Monitoring Forecasting Systems – Revisit Trigg's Tracking Signal

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Abstract: Time series forecast monitoring methods, such as the Brown and Trigg tracking signal, have been widely used to ensure that the underlying forecasting systems remain in control. After Brown's initiation in 1959, Trigg (1964) improved the tracking signal. First, if the forecasting system starts to give exceptionally accurate forecasts, the tracking signal will not continue to degrade. Second, Trigg claimed the new tracking signal no longer needed resetting in order to avoid a false warning signal. However, in this study we found that Trigg's tracking signal may still have these shortcomings in some business cases. We propose a new tracking signal – Complementary Tracking Signal (CTS) and show that this new measure overcomes the shortcomings of Trigg's tracking signal. Secondarily, we prove that if the current absolute value of the relative error is less than the previous absolute value of the CTS, then the CTS is improving through time.

Key words: **Forecasting; Decision support systems; Time series**

Introduction

Time series monitoring methods, such as the Brown and Trigg's tracking signal, have been widely used in monitoring forecasting systems to ensure that the underlying systems remain in control. Brown (1959, 1962) developed a method to monitor forecasting systems. Brown's method is to compute a "tracking signal" which is defined as the simple cumulative sum of error (CUSUM) divided by a simple smoothed Mean Absolute Deviation. Trigg (1964) proposed an improved method that is defined as the simple smoothed error divided by a simple smoothed Mean Absolute Deviation. Both methods are widely used in business practice today.

Since the initiative of tracking signal by Brown and its early development by Trigg, much additional research has emerged to assess them and improve their implementation. In his initial specification, Trigg did not exclude separate smoothing parameters for the numerator and the denominator, but was in favor of using the same parameter. Golder and Settle (1976) and Gardner (1983) performed simulation studies by setting both of the time series monitoring method's smoothing parameters to be the same as the smoothing parameter used in the simple exponential smoothing method generating forecast errors. McKenzie (1978) and Gardner (1985) allow the time series monitoring method parameters to differ from the forecast smoothing parameter, but with the numerator and denominator smoothing parameters being equal. McClain (1988) finds that the time series monitoring method generally performs better if both parameters vary and recommends low values for the denominator smoothing parameter.

Note that both methods by Brown and Trigg were originally designed for forecasting systems which use a simple exponential smoothing algorithm as the forecasting method. That means these methods use one-step-ahead forecast error from simple exponential smoothing as the forecast error needed in the formulation of the tracking signal. Since the proposals of the tracking signal by Brown (1959, 1962) and Trigg (1964), automatic forecasting systems have evolved in two aspects: 1) advances in forecast methods --many sophisticated statistical methods are used based on some predetermined model selection criteria; 2) forecast error is no longer necessarily the difference between the actual and one-step-ahead forecast. The specific forecast lead period depends on business relevancy. Therefore, there is need to revisit tracking signal measures in today's forecasting systems.

Methods for monitoring forecasting systems

To facilitate discussion of these methods, we introduce following notations:

t – Current Time, $t = 1,2,\cdots$

y_t – Actual value at time t

f_{t-k} – k-period-ahead forecast value based on data up to time t-k

$e_t = y_t - f_{t-k}$ – Forecast error for period t

e_0 – Initial value of e_t

α – Smooth parameter

$\tilde{e}_t = \alpha e_t + (1-\alpha)\tilde{e}_{t-1}$ – Smoothed forecast error at time t

M_0 – Initial value of Mean Absolute Deviation

$\widetilde{M}_t = \alpha|e_t| + (1-\alpha)\widetilde{M}_{t-1}$ – Smoothed Mean Absolute Deviation at time t

For notation economy, we use BTS and TTS for Brown's and Trigg's tracking signal respectively. Thus,

$$BTS_t = \frac{\sum_1^t e_t}{\widetilde{M}_t} = \frac{\sum_1^t e_t}{\alpha|e_t| + (1-\alpha)\widetilde{M}_{t-1}} \tag{1}$$

$$TTS_t = \frac{\tilde{e}_t}{\widetilde{M}_t} = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha|e_t| + (1-\alpha)\widetilde{M}_{t-1}} \tag{2}$$

Proposed by Brown, BTS uses a cumulative sum of error (CUSUM) from the beginning of the series. Though it has merit of quickly detecting a structural break of demand pattern based on the one-step-ahead simple exponential smoothing system, Trigg (1964) pointed out there are two disadvantages: 1) unless resetting the CUSUM back to zero is done following a time series pattern change, BTS may continue to broadcast false alarm signals because the CUSUM has long memory for the large forecast error; 2) if the forecasting system starts to give exceptionally accurate forecasts BTS may still go out of limits. To improve BTS's performance, Trigg (1964) propose TTS. Unlike BTS, a distinguished feature of TTS is that TTS is bounded at [-1, 1]. Trigg (1964) gives out the asymptotically approximated control limits of TTS assuming a sufficiently small α. Though TTS has some merits over BTS it appears that it has the following disadvantages.

