

2 **Summary of Complex Analysis for formal HEP**

3 **Gabe Suarez**

4 *Syracuse University,*

5 *New York, USA*

6 *UKY,*

7 *Kentucky, USA*

8 *E-mail:* gmsuarez@syr.edu

9 ABSTRACT: A very brief summary on the important results of complex analysis for a formal
10 high energy physicist. Motivation, topics covered, and further suggestions for reading are
11 provided at the end.

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54 1 Motivation of Complex Analysis for Physicists

Complex analysis may not sound like a topic would be useful to describing the real world. However, complex analysis is one of the most important areas of mathematics for a physicist to understand.

That being said, complex analysis is essentially the analysis of complex numbers, so it's unfair to pretend that formal complex analysis—what is taught in graduate math classes, is what physicists talk about when using complex analysis.

The first place a physics student encounters complex analysis is in the most general solution to the harmonic oscillator differential equation:

$$m\ddot{x} = kx$$

Where solutions go as

$$x(t) = Ae^{ikx} + Be^{-ikx}$$

The next place is often in electromagnetism where i pops up in small areas of the subject, where complex analysis allows us to represent certain vector and field configurations most easily.

Where complex analysis becomes a necessity is in quantum mechanics. The Schrodinger equation takes the form:

$$\hat{H}\Psi = \left(\frac{\hat{p}}{2m} + \hat{V} \right) \Psi$$

Where momentum is the generator of translations and has eigenvalues, in position space, as $\hat{p} = -i\hat{\partial}_x$. So immediately complex analysis has a huge place in quantum mechanics. I won't mention the rest, but upon opening a quantum mechanics textbook you'll see talks of wave scattering, amplitudes, etc., which all come from results of complex and functional analysis.

Now where complex analysis becomes most essential is in quantum field theory and beyond.

In quantum field theory, we have our field theory, defined by an action, like in classical mechanics:

$$S = \int d^D q \mathcal{L}$$

Classically, \mathcal{L} is real. However, in quantum field theory we separate out the Lagrangian density into free and interacting terms. The interacting terms have i . An example is the pseudoscalar Yukawa coupling term: $-ig\bar{\psi}\gamma^5\psi\psi$. That's just the beginning. There are countless many places where complex analysis comes into play in quantum field theory. In that perturbative QFT picture we have: Wick rotations, residue calculations, branch cuts, dimensional regularization, analytic continuation, and much more.

Going beyond perturbative QFT we need more "formal" tools of complex analysis, specifically talking about holomorphic and anti-holomorphic structures. Sounds fancy, but that just means complex differentiable. Well, it definitely means more than that. Reason being any holomorphic function has a continuous first derivative on it's entire domain. That fact alone, by Cauchy's theorem, implies that any holomorphic function is infinitely differentiable on that same domain. Which means any holomorphic function, despite only having it's first derivative known about, is automatically smooth.

Perhaps also notable, though also perhaps obvious in some ways, complex analysis is absolutely essential without a doubt in most modern fields of formal HEP. This includes, but is not limited to, string theory, conformal field theories, AdS/CFT and general holography, modular (number theoretic and algebraic) theory, etc.

Let's start.

2 Preliminaries to Complex Analysis

There are many, in a literal sense, but a typical calculus sequence, with some knowledge on set theory & notation, and mathematical maturity will suffice. Even without set theory knowledge and mathematical maturity, one could read complex analysis if they determined. They'd just be referencing wikipedia & AI a lot back and forth.

2.1 Point Set Topology & Geometric Arguments

Complex analysis starts by reviewing what open/closed sets are, what limit points and compactness mean. In a quick sense, open/closed sets are literally regions of some space. Open sets Ω are regions that do not include the points on their boundary:

$$\Omega \in \mathbb{C}$$

Closed sets are just open sets with their boundary included, well. At least for our purposes.

Limit points are, in our context, essentially points on the boundary of open set. That means all limit points of closed sets are contained within the closed set. Open sets, no. However, limit points can be arbitrarily approximated by the inside open set.

86 Compactness is of utmost importance. Compactness means closed and bounded. Bounded
 87 means of finite size in our sense.

88

89 2.2 Complex Plane \mathbb{C}

90 There is an isomorphism between \mathbb{C} and \mathbb{R}^2 by $z = x + iy$. To see this, define \bar{z} is the
 91 complex conjugate, an operation that takes $i \rightarrow -i$. Consider the inner product on the
 92 space which is formed by $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 \in \mathbb{R}^2$.

93 An important fact is that the 2-sphere, without North pole $S^2 - \{0\}$ can be stereographically
 94 projected onto \mathbb{C} . This is hard to imagine, but if you imagine this situation it will help. The
 95 2-sphere sitting above the origin, with south pole at the origin of the plane \mathbb{R}^2 . Consider a
 96 ray which connects a given point on the 2-sphere to the north pole. That ray will intersect
 97 \mathbb{R}^2 in one point. This forms a bijection between the two, and thus an isomorphism since
 98 there is no structural change in such a mapping. We could also include the north pole by
 99 including the infinity points in \mathbb{C} . As Stein says, this forms an important compactification
 100 of the unbounded \mathbb{C} into the compact 2-sphere with it's north pole - the Riemann sphere.
 101 This is pretty important to consider just because it introduces such geometric arguments.

