Functional Boxplots

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Functional Data

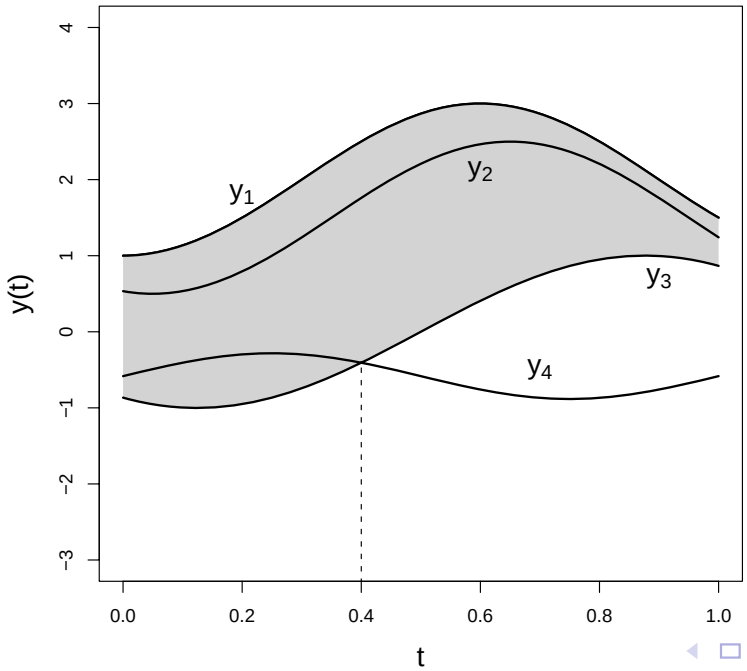
- The goal of this paper is
  - to develop functional boxplots for visualizing functional data.
- To do this, we need to
  - know the features of functional data,
  - generalize order statistics or ranks to the functional context.

Multivariate Ordering

- Basic ideas of depth in functional context
  - provides a method to order sample curves according to decreasing depth values,
  - $y_{[1]}$: the deepest (most central or median) curve,
  - $y_{[n]}$: the most outlying (least representative) curve,
  - $y_{[1]}, \ldots, y_{[n]}$: start from the center outwards.
- Usual order statistics: ordered from the smallest sample value to the largest.

Band Depth for Functional Data

- López-Pintado and Romo (2009) introduced the band depth (BD) concept through a graph-based approach.
- Grey area: band determined by two curves, y_1 and y_3.
- Completely contains the curve y_2, but only partly contains y_4.



Band Depth for Functional Data

- Population version of $BD^{(2)}$:

$$BD^{(2)}(y, P) = P\{G(y) \subset B(Y_1, Y_2)\}.$$

  - $G(y)$: graph of the curve y,
  - $B(Y_1, Y_2)$: band delimited by 2 random curves.

- The band could be delimited by more than 2 random curves,

$$BD_J(y, P) = \sum_{j=2}^{J} BD^{(j)}(y, P).$$

Sample Band Depth

- Population level: $BD^{(j)}(y, P)$ is a probability.
- Sample version of $BD^{(j)}(y, P)$

$$BD_n^{(j)}(y) = \binom{n}{j}^{-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} I\{G(y) \subseteq B(y_{i_1}, \ldots, y_{i_j})\},$$

  - $I\{\cdot\}$: the indicator function,
  - fraction of the bands completely containing the curve y.
- Sample BD: $BD_{n,J}(y) = \sum_{j=2}^{J} BD_n^{(j)}(y)$.

Modified Band Depth

- López-Pintado and Romo (2009) also proposed a more flexible definition, the modified band depth (MBD).

