Chapter 8

Interaction of Light and Matter

8.1 Electromagnetic Waves at an Interface

A beam of light (implicitly a plane wave) in vacuum or in an isotropic medium propagates in the particular fixed direction specified by its Poynting vector until it encounters the interface with a different medium. The light causes the charges (electrons, atoms, or molecules) in the medium to oscillate and thus emit additional light waves that can travel in any direction (over the sphere of 4π steradians of solid angle). The oscillating particles vibrate at the frequency of the incident light and re-emit energy as light of that frequency (this is the mechanism of light "scattering"). If the emitted light is "out of phase" with the incident light (phase difference $\cong \pm\pi$ radians), then the two waves interfere destructively and the original beam is attenuated. If the attenuation is nearly complete, the incident light is said to be "absorbed." Scattered light may interfere constructively with the incident light in certain directions, forming beams that have been reflected and/or transmitted. The constructive interference of the transmitted beam occurs at the angle that satisfies Snell's law; while that after reflection occurs for $\theta_{\text{reflected}} = \theta_{\text{incident}}$. The mathematics are based on Maxwell's equations for the three waves and the continuity conditions that must be satisfied at the boundary. The equations for these three electromagnetic waves are not difficult to derive, though the process is somewhat tedious. The equations determine the properties of light on either side of the interface and lead to the phenomena of:

1. Equal angles of incidence and reflection;

2. Snell's Law that relates the incident and refracted wave;

3. Relative intensities of the three waves;

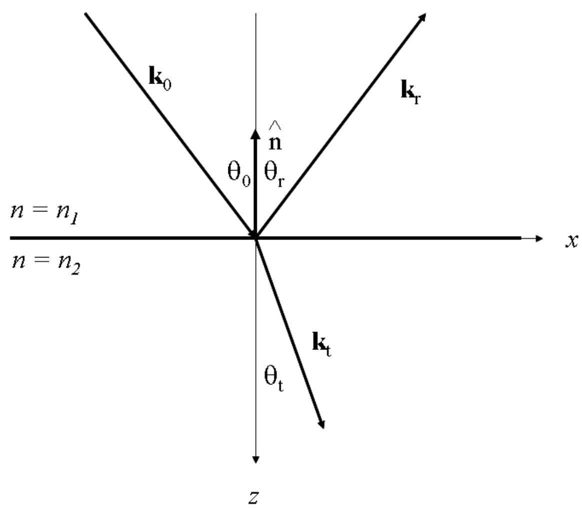
4. Relative phases of the three light waves; and

5. States of polarization of the three waves.

For simplicity, we consider only plane waves, so that the different beams are specified by single wavevectors $\underline{\mathbf{k}}_n$ that are valid at all points in a medium and that point in the direction of propagation. The lengths of the wavevectors are determined:

$$|\underline{\mathbf{k}}_n| = \frac{2\pi}{\lambda_n} = 2\pi\frac{n}{\lambda_0}$$

where λ_0 is the wavelength in vacuum and λ_n is the wavelength in the medium. The interface between the media is assumed to be the $x-y$ plane located at $z=0$. The incident wavevector $\underline{\mathbf{k}}_0$, the reflected vector $\underline{\mathbf{k}}_r$, the transmitted vector $\underline{\mathbf{k}}_t$ and the unit vector $\underline{\hat{\mathbf{n}}}$ normal to the interface are shown:



The $\underline{\mathbf{k}}$ vectors of the incident, reflected, and "transmitted" (refracted) wave at the interface between two media of index n_1 and n_2 (where $n_2 > n_1$ in the example shown).

The angles θ_0, θ_r, and θ_t are measured from the normal, so that θ_0, $\theta_t > 0$ and $\theta_r < 0$ as drawn.

The incident and reflected beams are in the same medium (with $n = n_1$) and so have the same wavelength:

$$|\underline{\mathbf{k}}_0| = |\underline{\mathbf{k}}_r| = \frac{\omega_0}{\mathrm{v}_1} = \frac{2\pi n_1}{\lambda_0}$$

$$\lambda_1 = \frac{2\pi n_1}{|\underline{\mathbf{k}}_0|} = \frac{2\pi n_1}{|\underline{\mathbf{k}}_r|}$$

The wavelength of the transmitted beam is different due to the different index of refraction:

$$\lambda_2 = \frac{2\pi n_2}{|\underline{\mathbf{k}}_t|}$$

As drawn, the normal to the surface is specified by the unit vector perpendicular to the interface; in this case, it points in the direction of the positive z-axis:

$$\underline{\hat{\mathbf{n}}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

(we could have defined $\underline{\hat{\mathbf{n}}}$ in the opposite direction).

The incident electric field is a sinusoidal oscillation that may be written in complex notation:

$$\underline{\mathbf{E}}_{incident} = \underline{\mathbf{E}}_0 \; \exp\left[+i\left(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t\right)\right]$$

where $\underline{\mathbf{r}} = [x, y, z]$ is the position vector of the location where the phase $\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t$ is measured; note that the phases measured at all positions in a plane perpendicular to the incident wavevector $\underline{\mathbf{k}}_0$ must be equal (because this is a plane wave).

The reflected and transmitted waves have the general forms:

$$\underline{\mathbf{E}}_{reflected} = \underline{\mathbf{E}}_r \; \exp\left[+i\left(\underline{\mathbf{k}}_r \bullet \underline{\mathbf{r}} - \omega_r t + \phi_r\right)\right]$$
$$\underline{\mathbf{E}}_{transmitted} = \underline{\mathbf{E}}_t \; \exp\left[+i\left(\underline{\mathbf{k}}_t \bullet \underline{\mathbf{r}} - \omega_t t + \phi_t\right)\right]$$

where we have yet to demonstrate that $\omega_r = \omega_t = \omega_0$. The constants ϕ_r and ϕ_t are the (perhaps different) initial phases of the reflected and transmitted waves.

8.1.1 Snell's Law for Reflection and Refraction

One boundary condition that must be satisfied is that the phases of all three waves must match at the interface ($z = 0$) at all times.

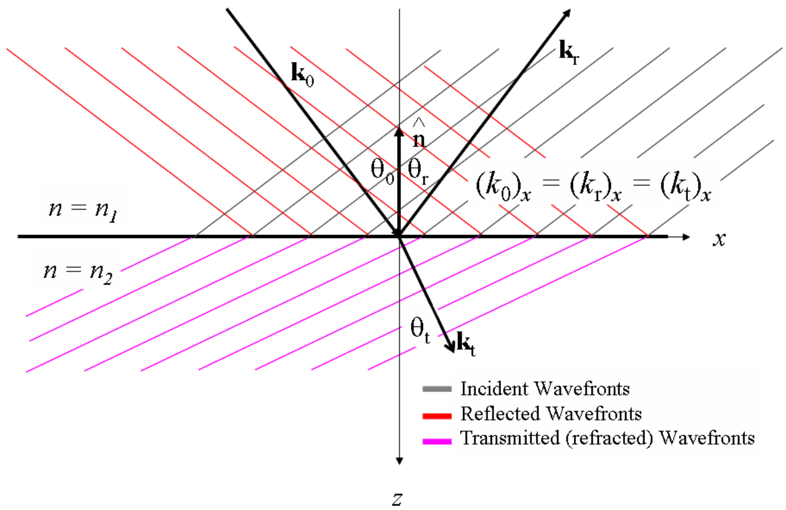
$$\left.\left(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t\right)\right|_{z=0} = \left.\left(\underline{\mathbf{k}}_r \bullet \underline{\mathbf{r}} - \omega t + \phi_r\right)\right|_{z=0} = \left.\left(\underline{\mathbf{k}}_t \bullet \underline{\mathbf{r}} - \omega t + \phi_t\right)\right|_{z=0}$$

This equivalence immediately implies that the temporal frequencies of the three waves must be identical (ω_0), because otherwise the phases would change by different amounts as functions of time. In words, the temporal frequency is invariant with medium, or the "color" of the light does not change as the light travels into a different medium. Therefore the spatial vectors must satisfy the conditions:

$$\left.\left(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}}\right)\right|_{z=0} = \left.\left(\underline{\mathbf{k}}_r \bullet \underline{\mathbf{r}} + \phi_r\right)\right|_{z=0} = \left.\left(\underline{\mathbf{k}}_t \bullet \underline{\mathbf{r}} + \phi_t\right)\right|_{z=0}$$

Since the scalar products of the three wavevectors with the same position vector $\underline{\mathbf{r}}$ must be equal, then the three vectors $\underline{\mathbf{k}}_0, \underline{\mathbf{k}}_r$ and $\underline{\mathbf{k}}_t$ must all lie in the same plane (call it the x-z plane, as shown in the drawing). The number of waves per unit length at any instant of time must be equal at the boundary for all three waves, as shown:

$$(k_0)_x = (k_r)_x = (k_t)_x$$



The x-components of the threee wavevectors (for the incident, reflected, and transmitted refracted waves) must match at the interface to ensure that each produces the same number of waves per unit length.

From the definitions of the vectors we can also see that:

$$(k_0)_x = |\underline{\mathbf{k}}_0| \cos\left[\frac{\pi}{2} - \theta_0\right] = |\underline{\mathbf{k}}_0| \sin[\theta_0]$$
$$(k_r)_x = |\underline{\mathbf{k}}_r| \cos\left[\frac{\pi}{2} - \theta_r\right] = |\underline{\mathbf{k}}_r| \sin[-\theta_r]$$

where the factor of -1 on the reflected angle is because the angle measured from the normal is clockwise, and hence negative. The equality of the lengths of the incident and reflected wavevectors immediately demonstrates that:

$$\begin{aligned}(k_0)_x = (k_r)_x = |\underline{\mathbf{k}}_0| \sin[\theta_0] &= |\underline{\mathbf{k}}_r| \sin[-\theta_r] \\ \Longrightarrow |\underline{\mathbf{k}}_0| \sin[\theta_0] &= |\underline{\mathbf{k}}_0| \sin[-\theta_r] \\ \Longrightarrow \sin[\theta_0] &= \sin[-\theta_r] \\ \Longrightarrow \theta_0 &= -\theta_r\end{aligned}$$

In words, *the angle of reflection is equal to the negative of the angle of incidence.* We usually ignore the sign of the angle and say that the angles of incidence and reflection are equal.

Now make the same observation for the transmitted wave:

$$(k_0)_x = |\underline{\mathbf{k}}_0| \sin[\theta_0] = \frac{2\pi n_1}{\lambda_0} \sin[\theta_0]$$
$$(k_t)_x = |\underline{\mathbf{k}}_t| \cos\left[\frac{\pi}{2} - \theta_t\right] = |\underline{\mathbf{k}}_t| \sin[\theta_t] = \frac{2\pi n_2}{\lambda_0} \sin[\theta_t]$$

We equate these to derive the relationship of the angles of the incident and transmitted wavevectors:

$$\frac{2\pi n_1}{\lambda_0}\sin[\theta_0] = \frac{2\pi n_2}{\lambda_0}\sin[\theta_t]$$
$$\implies n_1 \sin[\theta_0] = n_2 \sin[\theta_t]$$

We recognize this to be (of course) *Snell's law* for refraction.

