Advanced microeconomics problem set 2

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1 Exercise 1.39

1.1 Original question

Complete the proof of Theorem 1.9 by showing that $\mathbf{x}^h(\mathbf{p}, u) = \mathbf{x}(\mathbf{p}, e(\mathbf{p}, u))$.

1.2 Solution

For your convenience, I am restating Assumption 1.2 (p. 19) here: The consumer's preference relation $\succsim$ is complete, transitive, continuous, strictly monotonic, and strictly convex on $\mathbb{R}^n_+$. Therefore, by Theorems 1.1 and 1.3 it can be represented by a real-valued utility function u, that is continuous, strictly increasing, and strictly quasiconcave on $\mathbb{R}^n_+$.

So the utility maximization problem

$$v(\mathbf{p}, y) = \max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p}\cdot\mathbf{x} \leq y \tag{1}$$

has a solution and the solution is unique,
Also, the expenditure minimization problem

$$e(\mathbf{p}, u) = \min_{\mathbf{x}\in\mathbb{R}^n_+} \mathbf{p}\cdot\mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u \tag{2}$$

has a solution and the solution is unique.
Consequently, the Marshallian and Hicksian demand functions are well-defined.

At any given utility level u^0 and price vector $\mathbf{p}$, denote $\mathbf{x}^{h0} = x^h(\mathbf{p}, u^0)$ as the corresponding Hicksian demand, and $y^0 = e(\mathbf{p}, u^0)$ as the corresponding minimized cost. From the definition of $e(\mathbf{p}, u^0)$, we know that $u(\mathbf{x}^{h0}) = u^0$.
By theorem 1.8, $v(\mathbf{p}, e(\mathbf{p}, u^0)) = u^0$, or equivalently, $v(\mathbf{p}, y^0) = u^0$. So by definition of $v(\mathbf{p}, y^0)$ we can expect the Marshallian demand $x^0 = x(\mathbf{p}, y)$ satisfies $u(x^0) = u^0$. However, we also have shown $u(x^{h0}) = u^0$. So x^{h0}, solution to expenditure minimization problem, is also the solution to utility maximization problem.
We just argued that both expenditure minimization and utility maximization problems have unique solutions, so it can only be the case that $x^{h0} = x^0$, i.e. $x^h(\mathbf{p}, u^0) = x(\mathbf{p}, e(\mathbf{p}, u^0))$. Since it holds for any $\mathbf{p}$ and u^0, we conclude that $\mathbf{x}^h(\mathbf{p}, u) = \mathbf{x}(\mathbf{p}, e(\mathbf{p}, u))$.

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2 Exercise 1.51

2.1 Original question

Consider the utility function $u(\mathbf{x}) = (x_1)^{\frac{1}{2}} + (x_2)^{\frac{1}{2}}$

a) Compute the Marshallian demand functions.

b) Compute the substitution term in the Slutsky equation for the effects of x_1 of changes in p_2.

c) Classify x_1 and x_2 as (gross) complements or substitutes.

2.2 Solution

(a) The Marshallian demand function is given by

$$\mathbf{x}(\mathbf{p}, y) = \arg\max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p}\cdot\mathbf{x} \leq y \tag{3}$$

Let λ be a non-negative multiplier associated with the budget constraint. We form the Lagrangian

$$\mathcal{L} = u(\mathbf{x}) + \lambda(y - \mathbf{p}\cdot\mathbf{x}) \tag{4}$$

If utility is strictly increasing, then an interior solution $\mathbf{x}$ satisfies the first-order conditions

$$\frac{\partial\mathcal{L}}{\partial x_i} = u_i(\mathbf{x}) - \lambda p_i = 0 \quad \forall i \in \mathcal{I} \tag{5}$$

$$\frac{\partial\mathcal{L}}{\partial x_i} = y - \mathbf{p}\cdot\mathbf{x} = 0 \tag{6}$$

