Formulas

0.1 Multifactor Gaussian model

In a general d-factor *Gaussian* model, the T–bond price P follows

$$dP(t,T)/P(t,T) = r(t)dt - \sigma_P(t,T)^{\mathrm{T}} dW(t), \tag{0.1}$$

where $\sigma_P(t,T), dW(t)$ are d-dimensional column vectors and W is a $\mathbb{Q}$–Wiener. In terms of instantaneous forward rates, the above can equivalently be written in terms of instantaneous forward rates

$$df(t,T) = \sigma_f(t,T)^{\mathrm{T}}\sigma_P(t,T)dt + \sigma_f(t,T)^{\mathrm{T}} dW(t).$$

To prevent arbitrage, HJM condition must hold and we thus have based on the HJM drift restriction

$$df(t,T) = \sigma_f(t,T)^{\mathrm{T}} \int_t^T \sigma_f(t,u)du dt + \sigma_f(t,T)^{\mathrm{T}} dW(t).$$

To ensure that this model is Markovian, the following 'separability' condition must necessarily hold.

Theorem 0.1 (Forward rate volatility separability condition) *If the forward-rate volatility $\sigma_f(t,T)$ is separable such that*

$$\sigma_f(t,T) = g(t)h(T),$$

where g is a $d \times d$ deterministic matrix-valued function and h is a d–dimensional vector-valued function, then f is Markovian and

$$f(t,T) = f(0,T) + \Omega(t,T) + h(T)^{\mathrm{T}} z(t),$$

where

$$\begin{aligned} dz(t) &= g(t)^{\mathrm{T}} dW(t), \ \textit{given } z(0) = 0, \\ \Omega(t,T) &= h(T)^{\mathrm{T}} \int_0^t g(s)^{\mathrm{T}} g(s) \int_s^T h(u)duds. \end{aligned}$$

In particular we also have

$$r(t) = f(0,t) + \Omega(t,t) + h(t)^{\mathrm{T}} z(t).$$

Note that with this separability condition in effect, the HJM drift restriction implies

$$\sigma_P(t,T) = \int_t^T \sigma_f(t,u)du = g(t) \int_t^T h(u)du. \tag{0.2}$$

To formulate a mathematically tractable and economically meaningful model, it is best to define a particular functional form of mean reversion. One useful choice is to set

$$H(t) = \mathrm{diag}(h(t)).$$

If elements of h are non-zero, then H is invertible and we can define a 'mean-reversion' matrix

$$\varkappa(t) = -\frac{dH(t)}{dt} H(t)^{-1}. \tag{0.3}$$

It is also very useful to formulate the model in terms of state variables x such that

$$\begin{aligned} dx(t) &= (y(t)\mathbf{1} - \varkappa(t)x(t))\,dt + (g(t)H(t))^{\mathrm{T}} dW(t) \\ dy(t)/dt &= H(t)g(t)^{\mathrm{T}} g(t)H(t) - \varkappa(t)y(t) - y(t)\varkappa(t). \end{aligned} \tag{0.4}$$

Here x is a d-valued vector process with $x(0) = \mathbf{0}$ and $y(0)$ is a $d \times d$ matrix-valued deterministic function such that $y(0) = \mathbf{0}$. The differential equations for can be solved for x, y and this reads

$$\begin{aligned} x(t) &= H(t) \int\limits_0^t g(s)^{\mathrm{T}} g(s) \left(\int\limits_s^T h(u)du \right) ds + H(t)z(t) \\ y(t) &= H(t) \left(\int\limits_0^t g(u)^{\mathrm{T}} g(u)du \right) H(t). \end{aligned} \tag{0.5}$$

This formulation yields

$$f(t,T) = f(0,T) + M(t,T)^{\mathrm{T}} \left(x(t) + y(t) \int\limits_t^T M(t,u)du \right)$$

with

$$M(t,T) = H(T)H(t)^{-1}\mathbf{1}.$$

Notice that this results into a convenient representation of r in terms of the state variables x as

$$r(t) = f(t,t) = f(0,t) + \sum_{i=1}^{d} x_i(t).$$

Furthermore, letting

$$G(t,T) = \int\limits_t^T M(t,u)du,$$

we can define a useful bond reconstitution formula.

