t could be no larger than at least one of the following: the maximum level of utility she can achieve over B^1, or the maximum level of utility she can achieve over B^2.
- If our supposition is correct, therefore, we would know that

$$v(\mathbf{p}^t, y^t) \leq \max\left[v(\mathbf{p}^1, y^1), v(\mathbf{p}^2, y^2)\right] \text{ for all } t \in [0,1],$$

from which it follows that v is quasiconvex in $(\mathbf{p}, y)$.

- It will suffice, then, to show that our supposition on the budget sets is correct. We want to show that if $\mathbf{x} \in B^t$, then $\mathbf{x} \in B^1$ or $\mathbf{x} \in B^2$ for all $t \in [0,1]$. The result holds trivially if $t = 1$ or $t = 0$, so it remains to show that it holds for all $t \in (0,1)$.
- Suppose it were not true. Then we could find some $t \in (0,1)$ and some $\mathbf{x} \in B^t$ such that $\mathbf{x} \notin B^1$ and $\mathbf{x} \notin B^2$. If $\mathbf{x} \notin B^1$ and $\mathbf{x} \notin B^2$, then
$$\mathbf{p}^1 \cdot \mathbf{x} > y^1$$
and
$$\mathbf{p}^2 \cdot \mathbf{x} > y^2.$$
- So $t\mathbf{p}^1 \cdot \mathbf{x} > ty^1$ and $(1-t)\mathbf{p}^2 \cdot \mathbf{x} > (1-t)y^2$.

- Adding, we obtain

$$t\mathbf{p}^1 \cdot \mathbf{x} + (1-t)\mathbf{p}^2 \cdot \mathbf{x} = (t\mathbf{p}^1 + (1-t)\mathbf{p}^2) \cdot \mathbf{x} > ty^1 + (1-t)y^2$$

or

$$\mathbf{p}^t \cdot \mathbf{x} > y^t.$$

- But this final line says that $\mathbf{x} \notin B^t$, contradicting our original assumption. We must conclude, therefore, that if $\mathbf{x} \in B^t$, then $\mathbf{x} \in B^1$ or $\mathbf{x} \in B^2$ for all $t \in [0, 1]$. By our previous argument, we can conclude that v is quasiconvex in $(\mathbf{p}, y)$.

$\square$

An Example: CES Utility

- Suppose that the direct utility function takes the CES form: $u(x_1, x_2) \equiv (x_1^\rho + x_2^\rho)^{1/\rho}$ where $-\infty < \rho < 1$ and $\rho \neq 0$.
- We showed earlier that the Marshallian demand functions are:

$$x_1(\mathbf{p}, y) \equiv \frac{p_1^{r-1}\,y}{p_1^r + p_2^r}$$

$$x_2(\mathbf{p}, y) \equiv \frac{p_2^{r-1}\,y}{p_1^r + p_2^r}$$

where $r \equiv \rho/(\rho - 1)$.

- We can now form the indirect utility function:

$$\begin{aligned}
v(\mathbf{p}, y) &= (x_1^\rho + x_2^\rho)^{1/\rho} \\
&= (x_1(\mathbf{p}, y)^\rho + x_2(\mathbf{p}, y)^\rho)^{1/\rho} \\
&= \left(\left[\frac{p_1^{r-1} y}{p_1^r + p_2^r} \right]^\rho + \left[\frac{p_2^{r-1} y}{p_1^r + p_2^r} \right]^\rho \right)^{1/\rho} \\
&= \frac{y}{p_1^r + p_2^r} \left(\left[p_1^{r-1} \right]^\rho + \left[p_2^{r-1} \right]^\rho \right)^{1/\rho} \\
&= \frac{y}{p_1^r + p_2^r} \left(p_1^r + p_2^r \right)^{1/\rho} \\
&= y \left(p_1^r + p_2^r \right)^{(1/\rho)-1} \\
&= y \left(p_1^r + p_2^r \right)^{-1/r}
\end{aligned}$$

where $r \equiv \rho/(\rho - 1)$ implies that

$$r - 1 = \frac{\rho - (\rho - 1)}{\rho - 1} = \frac{1}{\rho - 1} \quad (r-1)\rho = r \quad -\frac{1}{r} = -\frac{\rho - 1}{\rho} = \frac{1}{\rho} - 1.$$

- Now that we have derived the indirect utility function

$$v(\mathbf{p}, y) = y\left(p_1^r + p_2^r\right)^{-1/r},$$

we check that $v(\mathbf{p}, y)$ is homogeneous of degree zero in prices and income.

- For any $t > 0$,

$$\begin{aligned} v(t\mathbf{p}, ty) &= ty\left((tp_1)^r + (tp_2)^r\right)^{-1/r} \\ &= tyt^{-1}\left(p_1^r + p_2^r\right)^{-1/r} \\ &= y\left(p_1^r + p_2^r\right)^{-1/r} \\ &= v(\mathbf{p}, y). \end{aligned}$$

- To see that $v(\mathbf{p}, y) = y\left(p_1^r + p_2^r\right)^{-1/r}$ is increasing in y and decreasing in prices,

$$\frac{\partial v(\mathbf{p}, y)}{\partial y} = \left(p_1^r + p_2^r\right)^{-1/r} > 0$$

$$\begin{aligned}\frac{\partial v(\mathbf{p}, y)}{\partial p_i} &= -\frac{1}{r}\, y\left(p_1^r + p_2^r\right)^{(-1/r)-1} r p_i^{r-1} \\ &= -y\left(p_1^r + p_2^r\right)^{(-1/r)-1} p_i^{r-1} < 0\end{aligned}$$

- Finally, to verify Roy's identity,

$$-\frac{\partial v(\mathbf{p}, y)/\partial p_i}{\partial v(\mathbf{p}, y)/\partial y} = -\frac{-y\left(p_1^r + p_2^r\right)^{(-1/r)-1} p_i^{r-1}}{\left(p_1^r + p_2^r\right)^{-1/r}} = \frac{p_i^{r-1} y}{p_1^r + p_2^r} = x_i(\mathbf{p}, y)$$

The Expenditure Function

- The indirect utility function is a neat and powerful way to summarize a great deal about the consumer's market behavior. A companion measure, called the **expenditure function**, is equally useful.
- To construct the indirect utility function, we fix market prices and income, and sought the maximum level of utility the consumer could achieve.
- To construct the expenditure function, we again fix prices, but we ask a different sort of question about the level of utility the consumer achieves. Specifically, we ask: What is the minimum level of money expenditure the consumer must make facing a given set of prices in order to achieve a given level of utility?

