Hybrid Modelling and Simulation of an Inverted Pendulum using Discrete State Space Analysis

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***Abstract*—Due to the nonlinear behavior of the inverted pendulum and its inherently unstable nature, its modelling turns out to be nontrivial and complex, specially when working with it as a nonlinear system [5], [6], [7]. As a result, the hybrid approach became a good alternative given the possibility of modelling complex systems by dividing them into several simple partitioning systems. Some of the hybrid solutions proposed include: representation in terms of the pendulum required energy [3], dividing the system according to the direction of its movement and focusing on the "suing up" movement [4]. In this paper, the proposed solution suggests dividing the system into 4 regions of behavior and, consequently, of control. The division is based on the small angle approximation of the sine and cosine for 0, 90, 180 and 270 degrees. By using Hybrid modelling and Discrete State Space Analysis, it is shown that it is possible to decompose the Inverted Pendulum in four main regions where linearity exists, facilitating the selection of the control strategy. Finally, by using computer simulation and a physical model implementation, it is shown that said approach can effectively balance an inverted pendulum.**

Keywords*—Inverted Pendulum, Hybrid Control, Discrete Time Hybrid Automata.

I. Introduction

The dynamics of the pendulum are a very well studied phenomena; its has been thoroughly studied and the nonlinear nature of its behavior has been already defined. The system known as an inverted pendulum refers to a two dimensional mechanical system where the pendulums axis of rotation is fastened to a cart that can be moved along an horizontal axis — the pendulum rotates freely along its axis.

Some direct approaches attack this problem by accepting the systems non-linearity and implement a complex type of control — Fuzzy logic control, neuronal networks control, nonlinear control, among others [5], [6], [7]. These methods result in complex algorithms difficult to implement and require high computational power. Hybrid control, on the other hand, is a stratified way of approaching systems that combine two main types of events: discrete and continuous. Hybrid control has shown to have a significant impact when studying complex systems by making them easier to understand and, therefore, easier to control [9].

The Discrete State Space Analysis, is an approach to hybrid system modelling that divides a whole process into several states of operation called 'Discrete States'. Along every discrete state [2], there's 1) a vector space that represents the continuous dynamic of each one of them, 2) a set of initial values for the continuous and discrete variables that the system involve, 3) a set that includes all of the possible transitions between states, 4) a set of conditions for each transition, 5) a set of maps for variables reallocation and 6) a set of restricting conditions that define the circumstances under which a mode of operation should endure.

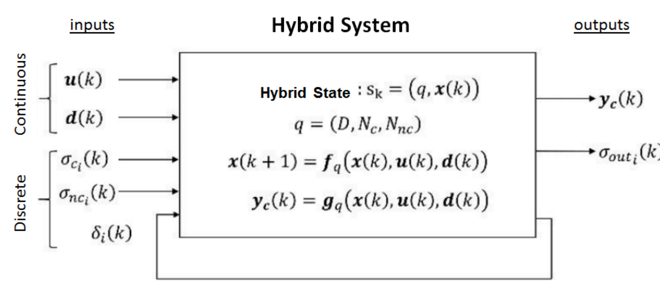

Fig. 1: Hybrid system main component.

The Hybrid systems are often represented as a diagram that contains all the influences in a certain process [1]; these diagrams include all the operating modes of a system placed in an axis that represents the type of event related to the transition to them. Fig. 2 shows the axis where the different states are placed according to the type of transition that leads to their initialization.

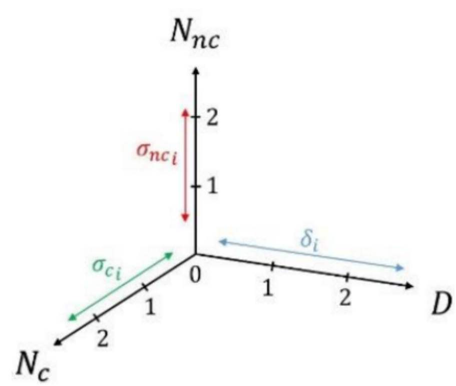

Fig. 2: Discrete State Space of a hybrid system.

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The axis shown in Fig. 2 is defined as follows:

1) Dimension D: Changes in this axis depend on δ_i which denote the discrete internal events of the system; any defined internal discrete change represents a state transition in this axis.

2) Dimension N_c: All changes in this axis are due to δ_{ci} (controllable discrete events). These events are controlled and allow us to modify the continuous dynamic of the system by transitioning to a parallel state.

3) Dimension N_{nc}: This state is associated with δ_{nci}, the discrete non-controllable events. these non controllable events include the effects of perturbations in the system.

A pendulum, in it's most simple form, consists of a rigid mass-less rod fastened to a hinge on one of its ends and to a certain weight on the other.

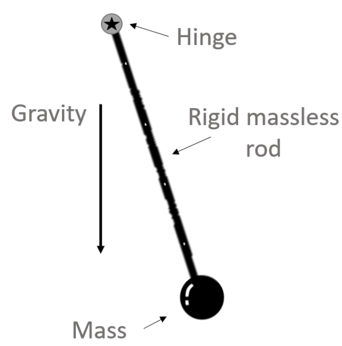

Fig. 3: Pendulum.

