Prepared for submission to JHEP

Complex Analysis for Formal Physicists. Motivated Survey & Summary

Gabe Suarez

Syracuse University,
New York, USA
UKY,
Kentucky, USA

E-mail: gmsuarez@syr.edu

Abstract: A rather brief, but mostly all-encompassing summary on the important results of complex analysis for a formal high energy physicist. Motivation, topics covered, and further suggestions for reading are provided at the end (or please email me if it's not, and you are looking for some readings and suggestions)

ArXiv ePrint: 1234.56789

Contents

---

56 1 Motivation of Complex Analysis for Physicists

Complex analysis may not sound like a topic would be useful to describing the real world. However, complex analysis is one of the most important areas of mathematics for a physicist to understand.

That being said, complex analysis is essentially the analysis of complex numbers, so it's unfair to pretend that formal complex analysis–what is taught in graduate math classes, is what physicists talk about when using complex analysis.

The first place a physics student encounters complex analysis is in the most general solution to the harmonic oscillator differential equation:

$$m\ddot{x} = kx$$

Where solutions go as

$$x(t) = Ae^{ikx} + Be^{-ikx}$$

The next place is often in electromagnetism where i pops up in small areas of the subject, where complex analysis allows us to represent certain vector and field configurations most easily.

Where complex analysis becomes a necessity is in quantum mechanics. The Schrodinger equation takes the form:

$$\hat{H}\Psi = \left(\frac{\hat{p}}{2m} + \hat{V}\right)\Psi$$

Where momentum is the generator of translations and has eigenvalues, in position space, as $\hat{p} = -i\hat{\partial}_x$. So immediately complex analysis has a huge place in quantum mechanics. I won't mention the rest, but upon opening a quantum mechanics textbook you'll see talks of wave scattering, amplitudes, etc., which all come from results of complex and functional analysis.

Now where complex analysis becomes most essential is in quantum field theory and beyond.

In quantum field theory, we have our field theory, defined by an action, like in classical mechanics:

$$S = \int d^D q \, \mathcal{L}$$

Classically, $\mathcal{L}$ is real. However, in quantum field theory we separate out the Lagrangian density into free and interacting terms. The interacting terms have i. An example is the pseudoscalar Yukawa coupling term: $-ig\bar{\psi}\gamma^5\psi\psi$. That's just the beginning. There are countless many places where complex analysis comes into play in quantum field theory. In that perturbative QFT picture we have: Wick rotations, residue calculations, branch cuts, dimensional regularization, analytic continuation, and much more.
Going beyond perturbative QFT we need more "formal" tools of complex analysis, specifically talking about holomorphic and anti-holomorphic structures. Sounds fancy, but that just means complex differentiable. Well, it definitely means more than that. Reason being any holomorphic function has a continuous first derivative on it's entire domain. That fact alone, by Cauchy's theorem, implies that any holomorphic function is infinitely differentiable on that same domain. Which means any holomorphic function, despite only having it's first derivative known about, is automatically smooth.

Perhaps also notable, though also perhaps obvious in some ways, complex analysis is absolutely essential without a doubt in most modern fields of formal HEP. This includes, but is not limited to, string theory, conformal field theories, AdS/CFT and general holography, modular (number theoretic and algebraic) theory, etc.
Let's start.

2 Preliminaries to Complex Analysis

There are many, in a literal sense, but a typical calculus sequence, with some knowledge on set theory & notation, and mathematical maturity will suffice. Even without set theory knowledge and mathematical maturity, one could read complex analysis if they determined. They'd just be referencing wikipedia & AI a lot back and forth.

2.1 Point Set Topology & Geometric Arguments

Complex analysis starts by reviewing what open/closed sets are, what limit points and compactness mean. In a quick sense, open/closed sets are literally regions of some space. Open sets Ω are regions that do not include the points on their boundary:

$$\Omega \subseteq \mathbb{C}$$

Closed sets are just open sets with their boundary included, well. At least for our purposes.
Limit points are, in our context, essentially points on the boundary of open set. That means all limit points of closed sets are contained within the closed set. Open sets, no. However, limit points can be arbitrarily approximated by the inside open set.

Compactness is of utmost importance. Compactness means closed and bounded. Bounded means of finite size in our sense.

2.2 Complex Plane $\mathbb{C}$

There is an isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$ by $z = x + iy$. To see this, define $\bar{z}$ is the complex conjugate, an operation that takes $i \to -i$. Consider the inner product on the space which is formed by $z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 \in \mathbb{R}^2$.
An important fact is that the 2-sphere, without North pole $S^2 - \{0\}$ can be stereographically projected onto $\mathbb{C}$. This is hard to imagine, but if you imagine this situation it will help. The 2-sphere sitting above the origin, with south pole at the origin of the plane $\mathbb{R}^2$. Consider a ray which connects a given point on the 2-sphere to the north pole. That ray will intersect $\mathbb{R}^2$ in one point. This forms a bijection between the two, and thus an isomorphism since there is no structural change in such a mapping. We could also include the north pole by including the infinity points in $\mathbb{C}$. As Stein says, this forms an important compactification of the unbounded $\mathbb{C}$ into the compact 2-sphere with it's north pole - the Riemann sphere. This is pretty important to consider just because it introduces such geometric arguments.

