Introductory Statistics

2024 Lectures
Part 4 - Sample Space

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Experiments

- goal: to define probability, a quantitative measure of uncertainty of outcomes to occur, or the degree of belief that some proposition/conjecture is true
- rational decision maker prefers to avoid statements “I do not know what will happen” or “I do not know whether the proposition is true or false”
- probability is a tool for distinguishing likely from unlikely states of affairs
- collected data ... past experience, observations, results of controlled process, etc.
- data are used to build a probabilistic model of the situation and to hypothesize about mechanisms of interest

Definition 1: Experiment is any process, possibly under partial control, that we may observe and for which the final state of affairs cannot be specified in advance, but for which a set containing all potential final states of affairs can be identified.

Outcomes and sample space

- analyzing an experiment we observe its outcome – a concept which has to be specified in every situation

Definition 2: Outcome is a final result, observation, or measurement occurring from an experiment.

- outcomes need to
  - exclude each other
  - exhaust all logical possibilities

Definition 3: The sample space, denoted by S, is the set of all outcomes of an experiment. The elements of the sample space are called elementary outcomes, or sample points.

Classification of sample spaces

- finite sample space (discrete sample space)
- infinitely countable sample space (discrete sample space)
- uncountable sample space (continuous sample space)

Sample space

Example 9: Consider an experiment consisting of two tosses of a regular die. Suppose that the only available information about the numbers, those that turn up on the upper faces of the die, is their sum. What is the sample space?

- outcomes can be described in various ways...the same experiment can be described via different sample spaces.

Example 10: Two persons enter a cafeteria and sit at a square table, with one chair on each of its sides. Suppose we are interested in the event "they sit at a corner" (as opposed to sitting across from one another). Construct a sample space.

Events

Definition 4: An event is a subset of the sample space S.
Elementary event is an event that is a singleton set.

Example 9 (cont.): An event such as “the sum equals 7” containing six outcomes (1, 6), (2, 5), (3, 4), (4, 3), (5, 2) and (6, 1) is a subset of the sample space S.

Terminology:

- We observe the outcome. If it belongs to a set representing the event A then we say that the event A has occurred.
- To denote events we use letter A, B, C... or A_1, A_2... or as $\{X = 1\}$ or $\{a < Z < b\}$, where X and Z are some functions on sample space S

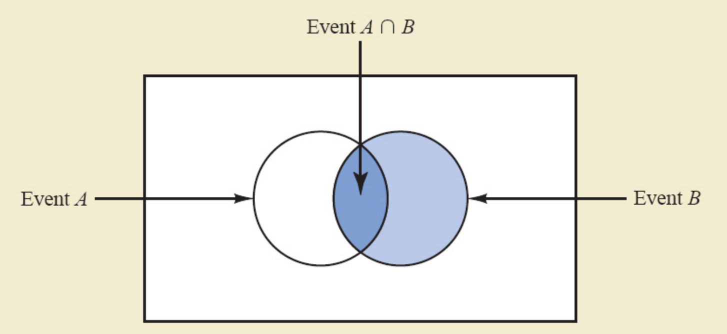
Basic operations and terminology

Definition 5: For events $A, B, C \subset S$ we say that

a) $A = \emptyset$ ... A is null or impossible event
b) $A = S$ ... A is sure event
c) $A \subset B$ ... A is a subevent of B (A is contained in B or A implies B)
d) $B = A^c = S \setminus A$ ... B is the complement of A in S
e) $C = A \cup B$ ... C is the union of A and B (called event A or event B)
f) $C = A \cap B (= AB)$ ... C is the intersection of A and B (called event A and event B)
g) $C = A - B$ ... C occurs when A occurs and event B does not
h) $\emptyset = A \cap B$ ... event A and B are disjoint/mutually exclusive.

Venn diagrams

- Venn diagrams can be used to check the validity of formulas. Picture does not constitute a proof, it may at least provide convincing evidence or suggest a method of proving the statement.
- in Venn diagram the sample space is usually represented by a rectangle and its subsets represent events.



Basic set operations

Example 11: Suppose that n shots are fired at a target, and let A_i, $i = 1, 2, \dots, n$, denote the event "the target is hit on the ith shot". Then the union $A_1 \cup \cdots \cup A_n$ is the event "the target is hit" (at least once). Its complement $(A_1 \cup \cdots \cup A_n)^c$ is the event "the target is missed" (on every shot), which is the same as the intersection $A_1^c \cap \cdots \cap A_n^c$.

Definition 6: (laws of operations on events)

a) Idempotence:

$$A \cup A = A, A \cap A = A$$

b) Double Complementation:

$$(A^c)^c = A$$

c) Absorption:

$$A \cup B = B \Leftrightarrow A \cap B = A \Leftrightarrow A \subset B$$

Basic set operations

d) Commutativity:

$$A \cup B = B \cup A, A \cap B = B \cap A$$

e) Associativity:

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

f) Distributivity:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

g) De Morgan's Laws:

$$(A_1 \cup \cdots \cup A_n)^c = A_1^c \cap \cdots \cap A_n^c$$
$$(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup \cdots \cup A_n^c$$

Example 12: (during Control Session 2)

Answer true or false.

(i) If A and B are distinct events (i.e., $A \neq B$) such that A and B^c are disjoint, then A^c and B are also disjoint.

(ii) If A and B are disjoint, and also B and C are disjoint, then A and C are disjoint.

(iii) If $A \cup B^c = B^c$, then $B \subset A^c$.

(iv) If A and B are both contained in C, then $C^c \subset A^c \cap B^c$.