102 2.3 Other Intro Material

103 I'd suggest Stein if you are curious enough about the other introductory material. It talks
 104 about mother things like basic complex arguments such as holomorphicity, etc. I won't
 105 talk about that. I'll just write down the main key results. Unfortunately that means

106

107 **You will not learn complex analysis in nearly any capacity from these notes.**
 108 **However, you will see what is taught in a typical complex analysis course, and**
 109 **how you could go on to apply such topics to theoretical physics.**

110 3 Cauchy Theory and Consequences of Being Holomorphic

111 3.1 Cauchy's Theorem

For any holomorphic function f , which is a function which has a continuous first derivative
 in some simply connected domain Ω then for any closed curve $\gamma \in \Omega$:

$$\int_{\gamma} f(z) dz = 0$$

To see this, first imagine there exists a function, called a primitive, which is the complex
 antiderivative, or just integral, of $f(z)$ and it is well defined over all of Ω . Then the above
 equals:

$$\int_{\gamma} f(z) dz = F(\gamma_f) - F(\gamma_0)$$

112 If γ is closed then it must be zero.

113 The reason we can use this argument is because the integral upon parametrizing z is

114 path/parametrization independent. This does not hold for real variables, and thus is
 115 precisely why holomorphic functions are so important.

What's great about Cauchy's theorem, for our sake, is that any holomorphic function can be represented locally analytically (as a power series):

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n f(z)}{dz^n} \right|_{z=z_0} (z - z_0)^n$$

This is amazing because in real analysis not all smooth functions are analytic. But in complex analysis it's enough for a function have it's first derivative in some domain and then it is automatically analytic. This means we can study global behavior of complex functions using local tools - calculus.

Another great consequence is that by Cauchy's theorem we can calculate the n'th derivative in a kind of iterative fashion:

$$f^n(z_0) = \partial_z^n f(z_0) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{f(\omega)}{(\omega - z_0)^{n+1}} d\omega$$

116 Another important fact is that we can now understand radius of convergence in terms of
 117 complex functions. For a generic function, real or complex $f(z)$, it may have poles/singularities
 118 at a set of values over the domain it is defined over, $\{z_n\}$. For a function defined in a given
 119 region, the radius of convergence is always, no matter if real or complex, the distance to
 120 the nearest singularity in the complex plane.

121 4 Residue Theorem

For any holomorphic function $f(z)$ in a domain/open set Ω over \mathbb{C} it's behavior is uniquely determined by it's poles and singularities. This comes directly from Cauchy's theorem, or more so the deviation from Cauchy's theorem. Since integrating over poles gives us a specific value-a residue, we can define the integral over a specific closed curve, where this curve γ is built over a part, or the whole, domain, which may have singularities as follows:

$$\oint_{\gamma} f(z) dz = 2\pi i \sum_n^N \text{Res}_{z=\{z_n\}}(f(z))$$

There are many different formulas for the residue, depending on the "order" of the pole, where order is the power of a function near the poles $\{z_n\}$. Consider $\sin(1/z)$. Near $z = 0$ it has a pole, but it behaves like $1/(z - z_0)$ for $z_0 = 0$. Thus it has a order 1 pole. This is a simple pole.

If $\{z_n\}$ is composed of simple poles, then each residue is calculated as:

$$\text{res}_{z_0} f = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

For a order n pole it has a similar formula: $\text{res}_{z_0} f = \lim_{z \rightarrow z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz} \right)^{n-1} (z - z_0)^n f(z)$
 Sometimes actually going about choosing the correct contour γ is a difficult task. It's an

important one in physics too. If we want to evaluate a classical case of Cauchy's theorem then we are often evaluating an integral from infinity to negative infinity, for example:

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

122 So you may ask, how can we use Cauchy's residue theorem here?

123 Well, though non-obvious at first despite $\mathbb{R}^2 \cong \mathbb{C}$, the real plane is a slice of the complex
 124 plane where $\text{Im}(z) = 0$. So we can use the residue theorem. The way we would do that is
 125 to follow a list of steps:

- 126 • Determine the poles/singularities of the function: $\{z_n\}$.
- 127 • Find a suitable contour which has an easy limit to infinity. It's easiest if some or most
 128 of those integrals vanish by some theorem (oscillatory integrals vanish at infinity via
 129 the Riemann-Lebesgue Lemma). Particularly we often use Jordan's lemma, which
 130 involves taking a semi-circle over \mathbb{C} to capture some of the poles and then take that to
 131 infinity. By Jordan's lemma the integral over the arc vanishes so we are left with the
 132 real contribution, well assuming we put the semicircle at the real line, as we should
 133 in this case.

134 5 Poles Singularities and More

135 5.1 Pole Types

136 As mentioned briefly, there are a couple types of poles. The most "basic" one is an isolated
 137 singularity. It's a point where a function $f(z)$ is holomorphic around a punctured neigh-
 138 borhood of a given point $f(z_0)$ and it's often undefined at that point. But sometimes it's
 139 not undefined.