2.3 Other Intro Material

I'd suggest Stein if you are curious enough about the other introductory material. It talks about mother things like basic complex arguments such as holomorphicity, etc. I won't talk about that. I'll just write down the main key results. Unfortunately that means

These notes are not a substitute for a rigorous analysis course, but are meant to give a motivated physicist an intuitive grasp of the key structures and where they show up in high-energy theory (You won't regret taking half an hour or so to look at your favorite topic in the T.O.C)

3 Cauchy Theory and Consequences of Being Holomorphic

3.1 Cauchy's Theorem

For any holomorphic function f, which is a function which has a continuous first derivative in some simply connected domain Ω then for any closed curve $\gamma : [a, a] \to \Omega$:

$$\int_\gamma f(z)dz = 0$$

To see this, first imagine there exists a function, called a primitive, which is the complex antiderivative, or just integral, of $f(z)$ and it is well defined over all of Ω. Then the above equals:

$$\int_\gamma f(z)dz = F(\gamma_f) - F(\gamma_0)$$

If γ is closed then it must be zero.
The reason we can use this argument is because the integral upon parametrizing z is

path/parametrization independent. This does not hold for real variables, and thus is precisely why holomorphic functions are so important.

What's great about Cauchy's theorem, for our sake, is that any holomorphic function can be represented locally analytically (as a power series):

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n f(z)}{dz^n}\bigg|_{z=z_0} (z - z_0)^n$$

This is amazing because in real analysis not all smooth functions are analytic. But in complex analysis it's enough for a function have it's first derivative in some domain and then it is automatically analytic. This means we can study global behavior of complex functions using local tools - calculus.
Another great consequence is that by Cauchy's theorem we can calculate the n'th derivative in a kind of iterative fashion:

$$f^n(z_0) = \partial_z^n f(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\omega)}{(\omega - z_0)^{n+1}} d\omega$$

Another important fact is that we can now understand radius of convergence in terms of complex functions. For a generic function, real or complex $f(z)$, it may have poles/singularities at a set of values over the domain it is defined over, $\{z_n\}$. For a function defined in a given region, the radius of convergence is always, no matter if real or complex, the distance to the nearest singularity in the complex plane.

4 Residue Theorem

For any holomorphic function $f(z)$ in a domain/open set Ω over $\mathbb{C}$ it's behavior is uniquely determined by it's poles and singularities. This comes directly from Cauchy's theorem, or more so the deviation from Cauchy's theorem. Since integrating over poles gives us a specific value-a residue, we can define the integral over a specific closed curve, where this curve γ is built over a part, or the whole, domain, which may have singularities as follows:

$$\oint_\gamma f(z)dz = 2\pi i \sum_{n}^{N} \mathrm{Res}_{z=\{z_n\}}(f(z))$$

There are many different formulas for the residue, depending on the "order" of the pole, where order is the power of a function near the poles $\{z_n\}$. Consider $\sin(1/z)$. Near $z = 0$ it has a pole, but it behaves like $1/(z - z_0)$ for $z_0 = 0$. Thus it has a order 1 pole. This is a simple pole.
If $\{z_n\}$ is composed of simple poles, then each residue is calculated as:

$$\mathrm{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z)$$

For a order n pole it has a similar formula: $\mathrm{res}_{z_0} f = lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz}\right)^{n-1} (z - z_0)^n f(z)$
Sometimes actually going about choosing the correct contour γ is a difficult task. It's an

imporant one in physics too. If we want to evaluate a classical case of Cauchy's theorem then we are often evaluating an integral from infinity to negative infinity, for example:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

So you may ask, how can we use Cauchy's residue theorem here?
Well, though non-obvious at first despite $\mathbb{R}^2 \cong \mathbb{C}$, the real plane is a slice of the complex plane where $\mathrm{Im}(z) = 0$. So we can use the residue theorem. The way we would do that is to follow a list of steps:

- Determine the poles/singularities of the function: $\{z_n\}$.

- Find a suitable contour which has an easy limit to infinity. It's easiest if some or most of those integrals vanish by some theorem (oscillatory integrals vanish at infinity via the Riemann-Lebesgue Lemma). Particularly we often use Jordan's lemma, which involves taking a semi-circle over $\mathbb{C}$ to capture some of the poles and then take that to infinity. By Jordan's lemma the integral over the arc vanishes so we are left with the real contribution, well assuming we put the semicircle at the real line, as we should in this case.

