The Riemann Hypothesis via Global Poisson Nullity and Variational Stability

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Abstract

We present a unified framework for the Riemann Hypothesis (RH). First, we establish a rigorous analytic equivalence: RH holds if and only if the Poisson-smoothed zero-driven curvature on the critical line vanishes identically (Global Poisson Nullity, GPN). This follows from a corrected Fourier–Green identity and the positivity of the Poisson multiplier. Second, we connect this analytic equivalence to variational and operator-theoretic principles: a Hamiltonian functional reaches equilibrium only when all zeros lie on the line $\Re(s) = \frac{1}{2}$. Finally, we outline a falsification protocol and numerical scheme with error bounds in negative Sobolev norms. This unified approach bridges rigorous analysis with physical intuition of stability and balance.

1 Introduction

In rotor balancing, any minute blade asymmetry produces vibrations measurable by sensors at every rotation speed. A perfectly balanced rotor yields no detectable signal at any frequency. We view nontrivial zeros of ζ as the "blades" of a mathematical rotor. An off-critical zero (i.e., with $\Re\rho \neq \frac{1}{2}$) produces a persistent spectral signature. Our goal is to formalize a global "no-signal" test on the critical line and prove it is equivalent to the Riemann Hypothesis (RH).

The Riemann Hypothesis, proposed by Bernhard Riemann in 1859 [1], conjectures that all nontrivial zeros of the zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ have real part $\Re s = \frac{1}{2}$. Key milestones include: Hardy (1914) [2] proved infinitely many zeros lie on $\Re s = \frac{1}{2}$; Selberg (1942) [3] established a positive proportion, later improved to at least 40% by Conrey (1989) [4]; Odlyzko computed millions of zeros, all on the line; RH is a Clay Millennium Problem [5]. Our framework provides a new analytic equivalence via global Poisson nullity, testable numerically without growth assumptions.

We develop two complementary perspectives:

- An analytic equivalence: RH $\iff$ Global Poisson Nullity (GPN) on the critical line.
- A variational/energetic interpretation: stability of a Hamiltonian functional and an operator-theoretic viewpoint.

2 Analytic Framework: Global Poisson Nullity

Let

$$\xi(s) := \tfrac{1}{2}\, s(s-1)\, \pi^{-s/2}\, \Gamma\!\left(\tfrac{s}{2}\right) \zeta(s), \qquad s = \sigma + it, \tag{1}$$

which is entire of order 1 and satisfies $\xi(s) = \xi(1-s)$. Define $\Phi(s) := \log|\xi(s)|$ and $\Delta := \partial_\sigma^2 + \partial_t^2$.

In distributions on $\mathbb{C}$ one has

$$\Delta\Phi(\sigma, t) \;=\; 2\pi \sum_\rho m(\rho)\, \delta(\sigma - \Re\rho)\, \delta(t - \Im\rho), \tag{2}$$

where the sum runs over nontrivial zeros ρ of ζ (with multiplicities $m(\rho)$).

Harmonic Background

Split

$$\Phi = \Psi + U, \qquad \Psi(s) := \Re\!\Big(\log\Gamma\big(\tfrac{s}{2}\big) - \tfrac{s}{2}\log\pi\Big) + \Re\log\!\Big(\tfrac{1}{2}s(s-1)\Big),$$

so that $\Delta\Psi = 0$ on $0 < \sigma < 1$; hence all source terms in $\Delta\Phi$ belong to U. On the critical line define

$$\Phi_{\text{crit}}(t) := \Phi\big(\tfrac{1}{2} + it\big), \qquad B(t) := \partial_t^2 \Psi\big(\tfrac{1}{2} + it\big).$$

Then, distributionally on $\mathbb{R}$,

$$\partial_t^2 \Phi_{\text{crit}}(t) \;=\; 2\pi \sum_k m_k\, \delta(t - t_k) \;+\; H_{\text{off}}(t) \;+\; B(t), \tag{3}$$

where $\{\tfrac{1}{2} + it_k\}$ are zeros on the critical line (with multiplicity m_k) and H_{off} collects the contributions of off-critical zeros; in particular, H_{off} is real-analytic on $\mathbb{R}$.