- We define the **expenditure function** as the minimum-value function

$$e(\mathbf{p}, u) \equiv \min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u$$

  for all $\mathbf{p} \gg \mathbf{0}$ and all attainable utility levels.
- For future reference, let $\mathcal{U} \equiv \{u(\mathbf{x}) \,|\, \mathbf{x} \in \mathbb{R}^n_+\}$ denote the set of attainable utility levels. Thus, the domain of $e(\cdot)$ is $\mathbb{R}^n_{++} \times \mathcal{U}$.
- Note that $e(\mathbf{p}, u)$ is well-defined because for $\mathbf{p} \in \mathbb{R}^n_{++}$ and $\mathbf{x} \in \mathbb{R}^n_+$, $\mathbf{p} \cdot \mathbf{x} \geq \mathbf{0}$. Hence, the set of numbers

$$\{e \,|\, e = \mathbf{p} \cdot \mathbf{x} \text{ for some } \mathbf{x} \text{ with } u(\mathbf{x}) \geq u\}$$

  is bounded below by zero. Moreover because $\mathbf{p} \gg \mathbf{0}$, this set can be shown to be closed. Hence it contains a smallest number. The value $e(\mathbf{p}, u)$ is precisely this smallest number.

- Note that any solution vector for this minimization problem will be nonnegative and will depend on the parameters $\mathbf{p}$ and u. Notice also that if $u(\mathbf{x})$ is continuous and strictly quasiconcave, the solution will be unique (suppose there are two solutions, take a convex combination), so we can denote the solution as the function $\mathbf{x}^h(\mathbf{p}, u) \geq \mathbf{0}$, and it satisfies

$$e(\mathbf{p}, u) = \mathbf{p} \cdot \mathbf{x}^h(\mathbf{p}, u).$$

- The hypothetical demand functions we are describing are often called **compensated demand functions**. However, because John Hicks (1939, *Value and Capital*) was the first to write about them in quite this way, these hypothetical demand functions are most commonly known as **Hicksian demand functions**.

- The expenditure function contains within it some important information on the consumer's Hicksian demands. Although the analytic importance of this construction will only become evident a bit later, we can take note here of the remarkable ease with which that information can be extracted from a knowledge of the expenditure function.
- The consumer's Hicksian demands can be extracted from the expenditure function by means of simple differentiation.

- **THEOREM (Properties of the Expenditure Function)**: If $u(\cdot)$ is continuous and strictly increasing, then $e(\mathbf{p}, u)$ is

  1. Zero when u takes on the lowest level of utility in $\mathcal{U}$.
  2. Continuous on its domain $\mathbb{R}^n_{++} \times \mathcal{U}$.
  3. For all $\mathbf{p} \gg \mathbf{0}$, strictly increasing and unbounded above in u.
  4. Increasing in $\mathbf{p}$.
  5. Homogeneous of degree 1 in $\mathbf{p}$.
  6. Concave in $\mathbf{p}$.

  If, in addition, $u(\cdot)$ is strictly quasiconcave, we have

  7. Shephard's lemma: $e(\mathbf{p}, u)$ is differentiable in $\mathbf{p}$ and

$$\frac{\partial e(\mathbf{p}^0, u^0)}{\partial p_i} = x_i^h(\mathbf{p}^0, u^0), \quad i = 1, \ldots, n.$$

- **Proof**: To prove property 1 (e is zero when u takes on the lowest level of utility in $\mathcal{U}$), note that the lowest value in $\mathcal{U}$ is $u(\mathbf{0})$ because $u(\cdot)$ is strictly increasing on $\mathbb{R}^n_+$. Consequently, $e(\mathbf{p}, u(\mathbf{0})) = 0$ because $\mathbf{x} = \mathbf{0}$ attains utility $u(\mathbf{0})$ and requires an expenditure of $\mathbf{p} \cdot \mathbf{0} = 0$.

- Property 2 (continuity) follows once again from the **Theorem of the Maximum**.

$$e(\mathbf{p}, u) \equiv \min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u$$

$$-e(\mathbf{p}, u) \equiv \max_{\mathbf{x} \in \mathbb{R}^n_+} -\mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u - u(\mathbf{x}) \leq 0$$

- Although property 3 (for all $\mathbf{p} \gg \mathbf{0}$, e is strictly increasing and unbounded above in u) holds without any further assumptions, we shall be content to demonstrate it under the additional assumptions that $\mathbf{x}^h(\mathbf{p}, u) \gg \mathbf{0}$ is differentiable for all $\mathbf{p} \gg \mathbf{0}$, $u > u(\mathbf{0})$, and that u is differentiable with $\partial u(\mathbf{x})/\partial x_i > 0$, for all i on $\mathbb{R}^n_{++}$.

- Now, because $u(\cdot)$ is continuous and strictly increasing, and $\mathbf{p} \gg \mathbf{0}$, the constraint $u(\mathbf{x}) \geq u$ must be binding. For if $u(\mathbf{x}^1) > u$, there is a $t \in (0, 1)$ close enough to 1 such that $u(t\mathbf{x}^1) > u$. Moreover, $u \geq u(\mathbf{0})$ implies $u(\mathbf{x}^1) > u \geq u(\mathbf{0})$, so that $\mathbf{x}^1 \neq \mathbf{0}$. Therefore, $\mathbf{p} \cdot (t\mathbf{x}^1) < \mathbf{p} \cdot \mathbf{x}^1$, because $\mathbf{p} \cdot \mathbf{x}^1 > 0$. Consequently, when the constraint is not binding, there is a strictly cheaper bundle that also satisfies the constraint. Hence, at the optimum, the constraint must bind.

- Consequently, we may write the problem as

$$-e(\mathbf{p}, u) \equiv \max_{\mathbf{x} \in \mathbb{R}^n_+} -\mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u - u(\mathbf{x}) = 0$$

  and the Lagrangian for this problem is

$$\mathcal{L}(\mathbf{x}, \lambda) \equiv -\mathbf{p} \cdot \mathbf{x} - \lambda \left[u - u(\mathbf{x}) \right].$$

- Now for $\mathbf{p} \gg \mathbf{0}$ and $u > u(\mathbf{0})$, we have that $\mathbf{x}^* = \mathbf{x}^h(\mathbf{p}, u) \gg \mathbf{0}$ by the earlier assumptions. So, by the Lagrange's theorem, there is a λ^* such that

$$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_i} = -p_i + \lambda^* \frac{\partial u(\mathbf{x}^*)}{\partial x_i} = 0, \quad i = 1, \ldots, n.$$

- $$\frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial x_i} = -p_i + \lambda^* \frac{\partial u(\mathbf{x}^*)}{\partial x_i} = 0, \quad i = 1, \ldots, n.$$

  Note then that because p_i and $\partial u(\mathbf{x}^*)/\partial x_i$ are strictly positive, so too is λ^*. Under our additional hypotheses, we can now use the Envelope theorem to show that $e(\mathbf{p}, u)$ is strictly increasing in u.