5 Poles Singularities and More

5.1 Pole Types

As mentioned briefly, there are a couple types of poles. The most "basic" one is an isolated singularity. It's a point where a function $f(z)$ is holomorphic around a punctured neighborhood of a given point $f(z_0)$ and it's often undefined at that point. But sometimes it's not undefined.
Removable. If $f(z)$ remains bounded as we head to the singularity, like a mountain top, or the limit toward it exists, then we can extend $f(z)$ to be holomorphic over z_0 as well.
A interesting example is the necessity for counter-terms in QFT Lagrangians. We need them to remove the singular behavior of vacuum correlators, loop calculations, etc.
Poles. I keep saying poles, but what a pole really is, is when the function tends to plus or minus infinity in the limit as $z \to z_0$, AND $f(z)$ is writtable as $f(z) = g(z)/(z - z_0)^n$ for some holomorphic function $g(z)$, which isn't necessarily holomorphic over z_0, but it must be non-zero. n is the pole order at z_0.
Essential!. Essential singularities are ones that cannot be removed, and they are not a pole. Think of a function which tends to infinity and minus infinity from different directions, but toward z_0. That is essential. Stein gives the example of $e^{1/z}$.
A fun, but perhaps not particularly useful fact about holomorphic functions with essential singularities is that they are dense in the complex plane in their domain. Always. It's the Casorati-Weierstrass theorem.

5.2 Laurent Series and Meromorphic Functions

A function $f(z)$ which is holomorphic except at the points $\{z_n\}$ is holomorphic over a punctured domain $\Omega' = \Omega - \{z_n\}$. Such an f has an analytic expression as well:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

The principal part, $\sum_{n<0}$, gives information on the singularities, since those blow up for $z \to z_0$. The behavior of those singularities does heavily depend on the coefficients a_n. So important we gave good behaving functions a name:meromorphic. If the principle part $f(z) = \sum_{n<0} a_n(z - z_0)^n$ is infinite, then $f(z)$ is meromorphic at all $\{z_n\}$. Meromorphic functions can thus be thought as holomorphic functions at any point away from the isolated poles. This is very! useful in physics.

5.3 Other Pole-y Results

A fascinating consequence of this, is the Argument Principle. It states that any meromorphic function f in a domain bounded by a simple closed curve γ with no zeros or poles specifically on γ has:

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz = \mathcal{N} - \mathcal{P}$$

For $\mathcal{N}$ as the number of zeroes of $f(z)$ inside γ, and $\mathcal{P}$ being the number of poles. It's essentially a winding number for $f(z)$ as it goes around γ.
Lastly is Rouche's Theorem. It says for any two holomorphic functions $f(z)$ and $g(z)$ if they are holomorphic on and over some closed curve γ and the modulus of $g(z)$ is always less than that of $f(z)$, for all z on γ then they have the same number of zeros in γ

$$\exists \ g(z) \ \text{s.t.} \ |g(z)| < |f(z)| \in \gamma \ \text{then} \ \mathcal{N}(\{) = \mathcal{N}(\})$$

6 An Obsession with Singularities

6.1 Entire and Failure to be Entire

Mathematician might have an issue. They are addicted to all the things that cause them issues.
Consider a function $f : \mathbb{C} \to \mathbb{C}$ which is holomorphic everywhere. That's an entire function. These are essentially real analytic functions over $\mathbb{R}$, except it's over $\mathbb{C}$ now, and as a result of being holomorphic it's globally defined.
It's a bit abstract-hence the obsession-but important. Some examples suffice. $e^z, \sin(z), \cos(z), \Gamma(\mathbb{Z} < 0)$, etc.
An important pure mathy result for us is Liouville's theorem. It says that any function which doesn't blow up at infinity, must be constant inside. More fancifully, if f is entire and bounded, then f is constant.

6.2 Functions from Zeros

Now that being said, just as we did with poles, we want to classify functions based on their growth. For that reason, we can define the order of an entire function as it tends to infinity by:

$$\rho := \lim\sup_{r\to\infty} \frac{loglogM(r)}{log(r)}$$

For $M(r) = \max_{|z|=r} |f(z)|$ $\rho = 1$ functions are exponential. $\rho = 2$ are Gaussian like $\cos(z^2)$. $\rho = 0$ are polynomial growth.
Fascinatingly, any function which is entire then should be able to have it's zeros bounded by the growth of the function. The more the growth, the more poles-essentially. We can formalize this via:

$$\sigma := \lim\sup_{r\to\infty} \frac{logM(r)}{r^\rho}$$

This is an important bound for physical situations, but I'll be in honest in saying I've never really used it before.
Because complex functions are amazing, we might expect to do to the converse as well. That is, get functions from their zeros. Fascinatingly we can in fact do so by Jensen's formula. It states that for any entire function, which zeroes inside some disc of radius $|z| < R$, we can write f, where $f(0) \neq 0$ as:

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta$$

So we can get the modulus by the zeroes. Pretty great.
We can abstract this a ton via Factorization Theorems.

