Date: 8/13/10

To: SG2

From: Noel Cressie

Re: Grain-germ (growth-attrition) model (2 pages)

Consider $\mathbf{s} \in D$, a spatial domain of interest (e.g., North America) and $t = t_0, t_0 + 1, \ldots,$ where t_0 is the start time of the process (e.g., $t_0 = 8402$, the first of a sequence of 3-hourly intervals in 1970). Let $P_t \subset D$ denote the union of all pixels above the thresholds, $\{k(\mathbf{s})\colon \mathbf{s} \in D\}$. That is, if $Y_t(\mathbf{s})$ is the rainfall in pixel $\mathbf{s}$ at time t, then

$$P_t \equiv \{\mathbf{s}\colon \mathbf{s} \in D \text{ and } I(Y_t(\mathbf{s}) > k(\mathbf{s})) = 1\}\,,$$

where $k(\cdot)$ is given and $I(\cdot)$ is the indicator function. We would initially like to model

$$\{P_t\colon t = t_0, t_0 + 1, \ldots\}\,,$$

but later we could also model rainfall *amounts* in these exceedance regions.

Let $\{\mathbf{s}_{ti}\colon i = 1, 2, \ldots, n_t\}$ denote the individual pixels in P_t. That is, $\mathbf{s}_{ti} \in P_t$ iff

$$I(Y_t(\mathbf{s}_{ti}) > k(\mathbf{s}_{ti})) = 1\,; \qquad i = 1, \ldots, n_t\,.$$

Let $\{\mathbf{v}_t(\mathbf{s})\colon \mathbf{s} \in D\}$ denote the velocity vectors (in km/hr) given by NARCCAP at time t. Then

$$\mathbf{u}_t(\mathbf{s}) \equiv 3\mathbf{v}_t(\mathbf{s})$$

is a displacement vector. Define the set,

$$L_t(\mathbf{s}) = \{c\mathbf{u}_t(\mathbf{s}) \oplus \mathbf{s}\colon\ -1 \le c \le 1\}$$

Now use “Bernoulli thinning” on

$$\cup\{L_t(\mathbf{s}_{ti})\colon i = 1, \ldots, n_t\}\,.$$

After thinning, denote the resulting set as Q_{t+1}; that is, each pixel is or is not in Q_{t+1} according to iid Bernoulli random variables with probability p_{t+1}; $t = t_0, t_0 + 1, \ldots.$ Denote

$$Q_{t+1} \equiv \{\mathbf{s}'_{t+1,j}\colon j = 1, \ldots, |Q_{t+1}|\}\,,$$

where $\{\mathbf{s}'_{t+1,j}\}$ are the “germs” (foci of growth). The “grains,” $\{S_{t+1,j}\}$, are iid random sets (here, they are possibly small ellipsoids with major axis in the direction of the rainfall front). That is,

$$\{S_{t+1,j}\colon j = 1, \ldots, |Q_{t+1}|\} \text{ are iid as } S_{t+1}\,.$$

Finally,

$$P_{t+1} \equiv \cup\{S_{t+1,j} \oplus \mathbf{s}'_{t+1,j}\colon j = 1, \ldots, |Q_{t+1}|\}\,,$$

which we can write as $P_{t+1} \equiv \{\mathbf{s}_{t+1,i} \colon i = 1, \ldots, n_{t+1}\}$. (Define $A \oplus B \equiv \{\mathbf{a} + \mathbf{b} \colon \mathbf{a} \in A,\ \mathbf{b} \in B\}$.) Hence, attrition can occur if p_t is small and $|S_{t+1}|$ is small. Growth can occur if $|S_{t+1}|$ is large. Notice that the displacement field is used to *expand* the set of possible pixels, and then the thinning field is used to *randomly delete* some of these pixels. This describes a type of "birth and death" process.

Notes

1. Data are the sequence, $P_{t_0}, P_{t_0+1}, \ldots$.
2. Given the dynamical nature of the model, we can estimate p_t, $E(|S_{t+1}|)$, and $\mathrm{var}(|S_{t+1}|)$ from the pair (P_t, P_{t+1}); $t = t_0, t_0 + 1, \ldots$.
3. A reference for this type of model in a cancer-cell-growth context is:
   Cressie, N. and Hulting, F.L. (1992). A spatial statistical analysis of tumor growth. *Journal of the American Statistical Association*, **87**, 272-283.