- Returning to the Lagrangian function is

  $$\mathcal{L}(\mathbf{x}, \lambda) \equiv -\mathbf{p} \cdot \mathbf{x} - \lambda \left[u - u(\mathbf{x})\right]$$

  and applying the Envelope Theorem yields

  $$\frac{\partial V(\mathbf{a})}{\partial a_j} = \frac{\partial - e(\mathbf{p}, u)}{\partial u} = \left.\frac{\partial \mathcal{L}}{\partial a_j}\right|_{\mathbf{x}(\mathbf{a}), \lambda(\mathbf{a})} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial u} = -\lambda^* < 0.$$

  Because this holds for all $u > u(\mathbf{0})$ and because $e(\cdot)$ is continuous, we may conclude that for all $\mathbf{p} \gg \mathbf{0}$, $e(\mathbf{p}, u)$ is strictly increasing in u on $\mathcal{U}$ (which includes $u(\mathbf{0})$).

- That e is unbounded in u can be shown to follow from the fact that $u(\mathbf{x})$ is continuous and strictly increasing.
- To prove property 7 (Shephard's lemma), we again appeal to the Envelope theorem, but now differentiate with respect to p_i. Returning to the Lagrangian function is

$$\mathcal{L}(\mathbf{x}, \lambda) \equiv -\mathbf{p} \cdot \mathbf{x} - \lambda \left[u - u(\mathbf{x}) \right]$$

and applying the Envelope Theorem yields

$$\frac{\partial V(\mathbf{a})}{\partial a_j} = \frac{\partial - e(\mathbf{p}, u)}{\partial p_i} = \left. \frac{\partial \mathcal{L}}{\partial a_j} \right|_{\mathbf{x}(\mathbf{a}), \lambda(\mathbf{a})} = \frac{\partial \mathcal{L}(\mathbf{x}^*, \lambda^*)}{\partial p_i} = -x_i^*$$

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_i} = x_i^* = x_i^h(\mathbf{p}, u)$$

- Property 4 (e increasing in $\mathbf{p}$) follows from property 7 $[\partial e(\mathbf{p}, u)/\partial p_i = x_i^h(\mathbf{p}, u) \geq 0]$.

- Property 5 (e homogeneous of degree 1 in $\mathbf{p}$) will be left as an exercise.

- For property 6, we must prove that $e(\mathbf{p}, u)$ is a concave function of prices. We begin by recalling the definition of concavity.

- **ASSUMPTION**: In the following definitions, whenever $f : D \to \mathbb{R}$ is a real-valued function, we will assume $D \subset \mathbb{R}^n$ is a convex set. When we take $\mathbf{x}^1 \in D$ and $\mathbf{x}^2 \in D$, we will let $\mathbf{x}^t \equiv t\mathbf{x}^1 + (1-t)\mathbf{x}^2$, for $t \in [0,1]$, denote the convex combination of $\mathbf{x}^1$ and $\mathbf{x}^2$. Because D is convex, we know that $\mathbf{x}^t \in D$.
- **DEFINITION**: $f : D \to \mathbb{R}$ is **concave** if, for all $\mathbf{x}^1, \mathbf{x}^2 \in D$,

$$f(\mathbf{x}^t) \geq t\, f(\mathbf{x}^1) + (1-t)\, f(\mathbf{x}^2) \text{ for all } t \in [0,1].$$

- **DEFINITION**: $f : D \to \mathbb{R}$ is **strictly concave** if, for all $\mathbf{x}^1 \neq \mathbf{x}^2$ in D,

$$f(\mathbf{x}^t) > t\, f(\mathbf{x}^1) + (1-t)\, f(\mathbf{x}^2) \text{ for all } t \in (0,1).$$

- Let $\mathbf{p}^1$ and $\mathbf{p}^2$ be any two strictly positive price vectors $(\mathbf{p} \gg \mathbf{0}$ assumed in $e(\mathbf{p}, u))$, and let $t \in [0, 1]$, and let $\mathbf{p}^t \equiv t\mathbf{p}^1 + (1 - t)\mathbf{p}^2$ be any convex combination of $\mathbf{p}^1$ and $\mathbf{p}^2$. Then the expenditure function will be concave in prices if

$$e(\mathbf{p}^t, u) \geq t\, e(\mathbf{p}^1, u) + (1 - t)\, e(\mathbf{p}^2, u).$$

- Suppose that $\mathbf{x}^1$ minimizes expenditure to achieve u when prices are $\mathbf{p}^1$, that $\mathbf{x}^2$ minimizes expenditure to achieve u when prices are $\mathbf{p}^2$, and that $\mathbf{x}^t$ minimizes expenditure to achieve u when prices are $\mathbf{p}^t$.

- Then the cost of $\mathbf{x}^1$ at prices $\mathbf{p}^1$ must be no more than the cost at prices $\mathbf{p}^1$ of any other bundle $\mathbf{x}$ that achieves utility u:

$$\mathbf{p}^1 \cdot \mathbf{x}^t \geq \mathbf{p}^1 \cdot \mathbf{x}^1$$

$$\mathbf{p}^2 \cdot \mathbf{x}^t \geq \mathbf{p}^2 \cdot \mathbf{x}^2$$

- It follows that

$$\begin{aligned} t\,\mathbf{p}^1 \cdot \mathbf{x}^t + (1-t)\,\mathbf{p}^2 \cdot \mathbf{x}^t &\geq t\,\mathbf{p}^1 \cdot \mathbf{x}^1 + (1-t)\,\mathbf{p}^2 \cdot \mathbf{x}^2 \\ \mathbf{p}^t \cdot \mathbf{x}^t &\geq t\,\mathbf{p}^1 \cdot \mathbf{x}^1 + (1-t)\,\mathbf{p}^2 \cdot \mathbf{x}^2 \\ e(\mathbf{p}^t, u) &\geq t\,e(\mathbf{p}^1, u) + (1-t)\,e(\mathbf{p}^2, u) \end{aligned}$$

as we intended to show.