It follows that an interior optimum $\mathbf{x}$ is characterized by the two conditions

$$\frac{u_i(\mathbf{x})}{u_j(\mathbf{x})} = \frac{p_i}{p_j} \tag{7}$$

$$\mathbf{p}\cdot\mathbf{x} = y \tag{8}$$

Now assume that $n = 2$ and therefore the index set is $\mathcal{I} = \{1, 2\}$. If the utility function takes the form $u(\mathbf{x}) = (x_1)^{\frac{1}{2}} + (x_2)^{\frac{1}{2}}$, its partial derivatives are given by

$$u_i(\mathbf{x}) = \frac{1}{2}x_i^{-\frac{1}{2}} \tag{9}$$

Hence, if we plug this into equation (7), and cancel terms, we get that a solution to the consumer's problem with $n = 2$ and utility as specified above solves the system of equations

$$\left(\frac{x_2}{x_1}\right)^{\frac{1}{2}} = \frac{p_i}{p_j} \tag{10}$$

$$p_1x_1 + p_2x_2 = y \tag{11}$$

The solution to this system of equations gives us the Marshallian demand functions

$$x_1(\mathbf{p}, y) = \frac{p_2 y}{p_1(p_1+p_2)} \qquad x_2(\mathbf{p}, y) = \frac{p_1 y}{p_2(p_1+p_2)} \tag{12}$$

And a indirect utility function $v(\mathbf{p}, y) = [\frac{p_2 y}{p_1(p_1+p_2)}]^{\frac{1}{2}} + [\frac{p_1 y}{p_2(p_1+p_2)}]^{\frac{1}{2}}$ (which I will not compute here).

(b) The Slutsky equation for the effects on x_1 of changes in p_2 is given by

$$\frac{\partial x_1(\mathbf{p},y)}{\partial p_2} = \frac{\partial x_1^h(\mathbf{p},u^\star)}{\partial p_2} - x_2(\mathbf{p},y)\frac{\partial x_1(\mathbf{p},y)}{\partial y}, \tag{13}$$

where $u^* = v(\mathbf{p},y)$.
Since we have computed the Marshallian demand function in the previous exercise, the easiest way to get at the substitution effect is to rearrange the previous equation

$$\frac{\partial x_1^h(\mathbf{p},u^\star)}{\partial p_2} = \frac{\partial x_1(\mathbf{p},y)}{\partial p_2} + x_2(\mathbf{p},y)\frac{\partial x_1(\mathbf{p},y)}{\partial y} \tag{14}$$

and then plug in our solutions for the Marshallian demand.
First, observe that

$$\frac{\partial x_1(\mathbf{p},y)}{\partial p_2} = \frac{y}{p_1(p_1+p_2)} - \frac{p_2 y}{p_1(p_1+p_2)^2} = \frac{y}{(p_1+p_2)^2} \tag{15}$$

The second term on the right-hand side of (13) features the derivative of the Marshallian demand with respect to income y, which can be expressed explicitly as

$$\frac{\partial x_1(\mathbf{p},y)}{\partial y} = \frac{p_2}{p_1(p_1+p_2)} \tag{16}$$

Multiplying by $x_2(\mathbf{p},y)$ and cancelling terms gives

$$x_2(\mathbf{p},y)\frac{\partial x_1(\mathbf{p},y)}{\partial y} = \frac{y}{(p_1+p_2)^2} \tag{17}$$

Equations (15) and (17) imply that the left-hand side of (14) satisfies

$$\frac{\partial x_1^h(\mathbf{p},u^\star)}{\partial p_2} = \frac{2y}{(p_1+p_2)^2} > 0 \quad \text{where} \quad u^* = v(\mathbf{p},y) \tag{18}$$

The substitution effect is positive, so an increase in p_2 raises x_1^h.

(c) We are asked to classify x_1 and x_2 as either (gross) compliments or substitutes. We will apply the following criterion

(a) If $\partial x_1/\partial p_2 > 0$ and $\partial x_2/\partial p_1 > 0$, then the two goods are *gross substitutes*.
(b) If $\partial x_1/\partial p_2 < 0$ and $\partial x_2/\partial p_1 < 0$, then the two goods are *gross complements*.