6.3 Weierstrass Factorization Theorem

Any entire function f with zeros z_k can be written as:

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{z_k}\right)$$

Where m is the multiplicity of the zero at 0, $g(z)$ is a holomorphic function which can be bootstrapped (determinable), and $E_p(z)$ is taken generally as: $E_p(z) = (1-z)\exp(z + z^2/2 + ...z^p/p)$ to ensure $f(z)$ converges for all $z \in \mathbb{C}$.
Time for an example because this is pretty intuitive, but not useful without example perhaps:

$$\frac{sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

So sin is built entirely from it's zeros and modulus at 0. Maybe that obsession with zeros wasn't so ill-founded.
Even more, if p, the order, is finite, then we have Hadamard's Theorem which says $g(z)$ in the above formula is a polynomial of degree k where $k \leq p$.

Overall this shows that an entire function is essentially determined by its zeros, up to the exponential factor in the front. Like a spectral decomposition of a function. Fitting for it being complex analysis. A great abstraction of this theory is that of Nevanlinna. It measures the rate of growth of a meromorphic-not entire function.

7 Special Functions & Analytic Continuation Basics

7.1 The Gamma Function

Special functions are useful in physics. No doubt about that. Sometimes though, they have weird singularities and poles. Good to analyze with complex analysis. The Gamma function is the first such function:

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$$

For all $z > 0$.
Now you say, well I don't want it for just $\Re z > 0$ ($\Re = \mathrm{Re}(\cdot)$). We want it for all z! Well you're lucky. Though non-obvious, $\Gamma(z)$ satisfies the following recursive formula:

$$\Gamma(z+1) = z\Gamma(z)$$

Which allows for an analytic continuation of Γ into all z since now $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ and that's holomorphic for $\Re z > -1$. Just iterate down to infinity and then $\Gamma(z)$ is a meromorphic function on $\mathbb{C}$, with singularities at the negative integers $s = 0, -1, ...$ (In the words of Stein almost directly).
The idea here is that we can also get the residues via the above formula as $(-1)^n/n!$.
An important formula is that

$$\Gamma(z)\Gamma(1-z) = \pi/(sin(\pi z)$$

Which can be proven by using the product decomposition formula for Γ and $\frac{\sin(\pi z)}{\pi z}$ which was written earlier (not the one for Gamma-that's below). One can also prove it using the original integral definition and then a change of variables.

7.2 The Zeta Function

A fascinating result of this analysis, but in the sense of the Riemann Zeta function, which is defined for $\Re s > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

It has analytic continuation by a connection to the Gamma function:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \ \forall \Re s > 1$$

Which might be hard to see but realize that:

$$\sum_{n=1}^{\infty}\int_0^{\infty} t^{s-1}e^{-nt}dt = \Gamma(s)\sum_{n=1}^{\infty}\frac{1}{n^s}$$

Immediately gives $\zeta(s)$ since the sum is a geometric series of value $\frac{1}{e^t-1}$.
With this motivation, someone once came up with the completed zeta function which is written as:

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

Which allows for Γ to have a meromorphic extension into $\mathbb{C}$ by

$$\xi(s) = \xi(1-s)$$

This also allows us to write $\zeta(s)$ as:

$$\zeta(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s)$$

Which is called the functional equation, or sometimes referred to as the Reflection formula, since it reflects across $\Re s = 0.5$.
There are also very many more special functions and their analytic continuations, but you get the jist.

8 Fourier and Laplace Transform in $\mathbb{C}$

8.1 Fourier Transform

A general Fourier transform of functions which are rapidly decaying and smooth. In complex analysis smooth means holomorphic:

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$$

What's good about this for us is that the Fourier transformed f, $\hat{f}$ is an isomorphism. So the space $\hat{\cdot}$ maps to is the same space–that of rapidly decaying and smooth functions. That's a Schwartz space S. Seeing as it's an isomorphism we can invert to get:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

This is very uesful in physics because acting derivatives on the Fourier transform just gives ξ. So derivatives are polynomials in ξ, in the Fourier transformed space. To reiterate, if the function is rapidly decaying, then the Fourier transformed space is the original space. This shows up a lot in QM, QFT. We just take derivatives on the Fourier transformed function and get eigenvalues. Position and momentum/frequency space representations being Fourier transformations of each other is the canonical example in theoretical physics.

8.2 Laplace Transform

The old forsaken Laplace Transform from that introductory differential equations class makes a come back–and now for a great reason. Just as derivatives in Fourier space become polynomials in the auxillary variable ξ, now we have a similar story, except we primarly get polynomials in terms of time evolving functions. That is, functions on the first Lebesgue space $L^1([0,\infty))$. That is functions which have a finite integral over that domain:

$$f \in L^1([0,\infty)) := \int_0^\infty |f(x)|dx < \infty$$

And also for any subregion within $[0,\infty)$.
Anyways, the Laplace transform is:

$$\mathcal{L}\{f\}s := \int_0^\infty e^{-st} f(t)dt$$

With real S greater than some σ_0 $\Re s > \sigma_0$.
Generally, the Lpalce transform is useful. THe Bromwich integral is particular yos since it gives information on the poles of the function $f(t)$. Also the Laplace transform can be made into the fourier transform by a transformation $s = iw$ for real w.
It's very useful in QFT and even statistical physics at a high level since it's related to transformations on partition functions, propagators, etc.

