

ECON 600: Industrial Organization

Charles Hodgson

Dynamic Games

From Single Agent Problems to Games

- ▶ So far we have studied *single agent* dynamic problems.
 - ▶ Trading off payoffs today with payoffs in the future.
 - ▶ E.g. dynamic demand.
- ▶ The approaches we have studied can be extended to the analysis of *dynamic games*.
 - ▶ Dynamic incentives + strategic interaction.
- ▶ Best reference is Aguirregabiria, Collard-Wexler, and Ryan (2021) in Handbook of IO

From Single Agent Problems to Games

- ▶ Examples:
- ▶ Entry and exit games.
 - ▶ Allow for simultaneous entry and exit.
 - ▶ Can distinguish sunk costs and fixed costs.
 - ▶ Rationalizes evolution of market structure.
 - ▶ Contrast to static entry literature (Bresnahan and Reiss (1991), Berry (1992), Seim (2006) etc.
- ▶ Investment in capacity/quality etc.
- ▶ Learning by doing.
- ▶ Firm pricing in response to consumer dynamics (e.g. durable goods).

Framework: A Simple Dynamic Game

- ▶ Let's think about a dynamic entry/exit and investment game.
- ▶ In the spirit of Pakes and McGuire (1994) and Ericson and Pakes (1995).
- ▶ Suppose we have a market with firms $i \in \{1, \dots, N\}$.
- ▶ Firm i 's state given by $s_{it} \in S$.
 - ▶ Let's call it a "quality level".
 - ▶ E.g. could be $S = \{1, 2, \dots, 10\}$.
- ▶ s_{-it} indicates the vector of "everyone else's" quality.
- ▶ $s_t = (s_{it}, s_{-it})$ is the "industry state": vector of all qualities in the market.
 - ▶ Assumption of "exchangability"/"symmetry" (all rivals look alike).

Framework: A Simple Dynamic Game

- ▶ At date t , flow payoff is given by

$$\pi(s_{it}, s_{-it}; \theta)$$

- ▶ This is a “reduced form” profit function of some underlying static game.
- ▶ For instance, we could have a static demand and supply model in which demand (and marginal costs) depends on s_{it} .
- ▶ Firms set prices in static equilibrium.
- ▶ This generates some profit, $\pi(s_{it}, s_{-it}; \theta)$.
- ▶ E.g. we could estimate BLP conditional on s_t and then generate the payoff mapping.
 - ▶ Or do something simpler e.g. log-linear demand for a homogeneous good.

Framework: A Simple Dynamic Game

- ▶ In each period t , incumbent firms make two choices:

1. Whether to stay in the market:

$$d_{it} \in \{0, 1\}$$

- ▶ If exit, get $\pi(s_{it}, s_{-it})$ this period. Get some “scrap value” ϕ next period that the exit forever ($V = 0$).

2. If $d_{it} = 1$, choose whether to invest in quality:

$$a_{it} \in \{0, 1\}$$

- ▶ Evolution of quality: $s_{it+1} \sim F(s_{it+1} | s_{it}, a_{it}; \theta)$.
- ▶ This transition function should allow for “depreciation”.
- ▶ e.g. “falling behind” industry-level trend.
- ▶ This ensures the stationary distribution of s (recurrent class of states) is not degenerate.

Framework: A Simple Dynamic Game

- Dynamic Problem:

$$V(s_{it}, s_{-it}; \theta) = E_{\epsilon_{ait}, \phi_{it}} [\max\{\pi(s_{it}, s_{-it}; \theta^\pi) + \beta(\phi + \phi_{it}), \\ \max_{a_{it} \in \{0,1\}} \{\pi(s_{it}, s_{-it}; \theta^\pi) - ca_{it} + \epsilon_{ait} + \beta E[V(s_{it+1}, s_{-it+1}; \theta) | s_t, a_{it}; \theta]\}\}]$$

- c is the investment cost.
- Allowing random shocks ϕ_{it} and $\epsilon_{it} = (\epsilon_{0it}, \epsilon_{1it})$.
 - iid over t and i .
 - Private information.