$\square$

An Example: CES Utility

- Suppose that the direct utility function takes the CES form: $u(x_1, x_2) \equiv (x_1^\rho + x_2^\rho)^{1/\rho}$ where $-\infty < \rho < 1$ and $\rho \neq 0$.
- Because preferences are strictly monotonic, we can formulate the expenditure minimization problem as

$$\max_{x_1, x_2} -(p_1 x_1 + p_2 x_2) \ \text{ s.t. } \ u - (x_1^\rho + x_2^\rho)^{1/\rho} = 0, \ x_1 \geq 0, \ x_2 \geq 0,$$

  and its Lagrangian,

$$\mathcal{L}(x_1, x_2, \lambda) = -(p_1 x_1 + p_2 x_2) - \lambda \left[u - (x_1^\rho + x_2^\rho)^{1/\rho}\right]$$

- Assuming an interior solution in both goods, the first-order conditions for a maximum subject to the constraint ensure that the solution values x_1, x_2 and λ satisfy the equations:

- $$\frac{\partial \mathcal{L}}{\partial x_1} = -p_1 + \lambda \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{(1/\rho)-1} \rho x_1^{\rho-1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = -p_2 + \lambda \frac{1}{\rho}(x_1^\rho + x_2^\rho)^{(1/\rho)-1} \rho x_2^{\rho-1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -u + (x_1^\rho + x_2^\rho)^{1/\rho} = 0$$
- By eliminating λ, these can be reduced to two equations in two unknowns:
$$\frac{p_1}{p_2} = \left(\frac{x_1}{x_2}\right)^{\rho-1}$$
$$u = (x_1^\rho + x_2^\rho)^{1/\rho}$$

- It follows that

$$(p_1/p_2)^{1/(\rho-1)} = x_1/x_2$$

$$u = (x_1^\rho + x_2^\rho)^{1/\rho}$$

$$u = \left(\left[x_2\left(\frac{p_1}{p_2}\right)^{1/(\rho-1)}\right]^\rho + x_2^\rho\right)^{1/\rho}$$

$$u = x_2\left(\left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)} + 1\right)^{1/\rho}$$

$$\left(\frac{u}{x_2}\right)^\rho = \left(\frac{p_1}{p_2}\right)^{\rho/(\rho-1)} + 1$$

$$\left(\frac{u}{x_2}\right)^\rho p_2^{\rho/(\rho-1)} = p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}$$

$$\left(\frac{x_2}{u}\right)^\rho = \frac{p_2^{\rho/(\rho-1)}}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}$$

- $$\left(\frac{x_2}{u}\right)^\rho = \frac{p_2^{\rho/(\rho-1)}}{p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}}$$

$$\frac{x_2}{u} = \frac{p_2^{1/(\rho-1)}}{\left(p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\right)^{1/\rho}}$$

$$x_2 = u\, p_2^{1/(\rho-1)} \left(p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\right)^{-1/\rho}$$

Now using $r \equiv \rho/(\rho-1)$, which implies that

$$r - 1 = \frac{\rho - (\rho-1)}{\rho - 1} = \frac{1}{\rho - 1} \qquad \frac{1}{r} = \frac{\rho - 1}{\rho} = 1 - \frac{1}{\rho}$$

$$x_2 = x_2^h(\mathbf{p}, u) = u\, p_2^{r-1} \left(p_1^r + p_2^r\right)^{(1/r)-1}$$

We have derived the Hicksian demand function for good 2.

- We can derive the corresponding Hicksian demand function for good 1 as follows:

$$(p_1/p_2)^{1/(\rho-1)} = x_1/x_2$$

$$x_1 = x_2\,(p_1/p_2)^{1/(\rho-1)}$$

$$x_1 = u\,p_2^{1/(\rho-1)}\left(p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\right)^{-1/\rho}(p_1/p_2)^{1/(\rho-1)}$$

$$x_1 = u\,p_1^{1/(\rho-1)}\left(p_1^{\rho/(\rho-1)} + p_2^{\rho/(\rho-1)}\right)^{-1/\rho}$$

$$x_1 = x_1^h(\mathbf{p}, u) = u\,p_1^{r-1}\,(p_1^r + p_2^r)^{(1/r)-1}$$

- Finally, to solve for the expenditure function

$$\begin{aligned} e(\mathbf{p}, u) &= p_1 x_1 + p_2 x_2 \\ &= p_1 x_1^h(\mathbf{p}, u) + p_2 x_2^h(\mathbf{p}, u) \\ &= p_1 u\, p_1^{r-1} \left(p_1^r + p_2^r\right)^{(1/r)-1} + p_2 u\, p_2^{r-1} \left(p_1^r + p_2^r\right)^{(1/r)-1} \\ &= u\, p_1^r \left(p_1^r + p_2^r\right)^{(1/r)-1} + u\, p_2^r \left(p_1^r + p_2^r\right)^{(1/r)-1} \\ &= u \left(p_1^r + p_2^r\right) \left(p_1^r + p_2^r\right)^{(1/r)-1} \end{aligned}$$

so the expenditure function takes a particularly simple form:

$$e(\mathbf{p}, u) = u \left(p_1^r + p_2^r\right)^{1/r}.$$

- In this expenditure function, the term $\left(p_1^r + p_2^r\right)^{1/r}$ is the CES price index. (one nice benefit of solving for the expenditure function is you find out what is the correct price index)

Relations between the Two

- Though the indirect utility function and the expenditure function are conceptually distinct, there is obviously a close relationship between them. The same can be said for the Marshallian and Hicksian demand functions. (not so obvious)

- Fix $(\mathbf{p}, y)$ and let $u = v(\mathbf{p}, y)$. By the definition of v, this says that at prices $\mathbf{p}$, utility level u is the maximum that can be attained when the consumer's income is y.
- Consequently, at prices $\mathbf{p}$, if the consumer wished to attain level of utility at least u, then income y would be certainly large enough to achieve this. But recall now that $e(\mathbf{p}, u)$ is the smallest expenditure needed to attain level of utility at least u. So $e(\mathbf{p}, u) \leq y$.
- Consequently

$$e(\mathbf{p}, v(\mathbf{p}, y)) \leq y, \text{ for all } (\mathbf{p}, y) \gg \mathbf{0}.$$