We have already computed $\partial x_1/\partial p_2$ in equation (15). This expression is clearly positive for all $y > 0$. Because of the symmetry of the problem, you can easily verify that $\partial x_2/\partial p_1 = \partial x_1/\partial p_2$, so the total effect on x_2 of changes in p_1 is also strictly positive. We conclude that the two goods are gross substitutes according to our definition above.

3 Exercise 1.54

3.1 Original question

The n-good Cobb-Douglas utility function is

$$u(\mathbf{x}) = A\prod_{i=1}^{n} x_i^{\alpha_i}, \tag{19}$$

where $A > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$.

a) Derive the Marshallian demand functions.

b) Derive the indirect utility function.

c) Compute the expenditure function.

d) Compute the Hicksian demands.

3.2 Solution

(a) Under the restrictions on α, the function $u(\cdot)$ is continuous and strictly quasiconcave, so the consumer's maximization problem has a solution and it is unique. Hence, the Marshallian demand function is well-defined and given by

$$\mathbf{x}(\mathbf{p}, y) = \arg\max_{\mathbf{x}\in\mathbb{R}^n_+} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p}\cdot\mathbf{x} \leq y \tag{20}$$

with $u(\mathbf{x})$ being defined as in (19). Let λ be a non-negative multiplier associated with the budget constraint. We form the Lagrangian

$$\mathcal{L} = u(\mathbf{x}) + \lambda(y - \mathbf{p}\cdot\mathbf{x}) \tag{21}$$

Since $u(\cdot)$ is strictly increasing and $\mathbf{p} \gg 0$ by assumption, the budget constraint must bind and an interior solution satisfies the first-order conditions

$$\frac{\partial\mathcal{L}}{\partial x_i} = u_i(\mathbf{x}) - \lambda p_i = 0 \quad \forall i \in \mathcal{I} \tag{22}$$

$$\frac{\partial\mathcal{L}}{\partial\lambda} = y - \mathbf{p}\cdot\mathbf{x} = 0 \tag{23}$$

Equation (22) implies that in optimum

$$\frac{u_i(\mathbf{x})}{u_j(\mathbf{x})} = \frac{p_i}{p_j} \quad \forall i, j \in \mathcal{I} \tag{24}$$

Now observe that for the utility functional assumed in (19), the partial derivative with respect to good i satisfies

$$u_i(\mathbf{x}) = \alpha_i x_i^{\alpha_i - 1} A \left(\prod_{j=1, j\neq i}^{n} x_j^{\alpha_j}\right) = \alpha_i x_i^{-1} A \left(\prod_{j=1}^{n} x_j^{\alpha_j}\right) \tag{25}$$

which can be written as

$$u_i(\mathbf{x}) = \alpha_i x_i^{-1} u(\mathbf{x}) \tag{26}$$

Plugging this into equation (24) and rearranging gives

$$p_j x_j = \left(\frac{\alpha_j}{\alpha_i}\right) p_i x_i \tag{27}$$

Now observe that in optimum total expenditure $\mathbf{p} \cdot \mathbf{x}$ satisfies

$$\begin{aligned}\mathbf{p} \cdot \mathbf{x} &= \sum_{j=1}^{n} p_j x_j \\ &= \sum_{j=1}^{n} [\left(\frac{\alpha_j}{\alpha_i}\right) p_i x_i] \\ &= \frac{p_i x_i}{\alpha_i} \left(\sum_{j=1}^{n} \alpha_j\right) \\ &= \frac{p_i x_i}{\alpha_i}\end{aligned}$$

where the first equality is simply the definition of the inner product, the second equality uses equation (27) which we have just derived, and the third equality uses the fact that we are summing over j, so all variables indexed by i are constants. The fourth and last equality follows from the parametric assumption imposed in (19). Using the expression above to substitute for $\mathbf{p} \cdot \mathbf{x}$ in the budget constraint in (23) and rearranging gives the Marshallian demand for good i

$$x_i(\mathbf{p}, y) = \frac{\alpha_i y}{p_i} \quad \forall i \in \mathcal{I} \tag{28}$$

