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# Worked-Out PDEs

## 1 Electric potential inside long rectangular tube

Problem statement:

> Within a very long (extends in the z-direction), rectangular, hollow pipe, there are no electric charges. It has width $a$ and height $b$. The walls of this pipe are kept at a known voltage. The walls at $x = 0, a$ are kept at constant potential $V_1$. Meanwhile, the walls at $y = \pm b/2$ are kept at constant potential $V_2$. Find potential inside the pipe.

Solution:

We need to solve the 2D Laplacian $\nabla^2 V(x, y) = 0$ with the following boundary conditions:

i $V(0, y) = V(a, y) = V_1$

ii $V(x, b/2) = V(x, -b/2) = V_2$

One way to solve this is to break into the following two subproblems:

1. Solve $\nabla^2 V = 0$ with boundary conditions: $V(0, y) = V(a, y) = V_1$ and $V(x, b/2) = V(x, -b/2) = 0$.
2. Solve $\nabla^2 V = 0$ with boundary conditions: $V(0, y) = V(a, y) = 0$ and $V(x, b/2) = V(x, -b/2) = V_2$.

But we can be more clever by changing things a bit:

1. Solve $\nabla^2 V = 0$ with boundary conditions: $V(0, y) = V(a, y) = V_1 - V_2$ and $V(x, b/2) = V(x, -b/2) = 0$.
2. Solve $\nabla^2 V = 0$ with boundary conditions: $V(0, y) = V(a, y) = V_2$ and $V(x, b/2) = V(x, -b/2) = V_2$.

### Subproblem 1

By separating variables using $V(x, y) = f(x)g(y)$:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{1}$$

$$f''(x)g(y) + f(x)g''(y) = 0 \tag{2}$$

$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = 0 \tag{3}$$

$$C_1 + C_2 = 0 \tag{4}$$

The assignment of constants will depend on boundary conditions. One of them will be positive and another will be negative. In order to satisfy the symmetric boundary condition $V(0, y) =$

$V(a,y) = V_1 - V_2$, we will use hyperbolics for x-equation ($C_1 = k^2$). On the other hand, we will use sinusoidals for y-equation ($C_1 = -k^2$) to satisfy the vanishing boundary conditions $V(x, b/2) = V(x, -b/2) = 0$. The respective eigenvalue problems are:

$$f''(x) = k^2 f(x) \tag{5}$$

$$g''(y) = -k^2 g(y) \tag{6}$$

As usual, the general solutions to these ODEs are:

$$f(x) = A\cosh(kx) + B\sinh(kx) \tag{7}$$

$$g(y) = C\cos(ky) + D\sin(ky) \tag{8}$$

The boundary conditions demand that we need to shift coordinates:

$$f(x) = A\cosh\left(k\left(x - \frac{a}{2}\right)\right) \tag{9}$$

$$g(y) = C\cos\left(k\left(y + \frac{b}{2}\right)\right) + D\sin\left(k\left(y + \frac{b}{2}\right)\right) \tag{10}$$

Note that in $f(x)$, the constant $B$ is set to zero since sinh is an odd function which is not suitable for even symmetry.

The boundary conditions for y-equation are $g(b/2) = g(-b/2) = 0$:

$$g\left(-\frac{b}{2}\right) = C\cos(0) + D\sin(0) = C = 0 \tag{11}$$

$$g\left(\frac{b}{2}\right) = D\sin(kb) = 0 \tag{12}$$

$$kb = n\pi \tag{13}$$

$$k = \frac{n\pi}{b} \tag{14}$$

For $n = 1, 2, 3, \ldots$. The solutions to ODEs are:

$$f_n(x) = A_n\cosh\left(\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right) \tag{15}$$

$$g_n(y) = D_n\sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \tag{16}$$

The general solution is just a linear combination of $f_n(x)g_n(y)$:

$$V(x,y) = \sum_{n=1}^{\infty} f_n(x)g_n(y) = \sum_{n=1}^{\infty} C_n\cosh\left(\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right)\sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \tag{17}$$

