Requirements for the Matrix Decompositions

LDU decomposition

1. $A \in \mathbb{C}^{n\times n}$, $A(1:k, 1:k)$ is nonsingular for all $k = 1, ..., n$.

$\mathbf{LDL^H}$ decomposition

1. $A \in \mathbb{C}^{n\times n}$, $A(1:k, 1:k)$ is nonsingular
2. A is Hermitian.

Cholesky decomposition

1. A is HPD.

Schur decomposition

1. $A \in \mathbb{C}^{n\times n}$. There are no requirements (except that A must be square).

Singular value decomposition

1. $A \in \mathbb{C}^{n\times m}$.

Polar decomposition

1. $A \in \mathbb{C}^{n\times n}$.

QR decomposition

1. $A \in \mathbb{C}^{n\times m}$ with $n \geq m$.

Arnoldi decomposition

1. $A \in \mathbb{C}^{n\times n}$.

Chapter 0: Basic Definitions and Matrix Classes

Theorems

1. $\forall$ Hermitian $A \in \mathbb{C}^{n\times n}, \forall x \in \mathbb{C}^n : x^H Ax \in \mathbb{R}$
2. The ordering $\geq$ defines a Loewner partial ordering on the set of Hermitian matrices $C \in \mathbb{C}^{n\times n}$.
   a) $A \geq A$ (reflexivity)
   b) $A \geq B \wedge B \geq A \implies A = B$ (anti symmetrie)
   c) $A \geq B, B \geq C \implies A \geq C$ (transitivity)

   What is missing so that it is a total ordering? Can you give an example why the missing property does not hold?
3. Some useful properties of HPSD/ PSD matrices.
   a) If A is HPSD, then $\lambda \geq 0$. If A is SPD, then $\lambda > 0$. $\det A \geq 0$ or $\det A > 0$ respectively.
   b) If A is HPSD, then $X^H AX$ is HPSD. If A is SPD and $X \in \mathbb{C}^{n\times k}$ has full column rank ($rank(X) = k$), then $X^H AX$ is SPD.
   c) If A is HPSD or SPD, then $A(1:k, 1:k)$ is HPSD or SPD for $\forall k = 1, ..., n$.
4. Inverse of complex matrices (applies *not only* for HPSD or PSD matrices).
   a) If A is nonsingular then the inverse is unique.
   b) If $AB = I$ or $BA = I$ then A is nonsingular and $B = A^{-1}$. We only need to check one equation.

Proofs

1. **Idea:** Show $x^H Ax = \overline{x^H Ax} \implies x^H Ax \in \mathbb{R}$.

   Proof. $x^H Ax = x^H A^H x = (x^H Ax)^H = \overline{(x^H Ax)^T} = \overline{x^T A^T \overline{x}} = \overline{x^H Ax}$. □
2. Statement (a) is easy to show.

   For statement (b) consider the matrix $D = A - B$ and calculate $e_j De_j$ and $e_j(-D)e_j$. You will see that $d_{jj} \geq 0$ and $d_{jj} \leq 0$. Hence, $d_{jj} = 0$. Then, consider $(e_i+e_j)(\pm D)(e_i+e_j)$ which will proove that $\Re(d_{ij}) = 0$. To show that $\Im(d_{ij}) = 0$ consider $(e_i+e_j)D(e_i-e_j)$.

   For a counterexample of the totality of $\geq$ consider the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
3. To show: $\lambda \geq 0$. Consider an eigenvektor $x \neq 0$. It holds: $x^H Ax \geq 0 \iff \lambda x^H x \geq 0 \implies \lambda \geq 0$ since $x^H x > 0$ $(x \neq 0)$.

   Next, $X^H AX$ is SPD. Substitute $z = Xy$. Then $z \neq 0$ because $rank(X) = k$. Thus, $y^H X^H AXy = z^H Az > 0$ for $z \neq 0$.

   Chose $X = [e_1, ..., e_k]$.
4. Let $AB = BA = I = AC = CA$. Then $B = BI = BAC = IC = C$.

