9.14 Classification of Surfaces

Definition 9.14.1 *A* surface *is a 2-dimensional manifold.*

Definition 9.14.2 *Let S_1 and S_2 be two manifolds of dimension n. The* connected sum $S_1 \# S_2$ *is the manifold obtained by removing a disk D^n from S_1 and S_2 and gluing the resulting manifold with boundary $S^1 \amalg S^1$ to the cylinder $S^1 \times [0,1]$.*

Theorem 9.14.3 *(a) Any compact orientable surface is homeomorphic to a sphere, or to the connected sum*

$$T^2 \# \ldots \# T^2$$

.

(b) Any compact nonorientable surface is homeomorphic to the connected sum

$$P \# \ldots P$$

where P is the projective plane $\mathbb{R}P^2$.

Alternative version of part (b) of Theorem 9.14.3:

Theorem 9.14.4 *Any compact orientable surface is homeomorphic to the connected sum of an orientable surface with either one copy of the projective plane P or one copy of the Klein bottle K.*

Proof of Theorem 9.14.3:

Definition 9.14.5 Euler Characteristic

The Euler characteristic of a topological space M is the alternating sum of the dimensions of the homology groups (with rational coefficients):

$$\chi(M) = h_0(M) - h_1(M) + \ldots$$

where $h_j(M) = \dim H_j(M; \mathbb{Q})$.

For a manifold of dimension 2 equipped with a triangulation, the Euler characteristic is given by

$$\chi(M) = V - E + F$$

where V is the number of vertices, E the number of edges and F the number of faces. The Euler characteristic is independent of the choice of triangulation.

Proposition 9.14.6 *The Euler characteristic of a connected sum of surfaces S_1 and S_2 is given by*

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$$

(This is proved by counting the number of vertices, edges and faces in a natural triangulation of the connected sum.)

Lemma 9.14.7 *The Euler characteristics of surfaces are as follows:*

genus = 0

$$\chi(S^2) = 2$$

genus = g

$$\chi(T^2 \# \dots T^2) = 2 - 2g$$

(the genus is the number of copies of T^2)

(connected sum of n copies of the projective plane)

$$\chi(P \# \dots \# P) = 2 - n$$

(connected sum of K with genus g orientable surface)

$$\chi(K \# T^2 \# \dots \# T^2) = -2g$$

(connected sum of P with genus g orientable surface)

$$\chi(P \# T^2 \# \dots \# T^2) = 1 - 2g$$

Lemma 9.14.8 *Surfaces are classified by:*
(i) whether they are orientable or nonorientable
(ii) their Euler characteristic

Proof of Theorem 9.14.3:

1. Take a triangulation of the surface S. Glue together some (not all) of the edges to form a surface D which is a closed disk. (This comes from a Lemma which asserts that if we glue together two disks along a common segment of their boundaries, the result is again a disk.) The edges along the boundary of D form a word where each edge is designated by a letter x_1 or x_2, with the same letter used to designate edges that are glued.

2. We now have a polygon D whose edges must be identified in pairs to obtain S. We subdivide the edges as follows.

(i) Edges *of the first kind* are those for which the letter designating the edge appears with both exponents $+1$ and -1.

(ii) Edges *of the second kind* are those for which the letter designating the edge appears with only one exponent ($+1$ or -1)

Adjacent edges of the first kind can be eliminated if there are at least four edges. (See Figure 1.17, p. 22, figure #2.)

3. Identify all vertices to a single vertex. If there are at least 2 different equivalence classes, then the polygon must have an adjacent pair of vertices which are not equivalent, call them P and Q.

Cut along the edge c from Q to the other vertex of a. Then glue together the two edges labelled a. The new polygon has one less vertex in the equivalence class of P. (See Figure 1.18, p. 23, figure #3.)

Perform step 2 again if possible (eliminate adjacent edges). Then perform step 3 again, reducing the number of vertices in the equivalence class of P. If more than one equivalence class of vertices remains, repeat the procedure to reduce the number of equivalence classes of vertices to 1, in other words we reduce to a polygon where all vertices are to be identified to a single vertex.

4. Make all pairs of edges of the second king adjacent. (See Figure 1.19, p. 24, #4.) Thus if there are no pairs of edges of the first kind, the symbol becomes

$$x_1x_1x_2x_2\ldots x_nx_n$$

In this case the surface is

$$S = P\#\ldots\#P$$

(the connected sum of n copies of P).

Otherwise there is at least one pair of edges of the first kind (label these c) One can argue that there is a second pair of edges of the first kind interspersed (label these d. It is possible to transform these so they are consecutive, so the symbol includes

$$cdc^{-1}d^{-1}$$

This corresponds to the connected sum of one copy of T^2 with a surface with fewer edges in its triangulation. (See Figure 1.21, p. 25, #5.) □

Lemma 9.14.9

$$T^2\#P \cong P\#P\#P$$

Remark 9.14.10 *$P\#P \cong K$ This is because we can carve up the diagram representing the Klein bottle, a square with two parallel edges identified in the same direction, and the two remaining parallel edges identified in opposite directions. (See Figure 1.5, p. 10, #1) This is the union of two copies of the Möbius strip along their boundary, using the fact that a Möbius strip is the same as the complement of a disk in the real projective plane.*

This reduces the proof of Lemma 9.14.9 to proving

Lemma 9.14.11 $P\#K \cong P\#T$

This is proved by decomposing a torus and a Klein bottle as the union of two rectangles. We excise a disk from one of the rectangles, and glue a Möbius strip to the boundary of the excised disk (to form the connected sum of P with the torus or Klein bottle). The text (Massey, see handout, Lemma 1.7.1) argues that the resulting objects are homeomorphic. Indeed, we can regard this as taking the connected sum of a Möbius strip with a torus or Klein bottle, and then gluing a disk to the boundary of the Möbius strip. The first step (connected sum of Möbius strip with torus or Klein bottle) yields two spaces that are manifestly homeomorphic. So they remain homeomorphic after gluing a disk to the boundary of the Möbius strip. See Figure 1.23, p. 27, #6. $\square$

References: 1. William S. Massey, *Algebraic Topology: An Introduction* (Harcourt Brace and World, 1967), Chapter 1.

(All figures are taken from Chapter 1 of Massey's book.)

2. James R. Munkres, *Topology* (Second Edition), Chapter 12.