Research Group
Applied Functional Analysis
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Functional Analysis I

Tutorial Assignment 5

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Exercise 1: Let E and F be normed spaces and $T \in L(E, F)$. Prove that $\|T^*\| = \|T\|$.
Hint: According to Corollary 4.8: $\|x\| = \sup\{|\ell(x)| : \ell \in E^*, \|\ell\| \leq 1\}$.

Exercise 2: Let E and F be normed spaces and let $T \in L(E, F)$. Show that

$$\overline{\operatorname{ran} T} = F \quad \implies \quad \ker T^* = \{0\}.$$

Now put $G = \overline{\operatorname{ran} T}$ and define $S \in L(E, G)$ by $Sx := Tx$ for $x \in E$. Prove that

$$\operatorname{ran} T^* = \operatorname{ran} S^*.$$

Exercise 3: The following statement is seemingly trivial, but in fact – without Hahn-Banach – it is not. It says: *If* $E \neq \{0\}$ *and* $F \neq \{0\}$ *are normed spaces then* $L(E, F) \neq \{0\}$. Prove it!

Exercise 4: Let E be a normed space. Show that E^* separates points, this means: for each pair $x, y \in E$, $x \neq y$, there exists $\ell \in E^*$ such that $\ell(x) \neq \ell(y)$.

The statements of the following exercises are needed to prove the strict separation theorem, which is shown in the big exercise.

Exercise 5: Let X be a vector space. For $A, B \subset X$ we define

$$A \pm B := \{a \pm b \ : \ a \in A, \, b \in B\}.$$

Show that if A and B are convex, then $A \pm B$ is also convex.

Exercise 6: Let E be a normed space, let $F \subset E$ be a linear subspace and let $f \in F^*$. Prove that

$$\mathcal{L} := \{\ell \in E^* : \ell|_F = f \ \text{ and } \ \|\ell\| = \|f\|\}$$

is convex.

Exercise 7: Let (X, d) be a metric space and $A, B \subset X$ non-empty and disjoint. Prove the following fact: If A is compact and B is closed, then

$$\mathrm{dist}(A, B) := \inf\{d(a, b) \ : \ a \in A,\, b \in B\} > 0.$$