1 Optimization problems

1.1 Existence of solutions

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be smooth (that means continuous). The problem is stated as

$$\min_{x \in \Omega} f(x), \qquad \Omega \subset \mathbb{R}^n.$$

Theorem 1.1: Weierstraß

If $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is continuous and $\Omega \subset D$ is compact then f attains its sup and inf.

Definition 1.1: Sub-level set

$f : D \subset \mathbb{R}^n \to \mathbb{R}$ then

$$\mathcal{N}(f, \alpha) = \{x \in D : f(x) \leq a\}$$

is called a sub-level set.

...

Young's inequality: $0 \leq (\sqrt{\delta}a - \frac{1}{2\sqrt{\delta}}b)^2 = \delta a^2 + \frac{1}{4\delta}b^2 - ab \implies ab \leq \delta a^2 + \frac{1}{4\delta}b^2$.

So

$$\frac{1}{2}x^T Hx + b^T x \geq \frac{\alpha}{2}|x|^2 - |b^T x| \overset{\delta=\frac{\alpha}{4}}{\geq} \frac{\alpha}{2}|x|^2 - \frac{\alpha}{4}|x|^2 - \frac{1}{\alpha}|b|^2 = \frac{\alpha}{4}|x|^2 + c \to \infty.$$