Definition 0.1 (Bond reconstitution formula) *Given the above settings in place, the bond price can be computed as*

$$P(t,T) \triangleq P(t,T,x) = \frac{P(0,T)}{P(0,t)} \exp\left(-G(t,T)^{\mathrm{T}} x(t) - \frac{1}{2} G(t,T)^{\mathrm{T}} y(t) G(t,T) \right).$$

0.2 Two factor Gaussian model

Of a particular interest is a two factor model with *constant* coefficients. Such a model will be fully determined by a mean reversion speeds $\varkappa_1, \varkappa_2$ and a diffusion matrix $\sigma(t) = \sigma$. Let us define g and h as

$$\begin{aligned} g(t) &= \begin{pmatrix} \sigma_{11}(t)e^{\int_0^t \varkappa_1(u)du} & \sigma_{12}(t)e^{\int_0^t \varkappa_2(u)du} \\ \sigma_{21}(t)e^{\int_0^t \varkappa_1(u)du} & \sigma_{22}(t)e^{\int_0^t \varkappa_2(u)du} \end{pmatrix} \equiv \begin{pmatrix} \sigma_{11}e^{\varkappa_1 t} & \sigma_{12}e^{\varkappa_2 t} \\ \sigma_{21}e^{\varkappa_1 t} & \sigma_{22}e^{\varkappa_2 t} \end{pmatrix} \\ h(t) &= \begin{pmatrix} e^{-\int_0^t \varkappa_1(u)du} \\ e^{-\int_0^t \varkappa_2(u)du} \end{pmatrix} \equiv \begin{pmatrix} e^{-\varkappa_1 t} \\ e^{-\varkappa_2 t} \end{pmatrix}. \end{aligned}$$

Having a look at the diffusion coefficient (matrix) of x in (0.4) gives

$$g(t)\mathrm{diag}(h(t)) = g(t)H(t) = \left(\begin{array}{cc} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{array}\right) = \sigma(t) \equiv \sigma.$$

Furthermore following (0.3) we also get a mean reversion matrix

$$\varkappa(t) = \varkappa = \left(\begin{array}{cc} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{array}\right).$$

Under this setup, the state variables $x = (x_1, x_2)^{\mathrm{T}}$ in (0.4) follow

$$\begin{aligned} \left(\begin{array}{c} dx_1(t) \\ dx_2(t) \end{array}\right) &= \left(\begin{array}{c} y_{11}(t) + y_{12}(t) - \varkappa_1 x_1(t) \\ y_{21}(t) + y_{22}(t) - \varkappa_2 x_2(t) \end{array}\right) dt + \sigma^{\mathrm{T}} dW(t) \\ &= \left(\begin{array}{c} v_1(t) - \varkappa_1 x_1(t) \\ v_2(t) - \varkappa_2 x_2(t) \end{array}\right) dt + \left(\begin{array}{cc} \sigma_{11} & \sigma_{21} \\ \sigma_{12} & \sigma_{22} \end{array}\right) \left(\begin{array}{c} dW_1(t) \\ dW_2(t) \end{array}\right), \end{aligned}$$

where $v_1(t) = y_{11}(t) + y_{12}(t), v_2(t) = y_{21}(t) + y_{22}(t)$. It is convenient to write the above as two differentials

$$\begin{aligned} dx_1(t) &= (v_1(t) - \varkappa_1 x_1(t))\, dt + \sigma_{11} dW_1(t) + \sigma_{21} dW_2(t) \\ dx_2(t) &= (v_2(t) - \varkappa_2 x_2(t))\, dt + \sigma_{12} dW_1(t) + \sigma_{22} dW_2(t). \end{aligned}$$

