1 The choo and siow model, matching estimation and inverse optimal transport

The Choo-Siow model.

Heterosexual marriage market men's characteristics $x \in X$ women's $y \in Y$.

n_x men of type x and m_y women of type y

If x and y match, then they get respectively

$$\alpha_{xy} + \tau_{xy} + \varepsilon_y \text{ and } \gamma_{xy} - \tau_{xy} + \eta_x$$

where τ_{xy} is a transfer from y to x. If partners remain unmatched, they get respectively

$$\varepsilon_0 \text{ and } \eta_0.$$

Assume $(\varepsilon_y)_{y \in Y \cup \{0\}}$ is iid Gumbel and $(\eta_x)_{x \in X \cup \{0\}}$ is the same. In this logit framework, we have

$$\frac{\mu_{xy}}{n_x} = \frac{\exp(\alpha_{xy} + \tau_{xy})}{\sum_{y'} \cdots} = \exp(\alpha_{xy} + \tau_{xy} - u_x)$$

where $u_x = \log\left(1 + \sum_{y \in Y} \exp(\alpha_{xy} + \tau_{xy})\right)$.

$$\frac{\mu_{x0}}{n_x} = \frac{1}{\sum_{y'} \cdots} = \exp(-u_x).$$

Thus we have

$$\frac{\mu_{xy}}{n_x} = \frac{\mu_{x0}}{n_x} \exp(\alpha_{xy} + \tau_{xy})$$

that

$$\mu_{xy} = \mu_{x0} \exp(\alpha_{xy} + \tau_{xy}).$$

Similarly on the other side of the market we have

$$\mu_{xy} = \mu_{0y} \exp(\gamma_{xy} - \tau_{xy}).$$

Multiplying term by term to eliminate τ, we get

$$\begin{aligned} \mu_{xy}^2 &= \mu_{x0}\mu_{0y} \exp(\alpha_{xy} + \gamma_{xy}) \\ &= \mu_{x0}\mu_{0y} \exp(\Phi_{xy}) \end{aligned}$$

Thus

$$\mu_{xy} = \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right).$$

This is a matching function. μ_{x0} and μ_{0y} are determined by

$$\begin{aligned} n_x &= \sum_{y \in Y} \mu_{xy} + \mu_{x0} \\ m_y &= \sum_{x \in X} \mu_{xy} + \mu_{0y} \end{aligned}$$

that is

$$n_x = \sum_{y \in Y} \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right) + \mu_{x0}$$

$$m_y = \sum_{x \in X} \sqrt{\mu_{x0}\mu_{0y}} \exp\left(\frac{\Phi_{xy}}{2}\right) + \mu_{0y}$$

Note that $\mu_{x0} = n_x \exp\left(-u_x\right)$ and $\mu_{0y} = m_y \exp\left(-v_y\right)$ and therefore

$$n_x = \sum_{y \in Y} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) + n_x \exp\left(-u_x\right)$$

$$m_y = \sum_{x \in X} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) + m_y \exp\left(-v_y\right)$$

My question to you: of what minimization problem are these the first order conditions?

$$\min_{u_x, v_y} \sum_x n_x u_x + \sum_y m_y v_y + 2 \sum_{xy} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) + \sum_x n_x \exp\left(-u_x\right) + \sum_y m_y \exp\left(-v_y\right)$$

Note that at equilibrium, the quantity

$$2 \sum_{xy} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy} - u_x - v_y}{2}\right) + \sum_x n_x \exp\left(-u_x\right) + \sum_y m_y \exp\left(-v_y\right)$$

$$= 2 \sum_{xy} \mu_{xy} + \sum_x \mu_{x0} + \sum_y \mu_{0y}$$

is nothing else than the sum of men and women, that is $\sum_x n_x + \sum_y m_y$.

2 Back to a model with heterogeneity in preferences in finite population

Assume that one inidividual man $i \in I$ has observable characteristics $x_i \in X$
and one inidividual woman $j \in J$ has observable characteristics $y_j \in Y$

If i and j match, then their joint utility is

$$\phi_{ij} = \Phi_{x_i y_j} + \varepsilon_{iy_j} + \eta_{x_i j}$$

and if they remain unmatched, they get respectively

$$\varepsilon_{i0} \text{ and } \eta_{0j}.$$

Dual of the optimal matching problem

$$\min_{u_i, v_j} \quad \sum_i u_i + \sum_j v_j$$

$$s.t. \quad u_i + v_j \geq \phi_{ij}$$

$$u_i \geq \varepsilon_{i0}$$

$$v_j \geq \eta_{0j}$$

Solve this knowing $\phi_{ij} = \Phi_{x_i y_j} + \varepsilon_{iy_j} + \eta_{x_i j}$. We have

$$u_i + v_j \geq \Phi_{x_i y_j} + \varepsilon_{iy_j} + \eta_{x_i j}$$

For all i such that $x_i = x$ and j such that $y_j = y$, we have

$$u_i + v_j \geq \Phi_{xy} + \varepsilon_{iy} + \eta_{xj}$$

that is

$$u_i - \varepsilon_{iy} + v_j - \eta_{xj} \geq \Phi_{xy}$$

that

$$\min_{i:x_i=x} \{u_i - \varepsilon_{iy}\} + \min_{j:y_j=y} \{v_j - \eta_{xj}\} \geq \Phi_{xy}$$