The reflection law may be cast into the form of Snell's refraction law by assuming that the index of refraction is negative *for the reflected beam:*

$$n_1 \sin[\theta_0] = -n_1 \sin[\theta_r]$$
$$\implies \sin[\theta_r] = -\sin[\theta_0]$$
$$\implies \theta_r = -\theta_0$$

Note that these laws were derived without having to consider the vector nature of the electric and magnetic fields, but rather just the spatial frequencies of the waves at the boundaries. The next task is not quite this simple.....

8.1.2 Boundary Conditions for Electric and Magnetic Fields

We've determined the angles of the reflected and transmitted (refracted) plane waves in the form of Snell's law(s). We also need to evaluate the "quantity" of light reflected and refracted due to the boundary. Since the geometries of the fields will depend on the directions of the electric field vectors, we will have to consider this aspect in the derivations. In short, this discussion will depend on the "polarization" of the electric field (different from the "polarizability" of the medium). We will again have to match appropriate boundary conditions at the boundary, but these conditions apply to the vector components of the electric and magnetic fields on each side of the boundary. We use the same notation as before for amplitudes of the electric fields of the incident, reflected, and transmitted (refracted) waves. Faraday's and Ampere's laws (the Maxwell equations involving curl) for plane waves can be recast into forms that are more useful for the current task:

$$\nabla\times\underline{\mathbf{E}} \propto -\frac{\partial\underline{\mathbf{B}}}{\partial t}$$
$$\nabla\times\underline{\mathbf{B}} \propto +\frac{\partial\underline{\mathbf{E}}}{\partial t}$$

We need the constants of proportionality in this derivation. Recall that they depend on the system of units. We will use the MKS system here:

$$\nabla\times\underline{\mathbf{E}} = -\frac{\partial\underline{\mathbf{B}}}{\partial t}$$
$$\nabla\times\underline{\mathbf{B}} = +\epsilon\mu\frac{\partial\underline{\mathbf{E}}}{\partial t}$$

where ϵ and μ are the permittivity and permeability of the medium, respectively and the phase velocity of light in the medium is:

$$\mathrm{v}_\phi = \sqrt{\frac{1}{\epsilon\mu}}$$

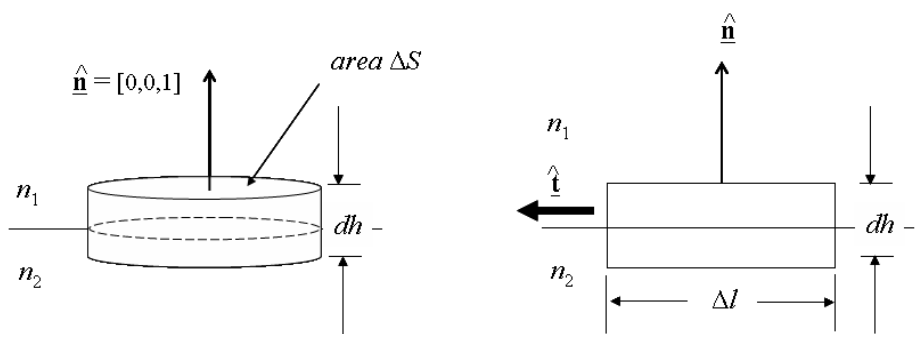
The incident field is assumed to be a plane wave of the form already mentioned:

$$\begin{aligned}
\underline{\mathbf{E}}_{incident}[x,y,z,t] &= \underline{\mathbf{E}}_0 \ \exp\left[+i\left(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t\right)\right] \\
&= \underline{\mathbf{E}}_0 \ \exp\left[+i\left([\underline{\mathbf{k}}_0]_x\, x + [\underline{\mathbf{k}}_0]_y\, y + [\underline{\mathbf{k}}_0]_z\, z - \omega_0 t\right)\right] \\
&= \underline{\mathbf{E}}_0 \ \exp\left[+i\left(k_{0x}x + k_{0y}y + k_{0z}z - \omega_0 t\right)\right] \\
&= \left(\underline{\hat{\mathbf{x}}}E_{0x} + \underline{\hat{\mathbf{y}}}E_{0y} + \underline{\hat{\mathbf{z}}}E_{0z}\right)\exp\left[+i\left(k_{0x}x + k_{0y}y + k_{0z}z - \omega_0 t\right)\right]
\end{aligned}$$

We know that $\underline{\mathbf{E}}_0 \perp \underline{\mathbf{k}}_0$. In our coordinate system, the incident wave vector lies in the $x - z$ plane (the plane defined by $\underline{\mathbf{k}}_0$ and $\underline{\hat{\mathbf{n}}}$), so that $k_{0y} = 0$:

$$\begin{aligned}
\underline{\mathbf{E}}_{incident}[x,y,z,t] &= \underline{\mathbf{E}}_0 \ \exp\left[+i\left(\underline{\mathbf{k}}_0 \bullet \underline{\mathbf{r}} - \omega_0 t\right)\right] \\
&= \left(\underline{\hat{\mathbf{x}}}E_{0x} + \underline{\hat{\mathbf{y}}}E_{0y} + \underline{\hat{\mathbf{z}}}E_{0z}\right)\exp\left[+i\left(k_{0x}x + k_{0z}z - \omega_0 t\right)\right]
\end{aligned}$$

The boundary conditions that must be satisfied by the electric fields and by the magnetic fields at the boundary are perhaps not obvious. Consider the figure on the left:



The boundary conditions on the electric and magnetic fields at the boundary are established from these situations.

We assume that there is no charge or current on the surface and within the cylinder that straddles the boundary. If the height of the cylinder is decreased towards zero, then Gauss' laws establish that the flux of the electric and magnetic fields through the top and bottom of the cylinder (the z components in this geometry) must cancel:

$$\begin{aligned}
&\epsilon_1 \underline{\mathbf{E}}_1 \bullet \underline{\hat{\mathbf{n}}} - \epsilon_2 \underline{\mathbf{E}}_2 \bullet \underline{\hat{\mathbf{n}}} = 0 \\
&\qquad\qquad \Longrightarrow \epsilon_1 E_{1z} = \epsilon_2 E_{2z}
\end{aligned}$$

$$\begin{aligned}
&\underline{\mathbf{B}}_1 \bullet \underline{\hat{\mathbf{n}}} - \underline{\mathbf{B}}_2 \bullet \underline{\hat{\mathbf{n}}} = 0 \\
&\qquad\qquad \Longrightarrow B_{1z} = B_{2z}
\end{aligned}$$

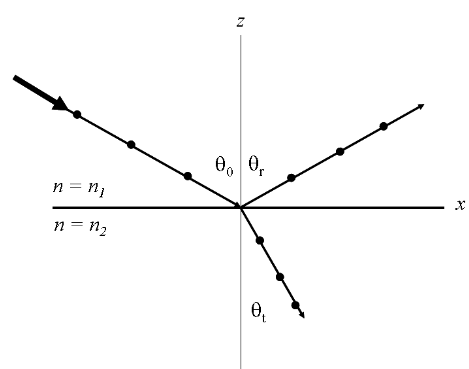
The flux of the electric field in a medium is the so-called "displacement" field $\underline{\mathbf{D}} = \epsilon\underline{\mathbf{E}}$ and the flux of the magnetic field is the field $\underline{\mathbf{B}}$ Thus Gauss' law determines that the normal components of $\underline{\mathbf{D}}$ and of $\underline{\mathbf{B}}$ are continuous across the boundary of the medium.

The figure on the right is a rectangular path (a "loop") that also straddles the boundary. The unit vector $\underline{\hat{\mathbf{t}}} \perp \underline{\hat{\mathbf{n}}}$ points along the surface. If the "height" of the loop $dh \to 0$, then the circulations of the electric and magnetic fields must cancel:

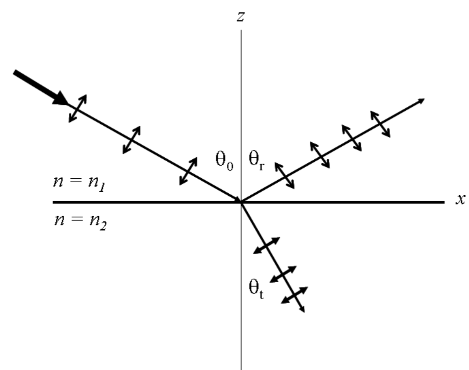
$$\underline{\mathbf{E}}_1 \bullet \underline{\hat{\mathbf{t}}} - \underline{\mathbf{E}}_2 \bullet \underline{\hat{\mathbf{t}}} = 0$$
$$\Longrightarrow E_{1x} = E_{2x}$$

$$\frac{\underline{\mathbf{B}}_1}{\mu_1} \bullet \underline{\hat{\mathbf{t}}} - \frac{\underline{\mathbf{B}}_2}{\mu_2} \bullet \underline{\hat{\mathbf{t}}} = 0$$
$$\Longrightarrow \frac{B_{1x}}{\mu_1} = \frac{B_{2x}}{\mu_2}$$

We now want to solve Maxwell's equations for an incident plane wave, which will depend on the incident angle θ_0 and on the vector direction of the electric field. It is convenient to evaluate these conditions in two cases of linearly polarized waves: (1) where the polarization is perpendicular to the plane of incidence defined by $\underline{\hat{\mathbf{n}}}$ and $\underline{\mathbf{k}}_0$ (the so-called "s" polarization or *transverse electric* (TE) waves), which also means that the electric field vector is "parallel" to the interface, and (2) the polarization is parallel to the plane of incidence defined by $\underline{\hat{\mathbf{n}}}$ and $\underline{\mathbf{k}}_0$ (the so-called "p" polarization or *transverse magnetic* (TM) waves). The two cases are depicted below:



The electric field perpendicular to the plane of incidence; this is the TRANSVERSE ELECTRIC field (TE, also called the "s" polarization).



Electric field parallel to plane of incidence ("p" polarization)

The electric field is parallel to the plane of incidence; this is the TRANSVERSE MAGNETIC field (TM, also called the "p" polarization).

