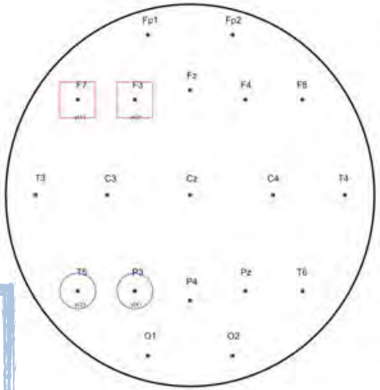
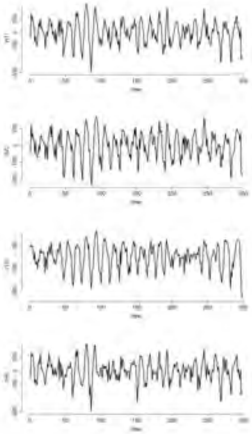
Bayesian Time Series Analysis & Forecasting

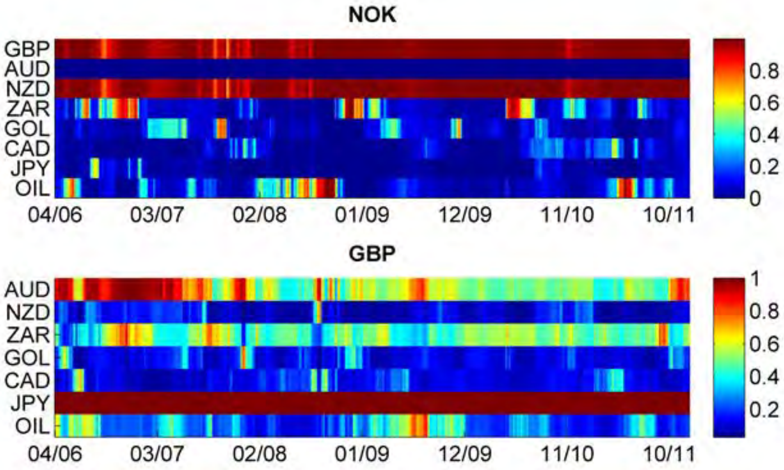
Neuroscience, Imaging,…



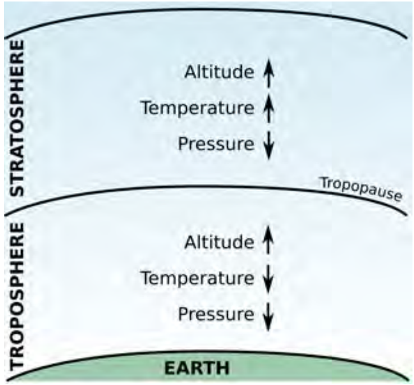


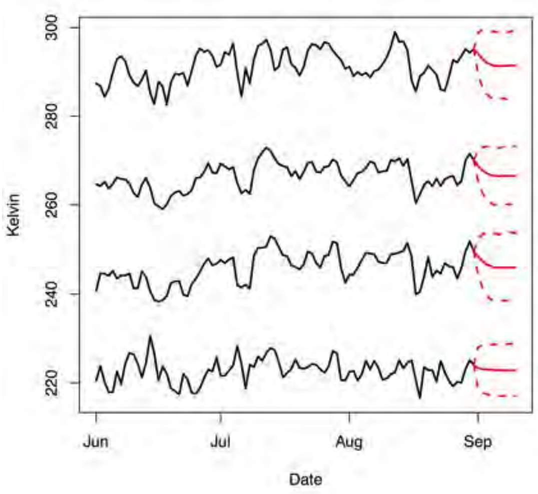
Business, Finance,
Econometrics,
Social Systems…

State Space
Dynamic Models



Environmental sciences





Marco Ferreira, Raquel Prado & Mike West

Course Structure

- **8:30-10:15 <u>Session 1</u>, 105 min, Prado**
  - Introduction - Dynamic linear models (DLMs)
  - Trend, seasonal & regression models; composition; discount factors
  - Sequential learning & forecasting; Retrospective analysis/ smoothing
  - Examples

- **10:15-10:30 Coffee break**

- **10:30-10:55, <u>Session 2</u>, 25 mins, Prado**
  - DLMs for AR models: Decompositions, ties to frequency analysis
  - EEG examples

Course Structure

- **10:55-12:30 <u>Session 3</u>, 95 min, West**
  - Non/locally-stationary models: time-varying AR (TVAR) models
  - Variance discount model of univariate stochastic volatility
  - Example: macroeconomic/inflation+business cycles
  - Entree to multivariate time series: coupling sets of univariate DLMs/TVAR models (subclass of TV-VAR models)

- **12:30-14:00 Lunch break**

- **14:00-14:35, <u>Session 4</u>, 35 mins, West**
  - More on a subclass of multivariate models: dynamic dependency network models (DDNMs)
  - Financial time series example

Course Structure

- **14:35-15:15 <u>Session 5</u>, 40 min, Ferreira**
  - Latent factor models with one factor, environmental time series example
  - MCMC in conditional DLMs: factor model example

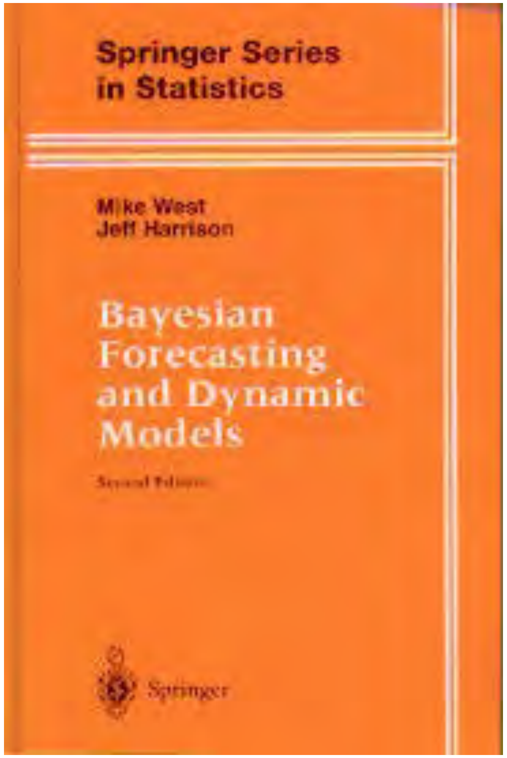
- **15:15-15:30 Tea break**

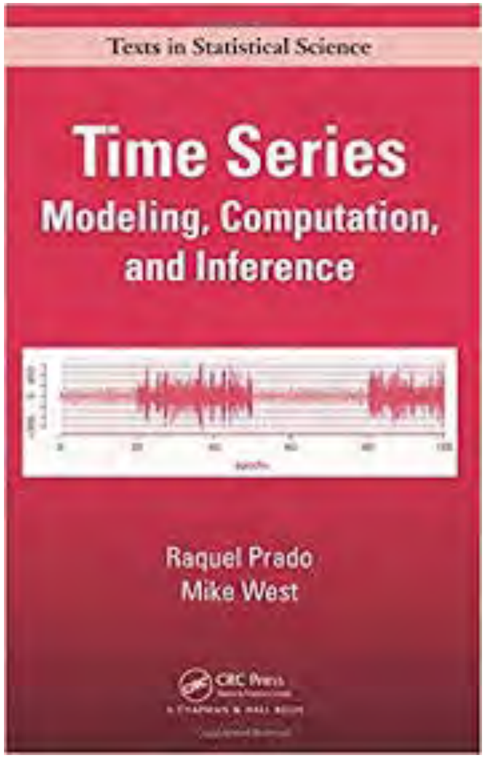
- **15:30-17:00, <u>Session 6</u>, 90 mins, Ferreira**
  - More general dynamic latent factor models
  - Factor stochastic volatility; financial time series example

Learning Objectives

- Exposure to basic ideas and approaches of Bayesian model-based time series analysis using key classes of dynamic models;
- An appreciation of the roles of computation — analytic — as well as simulation-based methods — for time series analysis in dynamic models, including filtering, parameter learning and smoothing;
- Awareness of texts, papers and software that will enable follow-on explorations and analysis;
- An appreciation of some of the breadth of application Bayesian dynamic modeling has had, and can have, in various applied fields.

Books & supporting material

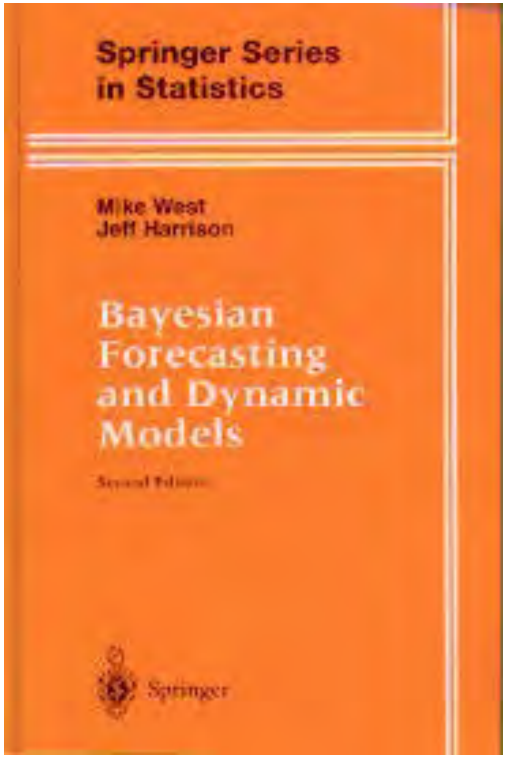


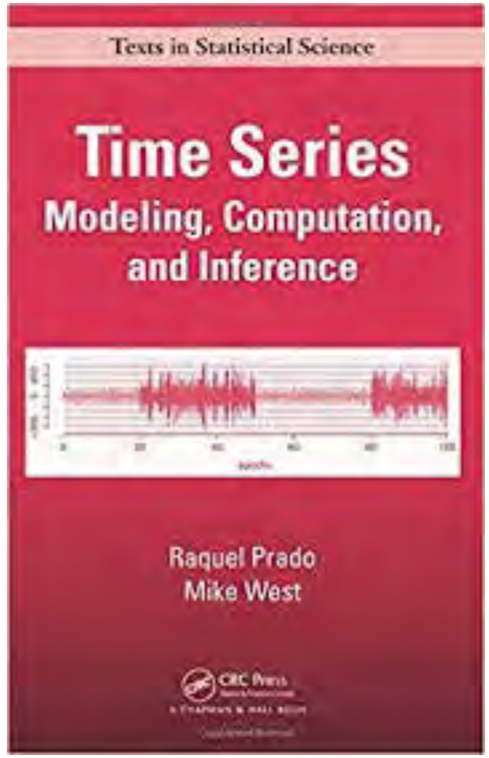


References + code + data at

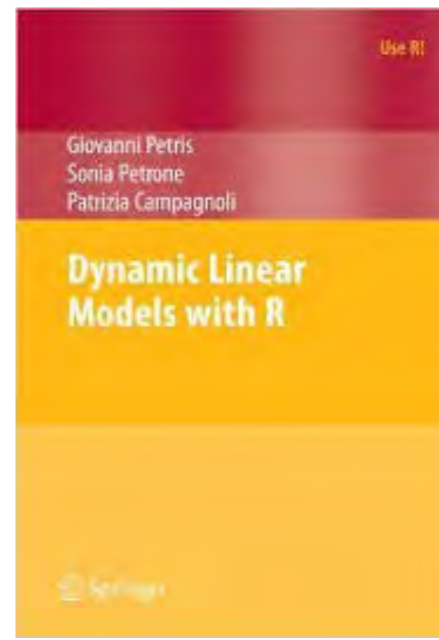
http://www2.stat.duke.edu/~mw/JSM2019CETimeSeries/

Books & supporting material





dlm R package by G. Petris



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Introduction - Dynamic linear models (DLMs)

$$y_t = \beta x_t + \epsilon_t, \quad t = 1, 2,$$

- β : Prior beliefs expressed through $p(\beta)$
- **Sequential learning:** $p(\beta | y_{1:t})$
- **Forecasting:** $p(y_{t+1} | y_{1:t}), \quad p(y_{t+h} | y_{1:t})$

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As time passes, we may want to characterize β as evolving to adapt to changes over time, e.g.,

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$$p(\beta_t | y_{1:T}), \quad T > t$$

$$p(y_{t+h} | y_{1:t})$$

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Filtering

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Filtering $p(\beta_t | y_{1:t})$

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Introduction - Dynamic linear models (DLMs)

Notation

- *Initial information set:* Denoted as $\mathcal{D}_0$. Represents all the available information used to form initial views about the future.

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- *Forecasting:*

$$(y_{t+h}|\mathcal{D}_t), \quad h > 0, \quad (y_s|\mathcal{D}_t), \quad s > t$$

General Dynamic Linear Model

General Dynamic Linear Model

$$
\begin{array}{rrcl}
\text{[Observation]} & y_t & = & \mathbf{F}_t'\boldsymbol{\theta}_t + \nu_t \\
\text{[System]} & \boldsymbol{\theta}_t & = & \mathbf{G}_t\boldsymbol{\theta}_{t-1} + \mathbf{w}_t \\
\text{[Initial info]} & \boldsymbol{\theta}_0 & \sim & p(\boldsymbol{\theta}_0|\mathcal{D}_0)
\end{array}
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- $\boldsymbol{\theta}_t$: p × 1 state vector

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- $\mathbf{F}_t$: p × 1 vector of constants

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Normal DLM (NDLM): observational & evolution noises independent, mutually independent

$\nu_t \sim N(0, v_t)$ $\mathbf{w}_t \sim N(\mathbf{0}, \mathbf{W}_t)$ and $(\boldsymbol{\theta}_0 | \mathcal{D}_0) \sim N(\mathbf{m}_0, \mathbf{C}_0)$

Introduction - Dynamic linear models (DLMs)

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- *Short hand notation:*

$$\{\mathbf{F}_t, \mathbf{G}_t, v_t, \mathbf{W}_t\}$$

$$y_t$$

Univariate DLM

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$$\mathbf{y}_t \quad n \times 1$$

Multivariate DLM

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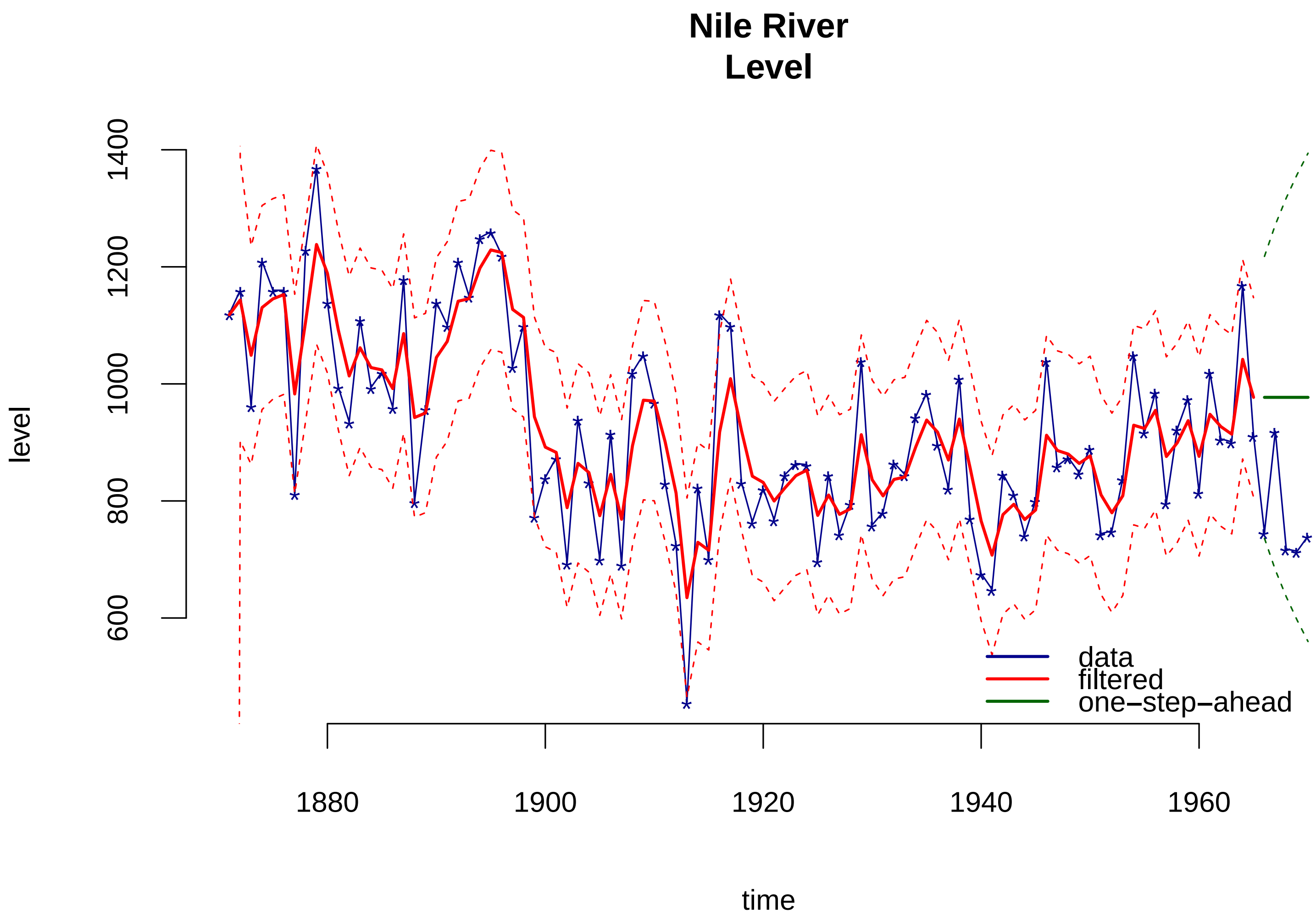
Multivariate DLM

- *h-step ahead forecast function h ≥ 1:*

$$f_t(h) = E(y_{t+h}|\mathcal{D}_t) = \mathbf{F}'_{t+h}\mathbf{G}_{t+h}\ldots\mathbf{G}_{t+1}E(\boldsymbol{\theta}_t|\mathcal{D}_t)$$

If $\mathbf{G}_t = \mathbf{G} \Rightarrow f_t(h) = \mathbf{F}'_{t+h}\mathbf{G}^h E(\boldsymbol{\theta}_t|\mathcal{D}_t)$

DLMs: Polynomial Trend Models



Forecast function: locally constant (first-order polynomial), time-varying level

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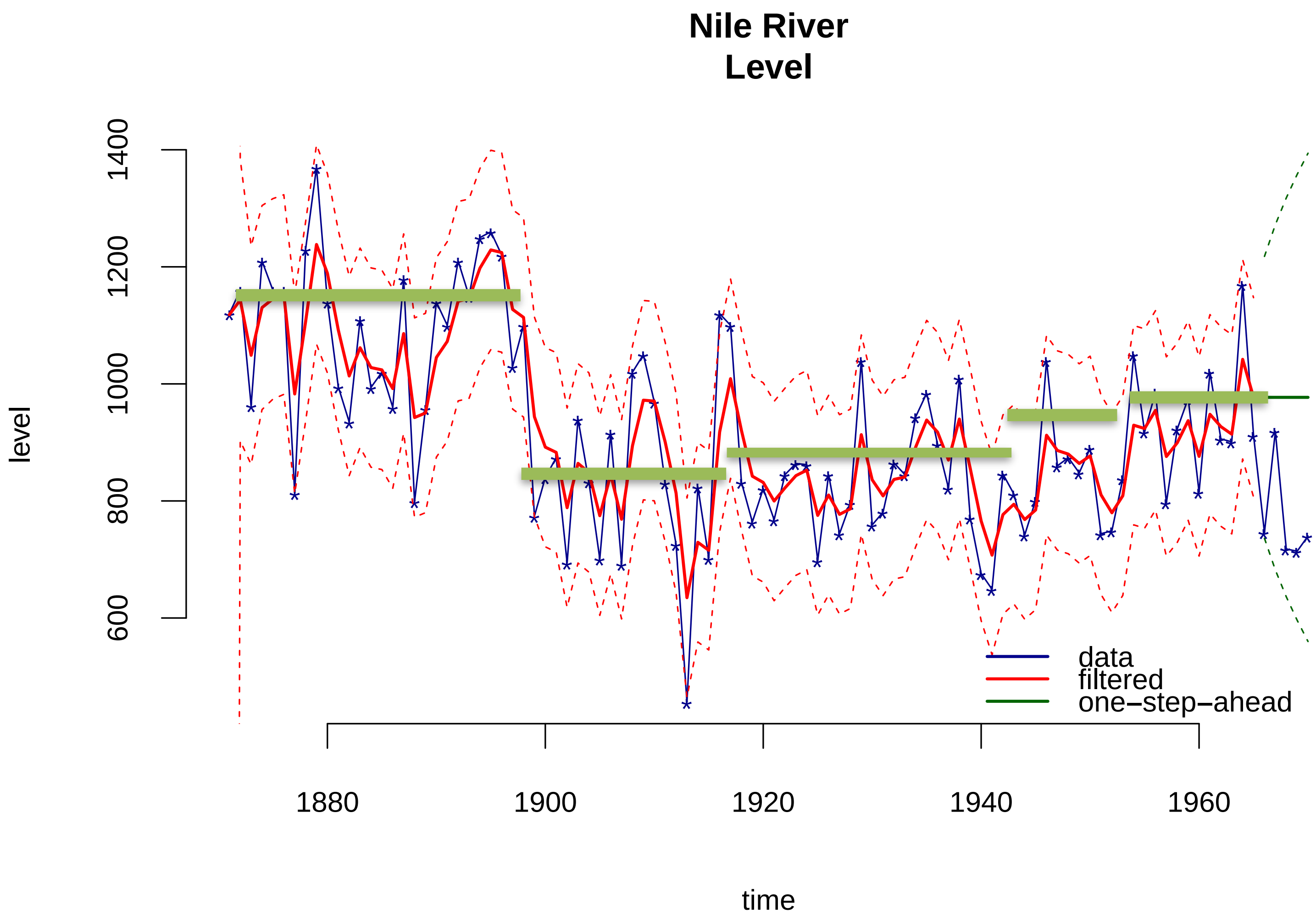

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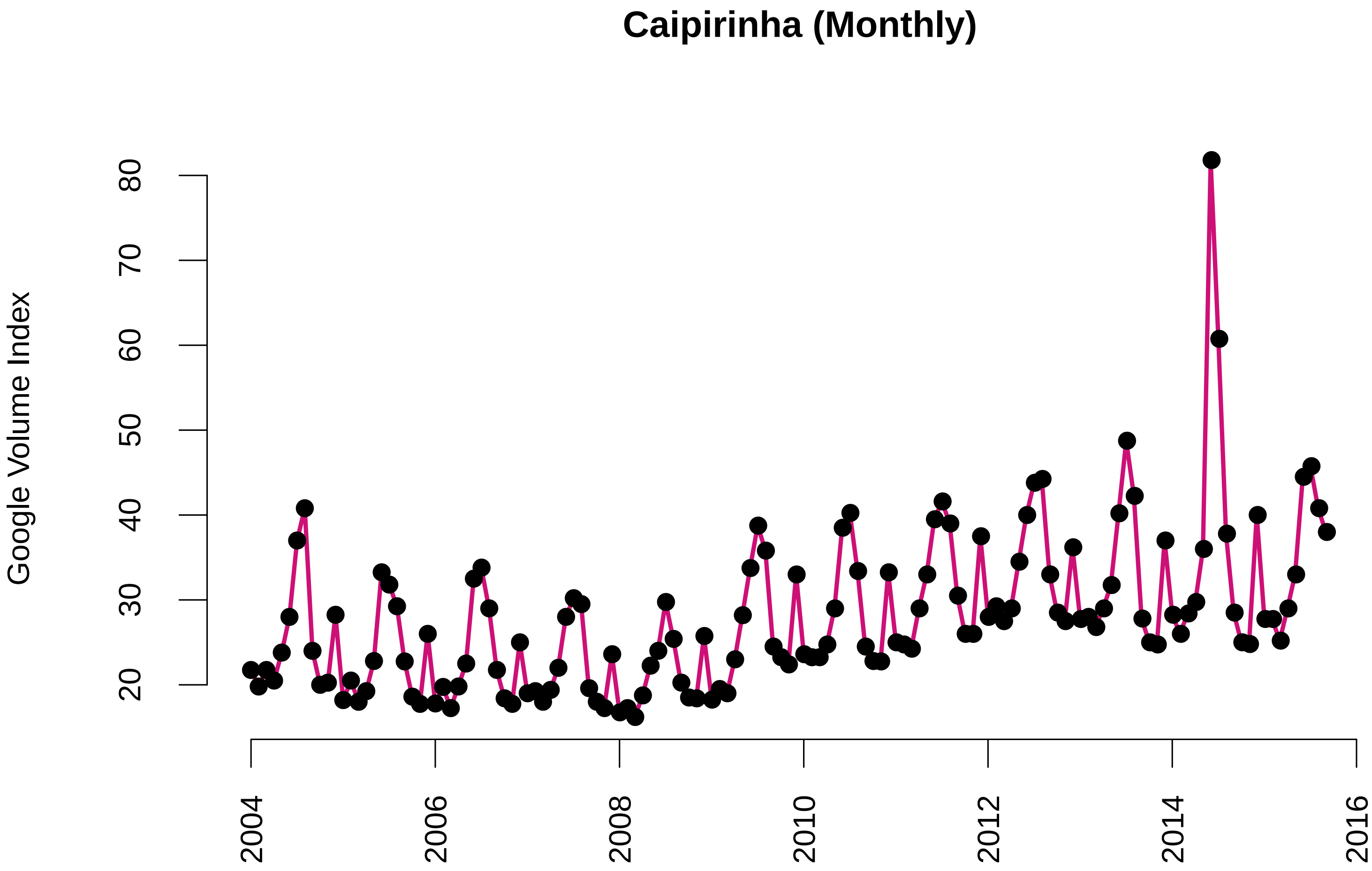