Case Study 1

TTS may remain out of limit even if a system comes back to normal (un-bias status). A primary impetus for the development of TTS is to avoid frequent intervention after a time series pattern change. However, we observed that in many cases, the intervention of resetting CUSUM back to zero should not be avoided.

Chart 1 shows the forecast accuracy of a forecasting system and the performances of TTS with and without resetting the error back to zero after detecting a level shift. The shipment data came from two rectangular distributions with a mean shift (See Table 1 in Appendix for computation details). For simplicity, we assume a three period simple moving average forecasting system. All TTSs are calculated with $\alpha = 0.1$. To assess the performance of TTSs with resetting, we start a new Reset-TTS if TTS >0.51 while continuing to monitoring the Reset-TTSs which were started in previous periods. Note that Error Measure, the first vertical axis is set for Forecast Error and TTS Measure; the second vertical axis is set for all TTSs. It is clear that at period 6, there is a spark of forecast error and TTS crosses the upper limit at period 6. That indicates that TTS appropriately captures the signal of level shift. However, the following points warrant our attention:

- Due to the adaptive nature of the exponential smoothing forecasting, the forecast error starts to decrease from period 7, and starting from period 9 the forecasting system is fully adaptive to the demand pattern shift with the forecast error returning to normal.
- From period 6, TTS stays in the warning zone (above upper limit) – it never returns to inside the limits.
- Reset-TTS1 is below TTS but above the upper limit.[1]
- Reset-TTS 2 is below Reset-TTS1 with only the first point above the upper limit.
- All points of Reset-TTS3 are identified inside the safe zone.

It seems that in this type of cases, once TTS has gone out of limit it will not necessarily return within limits even though the forecasting system comes back in control. Consequently, continuous resetting of TTS is necessary if future false alarms are to be avoided. Unfortunately, the prominent critique of BTS originally made by Trigg becomes true to TTS as well.

[1] Note that we exclude the first point for all Reset-TTS because of insufficient data.