Framework: A Simple Dynamic Game

- ▶ Expectation over tomorrow's value function:

$$E[V(s_{it+1}, s_{-it+1}; \theta) | s_t, a_{it}; \theta] = \int \int V(s_{it+1}, s_{-it+1}; \theta) f(s_{it+1} | s_{it}, a_{it}; \theta) ds_{it+1} g(s_{-it+1} | s_t)$$

- ▶ i 's state s_{it+1} evolves according to $f(s_{it+1} | s_{it}, a_{it}; \theta)$
 - ▶ Known (up to θ)
- ▶ From i 's perspective, other firms states evolve according to $g(s_{-it+1} | s_t)$.
 - ▶ Can't condition on a_{-it} (investments made simultaneously) and shocks $(\phi_{-it}, \epsilon_{-it})$ private info.
- ▶ $g(s_{-it+1} | s_t)$ includes distribution of other firms' actions conditional on today's industry state.
 - ▶ Plus stochastic evolution of other firms' states conditional on those actions.
 - ▶ This is the new part!

Framework: A Simple Dynamic Game

- ▶ Let's add an entry decision.
- ▶ Suppose at each t there is one “potential entrant”.
- ▶ Entrant's problem $e(s_t) \in \{0, 1\}$:

$$V^{ent}(s_t) = E_{\epsilon_t^{ent}} \max \left\{ \begin{array}{l} 0, \\ -\kappa + \epsilon_t^{ent} + \beta E[V(\bar{s}, s_{-it+1}; \theta) | s_t] \end{array} \right\}$$

- ▶ Pay κ to enter or stay out forever.
- ▶ Enter with quality \bar{s}
- ▶ Without this feature, the number of firms in the market would go to 0 at $T \rightarrow \infty$.
 - ▶ Why?

Markov Perfect Equilibrium

- Policy functions:

$$d(s_{it}, s_{-it}, \phi_{it}, \epsilon_{it}; \theta) \in \{0, 1\}$$

$$a(s_{it}, s_{-it}, \phi_{it}, \epsilon_{it}; \theta) \in \{0, 1\}$$

$$e(s_t, \epsilon_t^{ent}; \theta) \in \{0, 1\}$$

- Integrate out unobservable state variables (shocks) to get CCPs (e.g. logit shocks).
- Solve for policy functions in “Markov Perfect Equilibrium”.

Markov Perfect Equilibrium

- ▶ Idea of MPE (Maskin and Tirole, 1988):
 - ▶ Restrict strategies to only depend on “payoff-relevant” state variables.
 - ▶ e.g. we won't allow $d(s_t, s_{t-1}, s_{t-2}, \dots, \phi_{it}, \epsilon_{it}; \theta)$
- ▶ In single agent model, Markov policy functions followed from Markov state transitions.
 - ▶ Never optimal to condition on past states.
- ▶ In a dynamic game, there are equilibria with policy functions that condition on past states.
 - ▶ Even if state transitions are 1st order Markov.
 - ▶ E.g. collusive punishment equilibria.
 - ▶ Rules out Folk Theorem (anything is possible).

Markov Perfect Equilibrium

- ▶ Definition: in Markov Perfect Equilibrium:
 1. $d(s_{it}, s_{-it}, \phi_{it}, \epsilon_{it}; \theta)$, $a(s_{it}, s_{-it}, \phi_{it}, \epsilon_{it}; \theta)$, $e(s_t, \epsilon_t^{ent}; \theta)$ solve the dynamic problem conditional on beliefs about the evolution of other firms states, $g(s_{-it+1}|s_t)$.
 2. $g(s_{-it+1}|s_t)$ is generated by $d(\cdot)$, $a(\cdot)$, and $e(\cdot)$ for firms $-i$.
 - ▶ i.e. Symmetry: all firms have the same optimal policy function (although they may be at different states).
 - ▶ Rational expectations: Beliefs about other firms' state transitions are generated by optimal policy functions.

Solving the Model

- ▶ As with Rust (1987), if we can solve for the policy functions, we can write down a likelihood of the observed data.
- ▶ We are now solving for equilibrium.
 - ▶ Policies must be dynamically optimal given beliefs.
 - ▶ Beliefs must be correct given policy functions.
- ▶ Best response iteration.
 - ▶ Similar to Rust (1987) value function iteration, but embeds consistency of beliefs $g(s_{-it+1}|s_t)$.