We define the following inner product:

$$\langle f(y), g(y)\rangle = \int_{-\frac{b}{2}}^{\frac{b}{2}} f(y)g(y)dy \tag{18}$$

Applying the last boundary condition $V(0,y) = V(a,y) = V_1 - V_2$ and inner product:

$$V(0,y) = V(a,y) = \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi a}{2b}\right) \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) = V_1 - V_2 \tag{19}$$

$$\left\langle \sin\left(\frac{m\pi}{b}\left(y+\frac{b}{2}\right)\right), \sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi a}{2b}\right) \sin\left(\frac{n\pi}{b}\left(y+\frac{b}{2}\right)\right)\right\rangle = \left\langle \sin\left(\frac{m\pi}{b}\left(y+\frac{b}{2}\right)\right), V_1 - V_2\right\rangle \tag{20}$$

$$\sum_{n=1}^{\infty} C_n \cosh\left(\frac{n\pi a}{2b}\right) \left\langle \sin\left(\frac{m\pi}{b}\left(y+\frac{b}{2}\right)\right), \sin\left(\frac{n\pi}{b}\left(y+\frac{b}{2}\right)\right)\right\rangle = \left\langle \sin\left(\frac{m\pi}{b}\left(y+\frac{b}{2}\right)\right), V_1 - V_2\right\rangle \tag{21}$$

But notice that:

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} \sin\left(\frac{m\pi}{b}\left(y+\frac{b}{2}\right)\right) \sin\left(\frac{n\pi}{b}\left(y+\frac{b}{2}\right)\right) dy = \begin{cases} 0 & m \neq n \\ \frac{b}{2} & m = n \end{cases} \tag{22}$$

Applying orthogonality gives:

$$C_n \cosh\left(\frac{n\pi a}{2b}\right) \left\langle \sin\left(\frac{n\pi}{b}\left(y+\frac{b}{2}\right)\right), \sin\left(\frac{n\pi}{b}\left(y+\frac{b}{2}\right)\right)\right\rangle = \left\langle \sin\left(\frac{n\pi}{b}\left(y+\frac{b}{2}\right)\right), V_1 - V_2\right\rangle \tag{23}$$

$$C_n \cosh\left(\frac{n\pi a}{2b}\right) = \frac{2}{b}(V_1 - V_2)\left\langle \sin\left(\frac{n\pi}{b}\left(y+\frac{b}{2}\right)\right), 1\right\rangle \tag{24}$$

$$C_n \cosh\left(\frac{n\pi a}{2b}\right) = \begin{cases} \frac{4}{n\pi}(V_1 - V_2) & \text{odd n} \\ 0 & \text{even n} \end{cases} \tag{25}$$

Now the solution for subproblem 1 is:

$$V_{\text{s1}}(x,y) = \sum_{n=1,\text{odd}}^{\infty} \frac{4}{n\pi}(V_1 - V_2) \frac{\cosh\left(\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right)}{\cosh\left(\frac{n\pi a}{2b}\right)} \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \tag{26}$$

## Subproblem 2

This one is easy. It is a constant:

$$V_{\text{s2}}(x,y) = V_2 \tag{27}$$

## Solution

The final solution is just $V = V_{\text{s1}} + V_{\text{s2}}$:

$$V(x,y) = V_2 + \sum_{n=1,\text{odd}}^{\infty} \frac{4}{n\pi}(V_1 - V_2) \frac{\cosh\left(\frac{n\pi}{b}\left(x - \frac{a}{2}\right)\right)}{\cosh\left(\frac{n\pi a}{2b}\right)} \sin\left(\frac{n\pi}{b}\left(y + \frac{b}{2}\right)\right) \tag{28}$$

# 2 Electric potential inside and outside charged sphere

Problem statement:

> A hollow sphere of radius $R$ has a surface charge density $\sigma = \sigma_0 \sin^2\theta$, where $\sigma_0$ is a constant. Find potential inside and outside the sphere.