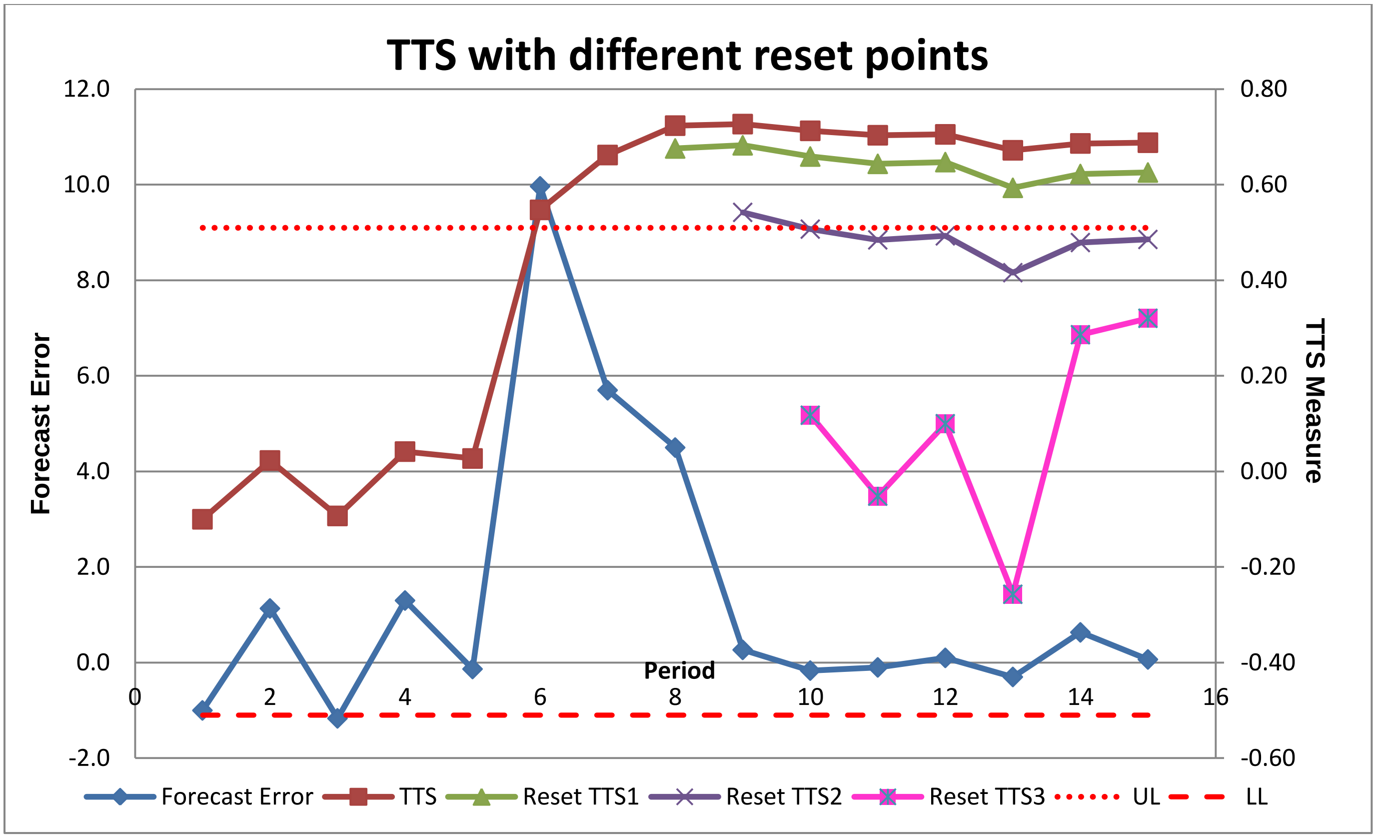

Chart 1: TTSs with different reset points in the case of level shift

Case study 2

In business practice, a biased forecasting system with a significantly smaller forecast error, or particularly, a forecasting system with slight over-forecast errors, may be preferred to an unbiased system with larger forecast errors. It is simply because the system may minimize risk of lost sale without too much over stock.

Chart 2 compares performances of TTS and Reset-TTSs for a forecasting system transitioned from an unbiased with large variance into a preferred bias system with slight over-forecast (See Table 2 in Appendix for computation details). From period 6, the system is turned to the preferred biased system; and consequently, the absolute forecast errors become minimal. Now let's examine the performance of TTS and Reset-TTSs. As expected, TTS crosses the alarm threshold, -0.51 at period 5; and unexpectedly it gets worse over time. On the other hand, all Reset-TTSs are worse than TTS and each Reset-TTS has tendency of getting worse. That clearly suggests that neither TTS nor Reset-TTS is appropriate for monitoring this system.