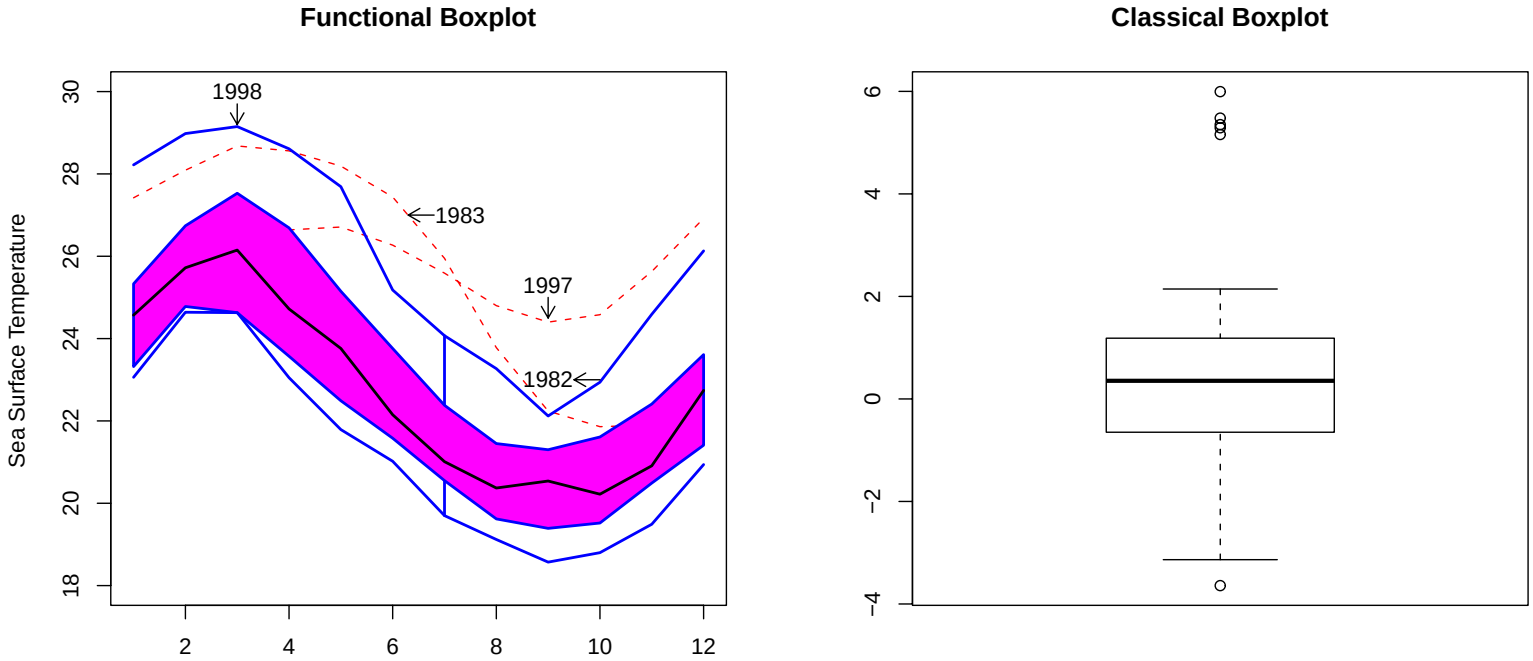
$$BD_n^{(j)}(y) = \binom{n}{j}^{-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} I\{G(y) \subseteq B(y_{i_1}, \ldots, y_{i_j})\},$$

$$MBD_n^{(j)}(y) = \binom{n}{j}^{-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} \lambda_r\{A(y; y_{i_1}, \ldots, y_{i_j})\}.$$

- $\lambda_r\{A(y; y_{i_1}, \ldots, y_{i_j})\}$ measures the proportion of time that a curve y is in the band.