The movement of the pendulum is given by the next differential equation:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{1}$$

Where g is the gravity force, l represents the length of of the rod and θ is the angle of rotation. We can imply from (1) that the system has two states of equilibrium, one at θ = 0 and the other at $\theta = \pi$, being the first one inherently unstable — the error tend to grow exponentially.

The Small angle linearization is a zero order linearization that is based on a Taylor Series approximation for small angles. As a result, the behavior of the pendulum can be linearized in four regions and it's represented as follows:

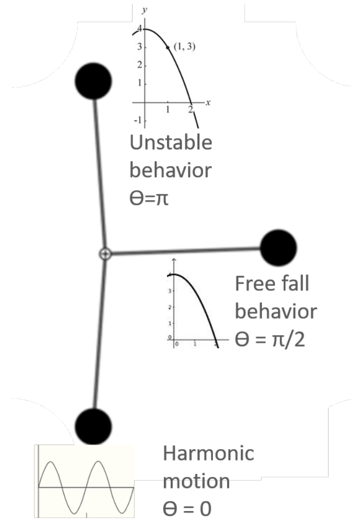

Fig. 4: Linearized behavior of a pendulum.

II. Inverted Pendulum Modeling

Knowing the equation that describes the dynamics of a single pendulum, the behavior of the inverted pendulum on a cart can be obtained considering it's acceleration caused by the single-axis movement. From the pendulum and the relative accelerations equations [2], the following model is obtained:

$$\ddot{\theta} = \frac{mgl\sin(\theta + D) - mal\cos(\theta + D)}{J} \tag{2}$$

θ is the angle of the pendulum (rad).
J is the moment of inertia of the pendulum.
m is the mass of the pendulum.
g is the acceleration due to gravity.
l is the length of the pendulum.
a is the acceleration of the cart.
D is the effect of the disturbances.

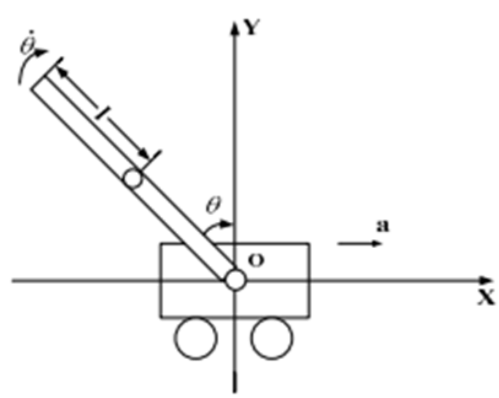

Fig. 5: Pendulum on a cart.

From (2) we can observe that our equation depends mainly on the acceleration of the cart, the angle of the pendulum with respect to the hinge and the effect of disturbances. These three variables will affect significantly how the pendulum behaves and how it shall be balanced in the upward position — The equation remains as a nonlinear one. The acceleration of the cart is limited to affect the horizontal axis of the cart and it is represented ignoring the effects of inertia of both the pendulum and the cart which later will be shown to be insignificant for the control strategy. Moreover, the disturbances include any outside effect caused by an external force, it being the hardware of the system, the environment or the user himself. An other important factor is that the road that represents the pendulum has an even mass, which means that the center of mass is located exactly at the half of its length.

A. Pendulum Linearization

Considering the areas where the pendulum behaves as a linear system (TABLE I), the system was partitioned in four state spaces: 1) the first one is around the upward position, the unstable equilibrium where the control system shall balance the pendulum for it to be inverted; 2) and 3) are the intermediate areas where the movement of the pendulum should be adjusted to reach the first state or to suing with more energy by undergoing region 4; 4) the swinging up region where the pendulum shall generate enough energy to move upwards. A scheme of the selected discrete steps is shown in Fig.6.

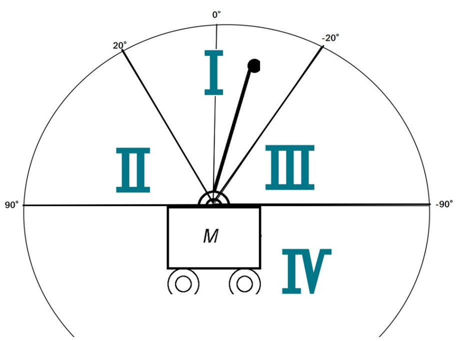

Fig. 6: Schematized Discrete Spaces.

It is important to notice that the regions where linearization is plausible are around $\pm 20^o$ ($\pm.35rad$). The regions selected for the discrete state spaces are clearly grater than the above mentioned range, therefore the control strategy shall be robust enough to be functional even when the pendulum's position oscillates far from the linear region. It can be observed from Fig. 6 that the ranges of the state spaces are bigger as the position of the pendulum approaches the stable regions and how they become thither as the pendulum gets closer to its unstable equilibrium — the desired position.

The reference point is selected to be the upward position — 0 rad; and the downward position to be π rad similarly.

The first region is the most sensitive one due to it's unstable nature, therefore, this region is strictly limited to the angles where the behavior can be linearized— from 20^o (.35rad approximately) to -20^o (-.35rad approximately). Figure 7 shows the linear behavior of the trigonometric function sine around 0 rad.