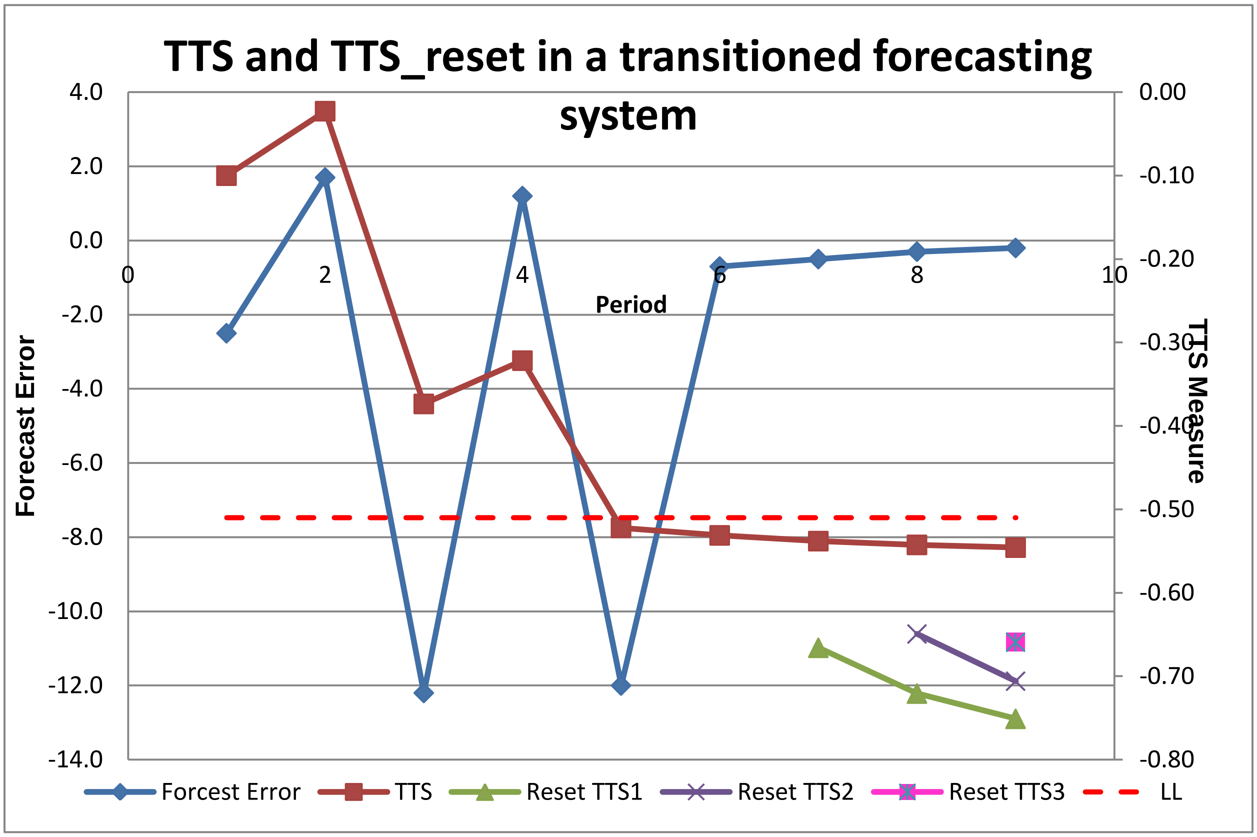

Chart 2: TTS and Reset-TTSs in a transitioned forecasting system

The disadvantages revealed in the two cases above may be attributed to TTS's tendency to accumulate tracking signals with the same error sign. That means if the current error has the same sign as the previous smoother errors, no matter how small the current error is, the current TTS is worse than the previous TTS. We state this unfavorable trait of TTS in proposition format.

Proposition 1.a Suppose that

(A.i) $e_t > 0$ and $\tilde{e}_{t-1} > 0$;

Then,

$TTS_t > TTS_{t-1}$.

Proposition 1.b Suppose that

(A.i) $e_t < 0$ and $\tilde{e}_{t-1} < 0$;

Then,

$TTS_t < TTS_{t-1}$.

It is worth noting that unlike BTS, in the case where a system begins to show a perfect forecast, TTS will not lead to infinity, however, it will stay the previous TTS level which will not show any improvement at all. In the perfect accuracy case $e_t = 0$,

$TTS_t = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha|e_t| + (1-\alpha)\tilde{M}_{t-1}} = \frac{(1-\alpha)\tilde{e}_{t-1}}{(1-\alpha)\tilde{M}_{t-1}} = \frac{\tilde{e}_{t-1}}{\tilde{M}_{t-1}} = TTS_{t-1}$. Surely, this is not desirable.

An new measure for monitoring forecasting systems

To overcome the shortcomings of TTS that we discussed before, we propose a complementary measure to TTS as:

$$CTS_t = \frac{\tilde{e}_t}{\tilde{y}_t} = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} \tag{3}$$

Unlike TTS, CTS has a good response to changes in the ratio of error to actual. We state this in the form of a proposition.

Proposition 2: Suppose that

(A.i) $y_t > 0,\ \forall t;$

(A.ii) $CTS_{t-1} = \left|\frac{\tilde{e}_{t-1}}{\tilde{y}_{t-1}}\right| = \theta > 0;$

(A.iii) $0 \le \left|\frac{e_t}{y_t}\right| < \theta;$

Then,

(C.i) $|CTS_t| < |CTS_{t-1}|$.

Proposition 2 can be interpreted as – if the current absolute value of the relative error is less than the previous absolute value of CTS, then CTS improves. Condition (A.i) is a natural assumption for demand. Recall the two examples we showed before, the absolute value of TTS continuously increases as the absolute relative error decreases. Apparently, Proposition 2 reveals an attractive feature of CTS. Condition (A.ii) is the absolute value of the previous CTS, and condition (A.iii) is the current absolute value of the relative error which the represents system's most recent performance. Conclusion (C.i) can be broken into two parts: (C.i.a) $CTS_t < CTS_{t-1}$; (C.i.b) $CTS_t > CTS_{t-1}$. Thus Proposition 2 can be further restated as following:

Proposition 2.a: suppose that

(A.i) $y_t > 0,\ \forall t;$

(A.ii.a) $\frac{\tilde{e}_{t-1}}{\tilde{y}_{t-1}} = \theta > 0;$

(A.iii.a) $0 \leq \frac{e_t}{y_t} < \theta$*; or*

(A.iii.b) $-\theta < \frac{e_t}{y_t} < 0.$

Then,

(C.i.a) $CTS_t < CTS_{t-1}$.

Proposition 2.b: suppose that

(A.i) $y_t > 0,\ \forall t;$

(A.ii.b) $\frac{\tilde{e}_{t-1}}{\tilde{y}_{t-1}} = -\theta < 0;$

(A.iii.a) $0 \leq \frac{e_t}{y_t} < \theta$*; or*

(A.iii.b) $-\theta < \frac{e_t}{y_t} < 0.$

Then,

(C.i.b) -$\theta < CTS_{t-1} < CTS_t < 0$.

It clear that Proposition 2.a represents the cases where CTS_t is a smaller positive number than CTS_{t-1}; in contrast, Proposition 2.b represents the cases where CTS_t is less negative than CTS_{t-1} . Now, we prove Proposition 2.a and 2.b.

In contrast to TTS, CTS does not have "same sign accumulation". The current CTS is determined by the current percentage error and previous smoothed percentage error. The current CTS will be reduced as long as the current percentage error is smaller than previous smoothed percentage error. This is a desirable feature of a measurement to be used to monitor forecasting systems. In the extreme cases where

$e_t = 0$, as we discussed before, $TTS_t = TTS_{t-1}$; on the other hand, applying Proposition 2, we have $|CTS_t| < |CTS_{t-1}|$. Clearly, CTS indicates the forecasting system improves with $e_t = 0$. It's interesting to point out that if Proposition 2 can be applied by k times and if k is large enough, CTS_t approaches zero.