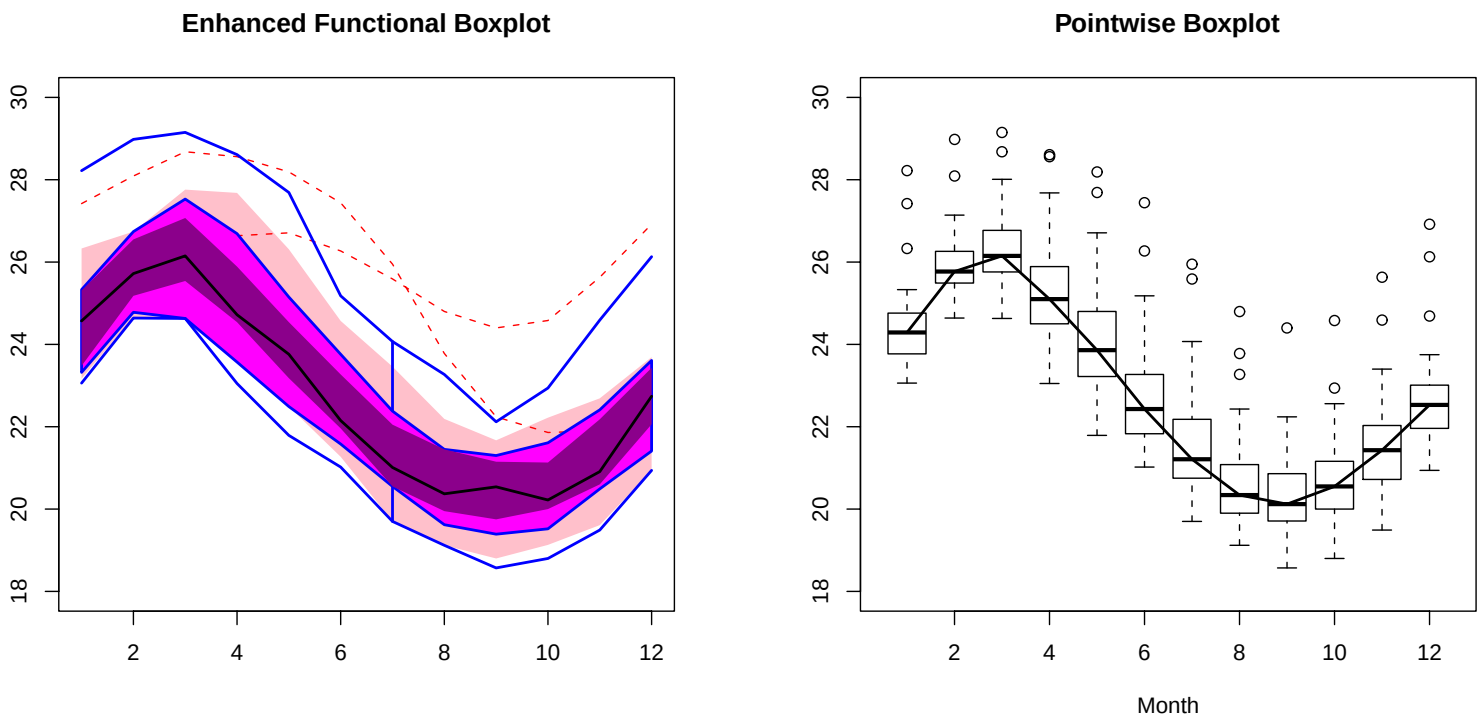
Functional Boxplots v.s. Classical Boxplots

- Comparing functional boxplots to classical boxplots.
- Functional boxplot: median, 50% central region, maximum non-outlying region, empirical outlier rule.



Enhanced Functional Boxplots

- The enhanced functional boxplot is a natural extension.
- Different from pointwise boxplots.

