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Worked-Out PDEs

1   Electric potential inside long rectangular tube

Problem statement:

> Within a very long (extends in the z-direction), rectangular, hollow pipe, there are no electric charges. It has width a and height b. The walls of this pipe are kept at a known voltage. The walls at $x = 0, a$ are kept at constant potential V_1. Meanwhile, the walls at $y = \pm b/2$ are kept at constant potential V_2. Find potential inside the pipe.

Solution:

We need to solve the 2D Laplacian $\nabla^2 V(x, y) = 0$ with the following boundary conditions:

i $V(0, y) = V(a, y) = V_1$

ii $V(x, b/2) = V(x, -b/2) = V_2$

One way to solve this is to break into the following two subproblems:

1. Solve $\nabla^2 V = 0$ with boundary conditions: $V(0, y) = V(a, y) = V_1$ and $V(x, b/2) = V(x, -b/2) = 0$.

2. Solve $\nabla^2 V = 0$ with boundary conditions: $V(0, y) = V(a, y) = 0$ and $V(x, b/2) = V(x, -b/2) = V_2$.

But we can be more clever by changing things a bit:

1. Solve $\nabla^2 V = 0$ with boundary conditions: $V(0, y) = V(a, y) = V_1 - V_2$ and $V(x, b/2) = V(x, -b/2) = 0$.

2. Solve $\nabla^2 V = 0$ with boundary conditions: $V(0, y) = V(a, y) = V_2$ and $V(x, b/2) = V(x, -b/2) = V_2$.

Subproblem 1

By separating variables using $V(x, y) = f(x)g(y)$:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1}$$

$$f''(x)g(y) + f(x)g''(y) = 0 \tag{2}$$

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = 0 C_1 + C_2 = 0 \tag{3}$$

The assignment of constants will depend on boundary conditions. One of them will be positive and another will be negative. In order to satisfy the symmetric boundary condition $V(0, y) = V(a, y) = V_1 - V_2$, we will use hyperbolics for x-equation ($C_1 = k^2$). On the other hand, we

will use sinusoidals for y-equation $(C_1 = -k^2)$ to satisfy the vanishing boundary conditions $V(x, b/2) = V(x, -b/2) = 0$. The respective eigenvalue problems are:

$$f''(x) = k^2 f(x) \qquad (4)$$
$$g''(y) = -k^2 g(y) \qquad (5)$$

As usual, the general solutions to these ODEs are:

$$f(x) = A \cosh(kx) + B \sinh(kx) \qquad (6)$$
$$g(y) = C \cos(ky) + D \sin(ky) \qquad (7)$$

The boundary conditions demand that we need to shift coordinates:

$$f(x) = A \cosh\left(k\left(x - \frac{a}{2}\right)\right) \qquad (8)$$
$$g(y) = C \cos\left(k\left(y + \frac{b}{2}\right)\right) + D \sin\left(k\left(y + \frac{b}{2}\right)\right) \qquad (9)$$

Note that in $f(x)$, the constant B is set to zero since sinh is an odd function which is not suitable for even symmetry.

The boundary conditions for y-equation are $g(b/2) = g(-b/2) = 0$:

$$g\left(-\frac{b}{2}\right) = C \cos(0) + D \sin(0) = C = 0 \qquad (10)$$

$$g\left(\frac{b}{2}\right) = D \sin(kb) = 0 \qquad (11)$$

$$kb = n\pi \qquad (12)$$
$$k = \frac{n\pi}{b} \qquad (13)$$

For $n = 1, 2, 3, \ldots$. The solutions to ODEs are:

$$f_n(x) = A_n \cosh\left(\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right) \qquad (14)$$
$$g_n(y) = D_n \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \qquad (15)$$

The general solution is just a linear combination of $f_n(x)g_n(y)$:

$$V(x, y) = \sum_{n=1}^{\infty} f_n(x)g_n(y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right) \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \qquad (16)$$

We define the following inner product:

$$\langle f(y), g(y) \rangle = \int_{-\frac{b}{2}}^{\frac{b}{2}} f(y)g(y)dy \qquad (17)$$

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Applying the last boundary condition $V(0, y) = V(a, y) = V_1 - V_2$ and inner product:

$$V(0, y) = V(a, y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi a}{2b}\right) \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) = V_1 - V_2 \tag{18}$$

$$\left\langle \sin\left(\frac{m\pi}{b}\left(y + \frac{b}{2}\right)\right), \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi a}{2b}\right) \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \right\rangle = \left\langle \sin\left(\frac{m\pi}{b}\left(y + \frac{b}{2}\right)\right), V_1 - V_2 \right\rangle \tag{19}$$

$$\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi a}{2b}\right) \left\langle \sin\left(\frac{m\pi}{b}\left(y + \frac{b}{2}\right)\right), \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \right\rangle = \left\langle \sin\left(\frac{m\pi}{b}\left(y + \frac{b}{2}\right)\right), V_1 - V_2 \right\rangle \tag{20}$$

But notice that:

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} \sin\left(\frac{m\pi}{b}\left(y + \frac{b}{2}\right)\right) \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) dy = \begin{cases} 0 & m \neq n \\ \frac{b}{2} & m = n \end{cases} \tag{21}$$

Applying orthogonality gives:

$$C_n \cosh\left(\frac{n\pi a}{2b}\right) \left\langle \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right), \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \right\rangle = \left\langle \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right), V_1 - V_2 \right\rangle \tag{22}$$

$$C_n \cosh\left(\frac{n\pi a}{2b}\right) = \frac{2}{b}(V_1 - V_2) \left\langle \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right), 1 \right\rangle \tag{23}$$

$$C_n \cosh\left(\frac{n\pi a}{2b}\right) = \begin{cases} \frac{4}{n\pi}(V_1 - V_2) & \text{odd n} \\ 0 & \text{even n} \end{cases} \tag{24}$$

Now the solution for subproblem 1 is:

$$V_{s1}(x, y) = \sum_{n=1,\text{odd}}^{\infty} \frac{4}{n\pi}(V_1 - V_2) \frac{\cosh\left(\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right)}{\cosh\left(\frac{n\pi a}{2b}\right)} \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \tag{25}$$

Subproblem 2

This one is easy. It is a constant:

$$V_{s2}(x, y) = V_2 \tag{26}$$

Solution

The final solution is just $V = V_{s1} + V_{s2}$:

$$V(x, y) = V_2 + \sum_{n=1,\text{odd}}^{\infty} \frac{4}{n\pi}(V_1 - V_2) \frac{\cosh\left(\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right)}{\cosh\left(\frac{n\pi a}{2b}\right)} \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \tag{27}$$

2 Electric potential inside and outside charged sphere

Problem statement:

> A hollow sphere of radius R has a surface charge density $\sigma = \sigma_0 \sin^2 \theta$, where σ_0 is a constant. Find potential inside and outside the sphere.

Solution:

We need to solve the Laplacian $\nabla^2 V(r, \theta) = 0$ with the following boundary conditions:

i $V_{in}(r = 0) = \text{finite}$

ii $V_{out}(r \to \infty) = 0$

iii $V_{in}(r = R) = V_{out}(r = R)$

iv $\left. \dfrac{\partial V_{out}}{\partial r} \right|_{r=R} - \left. \dfrac{\partial V_{in}}{\partial r} \right|_{r=R} = -\dfrac{\sigma_0 \sin^2 \theta}{\epsilon_0}$

The general solution for spherical Laplacian, assuming azimuthal symmetry, is:

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta) \tag{28}$$

where P_l are Legendre polynomials of order l.

From boundary conditions i and ii, V_{in} and V_{out} will be:

$$V_{in}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) \tag{29}$$

$$V_{out}(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) \tag{30}$$

Boundary condition iii gives:

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta) \tag{31}$$

Boundary condition iv gives:

$$\sum_{l=0}^{\infty} -(l+1)\frac{B_l}{R^{l+2}} P_l(\cos \theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos \theta) = -\frac{\sigma_0}{\epsilon_0} \sin^2 \theta \tag{32}$$

$$\sum_{l=0}^{\infty} \left(-(l+1)\frac{B_l}{R^{l+2}} - l A_l R^{l-1} \right) P_l(\cos \theta) = -\frac{\sigma_0}{\epsilon_0}(1 - \cos^2 \theta) = -\frac{\sigma_0}{\epsilon_0} \left(\frac{2}{3} - \frac{2}{3} P_2(\cos \theta) \right) \tag{33}$$

It seems like only polynomials of degrees $l = 0, 2$ matter, so from equations 31 and 32:

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$$A_0 + A_2 R^2 P_2(\cos\theta) = \frac{B_0}{R} + \frac{B_2}{R^3} P_2(\cos\theta) \tag{34}$$

$$-\frac{B_0}{R^2} + \left(-3\frac{B_2}{R^4} - 2A_2 R\right) P_2(\cos\theta) = -\frac{2}{3}\frac{\sigma_0}{\epsilon_0} + \frac{2}{3}\frac{\sigma_0}{\epsilon_0} P_2(\cos\theta) \tag{35}$$

Matching coefficients gives the following system to solve for constants A_0, A_2, B_0, B_2:

$$A_0 = \frac{B_0}{R} \tag{36}$$

$$A_2 R^2 = \frac{B_2}{R^3} \tag{37}$$

$$\frac{B_0}{R^2} = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} \tag{38}$$

$$-3\frac{B_2}{R^4} - 2A_2 R = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} \tag{39}$$

After a painful bit of algebra, the constants are:

$$A_0 = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} R \tag{40}$$

$$A_2 = -\frac{2}{15}\frac{\sigma_0}{\epsilon_0}\frac{1}{R} \tag{41}$$

$$B_0 = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} R^2 \tag{42}$$

$$B_2 = -\frac{2}{15}\frac{\sigma_0}{\epsilon_0} R^4 \tag{43}$$

The final solution is:

$$V_{in}(r,\theta) = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} R - \frac{2}{15}\frac{\sigma_0}{\epsilon_0}\frac{r^2}{R} P_2(\cos\theta) \tag{44}$$

$$V_{out}(r,\theta) = \frac{2}{3}\frac{\sigma_0}{\epsilon_0}\frac{R^2}{r} - \frac{2}{15}\frac{\sigma_0}{\epsilon_0}\frac{R^4}{r^3} P_2(\cos\theta) \tag{45}$$

