Volume 01

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Adressed to: Jan Macdonald

Functional Analysis

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With lyrics by Taylor Swift! Because she is awesome <3

We love Taylor Swift

I stay out too late, got nothin' in my brain
That's what people say, hmm, that's what people say, hmm

*From her masterwork **Shake it Off***

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Exercise 1

Written by Viet Duc Nguyen / Berlin, Germany / 23.04.2019, 09:30 AM

> To prove
>
> (X, d_1) and (X, d_2) are equivalent $\iff$ the convergent sequences concide

Proof. $\implies$ Let (X, d_1) and (X, d_2) be equivalent metric spaces. Let $(x_n)_{n\in\mathbb{N}}$ be any convergent sequence in (X, d_1), i.e. $x_n \to x \in X$ in (X, d_1).

Goal: We will show that $(x_n)_n$ converges in (X, d_2), i.e. $x_n \to x$ in (X, d_2).

Let $\epsilon > 0$. Due to the equivalence of U_1 and U_2, we find $r > 0$ such that $U_1(x, r) \subset U_2(x, \epsilon)$. Then, choose $N \in \mathbb{N}$ such that $x_n \in U_1(x, r)$ for all $n \geq N$; finding such N is possible since $(x_n)_n$ is convergent in (X, d) after the assumption. Thus, it follows that $x_n \in U_2(x, \epsilon)$ for all $n \geq N$. Per definition, $(x_n)_n$ converges in (X, d_2). Same argument applies to show that $(x_n)_n$ converges in (X, d_1) if $(x_n)_n$ is convergent in (X, d_2).

$\impliedby$ Assume that the convergent sequences in (X, d_1) and (X, d_2) are the same. We will prove the equivalence of the metric spaces by contradiction.

Goal: We will construct a sequence $(x_n)_n$ that converges in (X, d_1) but not in (X, d_2) if we assume that the metric spaces are *not* equivalent.

Let (X, d_1) and (X, d_2) be not equivalent. Hence, there is a $x \in X$ and $\epsilon > 0$ such that for all $r > 0$ it holds: $U_1(x, r) \not\subset U_2(x, \epsilon)$ or $U_2(x, r) \not\subset U_1(x, \epsilon)$. Consider the first case (the proof for the other case is identical). Thus, there exists x_r for each $r > 0$ such that $d_1(x_r, x) < r$ and $d_2(x_r, x) \geq \epsilon$. Construct the sequence $(y_n)_n$ with $y_n := x_{\frac{1}{n}}$. Finally, $y_n \to x$ in (X, d_1) but $y_n \not\to x$ in (X, d_2). Contradiction, for the sequences in both metric spaces conincide per assumption. The metric spaces must be therefore equivalent. $\square$

> To prove
>
> Show that $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric for a metric space (X, d).

Proof. Let $x, y, z \in X$. δ is zero if and only if the numerator is zero. The numerator is $d(x, y)$; thus, $\delta(x, y) = 0 \iff d(x, y) = 0 \iff x = y$. The ***symmetry*** follows from the symmetry of d. Observe that $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} =$

$\delta(y,x)$. Concerning the ***triangle inequality***:

$$\delta(x,z) \leq \frac{d(x,z)}{1+d(x,z)} \overset{(*)}{\leq} \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$
$$= \delta(x,y) + \delta(y,z),$$

where $(*)$ follows from the monotony of $x \mapsto \frac{x}{1+x}$ (remains to be shown). δ is therefore a norm. □

Lemma 0.1

$f(x) = \frac{x}{1+x}$ is monotonically increasing.

Proof. The derivative reads $\frac{1}{(1+x)^2}$. Since the derivative is strictly positive and f is continuous, f is monotonically increasing. □

To prove

Show that (X,d) and (X,δ) are equivalent metric spaces.

Proof. To show the claim we prove that the convergent sequences coincide. Let $(x_n)_n$ be a convergent sequence in (X,d) with limit $x \in X$. The convergence of $(x_n)_n$ in (X,δ) follows from $\delta(x,y) = \frac{d(x,y)}{\underbrace{1+d(x,y)}_{\geq 1}} \leq d(x,y)$ for all $x,y \in X$. Thus, for any $\epsilon > 0$ we will find a number $N \in \mathbb{N}$ (due to the convergence of $(x_n)_n$ in (X,d)) such that

$$\forall n \geq N : \delta(x_n,x) \leq d(x_n,y) < \epsilon,$$

which implies the convergence of $(x_n)_n$ in (X,δ).