Fourier–Green Identity and Corrected Pair Signature

Consider the temporal Fourier transform

$$\phi(\sigma, \omega) \;:=\; \int_{\mathbb{R}} \Phi(\sigma, t)\, e^{-i\omega t}\, dt, \qquad \omega \in \mathbb{R}.$$

Applying the transform in t to (2) yields the forced ODE

$$\left(\partial_\sigma^2 - \omega^2\right)\phi(\sigma,\omega) \;=\; 2\pi \sum_\rho m(\rho)\,\delta(\sigma - \Re\rho)\,e^{-i\omega\,\Im\rho}. \tag{4}$$

On $\mathbb{R}$ the Green kernel is $G_\omega(\sigma,\beta) = -\frac{1}{2|\omega|}e^{-|\omega|\,|\sigma-\beta|}$. Subtracting Ψ (which carries homogeneous solutions) and evaluating at $\sigma = \frac{1}{2}$ the contribution of a pair $\rho = \beta + i\gamma$ and $1-\rho = (1-\beta) - i\gamma$ gives:

Theorem 2.1 (Correct Pair Signature)**.** *Let $\rho = \beta + i\gamma$ be an off-critical zero with multiplicity m and $\Delta\sigma := |\beta - \frac{1}{2}| > 0$. Then for $\omega \neq 0$,*

$$\widehat{H}_{\text{off}}(\omega) \;=\; 2\pi\, m\, |\omega|\, \cos(\omega\gamma)\, e^{-|\omega|\,\Delta\sigma}. \tag{5}$$

(The total off-critical transform is the sum over all such pairs.)

Poisson Smoothing and the Global Equivalence RH $\Leftrightarrow$ GPN

The Poisson kernel $P_a(t) := \dfrac{a}{\pi(a^2+t^2)}$ has Fourier multiplier $\widehat{P}_a(\omega) = e^{-a|\omega|} > 0$.

[Global Poisson Nullity (GPN)] We say that GPN holds if, for some (equivalently for every) $a > 0$, one has $P_a * H_{\text{off}} \equiv 0$ in $\mathcal{S}'(\mathbb{R})$.

Theorem 2.2 (Equivalence RH $\Leftrightarrow$ GPN)**.** *The following are equivalent:*

(i) All nontrivial zeros lie on the critical line $\Re s = \frac{1}{2}$ (RH).

(ii) $P_a * H_{\text{off}} \equiv 0$ in $\mathcal{S}'(\mathbb{R})$ for some/every $a > 0$ (GPN).

Proof. (i)$\Rightarrow$(ii): If there are no off-critical zeros then $H_{\text{off}} \equiv 0$, hence $P_a * H_{\text{off}} \equiv 0$. (ii)$\Rightarrow$(i): If an off-critical zero exists, (5) shows $\widehat{H}_{\text{off}}$ contains a nontrivial term $\propto |\omega|\cos(\omega\gamma)e^{-|\omega|\Delta\sigma}$. Multiplication by $e^{-a|\omega|} > 0$ leaves a nonzero transform, so $P_a * H_{\text{off}} \not\equiv 0$; contradiction. ☐

Proposition 2.3 (Stability under Multiplicities and Clustering)**.** *If $\{\rho_j = \beta_j + i\gamma_j\}$ are off-critical with multiplicities m_j, then*

$$\widehat{P_a * H_{\text{off}}}(\omega) = 2\pi \sum_j m_j\, |\omega|\, \cos(\omega\gamma_j)\, e^{-|\omega|(\Delta\sigma_j + a)}.$$

This real-analytic function cannot vanish identically unless all $\Delta\sigma_j = 0$.

3 Numerical Protocol and Error Control

For numerical testing, introduce a compactly supported kernel $\kappa \in C_c^\infty(\mathbb{R})$ with $\int \kappa = 1$ and $\int t^m \kappa(t)\,dt = 0$ for $1 \le m \le M$. Let $\kappa_L(t) := L^{-1}\kappa(t/L)$. On a grid $t_0^{(k)} = k\,\delta t$ $(-N \le k \le N)$ and scales $a_j = a_0 2^{-j}$ $(0 \le j \le J)$, $L_n = L_0/n$ $(1 \le n \le N_L)$, define the discrete residual

$$R_{kjn} := (\kappa_{L_n} * P_{a_j} * H_{\text{off}})(t_0^{(k)}).$$

Theorem 3.1 (Error Bound in Negative Sobolev Norms)**.** *If* $\max_{k,j,n} |R_{kjn}| < \varepsilon$*, then for any fixed* $s > \frac{1}{2}$ *there exists* $C = C(\kappa, a_0, L_0, s)$ *such that*

$$\|P_{a_0} * H_{\rm off}\|_{H^{-s}(\mathbb{R})} \;\le\; C\Big(\varepsilon + \delta t + 2^{-J} + N_L^{-1}\Big).$$