- Next, fix $(\mathbf{p}, u)$ and let $y = e(\mathbf{p}, u)$. By the definition of e, this says that at prices $\mathbf{p}$, income y is the smallest income that allows the consumer to attain at least the level of utility u.
- Consequently, at prices $\mathbf{p}$, if the consumer's income were in fact y, then he could attain at least the level of utility u. Because $v(\mathbf{p}, y)$ is the largest utility level attainable at prices $\mathbf{p}$ and income y, this implies that $v(\mathbf{p}, y) \geq u$.
- Consequently

$$v(\mathbf{p}, e(\mathbf{p}, u)) \geq u, \text{ for all } (\mathbf{p}, u) \in \mathbb{R}^n_{++} \times \mathcal{U}.$$

- **THEOREM (Relations between Indirect Utility and Expenditure Functions)**: Let $v(\mathbf{p}, y)$ and $e(\mathbf{p}, u)$ be the indirect utility function and expenditure function for some consumer whose utility function is continuous and strictly increasing. Then for all $\mathbf{p} \gg \mathbf{0}$, $y \geq 0$ and $u \in \mathcal{U}$,

  1. $e(\mathbf{p}, v(\mathbf{p}, y)) = y$.
  2. $v(\mathbf{p}, e(\mathbf{p}, u)) = u$.

- **Proof**: Because $u(\cdot)$ is strictly increasing on $\mathbb{R}^n_+$, it attains a minimum at $\mathbf{x} = \mathbf{0}$ but does not attain a maximum. Moreover, because $u(\cdot)$ is continuous, the set $\mathcal{U}$ of attainable utility numbers must be an interval. Consequently, $\mathcal{U} = [u(\mathbf{0}), \bar{u})$ for $\bar{u} > u(\mathbf{0})$, and where $\bar{u}$ may be either finite or $+\infty$.

- To prove 1, fix $(\mathbf{p}, y) \in \mathbb{R}^n_{++} \times \mathbb{R}_+$. We have already established that $e(\mathbf{p}, v(\mathbf{p}, y)) \leq y$. We would like to show that equality must hold. So suppose not, that is, suppose $e(\mathbf{p}, u) < y$ where $u = v(\mathbf{p}, y)$.

- Note that by definition of $v(\cdot)$, $u \in \mathcal{U}$, so that $u < \bar{u}$. By the continuity of $e(\cdot)$ from the expenditure properties theorem, we may therefore choose $\epsilon > 0$ small enough so that $u + \epsilon < \bar{u}$ and $e(\mathbf{p}, u + \epsilon) < y$.

- Letting $y_\epsilon \equiv e(\mathbf{p}, u + \epsilon)$, $v(\mathbf{p}, e(\mathbf{p}, u)) \geq u$ implies that $v(\mathbf{p}, y_\epsilon) \geq u + \epsilon$.

- Because $y_\epsilon \equiv e(\mathbf{p}, u + \epsilon) < y$ and v is strictly increasing in income by the indirect utility properties theorem, $v(\mathbf{p}, y) > v(\mathbf{p}, y_\epsilon) \geq u + \epsilon$. But $u = v(\mathbf{p}, y)$ so this says that $u > u + \epsilon$, a contradiction. Hence, $e(\mathbf{p}, v(\mathbf{p}, y)) = y$.

- To prove 2, fix $(\mathbf{p}, u) \in \mathbb{R}^n_{++} \times [u(\mathbf{0}), \bar{u})$. We have already established that $v(\mathbf{p}, e(\mathbf{p}, u)) \geq u$. Again, to show that this must be an equality, suppose to the contrary that $v(\mathbf{p}, e(\mathbf{p}, u)) > u$. Letting $y = e(\mathbf{p}, u)$, we then have $v(\mathbf{p}, y) > u$. There are two cases to consider: $u = u(\mathbf{0})$ and $u > u(\mathbf{0})$. We shall consider the second case only, leaving the first as an exercise.

- Now, because $e(\mathbf{p}, u(\mathbf{0})) = 0$ and because $e(\cdot)$ is strictly increasing in utility by the expenditure properties theorem, $y = e(\mathbf{p}, u) > 0$. Because $v(\cdot)$ is continuous by the indirect utility properties theorem, we may choose $\epsilon > 0$ small enough so that $y - \epsilon > 0$ and $v(\mathbf{p}, y - \epsilon) > u$.

- Thus, income $y - \epsilon$ is sufficient, at prices $\mathbf{p}$, to achieve utility greater than u. Hence, we must have $e(\mathbf{p}, u) \leq y - \epsilon$. But this contradicts the fact that $y = e(\mathbf{p}, u)$.

$\square$

- Until now, if we wanted to derive a consumer's indirect utility and expenditure functions, we would have had to solve two separate constrained optimization problems: one a maximization problem and the other a minimization problem.
- This theorem, however, points to an easy way to derive either one from knowledge of the other, thus requiring us to solve only one optimization problem and giving us the choice of which one we care to solve.

- To see how this works, suppose that we have solved the utility-maximization problem and formed the indirect utility function. One thing we know about the indirect utility function is that it is strictly increasing in its income variable. But then, holding prices constant and viewing it only as a function of income, it must be possible to invert the indirect utility function in its income variable. From before,

$$v(\mathbf{p}, e(\mathbf{p}, u)) = u$$

so we can apply the inverse function to both sides and obtain

$$e(\mathbf{p}, u) = v^{-1}(\mathbf{p}, u)$$

Thus, starting from v we obtain e. v maps y to u, v^{-1} maps u to y

- Suppose, instead, that we have solved the expenditure-minimization problem and formed the expenditure function $e(\mathbf{p}, u)$. In this case, we know that $e(\mathbf{p}, u)$ is strictly increasing in u. Again, supposing prices constant, there will be an inverse of the expenditure function in its utility variable. Applying this inverse to both sides of

$$e(\mathbf{p}, v(\mathbf{p}, y)) = y$$

  we obtain

$$v(\mathbf{p}, y) = e^{-1}(\mathbf{p}, y).$$

  Thus, starting from e we obtain v.
  e maps u to y, e^{-1} maps y to u
- Both the indirect utility function and the expenditure function are simply the appropriately chosen *inverses* of each other.

- For a CES consumer, suppose that we have solved the maximization problem for the indirect utility function,

$$v(\mathbf{p}, y) = y \left(p_1^r + p_2^r\right)^{-1/r}.$$

  For an income level equal to $e(\mathbf{p}, u)$ dollars, therefore, we must have

$$u = v(\mathbf{p}, e(\mathbf{p}, u)) = e(\mathbf{p}, u) \left(p_1^r + p_2^r\right)^{-1/r}.$$

  Therefore,

$$e(\mathbf{p}, u) = u \left(p_1^r + p_2^r\right)^{1/r}.$$

- This is the same expression for the expenditure function obtained by directly solving the consumer's expenditure-minimization problem.

