ОМП θ:

$$L_x(\theta;\beta)=\prod_{i=1}^{n}\frac{\theta^{\beta}\cdot x_i^{\beta-1}}{\Gamma(\beta)}\cdot e^{-\theta x_i}=\left(\frac{\theta^{\beta}}{\Gamma(\beta)}\right)^{n}\cdot(x_1\cdots x_n)^{\beta-1}\cdot e^{-\theta\sum_i x_i}$$

$$l_x(\theta;\beta)=n\beta\ln\theta-n\ln\Gamma(\beta)+(\beta-1)\sum_{i=1}^{n}\ln x_i-\theta\sum_{i=1}^{n}x_i$$

$$\frac{\partial l_x(\theta;\beta)}{\partial\theta}=\frac{n\beta}{\theta}-\sum_{i=1}^{n}x_i\overset{\text{ОМП}}{=\!=}0$$

$$\frac{\beta}{\theta}=\overline{X}\qquad\boxed{\hat{\theta}=\frac{\beta}{\overline{X}}}$$

Асимптотическая дисперсия:

$$P_\theta(x)=\frac{\theta^{\beta}\cdot x^{\beta-1}}{\Gamma(\beta)}\cdot e^{-\theta x}$$

$$l_{x_1}\theta=\log p_\theta(x)=\beta\ln\theta+(\beta-1)\cdot\ln x-\theta x-\ln\Gamma(\beta)$$

$$\frac{\partial l_{x_1}}{\partial\theta}=\frac{\beta}{\theta}-x=\frac{\beta-\theta x}{\theta}$$

$$E_\theta\left(\frac{\partial l_x}{\partial\theta}\right)^2=E_\theta\left(\frac{\beta-\theta x}{\theta}\right)^2=\int_0^{\infty}\frac{(\beta-\theta x)^2}{\theta^2}\cdot\frac{\theta^{\beta}\cdot x^{\beta-1}}{\Gamma(\beta)}e^{-\theta x}dx=$$

$$=\int_0^{\infty}\frac{[\beta^2-2\theta x\beta+(\theta x)^2]\,\theta^{\beta-2}\cdot x^{\beta-1}}{\Gamma(\beta)}e^{-\theta x}dx=$$

$$=\underbrace{\beta^2\int_0^{\infty}\frac{\theta^{\beta-2}\cdot x^{\beta-1}}{\Gamma(\beta)}\cdot e^{-\theta x}dx}_{1)}-\underbrace{2\beta\int_0^{\infty}\frac{\theta^{\beta-1}x^{\beta}}{\Gamma(\beta)}e^{-\theta x}dx}_{2)}+\underbrace{\int_0^{\infty}\frac{\theta^{\beta}\cdot x^{\beta+1}}{\Gamma(\beta)}e^{-\theta x}dx}_{3)}=$$

$$\Gamma(p) = \int_0^\infty x^{p-1} \cdot e^{-x}\,dx \qquad \Gamma(p) = (p-1)!$$

$$\int_0^\infty \theta^{\beta-2} x^{\beta-1} e^{-\theta x}\,dx \overset{t=\theta x}{=} \frac{1}{\theta^2}\int_0^\infty t^{\beta-1} e^{-t}\,dt = \frac{1}{\theta^2}\cdot\Gamma(\beta) \Longrightarrow 1) = \frac{\beta^2}{\theta^2}$$

$$\int_0^\infty \theta^{\beta-1} \cdot x^{\beta} e^{-\theta x}\,dx = \frac{1}{\theta^2}\int_0^\infty (x\theta)^{\beta} \cdot e^{-\theta x}\,d(\theta x) = \frac{1}{\theta^2}\cdot\Gamma(\beta+1)$$

$$2) = -\frac{2\beta\cdot\Gamma(\beta+1)}{\theta^2\Gamma(\beta)} = -\frac{2\beta}{\theta^2}\cdot\frac{\beta!}{(\beta-1)!} = -\frac{2\beta^2}{\theta^2}$$

$$\int_0^\infty \theta^{\beta} \cdot x^{\beta+1} e^{-\theta x}\,dx = \frac{1}{\theta^2}\cdot\int_0^\infty t^{\beta+1} e^{-t}\,dt = \frac{1}{\theta^2}\cdot\Gamma(\beta+2)$$

$$3) = \frac{1}{\theta^2}\cdot\frac{\Gamma(\beta+2)}{\Gamma(\beta)} = \frac{(\beta+1)!}{(\beta-1)!}\cdot\frac{1}{\theta^2} = \frac{\beta\cdot(\beta+1)}{\theta^2}$$

$$= \frac{\beta^2}{\theta^2} - \frac{2\beta^2}{\theta^2} + \frac{(\beta+1)\cdot\beta}{\theta^2} = \left(\frac{\cancel{-\beta^2}+\cancel{\beta^2}+\beta}{\theta^2}\right) = \frac{\beta}{\theta^2}$$

$$\sigma^2(\theta) = i^{-1}(\theta)$$

$$i(\theta) = E_\theta\left(\frac{\partial l_x(\theta)}{\partial\theta}\right)^2 \qquad \Longrightarrow \qquad \boxed{\sigma^2(\theta) = \frac{\theta^2}{\beta}}$$