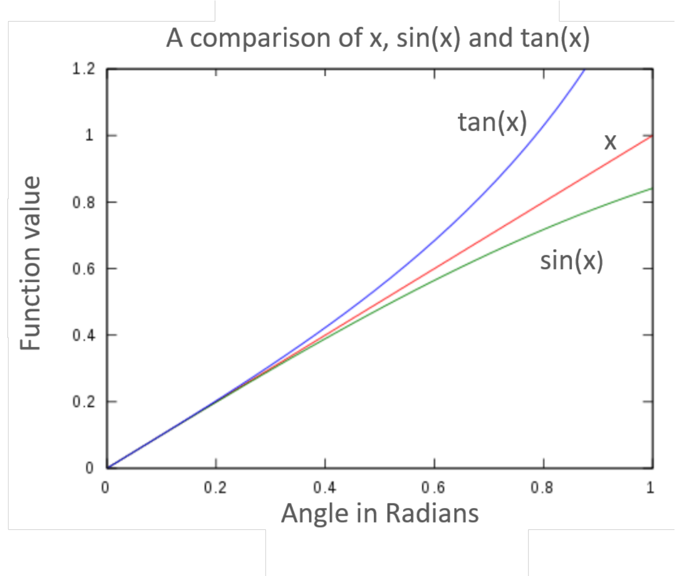

Fig. 7: Small angle behavior of sine(x) and tan(x).

Based on the Taylor series the function sine and cosine can be approximated to linear and quadratic behaviors around multiples of 90^o ($\frac{\pi}{2}$). TABLE I summarizes these approximations for Sine and Cosine.

$\theta \pm .35$	Sine(θ)	Cosine(θ)
0	θ	1-$\frac{\pi}{2}^2$
$\frac{\pi}{2}$	1-$\frac{(\theta-\frac{\pi}{2})^2}{2}$	$\frac{\pi}{2}-\theta$
π	$\pi-\theta$	-1+$\frac{(\theta-\pi)^2}{2}$
$\frac{3\pi}{2}$	-1+$\frac{(\theta-\frac{3\pi}{2})^2}{2}$	$-\frac{3\pi}{2}+\theta$

TABLE I: Linearization of sine and cosine.

Being a symmetrical system, the discrete spaces 2 and 3 follow the same logic. In region 2, at exactly $\frac{\pi}{4}$rad it is called the uncontrollable region, meaning that no matter how big the control action is, it can not affect the system whatsoever, therefore, region 2 was selected to start above this point.
Finally, the fourth region was given a wider range for it not being a critical region; the pendulum just needs to swing up anyhow.

III. States linearization

1) Fourth State: Firstly, the pendulum's position will be at π radians — it's stable equilibrium at region four. Thenceforth, with the linerization of the sine and cosine shown in Table I, we get the following behavior:

$$\ddot{\theta} = \frac{mgl(\pi - \theta + D) - mal(-1 + \frac{(\theta + D - \pi)^2}{2})}{J} \tag{3}$$

2) Second and third Discrete States: Consequently, after linearizing for the area near the horizontal position of the pendulum we get:

$$\ddot{\theta} = \frac{mgl(1 - \frac{(\theta + D - \frac{\pi}{2})^2}{2}) - mal(\frac{\pi}{2} - \theta + D)}{J} \tag{4}$$

$$\ddot{\theta} = \frac{mgl(-1 + \frac{(\theta + D - \frac{3\pi}{2})^2}{2}) - mal(-\frac{3\pi}{2} + \theta + D)}{J} \tag{5}$$

Equation (4) shows the dynamic at the second state (θ=$\frac{\pi}{2}$) and it is shown in equation (5) how the pendulum reacts in the thirs state (θ=$\frac{3\pi}{2}$).

3) First Discrete State: Finally, the first state is linearized as follows:

$$\ddot{\theta} = \frac{mgl(\theta + D) - mal(1 - \frac{\pi^2}{2})}{J} \tag{6}$$

As a result, we get the behavioral discrete state space analysis shown in Fig. 8.

Note: the regions where the linearization is accurate where manipulated in order to organize the system for the control strategy to be selected in the next section of this paper.

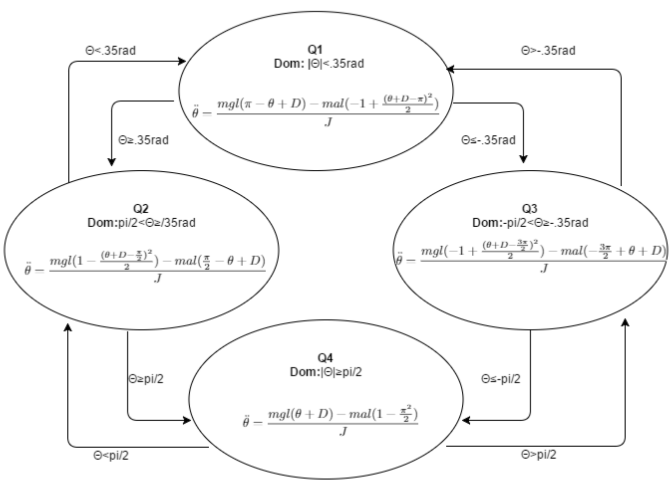

Fig. 8: Discrete State Spaces Diagram.