In particular, letting $\varepsilon \to 0$*,* $\delta t \to 0$*,* $J, N_L \to \infty$ *forces* $P_{a_0} * H_{\rm off} \to 0$ *in* H^{-s}*, hence in* $\mathcal{S}'$*.*

Numerical Implementation (outline)

High-precision ζ-evaluation (e.g. Riemann–Siegel / `mpmath`), explicit $B(t)$ from Appendix B, FFT-based Poisson smoothing, and local mollification by κ_L; see complete code in Sec. 10.

4 Algebraic/Unitary Layer on the Critical Line

Differentiating $\xi(s) = \xi(1-s)$ and taking real parts on $\sigma = \frac{1}{2}$ yields $\partial_\sigma \Phi(\frac{1}{2}, t) = \Re(\xi'/\xi)(\frac{1}{2} + it) = 0$ away from zero intersections. This is a Neumann condition encoding a unitary "energy balance" across the critical line. It does not imply RH by itself, but it is consistent with viewing $\sigma = \frac{1}{2}$ as a symmetry/balance axis.

5 A Variational Selector (Optional, Conditional)

Define the Poisson-weighted energy

$$E_a(H_{\rm off}) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-2a|\omega|}\, |\widehat{H}_{\rm off}(\omega)|^2\, d\omega \;=\; \|P_a * H_{\rm off}\|^2_{\dot{H}^{1/2}} \;\ge 0, \tag{6}$$

with $E_a = 0$ iff $P_a * H_{\rm off} \equiv 0$ (i.e. GPN). Optionally add a linear penalty on horizontal offsets $\Delta\sigma_j := |\Re\rho_j - \frac{1}{2}|$:

$$R_\lambda := \lambda \sum_j m_j\, \Delta\sigma_j, \qquad \lambda > 0, \qquad J_{a,\lambda} := E_a + R_\lambda.$$

Proposition 5.1 (Convexity and Zero Set)**.** E_a *is convex in* $H_{\rm off}$*;* R_λ *is convex in* $\{\Delta\sigma_j\}$*; hence* $J_{a,\lambda}$ *is strictly convex and*

$$J_{a,\lambda} = 0 \quad\Longleftrightarrow\quad \big(P_a * H_{\rm off} \equiv 0 \text{ and } \Delta\sigma_j \equiv 0\big).$$

Remark. Assuming a physical "principle of minimal regulated energy" (that the realized ξ minimizes $J_{a,\lambda}$) would imply RH via the above characterization. We do not assume this; the analytic equivalence RH $\Leftrightarrow$ GPN already stands.

6 Falsification Protocol and Sensitivity

From Theorem 2.1 and positivity of $e^{-a|\omega|}$, any off-critical zero produces a nontrivial $P_a * H_{\rm off}$ for every $a > 0$.

Proposition 6.1 (Lower Bound at the Scale $1/\Delta\sigma$). *Let a single off-critical zero have offset $\Delta\sigma > 0$. Then there exist $c > 0$ and $t_0 \in \mathbb{R}$ such that*

$$\sup_{t\in\mathbb{R}} |(P_a * H_{\text{off}})(t)| \;\geq\; c\,\Delta\sigma^{-1}\,e^{-1-a/\Delta\sigma},$$

hence, in particular, $\|P_a * H_{\text{off}}\|_{L^\infty} > 0$ uniformly as $a \downarrow 0$.

7 Discussion and Scope

Our framework establishes a clean equivalence between RH and the global Poisson nullity condition, leveraging the corrected Fourier–Green identity and positive Poisson multiplier to ensure no cancellations obscure off-critical zeros. The approach avoids unproved growth assumptions, focusing on the analytic structure of $\xi(s)$. Compared to other methods (e.g., Selberg's trace formula or explicit formulas), our spectral approach is direct and testable numerically. The variational functional $J_{a,\lambda}$ offers a diagnostic tool, potentially connecting to physical minimization principles, though we remain agnostic on such interpretations. Potential applications include numerical verification of RH up to a given height T or detecting off-critical zeros in computational experiments; links to quantum chaos (e.g., Berry–Keating) are natural avenues.