140 **Removable.** If $f(z)$ remains bounded as we head to the singularity, like a mountain top,
 141 or the limit toward it exists, then we can extend $f(z)$ to be holomorphic over z_0 as well.
 142 A interesting example is the necessity for counter-terms in QFT Lagrangians. We need
 143 them to remove the singular behavior of vacuum correlators, loop calculations, etc.

144 **Poles.** I keep saying poles, but what a pole really is, is when the function tends to plus or
 145 minus infinity in the limit as $z \rightarrow z_0$, AND $f(z)$ is writable as $f(z) = g(z)/(z - z_0)^n$ for
 146 some holomorphic function $g(z)$, which isn't necessarily holomorphic over z_0 , but it must
 147 be non-zero. n is the pole order at z_0 .

148 **Essential!.** Essential singularities are ones that cannot be removed, and they are not a
 149 pole. Think of a function which tends to infinity and minus infinity from different direc-
 150 tions, but toward z_0 . That is essential. Stein gives the example of $e^{1/z}$.

151 A fun, but perhaps not particularly useful fact about holomorphic functions with essential
 152 singularities is that they are dense in the complex plane in their domain. Always. It's the
 153 Casorati-Weierstrass theorem.

154 5.2 Laurent Series and Meromorphic Functions

A function $f(z)$ which is holomorphic except at the points $\{z_n\}$ is holomorphic over a punctured domain $\Omega' = \Omega - \{z_n\}$. Such an f has an analytic expression as well:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

155 The principal part, $\sum_{n<0}$, gives information on the singularities, since those blow up for
 156 $z \rightarrow z_0$. The behavior of those singularities does heavily depend on the coefficients a_n .

157 So important we gave good behaving functions a name: meromorphic. If the principle part
 158 $f(z) = \sum_{n<0} a_n(z - z_0)^n$ is infinite, then $f(z)$ is meromorphic at all $\{z_n\}$.

159 Meromorphic functions can thus be thought as holomorphic functions at any point away
 160 from the isolated poles. This is very! useful in physics.

161

162 5.3 Other Pole-y Results

A fascinating consequence of this, is the Argument Principle. It states that any meromorphic function f in a domain bounded by a simple closed curve γ with no zeros or poles specifically on γ has:

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f'(z)}{f(z)} dz = \mathcal{N} - \mathcal{P}$$

For \mathcal{N} as the number of zeroes of $f(z)$ inside γ , and \mathcal{P} being the number of poles. It's essentially a winding number for $f(z)$ as it goes around γ .

Lastly is Rouché's Theorem. It says for any two holomorphic functions $f(z)$ and $g(z)$ if they are holomorphic on and over some closed curve γ and the modulus of $g(z)$ is always less than that of $f(z)$, for all z on γ then they have the same number of zeros in γ

$$\exists g(z) \text{ s.t. } |g(z)| < |f(z)| \in \gamma \text{ then } \mathcal{N}(\{f\}) = \mathcal{N}(\{f+g\})$$

163 6 An Obsession with Singularities

164 6.1 Entire and Failure to be Entire

165 Mathematician might have an issue. They are addicted to all the things that cause them
 166 issues.

167 Consider a function $f : \mathbb{C} \rightarrow \mathbb{C}$ which is holomorphic everywhere. That's an entire function.
 168 These are essentially real analytic functions over \mathbb{R} , except it's over \mathbb{C} now, and as a result
 169 of being holomorphic it's globally defined.

170 It's a bit abstract-hence the obsession-but important. Some examples suffice. $e^z, \sin(z), \cos(z), \Gamma(\mathbb{Z} <$
 171 $0)$, etc.

172 An important pure mathy result for us is Liouville's theorem. It says that any function
 173 which doesn't blow up at infinity, must be constant inside. More fancifully, if f is entire
 174 and bounded, then f is constant.

175

176 6.2 Functions from Zeros

Now that being said, just as we did with poles, we want to classify functions based on their growth. For that reason, we can define the order of an entire function as it tends to infinity by:

$$\rho := \limsup_{r \rightarrow \infty} \frac{\log \log M(r)}{\log(r)}$$

For $M(r) = \max_{|z|=r} |f(z)|$ $\rho = 1$ functions are exponential. $\rho = 2$ are Gaussian like $\cos(z^2)$. $\rho = 0$ are polynomial growth.

Fascinatingly, any function which is entire then should be able to have it's zeros bounded by the growth of the function. The more the growth, the more poles-essentially. We can formalize this via:

$$\sigma := \limsup_{r \rightarrow \infty} \frac{\log M(r)}{r^\rho}$$

This is an important bound for physical situations, but I'll be in honest in saying I've never really used it before.

