Proof of Theorem 3.8

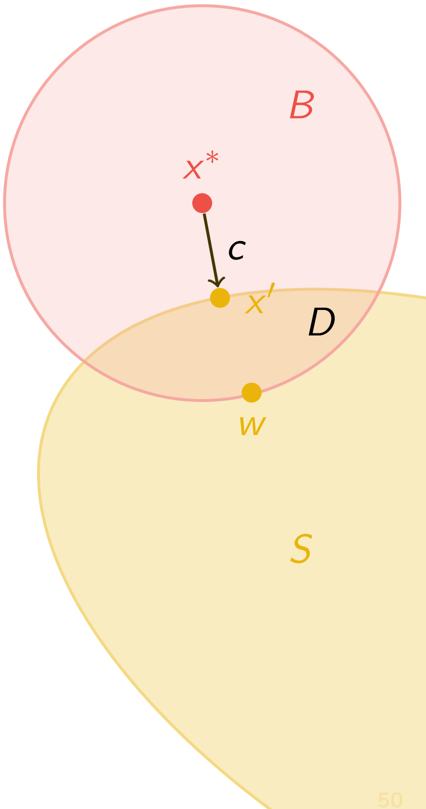
For arbitrary $w \in S$ let $B := \{x \mid \|x - x^*\| \leq \|w - x^*\|\}$ (Euclidean norm)

$\Longrightarrow \quad D := S \cap B$ closed, bounded, non-empty

$\Longrightarrow \quad \exists x' \in D: \ \|x' - x^*\| \leq \|x - x^*\| \quad \forall x \in D$

$\Longrightarrow \quad \|x' - x^*\| \leq \|x - x^*\| \quad \forall x \in S \qquad (*)$

Let $c := x' - x^* \neq 0$.



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$\Longrightarrow \quad \|x' - x^*\| \leq \|x - x^*\| \quad \forall x \in S \qquad (*)$

Let $c := x' - x^* \neq 0$. Claim: $c^T x > c^T x^* \ \forall x \in S$.

Proof:

For all $x \in S$ and $\lambda \in (0,1]: \quad x' + \lambda(x - x') \in S$

$$\Longrightarrow \quad \|x' - x^*\|^2 \overset{(*)}{\leq} \|x' + \lambda(x - x') - x^*\|^2$$

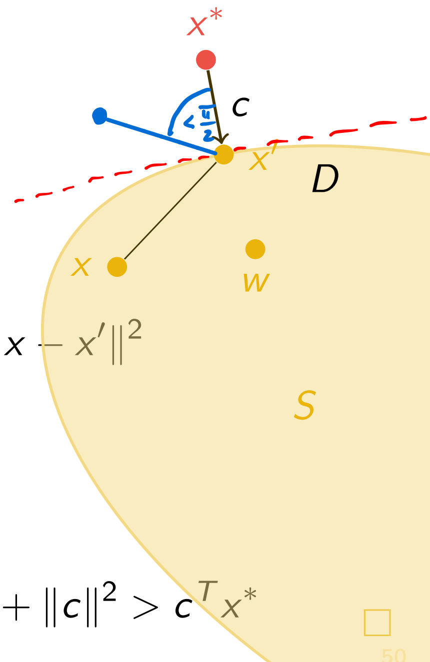
$$= \|x' - x^*\|^2 + 2\lambda(x' - x^*)^T(x - x') + \lambda^2\|x - x'\|^2$$

$$\Longrightarrow \quad 0 \leq (x' - x^*)^T(x - x') + \tfrac{1}{2}\lambda\|x - x'\|^2$$

$$\overset{\lambda \to 0}{\Longrightarrow} \quad 0 \leq (x' - x^*)^T(x - x') = c^T(x - x')$$

$$\Longrightarrow \quad c^T x \geq c^T x' = c^T x^* + c^T(x' - x^*) = c^T x^* + \|c\|^2 > c^T x^*$$

$\square$



Polyhedral Cones and Recession Cones

Definition 3.12.