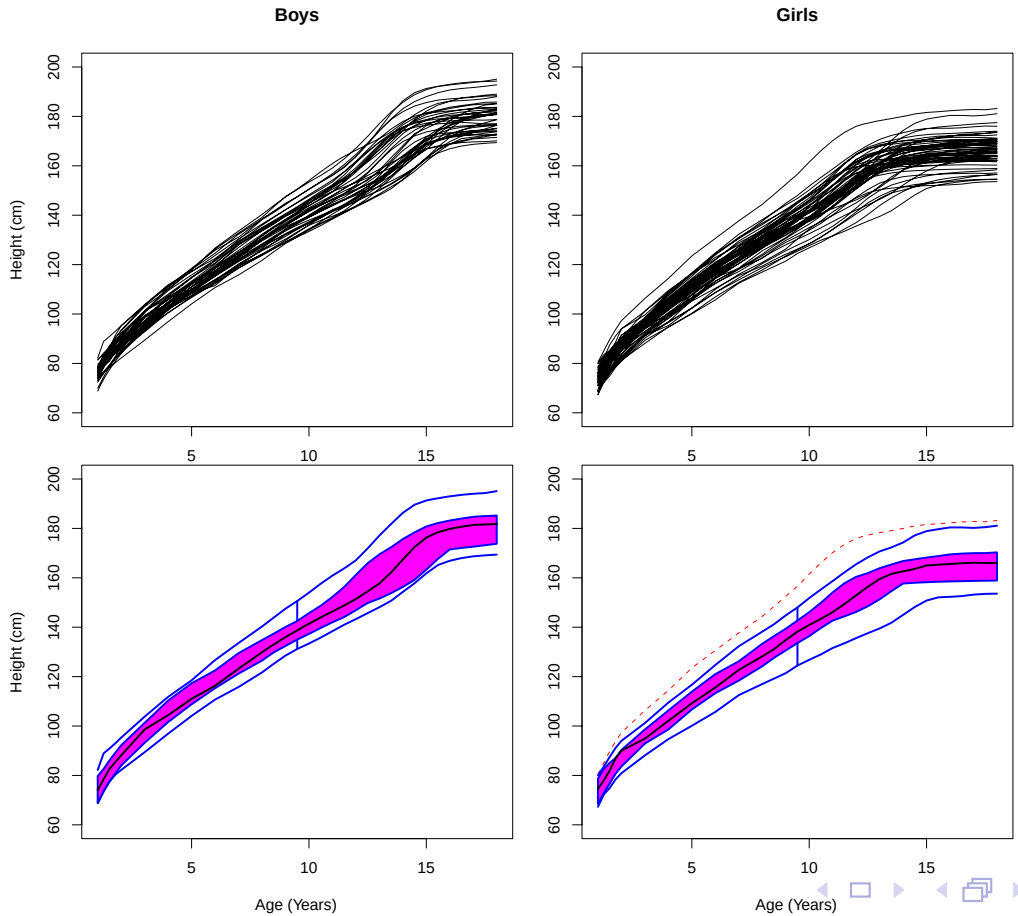


Functional Boxplot v.s. Functional Bagplot

- Functional boxplot:
  - uses MBD to order functional data,
  - visualizes functional data directly in the functional space,
  - a strong analog to the classical boxplot.
- Functional bagplot:
  - proposed by Hyndman and Shang (2010),
  - based on the first two robust principal component scores,
  - applies the bivariate bagplot (Rousseeuw et al., 1999),
  - maps the features of the bagplot into the functional space.

