Lecture 1: Review (Mostly)

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Packages for Today

Let's load some packages so that I can make some better looking plots:

```r
#always
library(tidyverse)
# for SE's
library(estimatr)
library(broom)
# for regressions
library(fixest)
library(sandwich)
```

Today's Plan

- Recap OLS and various forms of standard errors
- Standard errors are tedious but I guess you are supposed to know this stuff
- Hopefully first and last time we talk about this

Recap: Asymptotics for OLS and the Linear Model

OLS

$$y_i = \beta_0 + \beta x_i + u_i$$

Recall the three basic OLS assumptions

1. $\mathbb{E}(u_i|X_i) = 0$
2. (X_i, Y_i), $i = 1, \ldots, n$, are i.i.d.
3. Large outliers are rare $\mathbb{E}[Y^4] < \infty$ and $\mathbb{E}[X^4] < \infty$.

Unbiasedness and Consistency

- Unbiasedness means on average we don't over or under estimate $\widehat{\beta}$

$$\mathbb{E}[\widehat{\beta}] - \beta_0 = 0$$

- Consistency tells us that we approach the true β_0 as $n \to \infty$.

$$\widehat{\beta} \xrightarrow{p} \beta_0$$

- Example: $X_{(1)}$ is unbiased but not consistent for the mean.
- Example $\frac{n}{n-5}\overline{X}$ is consistent but biased for the mean.

Bias Variance Decomposition

We can decompose any estimator into two components

$$\underbrace{\mathbb{E}[(y - \hat{f}(x))^2]}_{MSE} = \underbrace{\left(\mathbb{E}[\hat{f}(x) - f(x)]\right)^2}_{Bias^2} + \underbrace{\mathbb{E}\left[\left(\hat{f}(x) - \mathbb{E}[\hat{f}(x)]\right)^2\right]}_{Variance}$$

- What minimizes MSE?

$$f(x_i) = \mathbb{E}[Y_i \mid X_i]$$

- In general we face a tradeoff between bias and variance.
- In OLS we minimize the variance among unbiased estimators assuming that the true $f(x_i) = X_i\beta$ is linear. (But is it?)

Outliers and Leverage

One way to find outliers is to calculate the leverage of each observation i. We begin with the hat matrix:

$$P = X(X'X)^{-1}X'$$

and consider the diagonal elements which for some reason are labeled h_{ii}

$$h_{ii} = x_i(X'X)^{-1}x_i'$$

This tells us how influential an observation is in our estimate of $\widehat{\beta}$.
Particularly important for $\{0, 1\}$ dummy variables with uneven groups.

Leave One Out Regression

- This is sometimes called the Jackknife
- Sometimes it is helpful to know what would happen if we omitted a single observation i
- Turns out we don't need to run N regressions

$$\begin{aligned} \widehat{\beta}_{-i} &= (X'_{-i}X_{-i})^{-1}X'_{-i}Y_{-i} \\ &= \widehat{\beta} - (X'X)^{-1}x_i\tilde{u}_i \quad \text{where } \tilde{u}_i = (1-h_{ii})^{-1}\hat{u}_i \end{aligned}$$

- $\tilde{u}_i$ has the interpretation of the LOO prediction error.
- high leverage observations move $\widehat{\beta}$ a lot.

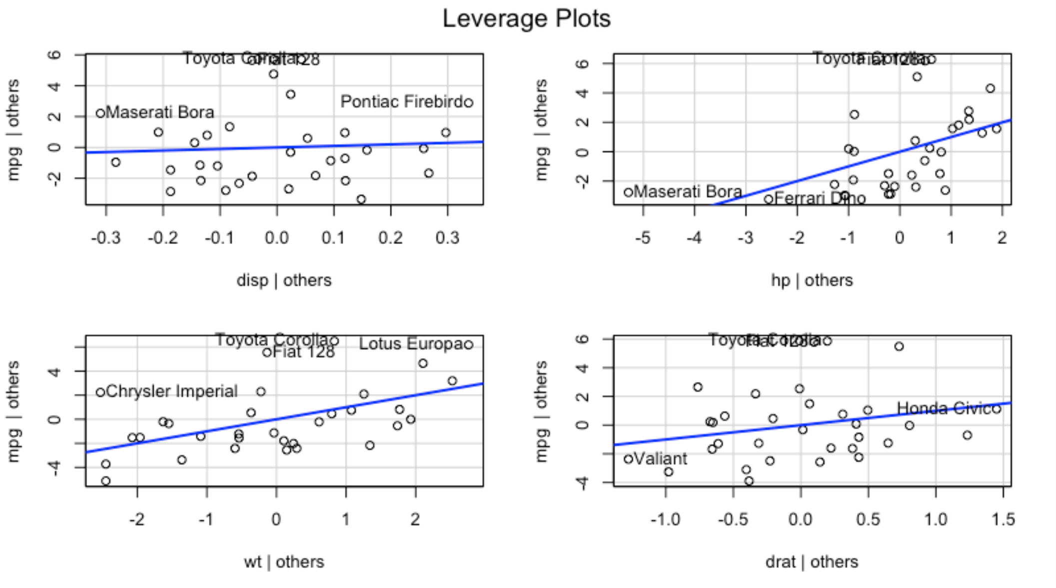
You can read more about this in Ch3 of Hansen. [Skip derivation]

