Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 4

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Exercise 1: **5 Points**

Let E be a normed space and let $\varphi : E \to \mathbb{K}$ be a linear functional $\varphi \neq 0$. Prove the following statements:

(i) There exists $x_0 \in E$ with $\varphi(x_0) = 1$.

(ii) $E = \ker \varphi \dotplus \operatorname{span}(x_0)$ (direct sum).

(iii) $\varphi \in E^*$ if and only if $\ker \varphi$ is closed.

(iv) $\varphi \notin E^*$ if and only if $\overline{\ker \varphi} = E$.

Exercise 2: **5 Points**

Let X be a non-trivial normed space and $P, Q : X \to X$ be linear such that

$$PQ - QP = \mathrm{Id}_X.$$

Then it is not possible that both P and Q are continuous. *Hint:* Show and use that for all $n \in \mathbb{N}$ there holds $PQ^n - Q^nP = nQ^{n-1}$.

Exercise 3: **5 Points**

Let $1 \leq p < \infty$ and let (c_n) be a sequence of numbers in $\mathbb{K}$. Let $\mathcal{L}_p := \{(x_n) \in \ell_p : (c_n x_n) \in \ell_p\}$ and define the linear operator

$$T_p : \ell_p \supset \mathcal{L}_p \to \ell_p, \quad T_p((x_n)) := (c_n x_n).$$

Prove that $T_p \in L(\ell_p)$ (this includes $\mathcal{L}_p = \ell_p$!) if and only if $(c_n) \in \ell_\infty$. Let $(c_n) \in \ell_\infty$. Show that $\|T_p\| = \|(c_n)\|_\infty$. Moreover, identifying ℓ_p^* with ℓ_q, prove that $T_p^* = T_q$.

Exercise 4: **5 Points**

Let X, Y, Z be Banach spaces and $T \in L(X, Y)$ such that the closure of $T(A)$ is compact in Y for every bounded set $A \subset X$. Furthermore, let $J \in L(Y, Z)$ such that J is injective. Show, that for all $\epsilon > 0$ there exists some $C_\epsilon > 0$ with

$$\|Tx\| \leq \epsilon\|x\| + C_\epsilon\|JTx\| \quad \text{for all } x \in X.$$

The following bonus exercise will only be corrected, if you have reached at least 15 points in Exercise 1-4.

Bonus Exercise*: **+5 Points**

Let $X := C[0,1]$ be endowed with the maximum norm. We define the *Volterra integral operator* $K : X \to X$ by

$$(Kf)(x) := \int_0^x f(y)\, dy, \quad x \in [0,1],\ f \in X.$$

Prove the following statements:

- $K \in L(X)$ with $\|K\| = 1$.
- The range (i.e., the image) of K is not closed in X.
- The perturbation operator of the identity Id by K, i.e. $T = \mathrm{Id} + K$, is surjective.

Please submit your homework in the BEGINNING of the big exercise on Monday, May 13th.