When implementing the system it is important to pay attention to the angle restriction. In this paper, the maximum angle permitted its 180^o degrees (πrad) after crossing this threshold, the angle is relocated to the minimum selected value: -179^o degrees ($\approx \pi$rad). if the systems restrictions are not well defined, the system may enter the so-called 'Zeno Behavior', which denotes the infinite loop caused when there occurs an infinite amount of jumps between states, or incur in other stability problem [8].

IV. Discrete State Space Analysis

Once the Process has been defined as the combination of linear and nonlinear behaviors in four main regions, A formal definition of the hybrid state space model can be established.

A. Process Description

The system was defined for convenience into four different stages that involve both discrete and continuous events along four different states that combine a linear region around a certain position of the pendulum and a nonlinear behavior at some intermediate points.

The controlled variable of the system is θ which represents the angle of the pendulum with respect to the hinge. It is directly dependent of the acceleration of the cart and the disturbances effects, both of them are also variables to consider in the system and hence, in the hybrid discrete state space analysis.

θ will vary according to a differential equation (2) in a nonlinear proportion according to the horizontal movement of the cart. Furthermore, it shall be directly affected by the perturbations of the environment, giving us immediate effects on the angle. In other words, the perturbations can affect the angle in a discrete manner with different magnitudes depending on the perturbation force.

All perturbations will be considered to be non-controllable. The internal discrete events of the systems (δ_i), as defined in section II, are characterized by the changes in the angle position. In addition to that, the initial position of θ was defined to be 180^o degrees or $\frac{3\pi}{2}$rad which means that the pendulum shall start in the fourth state.

B. Discrete State Space Definition

With this assumption, we define θ as the only variable for the discrete transitions and the only condition for a jump from one state to another. The transitions between states are then defined with respect to the changes in θ and shall be along the D axis due to the above made definition. Seemingly, the jumps in D will only involve whole numbers, in this case, from 0 to 3. There exist a special case with the second and third state (D= 1, 2); the system and its jump from the first to the fourth state will only undergo the second or the third state as intermediate states, never both. Hence the second and third state shall be called 'Parallel States' meaning that the system can only go throw one of them when before reaching the first or the last state.

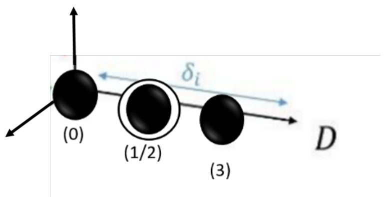

Fig. 9: Discrete State Space.

Figure 9 shows the jumps of state along the D axis with respect to the internal discrete events δ_i which is defined thusly.

1) Fourth State Transitions:

$$\theta \geq \frac{\pi}{2} \rightarrow \delta_i = 3 \rightarrow D = 3 \tag{7}$$

$$\theta \leq -\frac{\pi}{2} \rightarrow \delta_i = 3 \rightarrow D = 3 \tag{8}$$

2) Third State Transitions:

$$\theta > -\frac{\pi}{2} \rightarrow \delta_i = 2 \rightarrow D = 2 \tag{9}$$

$$\theta \leq -.35rad \rightarrow \delta_i = 2 \rightarrow D = 2 \tag{10}$$

3) Second State Transitions:

$$\theta < \frac{\pi}{2} \rightarrow \delta_i = 1 \rightarrow D = 1 \tag{11}$$

$$\theta \geq .35rad \rightarrow \delta_i = 1 \rightarrow D = 1 \tag{12}$$

4) First State Transitions:

$$\theta > .35rad \rightarrow \delta_i = 0 \rightarrow D = 0 \tag{13}$$

$$\theta > -.35rad \rightarrow \delta_i = 0 \rightarrow D = 0 \tag{14}$$

These transitions can be schematized in the D axis as shown in Figure 10, where δ_i=1 and δ_i=2 are two parallel states over the same axis and the same 'position'; both of them depend on the same discrete event, yet only one of them can be considered at a time and there exists no direct transition from one to the other.

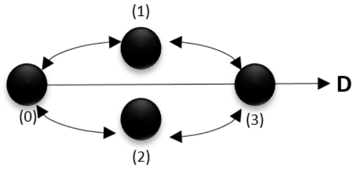

Fig. 10: Discrete State Space Jump Transitions.

C. Perturbations

There are several factors that can provoke disturbances in the inverted pendulum; basically, any fiscal disruption caused by an external force is considered a perturbation. In this paper, a disturbance is any kind of instantaneous effect on the inverted pendulum directly affecting the angle of the rod with respect to the hinge. This discrete force corresponds to the 'D' in (2) and is attributed to an external force from the environment.