Leverage and QQ plots

```r
library(car)
fit <- lm(mpg~disp+hp+wt+drat, data=mtcars)

# Assessing Outliers
outlierTest(fit) # Bonferonni p-value for most extreme obs
qqPlot(fit, main="QQ Plot") #qq plot for studentized resid
leveragePlots(fit) # leverage plots
```

Leverage Plot



Gauss Markov Theorem

Gauss Markov Adds two assumptions:

1. $E(u_i|X_i) = 0$
2. (X_i, Y_i), $i = 1, \ldots, n$, are i.i.d.
3. Large outliers are rare $\mathbb{E}[Y^4] < \infty$ and $\mathbb{E}[X^4] < \infty$.
4. $Var(u_i) = \sigma^2$ (homoskedasticity)
5. $u_i \sim N(0, \sigma^2)$ (normal errors)

Under these assumptions you learned that OLS is BLUE

Variance of $\widehat{\beta}$

Start with the variance of the residuals to form a diagonal matrix D:

$$\text{Var}(\mathbf{u}|\mathbf{X}) = \mathbb{E}\left(\mathbf{u}\mathbf{u}' \mid \mathbf{X}\right) = \mathbf{D}$$

$$\mathbf{D} = \text{diag}\left(\sigma_1^2, \ldots, \sigma_n^2\right) = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{pmatrix}$$

- $\mathbf{D}$ is diagonal because $\mathbb{E}[u_i u_j \mid X] = \mathbb{E}[u_i \mid x_i]\mathbb{E}[u_j \mid x_j] = 0$ (independence)
- The elements of D_i are given by $\mathbb{E}[u_i^2 \mid X] = \mathbb{E}[u_i^2 \mid x_i] = \sigma_i^2$.
- In the homoskedastic case $\mathbf{D} = \sigma^2 \mathbf{I}_n$.

Variance of $\widehat{\beta}$

A useful identity for linear algebra:

$$\mathrm{Var}(a\mathbf{Z}) = a^2\,\mathrm{Var}(\mathbf{Z})$$
$$\mathrm{Var}(A\mathbf{Z}) = A\,\mathrm{Var}(\mathbf{Z})A'$$

Recall that $\mathrm{Var}(\mathbf{Y}|\mathbf{X}) = \mathrm{Var}(\mathbf{u}|\mathbf{X})$ and also recall the formula for $\widehat{\beta}$:

$$\widehat{\beta} = \underbrace{(X'X)^{-1}X'}_{A}\,Y = A'Y$$

$$\mathbf{V}_{\widehat{\beta}} = \mathrm{Var}(\widehat{\beta}|X) = (X'X)^{-1}X'\,\mathrm{Var}(Y|X)X(X'X)^{-1}$$
$$= (X'X)^{-1}(X'\mathbf{D}X)(X'X)^{-1}$$

We have that $(X'\mathbf{D}X) = \sum_{i=1}^{N} x_i x_i' \sigma_i^2$. Under homoskedasticity $\mathbf{D} = \sigma^2\mathbf{I}_n$ and $\mathbf{V}_{\widehat{\beta}} = \sigma^2(X'X)^{-1}$.

