Proof of Theorem 3.20

(i) => (ii): x^* vertex $\Rightarrow \exists\, c \in \mathbb{R}^n : c^T x^* < c^T y \;\; \forall y \in P \setminus \{x^*\}$

By contradiction assume that x^* is not an extreme point, i.e.,

$$x^* = \lambda \cdot y + (1-\lambda) \cdot z \quad \text{with } y, z \in P \setminus \{x^*\},\ \lambda \in [0,1]$$

$$\Rightarrow c^T x^* = \lambda \cdot \underbrace{c^T y}_{> c^T x^*} + (1-\lambda)\, \underbrace{c^T z}_{> c^T x^*} > c^T x^* \quad ↯$$

(ii) => (iii): x^* extreme point, $I := \{ i \in M \cup N : a_i^T \cdot x^* = b_i \}$

Assume by contradiction that $\operatorname{rank} \{ a_i \mid i \in I \} < n$.

$\Rightarrow \exists\, d \in \mathbb{R}^n \setminus \{0\}$ with $a_i^T \cdot d = 0 \;\; \forall i \in I$

Let $x := x^* + \varepsilon \cdot d$ and $y := x^* - \varepsilon \cdot d$ for $\varepsilon > 0$.

Claim: $x, y \in P$ for $\varepsilon > 0$ small enough

(this is a contradiction to x^* extreme point)

Proof of Claim: For $i \in I$:

$$a_i^T \overset{y}{x} = a_i^T x^* \pm \varepsilon \cdot \underbrace{a_i^T \cdot d}_{=0} = a_i^T x^* = b_i .$$

For $i \notin I$:

$$a_i^T \cdot \overset{y}{x} = \underbrace{a_i^T x^*}_{> b_i} \pm \varepsilon \cdot a_i^T \cdot d \geq b_i \quad \text{for } \varepsilon \text{ small enough}$$

$\boxed{\text{Claim}}$

(iii) => (i): x^* basic feasible solution,

$I := \{ i \in M \cup N \mid a_i^T x^* = b_i \} \Rightarrow \operatorname{rank} \{ a_i \mid i \in I \} = n$

$$c := \sum_{i \in I} a_i, \quad c^T x^* = \sum_{i \in I} a_i^T x^* = \sum_{i \in I} b_i$$

$$\text{For } x \in P: \quad c^T x = \sum_{i \in I} \underbrace{a_i^T x}_{\geq b_i} \geq \sum_{i \in I} b_i$$

$$c^T x = \sum_{i \in I} b_i \iff a_i^T x = b_i \;\; \forall i \in I \iff x = x^* \qquad \square$$

Proof of Theorem 3.20

'(iii)$\Longrightarrow$(i)': x^* basic feasible solution, $I := \left\{i \in M \cup N \mid {a_i}^T \cdot x^* = b_i\right\}$

$$c := \sum_{i \in I} a_i, \qquad c^T \cdot x^* = \sum_{i \in I} {a_i}^T \cdot x^* = \sum_{i \in I} b_i.$$

For $x \in P$: $\quad c^T \cdot x = \sum_{i \in I} \underbrace{{a_i}^T \cdot x}_{\geq b_i} \geq \sum_{i \in I} b_i \quad$ and

$$c^T \cdot x = \sum_{i \in I} b_i \iff {a_i}^T \cdot x = b_i \ \forall\, i \in I \iff x = x^*$$

since there are n linearly independent vectors in $\{a_i \mid i \in I\}$.

'(i)$\Longrightarrow$(ii)': x^* vertex $\Longrightarrow \exists\, c \in \mathbb{R}^n: \ c^T \cdot x^* < c^T \cdot y \quad \forall\, y \in P \setminus \{x^*\}$

By contradiction assume that

$$x^* = \lambda y + (1-\lambda) z \qquad \text{with } y, z \in P \setminus \{x^*\},\ \lambda \in [0,1].$$

Then, $\quad c^T \cdot x^* = \lambda \underbrace{c^T \cdot y}_{> c^T \cdot x^*} + (1-\lambda) \underbrace{c^T \cdot z}_{> c^T \cdot x^*} > c^T \cdot x^*.$ ↯

Proof of Theorem 3.20 (Cont.)

'(ii)$\Longrightarrow$(iii)': x^* extreme point, $I := \left\{i \in M \cup N \mid {a_i}^T \cdot x^* = b_i\right\}$

Assume by contradiction that $\operatorname{rank}\{a_i \mid i \in I\} < n$.