Solution:

We need to solve the Laplacian $\nabla^2 V(r,\theta) = 0$ with the following boundary conditions:

i $V_{in}(r=0) = \text{finite}$

ii $V_{out}(r \to \infty) = 0$

iii $V_{in}(r=R) = V_{out}(r=R)$

iv $\left.\dfrac{\partial V_{out}}{\partial r}\right|_{r=R} - \left.\dfrac{\partial V_{in}}{\partial r}\right|_{r=R} = -\dfrac{\sigma_0 \sin^2\theta}{\epsilon_0}$

The general solution for spherical Laplacian, assuming azimuthal symmetry, is:

$$V(r,\theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \tag{29}$$

where $P_l$ are Legendre polynomials of order $l$.

From boundary conditions i and ii, $V_{in}$ and $V_{out}$ will be:

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \tag{30}$$

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \tag{31}$$

Boundary condition iii gives:

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \tag{32}$$

Boundary condition iv gives:

$$\sum_{l=0}^{\infty} -(l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) = -\frac{\sigma_0}{\epsilon_0}\sin^2\theta \tag{33}$$

$$\sum_{l=0}^{\infty} \left( -(l+1)\frac{B_l}{R^{l+2}} - l A_l R^{l-1} \right) P_l(\cos\theta) = -\frac{\sigma_0}{\epsilon_0}(1-\cos^2\theta) = -\frac{\sigma_0}{\epsilon_0}\left( \frac{2}{3} - \frac{2}{3}P_2(\cos\theta) \right) \tag{34}$$

It seems like only polynomials of degrees $l = 0, 2$ matter, so from equations 31 and 32:

$$A_0 + A_2 R^2 P_2(\cos\theta) = \frac{B_0}{R} + \frac{B_2}{R^3} P_2(\cos\theta) \tag{35}$$
$$-\frac{B_0}{R^2} + \left(-3\frac{B_2}{R^4} - 2A_2 R\right) P_2(\cos\theta) = -\frac{2}{3}\frac{\sigma_0}{\epsilon_0} + \frac{2}{3}\frac{\sigma_0}{\epsilon_0} P_2(\cos\theta) \tag{36}$$

Matching coefficients gives the following system to solve for constants $A_0, A_2, B_0, B_2$:

$$A_0 = \frac{B_0}{R} \tag{37}$$
$$A_2 R^2 = \frac{B_2}{R^3} \tag{38}$$
$$\frac{B_0}{R^2} = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} \tag{39}$$
$$-3\frac{B_2}{R^4} - 2A_2 R = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} \tag{40}$$

After a painful bit of algebra, the constants are:

$$A_0 = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} R \tag{41}$$
$$A_2 = -\frac{2}{15}\frac{\sigma_0}{\epsilon_0}\frac{1}{R} \tag{42}$$
$$B_0 = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} R^2 \tag{43}$$
$$B_2 = -\frac{2}{15}\frac{\sigma_0}{\epsilon_0} R^4 \tag{44}$$

The final solution is:

$$V_{in}(r,\theta) = \frac{2}{3}\frac{\sigma_0}{\epsilon_0} R - \frac{2}{15}\frac{\sigma_0}{\epsilon_0}\frac{r^2}{R} P_2(\cos\theta) \tag{45}$$
$$V_{out}(r,\theta) = \frac{2}{3}\frac{\sigma_0}{\epsilon_0}\frac{R^2}{r} - \frac{2}{15}\frac{\sigma_0}{\epsilon_0}\frac{R^4}{r^3} P_2(\cos\theta) \tag{46}$$

# 3 Electric potential for long cylinder

Problem statement:

> Two half-cylinders of radius $R$ made out of conducting material are separated by a thin air gap. The left half cylinder ($\pi/2 < \phi < 3\pi/2$) is held at potential $-V_0$ and the right half-cylinder ($0 < \phi < \pi/2$ or $3\pi/2 < \phi < 2\pi$) at potential $+V_0$. Find potential inside and outside the cylinder.

