Farkas Lemma

Base case $n = 1$

Let $A \in \mathbb{R}^{m\times 1}$, $b \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$.

$$A = \begin{pmatrix} | \\ a_j \\ | \end{pmatrix} \in \mathbb{R}^{m\times 1}, \quad b = \begin{pmatrix} | \\ b_j \\ | \end{pmatrix} \in \mathbb{R}^m, \quad y = \begin{pmatrix} | \\ y_j \\ | \end{pmatrix} \in \mathbb{R}^m$$

Proof by contradiction.

Assume $Ax \ngeq b$ for all $x \in \mathbb{R}$. So, there exists $x \in \mathbb{R}$ such that

$$a_i x \geq b_i \quad \text{and} \quad a_j x < b_j \qquad \text{for } i, j \in \{1, ..., m\}.$$

There are two cases for a contradiction:

1. There exists $a_i = 0$ for $i \in \{1, ..., m\}$ and $b_i > 0$. Then, we have $a_i x = 0 > b_i > 0$ which is false. Just let

$$y_k = \begin{cases} 0, & \text{if } k \neq i \\ 1 & \text{if } k = i \end{cases}.$$

   Then,

$$\begin{aligned} y^T a &= 0 \\ y^T b &= b_i > 0. \end{aligned}$$

2. Written in standard form $Ax \geq b$ it means that there is a contradiction:

$$a_i x \geq b_i \quad \text{and} \quad a_j x \geq b_j \qquad \text{for } a_i > 0, a_j < 0$$

   with

$$\frac{b_i}{a_i} > \frac{b_j}{a_j}. \tag{1}$$

   In other words, the lower bound $\frac{b_i}{a_i}$ is larger than the upper bound $\frac{b_j}{a_j}$.

   Now, consider

$$y_k = \begin{cases} 1, & \text{if } k = i \\ \frac{a_i}{-a_j} & \text{if } k = j \\ 0, & \text{else} \end{cases}.$$

3. Compute

$$y^T a = a_i y_i + a_j y_j = a_i \cdot 1 + a_j \cdot \frac{a_i}{-a_j} = 0.$$

Also, observe that

$$y^T b = b_i y_i + b_j y_j = b_i + b_j \frac{a_i}{-a_j} > 0$$

due to

$$\frac{y^T b}{a_i} = \frac{b_i}{a_i} + \frac{b_j}{-a_j} \overset{(1)}{>} 0 \quad \text{and} \quad a_i > 0.$$

Construction of y. Check if A contains a zero element. If $a_i = 0$ and $b_i > 0$, set

$$y_k = \begin{cases} 0, & \text{if } k \neq i \\ 1 & \text{if } k = i \end{cases}.$$

Otherwise, let

$$l = \text{argmax}_{i \in \{1,\ldots,m : a_i > 0\}} \frac{b_i}{a_i} \qquad \text{(lower bound)}$$

and

$$u = \text{argmin}_{j \in \{1,\ldots,m : a_j < 0\}} \frac{b_j}{a_j}. \qquad \text{(upper bound)}$$

Set

$$y_k = \begin{cases} 1, & \text{if } k = l \\ \frac{a_l}{-a_u} & \text{if } k = u \\ 0, & \text{else} \end{cases}.$$