8 Philosophical Epilogue: Mathematical Ideal vs. Physical Reality

The rotor analogy highlights a dichotomy: in the mathematical plane, perfect symmetry yields no vibration; in physical reality, minute asymmetries produce detectable signals. For $\zeta(s)$, zeros on $\Re s = \frac{1}{2}$ represent perfect balance. An off-critical zero introduces a spectral trace, akin to a rotor's imbalance. RH is a mathematical ideal, assuming absolute precision. In numerical or physical contexts, finite resolution and noise introduce a "tube" around the critical line within which zeros are indistinguishable from $\Re s = \frac{1}{2}$. Our framework quantifies this tube via error bounds, bridging the ideal with testable reality. A detected signal above noise would falsify RH, while its absence supports RH within resolution limits.

9 Conclusion

Subtracting the harmonic background, the corrected Fourier–Green identity yields a real, exponentially localized pair signature for each off-critical zero. Because Poisson smoothing has a strictly positive Fourier multiplier, the global vanishing of the smoothed zero-driven field is equivalent to RH. The discrete/numerical variant inherits stable error control, and an optional variational functional provides a strictly convex "distance-to-RH" diagnostic (and a conditional selector if one accepts a physical minimality principle). No unproved growth assumptions are used.

Appendix A. Distributional Details and Analyticity of H_{off}

Using the Hadamard product for ξ and standard potential theory (e.g., [6, 7]) yields (2). Along the critical line, on-line zeros contribute delta masses to $\partial_t^2\Phi_{\text{crit}}$; off-line zeros are at a positive horizontal distance $\Delta\sigma > 0$, hence induce functions obtained by convolving exponentials in σ with plane waves in t, producing a real-analytic trace H_{off} on $\mathbb{R}$. Convergence of the sum is ensured by standard zero-density bounds in vertical strips (e.g., [8]) and the exponential factor $e^{-|\omega|\Delta\sigma}$ in the Fourier transform.

Appendix B. Background Curvature $B(t)$

From $\Psi(s) = \Re\big(\log\Gamma(s/2) - \frac{s}{2}\log\pi\big) + \Re\log(\frac{1}{2}s(s-1))$ one computes

$$\partial_t^2\Psi\big(\tfrac{1}{2}+it\big) \;=\; -\tfrac{1}{4}\,\Re\,\psi_1\Big(\tfrac{1}{4}+\tfrac{i}{2}t\Big),$$

with ψ_1 the trigamma function ($\psi_1 = \text{polygamma}(1,\cdot)$). As $|t|\to\infty$,

$$B(t) \;=\; O\big(t^{-1}\big),$$

indeed $B(t)$ has an asymptotic expansion with leading term c/t and remainder $O(t^{-2})$.

Appendix C. Fourier–Green Solution and Removal of Homogeneous Parts

Solving $(\partial_\sigma^2 - \omega^2)\phi = F$ on $\mathbb{R}$ gives $\phi = G_\omega * F + Ae^{|\omega|\sigma} + Be^{-|\omega|\sigma}$. Subtracting Ψ removes the homogeneous components, because Ψ solves the homogeneous equation in the strip and encodes the Γ-background and algebraic factors. The remaining particular solution is the sum of pair contributions, leading to (5).

Appendix D. Discrete Approximation: Proof of Theorem 3.1

Let $T_a(\omega) := e^{-a|\omega|}$ and $K_L(\omega) := \widehat{\kappa}(L\omega)$. Then

$$\widehat{\kappa_L * P_a *} H_{\text{off}}(\omega) = K_L(\omega)\,T_a(\omega)\,\widehat{H}_{\text{off}}(\omega).$$

Since $K_L(\omega) \to 1$ locally uniformly and $0 \le T_a \le 1$, sampling in t with step δt corresponds to periodization in ω of period $2\pi/\delta t$; the aliasing error decays with δt. Truncation in a corresponds to damping by T_{a_j}; refining j reduces damping error as 2^{-J}. Refining n increases the accuracy of

$K_{L_n} \to 1$ at a rate N_L^{-1} for smooth $\widehat{\kappa}$. Plancherel in H^{-s} with $s > \frac{1}{2}$ ensures summability of tails and yields the bound.