Forecast function: Second order polynomial; time-varying slope & intercept

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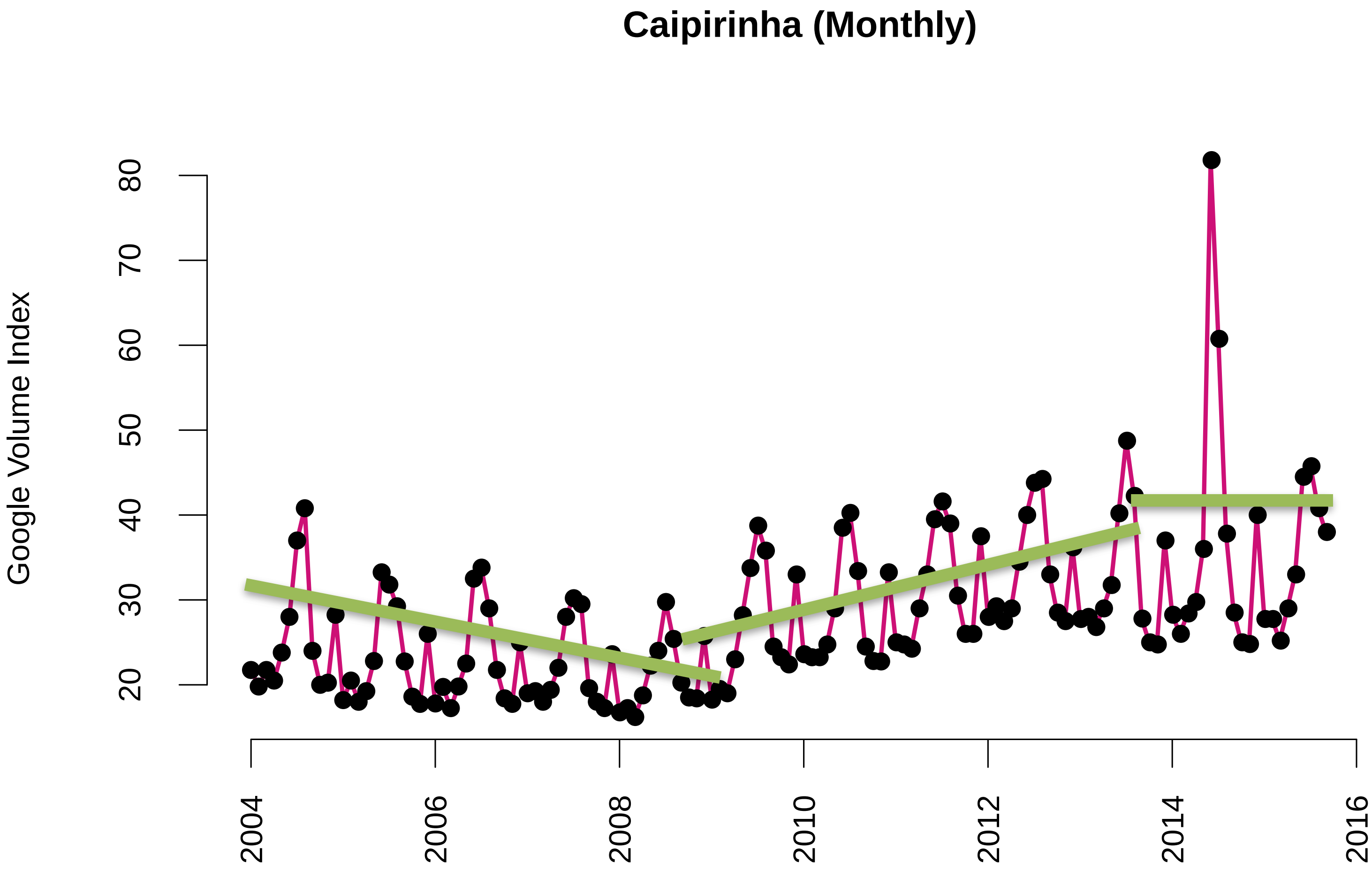

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DLMs: Polynomial Trend Models

- **First order polynomial model**

$$\begin{aligned} y_t &= \theta_t + \nu_t, \\ \theta_t &= \theta_{t-1} + w_t. \end{aligned}$$

Forecast function: $f_t(h) = a_{t,0}$

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- **Second order polynomial model**

$$\begin{aligned} y_t &= \theta_{t,1} + \nu_t, \\ \theta_{t,1} &= \theta_{t-1,1} + \theta_{t-1,2} + w_{t,1}, \\ \theta_{t,2} &= \theta_{t-1,2} + w_{t,2}. \end{aligned}$$

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<u>*Forecast function:*</u> $f_t(h) = a_{t,0} + a_{t,1} h$

Note that in this case we have $\{\mathbf{E}_2, \mathbf{J}_2(1), v_t, \mathbf{W}_t\}$ with $\mathbf{E}_2 = (1,0)'$ and

$$\mathbf{J}_2(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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- **p-th order polynomial model** (canonical representation):

$\{\mathbf{E}_p, \mathbf{J}_p(1), v_t, \mathbf{W}_t\}$ with

$$\mathbf{E}_p = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{J}_p(1) = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

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Forecast function: $f_t(h) = a_{t,0} + a_{t,1}h + \cdots + a_{t,p-1}h^{p-1}.$

DLMs: Polynomial Trend Models

- Alternative representation for the **p-th order polynomial DLM:** $\{\mathbf{E}_p, \mathbf{L}_p, \cdot, \cdot\}$ with

$$\mathbf{L}_p = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

DLMs: Polynomial Trend Models

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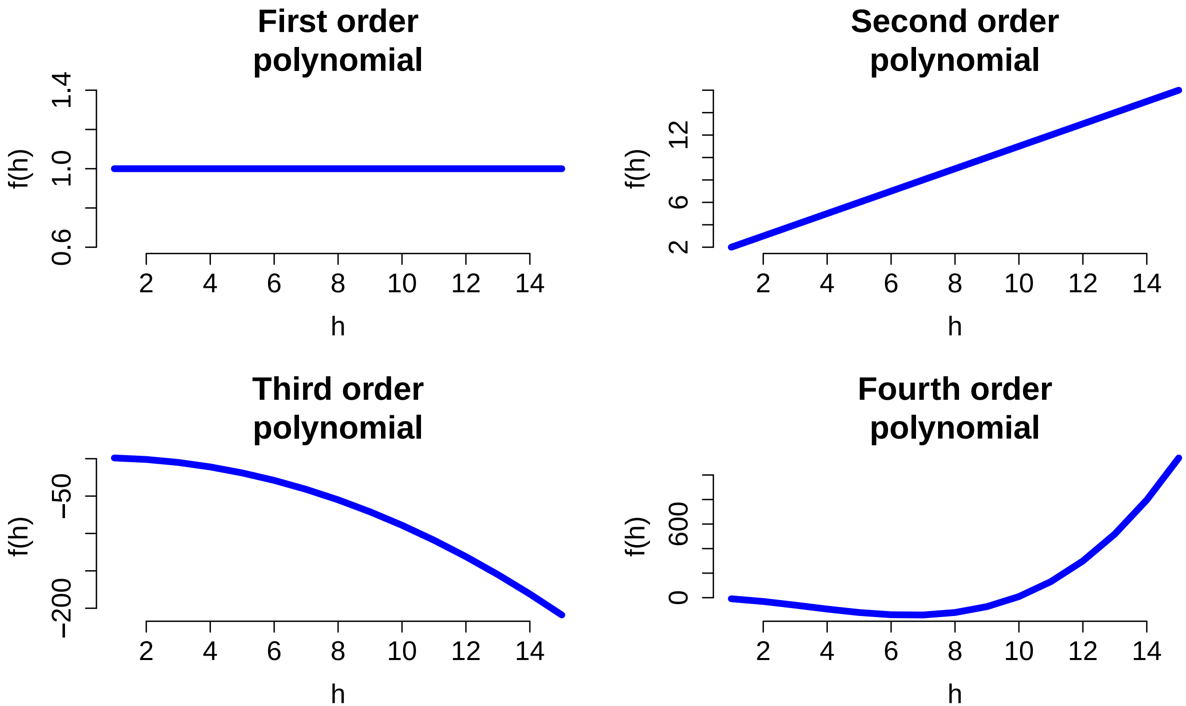


$$\mathcal{M}_1$$

$$\{\mathbf{E}_p, \mathbf{J}_p(1), v_t, \mathbf{W}_t\}$$

$$\mathcal{M}_2$$

$$\{\mathbf{E}_p, \mathbf{L}_p, \cdot, \cdot\}$$

$$\mathcal{M}_1 \qquad\qquad \mathcal{M}_2$$

$$\{\mathbf{E}_p, \mathbf{J}_p(1), v_t, \mathbf{W}_t\} \qquad \{\mathbf{E}_p, \mathbf{L}_p, \cdot, \cdot\}$$

Similar Models

$$\mathcal{M}_1 \sim \mathcal{M}_2$$

DLMs : Similar Models

$$\mathcal{M}_1 \qquad\qquad \mathcal{M}_2$$

$$\{\mathbf{E}_p, \mathbf{J}_p(1), v_t, \mathbf{W}_t\} \qquad\qquad \{\mathbf{E}_p, \mathbf{L}_p, \cdot, \cdot\}$$

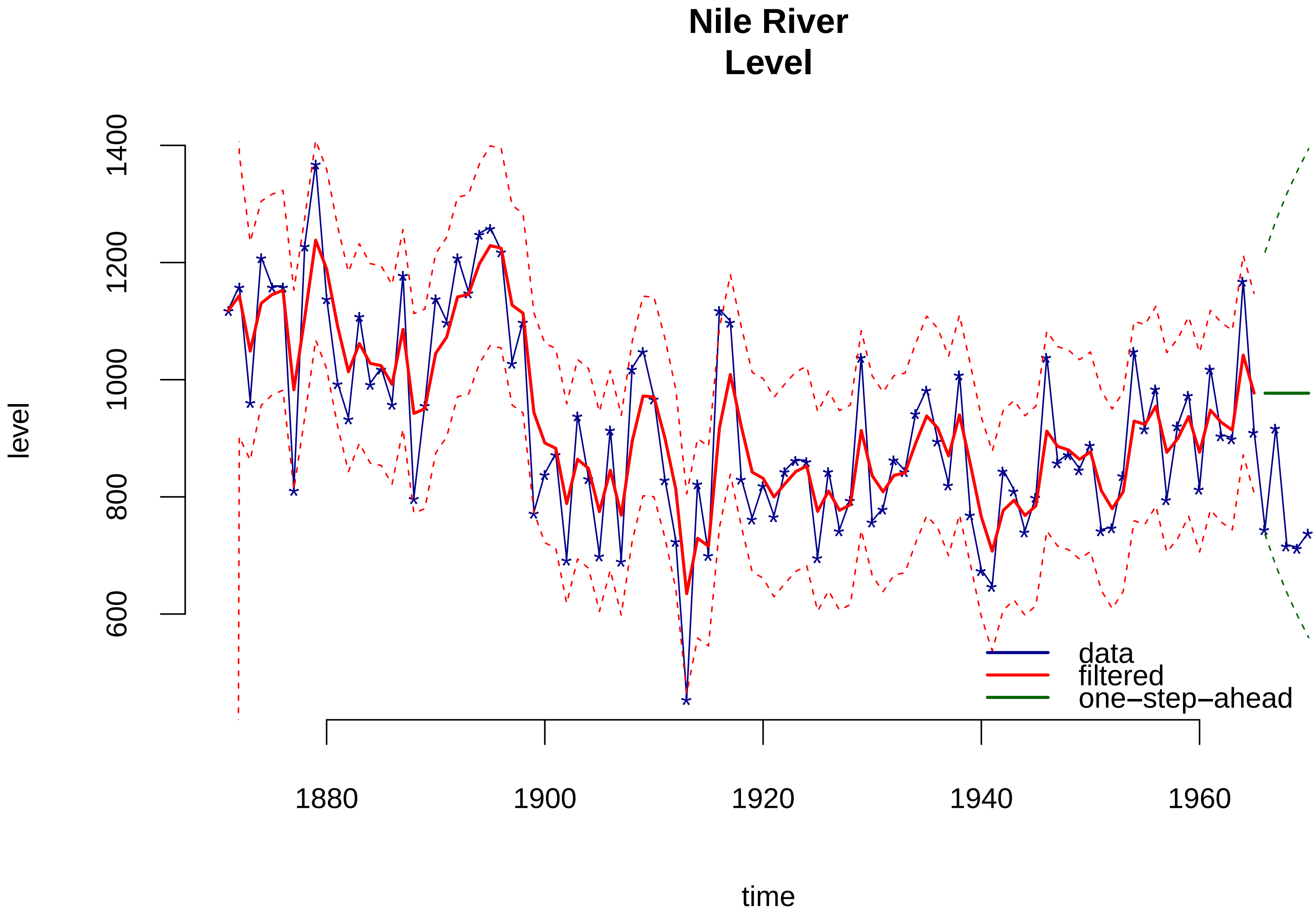
Similar Models

$$\mathcal{M}_1 \sim \mathcal{M}_2$$

$\mathbf{J}_p(1)$ and $\mathbf{L}_p$ are similar matrices

DLMs: Polynomial Trend Models

Example: Nile River Data (Petris et al., 2009)



Seasonal Factor Models

$$\{\mathbf{E}_p, \mathbf{P}, v_t, \mathbf{W}_t\}$$

- p: period
- $\boldsymbol{\theta}_t$: p × 1 state vector of seasonal levels; first component is the level at the current time.
- The evolution matrix is

$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Forecast function: $f_t(h) = \mathbf{E}_p' \mathbf{P}^h \mathbf{m}_t = m_{t,j}, \quad j = p|h$

DLMs: Seasonal Models

Seasonal Factor Models

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- $\boldsymbol{\theta}_t$: p × 1 state vector of seasonal levels; first component is the level at the current time.
- The evolution matrix is

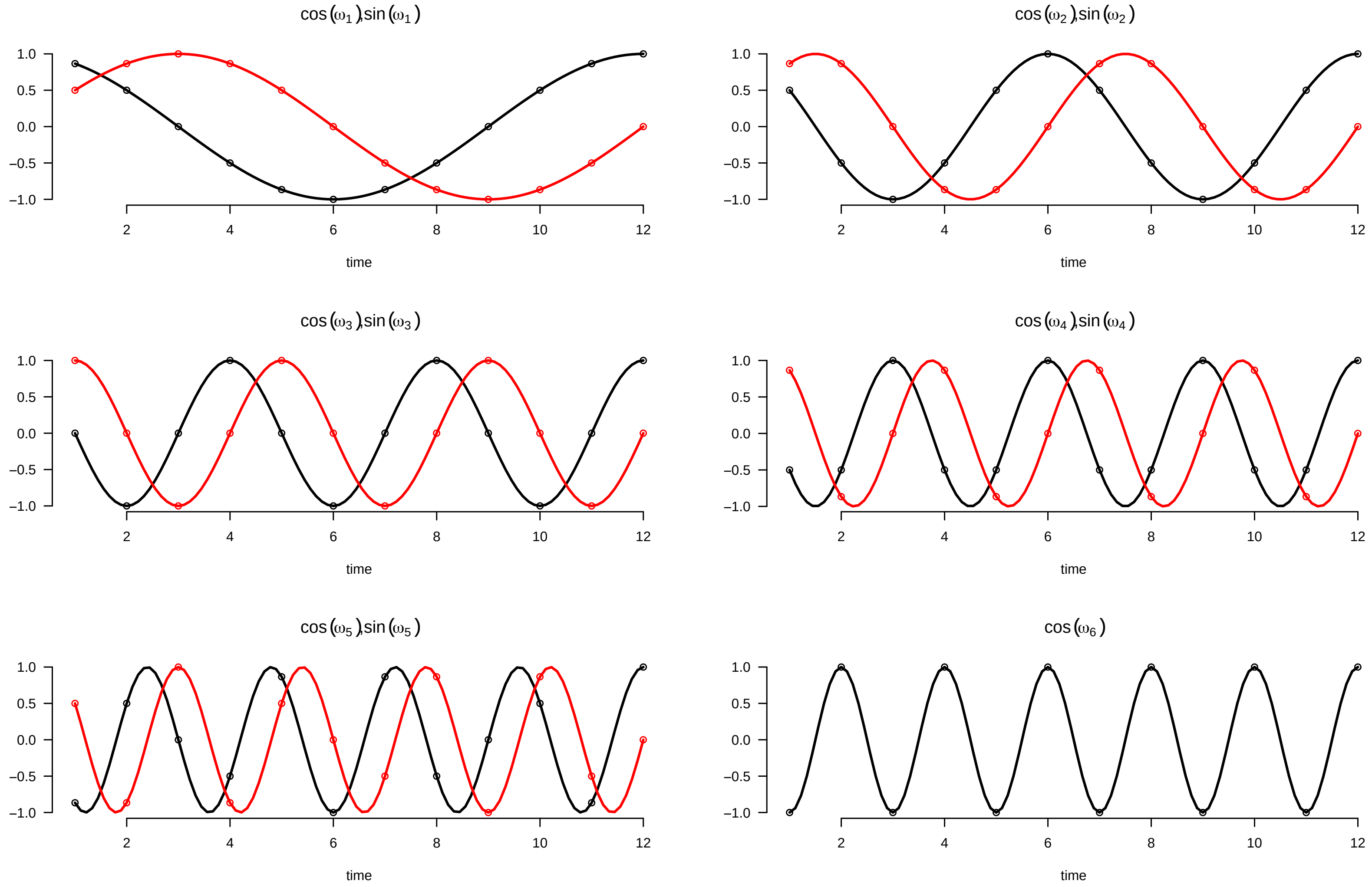
$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Forecast function: $f_t(h) = \mathbf{E}_p' \mathbf{P}^h \mathbf{m}_t = m_{t,j}, \quad j = p|h$

Seasonal effects: $\sum_{i=1}^{p} \theta_{t,i} = 0$

DLMs: Seasonal Models

E.g., p=12: the 12 seasonal factors can be represented in terms of 6 harmonic components related to the frequencies 2π/12, 2π/6, 2π/4, 2π/3, 5π/6, and π (Nyquist).



DLMs: Seasonal Models

- **Component Fourier Representation:** For a frequency ω in (0,π), a corresponding harmonic component DLM is with $\{\mathbf{E}_2, \mathbf{J}_2(1,\omega), \cdot, \cdot\}$

$$\mathbf{J}_2(1,\omega) = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix}.$$

For ω = π we have $\{1, -1, \cdot, \cdot\}$.

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For ω = π we have $\{1, -1, \cdot, \cdot\}$.

Forecast function:

$$f_t(h) = a_t \cos(\omega h) + b_t \sin(\omega h) = A_t \cos(\omega h + \gamma_t),$$

with A_t the amplitude and γ_t the phase for ω ∈ (0,π) and

$$f_t(h) = (-1)^h A_t$$

for ω=π.

Complete Fourier Representation

Complete Fourier Representation

- p=2m-1 odd:

$$\mathbf{F} = (\mathbf{E}_2', \ldots, \mathbf{E}_2')'$$

$$\mathbf{G} = \text{blockdiag}(\mathbf{G}_1, \ldots, \mathbf{G}_{m-1}), \quad \mathbf{G}_j = \mathbf{J}_2(1, \omega_j),$$

$$\omega_j = 2\pi j/p, \quad j = 1 : (m-1)$$

DLMs: Fourier Seasonal Models

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$$\omega_j = 2\pi j/p, \quad j = 1 : (m-1)$$

- p=2m even:

$$\mathbf{F} = (\mathbf{E}_2', \ldots, \mathbf{E}_2', 1)'$$
$$\mathbf{G} = \text{blockdiag}(\mathbf{G}_1, \ldots, \mathbf{G}_{m-1}, -1)$$

DLMs: Fourier Seasonal Models

Complete Fourier Representation

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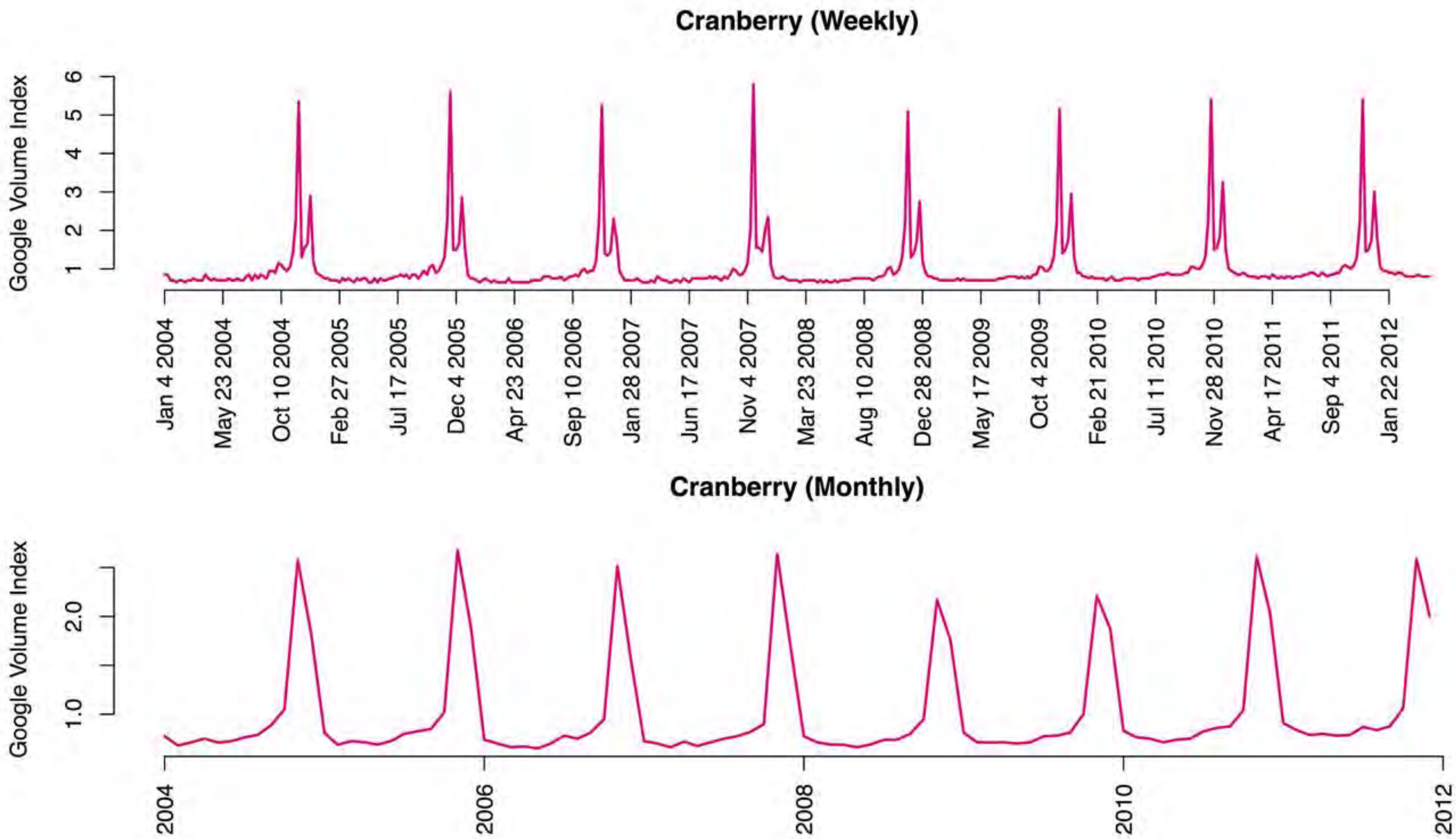
$$\mathbf{G} = \mathrm{blockdiag}(\mathbf{G}_1, \ldots, \mathbf{G}_{m-1}, -1)$$

Forecast function:

$$f_t(h) = \sum_{j=1}^{m-1} A_{t,j} \cos(\omega_j h + \gamma_{t,j}) + (-1)^h A_{t,m},$$

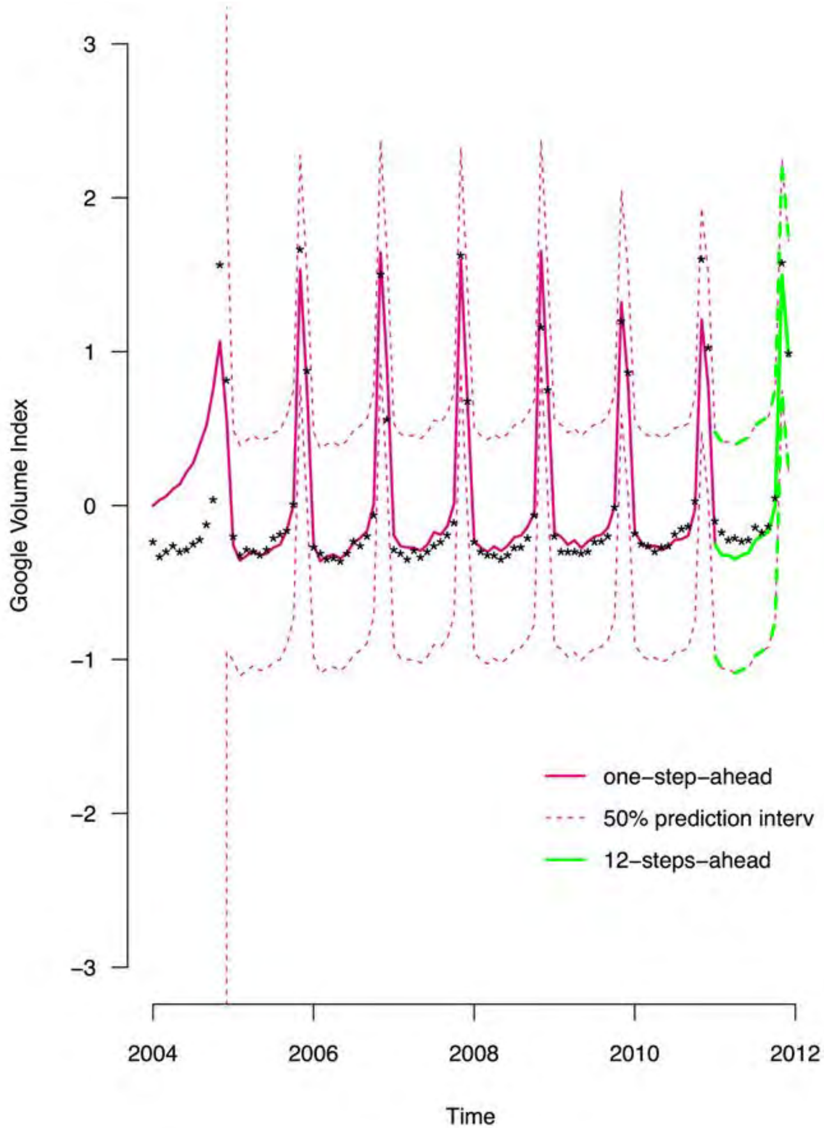
$A_{t,m} = 0$ if p odd.