(b) The indirect utility function is the solution to the consumer's maximization problem

$$v(\mathbf{p}, y) = \max_{\mathbf{x} \in \mathbb{R}_+^n} u(\mathbf{x}) \quad \text{s.t.} \quad \mathbf{p} \cdot \mathbf{x} \leq y \tag{29}$$

Since the Marshallian demand function for this problem is well-defined, the indirect utility function satisfies

$$\begin{aligned}v(\mathbf{p}, y) &= u(\mathbf{x}(\mathbf{p}, y)) & (30)\\ &= A \prod_{i=1}^{n} \left(\frac{\alpha_i y}{p_i}\right)^{\alpha_i} & (31)\\ &= A \left(\prod_{i=1}^{n} y^{\alpha_i}\right) \left(\prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}\right) & (32)\\ &= A y^{\sum_{i=1}^{n} \alpha_i} \left(\prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}\right) & (33)\\ &= A y \left(\prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i}\right)^{\alpha_i}\right) & (34)\end{aligned}$$

The second equality uses result for the Marshallian demand function in (28). The fifth equality again uses the restriction that α-coefficients sum up to one.

(c) In order to obtain the expenditure function, we can solve the cost-minimization problem. This will require us to compute the Hicksian demand functions, so we are also solving for d) in this way. Later, I will present a short-cut that utilizes our prior results afterwards.
The expenditure function is the solution to the cost-minimization problem

$$e(\mathbf{p}, u) = \min_{\mathbf{x} \in \mathbb{R}_+^n} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq u \tag{35}$$

Alternatively, we could also write the constraint as $u - u(\mathbf{x}) \leq 0$.
Now note that if it is our objective to minimize $\mathbf{p} \cdot \mathbf{x}$, then this is equivalent to maximizing $-(\mathbf{p} \cdot \mathbf{x})$. So instead of solving (35) for $e(\mathbf{p}, u)$, we could also solve for $-e(\mathbf{p}, u)$ defined by the maximization problem

$$-e(\mathbf{p}, u) = \max_{\mathbf{x} \in \mathbb{R}^n_+} -(\mathbf{p} \cdot \mathbf{x}) \quad \text{s.t.} \quad u - u(\mathbf{x}) \leq 0 \tag{36}$$

Because $u(\cdot)$ is continuous and strictly increasing, and further $\mathbf{p} \gg 0$, the constraint must be binding. Let λ be a non-negative multiplier associated with the constraint. We form the Lagrangian

$$\mathcal{L} = -(\mathbf{p} \cdot \mathbf{x}) + \lambda[u(\mathbf{x}) - u] \tag{37}$$

An interior solution satisfies the first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x_i} = -p_i + \lambda u_i(\mathbf{x}) = 0 \quad \forall i \in \mathcal{I} \tag{38}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = u(\mathbf{x}) - u = 0 \tag{39}$$

Because the constraint is binding, the multiplier is strictly positive $\lambda > 0$, and (38) implies the familiar condition for an interior optimum

$$\frac{u_i(\mathbf{x})}{u_j(\mathbf{x})} = \frac{p_i}{p_j} \tag{40}$$

You can see this condition is exactly the same as in the utility maximization problem we studied earlier. By our expressions for the partial derivative of u with respect to x_i given in (26), cancelling terms, and rearranging we get that in optimum

$$x_j = \left(\frac{\alpha_j}{\alpha_i}\right)\left(\frac{p_i}{p_j}\right) x_i \tag{41}$$