Solving the Model

1. Guess the value functions, $V^0(s_{it+1}, s_{-it+1})$, $V^{ent0}(s_t)$ and beliefs $g(s_{-it+1}|s_t)$.
2. Iterate Bellman equation holding $g(s_{-it+1}|s_t)$ fixed until $V(s_{it+1}, s_{-it+1})$, $V^{ent}(s_t)$ converge (Rust).
3. Converged functions imply policies $d(s_{it}, s_{-it}, \phi_{it}, \epsilon_{it}; \theta)$, $a(s_{it}, s_{-it}, \phi_{it}, \epsilon_{it}; \theta)$, $e(s_t, \epsilon_t^{ent}; \theta)$.
4. Policies + distribution of the “private states”/shocks imply new beliefs $g(s_{-it+1}|s_t)$:

► For example if $\epsilon_t^{ent} \sim N(0, 1)$:

$$\begin{aligned} e(s_t; \theta) &= P(-\kappa + \epsilon_t^{ent} + \beta E[V(\bar{s}, s_{-it+1}; \theta)] > 0) \\ &= \Phi(\beta E[V(\bar{s}, s_{-it+1}; \theta)] - \kappa) \end{aligned}$$

► = the probability a new firm enters tomorrow, conditional on today's state.

5. Repeat from 2 with new $g(s_{-it+1}|s_t)$ until convergence.

Solving the Model: Issues

- ▶ Unlike in the case of single agent dynamics, iteration is not guaranteed to converge!
 - ▶ Holding $g(s_{-it+1}|s_t)$ fixed, problem is just single agent.
 - ▶ Evolution of other firms states as-if “exogenous”.
 - ▶ This part is a contraction mapping.
 - ▶ In general, no guarantee of existence of equilibrium between V and g .
 - ▶ Existence proofs for specific cases (including the one presented here) (Ericson and Pakes, 1999; Gowrisankaran, 1999).
 - ▶ Key is the iid shocks ϵ , which generate “smooth” beliefs over other firms actions.
 - ▶ This algorithm can only find pure strategy equilibria.

Solving the Model: Issues

- ▶ Usually, the algorithm does converge
- ▶ More common problem is multiplicity!
- ▶ E.g. from Besanko et al. (2010) “homotopy algorithm”:

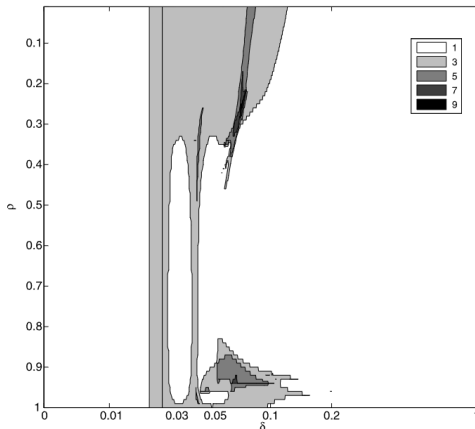


FIGURE 2.—Number of equilibria.

- ▶ Typical practice: try several initial V and g and check if you get the same equilibrium...

Solving the Model: Issues

- ▶ Computational curse of dimensionality.
 - ▶ size of the state space grows exponentially with the number of firms N .

$$(s_{1t}, s_{2t}, \dots, s_{Nt})$$

- ▶ In discrete cases (like above) with symmetry/exchangability, can write the state space as :
 - ▶ $\tilde{s}_t^1 = \sum_n s_{nt} = 1, \tilde{s}_t^2 = \sum_n s_{nt} = 2, \text{ etc.}$
 - ▶ i.e. keep track of counts of firms at each state.

Solving the Model: Issues

- ▶ Other approaches.
- 1. “Oblivious equilibrium” of Weintraub et al. (2008).
 - ▶ Firms optimize against stationary distribution of other firms’ states.
 - ▶ Rather than keep track of each firm, play against a distribution.
- 2. Limit the firm’s information set so they only condition their actions and beliefs on “moments” of the state vector.
 - ▶ e.g. the mean and variance of s_{it} .
 - ▶ See Gowrisankaran, Langer and Zhang (2023) “Moment Based Markov Equilibrium”
 - ▶ Context-specific assumptions: e.g. Hodgson (2021).
- ▶ Gowrisankaran, Langer and Zhang (2023) has an appendix that tries to sort out the equilibrium definition zoo.

Alternative Estimation Approaches

- ▶ If solving the model is so hard, can't we adopt a 2-step CCP approach to estimation?
 - ▶ Yes!
- ▶ Bajari, Benkard, and Levin (2007) and Pakes, Ostrovsky, and Berry (2007) adapt HM approach to dynamic games
 - ▶ Estimate CCPs.
 - ▶ Simulate value functions.
 - ▶ Find parameters that rationalize estimated CCPs.