8.1.3 Transverse Electric Waves, s Polarization

In the TE case in our geometry, the electric field is oriented along the y direction and the wavevector has components in the x and z directions:

$$\begin{aligned}\underline{\mathbf{E}}_{incident}[x,y,z,t] &= \left(\hat{\underline{\mathbf{x}}}\cdot 0+\hat{\underline{\mathbf{y}}}\cdot|\underline{\mathbf{E}}_0|+\hat{\underline{\mathbf{z}}}\cdot 0\right)\exp\left[+i\left(k_{0x}x+k_{0z}z-\omega_0 t\right)\right]\\ &= \hat{\underline{\mathbf{y}}}E_0\exp\left[+i\left(k_{0x}x+k_{0z}z-\omega_0 t\right)\right]\end{aligned}$$

The magnetic field is derived from the relation:

$$\underline{\mathbf{B}}=\frac{n}{c}\underline{\mathbf{k}}_0\times\underline{\mathbf{E}}$$

$$\begin{aligned}\underline{\mathbf{B}}_{incident}[x,y,z,t] = &\left(\left[-\cos[\theta_0]\cdot n_1\frac{|\underline{\mathbf{E}}_0|}{c}\right]\hat{\underline{\mathbf{x}}}+0\hat{\underline{\mathbf{y}}}+\left[+\sin[\theta_0]\cdot n_1\frac{|\underline{\mathbf{E}}_0|}{c}\right]\hat{\underline{\mathbf{z}}}\right)\\ &\cdot\exp\left[+i\left(k_{0x}x+k_{0z}z-\omega_0 t\right)\right]\end{aligned}$$

The reflected fields are:

$$\underline{\mathbf{E}}_{reflected}[x,y,z,t]=\hat{\underline{\mathbf{y}}}\cdot|\underline{\mathbf{E}}_0|\exp\left[+i\left(k_{rx}x+k_{rz}z-\omega_0 t\right)\right]$$

$$\begin{aligned}\underline{\mathbf{B}}_{reflected}[x,y,z,t] &= \left(\left[+\cos[-\theta_0]\cdot n_1\frac{|\underline{\mathbf{E}}_0|}{c}\right]\hat{\underline{\mathbf{x}}}+\left[-\sin[-\theta_0]\cdot n_1\frac{|\underline{\mathbf{E}}_0|}{c}\right]\hat{\underline{\mathbf{z}}}\right)\\ &\quad\cdot\exp\left[+i\left(k_{0x}x+k_{0z}z-\omega_0 t\right)\right]\\ &= \left(\left[+\cos[\theta_0]\cdot n_1\frac{|\underline{\mathbf{E}}_0|}{c}\right]\hat{\underline{\mathbf{x}}}+\left[\sin[\theta_0]\cdot n_1\frac{|\underline{\mathbf{E}}_0|}{c}\right]\hat{\underline{\mathbf{z}}}\right)\\ &\quad\cdot\exp\left[+i\left(k_{0x}x+k_{0z}z-\omega_0 t\right)\right]\end{aligned}$$

and the transmitted (refracted) fields are:

$$\underline{\mathbf{E}}_{transmitted}[x,y,z,t] = \hat{\underline{\mathbf{y}}} \cdot |\underline{\mathbf{E}}_t| \exp[+i(k_{tx}x + k_{tz}z - \omega_0 t)]$$

$$\underline{\mathbf{B}}_{transmitted}[x,y,z,t] = \left(\left[-\cos[\theta_t] \cdot n_2 \frac{|\underline{\mathbf{E}}_t|}{c}\right]\hat{\underline{\mathbf{x}}} + \left[\sin[\theta_t] \cdot n_2 \frac{|\underline{\mathbf{E}}_t|}{c}\right]\hat{\underline{\mathbf{z}}}\right) \cdot \exp[+i(k_{0x}x + k_{0z}z - \omega_0 t)]$$

The only components of the electric field at the interface are transverse, so the only boundary conditions to be satisfied are the tangential electric field:

$$E_0 + E_r = E_t \implies 1 + \frac{E_r}{E_0} = \frac{E_t}{E_0}$$

This is typically expressed in terms of the reflection and transmission coefficients for the *amplitude* of the waves (not the power of the waves; these are the reflectance R and transmittance T of the interface, which will be considered very soon):

$$r_{TE} \equiv \frac{E_r}{E_0}$$
$$t_{TE} \equiv \frac{E_t}{E_0}$$

where the subscripts denote the transverse electric polarization. The boundary condition for the normal magnetic field yields the expression:

$$\frac{n_1}{c}\sin[\theta_0](E_0 + E_r) = \frac{n_2}{c}\sin[\theta_t]E_t$$

while that for the tangential magnetic field:

$$\frac{n_1}{\mu_1 c}\cos[\theta_0](E_0 - E_r) = \frac{n_2}{\mu_2 c}\cos[\theta_t]E_t$$

These may be solved simultaneously for r and t to yield expressions in terms of the indices, permeabilities, and angles::

Reflectance Coefficient for TE Waves

$$r_{TE} = \frac{E_r}{E_0} = \frac{\frac{n_1}{\mu_1}\cos[\theta_0] - \frac{n_2}{\mu_2}\cos[\theta_t]}{\frac{n_1}{\mu_1}\cos[\theta_0] + \frac{n_2}{\mu_2}\cos[\theta_t]}$$

$$\boxed{r_{TE} = \frac{n_1\cos[\theta_0] - n_2\cos[\theta_t]}{n_1\cos[\theta_0] + n_2\cos[\theta_t]}} \; \textit{if} \; \mu_1 = \mu_2 \text{ (usual case)}$$

Transmission Coefficient for TE Waves

$$t_{TE} = \frac{E_t}{E_0} = \frac{+2\frac{n_1}{\mu_1}\cos[\theta_0]}{\frac{n_1}{\mu_1}\cos[\theta_0] + \frac{n_2}{\mu_2}\cos[\theta_t]}$$

$$\boxed{t_{TE} = \frac{+2n_1\cos[\theta_0]}{n_1\cos[\theta_0]+n_2\cos[\theta_t]}} \; \textit{if} \; \mu_1 = \mu_2$$

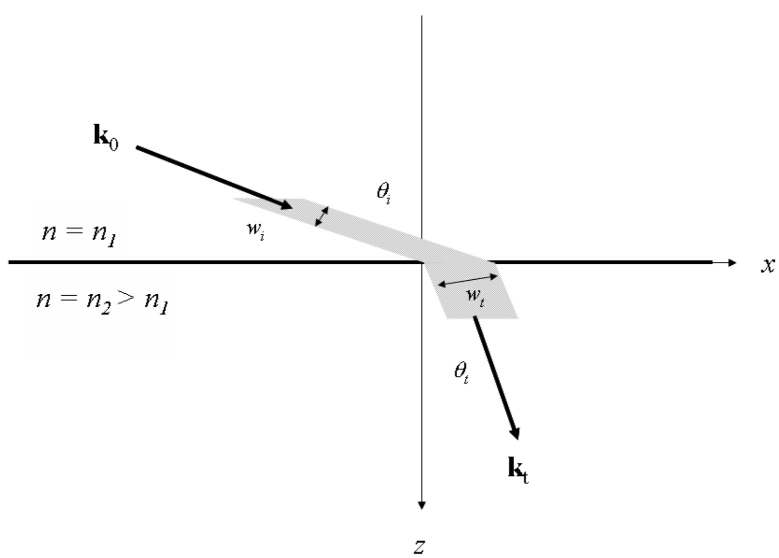
Again, these are the *amplitude* coefficients; the *reflectance* and *transmittance* of light at the surface relate the energies or powers. These measure the ratios of the reflected or transmitted *power* to the incident power. The power is proportional to the product of the the magnitude of the Poynting vector and the area of the beam. The areas of the beams before and after reflection are identical, which means that the reflectance is just the ratio of the magnitudes of the Poynting vectors. This reduces to the square of the amplitude reflection coefficient:

$$\boxed{R = r^2}$$

which reduces to this expression for the TE case:

$$R_{TE} = \left(\frac{n_1\cos[\theta_0] - n_2\cos[\theta_t]}{n_1\cos[\theta_0] + n_2\cos[\theta_t]}\right)^2$$

The transmission T is a bit more complicated to compute, because the refraction at the interface changes the "width" of the beam in one direction (along the x-axis in this example), so that the area of the transmitted beam is different from that of the incident beam. This is illustrated in the figure for a case with $n_1 > n_2$:



Demonstration that the areas of the beams differ in the two media. This must be accounted for in the calculation of the power transmission T.

The magnitude of the Poynting vector is proportional to the product of the index of refraction and the squared magnitude of the electric field:

$$|\underline{\mathbf{s}}_1| \propto n_1 |E_0|^2$$
$$|\underline{\mathbf{s}}_2| \propto n_2 |E_t|^2$$

The ratio of the powers is:

$$T = \frac{|\underline{\mathbf{s}}_2| A_2}{|\underline{\mathbf{s}}_1| A_1} = \frac{n_2 |E_t|^2 A_2}{n_1 |E_0|^2 A_1} = \frac{n_2}{n_1} \cdot t^2 \cdot \frac{A_2}{A_1}$$

The area of the transmitted beam changes in proportion to the dimension along the x-axis in this case, which allows us to see that:

$$\frac{A_2}{A_1} = \frac{w_2}{w_1} = \frac{\sin\left[\frac{\pi}{2} - \theta_t\right]}{\sin\left[\frac{\pi}{2} - \theta_0\right]} = \frac{\cos[\theta_t]}{\cos[\theta_0]}$$

which leads to the final expression for the transmission at the interface:

$$T = \frac{n_2}{n_1} \cdot t^2 \cdot \left(\frac{\cos[\theta_t]}{\cos[\theta_0]}\right)$$
$$\boxed{T = \left(\frac{n_2 \cos[\theta_t]}{n_1 \cos[\theta_0]}\right) \cdot t^2}$$

Snell's law gives a relationship between the incident and transmitted angles:

$$n_1 \sin[\theta_0] = n_2 \sin[\theta_t] \implies \sin[\theta_t] = \frac{n_1}{n_2} \sin[\theta_0]$$
$$\implies \cos[\theta_t] = \sqrt{1 - \sin^2[\theta_t]} = \sqrt{1 - \left(\frac{n_1}{n_2}\sin[\theta_0]\right)^2}$$

Thus we can write down the transmittance T in terms of the refractive indices and the incident angle:

$$T = \left(\frac{\sqrt{n_2^2 - n_1^2 \sin^2[\theta_0]}}{n_1 \cos[\theta_0]}\right) \cdot t^2$$

For the TE case, the transmission is:

$$T_{TE} = \left(\frac{\sqrt{n_2^2 - n_1^2 \sin^2[\theta_0]}}{n_1 \cos[\theta_0]}\right) \cdot \left(\frac{+2n_1 \cos[\theta_0]}{n_1 \cos[\theta_0] + n_2 \cos[\theta_t]}\right)^2$$

These will be plotted for some specific cases after we evaluate the coefficients for TM waves.