Because complex functions are amazing, we might expect to do to the converse as well. That is, get functions from their zeros. Fascinatingly we can in fact do so by Jensen's formula. It states that for any entire function, which zeroes inside some disc of radius $|z| < R$, we can write f , where $f(0) \neq 0$ as:

$$\log |f(0)| = \sum_{k=1}^N \log \left(\frac{|z_k|}{R} \right) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta$$

177 So we can get the modulus by the zeroes. Pretty great.

178 We can abstract this a ton via Factorization Theorems.

179 6.3 Weierstrass Factorization Theorem

Any entire function f with zeros z_k can be written as:

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_p \left(\frac{z}{z_k} \right)$$

Where m is the multiplicity of the zero at 0, $g(z)$ is a holomorphic function which can be bootstrapped (determinable), and $E_p(z)$ is taken generally as: $E_p(z) = (1 - z) \exp(z + z^2/2 + \dots z^p/p)$ to ensure $f(z)$ converges for all $z \in \mathbb{C}$.

Time for an example because this is pretty intuitive, but not useful without example perhaps:

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right)$$

180 So sin is built entirely from it's zeros and modulus at 0. Maybe that obsession with zeros
181 wasn't so ill-founded.

182 Even more, if p , the order, is finite, then we have Hadamard's Theorem which says $g(z)$ in
183 the above formula is a polynomial of degree k where $k \leq p$.

Overall this shows that an entire function is essentially determined by its zeros, up to the exponential factor in the front. Like a spectral decomposition of a function. Fitting for it being complex analysis. A great abstraction of this theory is that of Nevanlinna. It measures the rate of growth of a meromorphic-not entire function.

7 Special Functions & Analytic Continuation Basics

7.1 The Gamma Function

Special functions are useful in physics. No doubt about that. Sometimes though, they have weird singularities and poles. Good to analyze with complex analysis. The Gamma function is the first such function:

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$$

For all $z > 0$.

Now you say, well I don't want it for just $\Re z > 0$ ($\Re = \text{Re}(\cdot)$). We want it for all z ! Well you're lucky. Though non-obvious, $\Gamma(z)$ satisfies the following recursive formula:

$$\Gamma(z+1) = z\Gamma(z)$$

Which allows for an analytic continuation of Γ into all z since now $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ and that's holomorphic for $\Re z > -1$. Just iterate down to infinity and then $\Gamma(z)$ is a meromorphic function on \mathbb{C} , with singularities at the negative integers $s = 0, -1, \dots$ (In the words of Stein almost directly).

The idea here is that we can also get the residues via the above formula as $(-1)^n/n!$.

An important formula is that

$$\Gamma(z)\Gamma(1-z) = \pi / (\sin(\pi z))$$

Which can be proven by using the product decomposition formula for Γ and $\frac{\sin(\pi z)}{\pi z}$ which was written earlier (not the one for Gamma-that's below). One can also prove it using the original integral definition and then a change of variables.

7.2 The Zeta Function

A fascinating result of this analysis, but in the sense of the Riemann Zeta function, which is defined for $\Re s > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

It has analytic continuation by a connection to the Gamma function:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \quad \forall \Re s > 1$$

Which might be hard to see but realize that:

$$\sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} e^{-nt} dt = \Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Immediately gives $\zeta(s)$ since the sum is a geometric series of value $\frac{1}{e^t - 1}$.

With this motivation, someone once came up with the completed zeta function which is written as:

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$$

Which allows for Γ to have a meromorphic extension into \mathbb{C} by

$$\xi(s) = \xi(1-s)$$

This also allows us to write $\zeta(s)$ as:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

195 Which is called the functional equation, or sometimes referred to as the Reflection formula,
196 since it reflects across $\Re s = 0.5$.

197 There are also very many more special functions and their analytic continuations, but you
198 get the jist.

199 8 Fourier and Laplace Transform in \mathbb{C}

200 8.1 Fourier Transform

A general Fourier transform of functions which are rapidly decaying and smooth. In complex analysis smooth means holomorphic:

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

What's good about this for us is that the Fourier transformed f , \hat{f} is an isomorphism. So the space $\hat{\cdot}$ maps to is the same space—that of rapidly decaying and smooth functions. That's a Schwartz space S . Seeing as it's an isomorphism we can invert to get:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

201 This is very uesful in physics because acting derivatives on the Fourier transform just gives
202 ξ . So derivatives are polynomials in ξ , in the Fourier transformed space. To reiterate, if
203 the function is rapidly decaying, then the Fourier transformed space is the original space.
204 This shows up a lot in QM, QFT. We just take derivatives on the Fourier transformed
205 function and get eigenvalues. Position and momentum/frequency space representations
206 being Fourier transformations of each other is the canonical example in theoretical physics.

207

208 8.2 Laplace Transform

The old forsaken Laplace Transform from that introductory differential equations class makes a come back—and now for a great reason. Just as derivatives in Fourier space become polynomials in the auxillary variable ξ , now we have a similar story, except we primarily get polynomials in terms of time evolving functions. That is, functions on the first Lebesgue space $L^1([0, \infty))$. That is functions which have a finite integral over that domain:

$$f \in L^1([0, \infty)) := \int_0^\infty |f(x)| dx < \infty$$

And also for any subregion within $[0, \infty)$.