Consider a convergent sequence $(x_n)_n$ in (X,δ) with limit $x \in X$. We want to prove that $(x_n)_n$ also converges to the same limit x in (X,d). Again, take any arbitrary positive ϵ, and we need to find a number $N \in \mathbb{N}$ such that $d(x_n,x) < \epsilon$ for all $n \geq N$.

Define $f : X \to \mathbb{R}, u \mapsto \frac{u}{1+u}$. This function is continuous *(make sure this is clear to you!!!!!!)*, and injective (due to the strict monotonicity as stated in the previous lemma). We know that $f(d(x_n,x)) \to 0$ from the convergence of $(x_n)_n$

in (X, δ). Observe that $f(u) = 0 \iff u = 0$. By injectivity and monotonicity of f, we deduct

$$f(u_n) \to 0 \implies u_n \to 0, \quad \forall (u_n)_n \subset X.$$

Thus, $d(x_n, x) \to 0$ impliying the convergence of $(x_n)_n$ in (X, d). $\square$

We love Taylor Swift

> "I'm sorry, the old Taylor can't come to the phone right now"
> "Why?"
> "Oh, 'cause she's dead!"

From her masterwork ***Look what you made me do***

Aufgabe 2

Let X, d_1 and d_2 be as specified. We want to show that d_2-open sets are d_1-open and vice versa.

Let A be d_2-open and $x \in A$. Then there is an $\epsilon > 0$ with $U_2(x, \epsilon) \subset A$. Now let $y \in U_1(x, \epsilon)$. We obtain

$$\begin{aligned} \epsilon > d_1(x, y) = |\frac{1}{x} - \frac{1}{y}| = \frac{|x - y|}{|xy|} \\ \implies |x - y| < \epsilon xy < \epsilon \\ \implies y \in U_2(x, \epsilon) \subset A \end{aligned}$$

Which shows that A is also d_1-open.

For the second step, suppose that A is d_1-open. For each point $a \in A$ we want to find an open ϵ-Ball that is a subset of A.

Let $a \in A$ and let ϵ be so that $U_1(a, \epsilon) \subset A$ and $\epsilon \leq 1$. Now define $\epsilon' := \frac{a^2 \epsilon}{2}$ and let $x \in U_2(a, \epsilon')$. Notice that $\epsilon' \leq \frac{a}{2}$ which means $x > \frac{a}{2}$ for any $x \in U_2(a, \epsilon')$. Now we have:

$$d_2(a, x) < \epsilon' \implies |a - x| < \epsilon' \implies \frac{|a - x|}{ax} < \frac{\epsilon'}{ax} \implies |\frac{1}{a} - \frac{1}{x}| < \frac{2\epsilon'}{a^2}$$

Thus, $d_1(a, x) < \epsilon$ and furthermore $x \in A$.

Now regarding completeness:

The metric space (x, d_2) is obviously not complete as the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ is a Cauchy-sequence but not convergent in $(0, 1]$.

On the other hand, we can show that all d_1-Cauchy-sequences are d_2-Cauchy-sequences:

Let $(x_n)_{n\in\mathbb{N}}$ be a d_1-Cauchy-sequence. That means $\forall\epsilon > 0\, \exists N\in\mathbb{N}\, \forall k, l > N$: $|\frac{1}{x_k} - \frac{1}{x_l}| < \epsilon$ which implies $|x_k - x_l| < \epsilon x_k x_l < \epsilon$.

Now since we know that in $([0, 1], d_2)$ all Cauchy-sequences converge it is sufficient to show that any sequence $(x_n)_{n\in\mathbb{N}} \subset (0, 1]$ which converges to zero - in $([0, 1], d_2)$ - is not a Cauchy-sequence in $((0, 1], d_1)$. You can see that as follows:

Since $x_n \longrightarrow 0$ for $n \to \infty$ we know that for all $\epsilon > 0$ there is an index $N \in \mathbb{N}$ so that for all $k > N$: $x_k < \epsilon$. Now set $\epsilon = 1$ and let $N \in \mathbb{N}$ be arbitrary. Then, find another arbitrary $k > N$ with $\frac{1}{x_k} := \delta$. For this $\delta > 0$ we can find an $l > N$ with $|\frac{1}{x_l}| > \delta + 1$. Thus $|\frac{1}{x_k} - \frac{1}{x_l}| > 1$ and $(x_n)_{n\in\mathbb{N}}$ is not a Cauchy-sequence in $((0, 1], d_1)$.

