Proof. **Sufficiency:** We consider a candidate Lyapunov function: $F(x) = \langle x, Px\rangle, P \in GL(n, \mathbb{R})$ is positive, definitive, symmetric.

$F(x) > 0, \forall x \in \mathbb{R}^n \backslash \{0\}, F(0) = 0$. We need to prove that $\mathcal{L}_v F(x) < 0, \forall x \in \mathbb{R}^n \backslash \{0\}$.

$$v = \sum_{i=1}^{n} (Ax)_i \frac{\partial}{\partial x_i}, grad_n F(x) = 2P_x.$$

Then $(\mathcal{L}_v F)(x) = 2\langle Ax, Px\rangle = \langle x, A^T Px\rangle + \langle x, PAx\rangle = \langle x, (A^T P + PA)x\rangle = -\langle x, Qx\rangle < 0$.

Necessity: we assume $\Re\lambda_i < 0 \forall i = 1, ..., p$. We define $P = \int_0^\infty \exp(A^T t) Q \exp(At) dt$. We observe that P is positive definite and the $\int < +\infty$ because A is the matrix of an symptotically stable IVP. We reconstruct Lyapuniv equation:

$$\begin{aligned}
&- A^T(\int_0^\infty \exp(A^T t) Q \exp(At) dt) - (\int_0^\infty \exp(A^T t) Q \exp(At) dt) A \\
&= - \int_0^\infty (A^T \exp(A^T t) Q \exp(At) + \exp(A^T t) Q \exp(At) A) dt \\
&= - \int_0^\infty \frac{d}{dt} (\exp(A^T t) Q \exp(At)) dt = - \exp(A^T t) Q \exp(At) \Big|_0^\infty = Q
\end{aligned}$$

which shows that our P solves the Lyapunov equation. □

Consider the IVP:

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases} \quad f \in \mathcal{C}^k(M, \mathbb{R}^n), k \geqslant 1, \tilde{x} \text{ to be a fixed point } (f(\tilde{x}) = 0).$$

We can always assume that $\tilde{x} = 0$.

Idea: Linearization around $\tilde{x} = 0$ is $\begin{cases} \dot{x} = Ax, \\ x(0) = x_0 \end{cases}$ where $A = \frac{\partial f}{\partial x}\Big|_{\tilde{x}=0}$ (Jacobian matrix) with

$$\begin{gathered}
\dot{x} = f(x) = f(0) + Ax + g(x) \\
g(0) = 0, \|g(x)\| = O(\|x\|) \text{ as } \|x\| \to 0.
\end{gathered}$$

Remark: $\tilde{x} = 0$ is hyperbolic if $\tilde{x} = 0$ is hyperbolic for the linearized problem.

Theorem (Poincare-Lyapunov)**.** *Two statements:*

(1) If $\tilde{x} = 0$ is asymptotically stable for the linearization then $\tilde{x} = 0$ is asymptotically stable for the nonlinear problem.

(2) As abov, if $\tilde{x} = 0$ is unstable.

Proof. $\tilde{x} = 0$ is asymptotically stable for $\tilde{x} = Ax, A = \frac{\partial f}{\partial x}\Big|_0 \implies \Re\lambda_i < 0 \forall i$, From the previous Theorem, we know that $\exists P \in GL(n, \mathbb{R})$ positive, definitive, symmetric Matrix such that $F(x) = \langle x, Px\rangle$

is a strict Lyapunow function if $-PA - A^T P = Q$ is positive, definitive, symmetric matrix.

Let $q_i > 0$ be eigenvalues of Q, $p_i > 0$ egeinvalues of P, $p_0 = \max p_i > 0, q_0) \min q_i > 0$.

$$\|P_x\| \leqslant p_0\|x\|, \langle x, Qx\rangle \geqslant q_0\|x\|^2.$$

Look at F as a candidate strict Lyapunov function for $\dot{x} = f(x)$.

$$F(x) > 0, F(x) = 0 \iff x = 0.$$

The function g is such that $\forall\epsilon > 0 \exists\delta > 0$ such that $\|y(x)\| < \epsilon\|x\|$ for all $\|x\| < \delta$. Then

$$\begin{aligned}
(\mathcal{L}_v F)(x) = 2\langle f(x), Px\rangle &= \langle f(x), Px\rangle + \langle x, Pf(x)\rangle \\
&= \langle Ax + g(x), Px\rangle + \langle x, P(Ax + y(x))\rangle \\
&= -\langle x, Qx\rangle + 2\langle g(x), Px\rangle \\
&\leqslant -q_0\|x\|^2 + 2\|g(x)\|\|Px\| \\
&\leqslant -q_0\|x\|^2 + 2\epsilon p_0\|x\|^2 \\
&\leqslant (-q_0 + 2\epsilon p_0)\|x\|^2.
\end{aligned}$$

That is < 0 if we choose ϵ such that $-q_0 + 2\epsilon p_0 < 0$. □

Theorem (Poincare-Lyapunov). *Let $\lambda_1, ..., \lambda_p$ eigenvalues of $A = \frac{\partial f}{\partial x}\Big|_0$.*