We now apply CTS along with TTS to the previous two examples.

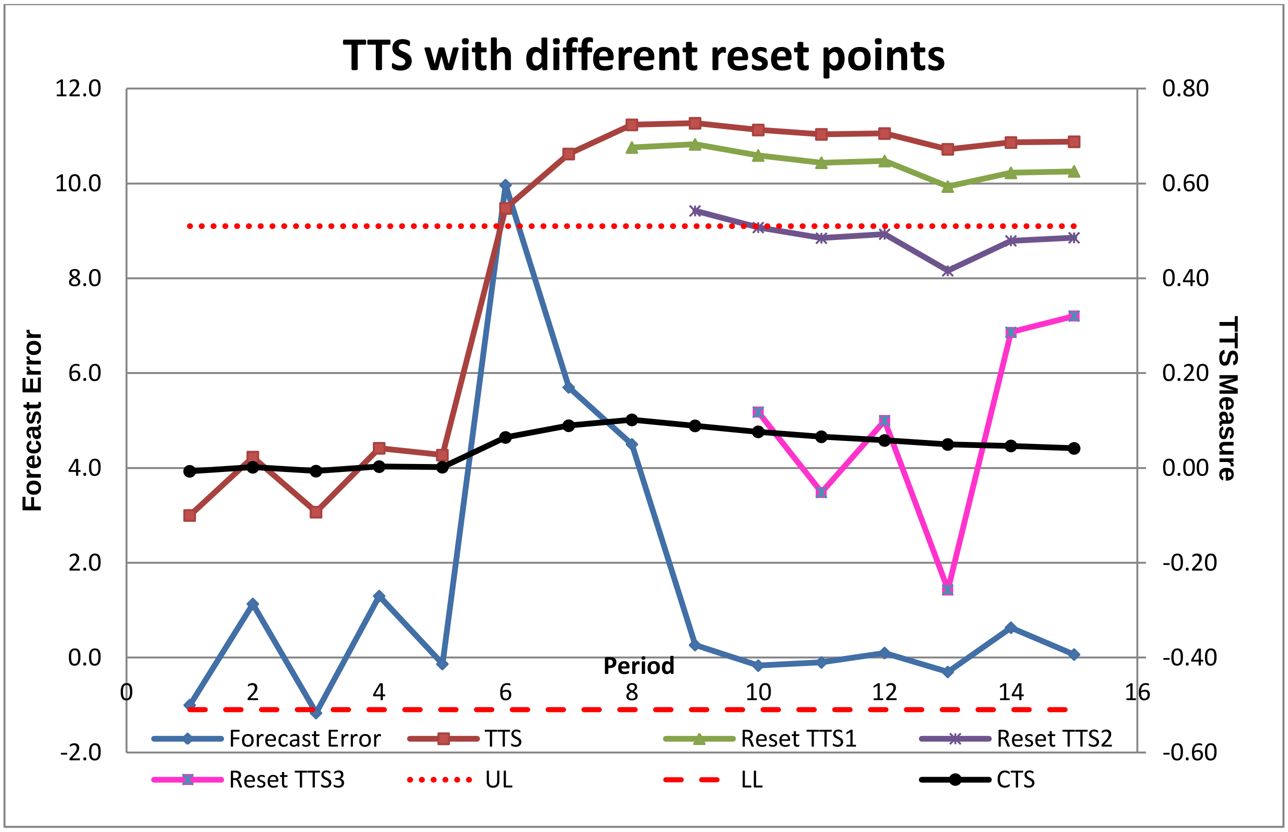

Chart 3: Comparison of CTS and TTS in the case of level shift

The trajectory of the forecast error reveals the fact that starting from period 9 the forecast error decreases around zero. That evidently suggests that the forecast system be fully adapted to the up-shifted demand. In contrast to TTS which still loiters well above the upper limit and releases a false warning signal after period 9, CTS does not affirm any caution indication. On the other hand, the continuous improvement of CTS (decline of CTS) intuitively implies a decrease of relative error. Note that a stable up-shift demand is the source to decrease relative error.