Even though the disturbances are a discrete events on the system, they might not have any effect in the dynamics of the system whatsoever — they obviously affect the angle of the pendulum but the State transition is not conditioned by the disturbance. Nevertheless it is important to consider its effect for the cases when it can actually change the dynamics of the system.
knowing that the jumps from state are only related to the variable θ, the disturbances shall only be significant when they are big enough to force the angle into the transition threshold otherwise, they will affect the angle but they should not be strong enough to force θ to cross any State Limits. Hence the Disturbance effect is represented with δ_d.

$$\delta_i \rightarrow \theta \rightarrow \delta_d \tag{15}$$

As shown in equation (15) there is a direct relationship between θ and the disturbances, an example of this phenomena is the effect of someone hitting the pendulum when it is being balances; this hit represents an immediate change of the angle and depending on the force of the hit, the pendulum might be able to apply a counter force in order to remain the balance of the pendulum or it will simply fall towards the downward position as the disturbance is to strong for the control to counteract it. As a result, the disturbances will only represent a jump of state if its magnitude is so that θ reaches any of the transitions (7), (8), (9), (10), (11), (12), (13) or (14).

Notice how in equation (2), and the other differential equations, the Disturbance effect was represented, for convenience, as a direct sum to the angle, when in reality, the angle is a function of the disturbances.

V. Hybrid Controller Design

In sections III and IV the complete system was partitioned in four different discrete states separated by the change of a discrete eve7 nt generated by the variable theta reaching certain threshold. in this section we now are given the task to generate a control strategy to finally full fill the purpose of an inverted pendulum, which is to balance it in the upward unstable position.

The four divisions of the system represent the divisions where the system behaves in distinct continuous dynamics. to create the control law we need to consider that the only control variable is the acceleration of the cart, so with the changes given to this acceleration, the variable θ (controlled variable) shall end with a value of 0. The behavior of theta with respect to the acceleration of the cart was described in section III equations (3) for the fourth state, (4) and (5) for the second and third state respectively and (6) for the fifth state. As a result, the proposed control strategy shall also be partitioned in a Discrete State Space to match the different behaviors of the system, in other words, four control strategies will be selected, one for each discrete state space of the system, to match the changes in the continuous dynamic of the plant.

A. Fourth State

Naturally, the first controlled stage shall be the stable equilibrium — when the pendulum is vertically down. The first assumption made at this stage is that the pendulums only task is to get out of this stage, which means that the discrete events depending on θ shall be so that the discrete internal events δ_d are forces to any of the transitions as depicted in equation (9) (changing into the third state by changing the value of δ_d to 2 and creating a jump to D=2) and equation (11) (a jump to the third state across D to δ_d =1 and D=1).

With the above mentioned objective in mind, and considering the oscillating dynamics of the pendulum around 180^o (π) as shown in Figure 4 and represented in the linearization of equation (3), the proposed solution involves a control strategy with the purpose of incrementing the peak to peak magnitude of the oscillations of the pendulum exponentially until any of the transitions (9) or (11) are reached.

Considering the preceding assumptions, the selected control law consists on a control signal based of an unstable system with positive conjugate poles that match the frequency of the pendulum's oscillations. The period of a pendulum is described by the equation below:

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{16}$$

T is the period of the pendulum in seconds.
g is the acceleration due to gravity.
L is the length of the rod of the pendulum.

Consequently, from equation (16) we can imply the equation for the angular frequency of the system as follows,

$$w = \sqrt{\frac{g}{L}} \tag{17}$$

w is the angular frequency of the pendulum in rad/sec.

It can be observed from equation (17) that the angular frequency of the pendulum is directly dependant on the force of the gravity and the length of the rod of the pendulum, this will be considered later on when the control strategy is chosen.

In order to generate an exponential increment of the oscillations of the pendulum, we observe in equation (3 that the linearized dynamic at this stage can be divided in to tow different equations that are summed together to get the full dynamic equation corresponding to the angular acceleration, and only one of them depends on the acceleration of the car denotes by a. this term multiplied by the acceleration is a combination of constants and the linearized behavior of the cosine as a quadratic wave.

$$\frac{mgl(\pi - \theta + D)}{J} \tag{18}$$

$$\frac{-mal(-1 + \frac{(\theta+D-\pi)^2}{2})}{J} \tag{19}$$

Equation (18) represents the inherently oscillating behavior of the pendulum and its angular frequency is described by (17).

The second part of the sum that equals the value of the angular acceleration is where our control variable takes place (19). Here, the control action shall create an exponential increase in the natural oscillations denoted by equation (18). As stated in section III this linearization comes from the function cosine which is 90^o degrees out of phase with respect to the sine represented in (18). As a result, the control strategy was selected to be a second order transfer function witch's frequency is controlled to full fill the requirements defined by the above mentioned assumptions.

$$\frac{Y(s)}{R(s)} = \frac{W_n^2}{s^2 + 2\zeta W_n s + W_n^2} \tag{20}$$

ζ is the damping ratio.
W_n is the natural frequency.

From equation (20) we shall pay special attention to the selection of W_n, the natural frequency of the transfer function, in order to match the established frequency requirements of the system. In addition to that, the damping ratio must also be considered because according to this value defines the poles placement an therefor the behavior of the system. For an exponential increase of the oscillating maneuvers, the poles shall be in the right side of the pole diagram and they shall be complex conjugate. The conditions are represented in equations causally,

$$W_n = \sqrt{\frac{g}{L}} + 2\pi \tag{21}$$

$$2\zeta W_n < 0 \tag{22}$$

The condition marked by (21) matches the frequency of our second order control transfer function with the pendulums innately oscillating behavior and (22) serves as a guide condition full filled by unstable complex conjugated poles yet the reciprocal of said condition is not always true (an argument in the range of (22) may not be a complex conjugate set of poles) therefore, special attention must be payed to the selected values and the generated poles.