Evaluating the utility function $u(\cdot)$ at this optimal bundle, one obtains

$$\begin{aligned}
u(\mathbf{x}) &= A \prod_{j=1}^{n} \left(\left(\frac{\alpha_j}{\alpha_i}\right)\left(\frac{p_i}{p_j}\right) x_i \right)^{\alpha_j} \\
&= A \left(\prod_{j=1}^{n} \left(\frac{p_i x_i}{\alpha_i}\right)^{\alpha_j} \right) \left(\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j} \right) \\
&= A \left(\frac{p_i x_i}{\alpha_i}\right)^{\sum_{j=1}^{n} \alpha_j} \left(\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j} \right) \\
&= A \left(\frac{p_i x_i}{\alpha_i}\right) \left(\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j} \right)
\end{aligned}$$

Using the last identity to substitute for $u(\mathbf{x})$ in (39) and solve for x_i as a function of prices, parameters and the utility level u. This function is the Hicksian demand for good i which we label as x_i^h and it is given by

$$x_i^h = \frac{\alpha_i u}{p_i A \left(\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j} \right)} \tag{42}$$

The expenditure function $e(\mathbf{p}, u)$ is simply the total expenditure if the bundle is chosen according to the Hicksian demand function. Using our previous results and some algebra yields

$$\begin{aligned}
e(\mathbf{p}, u) &= \mathbf{p} \cdot \mathbf{x}^h \\
&= \sum_{i=1}^{n} p_i x_i^h \\
&= \sum_{i=1}^{n} p_i \left(\frac{\alpha_i u}{p_i A \left(\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j} \right)} \right) \\
&= \sum_{i=1}^{n} p_i \left(\frac{\alpha_i u}{p_i A \left(\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j} \right)} \right) \\
&= \frac{(\sum_{i=1}^{n} \alpha_i) u}{A \left(\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j} \right)} \\
&= \frac{u}{A \left(\prod_{j=1}^{n} \left(\frac{\alpha_j}{p_j} \right)^{\alpha_j} \right)}
\end{aligned}$$

An alternative way: I will now present an alternative way of solving problem c) that builds on our results in b) and does not require us to compute the Hicksian demand functions. Theorem 1.8 tells us that under the assumption of continuity and strict monotonicity of $u(\cdot)$, which are both satisfied in the n-good Cobb-Douglas case, the following identity holds

$$v(\mathbf{p}, e(\mathbf{p}, u)) = u \tag{43}$$

For the indirect utility function we derived in (34), one obtains

$$A e(\mathbf{p}, u) \left(\prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i} \right) = u \tag{44}$$

which in turn implies

$$e(\mathbf{p}, u) = \frac{u}{A \left(\prod_{i=1}^{n} \left(\frac{\alpha_i}{p_i} \right)^{\alpha_i} \right)} \tag{45}$$

In order to compute the Hicksian demand function we can now use Shephard's Lemma (see Theorem 1.7 on page 37), which states that $e(\mathbf{p}, u)$ is differentiable in $\mathbf{p}$ at $(\mathbf{p}^0, u^0)$ with $\mathbf{p} \gg 0$ and its partial derivative with respect to p_i satisfies

$$\frac{\partial e(\mathbf{p}^0, u)}{\partial p_i} = x_i^h(\mathbf{p}^0, u^0) \quad \forall i \in \mathcal{I} \tag{46}$$

In order to apply this formula it is a little bit more convenient to write the expenditure function as

$$e(\mathbf{p}, u) = \frac{u \left(\prod_{i=1}^{n} \left(\frac{p_i}{\alpha_i} \right)^{\alpha_i} \right)}{A}$$

Taking the first derivative with regards to p_i and using the chain rule gives

$$\frac{\partial e(\mathbf{p},u)}{\partial p_i} = \alpha_i \frac{u\left(\prod_{j=1}^n \left(\frac{p_j}{\alpha_j}\right)^{\alpha_j}\right)}{p_i A}$$

or alternatively

$$\frac{\partial e(\mathbf{p},u)}{\partial p_i} = \alpha_i \frac{u}{p_i A\left(\prod_{j=1}^n \left(\frac{\alpha_j}{p_j}\right)^{\alpha_j}\right)} \tag{47}$$

4 Exercise 1.57

4.1 Original question

The **Stone-Geary utility function** has the form:

$$u(\mathbf{x}) = \prod_{i=1}^n (x_i - a_i)^{b_i} \tag{48}$$

where $b_i \geq 0$ and $\sum_{i=1}^n b_i = 1$. The $a_i \geq 0$ are often interpreted as 'subsistence' level of the respective commodities.