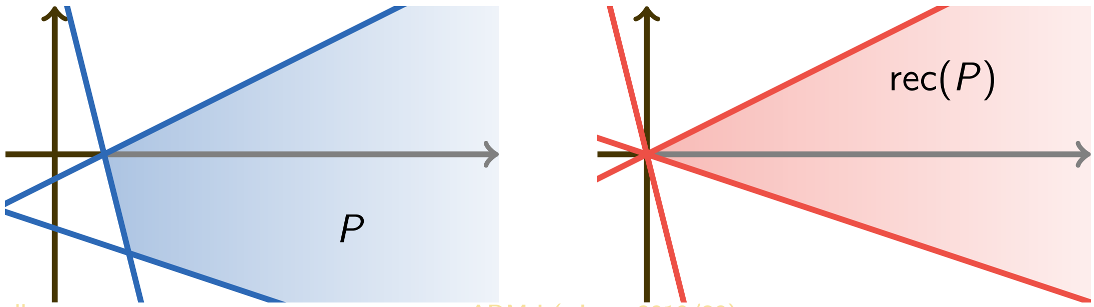
For $A \in \mathbb{R}^{m\times n}$ the polyhedron $\{x \in \mathbb{R}^n \mid A\cdot x \geq 0\}$ is a polyhedral cone.

Remark. Cone $\{0\}$ is the only polyhedral cone in $\mathbb{R}^n$ that is a polytope.

Definition 3.13.

For a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid A\cdot x \geq b\}$, the recession cone $\mathrm{rec}(P)$ is the polyhedral cone $\{x \in \mathbb{R}^n \mid A\cdot x \geq 0\}$. The non-zero elements of the recession cone are the rays of P.

Example in $\mathbb{R}^2$:



Characterization of Recession Cones

Lemma 3.14.

For a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid A\cdot x \geq b\}$ and $y \in P$,

$$\mathrm{rec}(P) = \{x \in \mathbb{R}^n \mid y + \lambda\cdot x \in P \text{ for all } \lambda \geq 0\} \quad .$$

In particular, if P is a polytope, $\mathrm{rec}(P) = \{0\}$.

Proof:

For a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid A\cdot x \geq b\}$ and $y \in P$,

$$\begin{aligned}\{x \in \mathbb{R}^n \mid y + \lambda\cdot x \in P\ \forall\lambda \geq 0\} &= \{x \in \mathbb{R}^n \mid A\cdot(y+\lambda\cdot x) \geq b\ \forall\lambda \geq 0\}\\ &= \{x \in \mathbb{R}^n \mid A\cdot x \geq 0\}.\end{aligned}$$ $\square$

$-A\cdot x \geq -b$
$A\cdot x \geq b$

Observation. The recession cone of $P = \{x \in \mathbb{R}^n \mid \underline{A\cdot x = b}, x \geq 0\}$ is

$$\{x \in \mathbb{R}^n \mid A\cdot x = 0,\ x \geq 0\}.$$

Extreme Points and Vertices of Polyhedra

Definition 3.15.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a) $x \in P$ is an extreme point of P if

$$x \neq \lambda \cdot y + (1-\lambda) \cdot z \qquad \text{for all } y, z \in P \setminus \{x\},\ 0 \leq \lambda \leq 1,$$

i.e., x is not a convex combination of two other points in P.

b) $x \in P$ is a vertex of P if there is some $c \in \mathbb{R}^n$ such that

$$c^T \cdot x < c^T \cdot y \qquad \text{for all } y \in P \setminus \{x\},$$

i.e., x is the unique optimal solution to the LP $\min\{c^T \cdot z \mid z \in P\}$.

Remark: The only possible vertex or extreme point of a polyhedral cone C is $0 \in C$.

Active and Binding Constraints

In the following, let $P \subseteq \mathbb{R}^n$ be a polyhedron defined by

$$\begin{aligned} {a_i}^T \cdot x &\geq b_i && \text{for } i \in M, \\ {a_i}^T \cdot x &= b_i && \text{for } i \in N, \end{aligned}$$

with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, for all i.

Definition 3.16.

If $x^* \in \mathbb{R}^n$ satisfies ${a_i}^T \cdot x^* = b_i$ for some i, then the corresponding constraint is active (or binding) at x^*.

Lemma 3.17.