3 Electric potential for long cylinder

Problem statement:

> Two half-cylinders of radius R made out of conducting material are separated by a thin air gap. The left half cylinder ($\pi/2 < \phi < 3\pi/2$) is held at potential $-V_0$ and the right half-cylinder ($0 < \phi < \pi/2$ or $3\pi/2 < \phi < 2\pi$) at potential $+V_0$. Find potential inside and outside the cylinder.

Solution:

We need to solve the Laplacian $\nabla^2 V(s,\phi) = 0$ for a cylinder with the following boundary conditions:

i $V_{in}(s=0) =$ finite

ii $V_{out}(s \to \infty) = 0$

iii $V_{in}(s = R) = V_{out}(s = R)$

iv $V(s = R) = \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases}$

The general solution for the cylindrical Laplacian is:

$$V(s, \phi) = A + B \ln s + \sum_{n=1}^{\infty} \left(A_n s^n + B_n s^{-n} \right) \left(C_n \cos(n\phi) + D_n \sin(n\phi) \right) \quad (46)$$

To satisfy boundary conditions i and ii, V_{in} and V_{out} will be:

$$V_{in}(s, \phi) = \sum_{n=1}^{\infty} A_n s^n \cos(n\phi) \quad (47)$$

$$V_{out}(s, \phi) = \sum_{n=1}^{\infty} B_n s^{-n} \cos(n\phi) \quad (48)$$

Note that only cos terms are used because boundary condition iv demands that V be negative when $\pi/2 < \phi < 3\pi/2$ which is satisfied by cos function.

We define the following inner product:

$$\langle f(\phi), g(\phi) \rangle = \int_0^{2\pi} f(\phi) g(\phi) d\phi \quad (49)$$

Boundary condition iv will be used to find constants A_n and B_n. We will start with V_{in}:

$$V_{in}(R, \phi) = \sum_{n=1}^{\infty} A_n R^n \cos(n\phi) = \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases} \quad (50)$$

Applying the inner product to both sides gives:

$$\left\langle \cos(m\phi), \sum_{n=1}^{\infty} A_n R^n \cos(n\phi) \right\rangle = \left\langle \cos(m\phi), \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases} \right\rangle \quad (51)$$

$$\sum_{n=1}^{\infty} A_n R^n \left\langle \cos(n\phi), \cos(m\phi) \right\rangle = \left\langle \cos(m\phi), \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases} \right\rangle \quad (52)$$

However notice that:

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$$\langle \cos(n\phi), \cos(m\phi) \rangle = \int_0^{2\pi} \cos(n\phi)\cos(m\phi)d\phi = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases} \tag{53}$$

Applying the orthogonality of $\cos(n\phi)$ to equation 7:

$$A_n R^n \langle \cos(n\phi), \cos(n\phi) \rangle = \left\langle \cos(n\phi), \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases} \right\rangle \tag{54}$$

$$A_n R^n \pi = \int_0^{\frac{\pi}{2}} V_0 \cos(n\phi)d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} V_0 \cos(n\phi)d\phi + \int_{\frac{3\pi}{2}}^{2\pi} V_0 \cos(n\phi)d\phi \tag{55}$$

$$A_n R^n = \frac{V_0}{\pi} \left[\int_0^{\frac{\pi}{2}} \cos(n\phi)d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(n\phi)d\phi + \int_{\frac{3\pi}{2}}^{2\pi} \cos(n\phi)d\phi \right] \tag{56}$$

$$A_n R^n = \frac{2V_0}{n\pi} \left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right) \right] \tag{57}$$

$$A_n R^n = \begin{cases} \frac{4V_0}{n\pi}(-1)^{\frac{n-1}{2}} & \text{odd n} \\ 0 & \text{even n} \end{cases} \tag{58}$$

Calculating constant B_n is similar:

$$B_n R^{-n} = \begin{cases} \frac{4V_0}{n\pi}(-1)^{\frac{n-1}{2}} & \text{odd n} \\ 0 & \text{even n} \end{cases} \tag{59}$$

The final solutions are now therefore:

$$V_{in}(s, \phi) = \sum_{n=1,\text{odd}}^{\infty} \frac{4V_0}{n\pi}(-1)^{\frac{n-1}{2}} \left(\frac{s}{R}\right)^n \cos(n\phi) \tag{60}$$

$$V_{out}(s, \phi) = \sum_{n=1,\text{odd}}^{\infty} \frac{4V_0}{n\pi}(-1)^{\frac{n-1}{2}} \left(\frac{R}{s}\right)^n \cos(n\phi) \tag{61}$$