thus if we set

$$U_{xy} = \min_{i:x_i=x} \{u_i - \varepsilon_{iy}\}$$

$$V_{xy} = \min_{j:y_j=y} \{v_j - \eta_{xj}\}$$

then the constraint rewrite

$$U_{xy} + V_{xy} \geq \Phi_{xy}$$

and

$$u_i \geq U_{xy} + \varepsilon_{iy}$$

$$v_j \geq V_{xy} + \eta_{xj}$$

The dual problem therefore equivalently rewrites as

$$\min_{u_i, v_j, U_{xy}, V_{xy}} \quad \sum_i u_i + \sum_j v_j$$

$$s.t. \quad U_{xy} + V_{xy} \geq \Phi_{xy}$$

$$u_i \geq U_{xy} + \varepsilon_{iy}$$

$$v_j \geq V_{xy} + \eta_{xj}$$

$$u_i \geq \varepsilon_{i0}$$

$$v_j \geq \eta_{0j}$$

But $u_i \geq U_{xy} + \varepsilon_{iy}$ and $u_i \geq \varepsilon_{i0}$ can be reexpressed as

$$u_i \geq \max_y \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\}$$

but at an optimum, this should hold as an equality. Thus the problem rewrites as

$$
\min_{U_{xy}, V_{xy}} \quad \sum_{i \in I} \max_y \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\} + \sum_{j \in J} \max_x \left\{ V_{xy} + \eta_{xj}, \eta_{0j} \right\}
$$
$$
s.t. \quad U_{xy} + V_{xy} = \Phi_{xy}
$$

This can be rewritten as

$$
\min_{U_{xy}, V_{xy}} \quad \sum_{x \in X} n_x \frac{1}{n_x} \sum_{i : x_i = x} \max_y \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\} + \sum_{y \in Y} m_y \frac{1}{m_y} \sum_{j : y_j = y} \max_x \left\{ V_{xy} + \eta_{xj}, \eta_{0j} \right\}
$$
$$
s.t. \quad U_{xy} + V_{xy} = \Phi_{xy}
$$

Let's now assume a large market.

$$
\frac{1}{n_x} \sum_{i : x_i = x} \max_y \left\{ U_{xy} + \varepsilon_{iy}, \varepsilon_{i0} \right\} \quad \rightarrow \quad E \left[\max_y \left\{ U_{xy} + \varepsilon_y, \varepsilon_0 \right\} \right]
$$
$$
\frac{1}{m_y} \sum_{j : y_j = y} \max_x \left\{ V_{xy} + \eta_{xj}, \eta_{0j} \right\} \quad \rightarrow \quad E \left[\max_x \left\{ V_{xy} + \eta_x, \eta_0 \right\} \right]
$$

Let's now make the logit assumption

$$
E \left[\max_y \left\{ U_{xy} + \varepsilon_y, \varepsilon_0 \right\} \right] = \log \left(1 + \sum_y \exp U_{xy} \right)
$$
$$
E \left[\max_x \left\{ V_{xy} + \eta_x, \eta_0 \right\} \right] = \log \left(1 + \sum_x \exp V_{xy} \right)
$$

Thus we have

$$
\min_{U_{xy}, V_{xy}} \quad \sum_{x \in X} n_x \log \left(1 + \sum_y \exp U_{xy} \right) + \sum_{y \in Y} m_y \log \left(1 + \sum_x \exp V_{xy} \right)
$$
$$
s.t. \quad U_{xy} + V_{xy} = \Phi_{xy}
$$

which rewrites into

$$
\min_{U_{xy}} \sum_{x \in X} n_x \log \left(1 + \sum_y \exp U_{xy} \right) + \sum_{y \in Y} m_y \log \left(1 + \sum_x \exp \left(\Phi_{xy} - U_{xy} \right) \right)
$$

By first order conditions, we have

$$
\mu_{xy} = n_x \frac{\exp U_{xy}}{1 + \sum_y \exp U_{xy}} = m_y \frac{\exp \Phi_{xy} - V_{xy}}{1 + \sum_x \exp \left(\Phi_{xy} - U_{xy} \right)}
$$

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and
$$\mu_{x0} = \frac{n_x}{1 + \sum_y \exp U_{xy}} \quad \text{and} \quad \mu_{0y} = \frac{m_y}{1 + \sum_x \exp\left(\Phi_{xy} - U_{xy}\right)}$$

therefore
$$\mu_{xy}^2 = \mu_{x0}\mu_{0y} \exp\left(\Phi_{xy}\right)$$

and we recover the Choo-Siow model.