8.1.4 Transverse Magnetic Waves (p polarization)

In the TM case in our geometry, the electric field is in the x-z plane and the wavevector has components in the x and z directions:

$$\begin{aligned}\underline{\mathbf{E}}_{incident}[x,y,z,t] &= \left(\hat{\underline{\mathbf{x}}}\cdot|\underline{\mathbf{E}}_0|\cos[\theta_0]+\hat{\underline{\mathbf{y}}}\cdot 0+\hat{\underline{\mathbf{z}}}\cdot|\underline{\mathbf{E}}_0|\sin[-\theta_0]\right)\exp[+i(k_{0x}x+k_{0z}z-\omega_0 t)]\\ &= (\hat{\underline{\mathbf{x}}}\cdot|\underline{\mathbf{E}}_0|\cos[\theta_0]-\hat{\underline{\mathbf{z}}}\cdot|\underline{\mathbf{E}}_0|\sin[\theta_0])\exp[+i(k_{0x}x+k_{0z}z-\omega_0 t)]\end{aligned}$$

The magnetic field is in the y-direction:

$$\underline{\mathbf{B}}_{incident}[x,y,z,t]=\left(n_1\frac{|\underline{\mathbf{E}}_0|}{c}\hat{\underline{\mathbf{y}}}\right)\exp[+i(k_{0x}x+k_{0z}z-\omega_0 t)]$$

The reflected fields are:

$$\begin{aligned}\underline{\mathbf{E}}_{reflected}[x,y,z,t] &= (\hat{\underline{\mathbf{x}}}\cdot-|\underline{\mathbf{E}}_0|\cos[\theta_0]-\hat{\underline{\mathbf{z}}}\cdot|\underline{\mathbf{E}}_0|\sin[\theta_0])\exp[+i(k_{0x}x+k_{0z}z-\omega_0 t)]\\ \underline{\mathbf{B}}_{reflected}[x,y,z,t] &= \left(n_1\frac{|\underline{\mathbf{E}}_r|}{c}\hat{\underline{\mathbf{y}}}\right)\exp[+i(k_{0x}x+k_{0z}z-\omega_0 t)]\end{aligned}$$

and the transmitted (refracted) fields are:

$$\begin{aligned}\underline{\mathbf{E}}_{transmitted}[x,y,z,t] &= (\hat{\underline{\mathbf{x}}}\cdot|\underline{\mathbf{E}}_0|\cos[\theta t]-\hat{\underline{\mathbf{z}}}\cdot|\underline{\mathbf{E}}_0|\sin[\theta_t])\exp[+i(k_{0x}x+k_{0z}z-\omega_0 t)]\\ \underline{\mathbf{B}}_{transmitted}[x,y,z,t] &= \left(n_2\frac{|\underline{\mathbf{E}}_t|}{c}\hat{\underline{\mathbf{y}}}\right)\exp[+i(k_{tx}x+k_{tz}z-\omega_0 t)]\end{aligned}$$

In the case, the boundary condition on the normal component of $\underline{\mathbf{B}}$ is trivial, but the other components are:

$$\begin{aligned}\mu_1\sin[\theta_0](E_0+E_r) &= \mu_2\sin[\theta_2]E_t\\ \cos[\theta_0](E_0-E_r) &= \cos[\theta_2]E_t\\ \frac{n_1}{\mu_1 c}(E_0+E_r) &= \frac{n_2}{\mu_1 c}E_t\end{aligned}$$

These are solved for the reflection and transmission coefficients:

Transverse Magnetic Waves

$$r_{TM}=\frac{+\frac{n_2}{\mu_2}\cos[\theta_0]-\frac{n_1}{\mu_1}\cos[\theta_t]}{+\frac{n_2}{\mu_2}\cos[\theta_0]+\frac{n_1}{\mu_1}\cos[\theta_t]}$$

which simplifies if the permeabilities are equal (as they usually are):

$$\boxed{r_{TM}=\frac{+n_2\cos[\theta_0]-n_1\cos[\theta_t]}{+n_2\cos[\theta_0]+n_1\cos[\theta_t]}}\ if\ \mu_1=\mu_2$$

The corresponding reflectance is:

$$R_{TM} = \left(\frac{+n_2 \cos[\theta_0] - n_1 \cos[\theta_t]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]} \right)^2$$

The amplitude transmission coefficient evaluates to:

$$t_{TM} = \frac{2\frac{n_1}{\mu_1} \cos[\theta_0]}{+\frac{n_2}{\mu_2} \cos[\theta_0] + \frac{n_1}{\mu_1} \cos[\theta_t]}$$

again, if the permeabilities are equal, this simplifies to:

$$\boxed{t_{TM} = \frac{2n_1 \cos[\theta_0]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]}} \quad if \;\; \mu_1 = \mu_2$$

The corresponding transmittance functionis:

$$T_{TM} = \left(\frac{\sqrt{n_2^2 - n_1^2 \sin^2[\theta_0]}}{n_1 \cos[\theta_0]} \right) \cdot \left(\frac{2n_1 \cos[\theta_0]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]} \right)^2$$

8.1.5 Comparison of Coefficients for TE and TM Waves

We should compare the coefficients for the two cases of TE and TM waves. The reflectance coefficients are:

$$r_{TE} = \frac{n_1 \cos[\theta_0] - n_2 \cos[\theta_t]}{n_1 \cos[\theta_0] + n_2 \cos[\theta_t]}$$
$$r_{TM} = \frac{+n_2 \cos[\theta_0] - n_1 \cos[\theta_t]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]}$$

where the angles are also determined by Snell's law:

$$n_1 \sin[\theta_0] = n_2 \sin[\theta_t]$$
$$\Longrightarrow \cos[\theta_t] = \sqrt{1 - \left(\frac{n_1}{n_2} \sin[\theta_0] \right)^2}$$

Note that angles and the indices for the TE case are "in" the same media, i.e., the index n_1 multiplies the cosine of θ_0, which is in the same medium. The same condition holdes for n_2 and θ_t. The opposite is true for the TM case: n_1 is applied to $\cos[\theta_t]$ and n_2 to $\cos[\theta_0]$. These same observations also apply to the corresponding transmission coefficients:

$$t_{TE} = \frac{+2n_1 \cos[\theta_0]}{n_1 \cos[\theta_0] + n_2 \cos[\theta_t]}$$
$$t_{TM} = \frac{+2n_1 \cos[\theta_0]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]}$$

Normal Incidence $(\theta_0 = 0)$

In the case of normal incidence where $\theta_0 = \theta_r = \theta_t = 0$, then the TE and TM equations evaluate to:

$$r_{TE}|_{\theta_0=0} = \frac{n_1 - n_2}{n_1 + n_2}$$
$$r_{TM}|_{\theta_0=0} = \frac{+n_2 - n_1}{+n_2 + n_1} = -\left(r_{TE}|_{\theta_0=0}\right)$$
$$t_{TE}|_{\theta_0=0} = \frac{+2n_1}{n_1 + n_2}$$
$$t_{TM}|_{\theta_0=0} = \frac{+2n_1}{n_1 + n_2} = t_{TE}|_{\theta_0=0}$$

cases are identical. Also, the areas of the incident and transmitted waves are identical so there is no area factor in the transmittance. The resulting formulas for reflectance and transmittance reduce to:

$$\textit{normal incidence} \ (\theta_0 = 0)$$
$$R_{TE}(\theta_0 = 0) = R_{TM}(\theta_0 = 0) \equiv \boxed{R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2}$$
$$\boxed{T = \frac{4n_1 n_2}{(n_1 + n_2)^2}}$$

Example: Rare-to-Dense Reflection If the input medium has a smaller refractive index n (a *rarer* medium) than the second (*denser*) medium, so that $n_1 < n_2$, then the coefficients are:

$$
\begin{aligned}
&n_1 = 1.0\\
&n_2 = 1.5\\
&\quad\Longrightarrow r_{TE} = \frac{1.0-1.5}{1.0+1.5} = -0.2 = 0.2e^{+i\pi}\\
&\quad\Longrightarrow r_{TM} = \frac{1.5-1.0}{1.0+1.5} = +0.2\\
&\quad\Longrightarrow t_{TE} = t_{TM} = \frac{2\cdot 1.0}{1.0+1.5} = +0.8\\
&\quad\Longrightarrow R_{TE} = R_{TM} = 0.04\\
&\quad\Longrightarrow T_{TE} = T_{TM} = 0.96\\
&\textit{for “rare-to-dense” reflection}
\end{aligned}
$$

In words, the phase of the reflected light is changed by π radians $= 180°$ if reflected at a “rare-to-dense” interface such as the usual air-to-glass case.

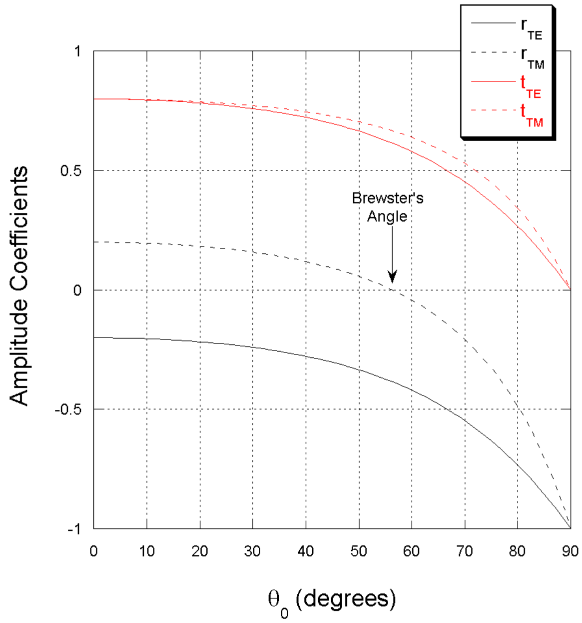
Example: Dense-to-Rare Reflection If the input medium is “denser” ($n_1 > n_2$), then these values are obtained:

$$
\begin{aligned}
&n_1 = 1.5\\
&n_2 = 1.0\\
&\quad\Longrightarrow r_{TE} = \frac{1.5-1.0}{1.5+1.0} = +0.2\\
&\quad\Longrightarrow r_{TM} = \frac{1.0-1.5}{1.0+1.5} = -0.2 = 0.2e^{+i\pi}\\
&\quad\Longrightarrow R_{TE} = R_{TM} = 0.04\\
&\quad\Longrightarrow T_{TE} = T_{TM} = 0.96
\end{aligned}
$$

There is no phase shift of the reflected amplitude in “dense-to-rare” reflection, commonly called “internal” reflection..

8.1.6 Angular Dependence of Reflection and Transmittance at “Rare-to-Dense” Interface

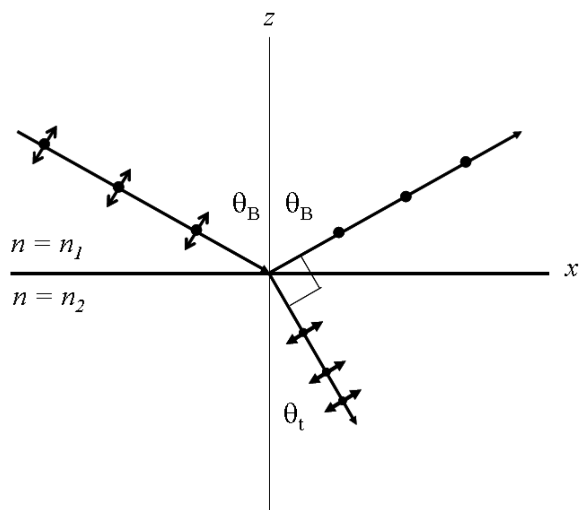
Consider the graphs of these coefficients for the cases of the “rare-to-dense” interface ($n_1 = 1 < n_2 = 1.5$). The reflection coefficients are plotted vs. incident angle measured in degrees from 0° (normal incidence) to 90° (grazing incidence).



Amplitude reflectance and transmittance coefficients for $n_1 = 1.0$(air) and $n_2 = 1.5$ (glass) for both TE and TM waves, plotted as functions of the incident angle from $\theta_0 = 0°$ (normal incidence) to $\theta_0 = 90°$ (grazing incidence). The reflectance coefficient $r_{TE} < 0$ for all θ, which means that there is a phase shift upon reflection, whereas $r_{TM} > 0$ for $\theta_0 < \theta_B$ (Brewster's angle). Also note that the transmittance coefficients are very similar functions.

Brewster's Angle – Angle of Complete Polarization

Note that $r_{TM} = 0$ at one particular angle ($\cong 60°$) in the TM case (parallel polarization), which means that no amplitude of this wave is reflected if incident at this angle. In other words, any light reflected at this angle must be the TE wave which is completely polarized perpendicular to the plane of incidence. This is *Brewster's angle*, the angle such that the reflected wave and the refracted wave are orthogonal (i.e., $\theta_0 + \theta_t = \frac{\pi}{2} \Longrightarrow \theta_t = \frac{\pi}{2} - \theta_0$). In this case, the electrons driven *in* the plane of the incidence will not emit radiation at the angle required by the law of reflection. This is sometimes called the *angle of complete polarization.* Note that the transmitted light contains both polarizations, though not in equal amounts.