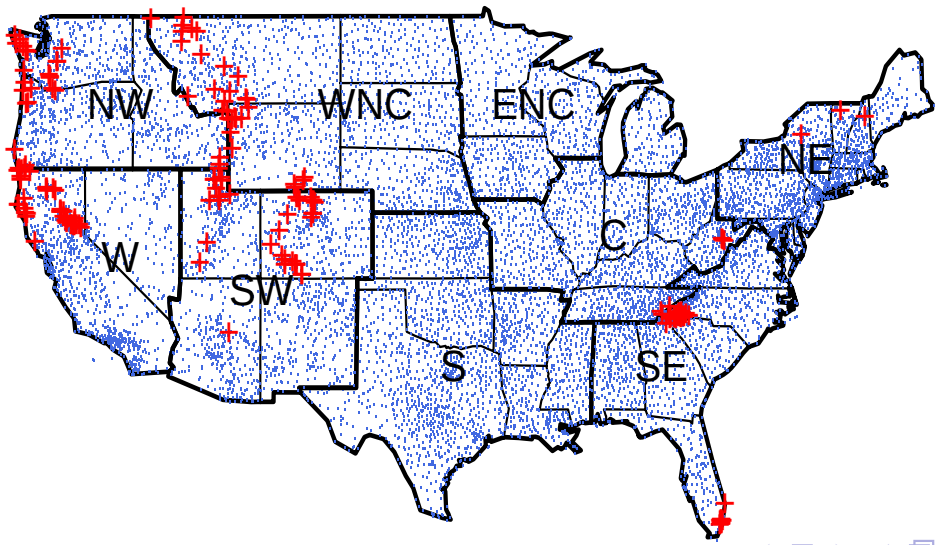
Children Growth Data

- The heights of 54 girls and 39 boys.



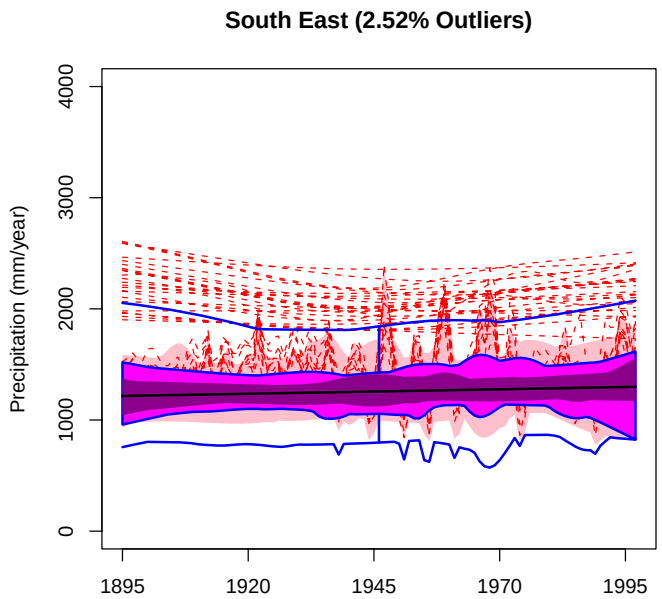
Spatio-Temporal Precipitation Data

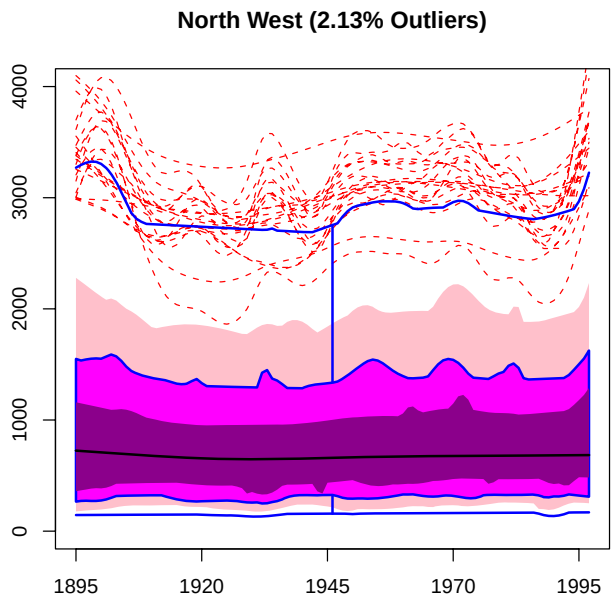
- Spatio-temporal precipitation data: annual total precipitation data for U.S. from 1895 to 1997 at 11,918 weather stations.
- Nine climatic regions for precipitation defined by National Climatic Data Center.
- Four areas of outliers detected by functional boxplots.