Accordingly, the frequency selected for said poles correspond to $\sqrt{\frac{g}{L}} + 2\pi$. consequently the poles can be graph as follows.

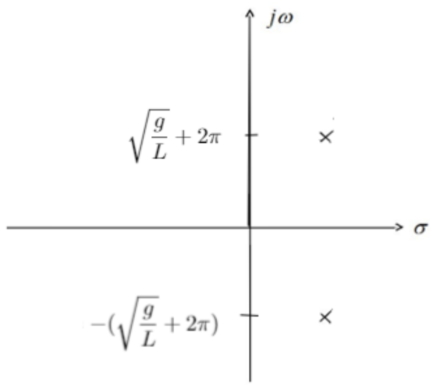

Fig. 11: Imaginary poles placement.

The value of the multiplication ζW_n was selected to be -1 so that 1 is the real value in the poles diagram around which the complex conjugate poles are placed, this value defines the growth rate of the exponential oscillating wave generated and may be changed if necessary. In like fashion, in this paper it will simply remain as one to simplify the calculations.

Likewise, the controller is represented by the following continuous transfer function:

$$G(s) = \frac{1}{s^2 - 2s + (\sqrt{\frac{g}{L}} + 2\pi)^2} \tag{23}$$

Notice how the numerator of the transfer function (23) is simply left as 1 instead of the value corresponding to W_n^2 as (20) commands. this value change simply implies a proportional gain of $\frac{1}{W_n^2}$ applied to (20).

Using Euler's forward method, the digital controller transfer function,

$$H(z) = G_c(s)|_{s=\frac{z-1}{T}} \tag{24}$$

$$= \frac{1}{(\frac{z-1}{T})^2 - 2\frac{z-1}{T} + (\sqrt{\frac{g}{L}} + 2\pi)^2} \tag{25}$$

$$= \frac{T^2}{z^2 - 2z + 1 - T(2z-2) + (\sqrt{\frac{g}{L}} + 2\pi)^2} \tag{26}$$

B. Second and third State

For the second and third stage the control strategy was selected in order to facilitate the movement of the pendulum to the first region. Knowing this objective and considering the behavior of the system around this area (4), (5), an hyperbolic tangent was selected in order to ease off the controllers actuation signal's change — the actuation must change signs to compensate the changing dynamics.

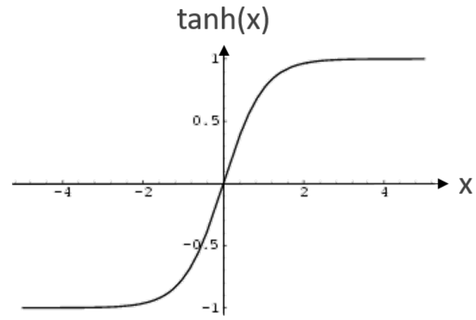

Fig. 13: Hyperbolic Tangent.

As shown in Figure 12, this function has two points of operation, one on the left side of the Y axis with a value of minus one, and the second one on the right side of the Y axis with a value of one. this two values of operation are separated by a diagonal transition in between near the Y axis when the value of x approaches 0, how ever small, this transition separates the hyperbolic function from a step from minus 1 to 1 at x=0.

We shall take advantage of this property of the hyperbolic tangent and use it as a weighing of the control variable, the acceleration. According to equation (4) in the case of the second state, we have that the behavior of the systems in terms of the acceleration is defined by the second part of the sum as follows:

$$\frac{-mal(\frac{\pi}{2} - \theta + D)}{J} \tag{27}$$

It is obvious from equation (27) that there exists a sign change when ever theta crosses the value of $\frac{\pi}{2}$ so the actuation of the acceleration is inverted at this point. in the other hand, we know that the first part of the equation remains with a positive sign due to the quadratic behavior of the sine function at this point of the process.

$$\frac{mgl(1 - \frac{(\theta + D - \frac{\pi}{2})^2}{2})}{J} \tag{28}$$

Seemingly, to keep the sign congruence at this point and to smoothen the transition, we apply the hyperbolic tangent, yet, it has to be modified in order for it to transition in the desired threshold. The hyperbolic tangent shall be centered at $\frac{\pi}{2}$, hence we shift the phase of the function as follows:

$$(-Tanh(\theta - \frac{\pi}{2})) * K \tag{29}$$

K is a proportional gain.

