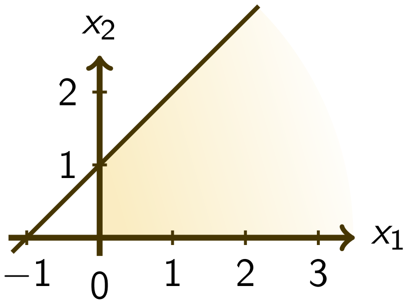
Graphical Representation and Solution (Cont.)

another 2D example:

$$\begin{array}{lrcrl} \min & c_1\,x_1 & + & c_2\,x_2 & \\ \text{s.t.} & -x_1 & + & x_2 & \leq 1 \\ & & & x_1,\, x_2 & \geq 0 \end{array}$$



- for $c = (1,1)^T$, the unique optimal solution is $x = (0,0)^T$
- for $c = (1,0)^T$, the optimal solutions are exactly the points
$$x = (0, x_2)^T \qquad \text{with } 0 \leq x_2 \leq 1$$
- for $c = (0,1)^T$, the optimal solutions are exactly the points
$$x = (x_1, 0)^T \qquad \text{with } x_1 \geq 0$$
- for $c = (-1,-1)^T$, the problem is unbounded, optimal cost is $-\infty$
- if we add the constraint $x_1 + x_2 \leq -1$, the problem is infeasible

Properties of the Set of Optimal Solutions

In the last example, the following 5 cases occurred:

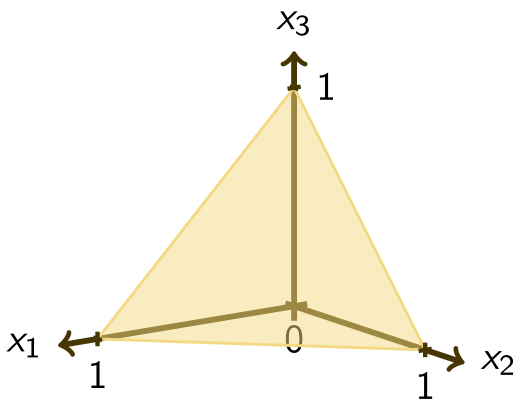
i. there is a unique optimal solution
ii. there exist infinitely many optimal solutions, but the set of optimal solutions is bounded
iii. there exist infinitely many optimal solutions and the set of optimal solutions is unbounded
iv. the problem is unbounded, i.e., the optimal cost is $-\infty$ and no feasible solution is optimal
v. the problem is infeasible, i.e., the set of feasible solutions is empty

These are indeed all cases that can occur in general (see also later).

Visualizing LPs in Standard Form

Example:
Let $A = (1, 1, 1) \in \mathbb{R}^{1\times 3}$, $b = (1) \in \mathbb{R}^1$ and consider the set of feasible solutions

$$P = \{x \in \mathbb{R}^3 \mid A \cdot x = b,\ x \geq 0\} \ .$$



More general:

- if $A \in \mathbb{R}^{m\times n}$ with $m \leq n$ and the rows of A are linearly independent, then
$$\{x \in \mathbb{R}^n \mid A \cdot x = b\}$$
is an $(n-m)$-dimensional affine subspace of $\mathbb{R}^n$.
- set of feasible solutions lies in this affine subspace and is only constrained by non-negativity constraints $x \geq 0$.

Chapter 3: The Geometry of Linear Programming

(cp. Bertsimas & Tsitsiklis, Chapter 2)

Linear, Conic, Affine, and Convex Combinations and Hulls

Definition 3.1.

Let $x^1, \dots, x^k \in \mathbb{R}^n$, $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$, and $x := \sum_{i=1}^k \lambda_i x^i$.

- a x is a linear combination of $x^1, \dots, x^k$.
- b If $\lambda \geq 0$, then x is a conic combination of $x^1, \dots, x^k$.
- c If $\sum_{i=1}^k \lambda_i = 1$, then x is an affine combination of $x^1, \dots, x^k$.
- d If $\lambda \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$, x is a convex combination of $x^1, \dots, x^k$.