Generalized to more general heterogeneity (outside of the logit case) https://arxiv.org/abs/2106.02371

3 Estimation of matching models

Idea = parameterize Φ_{xy} by a vector of parameters $\lambda \in R^K$. We have
$$\Phi_{xy}^\lambda = \sum_k \phi_{xy}^k \lambda_k$$

Example: $\phi_{xy}^1 = -|I_x - I_y|$ where I_x adn I_y are the partners' incomes
$\phi_{xy}^2 = -|E_x - E_y|$ where E_x adn E_y are the partners' educations
Recall that the social welfare is given by

$$W\left(\lambda\right) = \min_{u_x, v_y} \left\{ \begin{array}{c} \sum_x n_x u_x + \sum_y m_y v_y \\ +2\sum_{xy}\sqrt{n_x m_y}\exp\left(\frac{\Phi_{xy}^\lambda - u_x - v_y}{2}\right) + \sum_x n_x \exp\left(-u_x\right) + \sum_y m_y \exp\left(-v_y\right) \\ -\sum_x n_x - \sum_y m_y \end{array} \right\}$$

We have
$$\frac{\partial W}{\partial \lambda_k} = \sum_{xy}\sqrt{n_x m_y}\exp\left(\frac{\Phi_{xy}^\lambda - u_x - v_y}{2}\right)\phi_{xy}^k = \sum_{xy}\mu_{xy}^{u,v,\lambda}\phi_{xy}^k$$

Estimation strategy. Assume we observe $\hat{\mu}_{xy}$ and would like to estimate λ.
We are going to look for λ such that
$$\sum_{xy}\mu_{xy}^{u,v,\lambda}\phi_{xy}^k = \sum_{xy}\hat{\mu}_{xy}\phi_{xy}^k.$$

that is
$$\frac{\partial W}{\partial \lambda_k} = \sum_{xy}\hat{\mu}_{xy}\phi_{xy}^k$$

which means that
$$\frac{\partial}{\partial \lambda_k}\left\{ W\left(\lambda\right) - \sum_{xyk}\hat{\mu}_{xy}\phi_{xy}^k\lambda_k \right\} = 0$$

thus λ is found as

$$\min_{\lambda} \left\{ W\left(\lambda\right) - \sum_{xyk} \hat{\mu}_{xy} \phi_{xy}^k \lambda_k \right\}$$

that is

$$\min_{u_x,v_y,\lambda_k} \left\{ \begin{array}{c} \sum_x n_x u_x + \sum_y m_y v_y \\ +2 \sum_{xy} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy}^\lambda - u_x - v_y}{2} \right) + \sum_x n_x \exp\left(-u_x\right) + \sum_y m_y \exp\left(-v_y\right) \\ - \sum_{xyk} \hat{\mu}_{xy} \phi_{xy}^k \lambda_k \end{array} \right\}$$

Reformulate as a generalized linear model with log link function.

$$\max_{\theta} \left\{ \pi^\top X \theta - 1^\top \exp\left(X\theta\right) \right\}$$

Rewrite as

$$\max_{u_x,v_y,\lambda_k} \left\{ \begin{array}{c} -\sum_x n_x u_x - \sum_y m_y v_y + \sum_{xyk} \hat{\mu}_{xy} \phi_{xy}^k \lambda_k \\ -2 \sum_{xy} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy}^\lambda - u_x - v_y}{2} \right) - \sum_x n_x \exp\left(-u_x\right) - \sum_y m_y \exp\left(-v_y\right) \end{array} \right\}$$

that is

$$\max_{u_x,v_y,\lambda_k} \left\{ \begin{array}{c} -\sum_x \sum_y \hat{\mu}_{xy} u_x - \sum_x \hat{\mu}_{x0} u_x - \sum_y \sum_x \hat{\mu}_{xy} v_y - \sum_y \hat{\mu}_{0y} v_y + \sum_{xyk} \hat{\mu}_{xy} \phi_{xy}^k \lambda_k \\ -2 \sum_{xy} \sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy}^\lambda - u_x - v_y}{2} \right) - \sum_x n_x \exp\left(-u_x\right) - \sum_y m_y \exp\left(-v_y\right) \end{array} \right\}$$

that is

$$\max_{u_x,v_y,\lambda_k} \left\{ \begin{array}{c} +\sum_{xy} 2\sqrt{n_x m_y} \frac{\hat{\mu}_{xy}}{\sqrt{n_x m_y}} \left(\frac{\Phi_{xy}^\lambda - u_x - v_y}{2} \right) + \sum_x n_x \frac{\hat{\mu}_{x0}}{n_x} \left(-u_x\right) + \sum_y m_y \frac{\hat{\mu}_{0y}}{m_y} \left(-v_y\right) \\ -\sum_{xy} 2\sqrt{n_x m_y} \exp\left(\frac{\Phi_{xy}^\lambda - u_x - v_y}{2} \right) - \sum_x n_x \exp\left(-u_x\right) - \sum_y m_y \exp\left(-v_y\right) \end{array} \right\}$$

Can compute this by a GLM / Poisson of depednent variable

$$\frac{\hat{\mu}_{xy}}{\sqrt{n_x m_y}}, \frac{\hat{\mu}_{x0}}{n_x}, \frac{\hat{\mu}_{0y}}{m_y}$$

on regressors $\frac{1}{2}\phi_{xy}^k, -\frac{1}{2}\delta^x, -\frac{1}{2}\delta^y$
with weights $2\sqrt{n_x m_y}$, n_x, m_y.