Then, there exists $d \in \mathbb{R}^n \setminus \{0\}$ with ${a_i}^T \cdot d = 0$ for all $i \in I$.

Let $x := x^* + \varepsilon d$ and $y := x^* - \varepsilon d$ for some $\varepsilon > 0$.

Claim: $x, y \in P$ for $\varepsilon > 0$ small enough. ($\Longrightarrow$ contradiction!)

Proof: For $i \in I$, ${a_i}^T \cdot x = \underbrace{{a_i}^T \cdot x^*}_{= b_i} + \varepsilon \underbrace{{a_i}^T \cdot d}_{= 0} = b_i$;

for $i \notin I$, ${a_i}^T \cdot x = \underbrace{{a_i}^T \cdot x^*}_{> b_i} + \varepsilon\, {a_i}^T \cdot d \geq b_i$ for $\varepsilon > 0$ small.

The same holds for y instead of x. □

Existence of Extreme Points

Definition 3.21.

A polyhedron P is called pointed if it contains at least one vertex.

Definition 3.22.

A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if there is $x \in P$ and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$x + \lambda \cdot d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$

Theorem 3.23.

Consider non-empty $P \subseteq \mathbb{R}^n$ as on Slide 55. The following are equivalent:

- i P is pointed.
- ii P does not contain a line.
- iii $\operatorname{rank}\{a_i \mid i \in M \cup N\} = n$.

Proof of Theorem 3.23:

(i)=>(iii): $x^* \in P$ vertex $\Rightarrow$ x^* basic feas. solution

$\Rightarrow$ there are n linearly indep. constraints active at x^*

$\Rightarrow \operatorname{rank}\{a_i \mid a_i^T x^* = b_i\} = n$

(iii)=>(ii): Assume by contradiction that P contains a line, i.e., $x \in P$, $d \in \mathbb{R}^n \setminus \{0\}$ with

$x + \lambda \cdot d \in P \quad \forall \lambda \in \mathbb{R}$

$\Rightarrow a_i^T(x + \lambda d) = a_i^T x + \lambda a_i^T d \geq b_i$ for all $i \in M \cup N, \lambda \in \mathbb{R}$

$\Rightarrow a_i^T d = 0$ for all $i \in M \cup N$

$\Rightarrow \operatorname{rank}\{a_i \mid i \in M \cup N\} < n$ ↯

(ii) ⟹ (i): Choose $x \in P$ maximizing $|\underbrace{\{i \in M \cup N \mid a_i^T x = b_i\}}_{N \subseteq I :=}|$

If $\text{rank}\{a_i \mid i \in I\} = n$, then x is basic feas. sol. ✓

Otherwise, $\exists\, d \in \mathbb{R}^n \setminus \{0\}$ with $a_i^T d = 0 \;\forall i \in I$

$\Rightarrow a_i^T(x + \lambda \cdot d) = a_i^T x = b_i$ for all $i \in I, \lambda \in \mathbb{R}$

Since P does not contain a line, $x + \lambda' \cdot d \notin P$

for some $\lambda' \in \mathbb{R}$ $\Rightarrow a_i^T(x + \lambda' d) < b_i$ for some $i \in M \setminus I$

Assume w.l.o.g. that $\lambda' > 0$ (otherwise, replace d with $-d$)

Let $\lambda_0 := \max\{\lambda \mid a_i^T(x + \lambda d) \geq b_i \;\forall i \in M \cup N\} < \lambda'$

Notice that $\lambda_0 > 0$ since $a_i^T d = 0 \;\forall i \in I$ and $a_i^T \cdot x > b_i \;\forall i \in M \setminus I$.

Moreover, there is an $i \in M \setminus I$ with

$a_i^T(x + \lambda_0 d) = b_i$

$\Rightarrow$ At least $|I| + 1$ constraints are active at $x + \lambda_0 \cdot d \in P$ ↯ $\square$

Proof of Theorem 3.23

'(i)⟹(iii)': x^* extreme point $\Longrightarrow$ x^* basic feasible solution

$\Longrightarrow$ There are n linearly independent constraints that are active at x^*.

$\Longrightarrow$ There are n linearly independent vectors in $\{a_i \mid i \in M \cup N\}$.