- Suppose, instead, we begin with knowledge of the CES expenditure function

$$e(\mathbf{p}, u) = u\left(p_1^r + p_2^r\right)^{1/r}$$

and want to derive the indirect utility function. Then for utility level $v(\mathbf{p}, y)$, we will have

$$y = e(\mathbf{p}, v(\mathbf{p}, y)) = v(\mathbf{p}, y)\left(p_1^r + p_2^r\right)^{1/r}$$

and solving yields

$$v(\mathbf{p}, y) = y\left(p_1^r + p_2^r\right)^{-1/r}.$$

- This is the same expression for the indirect utility function obtained by directly solving the consumer's utility-maximization problem.

- **THEOREM (Relations between Marshallian and Hicksian Demand Functions)**: Given that the consumer's preference relation $\succsim$ is complete, transitive, continuous, strictly monotonic and strictly convex on $\mathbb{R}^n_+$, we have the following relations between the Hicksian and Marshallian demand functions for $\mathbf{p} \gg \mathbf{0}$, $y \geq 0$, $u \in \mathcal{U}$ and $i = 1, \ldots, n$,

  1. $x_i(\mathbf{p}, y) = x_i^h(\mathbf{p}, v(\mathbf{p}, y))$.
  2. $x_i^h(\mathbf{p}, u) = x_i(\mathbf{p}, e(\mathbf{p}, u))$.

- **Proof**: We will complete the proof of the first leaving the second as an exercise.
- Note that since $u(\cdot)$ is continuous and strictly quasiconcave, the solutions to both constrained optimization problems exist and are unique. Consequently, the Marshallian and Hicksian demand functions are well-defined.

- To prove the first relation, let $\mathbf{x}^0 = \mathbf{x}(\mathbf{p}^0, y^0)$ and let $u^0 = u(\mathbf{x}^0)$. Then $v(\mathbf{p}^0, y^0) = u^0$ by definition of $v(\cdot)$. Also $\mathbf{p}^0 \cdot \mathbf{x}^0 = y^0$ because $u(\cdot)$ is strictly increasing.
- By the previous result $e(\mathbf{p}, v(\mathbf{p}, y)) = y$, it follows that $e(\mathbf{p}^0, v(\mathbf{p}^0, y^0)) = y^0$, or equivalently, $e(\mathbf{p}^0, u^0) = y^0$.
- But because $u(\mathbf{x}^0) = u^0$ and $\mathbf{p}^0 \cdot \mathbf{x}^0 = y^0$, $e(\mathbf{p}^0, u^0) = y^0$ implies that $\mathbf{x}^0$ solves the expenditure minimization problem when $(\mathbf{p}, u) = (\mathbf{p}^0, u^0)$.
- Hence $\mathbf{x}^0 = \mathbf{x}^h(\mathbf{p}^0, u^0)$ and so $\mathbf{x}(\mathbf{p}^0, y^0) = \mathbf{x}^h(\mathbf{p}^0, v(\mathbf{p}^0, y^0))$.

$\square$

- Let us confirm this theorem for a CES consumer. From before, the Hicksian demand functions are

$$x_1^h(\mathbf{p}, u) = u\, p_1^{r-1} \left(p_1^r + p_2^r\right)^{(1/r)-1}$$

$$x_2^h(\mathbf{p}, u) = u\, p_2^{r-1} \left(p_1^r + p_2^r\right)^{(1/r)-1}$$

so

$$x_i^h(\mathbf{p}, u) = u\, p_i^{r-1} \left(p_1^r + p_2^r\right)^{(1/r)-1}, \quad i = 1, 2$$

and the indirect utility function is

$$v(\mathbf{p}, y) = y \left(p_1^r + p_2^r\right)^{-1/r}.$$

- Therefore

$$x_i^h(\mathbf{p}, v(\mathbf{p}, y)) = v(\mathbf{p}, y)\, p_i^{r-1} \left(p_1^r + p_2^r\right)^{(1/r)-1}$$

- $$\begin{aligned} x_i^h(\mathbf{p}, v(\mathbf{p}, y)) &= v(\mathbf{p}, y)\, p_i^{r-1}\, (p_1^r + p_2^r)^{(1/r)-1} \\ &= y\, (p_1^r + p_2^r)^{-1/r}\ p_i^{r-1}\, (p_1^r + p_2^r)^{(1/r)-1} \\ &= y\, p_i^{r-1}\, (p_1^r + p_2^r)^{-1}, \end{aligned}$$

  which is the same as the Marshallian demand functions that we solved for earlier:

  $$x_1(\mathbf{p}, y) \equiv \frac{p_1^{r-1}\, y}{p_1^r + p_2^r}$$

  $$x_2(\mathbf{p}, y) \equiv \frac{p_2^{r-1}\, y}{p_1^r + p_2^r}$$

- To confirm the second part of the theorem, suppose we know the Marshallian demands

$$x_i(\mathbf{p}, y) \equiv \frac{p_i^{r-1}\, y}{p_1^r + p_2^r} \quad i = 1, 2,$$

and the expenditure function

$$e(\mathbf{p}, u) = u\,(p_1^r + p_2^r)^{1/r}\,.$$

- Substituting yields the Hicksian demands

$$x_i(\mathbf{p}, e(\mathbf{p}, u)) \equiv \frac{p_i^{r-1}\, u\,(p_1^r + p_2^r)^{1/r}}{p_1^r + p_2^r}$$

$$= u\, p_i^{r-1}\,(p_1^r + p_2^r)^{(1/r)-1} = x_i^h(\mathbf{p}, u), \quad i = 1, 2$$

Properties of Consumer Demand

- **THEOREM (Homogeneity and Budget Balancedness)**: Given that the consumer's preference relation $\succsim$ is complete, transitive, continuous, strictly monotonic and strictly convex on $\mathbb{R}^n_+$, the consumer demand function $x_i(\mathbf{p}, y), i = 1, \ldots, n$, is homogeneous of degree zero in all prices and income, and it satisfies budget balancedness, $\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, y) = y$ for all $(\mathbf{p}, y)$.