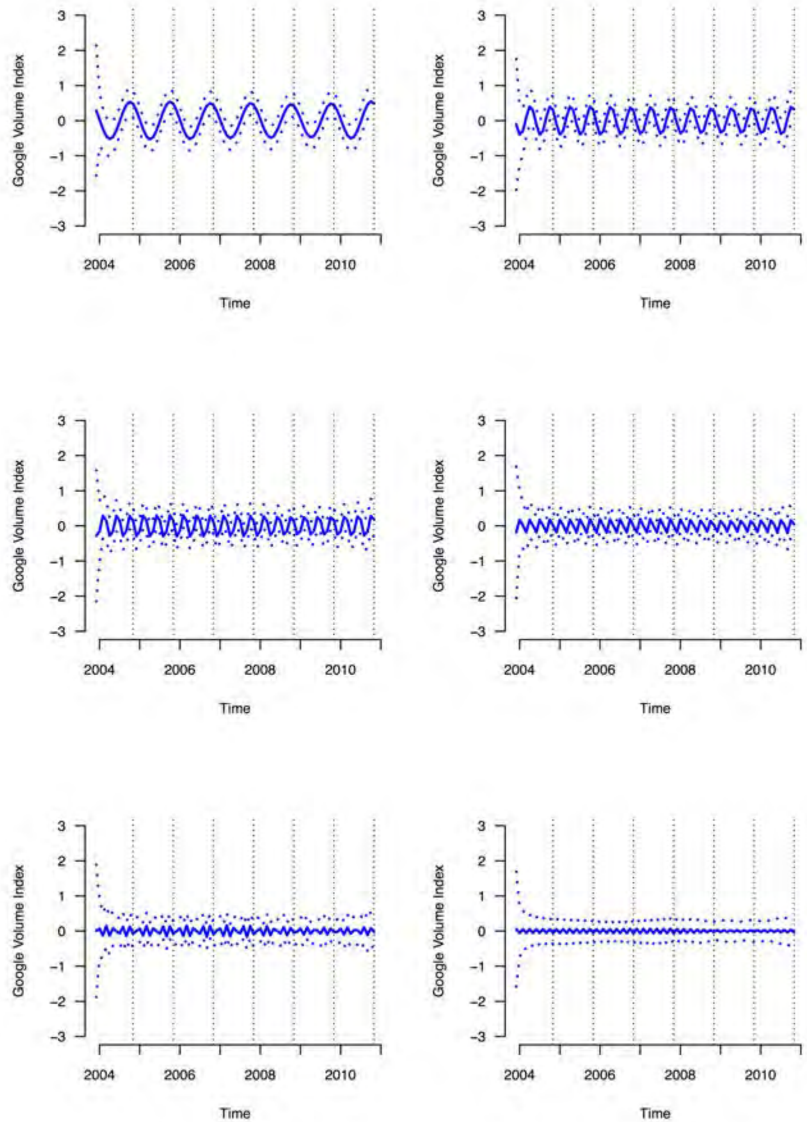
Examples: Google Trends



Examples: Google Trends

p=12, v=0.3, w=0.01





- **Simple dynamic regression**

$$
\begin{aligned}
y_t &= \alpha_t + \beta_t x_t + \nu_t, \\
\alpha_t &= \alpha_{t-1} + w_{t,1}, \\
\beta_t &= \beta_{t-1} + w_{t,2}.
\end{aligned}
$$

DLMs: Regression Models

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Forecast function: $f_t(h) = a_{t,0} + a_{t,1} x_{t+h}$

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- **Time-varying autoregression**

$$
\begin{aligned}
y_t &= \sum_{i=1}^{p} \phi_{t,i} y_{t-i} + \nu_t, \\
\phi_{t,i} &= \phi_{t-1,i} + w_{t,i}, \quad i = 1:p.
\end{aligned}
$$

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Forecast function: $f_t(h) = a_{t,1} y_{t+h-1} + \ldots + a_{t,p} y_{t+h-p}.$

DLMs: Regression Models

- **General regression DLM:** $\{\mathbf{F}_t, \mathbf{G}_t, v_t, \mathbf{W}_t\}$

$$\mathbf{F}_t = (x_{t,1}, \ldots, x_{t,k})', \quad \mathbf{G}_t = \mathbf{I}_k$$

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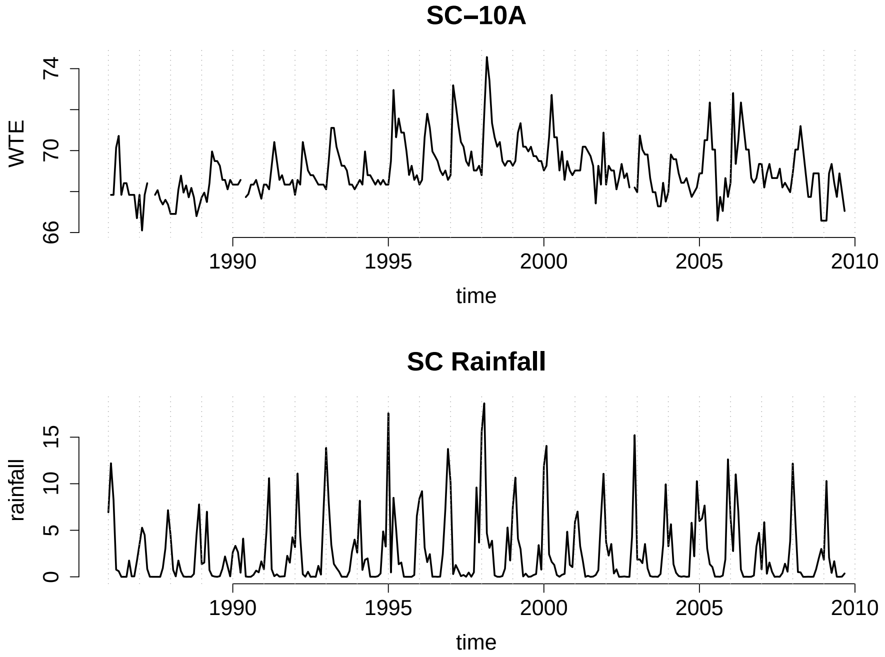
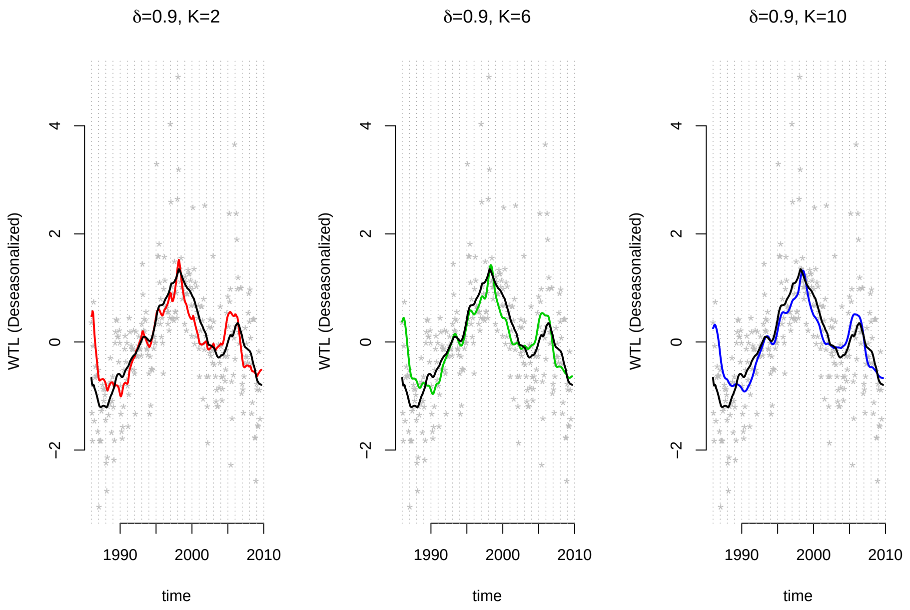
Example: Water table elevation (WTE) & rainfall, monitoring well in Santa Cruz, CA

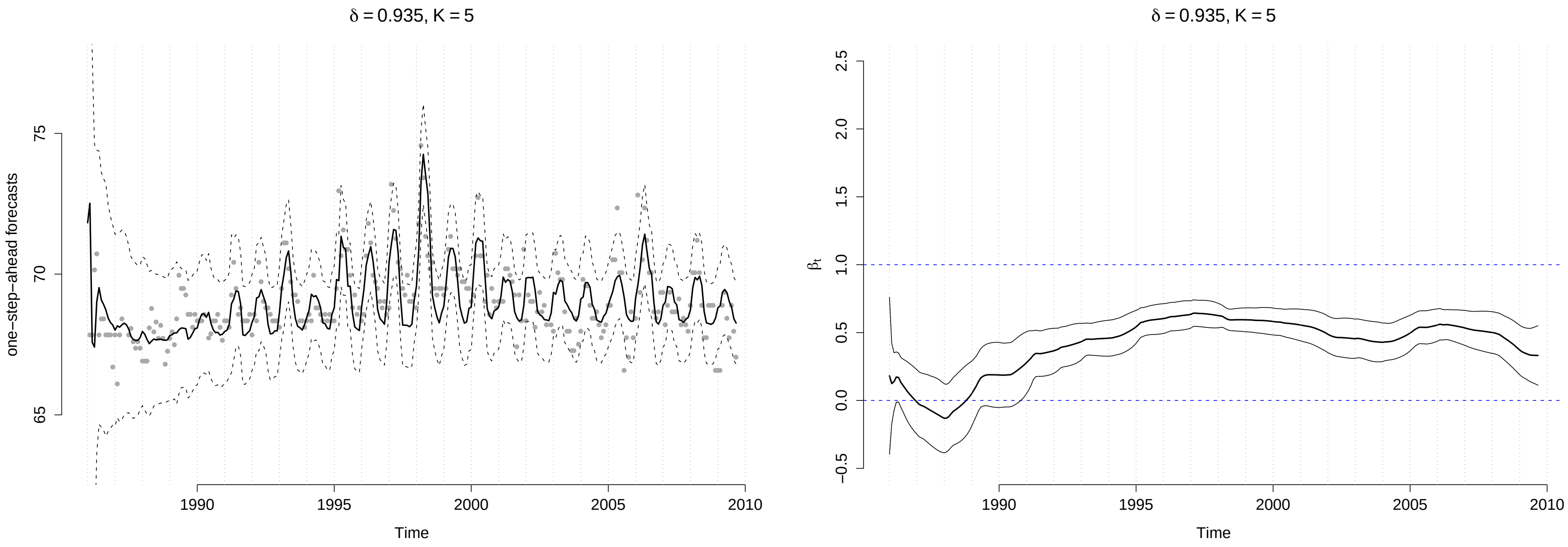




DLMs: Regression Models

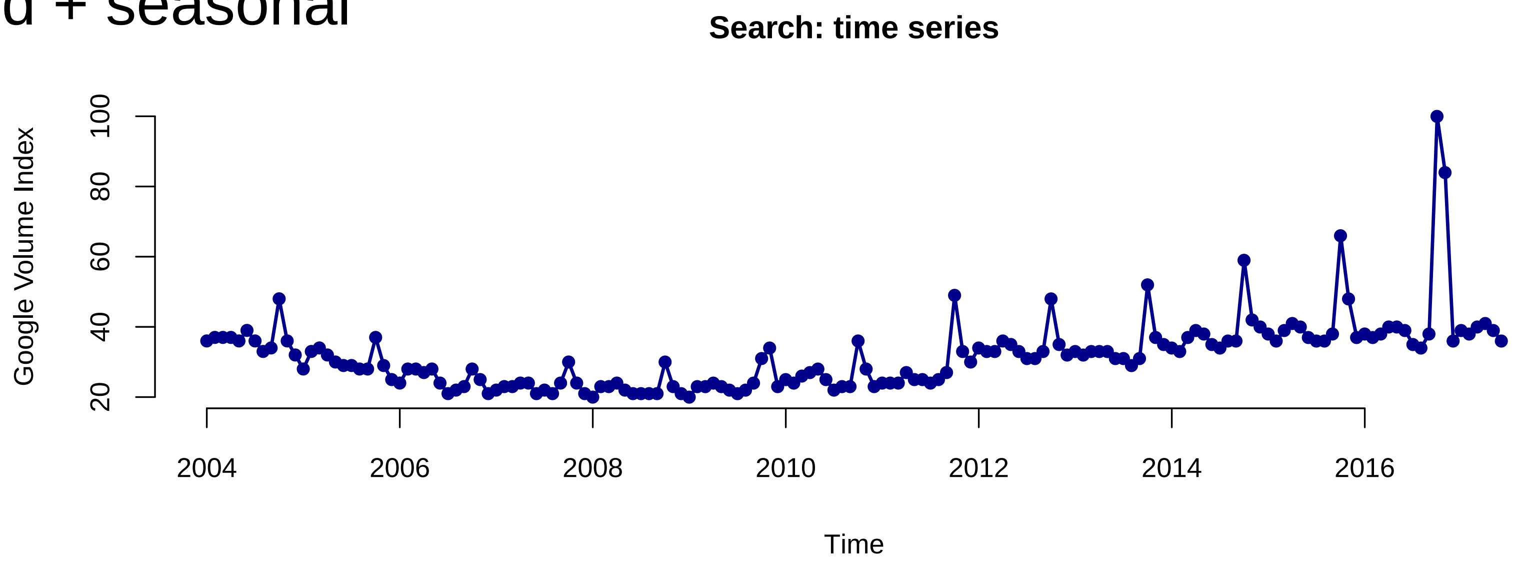
Example: **WTE at SC-10A**

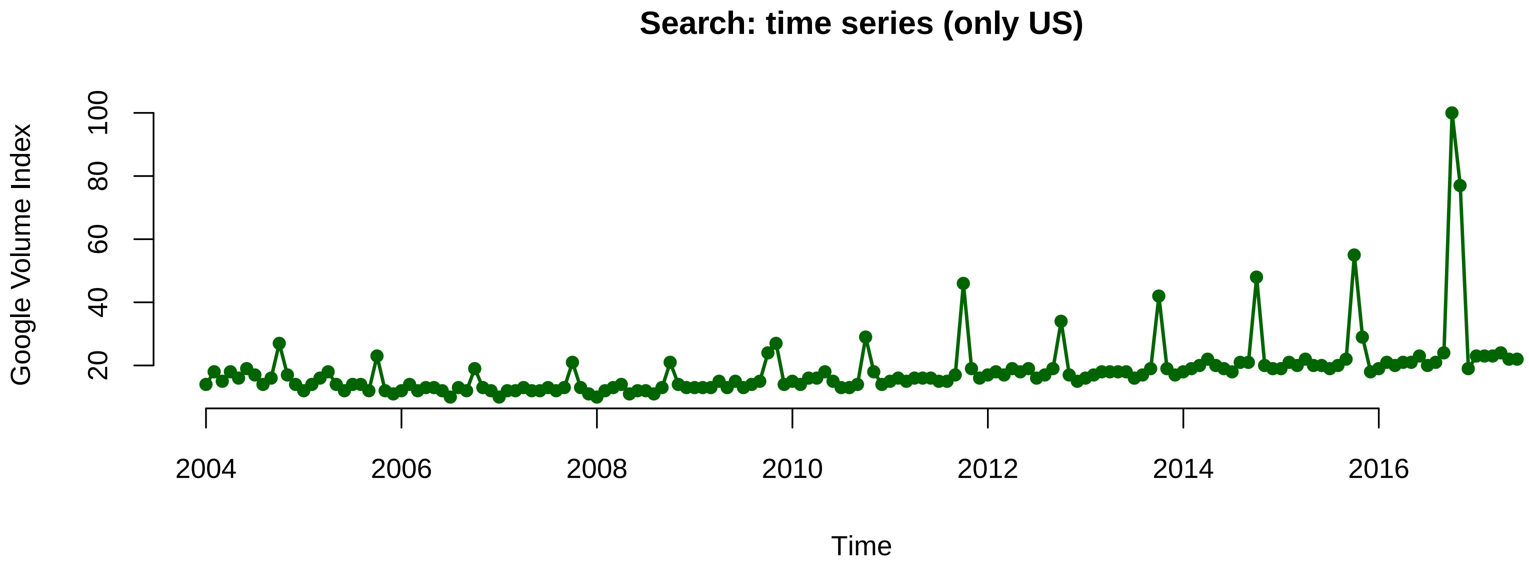
$$
\begin{aligned}
y_t &= \alpha + \beta_t \times f_t(\text{rainfall}) + \nu_t \\
\beta_t &= \beta_{t-1} + \omega_t
\end{aligned}
$$



DLMs: Superposition

How do we build a model with several components? E.g., trend + seasonal





DLMs: Superposition

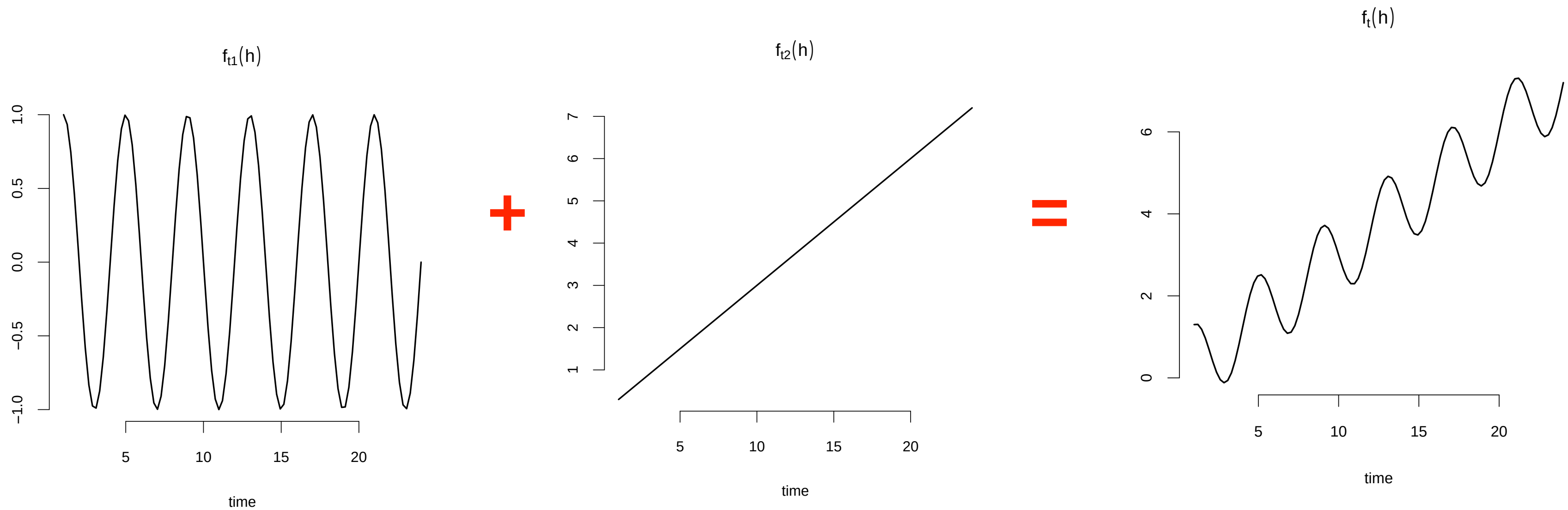
$$\{\mathbf{F}_{i,t}, \mathbf{G}_{i,t}, v_{i,t}, \mathbf{W}_{i,t}\} \text{ for } i = 1:m \Rightarrow y_t = \textstyle\sum_{i=1}^{m} y_{i,t}$$

has a DLM representation $\{\mathbf{F}_t, \mathbf{G}_t, v_t, \mathbf{W}_t\}$ given by

$$\mathbf{F}_t = (\mathbf{F}'_{1,t}, \ldots, \mathbf{F}'_{m,t})', \qquad \mathbf{G}_t = \text{blockdiag}(\mathbf{G}_1, \ldots, \mathbf{G}_m),$$

$$v_t = \sum_{i=1}^{m} v_{i,t}, \qquad \mathbf{W}_t = \text{blockdiag}(\mathbf{W}_{1,t}, \ldots, \mathbf{W}_{m,t}).$$

Forecast function: $f_t(h) = \sum_{i=1}^{m} f_{i,t}(h)$



Example: Linear Trend + Seasonal Component p=4

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$$\mathbf{F}_1 = (1, 0)' \qquad \mathbf{G}_1 = \mathbf{J}_2(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

DLMs: Superposition

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- Full seasonal component: $\{\mathbf{F}_2, \mathbf{G}_2, \cdot, \cdot\}$ with ω=2π/4=π/2, $\mathbf{F}_2 = (1,0,1)'$ and

$$\mathbf{G}_2 = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) & 0 \\ -\sin(\pi/2) & \cos(\pi/2) & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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DLMs: Superposition

The resulting DLM is a 5-dimensional model $\{\mathbf{F}, \mathbf{G}, \cdot, \cdot\}$ with

$$\mathbf{F} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

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Forecast function:

$$f_t(h) = (a_{t,1} + a_{t,2}h) + a_{t,3}\cos(\pi h/2) + a_{t,4}\sin(\pi h/2) + a_{t,5}(-1)^h$$

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- Seasonal model with p=12 including only harmonics 1, 3 and 4:

$$\mathbf{F}_2 = (\mathbf{E}_2', \mathbf{E}_2', \mathbf{E}_2')', \quad \mathbf{G}_2 = \mathrm{blockdiag}[\mathbf{G}_{h,1}, \mathbf{G}_{h,3}, \mathbf{G}_{h,4}]$$

$$\mathbf{G}_{h,r} = \begin{pmatrix} \cos(\pi r/6) & \sin(\pi r/6) \\ -\sin(\pi r/6) & \cos(\pi r/6) \end{pmatrix}, \quad r = 1, 3, 4$$