(a) Derive the associated expenditure and indirect utility functions. Note that the former is *linear* in utility, whereas the latter is proportional to the amount of 'discretionary income', $y - \sum_{i=1}^n p_i a_i$

(b) Show that b_i measures the share of this 'discretionary income' that will be spent on 'discretionary' purchase of goods x_i in excess of the subsistence level a_i

4.2 Solution

(a) The utility maximization problem is-

$$\max_{\mathbf{x}\in\mathbb{R}^n_+} \prod_{i=1}^n (x_i - a_i)^{b_i} \text{ subject to } \sum_{i=1}^n p_i x_i \leq y$$

We will first set up the Lagrangian function as usual-

$$\mathcal{L} = \prod_{i=1}^n (x_i - a_i)^{b_i} - \lambda(\sum_{i=1}^n p_i x_i - y)$$

The first order conditions with respect to x_i and λ are-

$$\frac{\partial L}{\partial x_i} = b_i(x_i - a_i)^{b_i - 1}\prod_{j\neq i}(x_j - a_j)^{b_j} - \lambda p_i = b_i(x_i - a_i)^{-1}\prod_{j=1}^n (x_j - a_j)^{b_j} - \lambda p_i = 0 \tag{49}$$

$$\frac{\partial L}{\partial \lambda} = -(\sum_i p_i x_i - y) = 0 \tag{50}$$

As before, we can assume that the budget constraint will bind with equality because the utility function is increasing in $\mathbf{x}$.

For any $i \neq j$, divide equation (49) with i over this equation with j, we can write the first order conditions as-

$$\frac{b_i}{b_j} \times \frac{x_j - a_j}{x_i - a_i} = \frac{p_i}{p_j}$$

$$\implies x_j = a_j + \frac{p_i}{p_j} \times \frac{b_j}{b_i}(x_i - a_i)$$

Substituting in the budget constraint we get an equation in terms of x_i-

$$p_i x_i + \sum_{j \neq i} p_j a_j + \frac{p_i(x_i - a_i)}{b_i} \sum_{j \neq i} b_j = y$$

Add and subtract $p_i(x_i - a_i)$ on the left hand side to get-

$$\sum_{i=1}^{n} p_i a_i + \frac{p_i(x_i - a_i)}{b_i} = y$$

$$\implies x_i = a_i + \frac{b_i(y - \sum_{i=1}^{n} p_i a_i)}{p_i}$$

Therefore, the indirect utility function will be-

$$v(\mathbf{p}, y) = (y - \sum_{i=1}^{n} p_i a_i) \prod_{i=1}^{n} (\frac{b_i}{p_i})^{b_i}$$

As stated in the question, the indirect utility function is proportional to discretionary income.

Since the utility function is continuous and strictly increasing, we can apply the theorem "Relations between indirect utility and expenditure functions" to derive the expenditure function as follows-

$$v(\mathbf{p}, e(\mathbf{p}, u)) = u$$

$$\implies (e(\mathbf{p}, u) - \sum_{i=1}^{n} p_i a_i) \prod_{i=1}^{n} (\frac{b_i}{p_i})^{b_i} = u$$

$$\implies e(\mathbf{p}, u) = \sum_{i=1}^{n} p_i a_i + u \prod_{i=1}^{n} (\frac{b_i}{p_i})^{(-b_i)}$$

The expenditure function is linear in u.

(b) We can use the expression for the Marshallian demand function of good x_i to write-

$$p_i(x_i(\mathbf{p}, y) - a_i) = b_i(y - \sum_{i=1}^{n} p_i a_i)$$

The left hand side of the equation is the amount of expenditure on discretionary amounts of good $x_i - a_i$. The right hand side gives us the required result that b_i proportion of discretionary income is spent on discretionary purchases of x_i.