In the same way, for the third state it can be appreciated from equation (5) that the continuous dynamics at this stage are a sum of two linearized trigonometric functions as well.

$$\frac{mgl(-1 + \frac{(\theta + D - \frac{3\pi}{2})^2}{2})}{J} \tag{30}$$

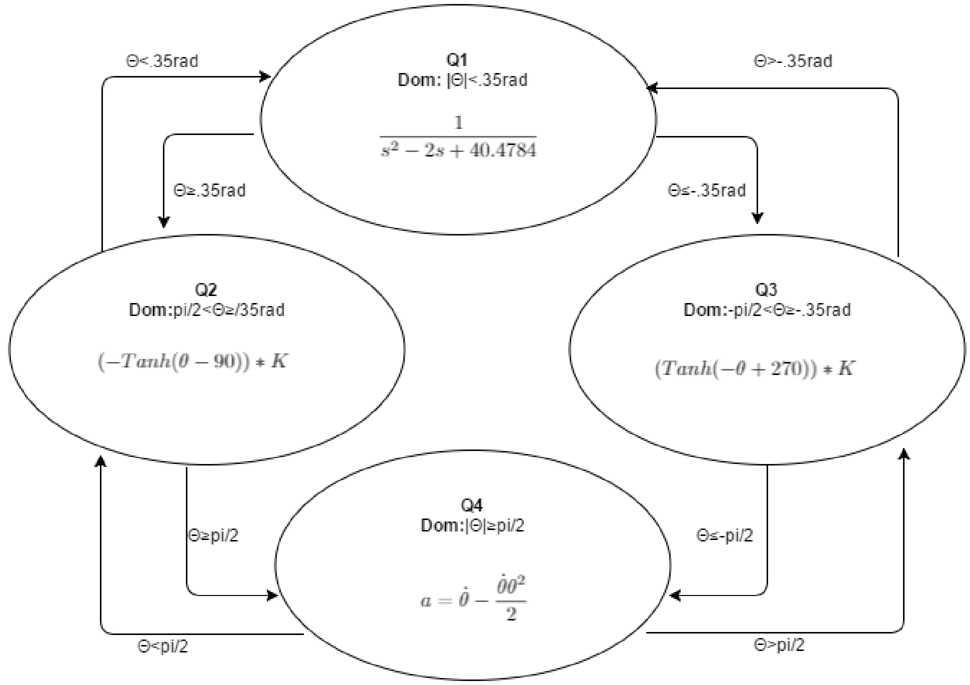

Fig. 12: Control Discrete State Space Diagram.

$$\frac{-mal(-\frac{3\pi}{2}+\theta+D)}{J} \tag{31}$$

Equation (31) tells us that the turning point at this stage occurs when the value of theta equals $\frac{3\pi}{2}$. As a result it is implied that the hyperbolic tangent for this case shall be shifted $\frac{3\pi}{2}$ rad out of phase:

$$(Tanh(-\theta+\frac{3\pi}{2}))*K \tag{32}$$

K is a proportional gain.

C. First State

Finally the first area's control strategy is based on the energy equation of the pendulum [3], by taking $\theta=0$ as the reference point for the energy transmission of the system. the energy is described by the following equation:

$$E=\frac{J\dot{\theta}^2}{2}+mgl\cos(\theta-1) \tag{33}$$

By differentiating equation (33) wit respect to the time, the change in the energy can be expressed as follows:

$$\dot{E}=J\dot{\theta}\ddot{\theta}-mgl\sin(\theta+D)\dot{\theta} \tag{34}$$

Substituting (2) into (34) the following is obtained:

$$\dot{E}=-mal\dot{\theta}\cos(\theta+D) \tag{35}$$

From equation (35), it can be assumed that the energy of the pendulum depends purely on the cosine and the angle derivative, therefore, following the linearized behavior of the cosine at 0^o shown in TABLE I,the control strategy for this state is chosen as follows:

$$a=\dot{\theta}-\frac{\dot{\theta}\theta^2}{2} \tag{36}$$

Consequently, the discrete state space diagram for the control system is constructed:

Where the transitions are the same as in the behavioral discrete state space shown in (7), (8), (9), (10), (11), (12), (13), (14).

VI. Simulation Results

The simulation was made in MATLAB using the state flow chart package to assemble the hybrid model and the "simscape" package to simulate the inverted pendulum:

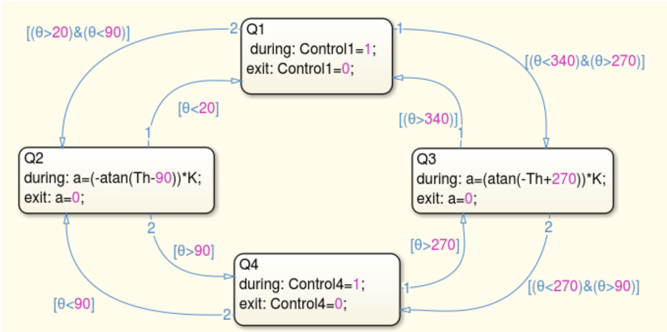

Fig. 14: State Flow Chart (Simulink).

Figure 14 shows the programming of the discrete state space diagram for the control strategy with it's different states and their corresponding transitions. The control law was not directly expressed in this diagram but rather an activating signal that will later in the code tell the corresponding controller when it is required. State flow chart can be used as a Simulink block with inputs and outputs to later interact with the further code in Simulink. the inputs of the block correspond to the angle θ which represents the conditioning variable for the state jumps and the outputs are the different activating signals for the four different control strategies.

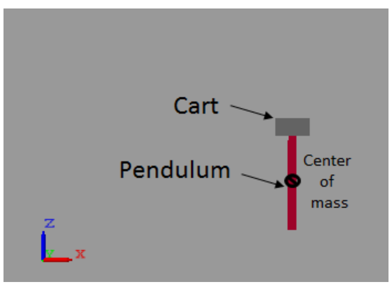

Fig. 15: Inverted Pendulum Model (Simscape).