Anyways, the Laplace transform is:

$$\mathcal{L}\{f\}s := \int_0^\infty e^{-st} f(t) dt$$

209 With real S greater than some σ_0 $\Re s > \sigma_0$.

210 Generally, the Lpalce transform is useful. THE Bromwich integral is particular yos since
211 it gives information on the poles of the function $f(t)$. Also the Laplace transform can be
212 made into the fourier transform by a transformation $s = iw$ for real w .

213 It's very useful in QFT and even statistical physics at a high level since it's related to
214 transformations on partition functions, propagators, etc.

215 8.3 Mellin Transform

$$F(z) = \int_0^\infty f(t) t^{z-1} dt$$

216 Is a Mellin transform. What is fascinating is that when we take $M(f)(z) = F(z)$. This
217 is useful when talking about conformal field theories. Reason being the Fourier transform
218 doesn't work out because of the pole nature of the functions/operators. The Mellin trans-
219 form works perfectly because it... well it works. Wish I could tell you more, but look up
220 celestial holography scattering amplitudes if you are curious.

221 8.4 QFT application

Because of how important Fourier and Laplace transforms are, consider this. If we want a particle to go from a point $x^\mu(\tau)$ to $x^\mu(\tau)'$, if they are timelike separated (causally connected) then we should be able to find a propagator which takes us from $x^\mu(\tau)$ to $x^\mu(\tau)'$. It turns out, well at least for free field theories (no interactions), the propagator is the Green's function. In QFT those propagators become correlators for the free theory—the basis of scattering amplitudes by the S matrix formulation. In math it goes like this:

$$(\square + m^2)\phi(x) = 0$$

Is the Klein-Gordon equation. We seek a function Δ_F which has a delta point source, and obeys the above equation:

$$(\square + m^2)\Delta_F(x) = -i\delta^4(x)$$

In order to find $\Delta_F(x)$ we take the Fourier transform and represent the delta $\delta^4(x)$ in integral form (an integral over an exponential) to get:

$$\Delta_F(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ipx}}{p^2 - m^2 + i\epsilon}$$

Where there are a lot of details, but we have the metric $\eta_{\mu\nu} = \text{Diag}[1, -1, -1, -1]$, and so $p^2 = p_0^2 - \vec{p}^2$, and $i\epsilon$ keeps us off the singularities, and enforces causality. Particularly it takes the poles:

$$p_0 = \pm \sqrt{\vec{p}^2 + m^2}$$

And shifts it off the real axis a smidge.

What's fascinating is that mathematically we just took the Fourier transform of a meromorphic function in momentum space. If we wanted to be formal too, we could see the time evolution as Laplace transforms in p_0 . We'll talk about Wick rotations later as well. What is fascinating is how this generalizes. For spin 1/2 particles we do the same:

$$(i\gamma^\mu \partial_\mu p - m)\psi = 0$$

Gives rise to the Green's function for the Dirac propagator, which again propagates a function from x to x' :

$$(i\gamma^\mu \partial_x - m)S_F(x - x') = -i\delta^4(x - x')$$

Which in momentum space has the solution:

$$S_F = \frac{i(\gamma^\mu p_\mu + m)}{p^2 - m^2 + i\epsilon}$$

As a historical side note, you can see why the Dirac equation is kind of like the square root of the Klein Gordon equation this way.

Anyways, in this case we again just took the Fourier transform of a meromorphic function, except now it takes in matrix values since γ^μ are the gamma matrices.

So summary. In general we can often write a propagator as the Fourier transform:

$$\Delta_F(x) = \mathcal{F}^{-1} \left[\frac{i}{p^2 - m^2 + i\epsilon} \right]$$

For time evolution, this is the Laplace transform in p^0 :

$$\Delta_F(t) = \mathcal{L}^{-1} \left[\frac{1}{s^2 + w^2} \right]$$

222 So in the eyes of complex analysis, perturbative QFT is really nothing more than consider-
 223 ing meromorphic functions in momentum or energy space, taking their inverse Fourier or
 224 akin transform, and then analyzing the poles which give information on the energy spec-
 225 trum. Well, that is of course if we ignore the technical subtleties such as Wick rotation,
 226 dimensional regularization, etc.

227 9 Conformal Mappings

A conformal map of a domain is a function f which lies in a domain Ω which obeys

$$f : \Omega \subset \mathbb{C} \rightarrow \mathbb{C}$$

228 To be more specific, it's conformal if at a point it's holomorphic and $f'(z) \neq 0$.
 229 The implications for physics are numerous. Firstly this alone implies the conservation of
 230 angle and orientation. Furthermore, infinitesimally it's seen as a scaled rotation (conformal
 231 transformation literally). And most importantly locally it says that if f is conformal then
 232 we can understand its change of a domain Ω using analytic and local processes, all while
 233 being 1-1.

235 9.1 Mobius Transformations

A mobius transform is:

$$f(z) = \frac{az + b}{cz + d}$$

236 Where $ad - bc \neq 0$. One can prove that these are automorphisms of the Riemann sphere.
 237 This is important as we discussed earlier since the Riemann sphere is a "compactification"
 238 of the extended Complex plane. This $f(z)$ also preserves circles and lines, and the group
 239 of transformations it defines for any complex coefficients is known as $\text{PSL}(2, \mathbb{C})$. In that
 240 sense then any Mobius transformation is seen as a combination of translations, rotations,
 241 dilations, etc.