All other Cauchy-sequences in $((0, 1], d_2)$ are convergent because they are Cauchy-sequences in $([0, 1], d_2)$ whose limit is not 0 but somewhere in $(0, 1]$ and thus in $((0, 1], d_2)$. This means that all Cauchy-sequences in $((0, 1], d_1)$ are also convergent. $\square$

We love Taylor Swift

> Nice to meet you, where you been?
> I could show you incredible things

*From her masterwork **Blank Space***

Aufgabe 3

Let $\mathcal{S}$ be the space of all sequences $(x_i)_{i=1}^{\infty}$ of complex numbers. Verify that

$$d(x, y) := \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - y_i|}{1 + |x_i - y_i|}, \quad x, y \in \mathcal{S}$$

provides a metric on the space S. Show that a sequence $(x^{(n)})_{n\in\mathbb{N}} \subset \mathcal{S}$ converges to $x \in \mathcal{S}$ in $(\mathcal{S}, d)$ if and only if $x_i^{(n)} \to x_i$ for all $i \in \mathbb{N}$. Prove that $\mathcal{S}$ is complete.

Lemma 0.2

d(x,y) provides a metric

Proof. Let $x, y \in \mathcal{S}$. Since for $x \in \mathbb{R}$ it holds that $\frac{x}{1+x} \leq 1$ we see:

$$d(x,y) = \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - y_i|}{1 + |x_i - y_i|} \leq \sum_{i=1}^{\infty} 2^{-i} = 1$$

This shows that $d : \mathcal{S} \times \mathcal{S} \mapsto \mathbb{R}$. We will now demonstrate that d satisfies the conditions for a metric as defined in Definition 1.1. Let $x, y, z \in \mathcal{S}$.

1. $d(x,y) = 0 \Leftrightarrow x = y$
   " $\Rightarrow$ " Let $d(x,y) = 0$. This results in $\sum_{i=1}^{\infty} \frac{2^{-i}|x_i - y_i|}{1+|x_i - y_i|} = 0$ Since all summands of this sum are positive, all of them have to be zero for the sum to be zero. This implies for $i \in \mathbb{N}$

   $$0 = \frac{2^{-i}|x_i - y_i|}{1 + |x_i - y_i|} \Rightarrow 0 = |x_i - y_i| \Rightarrow x_i = y_i$$

   For $i \in \mathbb{N}$ we see $x_i = y_i$ and therefor $x = y$.
   " $\Leftarrow$ " Let $x = y$. So for $i \in \mathbb{N}$ $x_i = y_i$ and $|x_i - y_i| = 0$. For d this results in:

   $$0 = \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - y_i|}{1 + |x_i - y_i|} = d(x,y)$$

2. $d(x,y) = d(y,x)$

   $$d(x,y) = \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - y_i|}{1 + |x_i - y_i|} = \sum_{i=1}^{\infty} \frac{2^{-i}|(-1)(y_i - x_i)|}{1 + |(-1)(y_i - x_i)|} = \sum_{i=1}^{\infty} \frac{2^{-i}|y_i - x_i|}{1 + |y_i - x_i|} = d(y,x)$$

3. $d(x,y) \leq d(x,z) + d(z,y)$ To show the triangle inequality we will use that the function $t \mapsto t/1 + t$ is monotonically increasing on $(0, \infty)$. We have

already shown this in Exercise 1.

$$\begin{aligned}
d(x,y) &= \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - y_i|}{1+|x_i-y_i|} \\
&\le \sum_{i=1}^{\infty} \frac{2^{-i}(|x_i - z_i| + |z_i - y_i|)}{1+|x_i-z_i|+|z_i-y_i|} \\
&= \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - z_i|}{1+|x_i-z_i|+|z_i-y_i|} + \frac{2^{-i}|z_i - y_i|}{1+|x_i-z_i|+|z_i-y_i|} \\
&\le \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - z_i|}{1+|x_i-z_i|} + \frac{2^{-i}|z_i - y_i|}{1+|z_i-y_i|} \\
&= \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - z_i|}{1+|x_i-z_i|} + \sum_{i=1}^{\infty} \frac{2^{-i}|z_i - y_i|}{1+|z_i-y_i|} \\
&= d(x,z) + d(z,y)
\end{aligned}$$

□

To prove

$(x^{(n)}) \subset \mathcal{S}$ converges to $x \in \mathcal{S}$ in $(\mathcal{S}, d)$ if and only if $x_i^{(n)} \to x_i$ for all $i \in \mathbb{N}$

Proof. " $\Rightarrow$ " Let $(x^{(n)}) \subset \mathcal{S}$ and $(x^{(n)})$ converges to $x \in \mathcal{S}$ in $(\mathcal{S}, d)$. Towards a contradiction we assume that there is $i \in \mathbb{N}$, such that $x_i^{(n)} \nrightarrow x_i$. We can then find $\delta > 0$ such that for every $N \in \mathbb{N}$ there is a $n \ge N$ with $|x_i - x_i^{(n)}| > \delta$. We choose $\varepsilon \le \frac{2^{-i}\delta}{1+\delta}$:

$$\varepsilon \le \frac{2^{-i}\delta}{1+\delta} \le \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - x_i^{(n)}|}{1+|x_i - x_i^{(n)}|} = d(x, x^{(n)})$$

which would be a contradiction of $(x^{(n)})$ converging to x.