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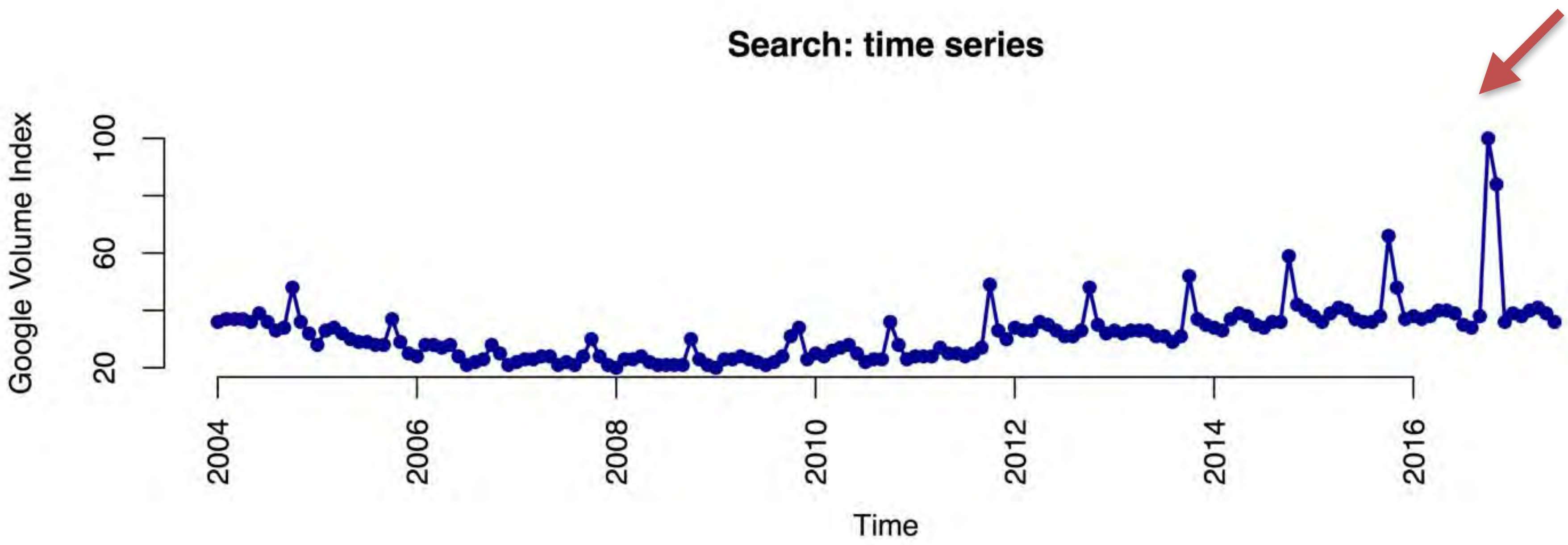
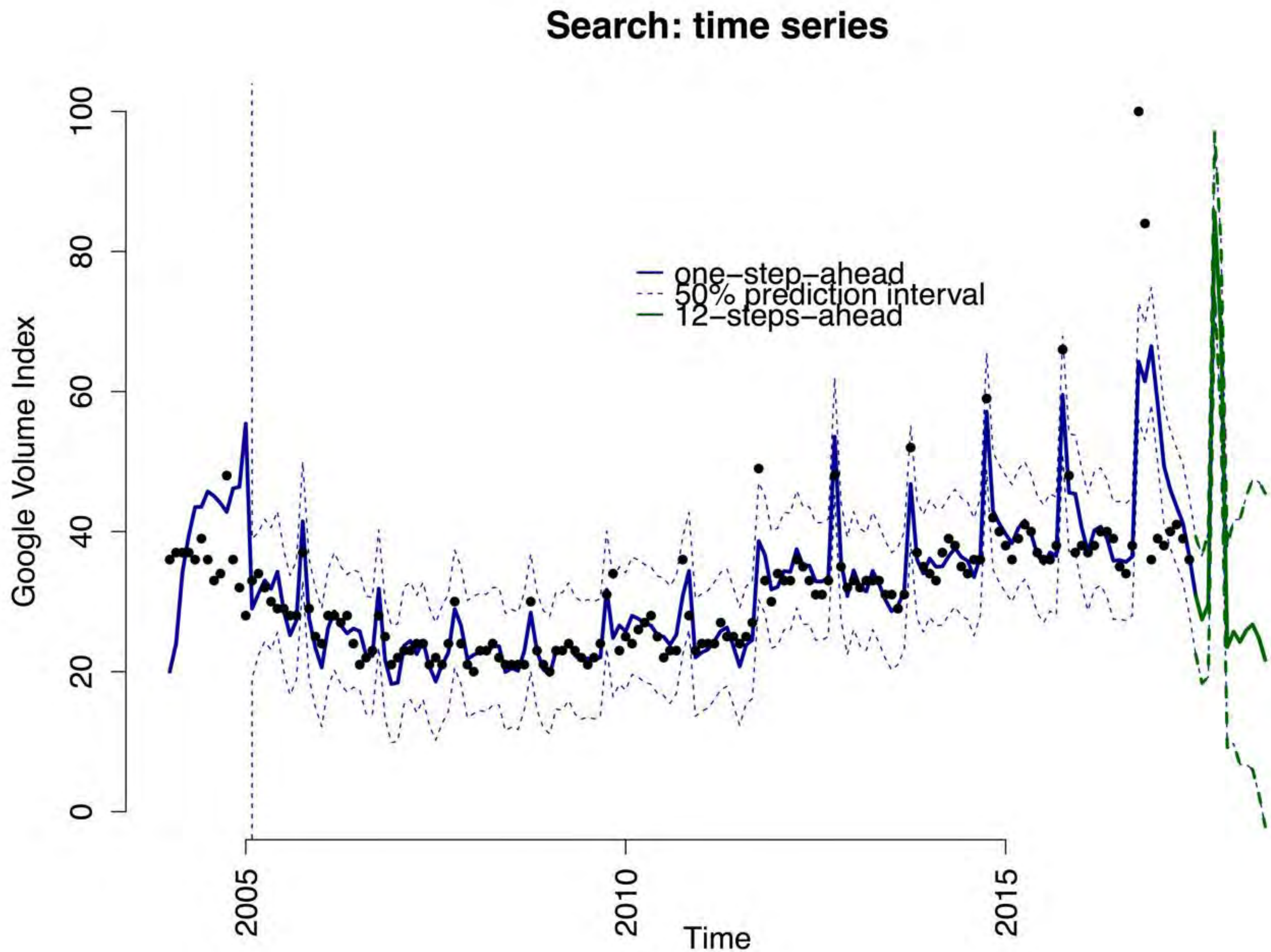
Full model:

$$\mathbf{F}_t = (\mathbf{F}_{1,t}', \mathbf{F}_2')', \quad \mathbf{G} = \mathrm{blockdiag}[\mathbf{G}_1, \mathbf{G}_2]$$

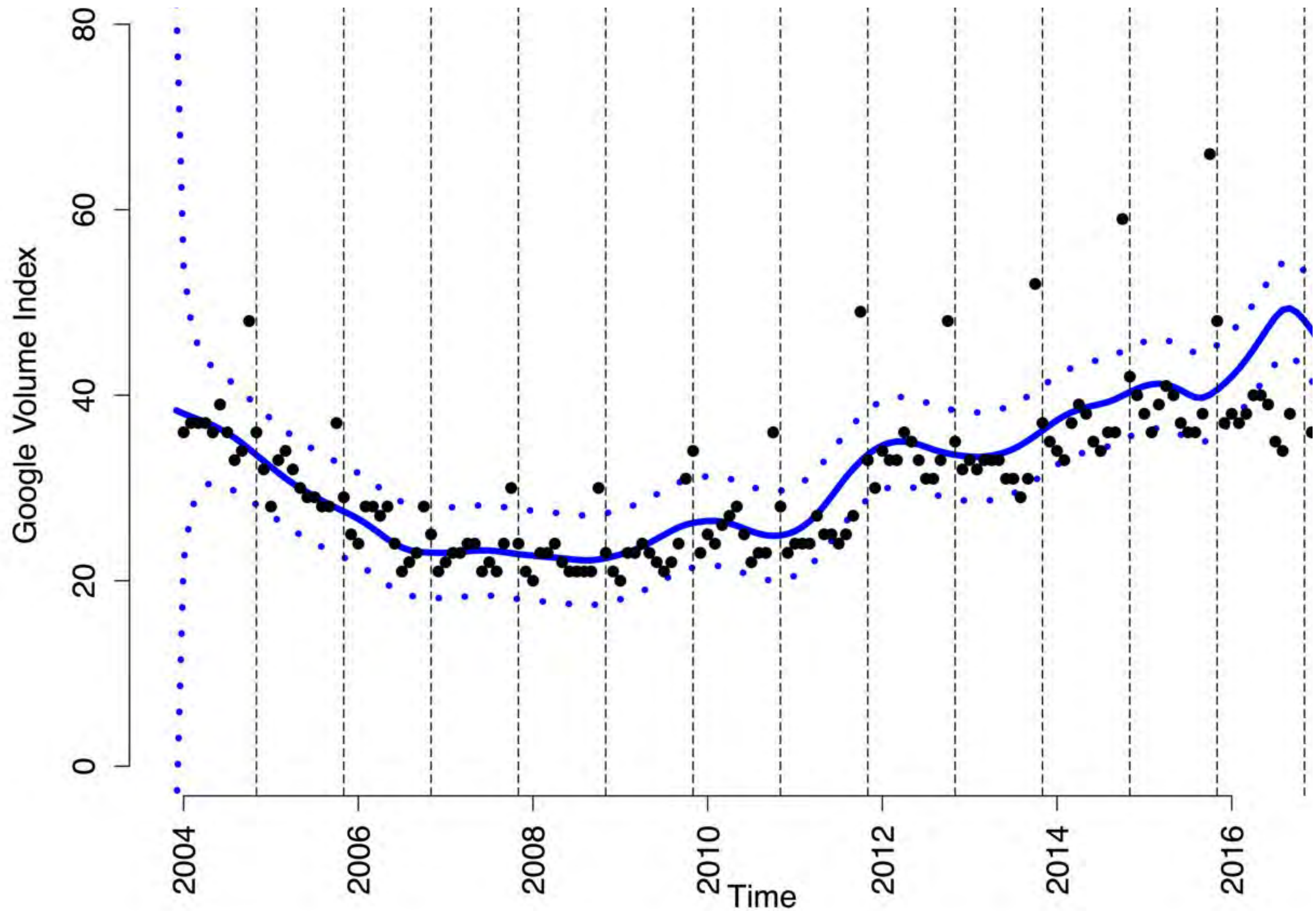
Google trends: “time series”



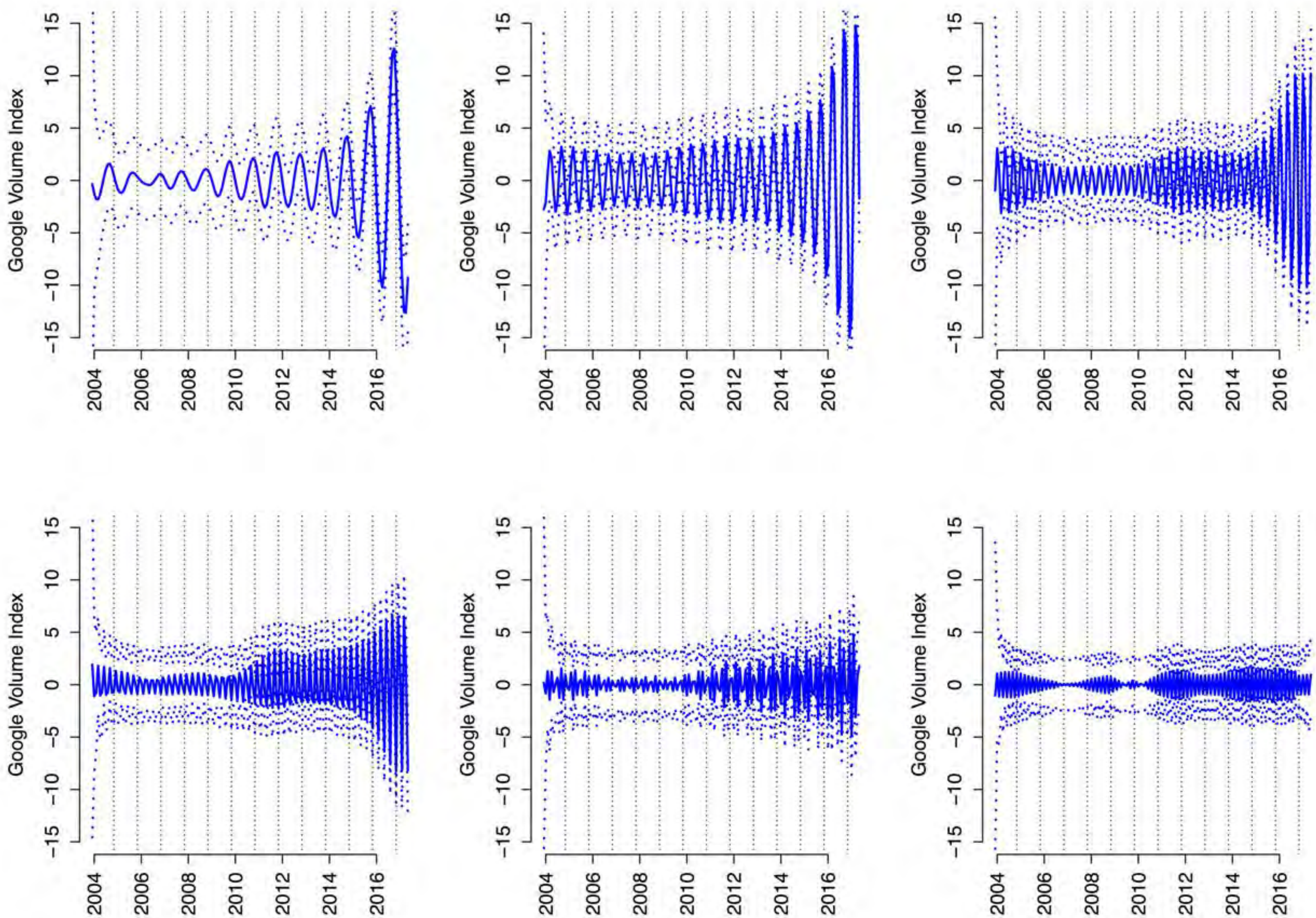
DLMs: Trend + Seasonal



Google index "time series": trend



Google index "time series": Seasonal



DLMs: Sequential Learning and Forecasting

Learning/Filtering

Case 1: observational variance <u>and</u> system covariance known

$$
\begin{aligned}
y_t &= \mathbf{F}_t' \boldsymbol{\theta}_t + \nu_t, \quad \nu_t \sim N(0, v_t), \\
\boldsymbol{\theta}_t &= \mathbf{G}_t \boldsymbol{\theta}_{t-1} + \mathbf{w}_t, \quad \mathbf{w}_t \sim N(0, \mathbf{W}_t)
\end{aligned}
$$

DLMs: Sequential Learning and Forecasting

Learning/Filtering

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$$\begin{array}{ccccccccccccccc} \boldsymbol{\theta}_0 & \rightarrow & \boldsymbol{\theta}_1 & \rightarrow & \boldsymbol{\theta}_2 & \rightarrow & \cdots & \rightarrow & \boldsymbol{\theta}_{t-1} & \rightarrow & \boldsymbol{\theta}_t & \rightarrow & \boldsymbol{\theta}_{t+1} & \rightarrow \\ & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & \\ & & y_1 & & y_2 & & & & y_{t-1} & & y_t & & y_{t+1} & \end{array}$$

DLMs: Sequential Learning and Forecasting

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\end{aligned}
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\begin{array}{ccccccccccccccc}
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 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & \\
 & & y_1 & & y_2 & & & & y_{t-1} & & y_t & & y_{t+1} &
\end{array}
$$

Begin with $(\boldsymbol{\theta}_0|\mathcal{D}_0) \sim N(\mathbf{m}_0, \mathbf{C}_0)$ and sequentially obtain the filtering and forecasting distributions via Bayes' rule:

$$
p(\boldsymbol{\theta}_t|\mathcal{D}_t) \propto p(y_t|\boldsymbol{\theta}_t) \times p(\boldsymbol{\theta}_t|\mathcal{D}_{t-1}),
$$

$$
p(\boldsymbol{\theta}_t|\mathcal{D}_{t-1}) = \int p(\boldsymbol{\theta}_t|\boldsymbol{\theta}_{t-1})p(\boldsymbol{\theta}_{t-1}|\mathcal{D}_{t-1})d\boldsymbol{\theta}_{t-1}.
$$

DLMs: Sequential Learning and Forecasting

- Evolution equation **+** $(\boldsymbol{\theta}_{t-1}|\mathcal{D}_{t-1})$ leads to

$$(\boldsymbol{\theta}_t|\mathcal{D}_{t-1}) \sim N(\mathbf{a}_t, \mathbf{R}_t), \quad \mathbf{a}_t = \mathbf{G}_t\mathbf{m}_{t-1}, \ \ \mathbf{R}_t = \mathbf{G}_t\mathbf{C}_{t-1}\mathbf{G}_t' + \mathbf{W}_t$$

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- Observation equation + $(\boldsymbol{\theta}_t|\mathcal{D}_{t-1})$ leads to

$$\underline{(y_t|\mathcal{D}_{t-1}) \sim N(y_t|f_t, q_t)}, \quad f_t = \mathbf{F}_t'\mathbf{a}_t, \;\; q_t = \mathbf{F}_t'\mathbf{R}_t\mathbf{F}_t + v_t$$

DLMs: Sequential Learning and Forecasting

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- After observing y_t the forecast error $e_t = y_t - f_t$ is obtained. Using Bayes' rule we have:

$$\underline{(\boldsymbol{\theta}_t|\mathcal{D}_t) \sim N(\boldsymbol{\theta}_t|\mathbf{m}_t, \mathbf{C}_t)},$$

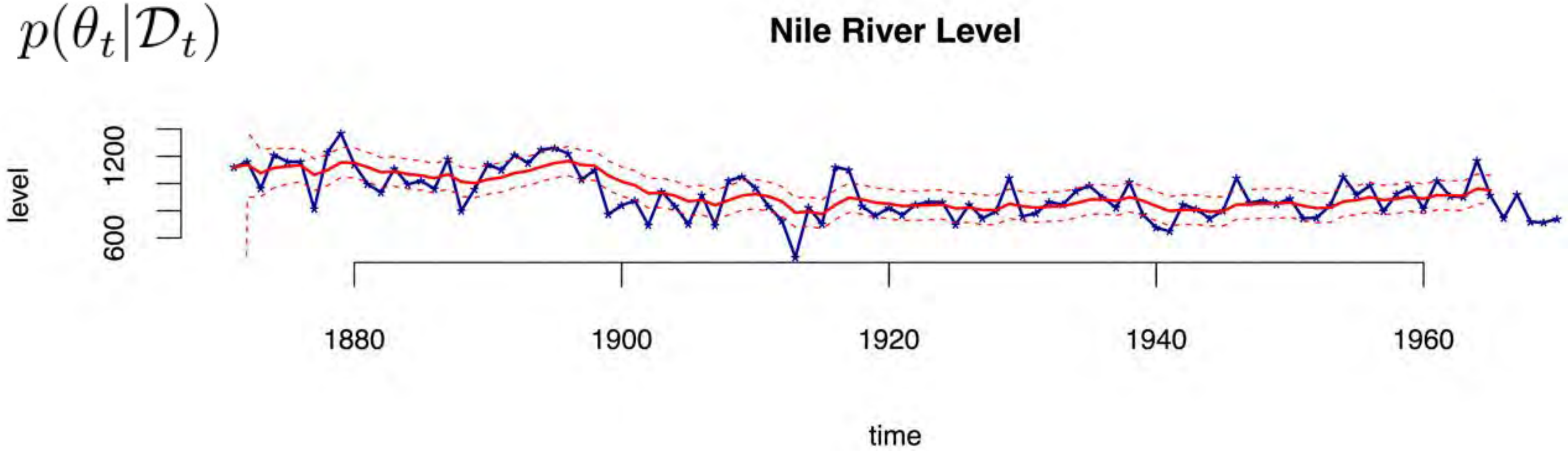
$$\mathbf{m}_t = \mathbf{a}_t + \mathbf{A}_t e_t, \quad \mathbf{C}_t = \mathbf{R}_t - \mathbf{A}_t\mathbf{A}_t' q_t, \quad \mathbf{A}_t = \mathbf{R}_t\mathbf{F}_t/q_t$$

DLMs: Sequential Learning and Forecasting

Nile River Level [P,P&C]: $m_0=0$, $C_0=10^7$

Model 1: {1, 1, 15100, 755}, w/v=0.05

$p(\theta_t|\mathcal{D}_t)$



DLMs: Sequential Learning and Forecasting

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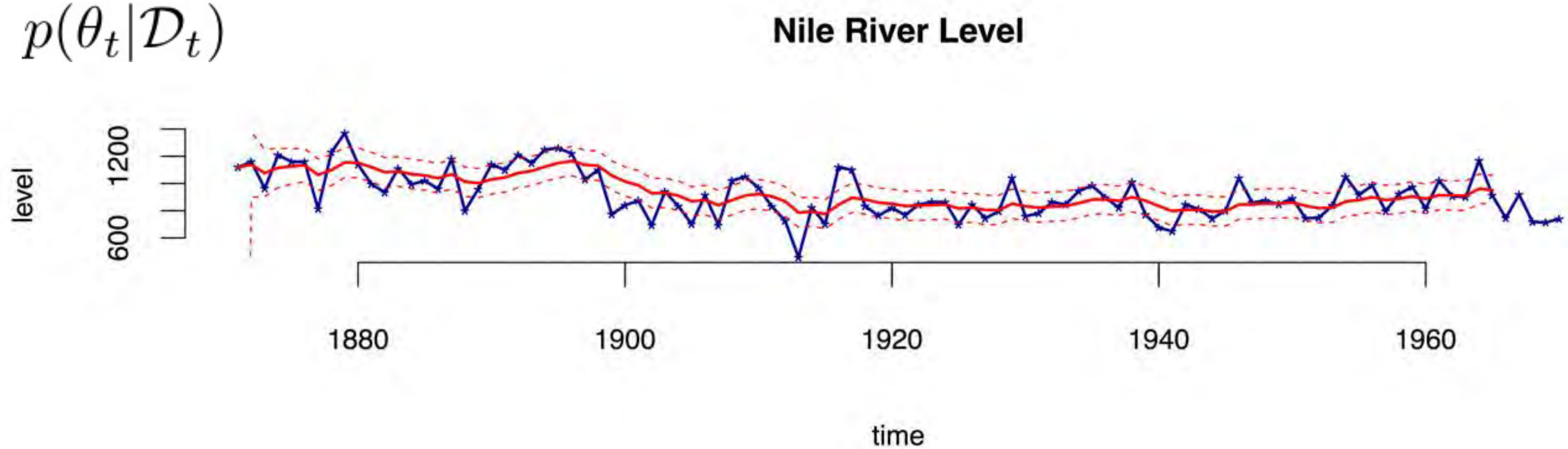
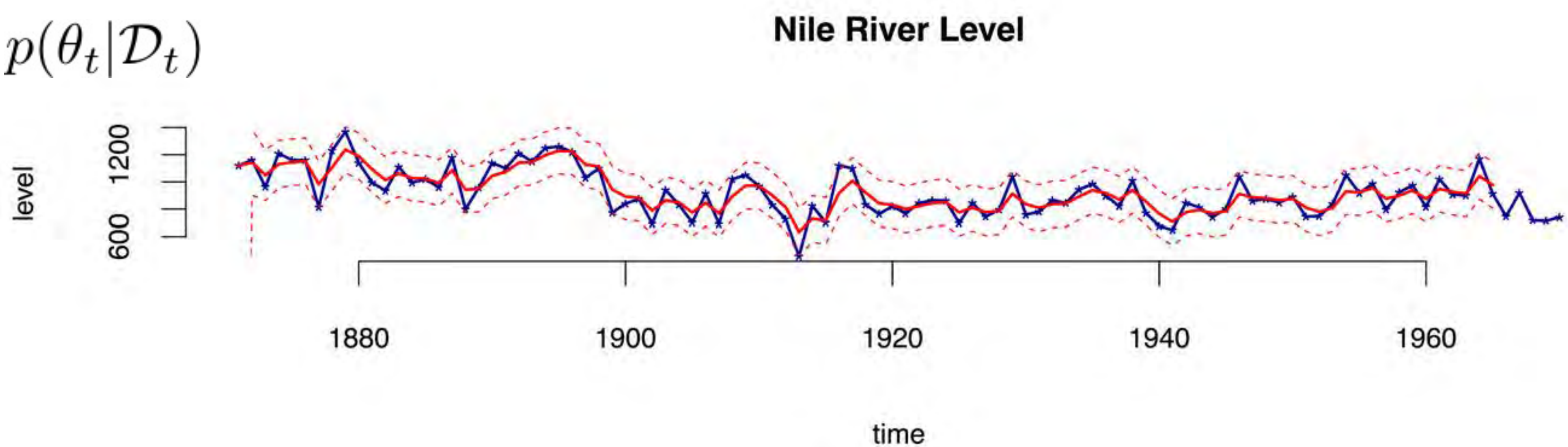


Model 2: {1, 1, 15100, 7550}, w/v=0.5

$p(\theta_t|\mathcal{D}_t)$



DLMs: Sequential Learning and Forecasting

Learning/Filtering

Case 2: observational variance constant but unknown <u>and</u> system covariance known

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\begin{aligned}
y_t &= \mathbf{F}_t'\boldsymbol{\theta}_t + v_t, \quad v_t \sim N(0, v), \\
\boldsymbol{\theta}_t &= \mathbf{G}_t\boldsymbol{\theta}_{t-1} + \mathbf{w}_t, \quad \mathbf{w}_t \sim N(\mathbf{0}, v\mathbf{W}_t^*),
\end{aligned}
$$

with prior:

$$
\begin{aligned}
(\boldsymbol{\theta}_0|\mathcal{D}_0) &\sim N(\mathbf{m}_0, v\mathbf{C}_0^*) \\
(\phi|\mathcal{D}_0) &\sim G(n_0/2, n_0S_0/2), \quad \phi = v^{-1}
\end{aligned}
$$

leading to the following distributions…

DLMs: Sequential Learning and Forecasting

- $(\boldsymbol{\theta}_t | \mathcal{D}_{t-1}) \sim T_{n_{t-1}}(\mathbf{a}_t, \mathbf{R}_t)$

- $(y_t | \mathcal{D}_{t-1}) \sim T_{n_{t-1}}(f_t, q_t)$ with $\quad q_t = \mathbf{F}_t' \mathbf{R}_t \mathbf{F}_t + S_{t-1}$

- $(\phi | \mathcal{D}_t) \sim G(n_t/2, n_t S_t/2)$

$$n_t = n_{t-1} + 1, \quad S_t = S_{t-1} + \frac{S_{t-1}}{n_t}\left(\frac{e_t^2}{q_t} - 1\right)$$

- $(\boldsymbol{\theta}_t | \mathcal{D}_t) \sim T_{n_t}(\mathbf{m}_t, \mathbf{C}_t)$

$$\mathbf{C}_t = \frac{S_t}{S_{t-1}}\left(\mathbf{R}_t - \mathbf{A}_t \mathbf{A}_t' q_t\right)$$