- **Proof**: We have already shown that the indirect utility function is homogeneous of degree zero, so that

$$v(\mathbf{p}, y) = v(t\mathbf{p}, ty) \text{ for all } t > 0.$$

This is equivalent to the statement

$$u(\mathbf{x}(\mathbf{p}, y)) = u(\mathbf{x}(t\mathbf{p}, ty)) \text{ for all } t > 0.$$

- Now, because the budget sets at $(\mathbf{p}, y)$ and $(t\mathbf{p}, ty)$ are the same, each of $\mathbf{x}(\mathbf{p}, y)$ and $\mathbf{x}(t\mathbf{p}, ty)$ was feasible when the other was chosen. Hence, the previous equality and the strict quasiconcavity of u implies that

$$\mathbf{x}(\mathbf{p}, y) = \mathbf{x}(t\mathbf{p}, ty) \text{ for all } t > 0,$$

so demand is homogeneous of degree zero in prices and income.

- Because $u(\cdot)$ is strictly increasing, $\mathbf{x}(\mathbf{p}, y)$ must exhaust the consumer's income. Otherwise, she could afford to purchase strictly more of every good and strictly increase her utility. So budget balancedness holds: $\mathbf{p} \cdot \mathbf{x}(\mathbf{p}, y) = y$ for all $(\mathbf{p}, y)$.

$\square$

- An important question in our model of consumer behavior concerns the response we should expect in quantity demanded when price changes. Ordinarily, we tend to think a consumer will buy more of a good when its price declines, and less when its price increases, other things being equal.
- That this need not always be the case is illustrated next.

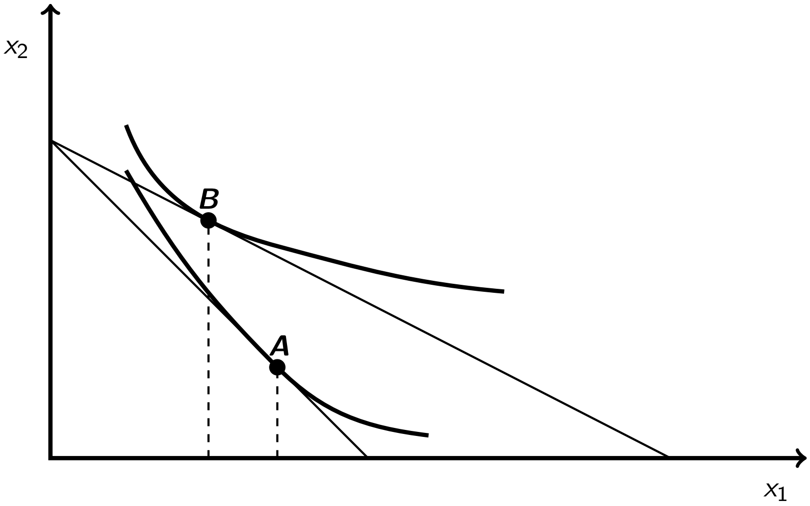

Figure: Response of quantity demanded to a decrease in price (Giffen good case)

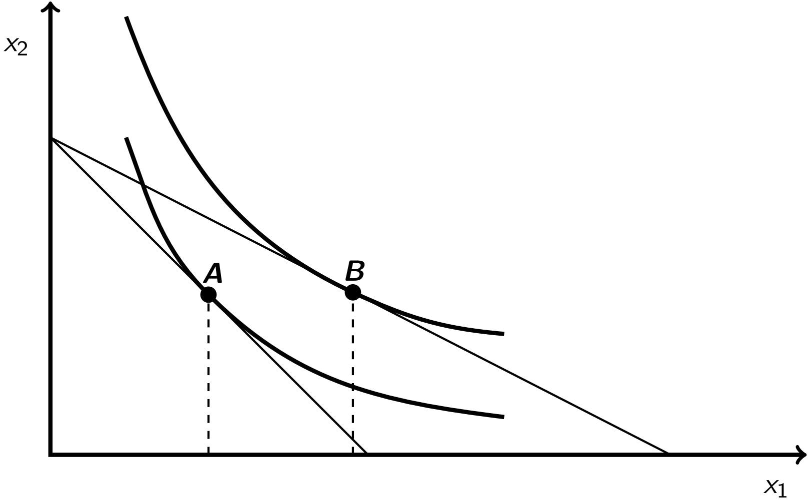

Figure: The downward sloping demand case

- When the price of a good declines, there are two conceptually separate reasons why we expect some change in the quantity demanded.
- First, that good becomes relatively cheaper compared to other goods. Because all goods are desirable, even if the consumer's total command over goods were unchanged, we would expect her to substitute more of the relatively cheaper good for less of the now relatively more expensive ones. This is called the **substitution effect**.
- At the same time, however, whenever a price of a good declines, the consumer's total command over all goods is effectively increased, allowing her to change her purchases of all goods in any way she sees fit. The effect on quantity demanded of this generalized increase in purchasing power is called the **income effect**.

- We follow the way of calculating substitution and income effects proposed by Hicks (1939).
- The substitution effect is that (hypothetical) change in consumption that would occur if relative prices were to change to their new levels but the maximum utility the consumer can achieve were kept the same as before the price change.
- The income effect is then defined as whatever is left of the total effect after the substitution effect.

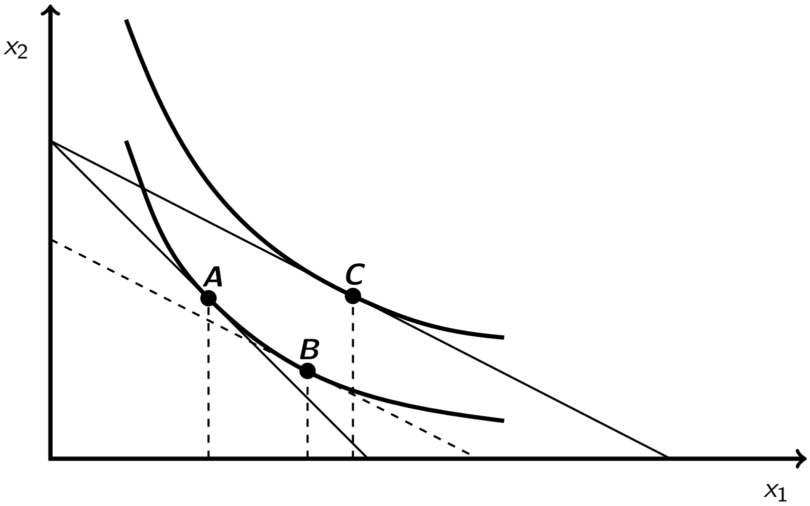

Figure: The Hicksian decomposition of a price change

- The Hicksian decomposition gives us a neat analytical way to isolate the two distinct forces working to change demand behavior following a price change. We can take these same ideas and express them much more precisely, much more generally, and in a form that will prove more analytically useful.
- The relationship between total effect (TE), substitution effect (SE) and income effect (IE) are summarized in the *Slutsky equation*. The Slutsky equation is sometimes called the "Fundamental Equation of Demand Theory," so what follows merits thinking about rather carefully.