Polarization of reflected light at Brewster's angle. The incident beam at $\theta_0 = \theta_B$ is unpolarized. The reflectance coefficient for light polarized in the plane (TM waves) is 0, and the sum of the incident and refracted angle is $90° = \frac{\pi}{2}$. Thus
$\theta_B + \theta_t = \frac{\pi}{2} \Longrightarrow \theta_t = \frac{\pi}{2} - \theta_B$.

From Snell's law, we have:

$$n_1 \sin[\theta_1] = n_2 \sin[\theta_2]$$

At Brewster's angle,

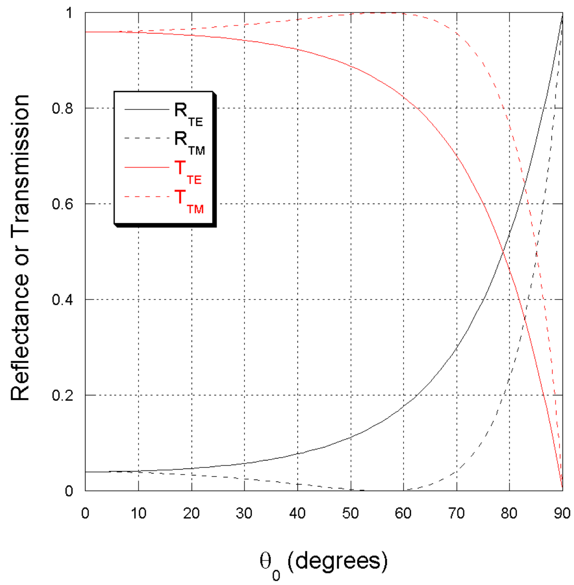
$$\begin{aligned} n_1 \sin[\theta_B] &= n_2 \sin\left[\frac{\pi}{2} - \theta_B\right] \\ &= n_2 \left(\sin\left[\frac{\pi}{2}\right]\cos[\theta_B] - \cos\left[\frac{\pi}{2}\right]\sin[\theta_B]\right) \\ &= +n_2 \cos[\theta_B] \\ n_1 \sin[\theta_B] &= n_2 \cos[\theta_B] \\ &\Longrightarrow \frac{n_2}{n_1} = \frac{\sin[\theta_B]}{\cos[\theta_B]} = \tan[\theta_B] \\ &\Longrightarrow \theta_B = \tan^{-1}\left[\frac{n_2}{n_1}\right] \end{aligned}$$

If $n_1 = 1$ (air) and $n_2 = 1.5$ (glass), then $\theta_B \cong 56.3°$. For incident angles larger than about 56°, the reflected light is plane polarized parallel to the plane of incidence. If the dense medium is water ($n_2 = 1.33$), then $\theta_B \cong 52.4°$. This happens at the interface with any dielectric. The reflection at Brewster's angle provides a handy means to determine the polarization axis of a linear polarizer – just look through a linear polarizer at light reflected at a shallow angle relative to the surface (e.g., a waxed floor).

Reflectance and Transmittance at "Rare-to-Dense" Interface

The reflectance and transmittance the two polarizations with $n_1 = 1.0$ and $n_2 = 1.5$ as functions of the incident angle θ_0 show the zero reflectance of the TM wave at

Brewster's angle.



Reflectance and transmittance for $n_1 = 1.0$ and $n_2 = 1.5$ for TE and TM waves. Note that $R_{TM} = 0$ and $T_{TM} = 1$ at one angle.

8.1.7 Reflection and Transmittance at "Dense-to-Rare" Interface, Critical Angle

At a "glass-to-air" interface where $n_1 > n_2$, the reflectance of the TM wave (s polarization) is:

$$r = \frac{-n_2 \cos[\theta_0] + n_1 \cos[\theta_t]}{+n_2 \cos[\theta_0] + n_1 \cos[\theta_t]}$$

The numerator evaluates to zero for a particular incident angle that satisfies:

$$n_2 \cos[\theta_0] = n_1 \cos[\theta_t]$$
$$\frac{n_1}{n_2} = \frac{\cos[\theta_0]}{\cos[\theta_t]}$$

This corresponds to the situation where Snell's law requires that:

$$\sin\left[\theta_t = \frac{\pi}{2}\right] = 1 = \frac{n_1}{n_2} \sin[\theta_0] \implies \sin[\theta_0] = \frac{n_2}{n_1}$$

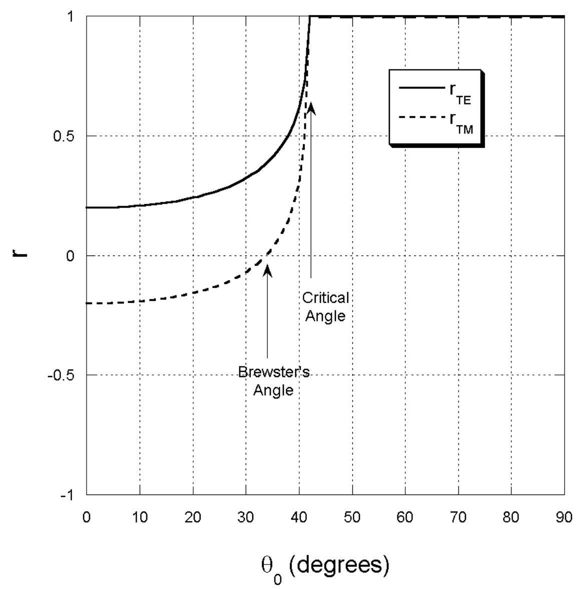
If $n_1 = 1.5$ and $n_2 = 1.0$, then

$$\sin[\theta_0] = \frac{2}{3} \implies \theta_0 \cong 0.73 \text{ radians } \cong 41.8^\circ \equiv \theta_c$$

If the incident angle exceeds this value θ_c, the *critical angle*, then the amplitude reflectance coefficients r_{TE} and r_{TM} are both unity, and thus so are the reflectances R_{TE} and R_{TM}. This means that light incident for $\theta_0 \geq \theta_c$ is *totally reflected*. This is the source of *total internal reflectance* ("internal" because the reflection is from glass back into glass). The phenomenon of TIR is the reason for the usefulness of optical fibers in communications.

The angular dependences of the amplitude reflection coefficients for the case $n_1 = 1.5$ (glass) and $n_2 = 1.0$ (air) are shown. Brewster's angle in this case satisfies:

$$\theta_B = \tan^{-1}\left[\frac{n_2}{n_1}\right] = \tan^{-1}\left[\frac{1}{1.5}\right] \cong 33.7^\circ$$



Amplitude reflectance coefficients for TE and TM waves if $n_1 = 1.5$ (glass) and $n_2 = 1.0$ (air). Both coefficients rise to $r = +1.0$ at the "critical angle" θ_c, for which $\theta_t = 90^\circ = \frac{\pi}{2}$. Also noted is Brewster's angle, where $r_{TM} = 0$. The situation for $\theta_0 > \theta_c$ can be interpreted as producing complex-valued r_{TE} and r_{TM}.

8.1.8 Practical Applications for Fresnel's Equations

The 4% normal reflectance of one surface of glass is the reason why windows look like mirrors at night when you're in the brightly lit room. Lasers incorporate end windows oriented at Brewster's angle to eliminate reflective losses at the mirrors (and also thus producing polarized laser light). Optical fibers use total internal reflection. Hollow fibers use high-incidence-angle near-unity reflections.

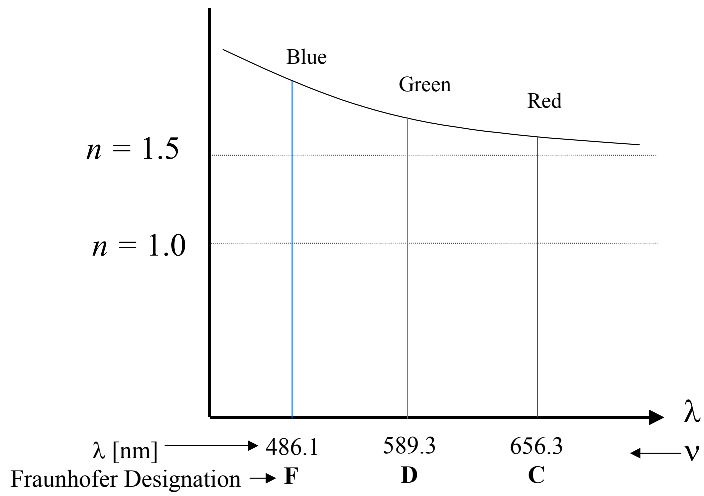
8.2 Index of Refraction of Glass

We have already stated that the index of refraction n relates the phase velocity of light in vacuum with that in matter:

$$n = \frac{c}{\mathrm{v}_\phi} \geq 1.$$

In a dispersive medium, the index n decreases with increasing λ, which ensures that the *phase velocity* $\frac{\omega}{k}$ (of the average wave) is larger than the *group velocity* $\frac{d\omega}{dk}$ (of the modulation wave).

Refraction is the result of the interaction of light with atoms in the medium and depends on wavelength because the refractive index is also; recall that the index decreases with increasing wavelength:



Typical dispersion curve for glass showing the decrease in n with increasing λ and the three spectral wavelengths used to specify "refractivity", "mean dispersion", and "partial dispersion".

To a first approximation, the index of refraction varies as λ^{-1}, which allows us to write an empirical expression for the *refractivity* of the medium $n-1$:

$$n\left[\lambda\right]-1 \cong a+\frac{b}{\lambda}$$

where a and b are parameters determined from measurements. The observation that the index decreases with increasing λ determines that $b>0$. Cauchy came up with an empirical relation for the refractivity more free parameters:

$$n\left[\lambda\right]-1 \cong A\left(1+\frac{B}{\lambda^2}+\frac{C}{\lambda^4}+\cdots\right)$$

Again, the behavior of normal dispersion ensures that A and B are both positive. Yet a better formula was proposed by Hartmann:

$$n\left[\lambda\right] \cong n_0+\frac{\alpha}{\left(\lambda-\lambda_0\right)^{1.2}}$$

where $\alpha>0$. The refractive properties of the glass are approximately specified by the refractivity and the measured differences in refractive index at the three Fraunhofer wavelengths *F, D,* and *C*:

Refractivity	$n_D - 1$	$1.75 \leq n_D \leq 1.5$
Mean Dispersion	$n_F - n_C > 0$	*differences between blue and red indices*
Partial Dispersion	$n_D - n_C > 0$	*differences between yellow and red indices*
Abbé Number	$\nu \equiv \frac{n_D - 1}{n_F - n_C}$	*ratio of refractivity and mean dispersion,* $25 \leq \nu \leq 65$

