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Mathematical Physics I - WS 2018/2019

http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/MP1/

Exercise Sheet 10

Exercise 1: (8 pts)

Consider a one-dimensional mechanical system describing the motion of a point (mass $m = 1$) under the influence of a potential energy

$$U(q) := \frac{\alpha}{q} + \log\left(\frac{q^2}{1+q^2}\right),$$

where $\alpha > 0$ is a parameter.

(i) Write down Newton equations and the corresponding dynamical system defined through a system of ODEs in $\mathbb{R}^2$.

(ii) Find the fixed points and investigate their stability. Draw the graph of the potential energy.

(iii) Make a qualitative analysis of the motion in the phase space $(q, \dot{q})$. Find the value(s) of α such that there exist periodic orbits.

Exercise 2: (6 pts)

Consider a particle of mass $m > 0$ in $\mathbb{R}^n$ with Lagrangian

$$\mathscr{L}(q, \dot{q}, t) := \mathrm{e}^{\alpha t/m}\left(\frac{1}{2}m\langle \dot{q}, \dot{q}\rangle - U(q)\right), \qquad \alpha > 0.$$

(i) Write down the Euler-Lagrange equations.

(ii) Fix $n = 1$ and $m = 1$ and consider $U(q) := -\beta q$, $\beta > 0$. Solve the Euler-Lagrange equation with $(q(0), \dot{q}(0)) = (q_0, 0)$, $q_0 > 0$. Compute $\lim_{t\to+\infty} \dot{q}$.

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Exercise 3: (6 pts)

Consider a point with mass $m = 1$ moving in $\mathbb{R}^3$ with Lagrangian

$$\mathscr{L}(q, \dot{q}) := \frac{1}{2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) + \alpha(q_1\,\dot{q}_2 - q_2\,\dot{q}_1), \qquad \alpha > 0.$$

(i) Write down the Euler-Lagrange equations.

(ii) Show that the system is invariant under rotations about the q_3-axis.

Hint: Construct the one-parameter Lie group of rotations about the q_3-axis, $\Psi_s : (s, q) \mapsto \widetilde{q} = \Psi_s(q)$, and its infinitesimal generator. The invariance condition of the Lagrangian is $\mathscr{L}(q, \dot{q}) = \mathscr{L}(\widetilde{q}, \dot{\widetilde{q}})$)

(iii) Use Noether‘s Theorem to find the integral of motion corresponding to the above symmetry. Check directly (using the equations of motion) that this function is an integral of motion.

Due Monday, January 14.