Solution:

We need to solve the Laplacian $\nabla^2 V(s,\phi) = 0$ for a cylinder with the following boundary conditions:

i $V_{in}(s=0) = \text{finite}$

ii $V_{out}(s \to \infty) = 0$

iii $V_{in}(s = R) = V_{out}(s = R)$

iv $V(s = R) = \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases}$

The general solution for the cylindrical Laplacian is:

$$V(s, \phi) = A + B \ln s + \sum_{n=1}^{\infty} \left( A_n s^n + B_n s^{-n} \right) \left( C_n \cos(n\phi) + D_n \sin(n\phi) \right) \tag{47}$$

To satisfy boundary conditions i and ii, $V_{in}$ and $V_{out}$ will be:

$$V_{in}(s, \phi) = \sum_{n=1}^{\infty} A_n s^n \cos(n\phi) \tag{48}$$

$$V_{out}(s, \phi) = \sum_{n=1}^{\infty} B_n s^{-n} \cos(n\phi) \tag{49}$$

Note that only cos terms are used because boundary condition iv demands that $V$ be negative when $\pi/2 < \phi < 3\pi/2$ which is satisfied by cos function.

We define the following inner product:

$$\langle f(\phi), g(\phi) \rangle = \int_0^{2\pi} f(\phi) g(\phi) d\phi \tag{50}$$

Boundary condition iv will be used to find constants $A_n$ and $B_n$. We will start with $V_{in}$:

$$V_{in}(R, \phi) = \sum_{n=1}^{\infty} A_n R^n \cos(n\phi) = \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases} \tag{51}$$

Applying the inner product to both sides gives:

$$\left\langle \cos(m\phi), \sum_{n=1}^{\infty} A_n R^n \cos(n\phi) \right\rangle = \left\langle \cos(m\phi), \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases} \right\rangle \tag{52}$$

$$\sum_{n=1}^{\infty} A_n R^n \langle \cos(n\phi), \cos(m\phi) \rangle = \left\langle \cos(m\phi), \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases} \right\rangle \tag{53}$$

However notice that:

$$\langle \cos(n\phi), \cos(m\phi) \rangle = \int_0^{2\pi} \cos(n\phi)\cos(m\phi) d\phi = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases} \tag{54}$$

Applying the orthogonality of $\cos(n\phi)$ to equation 7:

$$A_n R^n \langle \cos(n\phi), \cos(n\phi) \rangle = \left\langle \cos(n\phi), \begin{cases} V_0 & 0 < \phi < \pi/2 \\ -V_0 & \pi/2 < \phi < 3\pi/2 \\ V_0 & 3\pi/2 < \phi < 2\pi \end{cases} \right\rangle \tag{55}$$

$$A_n R^n \pi = \int_0^{\frac{\pi}{2}} V_0 \cos(n\phi) d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} V_0 \cos(n\phi) d\phi + \int_{\frac{3\pi}{2}}^{2\pi} V_0 \cos(n\phi) d\phi \tag{56}$$

$$A_n R^n = \frac{V_0}{\pi} \left[ \int_0^{\frac{\pi}{2}} \cos(n\phi) d\phi - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(n\phi) d\phi + \int_{\frac{3\pi}{2}}^{2\pi} \cos(n\phi) d\phi \right] \tag{57}$$

$$A_n R^n = \frac{2V_0}{n\pi} \left[ \sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{3n\pi}{2}\right) \right] \tag{58}$$

$$A_n R^n = \begin{cases} \frac{4V_0}{n\pi}(-1)^{\frac{n-1}{2}} & \text{odd n} \\ 0 & \text{even n} \end{cases} \tag{59}$$

Calculating constant $B_n$ is similar:

$$B_n R^{-n} = \begin{cases} \frac{4V_0}{n\pi}(-1)^{\frac{n-1}{2}} & \text{odd n} \\ 0 & \text{even n} \end{cases} \tag{60}$$

The final solutions are now therefore:

$$V_{in}(s,\phi) = \sum_{n=1,\text{odd}}^{\infty} \frac{4V_0}{n\pi}(-1)^{\frac{n-1}{2}} \left(\frac{s}{R}\right)^n \cos(n\phi) \tag{61}$$

$$V_{out}(s,\phi) = \sum_{n=1,\text{odd}}^{\infty} \frac{4V_0}{n\pi}(-1)^{\frac{n-1}{2}} \left(\frac{R}{s}\right)^n \cos(n\phi) \tag{62}$$