Nile River Level

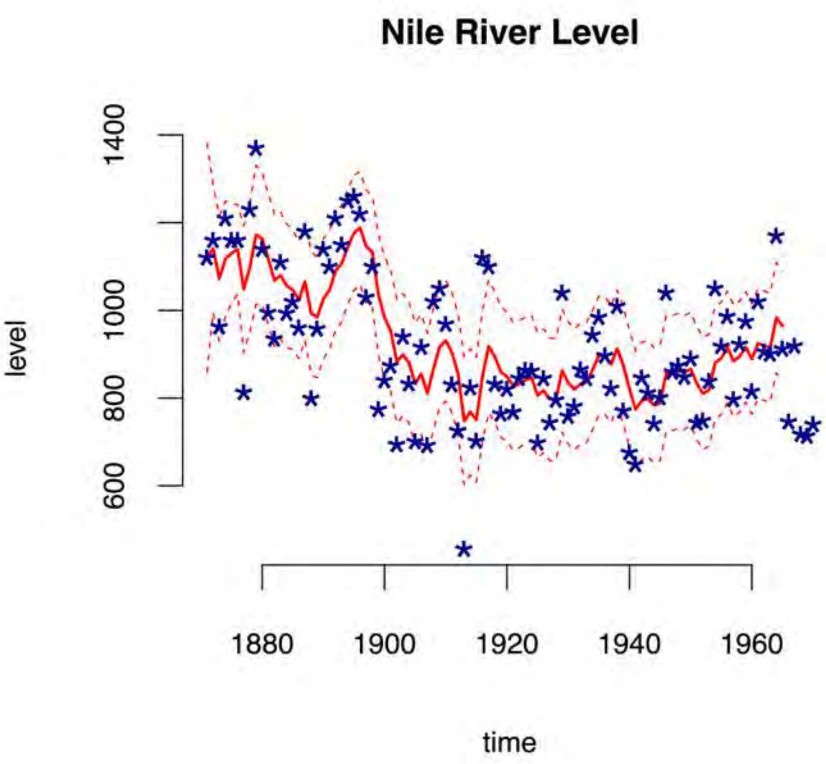
Nile River Level

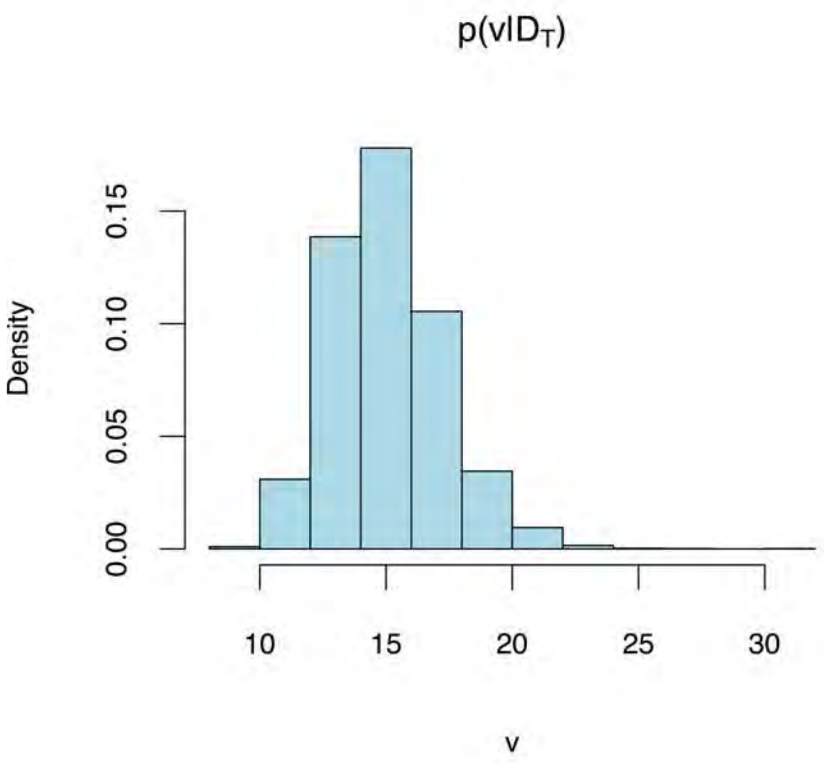
- **Model 3:** $\{1, 1, \hat{v}, \hat{w}\}$, $\hat{v}=15497.7$ and $\hat{w}=1213.5$ (MLEs)

DLMs: Sequential Learning and Forecasting

Nile River Level

- **Model 3:** {1, 1, v̂, ŵ}, v̂=15497.7 and ŵ=1213.5 (MLEs)
- **Model 4:** {1, 1, vV*, vW*}, m_0=0, C_0*=10^7, V*=1000, W*=100 , n_0=1, S_0=10





DLMs: Missing Data

If y_t is missing, $\mathcal{D}_t = \mathcal{D}_{t-1}$ and $p(\boldsymbol{\theta}_t|\mathcal{D}_t) = p(\boldsymbol{\theta}_t|\mathcal{D}_{t-1})$
Then,

$$\mathbf{m}_t = \mathbf{a}_t, \quad \mathbf{C}_t = \mathbf{R}_t, \quad n_t = n_{t-1} \quad S_t = S_{t-1}$$

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$$\begin{array}{ccccccccccc}
\boldsymbol{\theta}_0 & \rightarrow & \boldsymbol{\theta}_1 & \rightarrow & \boldsymbol{\theta}_2 & \rightarrow & \boldsymbol{\theta}_3 & \rightarrow & \boldsymbol{\theta}_4 & \rightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & y_1 & & y_2 & & \mathrm{NA} & & y_4 & &
\end{array}$$

$$\begin{array}{ccccccccccc}
\mathbf{m}_0 & \mathbf{a}_1 & \mathbf{m}_1 & \mathbf{a}_2 & \mathbf{m}_2 & \mathbf{a}_3 & \mathbf{m}_3 = \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{m}_4 & \mathbf{a}_5 & \cdots \\
\mathbf{C}_0 & \mathbf{R}_1 & \mathbf{C}_1 & \mathbf{R}_2 & \mathbf{C}_2 & \mathbf{R}_3 & \mathbf{C}_3 = \mathbf{R}_3 & \mathbf{R}_4 & \mathbf{C}_4 & \mathbf{R}_5 & \cdots \\
n_0 & & n_1 & & n_2 & & n_3 = n_2 & & n_4 & & \cdots \\
S_0 & & S_1 & & S_2 & & S_3 = S_2 & & S_4 & & \cdots
\end{array}$$

Forecasting

- $v_t, \mathbf{W}_t$ known:

$$(\boldsymbol{\theta}_{t+h}|\mathcal{D}_t) \sim N(\mathbf{a}_t(h), \mathbf{R}_t(h)), \quad (y_{t+h}|\mathcal{D}_t) \sim N(f_t(h), q_t(h)),$$

$$\begin{aligned} \mathbf{a}_t(h) &= \mathbf{G}_{t+h}\mathbf{a}_t(h-1), \ \ \mathbf{a}_t(0) = \mathbf{m}_t, \\ \mathbf{R}_t(h) &= \mathbf{G}_{t+h}\mathbf{R}_t(h-1)\mathbf{G}'_{t+h} + \mathbf{W}_{t+h}, \ \ \mathbf{R}_t(0) = \mathbf{C}_t \end{aligned}$$

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DLMs: Sequential Learning and Forecasting

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$$f_t(h) = \mathbf{F}'_{t+h}\mathbf{a}_t(h), \;\; q_t(h) = \mathbf{F}'_{t+h}\mathbf{R}_t(h)\mathbf{F}_{t+h} + v_{t+h}.$$

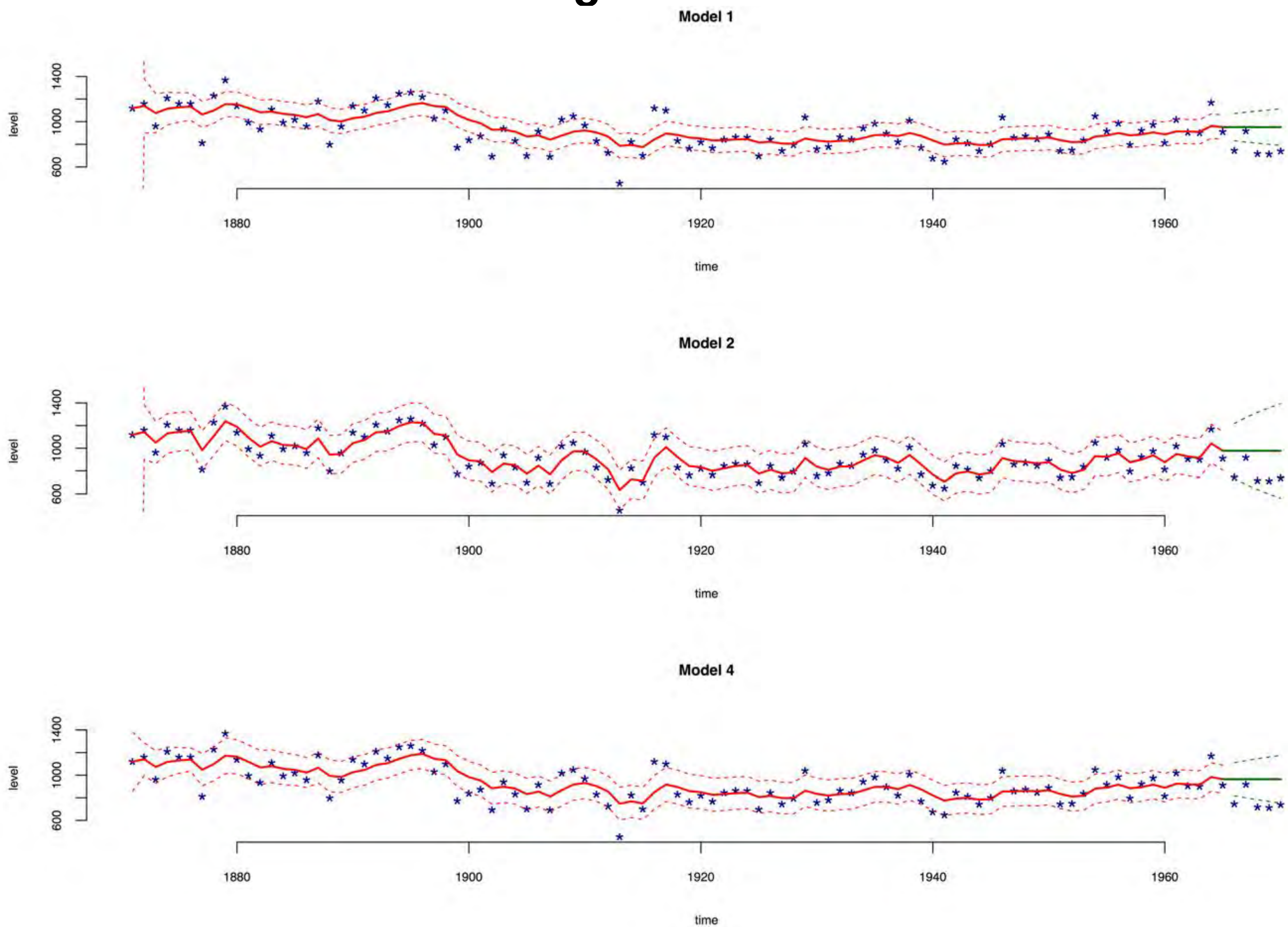
- $v_t = v$ unknown and $\mathbf{W}_t$ known:

$$\begin{aligned} (\underline{\boldsymbol{\theta}_{t+h}|\mathcal{D}_t}) &\sim T_{n_t}(\mathbf{a}_t(h), \mathbf{R}_t(h)), \\ (\underline{y_{t+h}|\mathcal{D}_t}) &\sim T_{n_t}(f_t(h), q_t(h)), \end{aligned}$$

with $q_t(h) = \mathbf{F}'_{t+h}\mathbf{R}_t(h)\mathbf{F}_{t+h} + S_t$

DLMs: Sequential Learning and Forecasting

Nile River Level: Forecasting



DLMs: Retrospective Analysis/Smoothing

- **Smoothing, Case 1:** observational variances and system covariances known. Let t < T and $\mathbf{a}_T(0) = \mathbf{m}_T, \ \ \mathbf{R}_T(0) = \mathbf{C}_T.$ Then, for t=(T-1), (T-2),…

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$$(\boldsymbol{\theta}_t | \mathcal{D}_T) \sim N(\mathbf{a}_T(t-T), \mathbf{R}_T(t-T))$$

$$\begin{aligned} \mathbf{a}_T(t-T) &= \mathbf{m}_t - \mathbf{B}_t[\mathbf{a}_{t+1} - \mathbf{a}_T(t-T+1)] \\ \mathbf{R}_T(t-T) &= \mathbf{C}_t - \mathbf{B}_t[\mathbf{R}_{t+1} - \mathbf{R}_T(t-T+1)]\mathbf{B}_t' \end{aligned}$$

with $\mathbf{B}_t = \mathbf{C}_t \mathbf{G}_{t+1}' \mathbf{R}_{t+1}^{-1}.$

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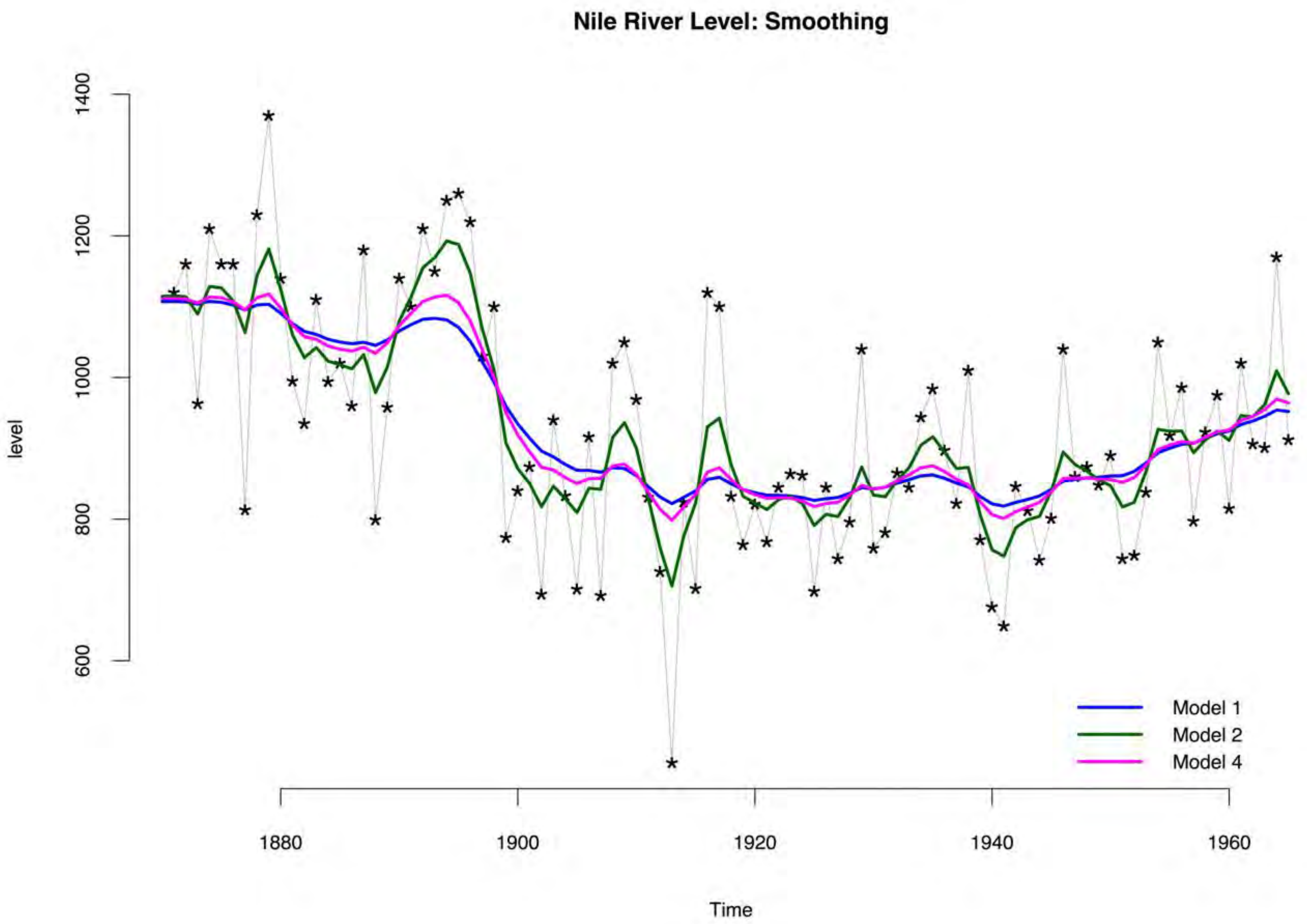
with $\mathbf{B}_t = \mathbf{C}_t \mathbf{G}_{t+1}' \mathbf{R}_{t+1}^{-1}.$

- **Smoothing, Case 2:** observational variance unknown but fixed and system covariances known

$$(\boldsymbol{\theta}_t|\mathcal{D}_T) \sim T_{n_T}(\mathbf{a}_T(t-T), \mathbf{R}_T(t-T)S_T/S_t)$$

DLMs: Retrospective Analysis/Smoothing

Nile River Data: Smoothing $SNR_1=0.05$, $SNR_2=0.5$, $SNR_4=0.1$



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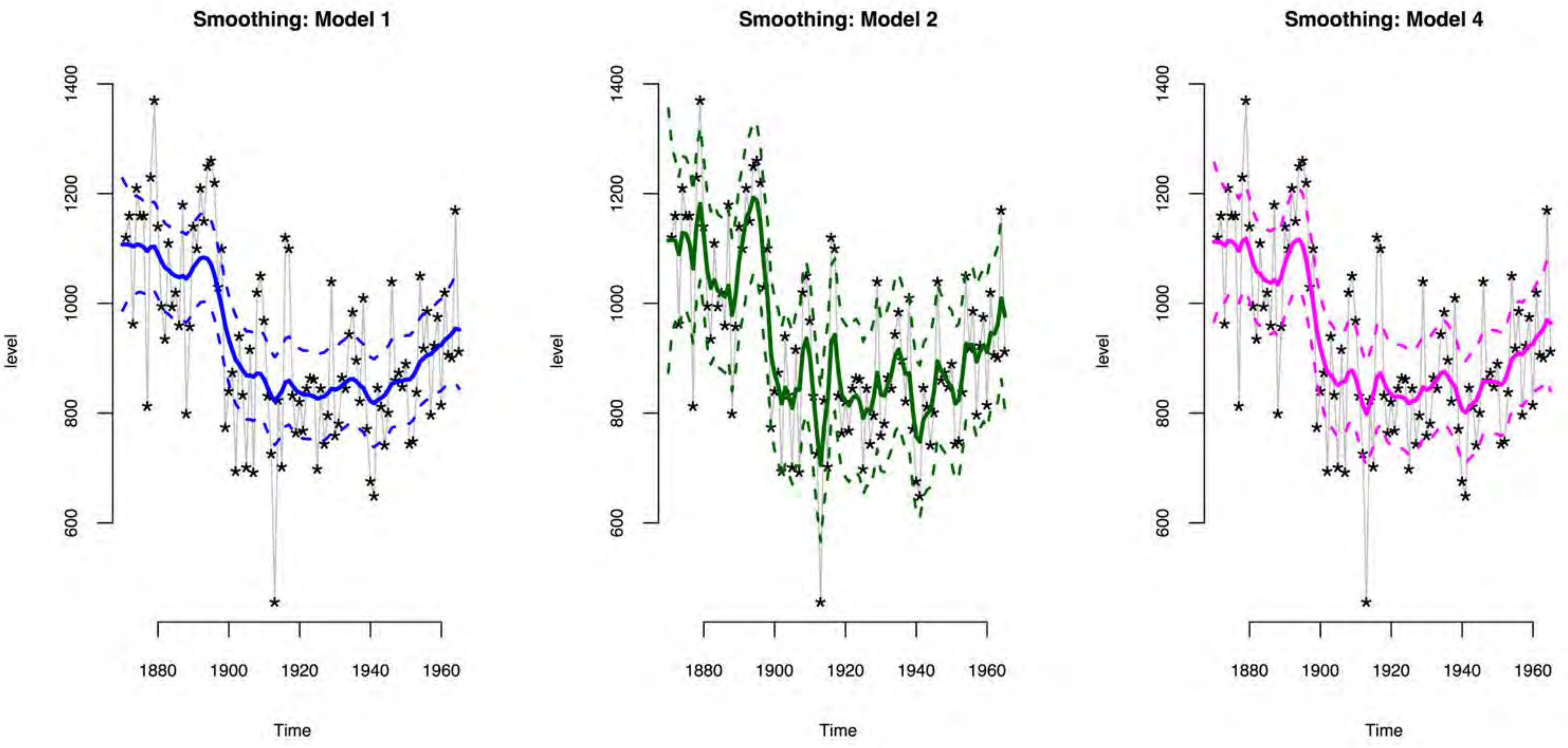

DLMs: Discount Factors

Discounting at the system level

Prior variance when $\mathbf{W}_t = \mathbf{0}$

$$\mathbf{R}_t = V(\boldsymbol{\theta}_t | \mathcal{D}_{t-1}) = \mathbf{P}_t + \mathbf{W}_t,$$

where $\mathbf{P}_t = \mathbf{G}_t \mathbf{C}_{t-1} \mathbf{G}_t'$.

Assume

$$\mathbf{R}_t = \frac{\mathbf{P}_t}{\delta}, \quad \delta \in (0, 1].$$

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Assume

$$\mathbf{R}_t = \frac{\mathbf{P}_t}{\delta}, \quad \delta \in (0,1].$$

$$\mathbf{W}_t = \frac{(1-\delta)}{\delta} \mathbf{P}_t.$$

Note that $\mathbf{W}_t$ is identified for all t given δ and $\mathbf{C}_0$.

Choosing δ

- Values of $\delta \in (0.8, 1]$ are typically relevant in practice.
- $\delta=1$ corresponds to a static model; smaller values of δ lead to larger variation over time.
- δ can be chosen to maximize

$$\log(\delta) \equiv \log[p(y_{1:T}|\mathcal{D}_0, \delta)] = \sum_{t=1}^{T} \log[p(y_t|\mathcal{D}_{t-1}, \delta)],$$

or to minimize the MSE/MAD.

- Smoothing distributions are available and depend on δ.

Component Discounting

If m DLMs are superposed, $\{\mathbf{F}_{i,t}, \mathbf{G}_{i,t}, v_{i,t}, \mathbf{W}_{i,t}\}, \quad i = 1:m,$
a component discount DLM can be considered defining

$$\mathbf{W}_{i,t} = \frac{(1-\delta_i)}{\delta_i}\mathbf{P}_{i,t},$$

with $\delta_i \in (0,1]$.

This allows greater flexibility, e.g., trend components may require more/less variation over time than seasonal components

Example: Google Cranberry Data

Model: $\{\mathbf{F}_t, \mathbf{G}_t, v, \mathbf{W}_t\}$ with $\mathbf{G}_t = \text{blockdiag}(1, \mathbf{G}_t^s)$,
$\mathbf{F}_t' = (1, \mathbf{E}_2', \mathbf{E}_2', \mathbf{E}_2') = (1, 1, 0, 1, 0, 1, 0)'$,

$$\mathbf{G}_t^s = \begin{pmatrix} \mathbf{J}_2(1, 2\pi/12) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2(1, 2\pi/6) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_2(1, 2\pi/4) \end{pmatrix},$$

where

$$\mathbf{J}_2(1, \omega) = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix}.$$

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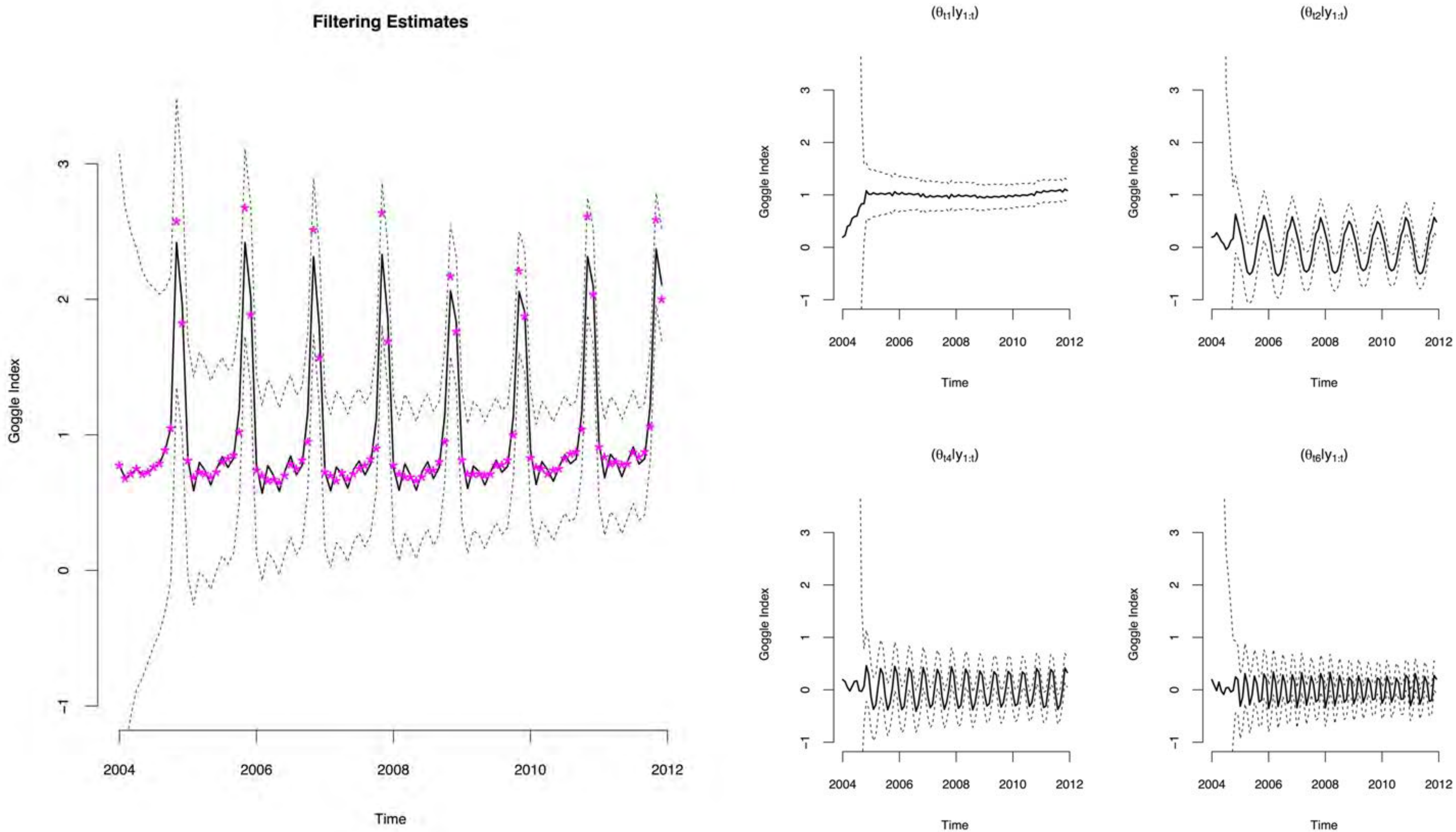
where

$$\mathbf{J}_2(1, \omega) = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix}.$$

Two options: (i) a single discount factor and (ii) two discount factors, one for the trend and another one for the seasonal components.