Glasses are specified by six-digit numbers *abcdef*, where $n_D = 1.abc$, to three decimal places, and $\nu = de.f$. Note that larger values of the refractivity mean that the refractive index is larger and thus so is the deviation angle in Snell's law. A larger Abbé number means that the mean dispersion is smaller and thus there will be a smaller difference in the angles of refraction. Such glasses with larger Abbé numbers and smaller indices and less dispersion are *crown* glasses, while glasses with smaller Abbé numbers are *flint* glasses, which are "denser". Examples of glass specifications include Borosilicate crown glass (BSC), which has a specification number of 517645, so its refractive index in the D line is 1.517 and its Abbé number is $\nu = 64.5$. The specification number for a common flint glass is 619364, so $n_D = 1.619$ (relatively large) and $\nu = 36.4$ (smallish). Now consider the refractive indices in the three lines for two different glasses: "crown" (with a smaller n) and "flint:"

Line	λ [nm]	n for Crown	n for Flint
C	656.28	1.51418	1.69427
D	589.59	1.51666	1.70100
F	486.13	1.52225	1.71748

The glass specification numbers for the two glasses are evaluated to be:

For the crown glass:

$$\textit{refractivity:}\ n_D - 1 = 0.51666 \cong 0.517$$

$$\textit{Abbé number}:\ \nu = \frac{1.51666 - 1}{1.52225 - 1.51418} \cong 64.0$$

$\boxed{\textit{Glass number} = 517640}$

For the flint glass:

$$\textit{refractivity:L}\ n_D - 1 = 0.70100 \cong 0.701$$

$$\textit{Abbé number}:\ \nu = \frac{0.70100 - 1}{1.71748 - 1.69427} \cong 30.2$$

$\boxed{\textit{Glass number} = 701302}$

8.2.1 Optical Path Length

Because the phase velocity of light in a medium is less than that in vacuum, light takes longer to travel through a given thickness of material than through the same "thickness" of vacuum. For a fixed distance d, we know that:

$$\begin{aligned} d &= \mathrm{v} \cdot t \ (\textit{distance} = \textit{velocity} \times \textit{time}) \\ &= c \cdot t_1 \quad (\textit{in vacuum}) \\ &= \frac{c}{n} \cdot t_2 \quad (\textit{in medium of index } n) \\ &\Longrightarrow t_1 = \frac{t_2}{n} \Longrightarrow t_2 > t_1 \end{aligned}$$

In the time t_2 required for light to travel the distance d in a material of index n, light would travel a longer distance $nd = ct_2$ in vacuum. The distance nd traveled in vacuum in the equivalent time is the *optical path length* in the medium.

8.3 Polarization

Maxwell's equations demonstrated that light is a *transverse* wave (as opposed to longitudinal waves, e.g., sound). Both the $\underline{\mathbf{E}}$ and $\underline{\mathbf{B}}$ vectors are perpendicular to the direction of propagation of the radiation. Even before Maxwell, Thomas Young inferred the transverse character of light in 1817 when he passed light through a calcite crystal (calcium carbonate, $CaCO_3$). Two beams emerged from the crystal, which Young brilliantly deduced were orthogonally polarized, i.e., the directions of the $\underline{\mathbf{E}}$ vectors of the two beams are orthogonal. The two components of an electromagnetic wave are the electric field $\underline{\mathbf{E}}\ \left[\frac{\mathrm{V}}{\mathrm{m}}\right]$ and the magnetic field $\underline{\mathbf{B}}\ \left[tesla = \frac{webers}{m^2}\right]$.

The *polarization* of radiation is defined as the plane of vibration of the electric vector $\underline{\mathbf{E}}$, rather than of $\underline{\mathbf{B}}$, because the effect of the $\underline{\mathbf{E}}$ field on a free charge (an electron) is much greater than the effect of $\underline{\mathbf{B}}$. This is seen from the Lorentz equation, or the Lorentz force law:

$$\underline{\mathbf{F}} \propto q_0 \left(\underline{\mathbf{E}} + \frac{\underline{\mathbf{v}}}{c} \times \underline{\mathbf{B}} \right)$$

$$\begin{aligned} q_0 &= \text{charge [coulombs]} \\ \underline{\mathbf{F}} &= \text{force on the charge [newtons, } 1\,\mathrm{N} = 1\tfrac{\mathrm{kg-m}}{\mathrm{s}^2}] \\ \underline{\mathbf{v}} &= \text{velocity of the charge } q_0\text{, measured in } [\tfrac{\mathrm{m}}{\mathrm{s}}] \\ c &= \text{velocity of light } [3 \cdot 10^8 \tfrac{\mathrm{m}}{\mathrm{s}}] \end{aligned}$$

The factor c^{-1} ensures that the force on the electron due to the magnetic field is usually much smaller than the electric force.

8.3.1 Plane Polarization = Linear Polarization

The most familiar type of polarization is linear polarization, where the $\underline{\mathbf{E}}$-vector oscillates in the same plane at all points on the wave.

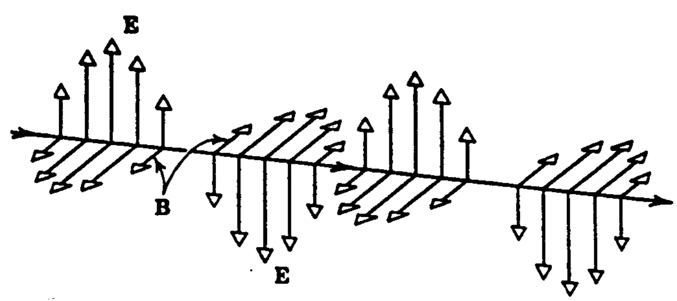
Any state of linear polarization can be expressed as a linear combination (sum) of two orthogonal states (*basis states*), e.g., the x- and y-components of the $\underline{\mathbf{E}}$-vector for a wave traveling toward $z = \pm\infty$:

$$\underline{\mathbf{E}} = \underline{\mathbf{E}}[\underline{\mathbf{r}}, t] = [\hat{\underline{\mathbf{x}}} E_x + \hat{\underline{\mathbf{y}}} E_y] \cos[kz - \omega t]$$
$$\hat{\underline{\mathbf{x}}}, \hat{\underline{\mathbf{y}}} = \textit{unit vectors along } x \textit{ and } y$$
$$E_x, E_y = \textit{amplitudes of the } x\textit{- and } y\textit{-components of } \underline{\mathbf{E}}.$$

For a wave of amplitude E_0 polarized at an angle θ relative to the x-axis:

$$E_x = E_0 \cos[\theta]$$
$$E_y = E_0 \sin[\theta]$$

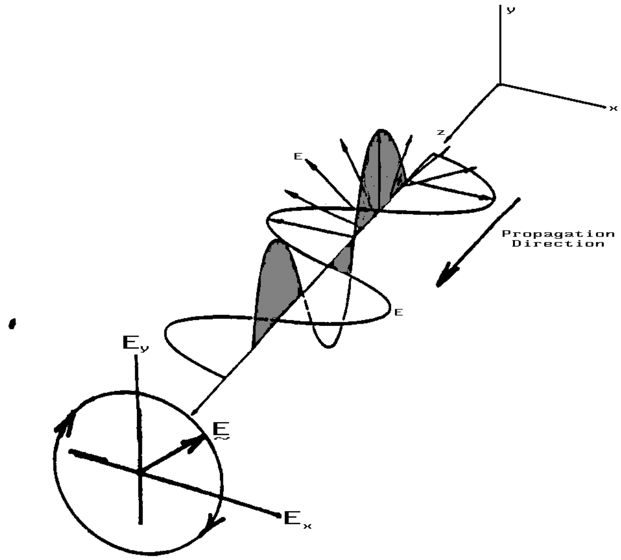
Linearly polarized radiation oscillates in the same plane at all times and at all points in space. Especially note that $E_x$and $E_y$are *in phase* for linearly polarized light, i.e., both components have zero-crossings at the same point in time and space.



Electric field vector **E** *and magnetic field vector* **H** *of a plane-polarized wave*

8.3.2 Circular Polarization

If the $\underline{\mathbf{E}}$-vector describes a helical (i.e., screw-like) motion in space, the projection of the $\underline{\mathbf{E}}$-vector onto a plane normal to the propagation direction $\underline{\mathbf{k}}$ exhibits circular motion over time, hence the polarization is *circular*:



Circular polarization occurs when the electric fields along orthogonal axes have the same amplitude by their phases differ by $\pm\frac{\pi}{2}$ radians.

If we sit at a fixed point in space $z = z_0$, the motion of the $\underline{\mathbf{E}}$-vector is the sum of two orthogonal linearly polarized states, but with one component out-of-phase by $90° = \frac{\pi}{2}$ radians. The math is identical to that used to describe oscillator motion as the projection of rotary motion:

$$\textit{motion} = \underline{\hat{\mathbf{x}}}\cos[\omega t] + \underline{\hat{\mathbf{y}}}\cos\left[\omega t \mp \frac{\pi}{2}\right] = \underline{\hat{\mathbf{x}}}\cos[\omega t] \pm \underline{\hat{\mathbf{y}}}\sin[\omega t]$$

For a traveling wave:

$$\begin{aligned}\underline{\mathbf{E}} = [E_x, E_y] &= \left[E_0\cos[kz - \omega t], E_0\cos\left[kz - \omega t \mp \frac{\pi}{2}\right]\right] \\ &= [E_0\cos[kz - \omega t], \pm E_0\sin[kz - \omega t]]\end{aligned}$$

where the upper sign applies to right-handed circular polarization (angular momentum convention)

8.3.3 Nomenclature for Circular Polarization

Like linearly polarized light, circularly polarized light has two orthogonal states, i.e., clockwise and counterclockwise rotation of the $\underline{\mathbf{E}}$-vector. These are termed *right-handed* (RHCP) and *left-handed* (LHCP). There are two conventions for the nomenclature:

1. Angular Momentum Convention (my preference): Point the thumb of the $\left\{\begin{array}{c}\text{right}\\\text{left}\end{array}\right\}$ hand in the direction of propagation. If the fingers point in the direction of ro-

tation of the $\underline{\mathbf{E}}$-vector, then the light is $\left\{ \begin{array}{c} \text{RHCP} \\ \text{LHCP} \end{array} \right\}$.

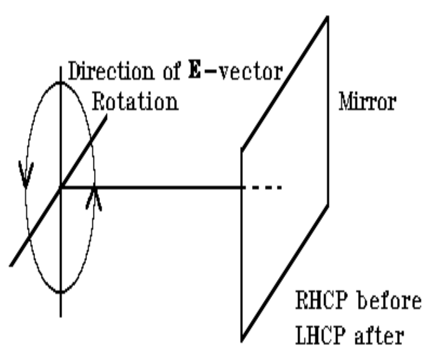
2. Optics (also called *screwy*) Convention: The path traveled by the $\underline{\mathbf{E}}$-vector of RHCP light is the same path described by a right-hand screw. Of course, the natural laws defined by Murphy ensure that the two conventions are opposite: RHCP light by the angular momentum convention is LHCP by the screw convention.

8.3.4 Elliptical Polarization, Reflections

If the amplitudes of the x-and y-components of the E-vector are not equal, or if the phase difference is not $\pm\frac{\pi}{2} = \pm 90°$, then the projection of the path of the $\underline{\mathbf{E}}$-vector is not a circle, but rather an ellipse. This results in *elliptical polarization.* Note that elliptical polarization may be either right- or left-handed, as defined above.

8.3.5 Change of Handedness on Reflection

By conservation of angular momentum, the direction of rotation of the $\underline{\mathbf{E}}$-vector does not change on reflection. Since the direction of propagation reverses, the handedness of the circular or elliptical polarization changes:



Change in "handedness" of a circularly polarized wave upon reflection by a mirror.