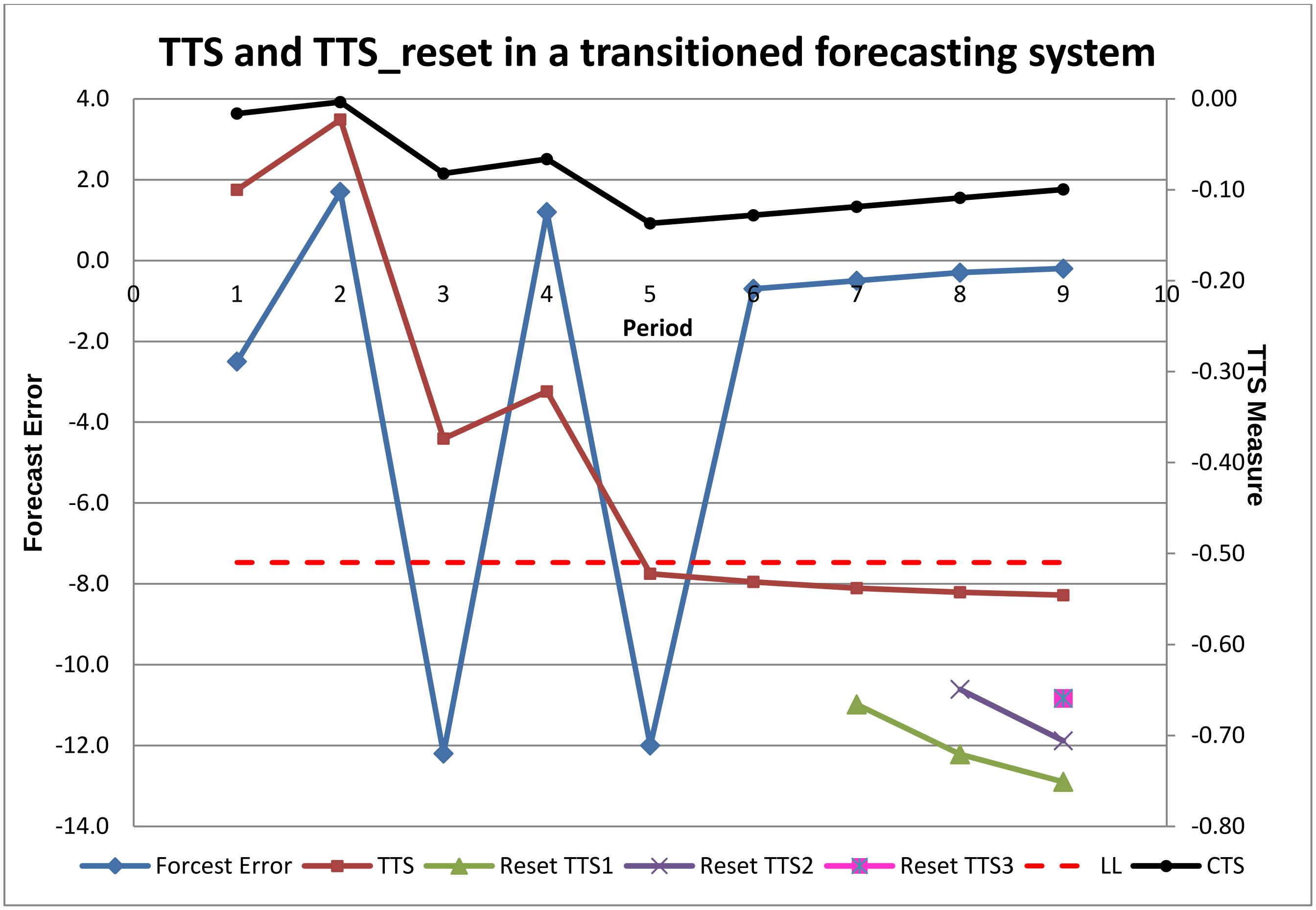

Chart 4: Comparison of CTS and TTS in a transitioned forecasting system

Chart 4 exhibits the different performances of TTS and CTS. It's interesting to point out that once the system changes at period 5, TTS and CTS go in opposite directions: TTS is getting worse and CTS is showing signs of improving. Apparently, no matter if there is reset or not, TTS provides false alarming signals. Meantime CTS releases the signal indicating the improvement of relative forecast error which is attributed to the process change.

Conclusions

In this study, we show that despite the effectiveness of TTS in issuing the first warning sign, its response can diverge further and further from the correct one over time. That may be attributed to "Tracking Signal Accumulation with same error sign". To surmount this shortcoming, we propose a new tracking signal measure using CTS and show that if a current absolute relative error is smaller than the previous smoothed CTS, the current CTS improves. We further suggest that using CTS along with TTS is an attractive approach to monitoring forecasting systems.

The benefits of applying CTS to a TTS based forecasting monitoring system include but are not limited to 1) removal of the false warning signals generated from a TTS system for groups of forecasting processes; 2) the concept is easily understandable because of its close relationship with the relative forecast error; 3) exhibiting dynamics of the relative forecast error. The avenue of future research may be i) explore the analytics of warning limits; ii) comparison study with other monitoring approaches in the lines of both empirical and large scale simulation.