To efficiently use $y(t)$ it is useful to borrow from (0.5) and analytically evaluate the integral

$$\bar{g}(t) = \int_0^t g(u)^{\mathrm{T}} g(u) du = \left(\begin{array}{cc} \frac{(e^{2\varkappa_1 t}-1)(\sigma_{11}^2+\sigma_{21}^2)}{2\varkappa_1} & \frac{(e^{(\varkappa_1+\varkappa_2)t}-1)(\sigma_{11}\sigma_{12}+\sigma_{21}\sigma_{22})}{\varkappa_1+\varkappa_2} \\ \frac{(e^{(\varkappa_1+\varkappa_2)t}-1)(\sigma_{11}\sigma_{12}+\sigma_{21}\sigma_{22})}{\varkappa_1+\varkappa_2} & \frac{(e^{2\varkappa_2 t}-1)(\sigma_{12}^2+\sigma_{22}^2)}{2\varkappa_2} \end{array}\right).$$

Then, we can calculate inexpensive matrix multiplication $y(t) = H(t)\bar{g}(t)H(t)$.

It is also useful to analytically compute M and G which gives

$$M(t,T) = \left(\begin{array}{c} e^{-\varkappa_1(T-t)} \\ e^{-\varkappa_2(T-t)} \end{array}\right), G(t,T) = \left(\begin{array}{c} \frac{1-e^{-\varkappa_1(T-t)}}{\varkappa_1} \\ \frac{1-e^{-\varkappa_2(T-t)}}{\varkappa_2} \end{array}\right).$$

For practical calculations, we often prefer to use $\mathbb{Q}_T$–measure such that

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)du} V(x(T),T) \middle| \mathcal{F}(t) \right] = P(t,T)\mathbb{E}^{\mathbb{Q}_T}\left[V(x(T),T) \middle| \mathcal{F}(t) \right].$$

To do this, we need to use a T–bond $P(\cdot,T)$ as a numeraire asset. A bond price has the dynamics (0.1) and σ_P is necessary. We can compute σ_P analytically and it gives

$$\sigma_P(t,T) = \left(\begin{array}{c} \frac{1-e^{-\varkappa_1(T-t)}}{\varkappa_1}\sigma_{11} + \frac{1-e^{-\varkappa_2(T-t)}}{\varkappa_2}\sigma_{12} \\ \frac{1-e^{-\varkappa_1(T-t)}}{\varkappa_1}\sigma_{21} + \frac{1-e^{-\varkappa_2(T-t)}}{\varkappa_2}\sigma_{22} \end{array}\right) = \left(\begin{array}{c} G_1(t,T)\sigma_{11} + G_2(t,T)\sigma_{12} \\ G_1(t,T)\sigma_{21} + G_2(t,T)\sigma_{22} \end{array}\right).$$

Thus, we have

$$\begin{aligned} dP(t,T)/P(t,T) &= r(t)dt - \sigma_P(t,T)^{\mathrm{T}} dW(t) \\ &= r(t)dt - \sigma_{P,1}(t,T)dW_1(t) - \sigma_{P,2}(t,T)dW_2(t) \\ &= r(t)dt - (G_1(t,T)\sigma_{11} + G_2(t,T)\sigma_{12})\, dW_1(t) - (G_1(t,T)\sigma_{21} + G_2(t,T)\sigma_{22})\, dW_2(t). \end{aligned}$$

Knowing the diffusion term of the Numeraire P, we can use Girsanov to switch to $\mathbb{Q}$ to $\mathbb{Q}_T$, i.e.