Step One

- Estimate nonparametrically:

$$\hat{d}(s_{it}, s_{-it}) \in (0, 1)$$

$$\hat{a}(s_{it}, s_{-it}) \in (0, 1)$$

$$\hat{e}(s_t) \in (0, 1)$$

$$\hat{f}(s_{it+1} | s_{it}, a_{it})$$

- If s_{it} is discrete we can do this for every possible state value.
- Otherwise, have to use some functional approximation (basis functions, splines, polynomial etc...)
- As with single agent problems, be as flexible as possible, because any arbitrary functional form assumption here might be inconsistent with the policy functions implied by the model.
 - But we never have infinite data...

Step Two: Inversion

- ▶ “Invert” choice probabilities.
- ▶ As before there are multiple ways to do this depending on the structure of the model.
- ▶ i.e. can make use of terminal states and renewal actions: “finite dependence” properties.
- ▶ For finite state space, Pakes, Ostrovsky and Berry show that you can do a matrix inversion similar to Aguirregabiria and Mira (2002).

$$V = (I - \beta F_0)^{-1} \psi$$

- ▶ Where now F_0 is a matrix that includes transition of own-firm and other-firms states.
 - ▶ See POB for details.

Step Two: Inversion

- ▶ BBL: If you have continuous state space simulate value functions.
- ▶ Just like in HMSS, but we have to simulate the actions of all firms.
- ▶ For a guess of θ :
 1. Start at state s_t , action a_t .
 2. Record flow utility, $\pi(s_{it}, s_{-it}; \theta)$
 3. Draw a new own-state using $\hat{f}(s_{it+1}|s_{it}, a_{it})$.
 4. Draw other firms' actions using $\hat{d}(\cdot), \hat{a}(\cdot), \hat{e}(\cdot)$
 5. Draw new other-firm states using $\hat{f}(s_{jt+1}|s_{jt}, a_{jt})$ for $j \neq i$.
 6. Now we are at s_{t+1} .
 7. Draw a new own-action using $\hat{d}(\cdot), \hat{a}(\cdot), \hat{e}(\cdot)$.
 8. Repeat from 2.
- ▶ Note: also have to record the expected value of the shocks, ϵ , at every step.
- ▶ For T1EV, $E(\epsilon_a|a_t = a) = \lambda - \log(P(a))$

Step Two: Inversion

- ▶ Simulate R times to obtain simulated choice-specific value function:
 $\tilde{v}_a(s_{it}, s_{-i}; \theta) = \frac{1}{R} \sum_{r=1}^R v_a^r(s_{it}, s_{-i}; \theta).$
- ▶ So the same thing with entry and exit choices...
- ▶ Now we can construct minimum distance estimators, just like in the single agent case:

$$\hat{\theta} = \arg \min_{\theta} \left\| \left(\hat{v}_a(s_{it}, s_{-i}) - \hat{v}_0(s_{it}, s_{-i}) \right) - \left(\tilde{v}_a(s_{it}, s_{-i}; \theta) - \tilde{v}_0(s_{it}, s_{-i}; \theta) \right) \right\|$$

- ▶ Where $\left(\hat{v}_a(x_t) - \hat{v}_0(x_t) \right)$ is the inversion of the 1st step CCPs.

Step Two: Objective Function

- ▶ Different papers have used different objective functions.
- ▶ Pakes, Ostrovsky and Berry construct moments that look like:

$$\frac{1}{S} \sum_s \hat{d}(s_{it}, s_{-it}) - \tilde{d}(s_{it}, s_{-i}; \theta) = 0$$

$$\frac{1}{S} \sum_s \hat{a}(s_{it}, s_{-it}) - \tilde{a}(s_{it}, s_{-i}; \theta) = 0$$

$$\frac{1}{S} \sum_s \hat{e}(s_t) - \tilde{e}(s_t; \theta) = 0$$

- ▶ Average the difference between the 1st step CCPs and the CCPs implied by simulated value functions.
- ▶ Averaging over states reduces the effect of 1st stage approximation error.
 - ▶ But might not want to use this with simulated \tilde{d} because simulation error enters non-linearly.
- ▶ Notice in this simple model, we'd only need these three moments to identify the parameters (c, κ, ϕ) .