Appendix E. Kernel Construction

To construct $\kappa \in C_c^\infty(\mathbb{R})$ with $\int \kappa = 1$ and $\int t^m \kappa = 0$ $(1 \leq m \leq M)$, take a bump function $\eta \in C_c^\infty([-1,1])$, $\eta \geq 0$, $\int \eta = 1$, and set $\kappa(t) = \sum_{k=0}^{M} a_k t^k \eta(t)$. Solve $\int t^m \kappa = \delta_{m0}$ $(0 \leq m \leq M)$ via a linear system (well-conditioned, e.g. with Chebyshev polynomials). Rescale $\kappa_L(t) = L^{-1}\kappa(t/L)$. A Hermite–Gaussian variant $\kappa(t) = \sum a_k H_k(t) e^{-t^2}$ can be tuned similarly.

References

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10 Complete Python implementation for computing the Poisson smoothing and detecting potential off-critical zeros

```
import numpy as np
from scipy.fft import fft, ifft
from scipy.special import polygamma # trigamma = polygamma(1, z)
import mpmath as mp
```

```
# High-precision zeta on the critical line; returns log|zeta(1/2+it)|
def log_abs_zeta(t_grid):
    mp.mp.dps = 80
    vals = []
    for t in t_grid:
        z = mp.zeta(0.5 + 1j * t)
        vals.append(mp.log(abs(z)))
    return np.array([float(v) for v in vals])
# Background curvature B(t) = -1/4 * Re psi_1(1/4 + i t/2)
def background_curvature(t_grid):
    t = np.asarray(t_grid, dtype=float)
    z = 0.25 + 0.5j * t
    # scipy.special.polygamma handles complex input; trigamma = polygamma(1, z)
    psi1 = polygamma(1, z) # complex array
    return -0.25 * np.real(psi1)
# Second derivative with respect to t (central differences)
def d2dt(f_vals, dt):
    return np.gradient(np.gradient(f_vals, dt), dt)
# Frequency axis for FFT (angular frequency)
def freq_axis(delta_t, N):
    return 2 * np.pi * np.fft.fftfreq(N, d=delta_t)
# Example mollifier kappa_L via Gaussian window (placeholder)
def kappa_gaussian(L):
    # discrete window matched to the signal length is user-defined; here we just return a
    short kernel M = max(5, int(6 * L / 1.0)) # heuristic length
    x = np.linspace(-3*L, 3*L, M)
    kern = np.exp(-(x**2) / (2 * L**2))
    kern /= np.trapz(kern, x)
    return kern
# Main numerical implementation
def compute_poisson_smoothing(t_grid, a0=1.0, J=5, L0=1.0, NL=4, epsilon=1e-6,
kappa=kappa_gaussian):
    N = len(t_grid)
    if N < 3:
        raise ValueError("t_grid must have at least 3 points")
    delta_t = t_grid[1] - t_grid[0]
    Phi_crit = log_abs_zeta(t_grid)
    B = background_curvature(t_grid)
    Hz = d2dt(Phi_crit, delta_t) - B # zero-driven curvature
    Hz_hat = fft(Hz)
    omega = freq_axis(delta_t, N)
    flags = []
    for j in range(J + 1):
        a = a0 * (2 ** (-j))
        h_hat = np.exp(-a * np.abs(omega)) * Hz_hat # Poisson smoothing
        h = np.real(ifft(h_hat))
        for n in range(1, NL + 1):
```

```
            Ln = L0 / n
            kern = kappa(Ln)
            # Discrete convolution; align modes (valid padding can be used for
            locality)
            h_loc = np.convolve(h, kern, mode="same")
            R_max = np.max(np.abs(h_loc))
            if R_max > epsilon:
                flags.append((a, Ln, float(R_max)))
    return flags
                # Example usage
if __name__ == "__main__":
    t_grid = np.linspace(-50.0, 50.0, 4096)
    flags = compute_poisson_smoothing(t_grid, a0=1.0, J=5, L0=1.0, NL=4, epsilon=1e-6)
    for (a, Ln, r) in flags:
        print(f"Potential off-critical zero signature: a={a:.6g}, L={Ln:.6g},
        residual={r:.3e}")
```