If neither $\lambda = 0$ nor $\lambda = e_i$ for some $i \in \{1, \dots, n\}$, then x is a proper linear / conic / affine /convex combination.

Definition 3.2.

For a non-empty subset $S \subseteq \mathbb{R}^n$, the linear / conic / affine / convex hull of S, denoted by $\text{lin}(S)$ / $\text{cone}(S)$ / $\text{aff}(S)$ / $\text{conv}(S)$, is the set of all vectors that can be written as a linear / conic / affine / convex combination of finitely many vectors from S.
Moreover, let $\text{lin}(\emptyset) := \text{cone}(\emptyset) := \{0\}$ and $\text{aff}(\emptyset) := \text{conv}(\emptyset) := \emptyset$.

Linear Subspace, Cone, Affine Subspace, Convex Subset

Definition 3.3.

- a $S \subseteq \mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$, if $S = \text{lin}(S)$.
- b $S \subseteq \mathbb{R}^n$ is a (convex) cone, if $S = \text{cone}(S)$.
- c $S \subseteq \mathbb{R}^n$ is an affine subspace of $\mathbb{R}^n$, if $S = \text{aff}(S)$.
- d $S \subseteq \mathbb{R}^n$ is called convex, if $S = \text{conv}(S)$ (cp. Definition 1.1).

Lemma 3.4.

- a Let $S \subseteq \mathbb{R}^n$. Then $\text{lin}(S)$ / $\text{cone}(S)$ / $\text{aff}(S)$ / $\text{conv}(S)$ is the inclusion-wise smallest linear subspace / cone / affine subspace / convex subset containing S.
- b The sets of linear subspaces / cones / affine subspaces / convex sets in $\mathbb{R}^n$ are closed under taking intersections.
- c Every linear subspace and every cone in $\mathbb{R}^n$ contains the zero-vector 0.

Proof of Lemma 3.4:

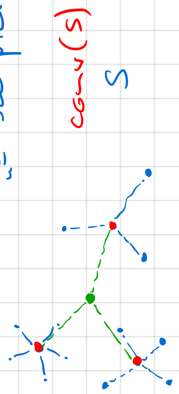
a) (for conv)

(i) conv(S) is convex, because:

$\mathrm{conv}(\mathrm{conv}(S)) = \mathrm{conv}(S)$

"$\supseteq$" clear

"$\subseteq$" see picture



For any convex set $X \supseteq S$:

$X = \mathrm{conv}(X) \supseteq \mathrm{conv}(S)$

b) (for cones)

For $j \in J$, let $C_j \subseteq \mathbb{R}^n$ with $C_j = \mathrm{cone}(C_j)$

$\mathrm{cone}\left(\bigcap_{j\in J} C_j\right) \subseteq \bigcap_{j\in J} \mathrm{cone}(C_j) = \bigcap_{j\in J} C_j$

c) Notice that a linear subspace or cone is always non-empty (as $\mathrm{lin}(\emptyset) = \mathrm{cone}(\emptyset) = \{0\}$)

Take some element x, multiply with 0 .... $\square$

Proof of Lemma 3.4

a) $\mathrm{conv}(S)$ is a convex set since $\mathrm{conv}\big(\mathrm{conv}(S)\big) = \mathrm{conv}(S)$. (Why?)

For any convex set $X \supseteq S$, it holds that $X = \mathrm{conv}(X) \supseteq \mathrm{conv}(S)$.