Step Two

- ▶ BBL uses “perturbations” of the optimal strategy.
- ▶ Simulate using $\hat{d}(\cdot)$, $\hat{a}(\cdot)$, $\hat{e}(\cdot)$ (as above) to get simulated expected value functions, $\tilde{V}(s_{it}, s_{-i}; \theta)$ (not choice-specific here).
- ▶ Now, “perturb” firm i ’s policy function to $\tilde{d}_i(\cdot)$, $\tilde{a}_i(\cdot)$, $\tilde{e}_i(\cdot)$.
- ▶ Forward simulate again, using $\hat{d}(\cdot)$, $\hat{a}(\cdot)$, $\hat{e}(\cdot)$ for firms $-i$ and $\tilde{d}_i(\cdot)$, $\tilde{a}_i(\cdot)$, $\tilde{e}_i(\cdot)$ for firm i , get $\tilde{\tilde{V}}(s_{it}, s_{-i}; \theta)$
- ▶ If $\hat{d}(\cdot)$, $\hat{a}(\cdot)$, $\hat{e}(\cdot)$ are in equilibrium given parameters θ , it should be:

$$\tilde{V}(s_{it}, s_{-i}; \theta) \geq \tilde{\tilde{V}}(s_{it}, s_{-i}; \theta)$$

- ▶ Otherwise there is a profitable deviation from the observed strategies, and we can't have the right θ .

Step Two

- Define violations of these inequalities:

$$g(s_{it}, s_{-i}, \tilde{a}, \tilde{d}, \tilde{e}; \theta) = \left[\min \left\{ \tilde{V}(s_{it}, s_{-i}; \theta) - \tilde{V}(s_{it}, s_{-i}; \theta), 0 \right\} \right]^2$$

- Objective function:

$$\hat{\theta} = \arg \min_{\theta} \sum_s \sum_{(\tilde{a}, \tilde{d}, \tilde{e})} g(s_{it}, s_{-i}, \tilde{a}, \tilde{d}, \tilde{e}; \theta)$$

- Inner sum is over a set of different perturbations of the policy function.

Step Two

$$\hat{\theta} = \arg \min_{\theta} \sum_s \sum_{(\tilde{a}, \tilde{d}, \tilde{e})} g(s_{it}, s_{-i}, \tilde{a}, \tilde{d}, \tilde{e}; \theta)$$

- ▶ Notice that since $g = 0$ is possible we can have set identification.
- ▶ i.e. multiple values of θ that generate 0 violations of the inequalities.
- ▶ Adding more perturbations to the inner sum will, in general, make the identified set smaller.
- ▶ How to pick perturbation strategies?
 - ▶ No clear rules.
 - ▶ BBL pick a small set of strategies at random:
 - ▶ e.g. $\tilde{d} = \hat{d}(\cdot) + \eta$, where $\eta \sim N(0, \sigma)$

Step Two

$$\hat{\theta} = \arg \min_{\theta} \sum_s \sum_{(\tilde{a}, \tilde{d}, \tilde{e})} g(s_{it}, s_{-i}, \tilde{a}, \tilde{d}, \tilde{e}; \theta)$$

- ▶ Why would you use this BBL objective function rather than:

$$\hat{\theta} = \arg \min_{\theta} \left\| \left(\hat{v}_a(s_{it}, s_{-i}) - \hat{v}_0(s_{it}, s_{-i}) \right) - \left(\tilde{v}_a(s_{it}, s_{-i}; \theta) - \tilde{v}_0(s_{it}, s_{-i}; \theta) \right) \right\|$$

- ▶ The one good reason I can think of is that BBL can handle *continuous control variables* more easily.
- ▶ In this case, $\hat{a}(s_{it}, s_{-it})$ is the empirical distribution of $a_t \in [0, \bar{a}]$ (for example) conditional on s_t .
- ▶ Can use this to simulate value functions.
 - ▶ But we can't necessarily do Hotz-Milelr inversion to get $\hat{v}_a(s_{it}, s_{-i}) - \hat{v}_0(s_{it}, s_{-i})$.

Issues

- ▶ Standard errors?
 - ▶ Bootstrap.
- ▶ Bad functional form assumptions in the 'nonparametric' part can make it difficult to fit the simulated choice probabilities:

$$\frac{1}{S} \sum_s \hat{d}(s_{it}, s_{-it}) - \tilde{d}(s_{it}, s_{-i}; \theta) = 0$$

- ▶ Counterfactual analysis requires solving for equilibrium!
 - ▶ But we avoid this in estimation.
- ▶ Unobserved heterogeneity is difficult to deal with.
 - ▶ e.g. serially correlated unobservables.
 - ▶ See references in syllabus.

Next Time

- ▶ We'll look at an application:
- ▶ Sweeting (2013), "Dynamic Product Positioning in Differentiated Product Markets"