Variance of $\widehat{\beta}$

$$\mathbf{D} = \operatorname{diag}\left(\sigma_1^2, \ldots, \sigma_n^2\right) = \mathbb{E}\left(u_i u_i' \mid \mathbf{X}\right) = \mathbb{E}\left(\widetilde{\mathbf{D}} \mid \mathbf{X}\right)$$

We can estimate $\widehat{\mathbf{V}}_{\widehat{\beta}}$ by plugging in $\mathbf{D} \rightarrow \widetilde{\mathbf{D}}$:

$$\begin{aligned}
\mathbf{V}_{\widehat{\beta}} &= (X'X)^{-1}(X'\widetilde{\mathbf{D}}X)(X'X)^{-1} \\
&= (X'X)^{-1}\left(\sum_{i=1}^{N} x_i x_i' u_i^2\right)(X'X)^{-1}
\end{aligned}$$

The expectation shows us this estimator is unbiased:

$$\begin{aligned}
\mathbb{E}[\mathbf{V}_{\widehat{\beta}} \mid X] &= (X'X)^{-1}\left(\sum_{i=1}^{N} x_i x_i' \, \mathbb{E}[u_i^2 | X]\right)(X'X)^{-1} \\
&= (X'X)^{-1}\left(\sum_{i=1}^{N} x_i x_i' \, \sigma_i^2\right)(X'X)^{-1} = (X'X)^{-1}(X'DX)(X'X)^{-1}
\end{aligned}$$

Heteroskedasticity Consistent (HC) Variance Estimates

What we need is a consistent estimator for $\hat{u}_i^2$.

$$\mathbf{V}_{\widehat{\beta}}^{HC0} = (X'X)^{-1}\left(\sum_{i=1}^{N} x_i x_i' \hat{u}_i^2\right)(X'X)^{-1}$$

$$\mathbf{V}_{\widehat{\beta}}^{HC1} = (X'X)^{-1}\left(\sum_{i=1}^{N} x_i x_i' \hat{u}_i^2\right)(X'X)^{-1} \cdot \left(\frac{n}{n-k}\right)$$

Could use leave one out variance estimate:

$$\mathbf{V}_{\widehat{\beta}}^{HC2} = (X'X)^{-1}\left(\sum_{i=1}^{N} (1-h_{ii})^{-1} x_i x_i' \hat{u}_i^2\right)(X'X)^{-1}$$

$$\mathbf{V}_{\widehat{\beta}}^{HC3} = (X'X)^{-1}\left(\sum_{i=1}^{N} (1-h_{ii})^{-2} x_i x_i' \hat{u}_i^2\right)(X'X)^{-1}$$

Heteroskedasticity Consistent (HC) Variance Estimates

- We know that $\mathbf{V}_{\widehat{\beta}}^{HC3} > \mathbf{V}_{\widehat{\beta}}^{HC2} > \mathbf{V}_{\widehat{\beta}}^{HC0}$ because $(1 - h_{ii}) < 1$.
- $HC3$ are the most conservative and also place the most weight on potential outliers.
- `Stata` uses $HC1$ as the default and it is what most people refer to when they say robust standard errors.
- These are often called White (1980) SE's or Eicher-Huber-White SE's.
- In small sample some evidence that $HC2$ has better coverage, (what is that?)

Heteroskedasticity Consistent (HC) Variance Estimates

To read about SE's in estimatr:
https://declaredesign.org/r/estimatr/articles/mathematical-notes.html or fixest
https://lrberge.github.io/fixest/articles/fixest_walkthrough.html#the-vcov-argument-1

```
dat <- data.frame(X = matrix(rnorm(2000*5), 2000), y = rnorm(2000))
res<-feols(y ~ X.1+X.2+X.3+X.4+X.5, data = dat)
hc0<-summary(res, vcov = sandwich::vcovHC, type = "HC0")$se
hc1<-summary(res, vcov = sandwich::vcovHC, type = "HC1")$se
hc2<-summary(res, vcov = sandwich::vcovHC, type = "HC2")$se
hc3<-summary(res, vcov = sandwich::vcovHC, type = "HC3")$se
all(hc2 > hc0 )
[1] TRUE
all(hc3> hc2 )
[1] TRUE
```

What is Clustering?

Suppose we want to relax our i.i.d. assumption:

- Each observation i is a villager and each group g is a village
- Each observation i is a student and each group g is a class.
- Each observation t is a year and each entity i is a state.
- Each observation t is a week and each entity i is a shopper.

We might expect that $\text{Cov}(u_{g1}, u_{g2}, \ldots, u_{gN}) \neq 0 \rightarrow$ independence is a bad assumption.

Clustering: Intuition

The groups (villages, classrooms, states) are independent of one another, but within each group we can allow for arbitrary correlation.