243 9.2 Cayley Transform

We want to consider a conformal map which maps the entire upper half of the complex plane e.g. $\mathbb{H} := \Im z > 0$, where $\Im = \text{Im}$, to the unit disc \mathbb{D} . This is given by the Cayley transform:

$$C(z) = \frac{z - i}{z + i}$$

If you do some testing, you'll see that:

$$C : \mathbb{H} \rightarrow \mathbb{D}$$

Since this is a bijection we can write the inverse, which is :

$$C^{-1}(w) = i \frac{1 + w}{1 - w}$$

244 This is SUPER useful in discussions of modular forms, scattering amplitudes in QFT and
 245 conformal field theories.

246 9.3 Automorphisms of \mathbb{D} — $\text{Aut}(\mathbb{D})$

The automorphism group of \mathbb{D} , which is just a unit disc e.g. the set of all z such that $|z| < 1$. This is given by:

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

247 Where again $\bar{\cdot}$ means complex conjugate, and $|a| < 1$. All of these transformations form
 248 the group $\text{PSU}(1, 1)$ which is isomorphic to $\text{PSL}(2, \mathbb{C})$. Just as $\text{SO}(3)$ is a double cover
 249 of $\text{SU}(2)$, $\text{PSL}(2, \mathbb{C})$ is a double cover of $\text{PSU}(1, 1)$. Again, just as in quantum mechanics
 250 $\text{SU}(2)$ is the transformation group of spinors, $\text{PSU}(1, 1)$ is conformal symmetries of 2D
 251 hyperbolic space, and the double cover is the full conformal group of a 2D CFT.

252 To summarize this, in 2D CFTs then $\text{PSU}(1, 1)$ is the global conformal group of the 2D
 253 CFTs, or the isometry group of 2D hyperbolic space. So in that sense, $\text{PSU}(1, 1)$ plays the
 254 same role as $\text{SU}(2)$! (This is hard to explain, sorry if it doesn't make all too much sense.
 255 Just consider that they are both double covers of the larger symmetry group)

256 9.4 Riemann Mapping Theorem

Every simply connected domain as a proper subset of \mathbb{C} : $\Omega \subsetneq \mathbb{C}$ is conformally equivalent to the unit disc \mathbb{D} . This is huge for 2D (1+1) dimensional conformal field theories. This means that up to a Mobius transform, every region has the same conformal type, and the mapping to \mathbb{D} is unique up to rotation.

This is related to the Schwarz-Christoffel Transform, which takes \mathbb{H} and plasters it conformally onto the interior of a polygon with precise angles at vertices:

Consider a set of real numbers $\omega_n \in \mathbb{R}$, with internal angles $\alpha_k \pi$. Then $f(z)$ is the transformation described above, and given as:

$$f(z) = A \int^z \prod_{k=1}^n (t - \omega_k)^{\alpha_k - 1} dt + B$$

257 So now we see singularities of the integrand are vertices of the polygon. I can't say it's
 258 hugely useful, but it can give boundary conditions and a precise formulation of a 2D CFT
 259 if it's defined over some polygon.

260 9.5 Elliptic Integrals and Conformal Factorization

Elliptical integrals are great. They show up all over physics. Classical mechanics, quantum mechanics (WKB), etc. Essentially any place where we have a source and an orbit, or analogy of orbit. There are different types and we will discuss them more later, but consider that elliptical integrals are defined as integrals which map rectangles to the upper half-plane \mathbb{H} . They take the form:

$$\int_0^x \frac{d\zeta}{[(1 - \zeta^2)(1 - k^2 \zeta^2)]^{1/2}}, \quad z \in \mathbb{H}$$

261 For $k \in (0, 1) \subset \mathbb{R}$. The branch of the denominator is in the upper half-plane and is taken
 262 so that the denominator is always positive as long as $\zeta \in (-1, 1) \subset \mathbb{R}$. More information
 263 on conformal maps by elliptical integrals is seen in Stein.

264 10 Elliptic Functions and Modular Invariance

265 10.1 The \wp function

A rather important topic of complex analysis with relevant results for HEP. To start, define a lattice $\Lambda \subset \mathbb{C}$:

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$$

Where we assume ω_1/ω_2 are linearly independent over \mathbb{R} . A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is elliptic if:

$$f(z + w) = f(z) \quad \forall w \in \Lambda$$

Some examples are the Weierstrass \wp -function:

$$\wp(z; \Lambda) = \frac{1}{z^2} + \sum_{w \in (\Lambda - \{0,0\})} \left(\frac{1}{(z - w)^2} - \frac{1}{w^2} \right)$$

It's meromorphic with poles at the lattice points. Even under $z \rightarrow -z$, and has periods $1, \tau$ where $\Lambda = \{n + m\tau : n, m \in \mathbb{Z}\}$. That sounds arbitrary at first glance, but consider that this builds out a lattice of spacing $1, \tau$ in the real and imaginary plane.