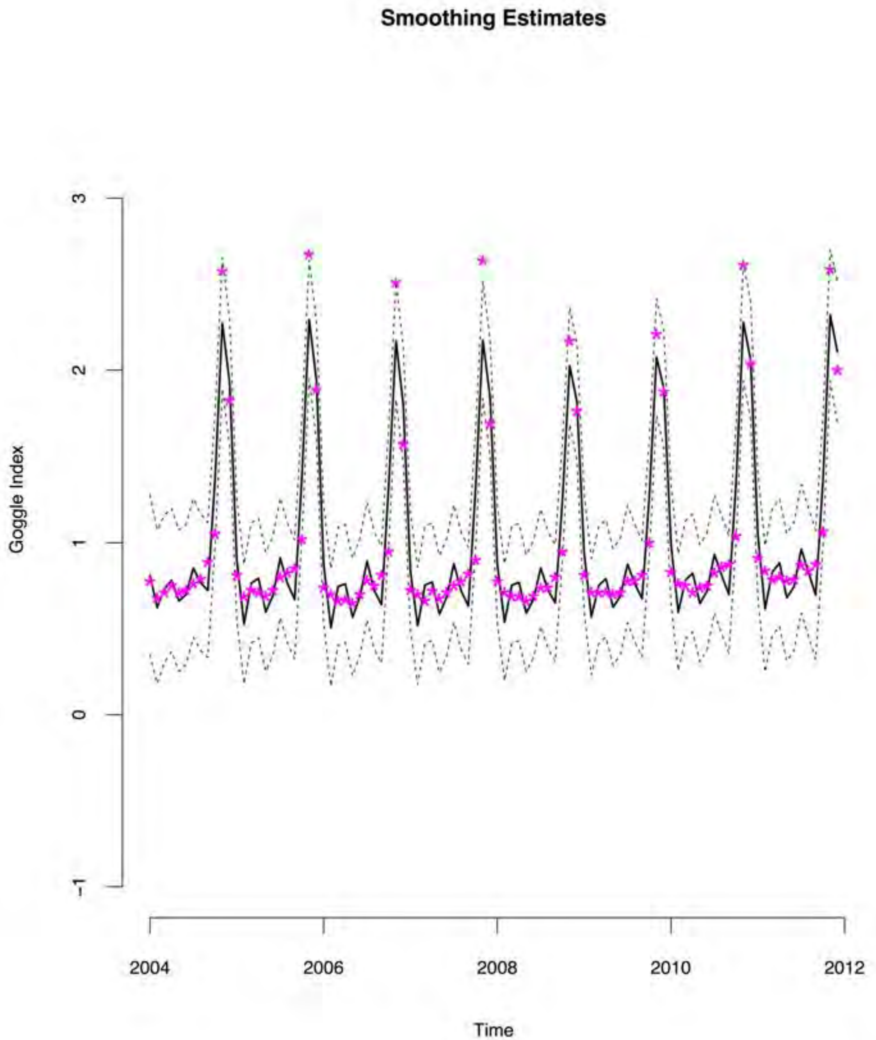
DLMs: Discount Factors

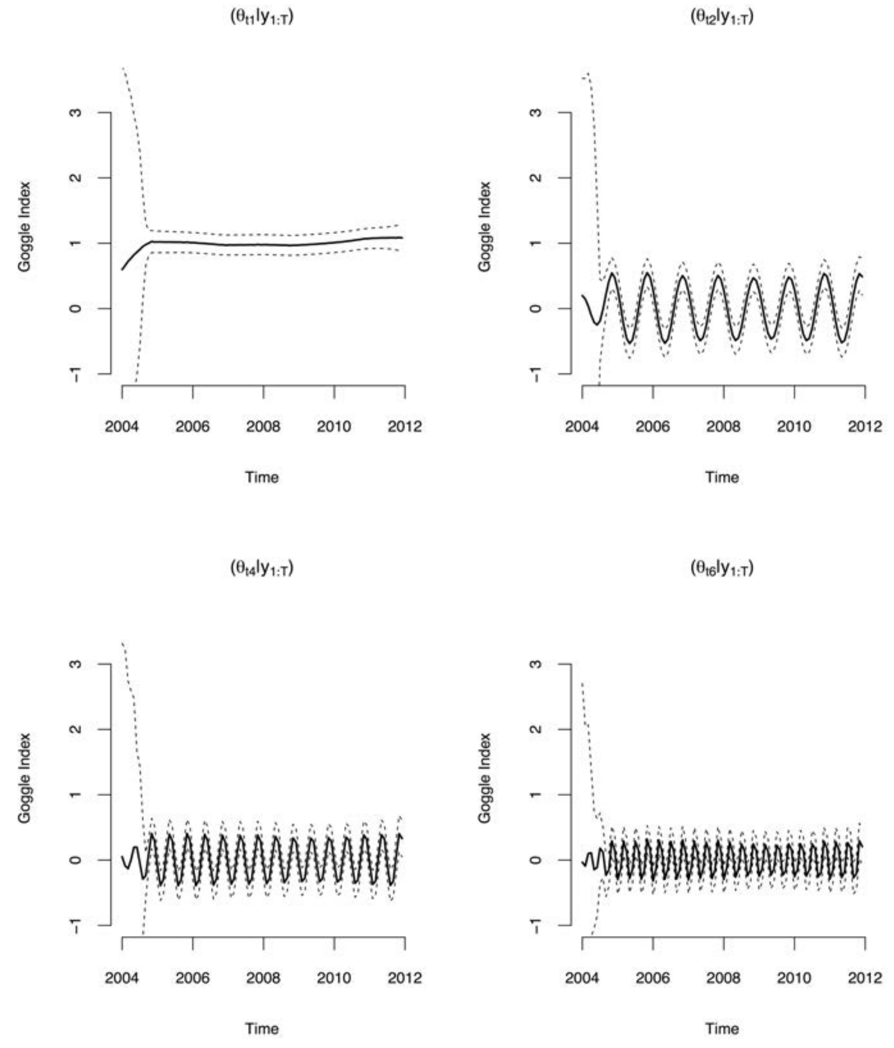
Filtering δ=0.87, SNR=1, n_0=1, S_0=1, m_0=0, C_0*=I



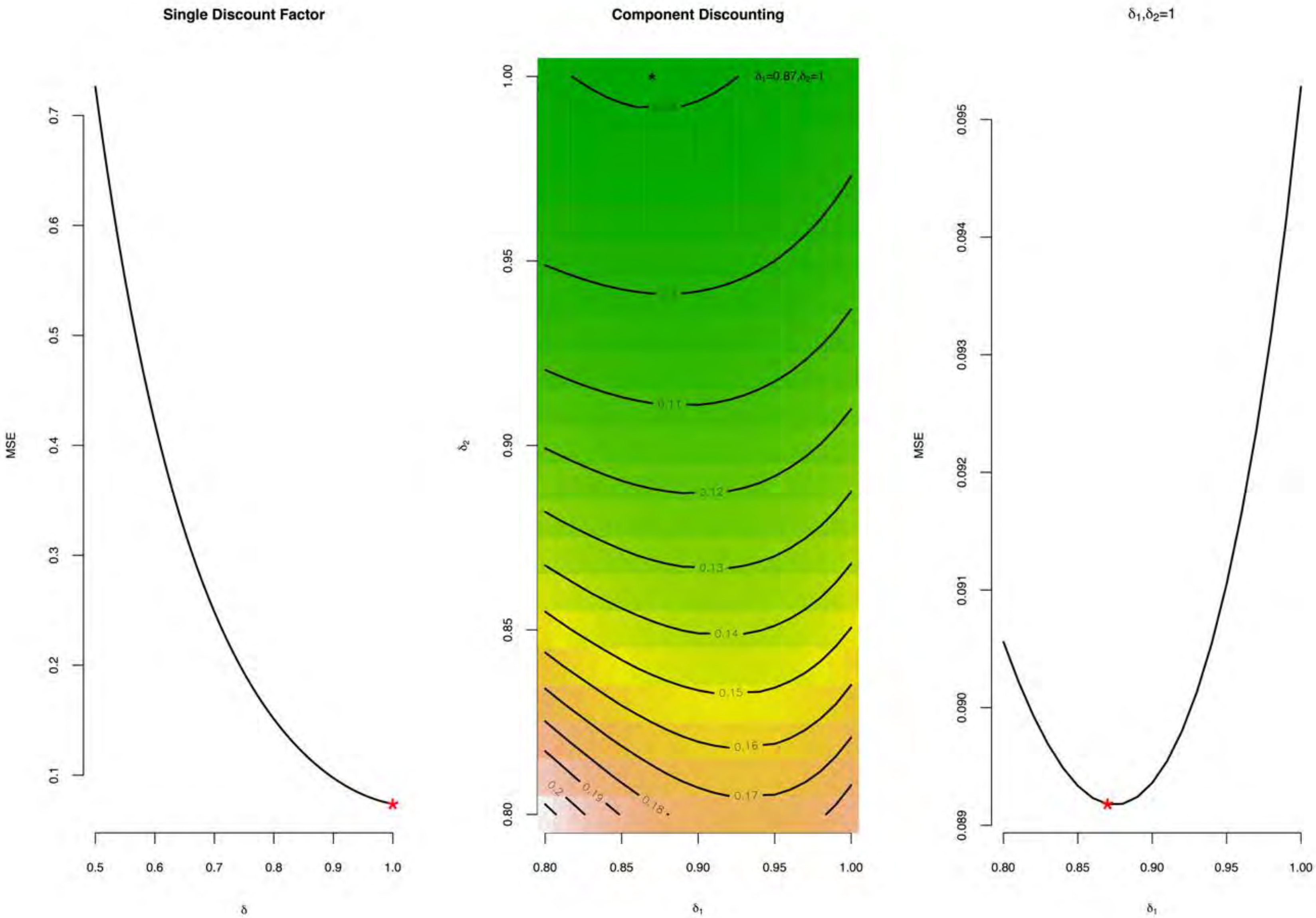
DLMs: Discount Factors

Smoothing δ=0.87



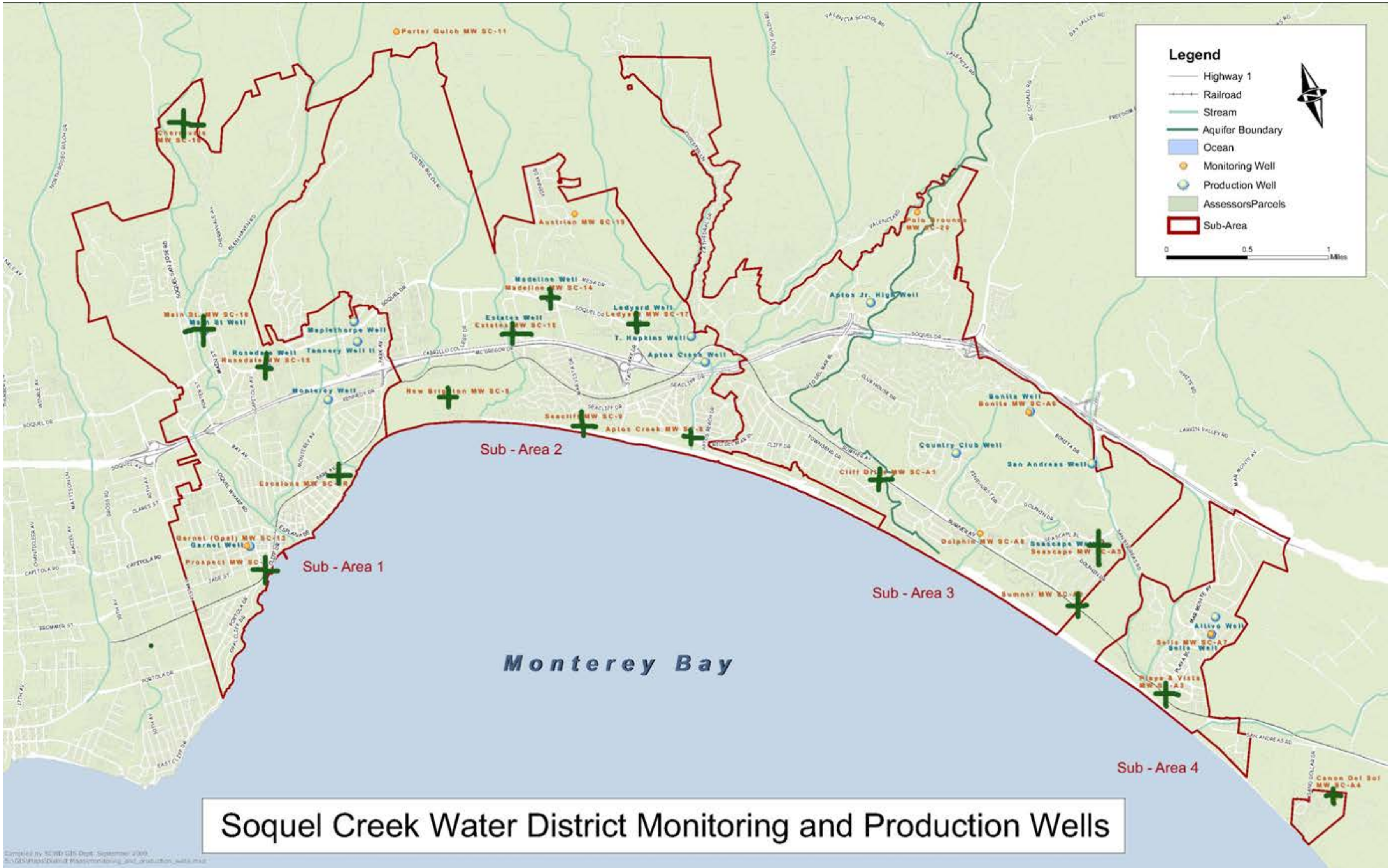


Discount factors

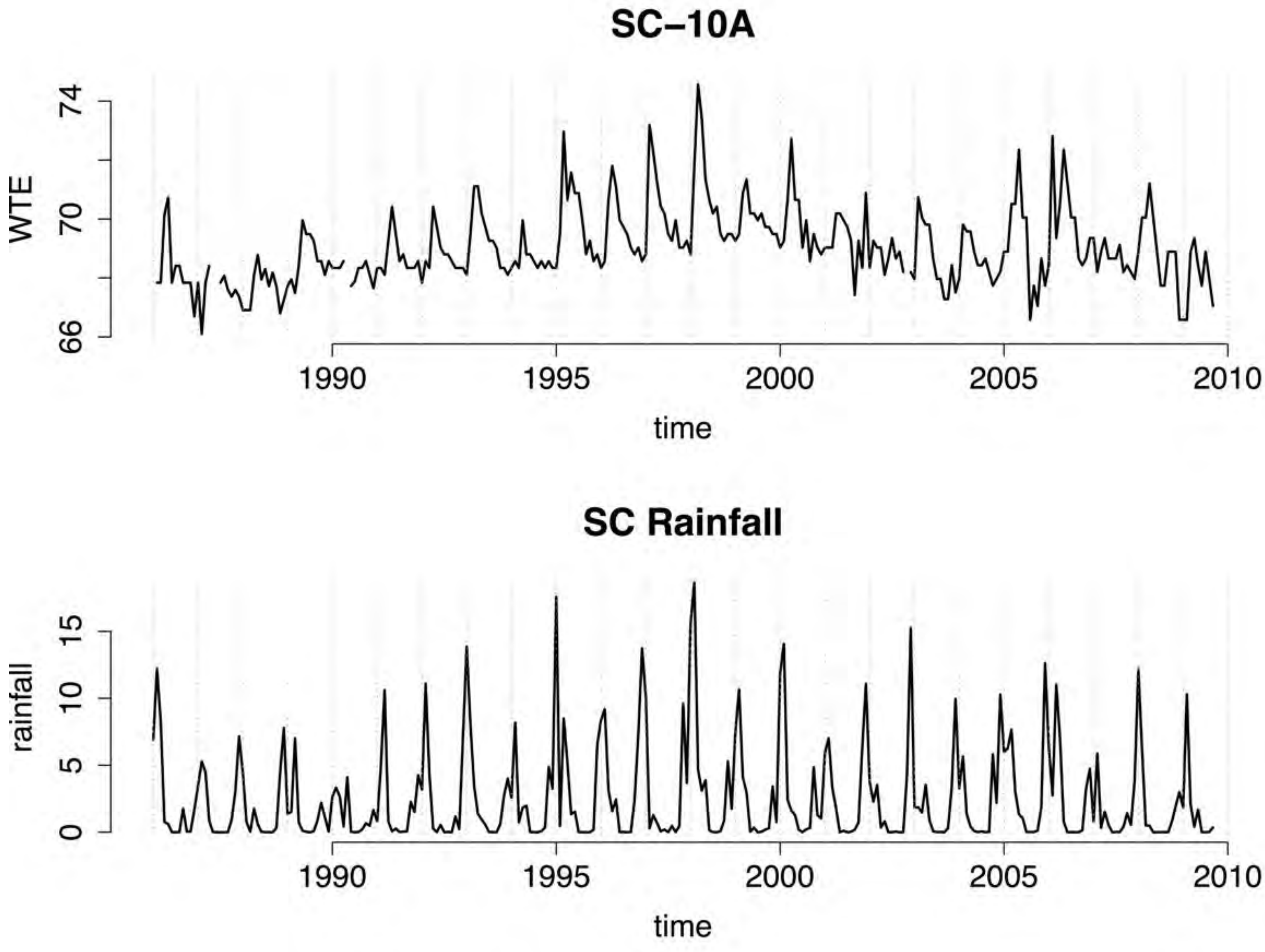


DLMs: Additional Examples

WTE: Selected Monitoring Wells

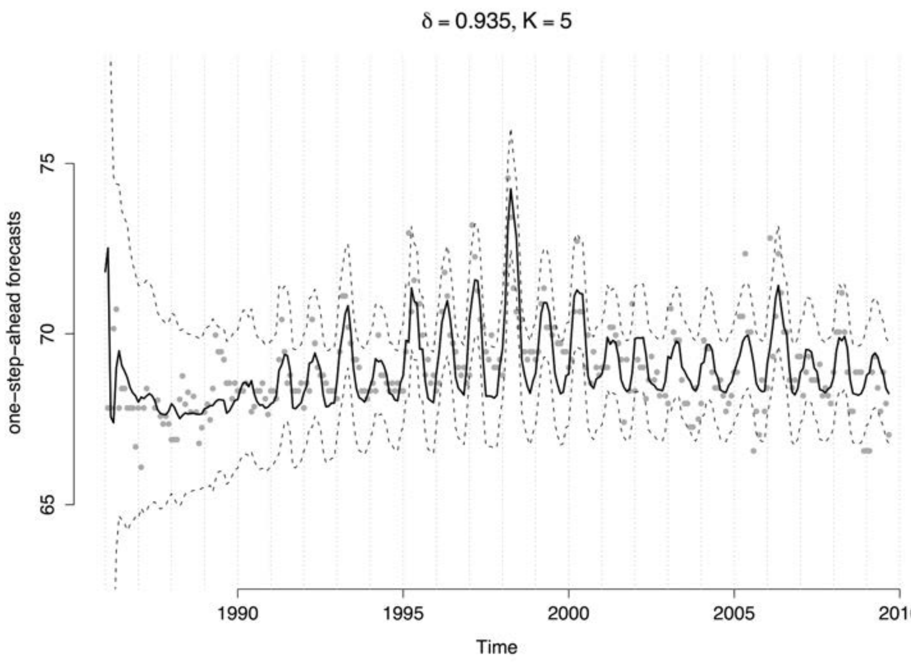


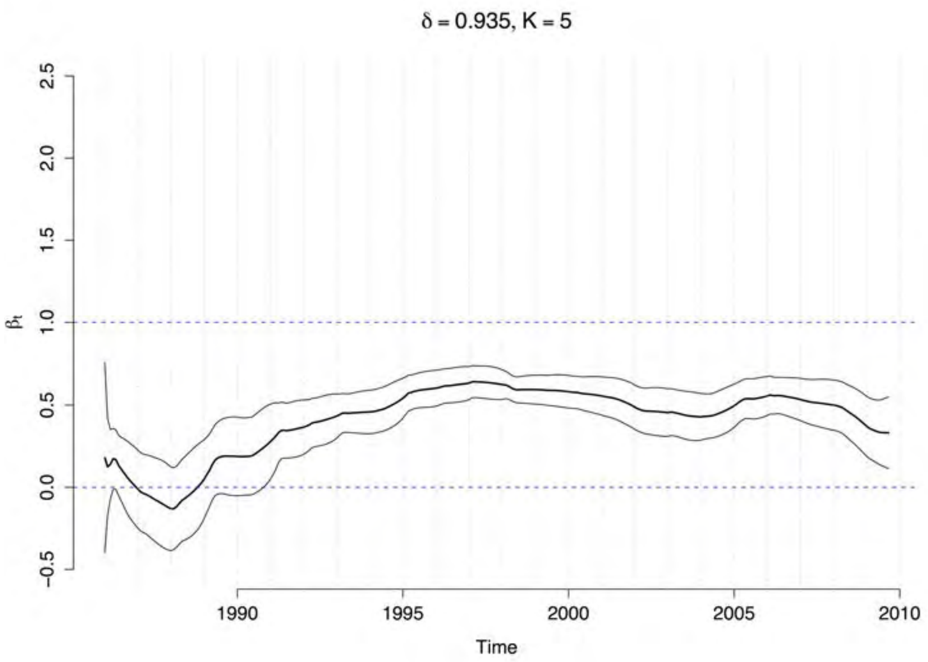
WTE Data: SC-10



WTE & rainfall

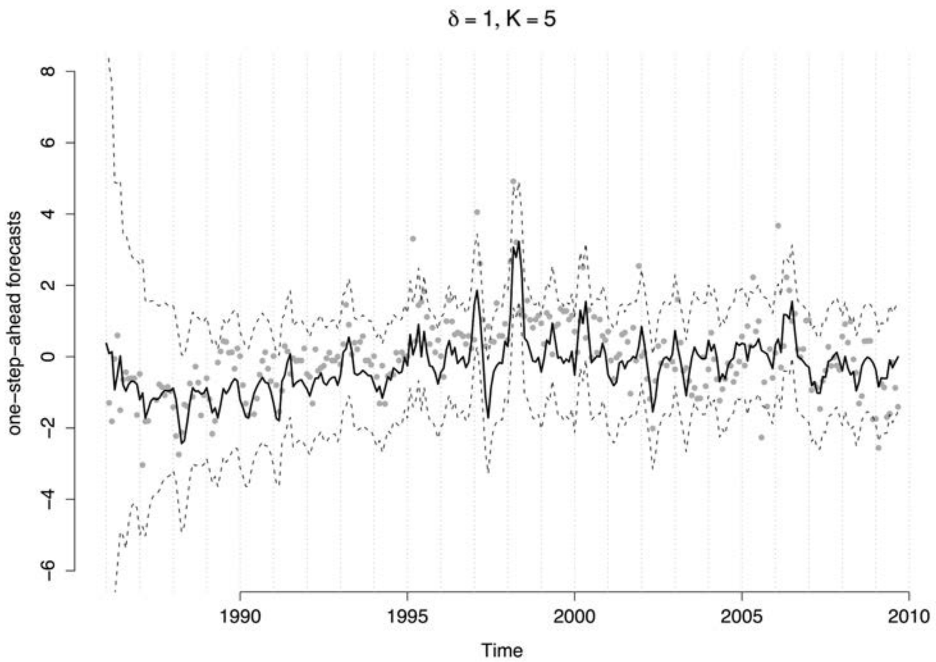
$$
\begin{aligned}
y_t &= \alpha + \beta_t \times f(\mathrm{rainfall}_t) + \epsilon_t, \quad \epsilon_t \sim N(0, v) \\
\beta_t &= \beta_{t-1} + w_t(\delta), \quad \delta \in (0, 1] \\
f(\mathrm{rainfall}_t) &= \frac{\sum_{j=0}^{5} \mathrm{rainfall}_{t-j}}{6}.
\end{aligned}
$$

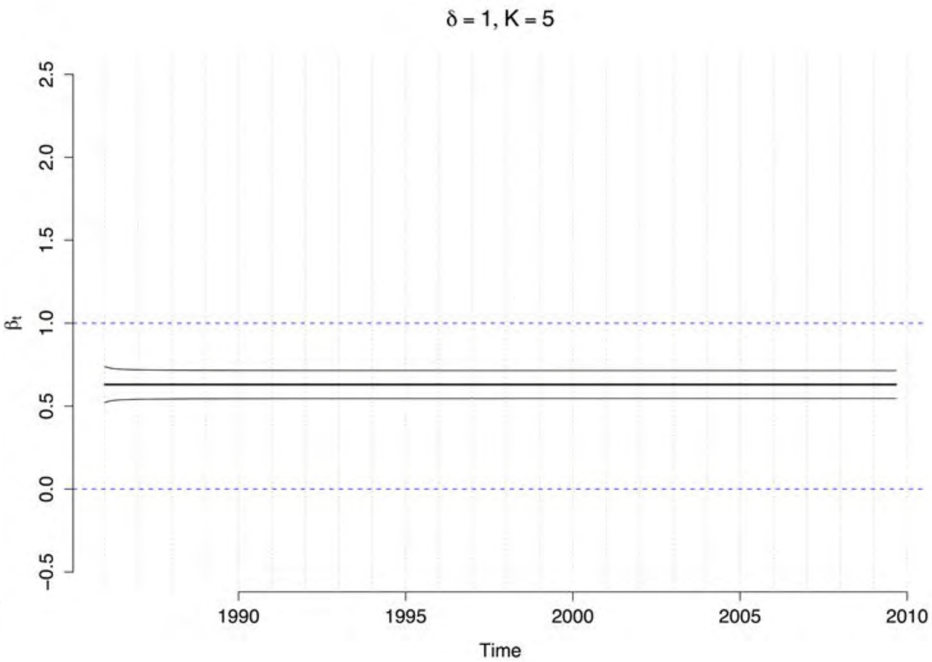




WTE & rainfall: anomalies

$$\begin{aligned} y_t^* &= \alpha^* + \beta_t^* \times f^*(\text{rainfall anomalies}_t) + \epsilon_t^*, \quad \epsilon_t^* \sim N(0, v) \\ \beta_t^* &= \beta_{t-1}^* + w_t^*(\delta^*), \quad \delta^* \in (0, 1]. \end{aligned}$$





Autoregression of order p

$$y_t = \sum_{j=1}^{p} \phi_j y_{t-j} + \epsilon_t, \quad \epsilon_t \sim N(0, v)$$

- Full likelihood:

$$p(y_{1:T}|\boldsymbol{\phi}, v) = p(y_{1:p}|\boldsymbol{\phi}, v) \times \prod_{t=(p+1)}^{T} p(y_t|y_{t-1}, \ldots, y_{t-p}, \boldsymbol{\phi}, v)$$

- Conditional likelihood:

$$p(y_{(p+1):T}|y_{1:p}\boldsymbol{\phi}, v) \quad = \quad \prod_{t=(p+1)}^{T} p(y_t|y_{t-1}, \ldots, y_{t-p}, \boldsymbol{\phi}, v)$$

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$$N(\mathbf{y}|\mathbf{F}'\boldsymbol{\phi}, v\mathbf{I}),$$

$$\mathbf{y} = (y_T, \ldots, y_{p+1})', \quad \boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)'$$

$$\mathbf{F} = [\mathbf{f}_T, \ldots, \mathbf{f}_{p+1}] \quad \mathbf{f}_t = (y_{t-1}, \ldots, y_{t-p})'$$

- **AR characteristic polynomial:**

$$\Phi(u) = 1 - \phi_1 u - \ldots - \phi_p u^p = \prod_{i=1}^{p} (1 - \alpha_i u),$$

$\alpha_1, \ldots, \alpha_p$ are the reciprocal roots of this polynomial.