Natural Light

The superposition of emissions from a large number of thermal source elements (as in a light bulb) has a random orientation of polarizations. The state of polarization of the resulting light changes direction randomly over very short time intervals ($\cong 10^{-8}$ seconds). The radiation is termed *unpolarized*, even though it *is* polarized when viewed within this short time period. Natural light is neither totally polarized nor totally unpolarized; rather, we speak of partial polarization.

8.4 Description of Polarization States

8.4.1 Jones Vector

The components of the electric field in the two orthogonal directions may used to represent a vector with complex components. This is called a *Jones vector*, which is useful *only* for completely polarized light.

$$
\begin{aligned}
\underline{\mathbf{E}} &= \mathrm{Re}\{\underline{\mathbf{E}}_0 e^{i[kz-\omega t]}\} = \left[\mathrm{Re}\left\{E_x e^{i[kz-\omega t]}\right\}, \mathrm{Re}\left\{E_y e^{i(kz-\omega t-\delta)}\right\}\right] \\
&= \mathrm{Re}\left\{[E_x, E_y e^{-i\delta}] e^{i[kz-\omega t]}\right\} \\
&\Longrightarrow \text{Jones Vector } \mathcal{E} = \begin{bmatrix} E_x \\ E_y e^{-i\delta} \end{bmatrix}
\end{aligned}
$$

Examples:

1. Plane-polarized light along x-axis

$$
\mathcal{E} = \begin{bmatrix} E_0 \\ 0 \end{bmatrix}
$$

2. Plane-polarized light along y-axis:

$$
\mathcal{E} = \begin{bmatrix} 0 \\ E_0 \end{bmatrix}
$$

3. Plane-polarized light at angle θ to x-axis:

$$
\mathcal{E} = \begin{bmatrix} E_0 \cos[\theta] \\ E_0 \sin[\theta] \end{bmatrix}
$$

4. RHCP

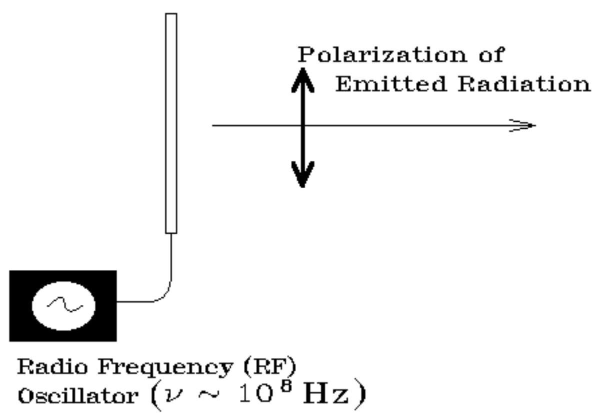
$$
\begin{aligned}
\underline{\mathbf{E}} &= \underline{\hat{\mathbf{x}}} E_0 \cos[kz-\omega t] + \underline{\hat{\mathbf{y}}} E_0 \sin[kz-\omega t] \\
&= \underline{\hat{\mathbf{x}}} E_0 \cos[kz-\omega t] + \underline{\hat{\mathbf{y}}} E_0 \cos\left[kz-\omega t-\frac{\pi}{2}\right] \\
&= \mathrm{Re}\left\{\begin{bmatrix} E_0 \\ E_0 \exp\left[-\frac{i\pi}{2}\right] \end{bmatrix} e^{i[kz-\omega t]}\right\}\} \Longrightarrow \mathcal{E} = \mathrm{Re}\left\{E_0 \begin{bmatrix} 1 \\ \exp\left[-\frac{i\pi}{2}\right] \end{bmatrix} e^{i[kz-\omega t]}\right\}
\end{aligned}
$$

Other representations of the state of polarization are available (e.g., Stokes' parameters, coherency matrix, Mueller matrix, Poincare sphere). They are more complicated, and hence more useful, i.e., they can describe partially polarized states. For more information, see (for example), *Polarized Light* by Shurcliff.

8.5 Generation of Polarized Light

8.5.1 Selective Emission:

If all emitting elements of a source (e.g., electrons in a bulb filament), vibrate in the same direction, the radiated light will be polarized in that direction. This is difficult to achieve at optical frequencies ($\Delta t \lessapprox 10^{-14}\,\mathrm{s} \implies \nu \gtrapprox 10^{14}\,\mathrm{Hz}$), but is easy at radio or microwave frequencies ($\nu \lesssim 10^{8}\,\mathrm{Hz}$) by proper design of the antenna that radiates the energy. For example, a radio-frequency oscillator attached to a simple antenna forces the free electrons in the antenna to oscillate along the long (vertical) dimension of the antenna. The emitted radiation is therefore mostly oscillating in the vertical direction; it is vertically polarized.



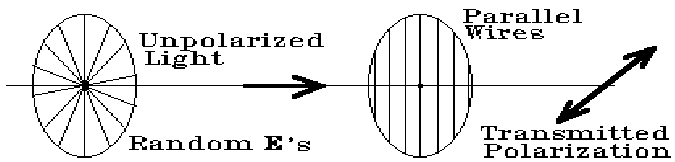
"Light" (electromagnetic radiation) emitted by a "dipole" radiator is polarized in the direction of motion of the emitting electrons (vertical, in this case).

Rather than generating polarized light at the source, we can obtain light of a selected polarization from natural light by removing unwanted states of polarization. This is the mechanism used in the next section.

8.5.2 Selective Transmission or Absorption

A manmade device for selecting a state of polarization by selective absorption is *polaroid*. This operates like the microwave-polarizing skein of wires. The wires are parallel to the y-axis in the figure. Radiation incident on the wires drives the free electrons in the wires in the direction of polarization of the radiation. The electrons driven in the y-direction along the surface of the wire and strike other such electrons, thus dissipating the energy in thermal collisions. What energy that is reradiated by such electrons is mostly directed back toward the source (reflected). The x-component of the polarization is not so affected, since the electrons in the wire are constrained against movement in that direction. The x-component of the radiation therefore passes nearly unaffected.

Common polaroid sheet acts as a skein of wires for optical radiation. It is made from clear polyvinyl acetate which has been stretched in one direction to produce long chains of hydrocarbon molecules. The sheet is then immersed in iodine to supply lots of free electrons.

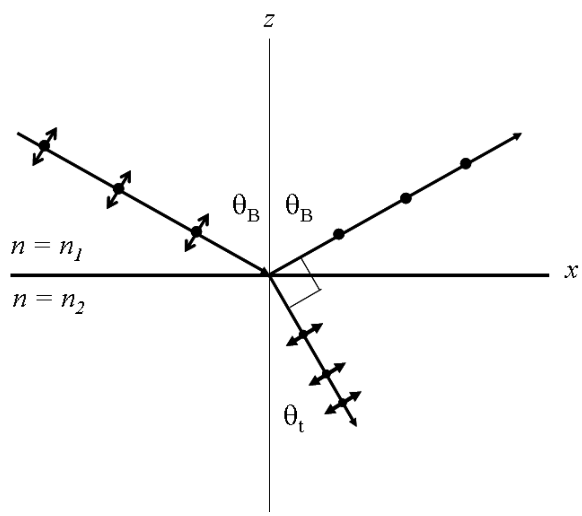


Polarization by "skein of wires" – the radiation polarized parallel to the direction of the wires in the skein is absorbed, so the radiation polarized ***perpendicular to the wires*** *is transmitted.*

8.5.3 Generating Polarized Light by Reflection – Brewster's Angle

H§8.6

The two polarizations of light reflected from an interface between two different dielectric media (i.e., media with different real refractive indices) see the same configuration of the interface only with normal incidence (i.e., the light is incident perpendicular to the surface). Thus the two polarizations must be identically reflected. However, if the light is incident obliquely, one polarization "sees" the bound electrons of the surface differently and therefore is reflected differently. The reflected wave is polarized to some extent; the amount of polarization depends on the angle of incidence and the index of refraction n. The polarization mechanism is simply pictured as a *forced electron oscillator*. The bound electrons in the dielectric material are driven by the incident oscillating electric field of the radiation $\underline{\mathbf{E}} \exp\left[i\left(k_0 z_0 \pm \omega_0 t\right)\right]$, and hence vibrate at frequency $\nu_0 = \frac{\omega_0}{2\pi}$. Due to its acceleration, the vibrating electron reradiates radiation at the same frequency ν to produce the reflected wave. The state of polarization of the reflected radiation is a function of the polarization state of the incident wave, the angle of incidence, and the indices of refraction on either side of the interface. If the reflected wave and the refracted wave are orthogonal (i.e., $\theta_0 + \theta_t = 90° \Longrightarrow \theta_t = \frac{\pi}{2} - \theta_0$), then the reflected wave is completely plane polarized parallel to the surface (and thus polarized perpendicular to the plane of incidence). This angle appeared in the discussion of the reflectance coefficients in the previous section. In this case, the electrons driven *in* the plane of the incidence will not emit radiation at the angle required by the law of reflection. This angle of complete polarization is called *Brewster's Angle* θ_B, which we mentioned earlier during the discussion of the Fresnel equations.



Brewster's angle: the incident beam at $\theta_0 = \theta_B$ is unpolarized. The reflectance coefficient for light polarized in the plane (TM waves) is 0, and the sum of the incident and refracted angle is $90° = \frac{\pi}{2}$. Thus $\theta_B + \theta_t = \frac{\pi}{2} \Longrightarrow \theta_t = \frac{\pi}{2} - \theta_B$.

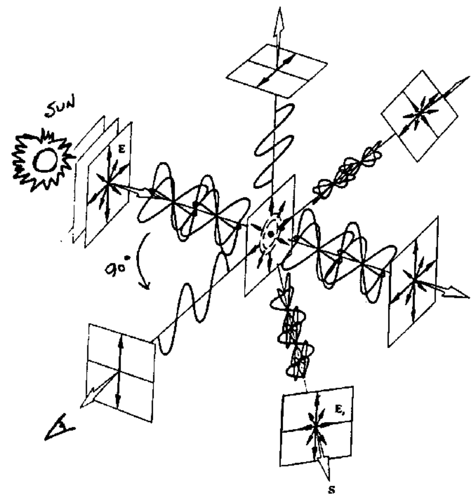
At Brewster's angle,

$$n_1 \sin[\theta_B] = n_2 \sin\left[\frac{\pi}{2} - \theta_B\right]$$
$$\Longrightarrow \theta_B = \tan^{-1}\left[\frac{n_2}{n_1}\right]$$

If $n_1 = 1$ (air) and $n_2 = 1.5$ (glass), then $\theta_B \cong 56.3°$. For incident angles larger than about 56°, the reflected light is plane polarized parallel to the plane of incidence. If the dense medium is water ($n_2 = 1.33$), then $\theta_B \cong 52.4°$. This happens at the interface with any dielectric. The reflection at Brewster's angle provides a handy means to determine the polarization axis of a linear polarizer – just look through the polarizer at light reflected at a steep angle.

8.5.4 Polarization by Scattering

Light impinging on an air molecule drives the electrons of the molecule in the direction of vibration of the electric field vector. This motion causes light to be *reradiated* in a dipole pattern; i.e., no light is emitted along the direction of electron vibration. If we look at scattered light (e.g., blue sky) at 90° from the source, the light is completely linearly polarized. Note that if the light is multiply scattered, as in fog, each scattering disturbs the state of polarization and the overall linear state is perturbed into unpolarized radiation.