Similar for $\mathrm{lin}(S)$, $\mathrm{cone}(S)$, and $\mathrm{aff}(S)$.

b) For $j \in J$, let $C_j \subseteq \mathbb{R}^n$ with $C_j = \mathrm{cone}(C_j)$; (family of cones)

then, $\mathrm{cone}\Big(\bigcap_{j\in J} C_j\Big) = \bigcap_{j\in J} C_j$ because:

'$\supseteq$': clear; '$\subseteq$': $\mathrm{cone}\Big(\bigcap_{j\in J} C_j\Big) \subseteq \bigcap_{j\in J} \mathrm{cone}(C_j) = \bigcap_{j\in J} C_j$

Similar for linear subspaces, affine subspaces, and convex sets.

c) A linear subspace or cone X is non-empty, as $\mathrm{lin}(\emptyset) = \mathrm{cone}(\emptyset) = \{0\}$.

Thus there is an element $x \in X$ and therefore also $0 = 0 \cdot x \in X$. $\square$

Linear and Affine Independence, Dimension

Definition 3.5.

A finite non-empty subset $S \subseteq \mathbb{R}^n$ is linearly (affinely) independent, if no element of S can be written as a proper linear (affine) combination of elements from S.

Definition 3.6.

The dimension $\dim(S)$ of a subset $S \subseteq \mathbb{R}^n$ is the largest cardinality of an affinely independent subset of S minus 1.

Remark.

- If $S \subseteq \mathbb{R}^n$ is a linear subspace, $\dim(S)$ according to Definition 3.6 is equal to the maximal number of linearly independent vectors in S.
- Adding the zero-vector to a linearly independent set of vectors yields an affinely independent set.

Hyperplanes and Halfspaces

Definition 3.7.

Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$:

a. set $\{x \in \mathbb{R}^n \mid a^T \cdot x = b\}$ is called hyperplane;
b. set $\{x \in \mathbb{R}^n \mid a^T \cdot x \geq b\}$ is called halfspace.

Remarks.

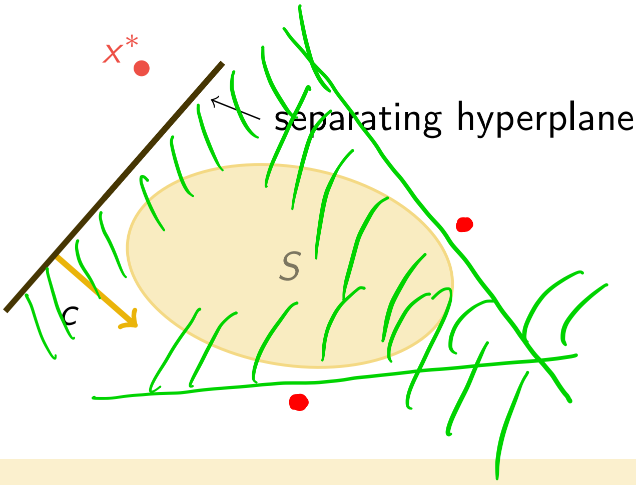
- A hyperplane is an affine subspace of dimension $n-1$.
- A hyperplane with right-hand side $b = 0$ is a linear subspace.
- Hyperplanes and halfspaces are cones if and only if $b = 0$.
- Hyperplanes and halfspaces are convex sets.

Separating Hyperplane Theorem for Convex Sets

Theorem 3.8.

Let $S \subseteq \mathbb{R}^n$ closed and convex, and let $x^* \in \mathbb{R}^n \setminus S$. There exists a vector $c \in \mathbb{R}^n$ such that $c^T \cdot x^* < c^T \cdot x$ for all $x \in S$.

Illustration:



Corollary 3.9.

Every closed and convex set is the intersection of a family of halfspaces.

Polyhedra and Polytopes

Definition 3.10.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:

- a set $\{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ is called polyhedron;
- b $\{x \mid A \cdot x = b,\ x \geq 0\}$ is polyhedron in standard form representation.

That is, a polyhedron is an intersection of finitely many halfspaces.

Definition 3.11.

- a Set $S \subseteq \mathbb{R}^n$ is bounded if there is $K \in \mathbb{R}$ such that

$$\|x\|_\infty \leq K \qquad \text{for all } x \in S.$$

- b A bounded polyhedron is called polytope.

Remark: The convex hull of finitely many points in $\mathbb{R}^n$ is a polytope (later).