$|\alpha_i| < 1$ for all i $\Rightarrow$ the process is <u>stationary</u>

DLMs: AR models

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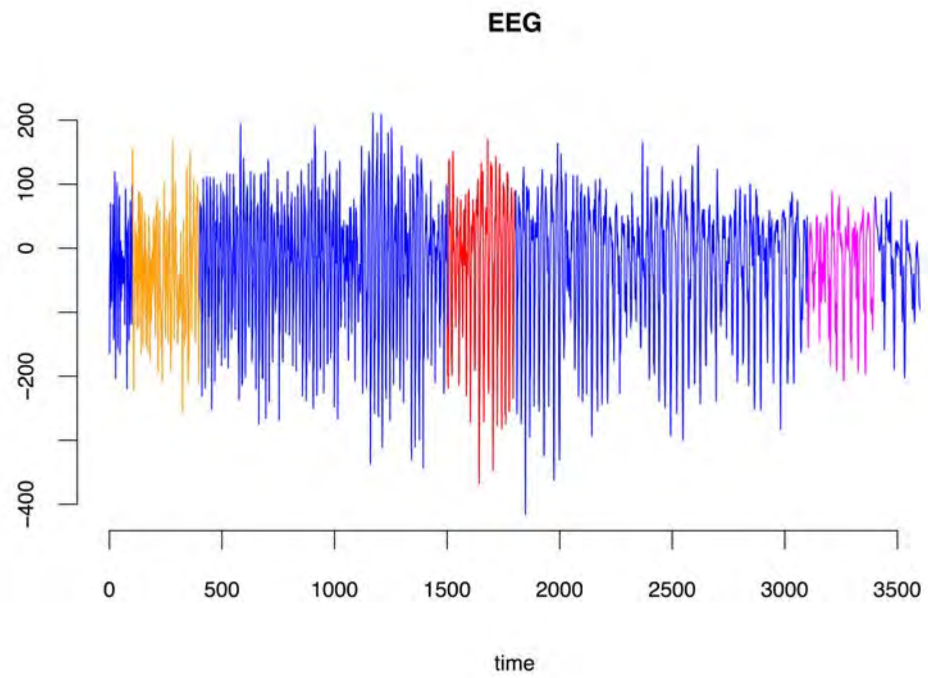

State-space (DLM) representations

- $\{\mathbf{F}_t, \mathbf{I}_p, v, \mathbf{0}\}, \quad \mathbf{F}'_t = (y_{t-1}, \ldots, y_{t-p})$

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- $\{\mathbf{E}_p, \mathbf{G}(\boldsymbol{\phi}), 0, \mathbf{W}\}, \quad \mathbf{E}_p = (1, 0, \ldots, 0)$

$$\boldsymbol{\theta}_t = (y_t, y_{t-1}, \ldots, y_{t-p+1})', \quad \mathbf{w}_t = (\epsilon_t, 0, \ldots, 0)',$$

$$\mathbf{G}(\boldsymbol{\phi}) = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

and $\mathbf{W} = \mathrm{diag}(v, 0, \ldots, 0)$

- The eigenvalues of $\mathbf{G}$ are the reciprocal roots of the AR characteristic polynomial, $\alpha_1, \ldots, \alpha_p$

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$$f_t(h) = E(y_{t+h}|y_{1:t}) = \mathbf{F}'\mathbf{G}^h\mathbf{x}_t = \sum_{j=1}^{p} c_{t,j}\alpha_j^h,$$

$c_{t,j} = d_j e_{t,j}, \quad d_j, \, e_{t,j}$ elements of

$\mathbf{d} = \mathbf{E}'\mathbf{F}, \quad \mathbf{e}_t = \mathbf{E}^{-1}\boldsymbol{\theta}_t$ where $\mathbf{E}$ is an eigenmatrix of $\mathbf{G}$

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This induces a decomposition of the observed time series y_t into latent components associated with the reciprocal roots…

DLM decompositions

$$y_t = x_t + \nu_t, \quad x_t = \mathbf{F}'\boldsymbol{\theta}_t, \quad \boldsymbol{\theta}_t = \mathbf{G}_t\boldsymbol{\theta}_{t-1} + \mathbf{w}_t$$

Assume:

- $\mathbf{G}_t$ p × p with p different eigenvalues: $\alpha_{t,1}, \ldots, \alpha_{t,p}$
  - **C** pairs of complex eigenvalues:

    $$r_{t,j}\exp(\pm i\omega_{t,j}), \quad j = 1 : C$$

  - **R=p-2C** real eigenvalues: $r_{t,j}, \quad j = (C+1) : (C+R)$

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$$\mathbf{G}_t = \mathbf{B}_t\mathbf{A}_t\mathbf{B}_t^{-1}$$

matrix of eigenvectors

diagonal matrix of eigenvalues

Decomposition: In the AR case we have

$$\mathbf{G} = \mathbf{B}\mathbf{A}\mathbf{B}^{-1}$$

with $\mathbf{A} = \mathrm{diag}(\alpha_1, \ldots, \alpha_p)$ which leads to:

$$y_t = \sum\nolimits_{j=1}^{C} x_{t,j} + \sum\nolimits_{j=1}^{R} z_{t,j},$$

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$$y_t = \sum_{j=1}^{C} x_{t,j} + \sum_{j=1}^{R} z_{t,j},$$

- $x_{t,j} \sim \mathrm{ARMA}(2,1)$ with <u>constant</u> moduli r_j and frequency ω_j for j=1, …, **C**
- $z_{t,j} \sim \mathrm{AR}(1)$ with <u>constant</u> moduli r_j for j=C+1,…, C+R

Example: AR(1)

$$y_t = \phi y_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, v)$$

- *Characteristic polynomial:* $\Phi(u) = 1 - \phi u$ if $|\phi| < 1$ the process is stationary.
- *Forecast function:* $f_t(h) = \phi^h y_t$
- *Autocorrelation function (ACF):* $\rho(h) = \mathrm{Corr}(y_t, y_{t-h}) = \phi^h, \quad h \geq 0.$

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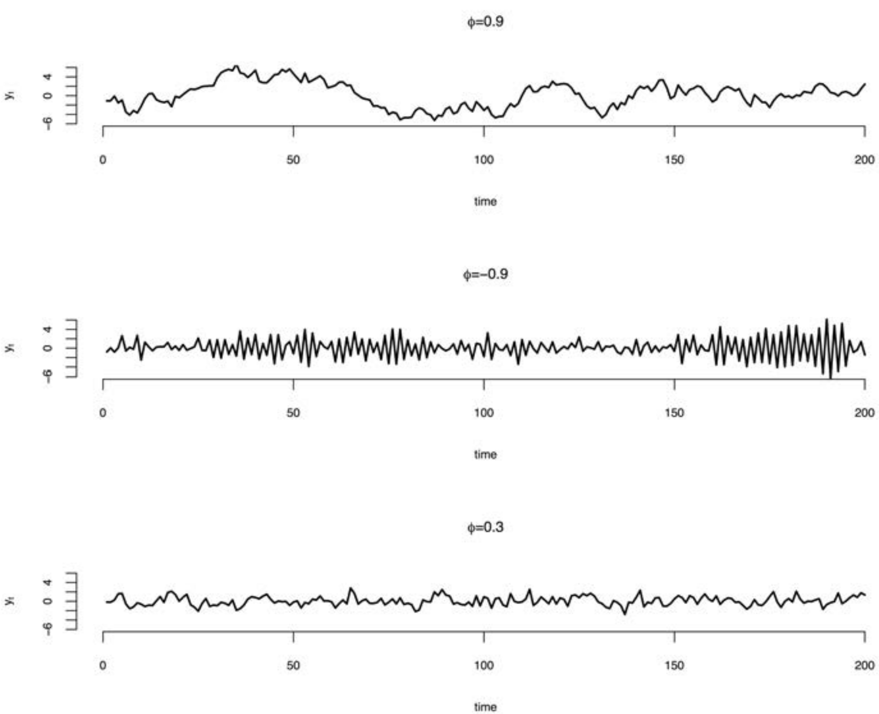

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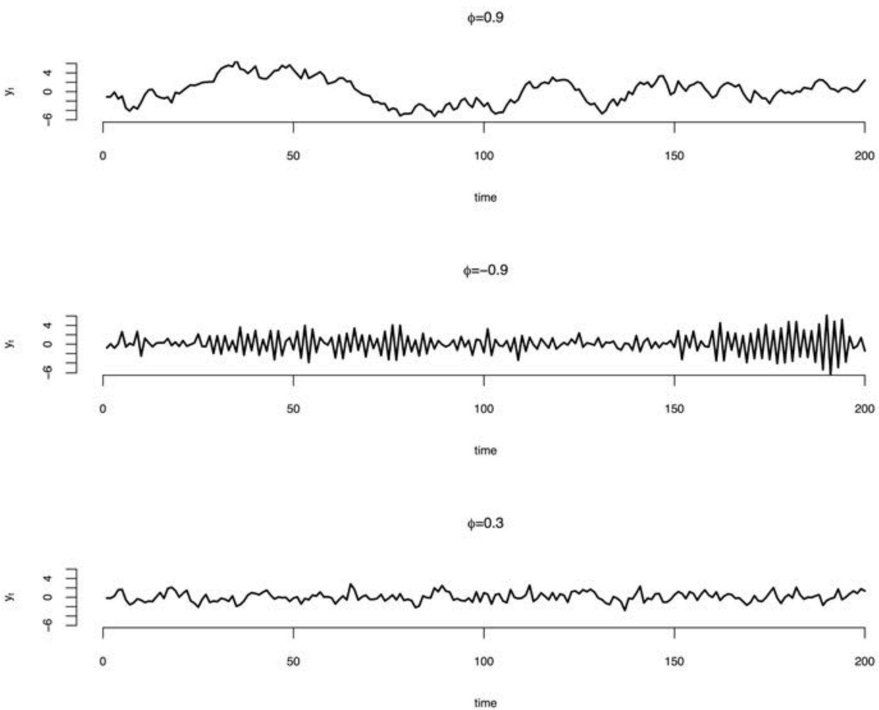
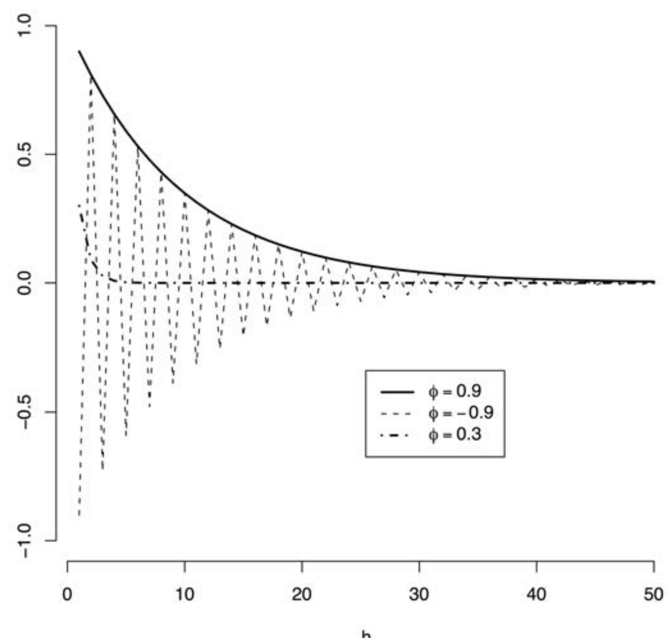
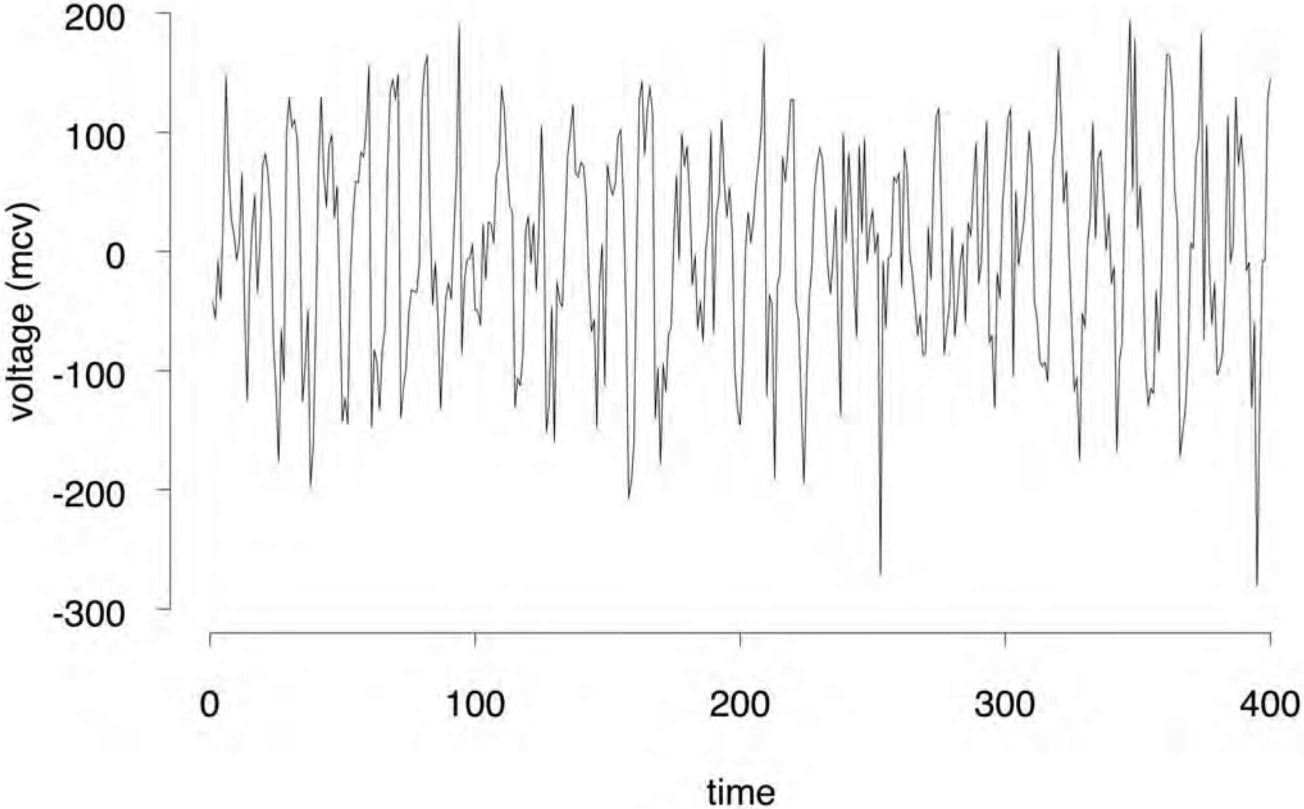




Bayesian inference in AR models

$$y_t = \sum_{i=1}^{8} \phi_i y_{t-i} + \epsilon_t, \quad \epsilon_t \sim N(0, v)$$

$$p(\boldsymbol{\phi}, v) \propto 1/v$$



$E(\hat{\phi} | y_{9:400})$: (0.27,0.07,-0.13,-0.15,-0.11,-0.15,-0.23,-0.14)'

AR models: EEG data

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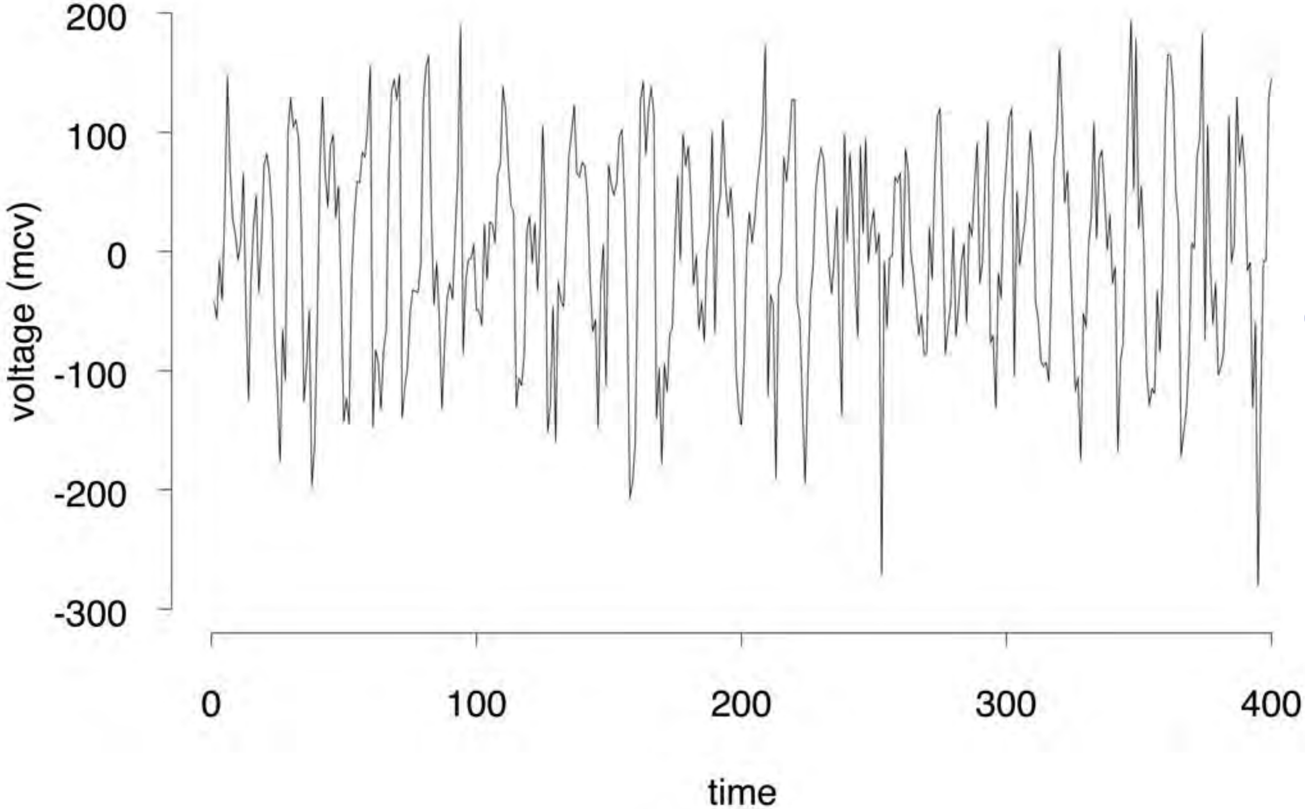


reciprocal roots, moduli and periods:

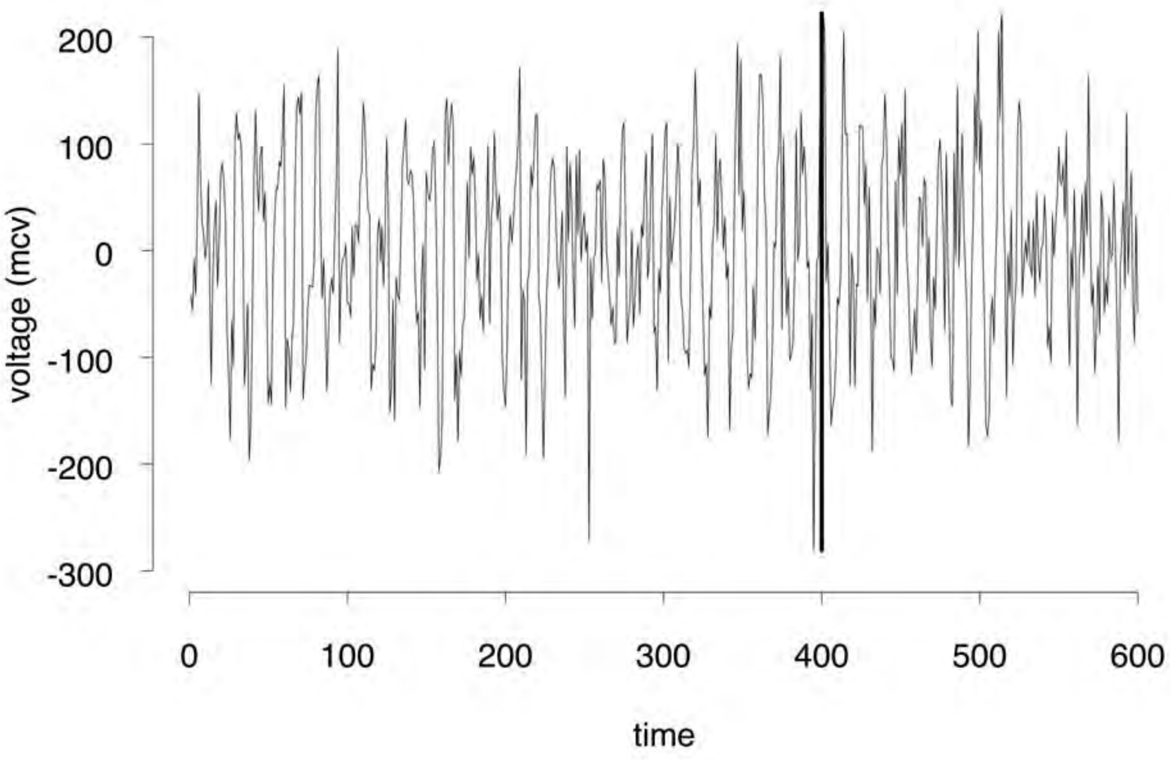
(097,12.7); (0.81,5.1)

(0.72,3.0); (0.63,2.2)

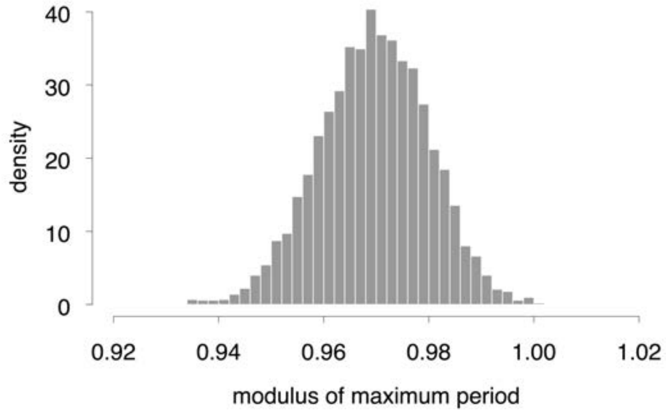
$E(\hat{\phi}|y_{9:400})$: (0.27,0.07,-0.13,-0.15,-0.11,-0.15,-0.23,-0.14)’

