TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Dr. Yuri B. Suris
Jan Techter

Mathematical Physics I - WS 2018/2019

http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/MP1/

Exercise Sheet 11

Exercise 1: (8 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^6$ consider a point of unit mass with Hamiltonian

$$\mathscr{H}(q,p) := \sqrt{\alpha^2 + \langle p, p \rangle} + \langle a, q \rangle, \qquad \alpha > 0,$$

where $a \in \mathbb{R}^3$ is a constant vector.

(i) Derive the canonical Hamilton equations.

(ii) Construct the Lagrangian.

(iii) Derive the Euler-Lagrange equations.

Exercise 2: (6 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^6$ consider a point of mass $m = 1$ with Hamiltonian

$$\mathscr{H}(q,p) := \frac{\langle p, p \rangle}{2} + U(\|q\|),$$

where $U = U(\|q\|)$ is a central potential. The *Runge-Lenz vector* is defined by the formula

$$A(q,p) := p \times \ell(q,p) + U(\|q\|)\, q,$$

where $\ell(q,p) := q \times p$ is the angular momentum.

(i) Prove that ℓ is an integral of motion.

(ii) Prove the following formula:

$$\dot{A}(q,p) = p\Big(\|q\| U'(\|q\|) + U(\|q\|)\Big).$$

(iii) Consider the *Kepler potential*

$$U(\|q\|) := -\frac{\alpha}{\|q\|}, \qquad \alpha > 0.$$

Prove that A is an integral of motion.

Exercise 3: (6 pts)

In $\mathbb{R}$ consider the system of N Newton equations

$$\ddot{q}_k = e^{q_{k+1}-q_k} - e^{q_k-q_{k-1}},$$

where $k = 1, \ldots, N$ and $q_{N+k} \equiv q_k$ (mod. N).

(i) Prove that the above equations of motion are Euler-Lagrange equations for the Lagrangian

$$\mathscr{L}(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N) := \sum_{k=1}^{N} \left(\frac{\dot{q}_k^2}{2} - e^{q_{k+1}-q_k} \right).$$

(ii) Construct the Hamiltonian and write down the canonical Hamilton equations.

(iii) Prove that the total linear momentum

$$P(p_1, \ldots, p_N) := \sum_{k=1}^{N} p_k$$

is an integral of motion.

Due Monday, January 21.