Discrete Geometrie I - Assignment 03

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Exercise 01

Let P_1 and P_2 be two polyhedron in $\mathbb{R}^n$. Consider the operator $+ : \mathbb{R}^{n \times n} \to \mathbb{R}^n, (x, y) \mapsto x + y$. That operator is linear.

1. $P_1 \times P_2$ is a polyhedron. To see this, let $A_1 x \leq b_1$ and $A_2 x \leq b_2$ denote the linear inequalities that define P_1 and P_2. Then,

$$P_1 \times P_2 = \{(x, y) \in \mathbb{R}^{n \times n} : A_1 x \leq b_1 \wedge A_2 y \leq b_2\},$$

   which shows that $P_1 \times P_2$ is a polyhedron.

2. $P_1 + P_2$ is the image of $P_1 \times P_2$ under the operator $+$. For applying a linear transformation on a polyhedra in $\mathbb{R}^n$ yields again a polyhedra, $P_1 + P_2$ is a polyhedra.

Exercise 02

First, we show that $T(P)$ solves the linear inequalities. Second, we show that the linear inequalities are contained in the image of P under the linear transformation T.

1. Let $Q = T(P)$ and $y \in Q$. Then there exists $x \in P$ such that $T(x) = y$, which implies

$$\langle c_i, y \rangle = \langle c_i, T(x) \rangle = \langle T^*(c_i), x \rangle = \langle a_i, x \rangle \leq b_i \quad \forall i = 1, ..., m.$$

2. Let $y \in \mathbb{R}^n$ such that $\langle c_i, y \rangle \leq b_i$ for all $i = 1, ..., m$. Then, there is $x \in \mathbb{R}^n$ such that $T(x) = y$ since T is bijective. We show that $x \in P$, and hence it follows that $y \in T(P)$. We claim $\langle a_i, x \rangle \leq b_i$ for all $i = 1, ..., m$.

$$\langle c_i, y \rangle = \langle c_i, T(x) \rangle = \langle T^*(c_i), x \rangle = \langle a_i, x \rangle \leq b_i \quad \forall i = 1, ..., m.$$

Exercise 03

Let $A \subset \mathbb{R}^d$.

1. **Let A be convex and open. Then, A is algebraically open.** Let $L = \{r + \tau s : \tau \in \mathbb{R}\}$ for any $r, s \in \mathbb{R}^d$. We want to show that

$$A \cap L = \{r + \tau s : \tau \in I\},$$

where I must be an open interval.

(a) I is an interval, because L and A are convex and the intersection of convex sets yields a convex set. Hence,

$$A \cap L = \{r + \tau s : \tau \in I\}$$

must be convex. This set is convex if and only if I is convex. Consequently, I must be an intervall for the only convex sets in $\mathbb{R}^1$ are intervals.

(b) Claim: I is open. Let $x = r + \tau_0 s \in A \cap L$ for some $\tau_0 \in \mathbb{R}$. To show that I is open, we need to find an open neighborhood $\tau_0 \in U_\epsilon(\tau_0)$ with radius ϵ such that $r + \tau s \in A \cap L$ for all $\tau \in U_\epsilon(\tau_0)$.

There exists an open neighborhood $U_\delta(x) \subset A$ with radius $\delta > 0$ since A is open. Let $\epsilon = \frac{\delta}{||s||}$. Then, it holds for all $|\tau| < \epsilon$ that

$$||x - (r + (\tau_0 + \tau)s)\,|| = ||\tau s|| < ||\frac{s}{||s||}\delta|| = \delta.$$

This implies that

$$r + (\tau_0 + \tau)s \in U_\delta(x) \subset A \cap L.$$

We found a neighborhood with the desired properties, and thus I is open.

We see that I is an open interval. Hence, A is algebraically open.

2. **Let A be algebraically open. Then, A is convex and open.** By definition, A is convex if A is algebraically open. So, A is convex. It remains to show that A is open.

Remember that A is a subset in $\mathbb{R}^d$. Let e_i denote the i-th unit vector for $i = 1, ..., d$. Let $x \in A$. We want to find an open neighborhood $U_\delta(x) \subset A$.

Consider straight lines

$$L_i = \{x + \tau e_i : \tau \in \mathbb{R}\}, \quad i = 1, ..., d,$$

that when intersected with A yields

$$A \cap L_i = \{x + \tau e_i : \tau \in I_i\}\,,$$

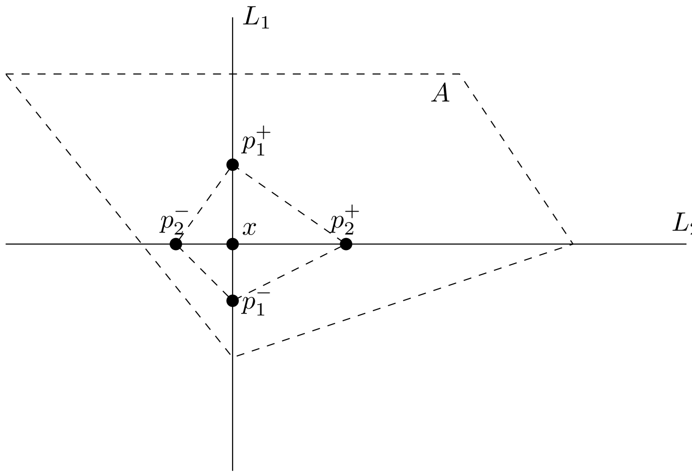
where $I_i = (\alpha, \beta)$ is an open interval with $\alpha \in [-\infty, 0)$ and $\beta \in (0, \infty]$. Let $p_i^+ = x + \tau e_i$ with $\tau \in (\alpha, 0)$ and $p_i^- = x + \tau e_i$ with $\tau \in (0, \beta)$. Then, consider the convex set

$$\operatorname{conv}\left(\bigcup_{i=1}^{d} \{p_i^+, p_i^-\}\right).$$

From the lecture we know that the interior of a convex set is convex. So,

$$\operatorname{int}\left(\operatorname{conv}\left(\bigcup_{i=1}^{d} \{p_i^+, p_i^-\}\right)\right) \tag{1}$$

is convex, too. That set is non-empty (the interior is non-empty, see exercise 5) and open. Furthermore, A contains (1) since A is convex. Thus, (1) is an open neighborhood in A containing x. We showed that A is open.