8.3 Mellin Transform

$$F(z) = \int_0^\infty f(t)t^{z-1}dt$$

Is a Mellin transform. What is fascinating is that when we take $M(f)(z) = F(z)$. This is useful when talking about conformal field theories. Reason being the Fourier transform doesn't work out because of the pole nature of the functions/operators. The Mellin transform works perfectly because it... well it works. Wish I could tell you more, but look up celestial holography scattering amplitudes if you are curious.

8.4 QFT application

Because of how important Fourier and Laplace transforms are, consider this. If we want a particle to go from a point $x^\mu(\tau)$ to $x^\mu(\tau)'$, if they are timelike separated (causually connected) then we should be able to find a propagator which takes us from $x^\mu(\tau)$ to $x^\mu(\tau)'$. It turns out, well at least for free field theories (no interactions), the propagator is the Green's function. In QFT those propagators become correlators for the free theory–the basis of scattering amplitudes by the S matrix formulation. In math it goes like this:

$$(\Box + m^2)\phi(x) = 0$$

Is the Klein-Gordon equation. We seek a function Δ_F which has a delta point source, and obeys the above equation:

$$(\Box + m^2)\Delta_F(x) = -i\delta^4(x)$$

In order to find $\Delta_F(x)$ we take the Fourier transform and represent the delta $\delta^4(x)$ in integral form (an integral over an exponential) to get:

$$\Delta_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ipx}}{p^2 - m^2 + i\epsilon}$$

Where there are a lot of details, but we have the metric $\eta_{\mu\nu} = \text{Diag}[1, -1, -1, -1]$, and so $p^2 = p_0^2 - \vec{p}^2$, and $i\epsilon$ keeps us off the singularities, and enforces causality. Particularly it takes the poles:

$$p_0 = \pm\sqrt{\vec{p}^2 + m^2}$$

And shifts it off the real axis a smidge.
What's fascinating is that mathematically we just took the Fourier transform of a meromorphic function in momentum space. If we wanted to be formal too, we could see the time evolution as Laplace transforms in p_0. We'll talk about Wick rotations later as well.
What is fascinating is how this generalizes. For spin 1/2 particles we do the same:

$$(i\gamma^\mu \partial_\mu p - m)\psi = 0$$

Gives rise to the Green's function for the Dirac propagator, which again propagates a function from x to x':

$$(i\gamma^\mu \partial_x - m)S_F(x - x') = -i\delta^4(x - x')$$

Which in momentum space has the solution:

$$S_F = \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon}$$

As a historical side note, you can see why the Dirac equation is kind of like the square root of the Klein Gordon equation this way.
Anyways, in this case we again just took the Fourier transform of a meromorphic function, except now it takes in matrix values since γ^μ are the gamma matrices.
So summary. In general we can often write a propagator as the Fourier transform:

$$\Delta_F(x) = \mathcal{F}^{-1}\left[\frac{i}{p^2 - m^2 + i\epsilon}\right]$$

For time evolution, this is the Laplace transform in p^0:

$$\Delta_F(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 + w^2}\right]$$

So in the eyes of complex analysis, perturbative QFT is really nothing more than considering meromorphic functions in momentum or energy space, taking their inverse Fourier or akin transform, and then analyzing the poles which give information on the energy spectrum. Well, that is of course if we ignore the technical subtleties such as Wick rotation, dimensional regularization, etc.

9 Conformal Mappings

A conformal map of a domain is a function f which lies in a domain Ω which obeys

$$f : \Omega \subset \mathbb{C} \to \mathbb{C}$$

To be more specific, it's conformal if at a point it's holomorphic and $f'(z) \neq 0$. The implications for physics as numerous. Firstly this alone implies the conservation of angle and orientation. Furthermore, infinitesmally it's seen as a scaled rotation (conformal transformation literally). And most importantly locally is says that if f is conformal then we can understand it's change of a domain Ω using analytic and local processes, all while being 1-1.

9.1 Mobius Transformations

A mobius transform is:

$$f(z) = \frac{az + b}{cz + d}$$

Where $ad - bc \neq 0$. One can prove that these are automorphisms of the Riemann sphere. This is important as we discussed earlier since the Riemann sphere is a "compactification" of the extended Complex plane. This $f(z)$ also preserves circles and lines, and the group of transformations it defines for any complex coefficients is known as $\mathrm{PSL}(2, \mathbb{C})$. In that sense then any Mobius transformation is seen as a combination of translations, rotations, dilations, etc.