- **THEOREM (The Slutsky Equation)**: Assume that the consumer's preference relation $\succsim$ is complete, transitive, continuous, strictly monotonic and strictly convex on $\mathbb{R}^n_+$, and assume that we can freely differentiate whenever necessary. Let $\mathbf{x}(\mathbf{p}, y)$ be the consumer's Marshallian demand system. Let u^* be the level of utility the consumer achieves at prices $\mathbf{p}$ and income y. Then

$$\underbrace{\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j}}_{TE} = \underbrace{\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j}}_{SE} \underbrace{-x_j(\mathbf{p}, y)\frac{\partial x_i(\mathbf{p}, y)}{\partial y}}_{IE}, \quad i, j = 1, \ldots, n.$$

- **Proof**: The proof of this remarkable theorem is quite easy, though you must follow it quite carefully to avoid getting lost.
- From one of the earlier theorems, we know that

$$x_i^h(\mathbf{p}, u^*) = x_i(\mathbf{p}, e(\mathbf{p}, u^*))$$

for any prices and level of utility u^*. Because this holds for all $\mathbf{p} \gg \mathbf{0}$, we can differentiate both sides with respect to p_j and the equality is preserved.
- Using the chain rule $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$, we obtain

$$\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j} = \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial p_j} + \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial y} \frac{\partial e(\mathbf{p}, u^*)}{\partial p_j}.$$

- $$\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j} = \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial p_j} + \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial y}\frac{\partial e(\mathbf{p}, u^*)}{\partial p_j}$$
- Next, note that $u^* = v(\mathbf{p}, y)$. Therefore, an earlier theorem implies that
$$e(\mathbf{p}, u^*) = e(\mathbf{p}, v(\mathbf{p}, y)) = y.$$
Then Shephard's lemma implies that
$$\frac{\partial e(\mathbf{p}, u^*)}{\partial p_j} = x_j^h(\mathbf{p}, u^*) = x_j^h(\mathbf{p}, v(\mathbf{p}, y)) = x_j(\mathbf{p}, y),$$
where the last equality follows from the theorem about the relations between Marshallian and Hicksian demand functions.

- Substituting into

$$\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j} = \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial p_j} + \frac{\partial x_i(\mathbf{p}, e(\mathbf{p}, u^*))}{\partial y} \frac{\partial e(\mathbf{p}, u^*)}{\partial p_j},$$

we obtain

$$\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j} = \frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} + \frac{\partial x_i(\mathbf{p}, y)}{\partial y} x_j(\mathbf{p}, y),$$

and rearranging terms, we have what we wanted to show:

$$\frac{\partial x_i(\mathbf{p}, y)}{\partial p_j} = \frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_j} - x_j(\mathbf{p}, y) \frac{\partial x_i(\mathbf{p}, y)}{\partial y}, \quad i, j = 1, \ldots, n.$$

□

- **THEOREM (Negative Own-Substitution Terms)**: Assume that the consumer's preference relation $\succsim$ is complete, transitive, continuous, strictly monotonic and strictly convex on $\mathbb{R}^n_+$, and assume that we can freely differentiate whenever necessary. Let $x_i^h(\mathbf{p}, u)$ be the Hicksian demand for good i. Then

$$\frac{\partial x_i^h(\mathbf{p}, u)}{\partial p_i} \leq 0, \quad i = 1, \ldots, n.$$

- **Proof**: Shephard's lemma tells us that for any $\mathbf{p}$ and u,

$$\frac{\partial e(\mathbf{p}, u)}{\partial p_i} = x_i^h(\mathbf{p}, u).$$

Differentiating again with respect to p_i shows that

$$\frac{\partial^2 e(\mathbf{p}, u)}{\partial p_i^2} = \frac{\partial x_i^h(\mathbf{p}, u)}{\partial p_i}, \quad i = 1, \ldots, n.$$

Having shown that the expenditure function is a concave function of $\mathbf{p}$, therefore all of its second-order own partial derivatives are nonpositive, proving the theorem.

- **THEOREM (Concavity, Convexity and Second-Order Own Partial Derivatives)**: Let $f : D \to \mathbb{R}$ be a twice differentiable function.

  1. If f is concave, then for all $\mathbf{x}$, $f_{ii}(\mathbf{x}) \leq 0, \quad i = 1, \dots, n$.

  1. If f is convex, then for all $\mathbf{x}$, $f_{ii}(\mathbf{x}) \geq 0, \quad i = 1, \dots, n$.

$\square$

- A good is called **normal** if consumption of it increases as income increases, holding prices constant. A good is called **inferior** if consumption of it declines as income increases, holding prices constant.
- **THEOREM (The Law of Demand)**: A decrease in the own price of a normal good will cause quantity demanded to increase. If an own price decrease causes a decrease in quantity demanded, the good must be inferior.

- **Proof**: For a normal good, look at the Slutsky equation:

$$\frac{\partial x_i(\mathbf{p}, y)}{\partial p_i} = \underbrace{\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_i}}_{\leq 0} - \underbrace{x_i(\mathbf{p}, y)}_{\geq 0} \underbrace{\frac{\partial x_i(\mathbf{p}, y)}{\partial y}}_{>0} \leq 0.$$

To see that a Giffen good must be inferior:

$$\frac{\partial x_i(\mathbf{p}, y)}{\partial p_i} = \underbrace{\frac{\partial x_i^h(\mathbf{p}, u^*)}{\partial p_i}}_{\leq 0} - \underbrace{x_i(\mathbf{p}, y)}_{\geq 0} \underbrace{\frac{\partial x_i(\mathbf{p}, y)}{\partial y}}_{<0} > 0.$$

□

History of Economic Ideas

- The argument for the existence of a utility function is based on Herman Wold (1943). A general theorem on the existence of a utility function can be found in Gerard Debreu (1964).
- The importance of the indirect utility function was first recognized by Rene Roy (1942).
- The expenditure function seems to be due to John Hicks (1939, *Value and Capital*). Hicks also introduced the distinction between substitution and income effects of price changes in this book.
- The proof of the Slutsky equation follows Lionel McKenzie (1957).