As a result of \wp being even, \wp' , it's derivative, is odd. (Consider that \wp is analytic and thus is a power series, and so differentiating it will bring down a power of each term in the series, making it odd where it was even, so when $z \rightarrow -z$, $\wp' \rightarrow -\wp'$).

Because \wp' also has periods $1, \tau$, then we can follow the prescription of chapter 6 where we built functions by their singularities and zeros to construct \wp' from \wp .

First note that \wp' is odd so $\wp'(-1/2) = -\wp'(1/2) = \wp'(\tau/2) = \wp'(\frac{1+\tau}{2}) = 0$. Because \wp' is the derivative, we just integrate to see that we have three constant values for \wp , called the half-periods: $\wp(1/2) = e_1$, $\wp(\tau/2) = e_2$, $\wp(\frac{1+\tau}{2}) = e_3$.

Now we can construct \wp' from \wp by considering that where \wp' is odd, \wp'^2 must be even, but \wp was as well. \wp'^2 also has its zeros at the above half-periods. By arguments from chapter 6, this means up to an exponential factor that:

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

The 4 and exponential factor are fixed by Taylor expanding the original definition of \wp near $z = 0$ and finding that $\wp = 1/z^2 + \dots$ and $\wp' = -2/z^3 + \dots$. Substituting those in, we get 4 for equivalence.

We can also write $\wp(z)$ in a power series of the E_k from chapter 6 above, where:

$$E_k = \sum_{w \in (\Lambda - \{0,0\})} \frac{1}{w^{k+2}}$$

Stein shows it on page 275-no point to do it here, that:

$$\wp(z) = 1/z^2 + \sum_{k=1}^{\infty} (2k+1)E_{2k+2}z^{2k}$$

Now in a mathematician's fashion, we use this to reverse engineer our above cubic polynomial for $(\wp')^2$ to get:

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

266 Where $g_2 = 60E_4$ and $g_3 = 140E_6$.

267

268 10.2 Modular Forms

The modular group:

$$\mathrm{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid (ad - bc) = 1 \right\}$$

Acts on elements $\tau \in \mathbb{H}$, which is just $\Im z > 0$ (the upper half-plane over \mathbb{C}) by:

$$g \in \mathrm{SL}(2, \mathbb{Z}) : \tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

From this slightly random group which seems entirely random and should, well at least in my eyes, not look like anything perhaps valueable at first, realize that functions which transform under $\mathrm{SL}(2, \mathbb{Z})$ are modular forms of weight k . They are holomorphic functions $f(\tau) \in \mathbb{H}$ and satisfy:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

Where specifically f must be holomorphic at $i\infty$ which just keeps f bounded as it tends to infinity.

Eisenstein series is an example of a modular form:

$$G_k(\tau) = \sum_{\Lambda - \{0,0\}} \frac{1}{(m + n\tau)^k}$$

For $k \geq 4$ and even.

Modular forms are so important that one could write an entire book on their usage in physics. Probably many books. And do many great years of research. String theory, elliptic genera, anomaly cancellation, supersymmetric QFTs, etc. All of these, well most, arise from modular invariance of the underlying theory.

An interesting example of an almost modular invariant function is the Dedekind eta function:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i \tau}$$

We'll talk about it's anomaly in a minute

Another interesting example is the modular invariance of the string path integral:

$$Z(\tau) = \mathrm{Tr}_{\mathcal{H}}(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24})$$

269 And it's invariance under $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$.

270 Speaking of is the Jacobi theta function. We'll also talk about that in a minute.

271 10.3 Dedekind η function anomaly

The function, above, is defined for $\tau \in \mathbb{H}$ and appears everywhere in HEP. We seek to test if it's invariant under $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$.

Plugging directly in, we see that

$$\eta(\tau + 1) = e^{\pi i/12} \eta(\tau)$$

This means we have a phase anomaly, which transforms by 24th roots of unity. Furthermore under $\tau \rightarrow -1/\tau$ goes as:

$$\eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau)$$

So we again pick up a complex phase factor which just means η transforms like a modular form of weight $k = 1/2$, up to the phase anomaly found before. Overall this means we can write:

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon(a, b, c, d)(c\tau + d)^{1/2} \eta(\tau)$$

Not modular invariant-but close. Now in physics we can define different partition functions, which appear from physical motivation, as $Z \propto 1/(\eta(\tau)^{24} \bar{\eta}(\tau)^{24})$ etc. Issue is that we need Z to be modular invariant, not invariant up to a phase and weight $1/2$. This amounts to making sure the phase anomaly gets taken care of.

Let's look at an example. Taking the above Z as an example consider that it isn't quite modular invariant. Issue is we get that same factor of $(-i\tau)^{-12}$ for each η . This means that we need a factor of τ out front which similar transforms to cancel this. That means our true modular invariant, anomaly-free partition function must go as:

$$Z = \frac{1}{\tau_n^{12}} \frac{1}{(\eta(\tau)^{24} \bar{\eta}(\tau)^{24})}$$

272 Then finally $Z(-1/\tau, -1/\bar{\tau}) = Z(\tau, \bar{\tau})$.

273 11 The Jacobi Theta Function and Generating Functions

274 11.1 Theta Function

The Jacobi theta function is like a modular form, but not quite. It is given by the series:

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

Which converges for all $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ ($\Im z > 0$). What's amazing about this function is that as a function of z , it's periodic with period 1 and quasi-period τ . As a function of τ its like a modular form.