Infinite sequence of events

Definitions of union and intersection can readily be extended to the case of infinite number of events:

- Let $A_1, A_2, \ldots$ be an infinite sequence of events. Then

  $$\cup_{i=1}^{\infty} A_i = A_1 \cup (A_2 \cup (\ldots))$$

  and

  $$\cap_{i=1}^{\infty} A_i = A_1 \cap (A_2 \cap (\ldots))$$

  are events "at least one A_i occurs" and "all A_i's occur."
- De Morgan's Laws can also be extended to infinite sequences of events
- If at least one event occurs then there is one that occurs first. We can rewrite

  $$\cup_{i=1}^{\infty} A_i = A_1 \cup (A_1^c \cap A_2) \cup (A_1^c \cap A_2^c \cap A_3) \cup \ldots$$

  as a union of disjoint events $B_i = A_1^c \cap \cdots \cap A_{i-1}^c \cap A_i$ reading "A_i is the first event in the sequence to occur"

Infinite sequence of events

Definition 7: Let $A_1, A_2, \ldots$ be an infinite sequence of events. Then

$$\limsup_{n\to\infty} A_n = \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} A_i, \quad \liminf_{n\to\infty} A_n = \cup_{k=1}^{\infty} \cap_{i=k}^{\infty} A_i$$

are the events that "infinitely many A_i's occur" and that "all except finitely many A_i's occur", respectively.

- Clearly, if all but finitely many event occur then infinitely many events occur, i.e.

$$\limsup_{n\to\infty} A_n \supset \liminf_{n\to\infty} A_n$$

Definition 8: If $\limsup_{n\to\infty} A_n \subset \liminf_{n\to\infty} A_n$ then we say that the sequence $\{A_n\}_{n=1}^{\infty}$ converges and the limit of events is given as

$$\lim_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$$

Theorem 1:

a) If the sequence of A_i's is increasing ($A_1 \subset A_2 \subset \dots$) then the limit $\lim_{n\to\infty} A_n$ exists, i.e.
$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \lim_{n\to\infty} A_n$, and

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

b) If the sequence of A_i's is decreasing ($A_1 \supset A_2 \supset \dots$) then then the limit $\lim_{n\to\infty} A_n$ exists, i.e.
$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \lim_{n\to\infty} A_n$, and

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

Example 13: Find $\lim_{n\to\infty} A_n$ if

(i) $A_n = \left\{x \middle| \frac{1}{n} \le x \le 3 - \frac{1}{n}\right\}, n = 1, 2, 3, \dots$.

(ii) $A_n = \{x | 2 < x \le 2 + \frac{1}{n}\}, n = 1, 2, 3, \dots$.

(iii) $A_n = \left\{(x, y) \middle| 0 \le x^2 + y^2 \le \frac{1}{n}\right\}, n = 1, 2, 3, \dots$.

Need for σ-algebras

- in order to define properly probability, we need to understand numerical functions on subsets of sample space
  - in case of finite or countable S we can consider the class of all subsets (and it will not cause any troubles for definition of probability)
  - the system of all subsets of uncountable S is too rich, it contains “special” sets with nonunique “length” and thus probability defined on all subsets may produce paradoxes – nonunique values of probability of some subsets called nonevents – we need restrictions to a system of subsets which does not contain such subsets and is closed under basic operations of unions, intersections and complements

Definition 9: (closure under operation)
We say that a class $\mathcal{A}$ of subsets of S is closed under a given operation if the sets resulting from performing this operation on elements of $\mathcal{A}$ are also elements of $\mathcal{A}$.

σ-algebra

Definition 10: Let S be a nonempty set. Then nonempty class $\mathcal{A}$ of subsets of S is called σ-algebra (σ-field), if

(i) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$ (closed on complements)

(ii) $A_n \in \mathcal{A}, n = 1, 2, \cdots \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{A}$ (closed on countable unions)

- Closeness on finite/countable intersections follows from de Morgan's laws.

Example 14: Let $S_1 = \{a\}$ and $S_2 = \{a, b\}$. Construct σ-algebras of all subsets.

Example 15: Let $S = \{1, 2, \ldots\}$ and let $\mathcal{A}$ consists of all subsets of S that are finite. Is $\mathcal{A}$ a σ-algebra?

Example 16: Let $S = \{1, 2, \ldots\}$, and let $\mathcal{A}$ be the class of all subsets A of S such that either A or A^c is finite. Is $\mathcal{A}$ a σ-algebra?

σ-algebra

Definition 11: Let S be any set.

(i) The class consisting of two sets, $\emptyset$ and S, is called the smallest σ-algebra.

(ii) The class of all subsets of S is called the largest (or maximal) σ-algebra.

- For any event A it is easy to check that the class $\mathcal{A}$, consisting of the four events $\emptyset, A, A^c, S$, is closed under any operations. This class is an example of a σ-algebra that contains the events A and A^c, and it is the smallest such σ-algebra.

σ-algebra

The following theorem ensures existence of σ-algebra to any collection of subsets.

Theorem 2: (without proof) For any nonempty class $\mathcal{K}$ of subsets of S there exists $\sigma(\mathcal{K})$, a unique smallest σ-algebra containing all sets in $\mathcal{K}$. It is called the σ-algebra generated by $\mathcal{K}$.

Why not to take the maximal σ-algebra in the case of uncountable S?

On a real line it is natural to consider “simple” events in the form of intervals. σ-algebra generated by all open intervals is called Borel σ-algebra $\mathcal{B}(\mathbb{R})$. It is of a rich structure. On the other hand, the maximal σ-algebra on $\mathbb{R}$ is too rich and contains “nonevents” - e.g. Vitali sets.