Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 8

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Exercise 1: **5 Points**

Prove the unproved claims of the lecture:

(i) Lemma 6.20: *The intersection of topologies on a set X is again a topology on X. If $\gamma \subset \mathcal{P}(X)$, then $\mathcal{T}(\gamma)$ is the smallest topology on X containing γ.*

(ii) Lemma 6.22: *Let X be a set, $(X_i, \mathcal{T}_i)$, $i \in I$, topological spaces. Further, let γ_i be a subbasis of $\mathcal{T}_i$ for every $i \in I$. Then the weak topology (see Definition 6.16) with respect to the functions $f_i : X \to X_i$, $i \in I$, is given by $\mathcal{T}(\gamma)$, where*

$$\gamma = \{f_i^{-1}(V) \ : \ V \in \gamma_i, \, i \in I\}.$$

Exercise 2: **5 Points**

Let $(E, \|\cdot\|)$ be a finite-dimensional normed space. Show that $\sigma(E, E^*)$ coincides with the usual topology on E induced by $\|\cdot\|$.

Exercise 3: **4 Points**

Let E be a normed space. Show, that a sequence $(x_n)_n \subset E$ is weakly convergent to $x \in E$ in the sense of Definition 6.1 if and only if $(x_n)_n$ converges to x with respect to $\sigma(E, E^*)$. Furthermore, show that a sequence $(f_n)_n \subset E^*$ is weak-*- convergent to $f \in E^*$ in the sense of Definition 6.3 if and only if $(f_n)_n$ converges to f with respect to $\sigma(E^*, E)$.

Exercise 4: **6 Points**

Let E be a reflexive space and $A \subset E$ be bounded and $\sigma(E, E^*)$-closed. Show that A is *weakly sequentially compact*, i.e. every sequence in A has a weakly convergent subsequence with limit in A. This statement is a special case of the Eberlein-Smulian Theorem, which you are not allowed to use in this task, since we did not prove it.

Since June, 10 is a public holiday, please submit your homework in a tutorial of your choice between June, 11 and June, 14.