5 Exercise 1.35, not in the homework, only for reference

5.1 Original question

Complete the proof of Theorem 1.7 by proving property 5.

5.2 Solution

The expenditure function is the solution to the expenditure-minimization problem

$$e(\mathbf{p}, u) = \min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq \bar{u} \tag{51}$$

Now we are asked to show that this function is homogeneous of degree 1 in $\mathbf{p}$, that is,

$$e(t\mathbf{p}, u) = te(\mathbf{p}, u) \tag{52}$$

for any scalar $t \in \mathbb{R}_+$.
Observe that the left-hand side of (52) is defined through

$$e(t\mathbf{p}, u) = \min_{\mathbf{x} \in \mathbb{R}^n_+} t\mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq \bar{u} \tag{53}$$

We will prove from contradiction. Assume that the claim in (52) does not hold. We consider two cases. First, consider $e(t\mathbf{p}, u) > te(\mathbf{p}, u)$. Since u is strictly increasing and continuous, we know that a solution to (51) exists. However, we have not assumed quasiconcavity up to this point, so we cannot guarantee uniqueness. Therefore, denote by $X(\mathbf{p}, u)$ the set of $\mathbf{x} \in \mathbb{R}^n_+$ that solves (51), i.e.

$$X(\mathbf{p}, u) = \arg\min_{\mathbf{x} \in \mathbb{R}^n_+} \mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq \bar{u} \tag{54}$$

By construction, if $\mathbf{x}^\star \in X(\mathbf{p}, e)$, it satisfies the constraint, $u(\mathbf{x}^\star) \geq \bar{u}$. Then $\mathbf{x}^\star$ also satisfies the constraint in (53). The expenditure function is given by the cost of the optimal bundle, i.e. $e(\mathbf{p}, u) = \mathbf{p} \cdot \mathbf{x}^\star$. Multiplying both sides by t gives $te(\mathbf{p}, u) = t\mathbf{p} \cdot \mathbf{x}^\star$. But if we assume that $te(\mathbf{p}, u) < e(t\mathbf{p}, u)$, we have now found a bundle that satisfies the constraint at price $t\mathbf{p}$ but results in lower expenditure. Since we define $e(t\mathbf{p}, u)$ as the least expenditure at price $t\mathbf{p}$, but now we have a specific bundle $\mathbf{x}^*$ that achieves the given utility level u with even lower expenditure. This is a contradiction.
We can rule out $te(\mathbf{p}, u) > e(t\mathbf{p}, u)$ in a similar way. Define

$$X^t(t\mathbf{p}, u) = \arg\min_{\mathbf{x} \in \mathbb{R}^n_+} t\mathbf{p} \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) \geq \bar{u} \tag{55}$$

If $\mathbf{x}^\star \in X^t(t\mathbf{p}, u)$, it satisfies the constraint in both problems[1], $u(\mathbf{x}^\star) \geq u$. By definition, $e(tp, u) = t\mathbf{p} \cdot \mathbf{x}^\star$. If $te(\mathbf{p}, u) > e(t\mathbf{p}, u)$, then $e(\mathbf{p}, u) > \mathbf{p} \cdot \mathbf{x}^\star$. So we have again found a bundle in $\mathbf{x}^\star$ that satisfies the constraint but results in a strictly lower expenditure than $e(\mathbf{p}, u)$, contradicting our initial assumption in (51) that $e(\mathbf{p}, u)$ is a minimum.
Since we have shown that both $te(\mathbf{p}, u) > e(t\mathbf{p}, u)$ and $te(\mathbf{p}, u) < e(t\mathbf{p}, u)$ result in contradictions, it must be that $te(\mathbf{p}, u) = e(t\mathbf{p}, u)$. Hence, the expenditure function is homogeneous of degree one in $\mathbf{p}$.

[1] This refers to the expenditure minimization at price $\mathbf{p}$ and $t\mathbf{p}$