" $\Leftarrow$ " For $i \in \mathbb{N}$ $x_i^{(n)} \to x_i$. Let $\tilde{\varepsilon} > 0$. We choose $m \in \mathbb{N}$ such that $\tilde{\varepsilon} > (\frac{1}{2})^{(m-1)}$. Let $\varepsilon = (\frac{1}{2})^{(m)}$. Since for $i \in \mathbb{N}$ $x_i^{(n)} \to x_i$ we can find for $i \in \{1, ..., m\}$ $N_{\varepsilon,i} \in \mathbb{N}$ such that for $n \ge N_{\varepsilon,i}$ $|x_i - x_i^{(n)}| < \varepsilon$. Set $N = \max\{N_{\varepsilon,1}, ..., N_{\varepsilon,m}\}$. For $n \ge N$

we obtain:

$$
\begin{aligned}
d(x, x^{(n)}) &= \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - x_i^{(n)}|}{1 + |x_i - x_i^{(n)}|} \\
&\leq \sum_{i=1}^{m} \frac{2^{-i}\varepsilon}{1+\varepsilon} + \sum_{i=m+1}^{\infty} \frac{2^{-i}|x_i - x_i^{(n)}|}{1 + |x_i - x_i^{(n)}|} \\
&\leq \sum_{i=1}^{m} 2^{-i}\varepsilon + \sum_{i=m+1}^{\infty} 2^{-i} \\
&\leq (\frac{1}{2})^m + (\frac{1}{2})^m \\
&= 2(\frac{1}{2})^m \\
&= (\frac{1}{2})^{m-1} \\
&< \tilde{\varepsilon}
\end{aligned}
$$

Hence $(x^{(n)}) \subset \mathcal{S}$ converges to $x \in \mathcal{S}$.

□

In order to prove that $(\mathcal{S}, d)$ is complete we will first prove the following lemma:

Lemma 0.3

Let $d_1(x, y) = \frac{|x-y|}{1+|x-y|}$ and $d_2(x, y) = |x - y|$. A Cauchy-sequence in (X, d_1) is also a Cauchy-sequence in (X, d_2).

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a d_1-Cauchy-sequence. In the first exercise we proofed that d_1 and d_2 are equivalent. Due to the equivalence of d_1 and d_2 for every $\varepsilon > 0$ exists $\delta > 0$ such that $U(d_1, \delta) \subset U(d_2, \varepsilon)$. Since $(x_n)_{n\in\mathbb{N}}$ is a d_1-Cauchy-sequence for $\delta > 0$ exists $N_\delta \in \mathbb{N}$ such that for all $n, m \geq N_\delta$ $d_1(x_n, x_m) < \delta$. With the inclusion $U(d_1, \delta) \subset U(d_2, \varepsilon)$ then follows:

$$d_1(x_n, x_m) < \delta \Rightarrow d_2(x_n, x_m) < \varepsilon$$

Therefor $(x_n)_{n\in\mathbb{N}}$ is a d_2-Cauchy-sequence . □

To prove

$(\mathcal{S}, d)$ is complete

Proof. Let $(x^{(n)})_{n\in\mathbb{N}}$ be a Cauchy-Sequence in $\mathcal{S}$. Since $(x^{(n)})_{n\in\mathbb{N}}$ is a Cauchy-Sequence, for every $\varepsilon > 0$ exists $N_\varepsilon \in \mathbb{N}$ such that for $n, m \geq N_\varepsilon$ $\varepsilon > d(x_n, x_m)$. From the definition of d then follows for $i \in \mathbb{N}$:

$$\varepsilon > d(x_n, x_m) = \sum_{i=1}^{\infty} \frac{2^{-i}|x_i^{(m)} - x_i^{(n)}|}{1 + |x_i^{(m)} - x_i^{(n)}|}$$
$$\Rightarrow \varepsilon > \frac{2^{-i}|x_i^{(m)} - x_i^{(n)}|}{1 + |x_i^{(m)} - x_i^{(n)}|} \Rightarrow \varepsilon 2^i > \frac{|x_i^{(m)} - x_i^{(n)}|}{1 + |x_i^{(m)} - x_i^{(n)}|}$$