If $P \neq \emptyset$ and none of the constraints ${a_i}^T \cdot x \geq b_i$ for $i \in M$ is active for all $x \in P$, then $\dim(P) = k := n - \operatorname{rank}\{a_i \mid i \in N\}$.

Proof of Lemma 3.17

'$\leq$': $\dim(P) \leq k$ since $\emptyset \neq P \subseteq \underbrace{\{x \in \mathbb{R}^n \mid {a_i}^T \cdot x = b_i \ \forall\, i \in N\}}_{\dim(\cdots)=k}$.

'$\geq$': Let $d_j \in \mathbb{R}^n$, $j = 1, \dots, k$, linearly indep. with ${a_i}^T \cdot d_j = 0 \ \forall\, i \in N$.

For $i \in M$, let $x^i \in P$ with ${a_i}^T \cdot x^i > b_i$; let $y := \frac{1}{|M|} \sum_{i \in M} x^i \in P$.

Notice that ${a_i}^T \cdot y > b_i$ for all $i \in M$. Thus, for $\varepsilon > 0$ small enough,

$$ {a_i}^T \cdot (y + \varepsilon\, d_j) \geq b_i \quad \text{for all } i \in M,\ j = 1, \dots, k. $$

Thus, $y + \varepsilon\, d_j \in P$ for $j = 1, \dots, k$, and

$$ \{y\} \cup \{y + \varepsilon\, d_j \mid j \in \{1, \dots, k\}\} \qquad \text{affinely independent.} $$

Therefore $\dim(P) \geq k$. $\square$

Reminder: Basic Facts from Linear Algebra

Theorem 3.18.

Let $x^* \in \mathbb{R}^n$ and $I = \{i \mid {a_i}^T \cdot x^* = b_i\}$. The following are equivalent:

i there are n vectors in $\{a_i \mid i \in I\}$ which are linearly independent;

ii the vectors in $\{a_i \mid i \in I\}$ span $\mathbb{R}^n$;

iii x^* is the unique solution to the system of equations ${a_i}^T \cdot x = b_i$, $i \in I$.

Vertices, Extreme Points, and Basic Feasible Solutions

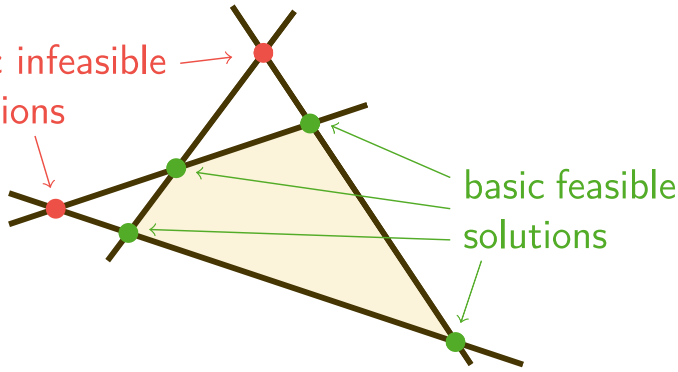
Definition 3.19.

a $x^* \in \mathbb{R}^n$ is a basic solution of P if

- all equality constraints are active and
- there are n linearly independent constraints that are active.

b A basic solution satisfying all constraints is a basic feasible solution.

Example:



Vertices, Extreme Points, and Basic Feasible Solutions

Definition 3.19.

a $x^* \in \mathbb{R}^n$ is a basic solution of P if

- all equality constraints are active and
- there are n linearly independent constraints that are active.

b A basic solution satisfying all constraints is a basic feasible solution.

Theorem 3.20.

For $x^* \in P$, the following are equivalent:

i x^* is a vertex of P;

ii x^* is an extreme point of P;

iii x^* is a basic feasible solution of P.

Existence of Extreme Points

Definition 3.21.

A polyhedron P is called pointed if it contains at least one vertex.

Definition 3.22.

A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if there is $x \in P$ and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$x + \lambda \cdot d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$

Theorem 3.23.

Consider non-empty $P \subseteq \mathbb{R}^n$ as on Slide 55. The following are equivalent:

i. P is pointed.
ii. P does not contain a line.
iii. $\text{rank}\{a_i \mid i \in M \cup N\} = n$.