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Appendix –Table 1: Case of level shift

Period	Shipment	Forecast	Forecast Error	TTS	Reset TTS1	Reset TTS2	Reset TTS3	Smthed Act	CTS
-2	17	-							
-1	14	-							
0	15.5	-							
1	14.5	15.5	-1.0	-0.10				14.5	-0.007
2	15.8	14.7	1.1	0.02				14.63	0.002
3	14.1	15.3	-1.2	-0.09				14.58	-0.007
4	16.1	14.8	1.3	0.04				14.73	0.003
5	15.2	15.3	-0.1	0.03				14.78	0.002
6	25.1	15.1	10.0	0.55				15.81	0.065
7	24.5	18.8	5.7	0.66	0.526			16.68	0.089
8	26.1	21.6	4.5	0.72	0.676	0.526		17.62	0.102
9	25.5	25.2	0.3	0.73	0.683	0.542	0.526	18.41	0.089
10	25.2	25.4	-0.2	0.71	0.659	0.507	0.118	19.09	0.076
11	25.5	25.6	-0.1	0.70	0.644	0.485	-0.051	19.73	0.066
12	25.5	25.4	0.1	0.71	0.647	0.493	0.100	20.31	0.058
13	25.1	25.4	-0.3	0.67	0.594	0.416	-0.257	20.79	0.050
14	26.0	25.4	0.6	0.69	0.622	0.479	0.286	21.31	0.047
15	25.6	25.5	0.1	0.69	0.626	0.486	0.321	21.74	0.041

Appendix --Table 2: TTS calculation: Case of transitioned forecasting system

Period	Shipment	Forecast	Forcest Error	TTS	Reset TTS1	Reset TTS2	Reset TTS3	Smthed Act	CTS
0									
1	15.5	18.0	-2.5	-0.10				15.5	-0.016
2	14.7	13.0	1.7	-0.02				15.42	-0.004
3	15.8	28.0	-12.2	-0.37				15.46	-0.082
4	15.2	14.0	1.2	-0.32				15.43	-0.066
5	16	28.0	-12.0	-0.52				15.49	-0.137
6	15.1	15.8	-0.7	-0.53	-0.53			15.45	-0.128
7	15.2	15.7	-0.5	-0.54	-0.67	-0.53		15.43	-0.119
8	15.3	15.6	-0.3	-0.54	-0.72	-0.65	-0.53	15.41	-0.109
9	14.9	15.1	-0.2	-0.55	-0.75	-0.71	-0.66	15.36	-0.100

Appendix -- Proof

Proof of Proposition 1.a:

Given (A.i), we have $TTS_t = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha|e_t| + (1-\alpha)\widetilde{M}_{t-1}} = \frac{\frac{\alpha}{(1-\alpha)}e_t + \tilde{e}_{t-1}}{\frac{\alpha}{(1-\alpha)}e_t + \widetilde{M}_{t-1}}$. It is trivial to show $\tilde{e}_{t-1} < \widetilde{M}_{t-1}$, or $\frac{\tilde{e}_{t-1}}{\widetilde{M}_{t-1}} < 1$. (A.i) also implies that $\frac{\alpha}{(1-\alpha)} e_t > 0$. Therefore, It is straightforward that $TTS_t = \frac{\frac{\alpha}{(1-\alpha)}e_t + \tilde{e}_{t-1}}{\frac{\alpha}{(1-\alpha)}e_t + \widetilde{M}_{t-1}} > \frac{\tilde{e}_{t-1}}{\widetilde{M}_{t-1}} = TTS_{t-1}$. It completes proof.

Proof of Proposition 1.b:

Given (A.i) of Proposition 2.b, immediately, we have $\alpha e_t + (1-\alpha)\tilde{e}_{t-1} < 0$ and $\alpha|e_t| + (1-\alpha)\widetilde{M}_{t-1} > 0$. Thus, there exist a , $\delta > 0$ and let $\delta = \left(-\frac{\widetilde{M}_{t-1}}{\tilde{e}_{t-1}} - 1\right) \cdot \alpha|e_t| > 0$, and

$$TTS_t = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha|e_t| + (1-\alpha)\widetilde{M}_{t-1}} < \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha|e_t| + \delta + (1-\alpha)\widetilde{M}_{t-1}} = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha|e_t| + \left(-\frac{\widetilde{M}_{t-1}}{\tilde{e}_{t-1}} - 1\right)\cdot\alpha|e_t| + (1-\alpha)\widetilde{M}_{t-1}}$$

$$= \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\left(-\frac{\widetilde{M}_{t-1}}{\tilde{e}_{t-1}}\right)\cdot\alpha|e_t| + (1-\alpha)\widetilde{M}_{t-1}} \tag{A.1}$$

To show right side of equation (A.1) equals to $\frac{\tilde{e}_{t-1}}{\widetilde{M}_{t-1}} = TTS_{t-1}$, we only show $\frac{\alpha e_t}{\left(-\frac{\widetilde{M}_{t-1}}{\tilde{e}_{t-1}}\right)\cdot\alpha|e_t|} = \frac{\tilde{e}_{t-1}}{\widetilde{M}_{t-1}}$.