AR models: EEG data

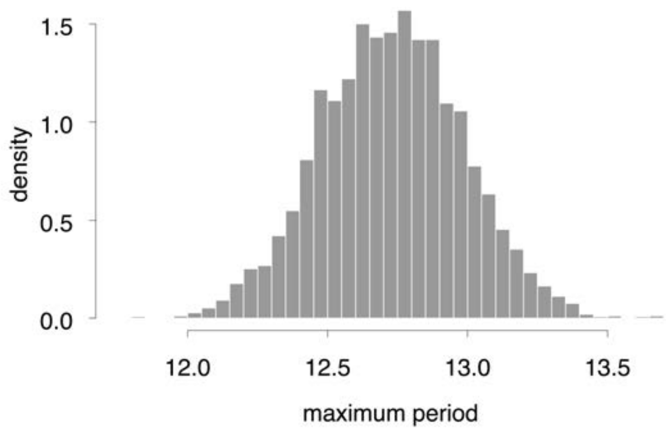
Predictive Distribution



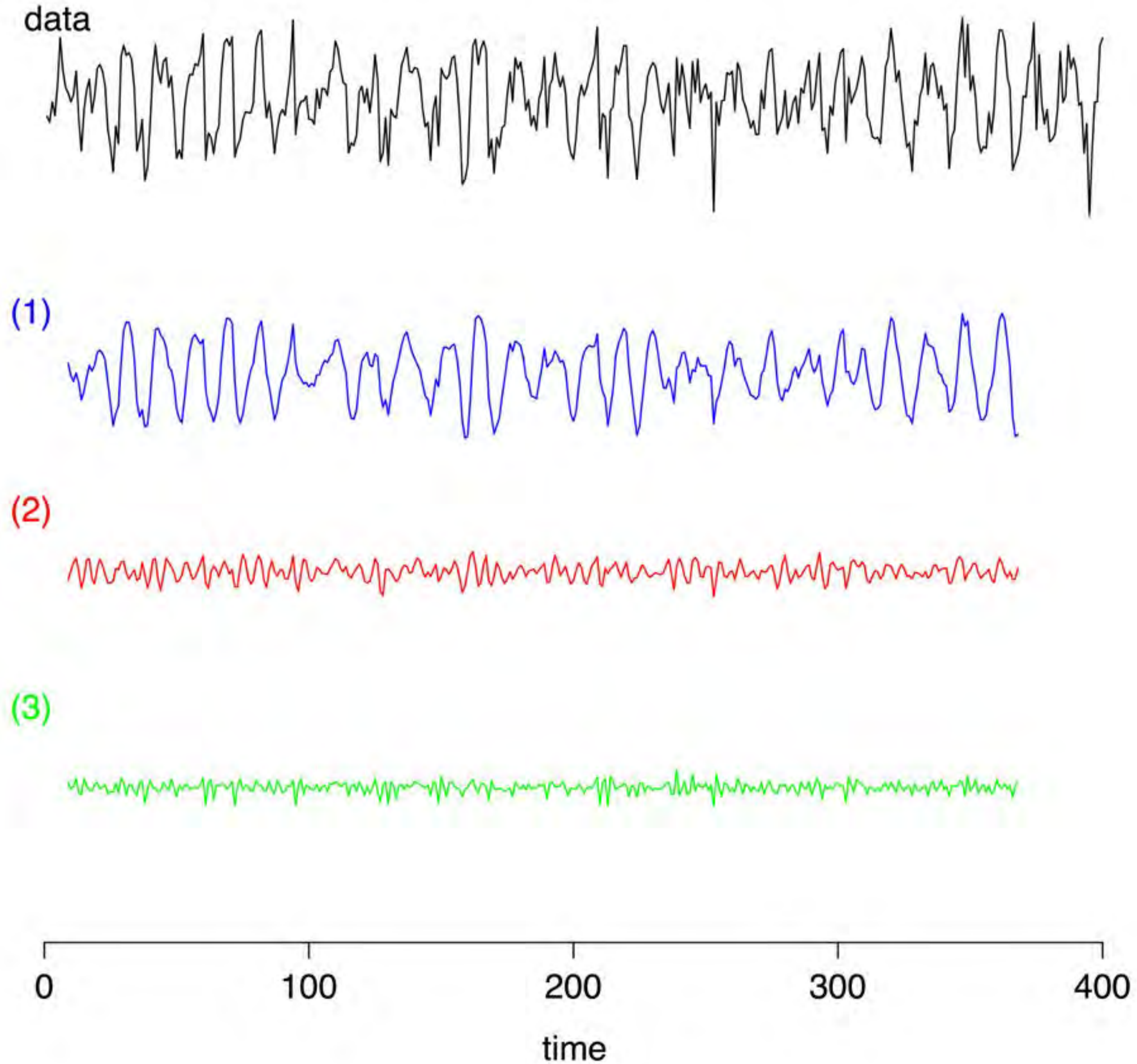
$p(r_{max}|y_{9:400})$



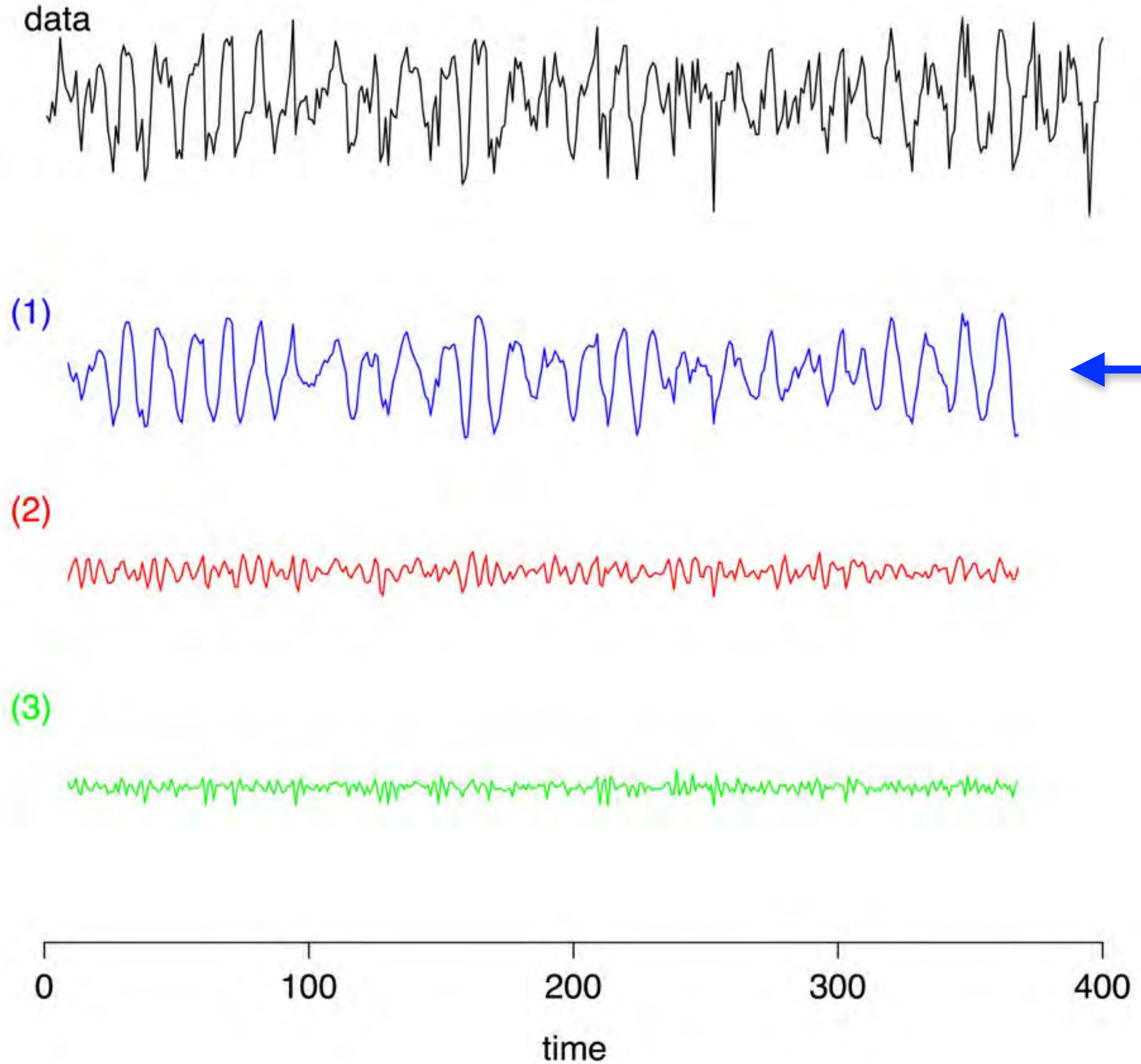
$p(\lambda_{max}|y_{9:400})$



AR models: EEG data

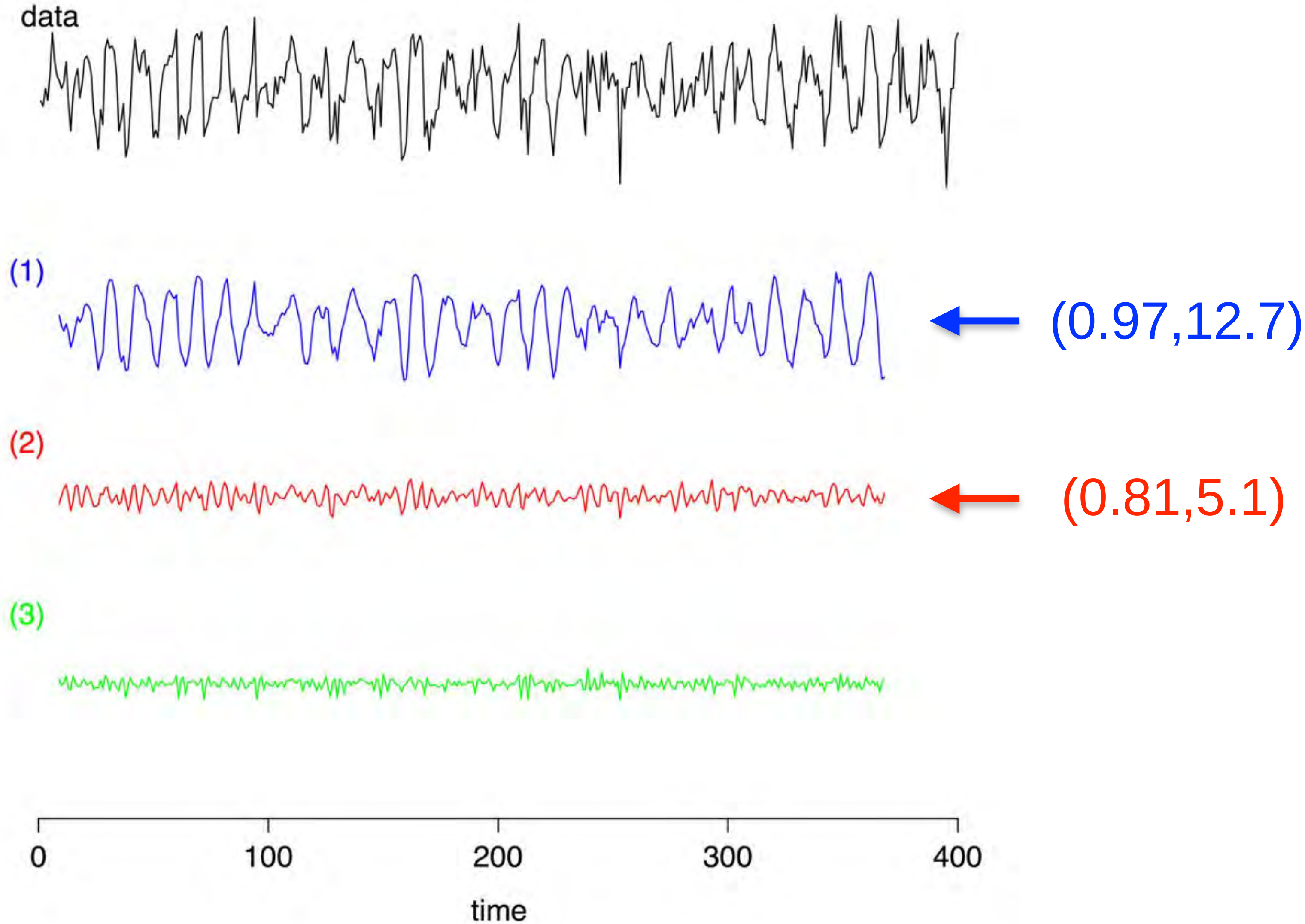


AR models: EEG data

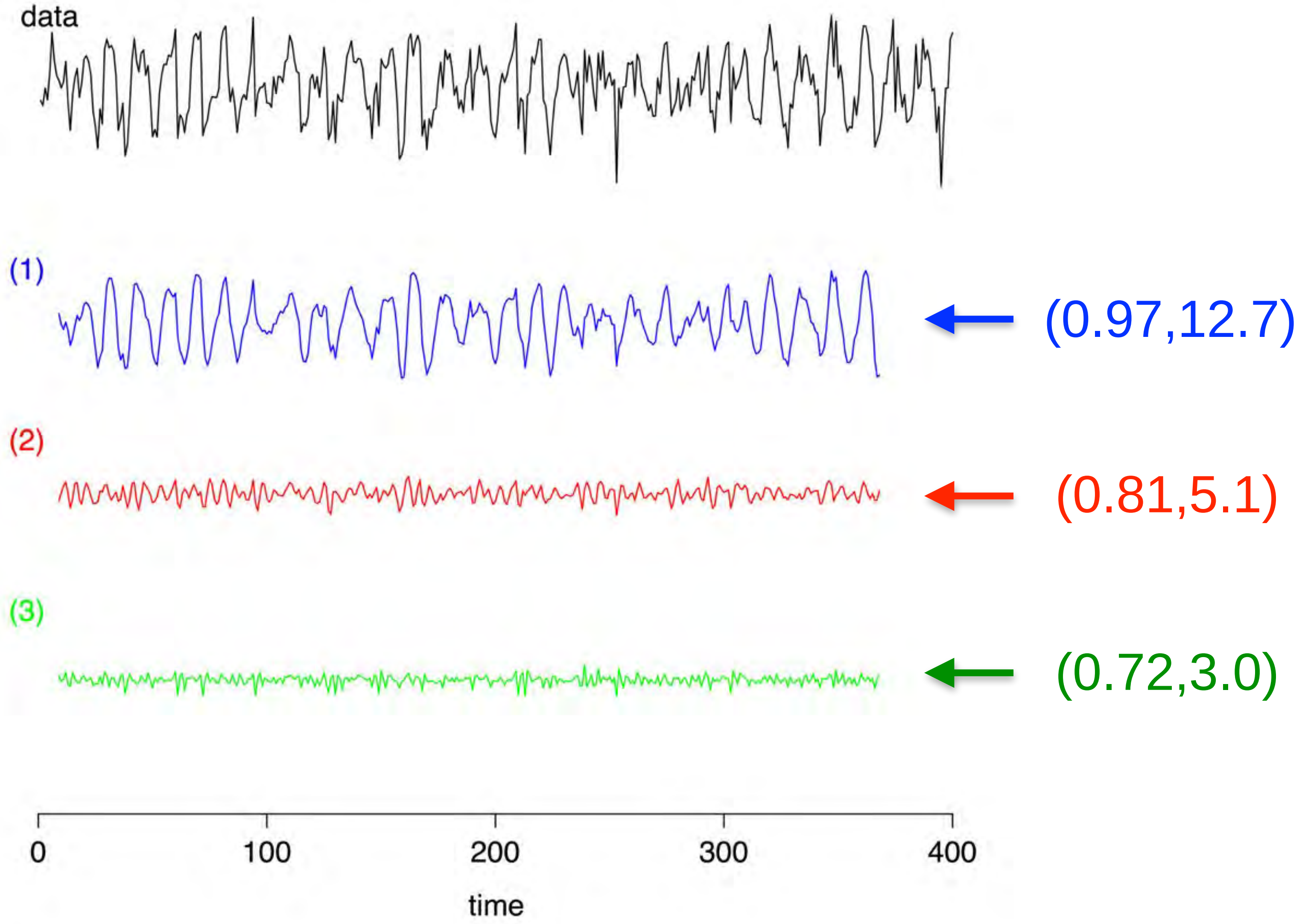


(0.97,12.7)

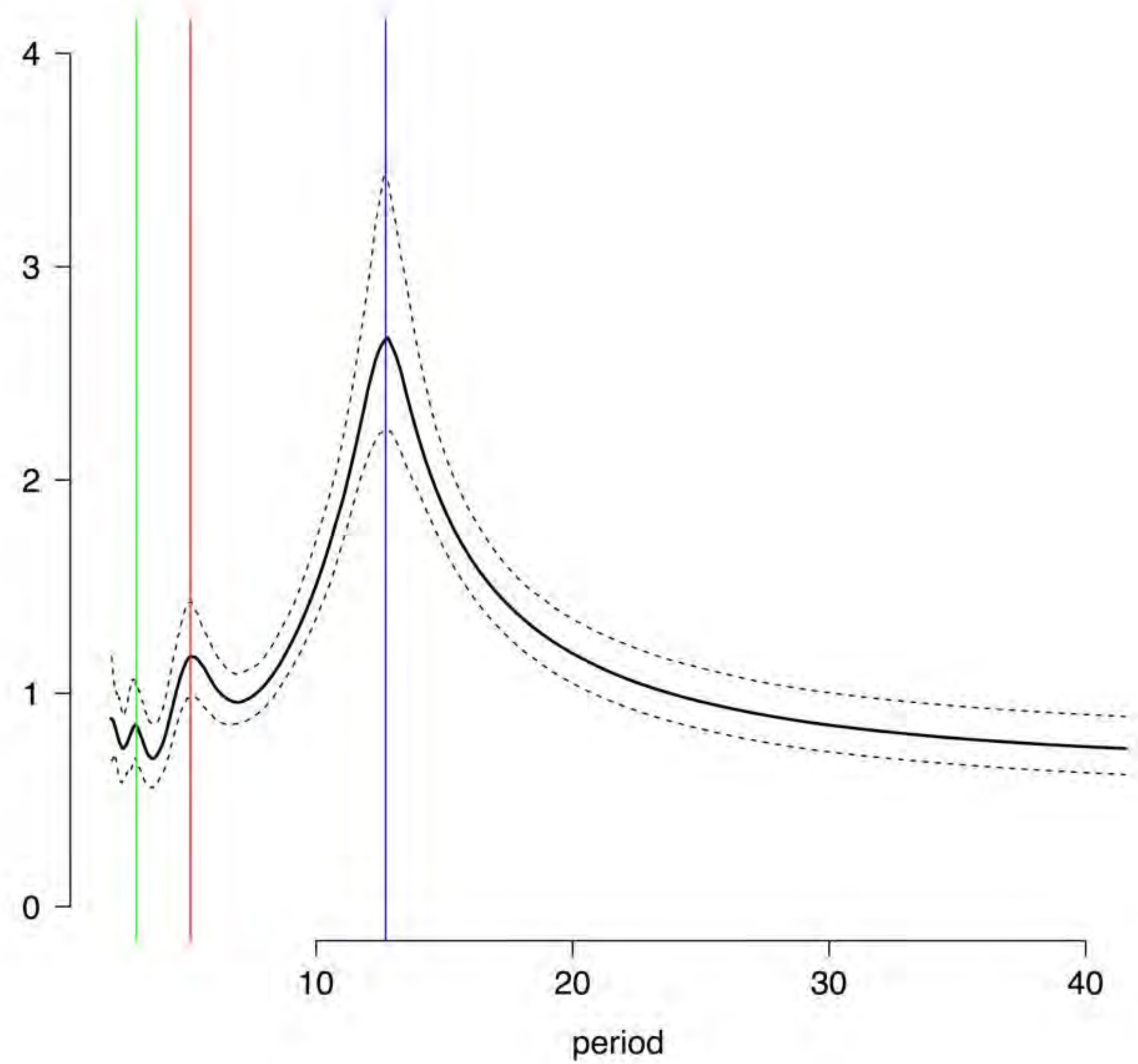
AR models: EEG data



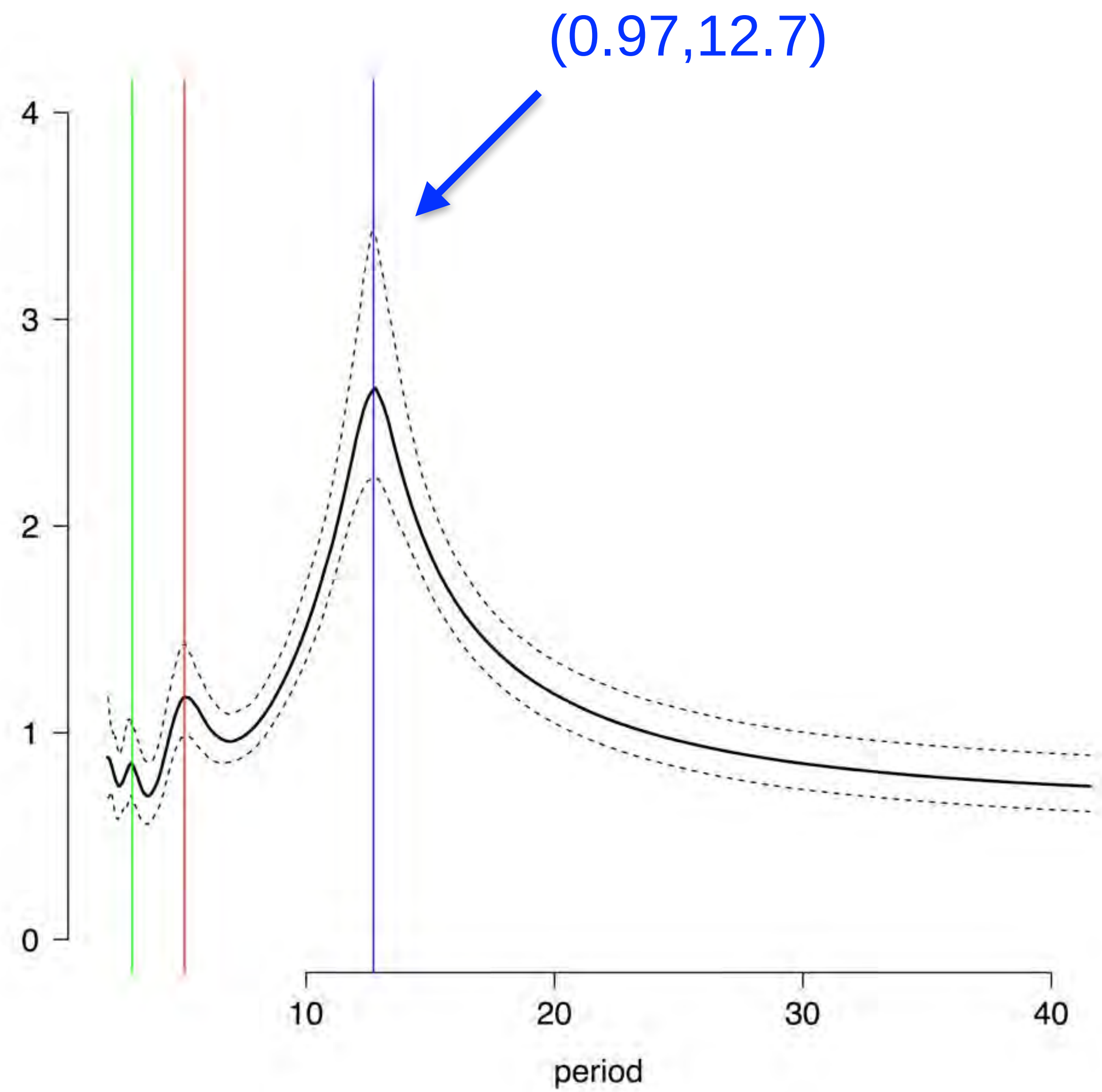
AR models: EEG data



AR models: EEG data



AR models: EEG data



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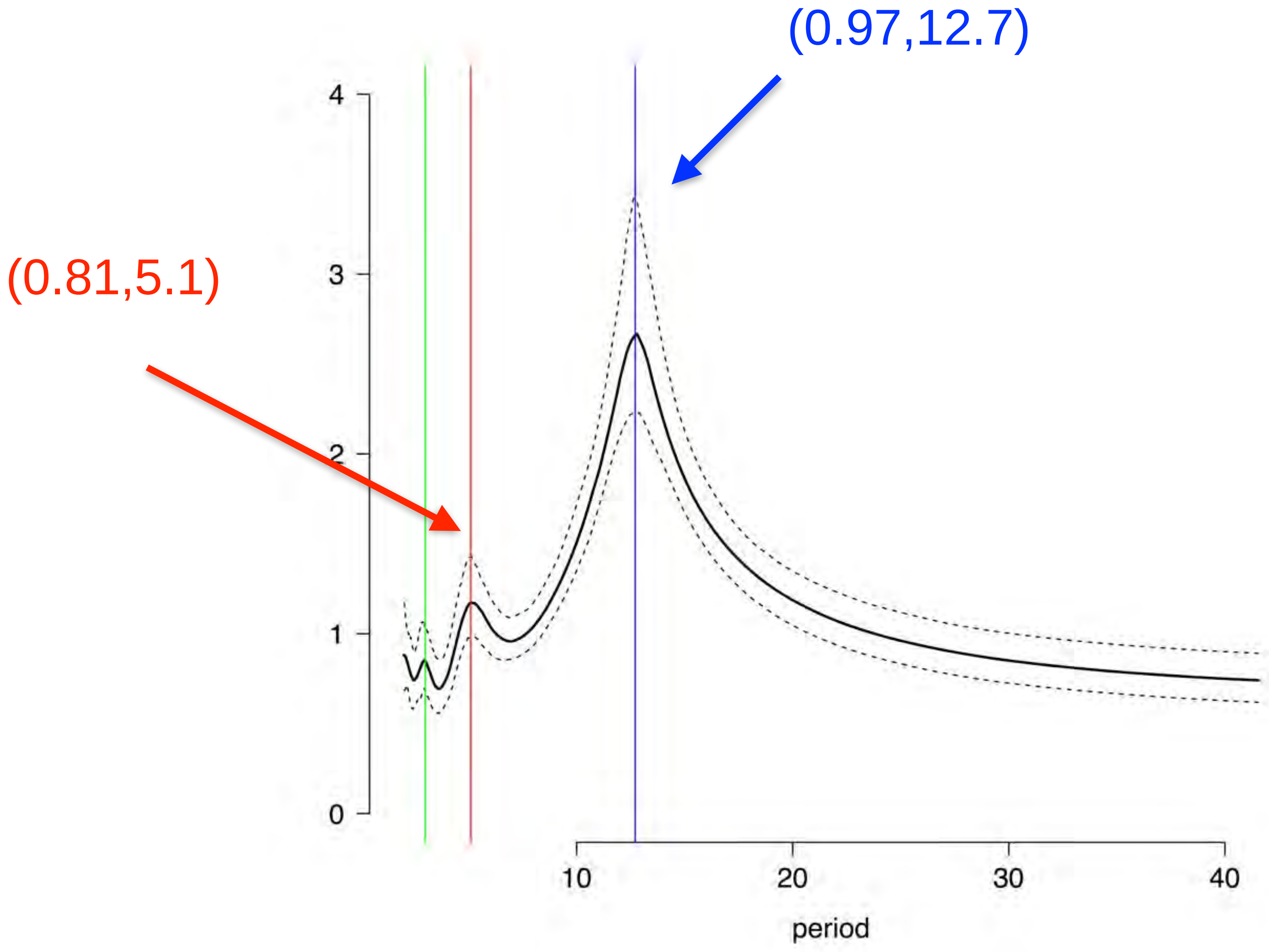

AR models: EEG data