Scattering of sunlight by atmospheric molecules.

8.6 Birefringence – Double Refraction

H§8.4

Many natural crystals and manmade materials interact with the two orthogonal polarizations differently. This is often due to an anistropy (nonuniformity) in the crystalline structure; such materials are called *dichroic* or *birefringent* Many crystals (e.g., calcite) divide a nonpolarized light wave into two components with orthogonal polarizations. The two indices of refraction are sometimes denoted n_f and n_s for *fast* and *slow* axes, where $n_f < n_s$. They are also denoted n_o and n_e for *ordinary* and *extraordinary* axes. The ordinary ray obeys Snell's law; the extraordinary ray does not.One is called the *ordinary ray*, because it obeys Snell's law for refraction. The second, or *extraordinary ray*, does not obey Snell. By dividing the incoming natural light into two beams in such a crystal, we can select one of the two polarizations.

8.6.1 Examples:

Refractive indices along the fast and slow axes at $\lambda = 589.3\,\text{nm}$

Material	n_s	n_f
Calcite ($CaCO_3$)	1.6584	1.4864
Crystalline Quartz (SiO_2)	1.5534	1.5443
Ice (crystalline H_2O)	1.313	1.309
Rutile (TiO_2)	2.903	2.616
Sodium Nitrate ($SiNO_3$)	1.5854	1.3369

The wavelength of light in a medium is $\lambda' = \frac{\lambda}{n}$, so light along the two polarization directions have different wavelengths:

$$\lambda'_s = \frac{\lambda}{n_s} < \lambda'_f = \frac{\lambda}{n_f}$$

8.6.2 Phase Delays in Birefringent Materials – Wave Plates

Consider light incident on a birefringent material of thickness d. The electric field as a function of distance z and time t is:

$$\underline{\mathbf{E}}[z,t] = \left(\hat{\underline{\mathbf{x}}}E_x + \hat{\underline{\mathbf{y}}}E_y\right)\ e^{i(kz-\omega t)}.$$

At the input face of the material ($z = 0$) and the output face ($z = d$), the fields are:

$$\begin{aligned}\underline{\mathbf{E}}[z=0,t] &= \left(\hat{\underline{\mathbf{x}}}E_x + \hat{\underline{\mathbf{y}}}E_y\right)\ e^{-i\omega t}\\ \underline{\mathbf{E}}[z=d,t] &= \left(\hat{\underline{\mathbf{x}}}E_x + \hat{\underline{\mathbf{y}}}E_y\right)\ e^{i(kd-\omega t)}\end{aligned}$$

If $n_x = n_s > n_y = n_f$,, then $\lambda_f > \lambda_s$ and :

$$k_s = k_x = \frac{2\pi n_s}{\lambda} > k_f = k_y = \frac{2\pi n_f}{\lambda}$$

The field at the output face ($z = d$) is therefore:

$$\begin{aligned}\underline{\mathbf{E}}[d,t] &= \left[\hat{\underline{\mathbf{x}}}E_x \exp\left[+\frac{i\,(2\pi d\cdot n_s)}{\lambda}\right] + \hat{\underline{\mathbf{y}}}E_y \exp\left[+i\frac{2\pi d\cdot n_f}{\lambda}\right]\right]\ e^{-i\omega t}\\ &= \left(\hat{\underline{\mathbf{x}}}E_x + \hat{\underline{\mathbf{y}}}E_y \exp\left[\frac{2\pi i}{\lambda}d\,(n_f - n_s)\right]\right)\ \exp\left[+i\frac{2\pi d n_f}{\lambda}\right]\end{aligned}$$

By defining a constant phase term $\delta \equiv \frac{2\pi}{\lambda}d\,(n_f - n_s)$, the electric field at the output face of the birefringent material can be expressed as:

$$\underline{\mathbf{E}}[d,t] = \left(\hat{\underline{\mathbf{x}}}E_x + \hat{\underline{\mathbf{y}}}E_y e^{i\delta}\right)\ \exp\left[+i\frac{2\pi d n_f}{\lambda}\right]$$

On emergence from the material, the y-component of the polarization has a different phase than the x-component; the phase difference is δ.

Example:

$\delta = +\frac{\pi}{2} \Longrightarrow (n_f - n_s)\,d = -\frac{\lambda}{4}$, and there is a phase difference of one quarter wavelength between the polarizations of the x- and the y-components of the wave. This is a *quarter-wave plate*. The required thickness d of the material is:

$$d = \frac{\lambda}{4\,(n_s - n_f)}$$

And the emerging field is:

$$\underline{\mathbf{E}}\left[d,t\right]=\left[\underline{\hat{\mathbf{x}}}E_x+\underline{\hat{\mathbf{y}}}E_ye^{+\frac{i\pi}{2}}\right]\ \exp\left[i\left(k_sd-\omega t\right)\right]$$

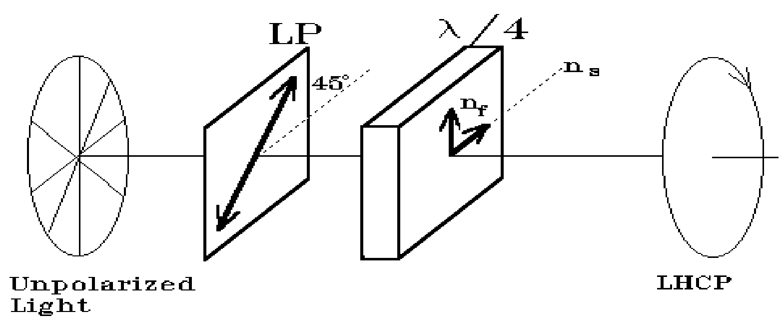
If $E_x=E_y$, (i.e., the incident wave is linearly polarized @ 45° to the x-axis), then the emerging wave is circularly polarized. This is the principle of the circular polarizer.

Example:

If $\delta=+\pi \implies d=\frac{\lambda}{2\left(n_s-n_f\right)}$, and the relative phase delay is 180°. Such a device is a *half-wave plate.* If the incident light is linearly polarized along the orientation midway between the fast and slow axes, the plane of polarization of the exiting linearly polarized light is rotated by 90°.

8.6.3 Circular Polarizer:

A circular polarizer is a sandwich of a linear polarizer and a $\frac{\lambda}{4}$ plate, where the polarizing axis is oriented midway between the fast and slow axes of the quarter-wave plate. The LP ensures that equal amplitudes exist along both axes of the quarter-wave plate, which delays one of the components to create circularly polarized light. Light incident from the back side of a circular polarizer is not circularly polarized on exit; rather it is linearly polarized. A circular polarizer can be recognized and properly oriented by placing it on a reflecting object (e.g.,.a dime). If the image of the coin is dark, the polarizer has the linear polarizer on top. This is because the handedness of the light is changed on reflection; the light emerging from the $\frac{\lambda}{4}$ plate is now linearly polarized perpendicular to the axis of the LP and no light escapes.



A circular polarizer is a sandwich of a linear polarizer and a quarter-wave plate.

8.7 Critical Angle – Total Internal Reflection

We also mentioned this phenomenon during the discussion of the Fresnel equations. From Snell, we have the relation:

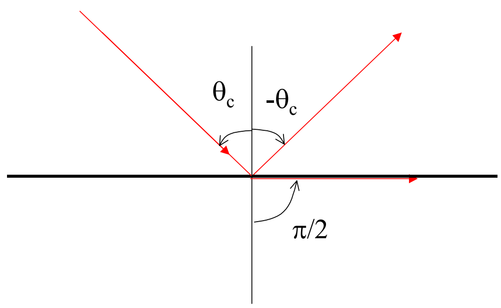
$$n_1 \sin[\theta_1] = n_2 \sin[\theta_2]$$

If $n_1 > n_2$ then a specific angle θ_1 satisfies the condition:

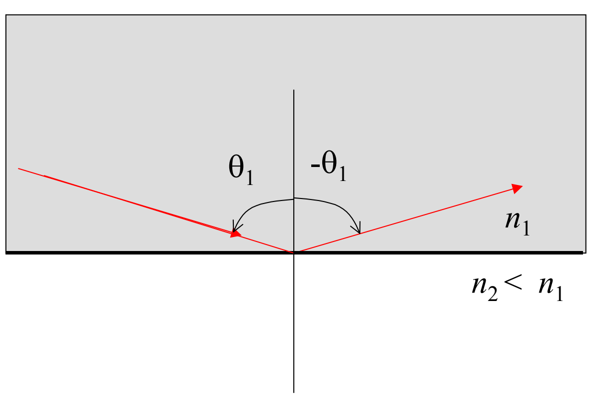
$$\frac{n_1}{n_2}\sin[\theta_1] = 1 \Longrightarrow \sin[\theta_1] = \frac{n_2}{n_1} < 1 \Longrightarrow \theta_2 = \frac{\pi}{2}$$

which means that the outgoing ray is refracted parallel to the interface ("surface"). The incident angle θ_1 that satisfies this condition is the *critical angle* θ_c

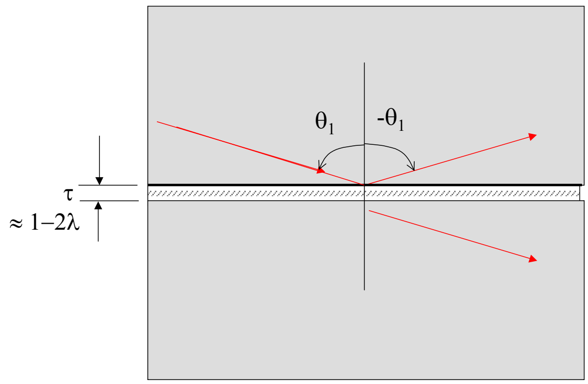
$$\theta_c = \sin^{-1}\left[\frac{n_2}{n_1}\right]$$



For crown glass with $n_d = 1.52$, the critical angle is $\sin^{-1}\left[\frac{1}{1.52}\right] \cong 0.718$ radians $\cong 41°$. For a common flint glass with $n_d = 1.70$, then $\theta_c \cong 0.629$ radians $\cong 36°$. If the incident angle $\theta_1 > \theta_c$ and $n_1 > n_2$ (e.g., the first medium is glass and the second is air), then no real-valued solution for Snell's law exists, and there is no refracted light. This is the well-known phenomenon of *total internal reflection* – all of the incident light is reflected at the interface.



This may be analyzed rigorously by applying Maxwell's equations to show that the refracted angle θ_2 is *complex valued* instead of real valued, so that the electromagnetic field is attenuated exponentially as it crosses the interface. In other words, the electric field decays so rapidly across the interface that no energy can flow across the boundary, and hence no light escapes. However, we can "frustrate" the total internal reflection by placing another medium (such as another piece of glass) within a few light wavelengths of the interface. If close enough to the boundary, then some electric field can get into the second glass and a refracted wave "escapes".



Schematic of "frustrated total internal reflection": some energy can "jump" across a small gap between two pieces of glass even though the incident angle exceeds the critical angle. As the width τ of the gap increases, then the quantity of energy coupled across the gap decreases very quickly.