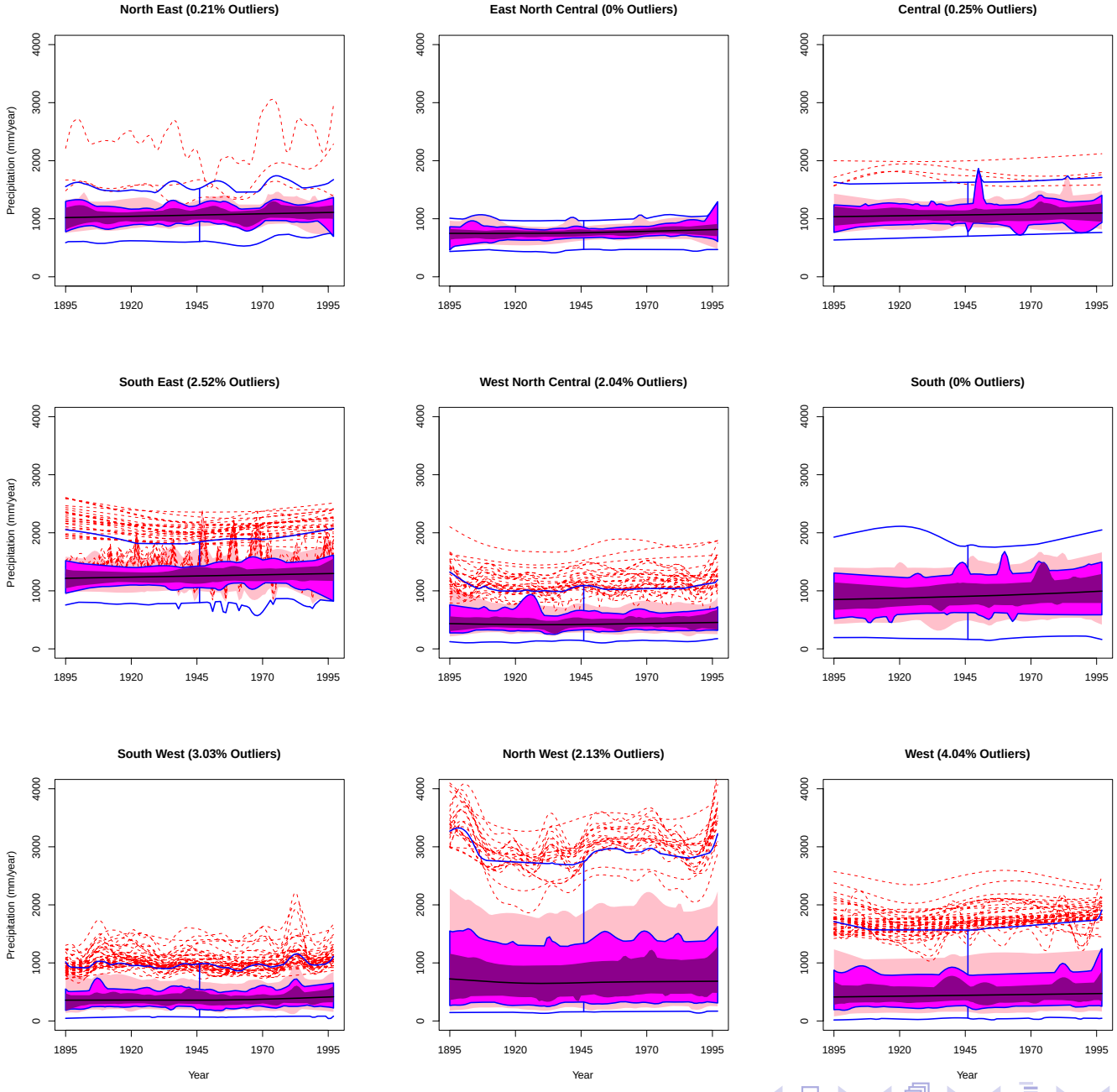
South East and North West

- Outliers at the southmost tip in Florida show an oscillatory pattern, which captures the hurricane effect.
- Outliers along the west coast have higher precipitation.
- Easy to compare two functional boxplots.





