Exercise 1

We have

$$f(x_1, x_2) = \begin{pmatrix} x_1 x_2 \\ x_2^2 \end{pmatrix} g(x_1, x_2) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} h(x_1, x_2) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$$

with $v_1 = \sum f_i \frac{\partial}{\partial x_i}, v_2 = \sum g_i \frac{\partial}{\partial x_i}, v_3 = \sum h_i \frac{\partial}{\partial x_i}$.

Now we calculate the Lie brackets:

$$[v_1, v_2] = (x_1 x_2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2}) x_1 \frac{\partial}{\partial x_1} - (x_1 \frac{\partial}{\partial x_1})(x_1 x_2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2}) = x_1 x_2 \frac{\partial}{\partial x_1} - x_1 x_2 \frac{\partial}{\partial x_1} = 0.$$

Φ^t and Ψ^s commute due to $[v_1, v_2] \neq 0$.

$$[v_1, v_3] = (x_1 x_2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2}) x_2 \frac{\partial}{\partial x_1} - (x_2 \frac{\partial}{\partial x_1})(x_1 x_2 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) = 0 + x_2^2 \frac{\partial}{\partial x_1} - x_2^2 \frac{\partial}{\partial x_1} - 0 = 0.$$

Φ^s and Θ^w commute.

$$[v_2, v_3] = (x_1 \frac{\partial}{\partial x_1}) x_2 \frac{\partial}{\partial x_1} - (x_2 \frac{\partial}{\partial x_1}) x_1 \frac{\partial}{\partial x_1} = -x_2 \frac{\partial}{\partial x_1} \neq 0.$$

Ψ^s and Θ^w do not commute.

Exercise 2

Characteristic equation:

$$\lambda^2 - 4\lambda + 2\alpha = 0.$$

Solution is $\lambda_{1,2} = 2 \pm \sqrt{4 - 2\alpha}$.

- For $\alpha \leqslant 2$ we obtain the solution

  $$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad c_1, c_2 \in \mathbb{R}.$$

  We know that $x(0) = 0$, so $c_1 = -c_2$. Additionally, $x(\pi) = 0$ which means

  $$c_1 e^{\lambda_1 \pi} - c_1 e^{\lambda_2 \pi} = 0 \implies e^{\lambda_1 \pi} = e^{\lambda_2 \pi}.$$

  That is false since $\lambda_1 \neq \lambda_2$ for $\alpha < 2$. For $\alpha = 2$, we obtain the trivial solution $x \equiv 0$. So for $\alpha \leqslant 2$, there exists only the trivial solution.

- For $\alpha > 2$, let $\beta := -(2 - \alpha)$. So we get $\lambda_{1,2} = 2 \pm i\sqrt{2\beta}$ with $\beta > 0$. One solution for the ODE is

  $$x(t) = ce^{2t}(\cos(t\sqrt{2\beta}) + i \sin(t\sqrt{2\beta})).$$

  The general, real solution for the ODE is

  $$x(t) = c_1 e^{2t} \cos(t\sqrt{2\beta}) + c_2 e^{2t} \sin(t\sqrt{2\beta}), \quad c_1, c_2 \in \mathbb{R}.$$

For $x(0) = 0$ we get $c_1 = 0$. For $x(\pi) = 0$ we get

$$x(\pi) = \underbrace{c_2 e^{2\pi}}_{\neq 0} \sin(\pi\sqrt{2\beta}) = 0, \quad c_2 \neq 0.$$

This means that $\sqrt{2\beta} \in \mathbb{N}$. This is the case if

$$\sqrt{2}\sqrt{\alpha - 2} = z \iff 2(\alpha - 2) = z^2 \iff \alpha = \frac{z^2}{2} + 2, \quad z \in \mathbb{N}$$

So for $\alpha > 2$ and $\alpha = \frac{z^2}{2} + 2, z \in \mathbb{N}$ there exists a non trivial solution.

Exercise 3

(i) The zeros of $f(x) = (x + \alpha)^2(x^2 - \alpha)$ are the fixed points. Fixed points only exist for $\alpha \geqslant 0$ since $(x^2 - \alpha) = 0$ for $x = \pm\sqrt{\alpha}$.

(ii) We see that the positive fixed is not stable since small perturbations move away from the fixed point. The negative fixed point is stable. If x is positive and smaller than the positive fixed point, it approaches the negative fixed point.