Hence for $i \in \mathbb{N}$ the sequence $(x_i^{(n)})_{n\in\mathbb{N}}$ is also a Cauchy-Sequence in $(\mathbb{C}, \tilde{d})$ with $\tilde{d}(x, y) := \frac{|x-y|}{1+|x-y|}$. From the first part of this exercise we already know that for a convergent sequence in $\mathcal{S}$ $x^{(n)} \to x$ is equivalent to $x_i^{(n)} \to x_i$ for all $i \in \mathbb{N}$. So if we show that for $i \in \mathbb{N}$ the Cauchy-sequence $(x_i^{(n)})_{n\in\mathbb{N}}$ converges to $x_i \in \mathbb{C}$ the Sequence $(x^{(n)})_{n\in\mathbb{N}}$ would also converge. Let $\dot{d}(x, y) = |x - y|$. From the previous Lemma we obtain that $(x_i^{(n)})_{n\in\mathbb{N}}$ is a Cauchy-Sequence in $(\mathbb{C}, \dot{d})$ and from Analysis 2 we know, that $(\mathbb{C}, \dot{d})$ is complete. So the sequence $(x_i^{(n)})_{n\in\mathbb{N}}$ converges in $(\mathbb{C}, \dot{d})$ and there exists a $x_i \in \mathbb{C}$, such that

$$\lim_{n\to\infty} x_i^{(n)} = x_i$$

With the continuity of $f : t \mapsto \frac{t}{1+t}$, which we already stated in exercise 1 then follows:

$$\lim_{n\to\infty} \dot{d}(x_i, x_i^{(n)}) = 0 \Rightarrow \lim_{n\to\infty} \tilde{d}(x_i, x_i^{(n)}) = \lim_{n\to\infty} \frac{\dot{d}(x_i, x_i^{(n)})}{1 + \dot{d}(x_i, x_i^{(n)})} = \lim_{n\to\infty} f(\dot{d}(x_i, x_i^{(n)})) = 0$$

So x_i is the limit of $(x_i^{(n)})_{n\in\mathbb{N}}$ in $(\mathbb{C}, \tilde{d})$. For $i \in \mathbb{N}$ we have shown that $x_i^{(n)} \to x_i$, which is equivalent to $(x^{(n)})$ converging in $\mathcal{S}$. Definition 1.3 implies then that $(\mathcal{S}, d)$ is complete.

□

We love Taylor Swift

It seems like one of those nights
This place is too crowded, too many cool kids uh uh, uh uh

*From her masterwork **22***

Aufgabe 4

Let $1 \leq p < q < \infty$. For $q = \infty$ not much is to be done as all sequences with finite sum have to be bounded. Now let $x = (x_n)_{n \in \mathbb{N}} \in \ell_p$ and define $A := \{n \in \mathbb{N} : x_n > 1\}$ and $B := \{n \in \mathbb{N} : x_n \leq 1\}$. Then:

$$\sum_{n \in \mathbb{N}} |x_n|^q = \sum_{n \in A} |x_n|^q + \sum_{n \in B} |x_n|^q$$

where the first sum is finite because A has to be finite, otherwise the sequence could not be in ℓ_p. But we also now have

$$\sum_{n \in B} |x_n|^q \leq \sum_{n \in B} |x_n|^p < \sum_{n \in \mathbb{N}} |x_n|^q < \infty$$

Thus, $\sum_{n \in \mathbb{N}} |x_n|^q$ is the sum of two finite terms and therefore it is finite itself.

There are $x \in \ell_q$ with $x \notin \ell_p$: For all $n \in \mathbb{N}$, let

$$x_n := \left(\frac{1}{n}\right)^{\frac{1}{p}}.$$

For $q > p$ we obtain

$$\sum_{n \in \mathbb{N}} |\left(\frac{1}{n}\right)^{\frac{1}{p}}|^q = \sum_{n \in \mathbb{N}} |\left(\frac{1}{n}\right)^{\frac{q}{p}}| < \infty$$

because $\frac{q}{p} > 1$ but

$$\sum_{n \in \mathbb{N}} |\left(\frac{1}{n}\right)^{\frac{1}{p}}|^p = \sum_{n \in \mathbb{N}} |\frac{1}{n}| = \infty.$$

So we have found a sequence which is in ℓ_q but not in ℓ_p.

A final note

We would like to thank everyone who has supported us over the years. A big thank you to **you**! Without your awesome tutorials this would not have been possible.

Stay tuned for next week's volume.

In love, Duc, Jacky & Lukas.