$$dW^{\mathbb{Q}_T}(t) = dW(t) + \sigma_P(t,T)dt.$$

After substituting for $dW(t)$, this means for dx that

$$\begin{aligned}
dx_1(t) &= \left(v_1^T(t) - \varkappa_1 x_1(t)\right) dt + \sigma_{11} dW_1^{\mathbb{Q}_T}(t) + \sigma_{21} dW_2^{\mathbb{Q}_T}(t) \\
dx_2(t) &= \left(v_2^T(t) - \varkappa_2 x_2(t)\right) dt + \sigma_{12} dW_1^{\mathbb{Q}_T}(t) + \sigma_{22} dW_2^{\mathbb{Q}_T}(t),
\end{aligned}$$

where

$$\begin{aligned}
v_1^T(t) &= v_1(t) - \sigma_{11}\sigma_{P,1}(t,T) - \sigma_{21}\sigma_{P,2}(t,T) \\
v_2^T(t) &= v_2(t) - \sigma_{12}\sigma_{P,1}(t,T) - \sigma_{22}\sigma_{P,2}(t,T).
\end{aligned}$$

Since option pricing is based on the (joint) distribution of the state variables x_1, x_2, under $\mathbb{Q}_T$, we note that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_T}[x_i(T)] &= \int_0^T e^{-(T-u)\varkappa_i} v_i^T(u) du = 0 \\
\mathbb{V}^{\mathbb{Q}_T}[x_i(T)] &= \int_0^T e^{-2(T-u)\varkappa_i} (\sigma_{1i}^2 + \sigma_{2i}^2) du = (\sigma_{1i}^2 + \sigma_{2i}^2) \frac{1 - e^{-2\varkappa_i T}}{2\varkappa_i} \\
\mathrm{Cov}^{\mathbb{Q}_T}[x_1(T), x_2(T)] &= \int_0^T e^{-(T-u)(\varkappa_1+\varkappa_2)} (\sigma_{11}\sigma_{12} + \sigma_{21}\sigma_{22}) du = (\sigma_{11}\sigma_{12} + \sigma_{21}\sigma_{22}) \frac{1 - e^{-(\varkappa_1+\varkappa_2)T}}{\varkappa_1 + \varkappa_2}.
\end{aligned}$$

Since x_1, x_2 are jointly normal, based on the properties of conditional normal variables we can obtain the x_2–conditional moments

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_T}\left[x_1(T) \middle| x_2(T) = x_2\right] &= \mathbb{E}^{\mathbb{Q}_T}\left[x_1(T)\right] + \frac{\mathrm{Cov}^{\mathbb{Q}_T}[x_1(T), x_2(T)]}{\mathbb{V}^{\mathbb{Q}_T}[x_2(T)]} \left(x_2 - \mathbb{E}^{\mathbb{Q}_T}\left[x_2(T)\right]\right) \\
&= \frac{\mathrm{Cov}^{\mathbb{Q}_T}[x_1(T), x_2(T)]}{\mathbb{V}^{\mathbb{Q}_T}[x_2(T)]} x_2 = \mu_1(T, x_2) \\
\mathbb{V}^{\mathbb{Q}_T}\left[x_1(T) \middle| x_2(T) = x_2\right] &= \mathbb{V}^{\mathbb{Q}_T}[x_1(T)] - \frac{\left(\mathrm{Cov}^{\mathbb{Q}_T}[x_1(T), x_2(T)]\right)^2}{\mathbb{V}^{\mathbb{Q}_T}[x_2(T)]} = s_1^2(T, x_2)
\end{aligned}$$

0.3 Two-dimensional Jamshidian decomposition

Overall, a T-expiry s–bond price V under the measure $\mathbb{Q}_T$ has the following pricing equation

$$V(0) = P(0,T)\mathbb{E}^{\mathbb{Q}_T}\left[V(T, x_1(T), x_2(T))\right] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} V(T, x_1, x_2) f_{x_1 x_2}(x_1, x_2) dx_1 \right) dx_2.$$

This isn't a very nice formula as it involves a two-dimensional integral whose calculation isn't computationally too friendly.