'(iii)⟹(ii)': By contradiction assume that $x \in P$, $d \in \mathbb{R}^n \setminus \{0\}$ with

$$x + \lambda\, d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$

$\Longrightarrow$ $\quad {a_i}^T \cdot (x + \lambda\, d) = {a_i}^T \cdot x + \lambda\, {a_i}^T \cdot d \geq b_i$ for $i \in M \cup N$, $\lambda \in \mathbb{R}$

$\Longrightarrow$ $\quad {a_i}^T \cdot d = 0$ for all $i \in M \cup N$.

$\Longrightarrow$ $\quad \text{rank}\{a_i \mid i \in M \cup N\} < n$. ↯

Proof of Theorem 3.23 (Cont.)

'(ii)$\Longrightarrow$(i)': Choose $x \in P$ maximizing $\big|\underbrace{\{i \in M \cup N \mid {a_i}^T \cdot x = b_i\}}_{N \subseteq I :=}\big|$.

If $\operatorname{rank}\{a_i \mid i \in I\} = n$, then x is an extreme point and we are done.

Otherwise, there is $d \in \mathbb{R}^n \setminus \{0\}$ with ${a_i}^T \cdot d = 0$ for all $i \in I$.

$\Longrightarrow \quad {a_i}^T \cdot (x + \lambda\, d) = {a_i}^T \cdot x = b_i \quad$ for all $i \in I$, $\lambda \in \mathbb{R}$.

Since P does not contain a line, $x + \lambda'\, d \notin P$ for some $\lambda' \in \mathbb{R}$.

$\Longrightarrow \quad {a_i}^T \cdot (x + \lambda'\, d) < b_i$ for some $i \in M \setminus I$.

Assume w.l.o.g. $\lambda' > 0$ (otherwise replace d with $-d$) and let

$$\lambda_0 := \max\{\lambda \mid {a_i}^T \cdot (x + \lambda\, d) \geq b_i \;\forall\, i \in M \cup N\} < \lambda'.$$

Notice: $\lambda_0 > 0$ since ${a_i}^T \cdot d = 0 \;\forall\, i \in I$ and ${a_i}^T \cdot x > b_i \;\forall\, i \in M \setminus I$.

Moreover, there is an $i \in M \setminus I$ with ${a_i}^T \cdot (x + \lambda_0\, d) = b_i$.

$\Longrightarrow$ At least $|I| + 1$ constraints active at $x + \lambda_0\, d \in P$. ↯ □

Existence of Extreme Points (Cont.)

Corollary 3.24.

- a) A non-empty polytope contains an extreme point.
- b) A non-empty polyhedron in standard form contains an extreme point.
- c) Polyhedron $P \neq \emptyset$ is pointed if and only if $\operatorname{rec}(P)$ is pointed.

Proof of b:

$$\begin{aligned} A \cdot x &= b \\ x &\geq 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \cdot x \geq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$ □

Example:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rcrl} x_1 & + & x_2 & \geq 1 \\ x_1 & + & 2\,x_2 & \geq 0 \end{array} \right\}$$

contains a line since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in P \quad$ for all $\lambda \in \mathbb{R}$.

Characterization of Polytopes

Theorem 3.25.

A polytope is equal to the convex hull of its vertices.

Proof: Follows from Theorem 3.20 (see exercise). $\square$

Theorem 3.26.

A set $P \subseteq \mathbb{R}^n$ is a polytope if and only if there exists a finite set $V \subseteq \mathbb{R}^n$ such that P is the convex hull of V.

Proof: '$\Longrightarrow$': Follows from Theorem 3.25 since a polytope has only finitely many basic feasible solutions (vertices).

'$\Longleftarrow$': See exercise session. $\square$

Valid Inequalities, Supporting Hyperplanes, Faces, Facets

Definition 3.27.

Let $c \in \mathbb{R}^n \setminus \{0\}$ and $\gamma \in \mathbb{R}$.

a The linear inequality $c^T \cdot x \geq \gamma$ is valid for P if $P \subseteq \{x \mid c^T \cdot x \geq \gamma\}$.

b $H = \{x \in \mathbb{R}^n \mid c^T \cdot x = \gamma\}$ is a supporting hyperplane of P if $c^T \cdot x \geq \gamma$ is valid for P and $P \cap H \neq \emptyset$.

c The intersection of P with a supporting hyperplane is a face of P. Also P and $\emptyset$ are faces of P; the others are called proper faces.

d The inclusion-wise maximal proper faces are called facets.

Remarks.

- Every face of P is itself a polyhedron.
- Every vertex of P is a 0-dimensional face of P.
- The optimal LP solutions form a face of the underlying polyhedron.