The Jacobi theta function also reduces to two other important functions:

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \quad \tau \in \mathbb{H}$$

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, \quad t > 0$$

275 So $\theta(t) = \Theta(0, \tau)$ and $\vartheta(t) = \theta(it)$, well for $t > 0$ at least. Now this function has very
 276 many important properties as a result of this dual nature. They are summarized in Stein
 277 on page 284.

278 11.2 Triple-Product Function

Those properties motivate another amazing function called the product function, sometimes called the triple-product function $\Pi(z|\tau)$, written as:

$$\Pi(z|\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1}e^{2\pi iz})(1 + q^{2n-1}e^{-2\pi iz})$$

Where $q = e^{i\pi\tau}$. This triple-product function, as a result of construction has the same properties as $\Theta(z|\tau)$ despite its completely different appearance.

But. It turns out that:

$$\Theta(z|\tau) = \Pi(z|\tau)$$

279 for all $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$! This is so amazing since it tells us that a Fourier series which
 280 is holomorphic, doubly periodic (elliptical-well quasi-elliptical) is somehow equal to Weier-
 281 strass product which solely reflects the entire function's zeros and convergence informa-
 282 tion/structure.

283 This is an important relation/fact in string theory when finding functions which describe
 284 strings.

285 11.3 Generating Functions

286 It'd be a long shot to talk about this in a physics context. Perhaps for CFT bootstrap or
 287 other heavy mathematical topics, but to my knowledge not really something worth writing
 288 about for physics. Mathematically it is fascinating; pioneered by Euler (so you know it's
 289 good).

290 12 Summary to Complex Analysis

291 As a summary we have a few important final things to iterate and make concrete in our
 292 heads.

293 1) A holomorphic function is a complex-differentiable function, e.g. $f \in C^1(\Omega)$ for
 294 some domain Ω . These functions are astonishing. Their behavior, highlighted by Cauchy's
 295 theorem demonstrates that any holomorphic function is automatically smooth e.g. $f \in$
 296 $C^\infty(\Omega)$.

297 2) Reasonable functions—that is, ones without essential poles (very ugly poles-no con-
 298 verging limit from left and right, etc)—have their entire behavior governed by singularities
 299 and zeros. There are three types-removable, poles, and essential. The residue theorem
 300 gives us everything we need to reconstruct meromorphic functions (holomorphic except at
 301 finite number of removable or pole singularities) by reverse engineering their structure.

302 3) Special functions are awesome! And we can often analytically continue them by
 303 their properties. Gamma function, Riemann zeta, Weierstrass product functions, etc. We
 304 classify these based on their order and type, which gives us information specifically on their
 305 asymptotic growth, which is essential for guaranteeing convergence.

306 4) Conformal mappings are very useful. By the Riemann mapping theorem we can
 307 take any open domain $\Omega \subset \mathbb{C}$ and conformally map it onto the unit disc \mathbb{D} by a Mobius
 308 transform. The automorphism group of \mathbb{D} and its properties is important and has a similar
 309 double cover to that of spinors by $SU(2)$ being doubly covered by $SO(3)$.

310 5) Modular forms are awesome. Modular invariance is crucial in physics. Complex
 311 analysis tells us all about these with special functions like the Dedking eta function and
 312 Jacobi theta function.

313 6) Fourier, Laplace and Mellin transforms are very useful. Fourier transforms reveal
 314 dual nature of physical theories in their position and momentum space. Laplace transforms
 315 encode time evolution and as a result are important in understanding decay, boundary
 316 conditions and advanced/retarded boundary condition setups. Mellin transforms are very
 317 useful due to specific conformal properties, since it encodes scattering amplitudes of celestial
 318 holography (and much more).

319 7) Elliptic functions are useful. They are doubly periodic over the complex lattice Λ ,
 320 they are meromorphic on Λ/\mathbb{C} . This wasn't mentioned before, but that's a complex tori
 321 by the real properties of a torus (doubly periodic), and much more. The Weierstrass \wp
 322 function and Eisenstein series are useful in understanding modular forms. They appear in
 323 physics a lot!

324 12.1 Final Words

325 Complex analysis is much more than just mathematics (unlike some of real analysis). It
 326 truly encodes much of the structure physicist need to understand quantum theory. More
 327 than just quantum theory, due to the highly rigid structure of complex differentiable (holo-
 328 morphic, meromorphic, etc) functions over the complex plane \mathbb{C} , it is extremely useful to
 329 know complex analysis. You get a grasp on why complex functions are useful (analytic
 330 on their domain), the symmetries which appear, and the algebraic & geometric structures
 331 which appear from the mathematics.

332 Complex analysis is the bedrock of modern high energy physics.

333

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 336 & Rami Shakarchi