Exercise 04

Let $V = \{(x_1, x_2, ...) \subset \mathbb{R} : x_i \neq 0 \text{ for finitely many } i\ \}$. Let $A \subset V$ denote the sequence whose last non-zero entry is strictly positive.

1. A **is convex.** Let $(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}} \in A$. Let α denote the last entry of (x_i) that is non-zero; let β denote the last entry of (y_i) that is non-zero. We want to show that for all $\lambda \in [0, 1]$ it holds

   $$(z_i) = \lambda(x_i) + (1 - \lambda)(y_i) \in A.$$

   Without loss of generality, let $\alpha \geq \beta$. Then, $y_\alpha \geq 0$ because $(y_i) \in A$. Furthermore, $x_\alpha > 0$. Hence, $z_\alpha = \lambda x_\alpha + (1 - \lambda)y_\alpha > 0$ for $\lambda \in [0, 1]$. For z_i with $i > \alpha$ it holds $z_i = \lambda x_i + (1 - \lambda)y_i = 0$. To conclude, the last non-zero entry of (z_i) is strictly positive, which implies $(z_i) \in A$.

2. **There is no isolating hyperplane (we show a stronger statement).** Assume there is a linear functional $f : V \to \mathbb{R}$, $f \not\equiv \mathbf{0}$ such that $f(\mathbf{x}) \geq \alpha$ for all $\mathbf{x} \in A$ for some fixed $\alpha \in \mathbb{R}$. Let $\mathbf{z} = (z_i) \in V$ be a sequence such that

   - $f(\mathbf{z}) \neq 0$
   - $z_{i^*} > 0$, where z_{i^*} is the last non-zero sequence member (if $z_{i^*} < 0$ just multiply the sequence with a negative scalar).

   For $\mathbf{z} \in A$, it holds
   $$f(\mathbf{z}) \geq \alpha.$$
   Now, consider the sequence $\tilde{\mathbf{z}} = (\tilde{z}_i) \in A$ defined as
   $$\tilde{z}_i = \begin{cases} z_i & i \leq i^* \\ 1 & i = i^* + 1 \\ 0 & \text{otherwise} \end{cases}$$
   $$\Downarrow$$
   $$(\tilde{z}_i) = (z_1, z_2, ..., z_{i^*}, 1, 0, 0, ...)\,.$$
   Again, it holds
   $$f(\tilde{\mathbf{z}}) \geq \alpha.$$
   Here lies a contradiction. We decompose $\tilde{\mathbf{z}}$ into
   $$\tilde{\mathbf{z}} = \mathbf{z} + \mathbf{e}_{i^*+1},$$
   where $\mathbf{e}_{i^*+1}$ denotes the (i^*+1)-th unit vector. Since f is linear, it follows
   $$f(\tilde{\mathbf{z}}) = f(\mathbf{z}) + f(\mathbf{e_{i^*+1}}) \geq \alpha.$$
   Here is the trick: we make use of the linearity of f by multiplying the component $f(\mathbf{z})$ by a scalar such that the inequality $f(\mathbf{z}) + f(\mathbf{e_{i^*+1}}) \geq \alpha$ does not hold anymore! Let $\iota \in \mathbb{R}$ denote that evil scalar, and define
   $$\hat{\mathbf{z}} = \iota\mathbf{z} + \mathbf{e_{i^*+1}}.$$
   By definition of f it holds
   $$f(\hat{\mathbf{z}}) \geq \alpha \quad \text{but} \quad f(\hat{\mathbf{z}}) \leq \alpha \text{ by construction of } \hat{\mathbf{z}}.$$
   To conclude, whenever we find a sequence $\mathbf{z} \in V$ such that $f(\mathbf{z}) > 0$, we can multiply $\mathbf{z}$ with a scalar such that it is a member of A. Then, we can construct a contradiction. This implies that it holds $f(\mathbf{z}) = 0$ for all $\mathbf{z} \in V$. Thus, there cannot exist an isolating hyperplane, for hyperplanes are defined by linear functionals that are not identical to zero!

3. **There is no hyperplane containing A.** This follows from 2., since if there is a hyperplane containing A then we found an isolating hyperplane. But we showed that there is no isolating hyperplane!

Exercise 05

Let $\Delta \subset \mathbb{R}^n$ be a simplex. Let $a_1, ..., a_{n+1}$ the affinely independent points that define the simplex. Without loss of generality, we can assume that one of the vertices is the origin, i.e. $a_{n+1} = 0 \in \mathbb{R}^n$.

1. We know from linear algebra that there exists a bijective linear transformation T which transforms the simplex Δ to an unit cube $Q \subset \mathbb{R}^d$.
2. The point $y = \sum_{i=1}^{n} \frac{1}{n} e_i$ lies in the unit cube.
3. There is an open ball $U_1(y) \subset Q$ with radius 1 which contains y.
4. T and T^{-1} are continous because they are bounded (known from functional analysis). Hence, $T^{-1}(U_1(y))$ is an open subset of the interior of Δ, which shows that $\mathrm{int}(\Delta) \neq \emptyset$.