- If correlation is within an individual over time we call it serial correlation or autocorrelation
- Just like in time-series→ we have fewer effective independent observations in our sample.
- Asymptotics now about the number of groups $G \to \infty$ not observations $N \to \infty$

Clustering

Begin by stacking up observations in each group $\mathbf{y}_g = [y_{g1}, \ldots, y_{gn_g}]$, we can write OLS three ways:

$$
\begin{aligned}
y_{ig} &= x_{ig}'\beta + u_{ig} \\
\mathbf{y}_g &= \mathbf{X}_g\beta + \mathbf{u}_g \\
\mathbf{Y} &= \mathbf{X}\beta + \mathbf{u}
\end{aligned}
$$

All of these are equivalent:

$$
\begin{aligned}
\widehat{\beta} &= \left(\sum_{g=1}^{G}\sum_{i=1}^{n_g} x_{ig}'x_{ig}\right)^{-1}\left(\sum_{g=1}^{G}\sum_{i=1}^{n_g} x_{ig}'y_{ig}\right) \\
\widehat{\beta} &= \left(\sum_{g=1}^{G} \mathbf{X}_g'\mathbf{X}_g\right)^{-1}\left(\sum_{g=1}^{G} \mathbf{X}_g'\mathbf{y}_g\right) \\
\widehat{\beta} &= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\mathbf{X}'\mathbf{Y}\right)
\end{aligned}
$$

Clustering (Continued)

The error terms have covariance within each cluster g as:

$$\boldsymbol{\Sigma}_g = \mathbb{E}\left(\mathbf{u}_g\mathbf{u}_g' \mid \boldsymbol{X}_g\right)$$

In order to calculate $\widehat{V}_{\widehat{\beta}}$ we replace the covariance matrix $\mathbf{D}$ with Ω and consider an estimator $\widehat{\Omega}_n$. We exploit independence across clusters:

$$\operatorname{var}\left(\left(\sum_{g=1}^{G}\boldsymbol{X}_g'\mathbf{u}_g\right) \mid \boldsymbol{X}\right) = \sum_{g=1}^{G}\operatorname{var}\left(\boldsymbol{X}_g'\boldsymbol{u}_g|\boldsymbol{X}_g\right) = \sum_{g=1}^{G}\boldsymbol{X}_g'\mathbb{E}\left(\boldsymbol{u}_g\boldsymbol{u}_g'|\boldsymbol{X}_g\right)\boldsymbol{X}_g = \sum_{g=1}^{G}\boldsymbol{X}_g'\boldsymbol{\Sigma}_g\mathbf{X}_g \equiv \Omega_N$$

And an estimate of the variance:

$$\boldsymbol{V}_{\widehat{\beta}} = \operatorname{var}(\widehat{\beta} \mid \boldsymbol{X}) = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\boldsymbol{\Omega}_n\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

Clustered SE's

$$
\begin{aligned}
\widehat{\Omega}_n &= \sum_{g=1}^{G} X_g' \widehat{\mathbf{u}}_g \widehat{\mathbf{u}}_g' X_g \\
&= \sum_{g=1}^{G} \sum_{i=1}^{n_g} \sum_{\ell=1}^{n_g} x_{ig} x_{\ell g}' \widehat{u}_{ig} \widehat{u}_{\ell g} \\
&= \sum_{g=1}^{G} \left(\sum_{i=1}^{n_g} x_{ig} \widehat{u}_{ig} \right) \left(\sum_{\ell=1}^{n_g} x_{\ell g} \widehat{u}_{\ell g} \right)'
\end{aligned}
$$

- First line makes explicit: independence over each of G clusters
- Last line easiest for computer

Clustered SE's

$$\widehat{\boldsymbol{V}}_{\hat{\beta}}^{\mathrm{CR1}} = \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\left(\sum_{g=1}^{G}\boldsymbol{X}_g'\widehat{u}_g\widehat{\boldsymbol{u}}_g'\boldsymbol{X}_g\right)\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}$$

$$\widehat{\boldsymbol{V}}_{\hat{\beta}}^{\mathrm{CR3}} = \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}\left(\sum_{g=1}^{G}\boldsymbol{X}_g'\widetilde{u}_g\widetilde{\boldsymbol{u}}_g'\boldsymbol{X}_g\right)\left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1}$$

- Can replace $\hat{\mathbf{u}}_g \rightarrow \tilde{\mathbf{u}}_g$ for leave-one out like *HC*3 (these are called *CR*3).

Clustering in R

```
lm_robust(y~ x1 + x2, data=df, se_type="CR0", cluster=group_id )
lm_robust(y~ x1 + x2, data=df, se_type="CR2", cluster=group_id )
lm_robust(y~ x1 + x2, data=df, se_type="CR1", cluster=group_id )
```

Most Asked PhD Student Econometric Question

How should I cluster my standard errors?

- Heck if I know.
- This is very problem specific
- It matters a lot → standard errors can get orders of magnitude larger.
- Do you believe across group independence or not? [this is the only thing that matters]
- If you include fixed effects probably you need at least clustering at that level.