Substituting $e_t = -|e_t|$ into it, we have the result immediately. It completes proof.

Proof of Proposition 2.a:

Now, we first show cases of (A.ii.a) and the inequality part of (A.iii.a). Given that, it implies there exist a positive number, $\delta > 0$, such that $\theta = \frac{e_t + \delta}{y_t}$, or

$$e_t = \theta y_t - \delta \tag{A.2}$$

Substituting (A.2) into equation (4), we have

$$CTS_t = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} = \frac{\alpha\cdot(\theta y_t - \delta) + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} < \frac{\alpha\theta y_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} \tag{A.3}$$

From (A.ii.a), we have $\frac{\tilde{e}_{t-1}}{\tilde{y}_{t-1}} = \theta = \frac{\theta y_t}{y_t}$. Let $\frac{y_t}{\tilde{y}_{t-1}} = \frac{\theta y_t}{\tilde{e}_{t-1}} = \mu$. Substituting $\theta y_t = \mu \tilde{e}_{t-1}$ and $y_t = \mu \tilde{y}_{t-1}$ into (A.3), we have $CTS_t < \frac{\alpha\theta y_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} = \frac{\alpha\mu\tilde{e}_{t-1} + (1-\alpha)\tilde{e}_{t-1}}{\alpha\mu\tilde{y}_{t-1} + (1-\alpha)\tilde{y}_{t-1}} = \frac{(\alpha\mu + 1 - \alpha)\tilde{e}_{t-1}}{(\alpha\mu + 1 - \alpha)\tilde{y}_{t-1}} = \frac{\tilde{e}_{t-1}}{\tilde{y}_{t-1}} = CTS_{t-1}$.

Next, it's simple to show the case of (A.ii.a) and the equality part of (A.iii.a). Since $e_t = 0$, we have $CTS_t = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} = \frac{(1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} < \frac{(1-\alpha)\tilde{e}_{t-1}}{(1-\alpha)\tilde{y}_{t-1}} = \frac{\tilde{e}_{t-1}}{\tilde{y}_{t-1}} = CTS_{t-1}$.

Thirdly, we show the cases that of (A.ii.a) and (A.iii.b). Multiplying both side of (A.iii.b) by -1, we get $\theta > \frac{-e_t}{y_t} = \frac{|e_t|}{y_t}$. This simply becomes (A.iii.a). It completes proof.

Proof of Proposition 2.b:

Now we first show the cases of (A.ii.b) and (A.iii.b). Given (A.iii.b), there exist a positive number, $\delta > 0$, such that $-\theta = \frac{e_t - \delta}{y_t}$, or

$$e_t = -\theta y_t + \delta \tag{A.4}$$

Substituting (A.4) into equation (3), we have

$$CTS_t = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} = \frac{\alpha \cdot (-\theta y_t + \delta) + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} \geq \frac{-\alpha\theta y_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} \tag{A.5}$$

From (A.ii.b), we have $\frac{\tilde{e}_{t-1}}{\tilde{y}_{t-1}} = -\theta = -\frac{\theta y_t}{y_t}$. Let $\frac{y_t}{\tilde{y}_{t-1}} = \frac{-\theta y_t}{\tilde{e}_{t-1}} = \mu$. Substituting $-\theta y_t = \mu \tilde{e}_{t-1}$ and $y_t = \mu \tilde{y}_{t-1}$ into (8), we have

$$CTS_t \geq \frac{-\alpha\theta y_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} = \frac{\alpha\mu\tilde{e}_{t-1} + (1-\alpha)\tilde{e}_{t-1}}{\alpha\mu\tilde{y}_{t-1} + (1-\alpha)\tilde{y}_{t-1}} = \frac{(\alpha\mu + 1 - \alpha)\tilde{e}_{t-1}}{(\alpha\mu + 1 - \alpha)\tilde{y}_{t-1}} = \frac{\tilde{e}_{t-1}}{\tilde{y}_{t-1}} = CTS_{t-1}.$$

Next, we show the cases of (A.ii.b) and the inequality part of (A.iii.a). Multiplying the inequality part of (A.iii.a) by -1, we have $-\theta < \frac{-e_t}{y_t} < 0$ which becomes (A.iii.b).

Finally, it's simple to show the case of (A.ii.b) and the equality part of (A.iii.a). Since $e_t = 0$, we have $CTS_t = \frac{\alpha e_t + (1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} = \frac{(1-\alpha)\tilde{e}_{t-1}}{\alpha y_t + (1-\alpha)\tilde{y}_{t-1}} > \frac{(1-\alpha)\tilde{e}_{t-1}}{(1-\alpha)\tilde{y}_{t-1}} = \frac{\tilde{e}_{t-1}}{\tilde{y}_{t-1}} = CTS_{t-1}$. This completes proof.