1. *If $\Re\lambda_i < -y \forall i = 1, ..., p$ with some $y > 0$ then $\exists$ neighbourhoog $\tilde{M}$ of $\tilde{x} = 0$ such that*
   (a) *Φ_t is such that $\Phi_t(x) \in \tilde{M} \forall x \in \tilde{M} \forall t \geqslant 0$*
   (b) *$\exists C > 0$ such that $\|\Phi_t(x)\| \leqslant C\exp(-y\frac{t}{2})\|x\| \quad \forall \tilde{x} \in \tilde{M}, t \geqslant 0$*
2. *If $\exists\lambda_k$ with $\Re\lambda_k > 0$ then $\tilde{x} = 0$ is unstable.*

Example:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_2(1 - x_1^2) - x_1 \end{cases} \qquad \text{(Van der Pol system)}$$

Detect stability of fixed point $(0, 0)$. Linearize this problem

$$\dot{x} = Ax, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

If this point $(0, 0)$ is hyperbolic, we can use Poincare-Lyapunov (?). We find the eigenvalues

$$\lambda_{1,2} = \frac{(1 \pm i\sqrt{3})}{2}$$

Unstable for the linearized problem, so for the origin problem.

Topological equivalence / conjugacy of dynamical systems

Idea: find some equivalence classes for dynamical systems.

Consider $\{\Phi_t, \mathbb{R}, M_1\}, M_1 \subset \mathbb{R}^n$ and $\{\Psi_t, \mathbb{R}, M_2\}, M_2 \subset \mathbb{R}^n$ with the vector fields $v = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ and $w = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i}$, $f(x) = \frac{d}{dt}\Big|_{t=0} \Phi_t(x)$ and $g(x) = \frac{d}{dt}\Big|_{t=0} \Psi_t(x)$

Definiton. *$\{\Phi_t, \mathbb{R}, M_1\}$ and $\{\Psi_t, \mathbb{R}, M_2\}$ are topologically equivalent if there exists a homeomorphism $\Delta : M_1 \to M_2$ which maps the orbits of $\{\Phi_t, \mathbb{R}, M_1\}$ onto the orbits of $\{\Psi_t, \mathbb{R}, M_2\}$ preserving the direction of time:*

$$(\Delta \circ \Phi_t)(x) = (\Psi_t \circ \Delta)(x), \tau : \mathbb{R} \to \mathbb{R} : t \mapsto \tau(t) \textit{ reapproximation of time.}$$

If Δ is C^k then $\{\Phi_t, \mathbb{R}, M_1\}$ and $\{\Psi_t, \mathbb{R}, M_2\}$ are C^k-diffeomorphic. If $\tau = id$ then $\{\Phi_t, \mathbb{R}, M_1\}$ and $\{\Psi_t, \mathbb{R}, M_2\}$ are conjugate.

Assume that $\{\Phi_t, \mathbb{R}, M_1\}$ and $\{\Psi_t, \mathbb{R}, M_2\}$ are C^k-diffeomorphics with $\tau(t) = t$.

$$\Delta(\Phi_t(x)) = \Psi_t(\Delta(x)) \text{ differentiate with respect to } t \text{ at } t = 0.$$

$$\frac{\partial \Delta}{\partial x} f(x) = y(\Delta(x)) \implies f(x) = (\frac{\partial \Delta}{\partial x})^{-1} g(\Delta(x)) \quad \text{Formular for transformating coordinates}$$

Example: Let us consider three dynamical systems.

$$\begin{cases} \dot{x}_1 = -2x_1 \\ \dot{x}_2 = -2x_2 \end{cases} \quad \begin{cases} \dot{x}_1 = -2x_1 + x_2 \\ \dot{x}_2 = -x_1 - 2x_2 \end{cases} \quad \begin{cases} \dot{x}_1 = -2x_1 \\ \dot{x}_2 = x_1 - 2x_2 \end{cases}$$

Eigenvalues:

$$\lambda_{1,2} = -2, \lambda_{1,2} = -2 \pm i, \lambda_{1,2} = -2$$

$$(x_1, x_2) = r(\sin(\sigma), \cos(\sigma))$$

$$\begin{cases} \dot{r} = -2r \\ \dot{\sigma} = 0 \end{cases} \quad \begin{cases} \dot{s} = -2s \\ \dot{\varphi} = -1 \end{cases}$$

$A : (s, \varphi) \mapsto (r, \sigma) = (s, \varphi - \frac{1}{2}\ln(s))$

Theorem (Hartman-Grobman)**.** *Any hyperbolic IVP in $\mathbb{R}^n$*

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}, A \in gl(n, \mathbb{R}) \textit{ can be non invertible}$$

is locally topologically conjugate in $\tilde{M}$ (neighbourhood of $\tilde{x} = 0$) to any nonlinear IVP in $\tilde{M}$:

$$\begin{cases} \dot{y} = Ay + f(y) \\ y(0) = x_0 \end{cases} \tag{1}$$

where $f \in C^1(\tilde{M}, \mathbb{R}^n)$ with $f(0) = 0, \frac{\partial f}{\partial y}\Big|_0 = 0$. Precisely, if Φ_t is the flow of (1) then there exists homeomorphism $x \mapsto y\Delta(x)$ for which

$$(\Delta \circ \exp(tA))(a) = (\Phi_t \circ \Delta)(x), \quad \forall x \in \tilde{M}.$$