Newey West Standard Errors (HAC)

- In serially correlated data we need to account for $\text{Cov}(u_t, u_{t-1}, \dots) \neq 0$.
- Clustering is one solution, but we may end up throwing away all of our data.
- Instead we could estimate the serial correlation.
- May also want standard errors that are heteroskedasticity AND autocorrelation consistent (HAC).
- Have to select a number of lags L

$$\widehat{\Omega}^{HAC}_{n,L} = \sum_{t=1}^{T} u_t^2 x_t x_t' + \sum_{l=1}^{L} \sum_{t=l+1}^{T} w_l u_t u_{t-l} \left(x_t x_{t-l}' + x_{t-l} x_t' \right)$$

$$w_l = 1 - \frac{l}{L+1}$$

What about β?

- All of the estimates above should produce identical point estimates
- We have just been talking about adjusting standard errors
- Should the presence of heteroskedasticity change our estimates of $\widehat{\beta}$ as well?

OLS and WLS

A simple extension is Weighted Least Squares (WLS)

- Different motivations
- Suppose we have sampling weights that are not $\frac{1}{n}$ from survey data, etc:
  - If my population is supposed to represent all US residents and my sample is 75% Women...
  - Relax LSA (2) (X_i, Y_i), $i = 1, \ldots, n$, are i.i.d.
- In this case, OLS is still unbiased and consistent, just inefficient

WLS

Can weight each observation as w_i so that $\sum_{i=1}^{N} w_i = 1$ instead of $w_i = \frac{1}{N}$.
Can define a diagonal matrix W with entries w_i.

$$\arg\min_{\beta} \sum_{i=1}^{N} w_i (y_i - X_i \beta)^2 = \arg\min_{\beta} \left\| W^{1/2} \, | Y - X\beta | \right\|$$

Can also consider a transformation of the data

$$\tilde{y}_i = \sqrt{w_i} y_i, \quad \tilde{x}_i = \sqrt{w_i} x_i$$
$$\tilde{Y} = W^{1/2} Y, \quad \tilde{X} = W^{1/2} X$$

A regression of $\tilde{Y}$ on $\tilde{X}$:

$$\widehat{\beta}_{WLS} = (\tilde{X}'\tilde{X})^{-1} \tilde{X}'\tilde{Y} = (X'WX)^{-1} X'WY$$

WLS

Also used as a solution to heteroskedasticity

- Relax LSA (2) (X_i, Y_i), $i = 1, \ldots, n$, are i.i.d.
- Relax LSA (4) $Var(u_i) = \sigma^2$ (homoskedasticity)

Why? We are minimizing weighted sum of squared residuals:

$$\sum_{i=1}^{N} w_i (y_i - \hat{y}_i)^2 = \sum_{i=1}^{N} w_i u_i^2$$

Suppose we have heteroskedasticity so that $Var(\varepsilon_i) = \sigma_i^2$ and $w_i \propto \frac{1}{\sigma_i^2}$.
In this setting WLS is BLUE.

WLS

Why does anyone ever run OLS instead of WLS?

- Problem is that σ_i^2 is unknown before we run our regression.
- We can estimate $\widehat{\sigma}_i^2$.

This procedure is known as Iteratively Re-weighted Least Squares IRLS

1. Intialize weights to identity matrix: $W = I$
2. Regress Y on X with weights W
3. Obtain $\widehat{u}_i$.
4. Update W with $w_{ii} = \frac{1}{\widehat{u}_i^2}$
5. Repeat until parameter estimates don't change

GLS and FGLS

There is no reason to require that W be diagonal. This gives us Generalized Least Squares

$$\widehat{\beta}_{GLS} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{Y} = (X'\Omega X)^{-1}\Omega' WY$$

The idea is to use the inverse covariance matrix of residuals. But this is high dimensional ($N \times N$) and estimating it is harder than our original problem!

Feasible Generalized Least Squares FGLS:

1. Intialize weights to identity matrix: $\widehat{\Omega} = I$
2. Regress Y on X with weighting matrix $\widehat{\Omega}$
3. Obtain $\widehat{u}_i$.
4. Construct $\mathbb{E}[u_i^2 \mid X, Z]$ via (nonlinear) regression: $\exp[\gamma_0 + \gamma_1 x_i + \gamma_2 z_i]$.
5. Update $\widehat{\Omega}$ with $\mathbb{E}[u_i^2 \mid X, Z]$
6. Repeat until parameter estimates don't change

Thanks!