The model of the inverted pendulum was directly used from the package 'Inverted Pendulum Example' of MATLAB, and it was used as a block in Simulink whose entrances correspond to the acceleration of the cart and the outputs where the angular position of the pendulum and the actual position of the cart in the horizontal axis. It is important to know the overall position of the cart later on for the real life implementation, due to the fact that there will be a limited space j which the cart can operate and going out of this space can be translated to damages in the system's infrastructure.

In order to restrict the movement of the cart, a PD cascade controller was places as a slave to have a certain control action on the overall position of the cart. This controller has as reference the point where the pendulum is exactly in the middle of the rail, meaning that, even when the pendulum has been successfully balances, the control action shall take it to the middle of the rail maintaining its balance.

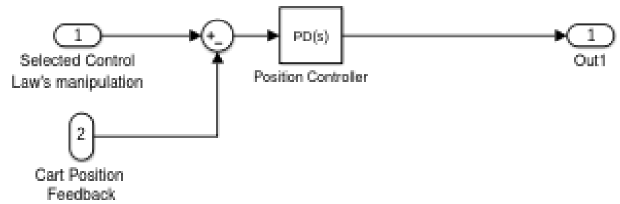

Fig. 16: Slave Cascade Controller.

The inertia of the simulated model was considered for a 1kg pendulum. the resulting Forces are as follows:

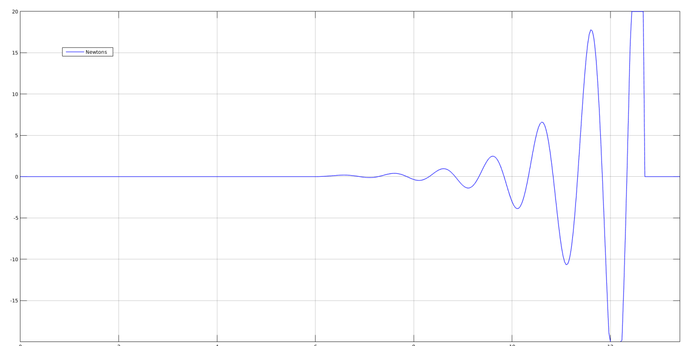

Fig. 17: Fourth State Forces.

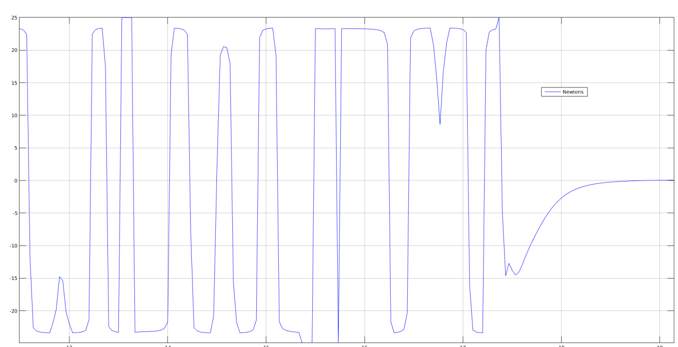

Fig. 18: First and Second State.

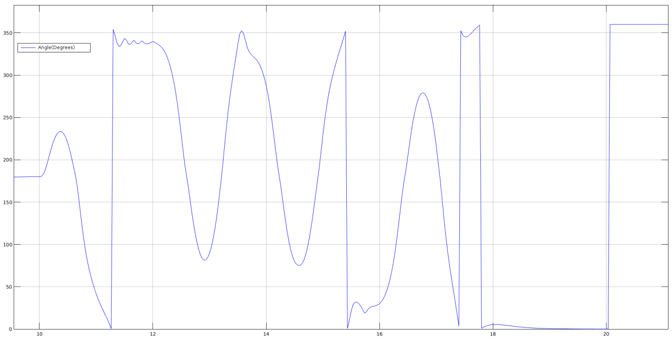

Fig. 19: theta (Degrees).

VII. Physical Implementation

For the physical implementation, an inverted pendulum was manufactured with the help of 3D printings and metal machining.A DC motor in one of the extremes of the pendulum drives the movement of the cart across two rails and in the hinge of the pendulum an encoder was placed to measure the angle of the pendulum to generate the feedback of the system:

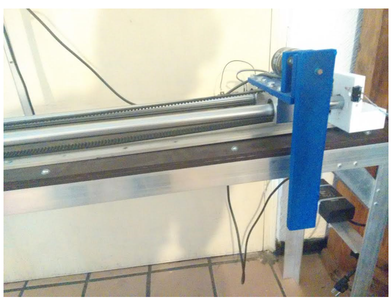

Fig. 20: Physical System.

VIII. Conclusion

In this paper, the presented Analysis of the pendulum based on a discrete state space analysis, describes a different way of modeling said system by dividing it in regions that facilitate its control. With this division we can take advantage of the simplicity of the different states and implement a control strategy that guarantees an effective suing up and a successful equilibrium of the pendulum in the upward ‘unstable’ position. Furthermore, by the construction of a simulation model in MATLAB it is presented that the model is able to full fill the desired task in a short period of time with a simple algorithm and basic programming.

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