9.2 Cayley Transform

We want to consider a conformal map which maps the entire upper half of the complex plane e.g. $\mathbb{H} := \Im z > 0$, where $\Im = \mathrm{Im}$, to the unit disc $\mathbb{D}$. This is given by the Cayley transform:

$$C(z) = \frac{z - i}{z + i}$$

If you do some testing, you'll see that:

$$C : \mathbb{H} \to \mathbb{D}$$

Since this is a bijection we can write the inverse, which is :

$$C^{-1}(w) = i\frac{1 + w}{1 - w}$$

This is SUPER useful in discussions of modular forms, scattering amplitudes in QFT and conformal field theories.

9.3 Automorphisms of $\mathbb{D}$ — Aut($\mathbb{D}$)

The automorphism group of $\mathbb{D}$, which is just a unit disc e.g. the set of all z such that $|z| < 1$. This is given by:

$$f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$$

Where again $\bar{\cdot}$ means complex conjugate, and $|a| < 1$. All of these transformations form the group PSU$(1,1)$ which is isomorphic to PSL$(2,\mathbb{C})$. Just as SO(3) is a double cover of $SU(2)$, PSL$(2,\mathbb{C})$ is a double cover of PSU$(1,1)$. Again, just as in quantum mechanics SU(2) is the transformation group of spinors, PSU$(1,1)$ is conformal symmetries of 2D hyperbolic space, and the double cover is the full conformal group of a 2D CFT.
To summarize this, in 2D CFTs then PSU$(1,1)$ is the global conformal group of the 2D CFTs, or the isometry group of 2D hyperbolic space. So in that sense, PSU$(1,1)$ plays the same role as SU(2)! (This is hard to explain, sorry if it doesn't make all too much sense. Just consider that they are both double covers of the larger symmetry group)

9.4 Riemann Mapping Theorem

Every simply connected domain as a proper subset of $\mathbb{C}$: $\Omega \subsetneq \mathbb{C}$ is conformally equivalent to the unit disc $\mathbb{D}$. This is huge for 2D (1+1) dimensional conformal field theories. This means that up to a Mobius transform, every region has the same conformal type, and the mapping to $\mathbb{D}$ is unique up to rotation.
This is related to the Schwarz-Christoffel Transform, which takes $\mathbb{H}$ and plasters it conformally onto the interior of a polygon with precise angles at vertices:
Consider a set of real numbers $\omega_n \in \mathbb{R}$, with internal angles $\alpha_k\pi$. Then $f(z)$ is the transformation described above, and given as:

$$f(z) = A\int^z \prod_{k=1}^{n}(t-\omega_k)^{\alpha_k - 1}dt + B$$

So now we see singularities of the intergrand are vertices of the polygon. I can't say it's hugely useful, but it can give boundary conditions and a precise formulation of a 2D CFT if it's defined over some polygon.

9.5 Elliptic Integrals and Conformal Factorization

Elliptical integrals are great. They show up all over physics. Classical mechanics, quantum mechanics (WKB), etc. Essentially any place where we have a source and an orbit, or analogy of orbit. There are different types and we will discuss them more later, but consider that elliptical integrals are defined as integrals which map rectangles to the upper half-plane $\mathbb{H}$. They take the form:

$$\int_0^x \frac{d\xi}{[(1-\zeta^2)(1-k^2\zeta^2)]^{1/2}},\ \ z \in \mathbb{H}$$

For $k \in (0,1) \subset \mathbb{R}$. The branch of the denominator is in the upper half-plane and is taken so that the denominator is always positive as long as $\zeta \in (-1,1) \subset \mathbb{R}$. More information on conformal maps by elliptical integrals is seen in Stein.

10 Elliptic Functions and Modular Invariance

10.1 The $\wp$ function

A rather important topic of complex analysis with relevant results for HEP. To start, define a lattice $\Lambda \subset \mathbb{C}$:

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

Where we assume w_1/w_2 are linearly independent over $\mathbb{R}$. A function $f : \mathbb{C} \to \mathbb{C}$ is elliptic if:

$$f(z+w) = f(z) \ \forall w \in \Lambda$$

Some examples are the Weierstrass $\wp$-function:

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{w \in (\Lambda - \{0,0\})} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

It's meromorphic with poles at the lattice points. Even under $z \to -z$, and has periods $1, \tau$ where $\Lambda = \{n + m\tau : n, m \in \mathbb{Z}\}$. That sounds arbitrary at first glance, but consider that this builds out a lattice of spacing $1, \tau$ in the real and imaginary plane.
As a result of $\wp$ being even, $\wp'$, it's derivative, is odd. (Consider that $\wp$ is analytic and thus is a power series, and so differentiating it will bring down a power of each term in the series, making it odd where it was even, so when $z \to -z$, $\wp' \to -\wp'$.
Because $\wp'$ also has periods $1, \tau$, then we can follow the prescription of chapter 6 where we built functions by their singularities and zeros to construct $\wp'$ from $\wp$.
First note that $\wp'$ is odd so $\wp'(-1/2) = -\wp'(1/2) = \wp'(\tau/2) = \wp'(\frac{1+\tau}{2}) = 0$. Because $\wp'$ is the derivative, we just integrate to see that we have three constant values for $\wp$, called the half-periods: $\wp(1/2) = e_1$, $\wp(\tau/2) = e_2$, $\wp(\frac{1+\tau}{2}) = e_3$.
Now we can construct $\wp'$ from $\wp$ by considering that where $\wp'$ is odd, $\wp'^2$ must be even, but $\wp$ was as well. $\wp'^2$ also has its zeros at the above half-periods. By arguments from chapter 6, this means up to an exponential factor that:

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

The 4 and exponential factor are fixed by taylor expanding the original definition of $\wp$ near $z = 0$ and finding that $\wp = 1/z^2 + ...$ and $\wp' = -2/z^3 + ...$ Substituting those in, we get 4 for equivalence.
We can also write $\wp(z)$ in a power series of the E_k from chapter 6 above, where:

$$E_k = \sum_{w \in (\Lambda - \{0,0\})} \frac{1}{w^{l+2}}$$

Stein shows it on page 275-no point to do it here, that:

$$\wp(z) = 1/z^2 + \sum_{k=1}^{\infty} (2k+1) E_{2k+2} z^{2k}$$

Now in a mathematician's fashion, we use this to reverse engineer our above cubic polynomial for $(\wp')^2$ to get:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

Where $g_2 = 60E_4$ and $g_3 = 140E_6$.

10.2 Modular Forms

The modular group:

$$\mathrm{SL}(2,\mathbb{Z}) = \{\begin{pmatrix} a\,b \\ c\,d \end{pmatrix} |(ad-bc) = 1\}$$

Acts on elements $\tau \in \mathbb{H}$, which is just $\Im z > 0$ (the upper half-plane over $\mathbb{C}$) by:

$$g \in \mathrm{SL}(2,\mathbb{Z}) : \tau \to \frac{a\tau + b}{c\tau + d}$$

From this slightly random group which seems entirely random and should, well at least in my eyes, not look like anything perhaps valueable at first, realize that functions which transform under $\mathrm{SL}(2,\mathbb{Z})$ are modular forms of weight k. They are holomorphic functions $f(\tau) \in \mathbb{H}$ and satisfy:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

Where specifically f must be holomorphic at $i\infty$ which just keeps f bounded as it tends to infinity.
Eisenstein series is an example of a modular form:

$$G_k(\tau) = \sum_{\Lambda - \{0,0\}} \frac{1}{(m+n\tau)^k}$$

For $k \geq 4$ and even.
Modular forms are so important that one could write an entire book on their usage in physics. Probably many books. And do many great years of research. String theory, elliptic genera, anomaly canncelation, supersymmetric QFTs, etc. All of these, well most, arise from modular invariance of the underlying theory.
An interesting example of an almost modular invariant function is the Dedekind eta function:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n),\; q = e^{2\pi i \tau}$$

We'll talk about it's anomaly in a minute
Another interseting example is the modular invariance of the string path integral:

$$Z(\tau) = Tr_{\mathcal{H}}(q^{L_0 - c/24} \bar{q}^{\bar{L_0} - c/24})$$

And it's invariance under $\tau \to \tau + 1$ and $\tau \to -1/\tau$.
Speaking of is the Jacobi theta function. We'll also talk about that in a minute.

274 ### 10.3 Dedekind η function anomaly

The function, above, is defined for $\tau \in \mathbb{H}$ and appears everywhere in HEP. We seek to test if it's invariant under $\tau \to \tau + 1$ and $\tau \to -1/\tau$.
Plugging directly in, we see that

$$\eta(\tau + 1) = e^{\pi i/12}\eta(\tau)$$

This means we have a phase anomaly, which transforms by 24th roots of unity. Furthermore under $\tau \to -1/\tau$ goes as:

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$

So we again pick up a complex phase factor which just means η transforms like a modular form of weight $k = 1/2$, up to the phase anomaly found before. Overall this means we can write:

$$\eta(\frac{a\tau + b}{c\tau + d}) = \epsilon(a, b, c, d)(c\tau + d)^{1/2}\eta(\tau)$$

Not modular invariant-but close. Now in physics we can define different partition functions, which appear from physical motivation, as $Z \propto 1/(\eta(\tau)^{24}\bar{\eta}(\tau)^{24})$ etc. Issue is that we need Z to be modular invariant, not invariant up to a phase and weight 1/2. This amounts to making sure the phase anomaly gets taken care of.
Let's look at an example. Taking the above Z as an example consider that it isn't quite modular invariant. Issue is we get that same factor of $(-i\tau)^{-12}$ for each η. This means that we need a factor of τ out front which similar transforms to cancel this. That means our true modular invariant, anomaly-free partition function must go as:

$$Z = \frac{1}{\tau_n^{12}} \frac{1}{(\eta(\tau)^{24}\bar{\eta}(\tau)^{24})}$$

275 Then finally $Z(-1/\tau, -1/\bar{\tau}) = Z(\tau, \bar{\tau})$.

276 ## 11 The Jacobi Theta Function and Generating Functions

277 ### 11.1 Theta Function

The Jacobi theta function is like a modular form, but not quite. It is given by the series:

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

Which converges for all $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ ($\Im z > 0$. What's amazing about this function is that as a function of z, it's periodic with period 1 and quasi-period τ. As a function of τ its like a modular form.
The Jacobi theta function also reduces to two other important functions:

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \;\; \tau \in \mathbb{H}$$

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, \ t > 0$$

So $\theta(t) = \Theta(0, \tau)$ and $\vartheta(t) = \theta(it)$, well for $t > 0$ at least. Now this function has very many important properties as a result of this dual nature. They are summarized in Stein on page 284.

11.2 Triple-Product Function

Those properties motivate another amazing function called the product function, sometimes called the triple-product function $\Pi(z|\tau)$, written as:

$$\Pi(z|\tau) = \prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n-1}e^{2\pi iz})(1 + q^{2n-1}e^{-2\pi iz})$$

Where $q = e^{i\pi\tau}$. This triple-product function, as a result of construction has the same properties as $\Theta(z|\tau)$ despite its completely different appearance.
But. It turns out that:

$$\Theta(z|\tau) = \Pi(z|\tau)$$

for all $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$! This is so amazing since it tells us that a Fourier series which is holomorphic, doubly periodic (elliptical-well quasi-elliptical) is somehow equal to Weierstrass product which solely reflects the entire function's zeros and convergence information/structure.
This is an important relation/fact in string theory when finding functions which describe strings.

11.3 Generating Functions

It'd be a long shot to talk about this in a physics context. Perhaps for CFT bootstrap or other heavy mathematical topics, but to my knowledge not really something worth writing about for physics. Mathematically it is fascinating; pioneered by Euler (so you know it's good).

12 Summary to Complex Analysis

As a summary we have a few important final things to iterate and make concrete in our heads.

1) A holomorphic function is a complex-differentiable function, e.g. $f \in C^1(\Omega)$ for some domain Ω. These functions are astonishing. Their behavior, highlighted by Cauchy's theorem demonstrates that any holomorphic function is automatically smooth e.g. $f \in C^\infty(\Omega)$.

2) Reasonable functions–that is, ones without essential poles (very ugly poles-no converging limit from left and right, etc)–have their entire behavior governed by singularities and zeros. There are three types-removable, poles, and essential. The residue theorem gives us everything we need to reconstruct meromorphic functions (holomorphic except at finite number of removable or pole singularities) by reverse engineering their structure.

3) Special functions are awesome! And we can often analytically continue them by their properties. Gamma function, Riemann zeta, Weierstrass product functions, etc. We classify these based on their order and type, which gives us information specifically on their asymptotic growth, which is essential for guaranteeing convergence.

4) Conformal mappings are very useful. By the Riemann mapping theorem we can take any open domain $\Omega \subset \mathbb{C}$ and conformally map it onto the unit disc $\mathbb{D}$ by a Mobius transform. The automorphism group of $\mathbb{D}$ and it's properties is important and has a similar double cover to that of spinors by SU(2) being doubly covered by SO(3).

5) Modular forms are awesome. Modular invariance is crucial in physics. Complex analysis tells us all about these with special functions like the Dedking eta function and Jacobi theta function.

6) Fourier, Laplace and Mellin transforms are very useful. Fourier transforms reveal dual nature of physical theories in their position and momentum space. Laplace transforms encode time evolution and as a result are important in understanding decay, boundary conditions and advanced/retarded boundary condition setups. Mellin transforms are very useful due to specific conformal properties, since it encodes scattering amplitudes of celestial holography (and much more).

7) Elliptic functions are useful. They are doubly periodic over the complex lattice Λ, they are meromorphic on $\Lambda/\mathbb{C}$. This wasn't mentioned before, but that's a complex tori by the real properties of a torus (doubly periodic), and much more. The Weierstrass $\wp$ function and Eisenstein series are useful in understanding modular forms. They appear in physics a lot!

12.1 Final Words

Complex analysis is much more than just mathematics (unlike some of real analysis). It truly encodes much of the structure physicist need to understand quantum theory. More than just quantum theory, due to the highly rigid structure of complex differentiable (holomorphic, meromorphic, etc) functions over the complex plane $\mathbb{C}$, it is extremely useful to know complex analysis. You get a grasp on why complex functions are useful (analytic on their domain), the symmetries which appear, and the algebraic & geometric structures which appear from the mathematics.
Complex analysis is the bedrock of modern high energy physics.

Acknowledgments

Summary from Princeton Lectures in Analysis II COMPLEX ANALYSIS Elias M. Stein & Rami Shakarchi