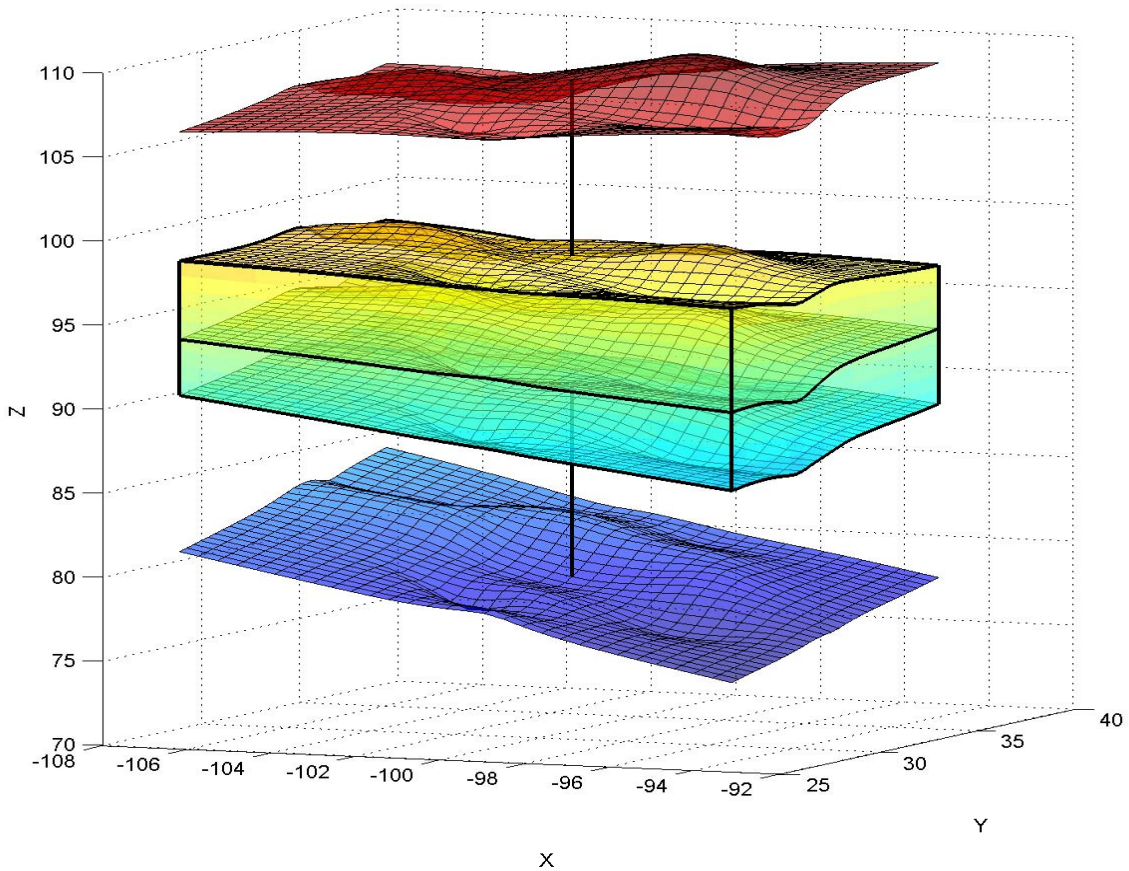
Functional Boxplots for Nine Climatic Regions



Surface Boxplots for Spatio-temporal Data

- We have viewed the information as a temporal curve at each spatial location.
- An alternative: treat the dataset as a spatial surface at each time.
- Define a volume-based surface band depth for a surface S by counting the proportion of surface bands determined by J different surfaces ($2 \leq J \leq n$) in $\mathbb{R}^3$, containing S.
- Lead to a three-dimensional surface boxplot.

Surface Boxplot



Conclusions

- BD: graph-based nonparametric ordering for functional data.
- Functional boxplot: visualize complex datasets, detect outliers.
- Precipitation data: provided by the Institute for Mathematics Applied to Geosciences (*http://www.image.ucar.edu/Data/US.monthly.met/*).
- R command: fbplot (*http://www.stat.tamu.edu/∼sunwards/publication.html*).