   Idea: Use the rank of AB and A (show that A is nonsingular). If $AB = I_n$ then $rank(AB) = n \leq rank(A) \implies A$ is nonsingular. Then $A^{-1} = A^{-1}I = A^{-1}AB = B$.

Chapter 1: Matrix Decompositions

Theorems

1. **LDU decomposition:** The following assertions are equivalent. Let $A \in \mathbb{C}^{n\times n}$.
   a) $A = LDU$ where L and U are *unit* lower/ upper triangular matrices and D is a *nonsingular* diagonal matrix. The decomposition is *unique*.
   b) $A(1:k,1:k)$ is nonsingular for all $k = 1,...,n$ (including n).

   Give an example where the LDU decomposition does not hold but the LU decomposition.

2. **$\mathbf{LDL^H}$ decomposition:** Given *Hermitian* $A \in \mathbb{C}^{n\times n}$ with $A(1:k,1:k)$ is nonsingular for all $k = 1,...,n$. We can find *real, nonsingular, unique, diagonal* $D \in \mathbb{R}^{n\times n}$ and *unique* unit lower triangular L such that $A = LDL^H$.

3. **Cholesky decomposition:** Given a HPD matrix $A \in \mathbb{C}^{n\times n}$ we can write it as $A = LL^H$ where L is a lower, unique, triangular matrix. L has positive entries on its diagonal.

4. **Schur decomposition:** Let $A \in \mathbb{C}^{n\times n}$. Then $A = URU^H$ where U is unitary and R is upper triangular matrix.

5. **Singular Value decomposition:** Let $A \in \mathbb{C}^{n\times m}$ with $rank(A) = r$. Then $A = U\Sigma_+V^H$ with $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$ unitary and $\Sigma_+ = diag(\sigma_1,...,\sigma_r), \sigma_1 \geq ... \geq \sigma_r > 0$, $\Sigma = \begin{pmatrix}\Sigma_+ & 0\\ 0 & 0\end{pmatrix}$. Alternatively, $A = \sum_{i=1}^r \sigma_i u_i v_i^H$.

Proofs

1. $\implies$: Given $A = LDU$ show that $A(1:k,1:k)$ is nonsingular. Consider

$$A = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix} = \begin{pmatrix}L_{11} & 0\\ L_{21} & L_{22}\end{pmatrix}\begin{pmatrix}D_{11} & 0\\ 0 & D_{22}\end{pmatrix}\begin{pmatrix}U_{11} & U_{12}\\ 0 & U_{22}\end{pmatrix} = LDU$$

   with $A_{11} = A(1:k,1:k)$. Since L_{11} and U_{11} are lower/ upper triangular matrices we obtain $\det(A_{11}) = \det(D_{11}) \neq 0$ because D and thus D_{11} is nonsingular.

   $\impliedby$: Show that there exists such a decomposition for $A \in \mathbb{C}^{n\times n}$ whose leading principal submatrix are all nonsingular. **Idea:** Proof by induction over n.

   (Beginning:) For $A = (a_{11}) = 1 * a_{11} * 1$ and $a_{11} \neq 0$ because A is nonsingular.

   (Assumption:) Assume for every $A \in \mathbb{C}^{(n-1)\times(n-1)}$ with nonsingular $A(1:k,1:k), k = 1,...,n-1$ there exists a decomposition L, D, U.

   (Step:) Consider a matrix $A \in \mathbb{C}^{n\times n}$ with nonsingular leading principle submatrix:

$$A = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix} = \begin{pmatrix}I_{n-1} & 0\\ A_{21}A_{11}^{-1} & 1\end{pmatrix}\begin{pmatrix}A_{11} & 0\\ 0 & s\end{pmatrix}\begin{pmatrix}I_{n-1} & A_{11}^{-1}A_{12}\\ 0 & 1\end{pmatrix}$$

   and nonsingular $A_{11} = A(n-1,n-1)$, $s = A_{22} - A_{21}A_{11}^{-1}A_{12}$. Thus, $s \neq 0$. Since A_{11} is nonsingular, we can use the induction hypothesis to show that $A_{11} = L_{n-1}D_{n-1}U_{n-1}$. So

$$A = \begin{pmatrix}I_{n-1} & 0\\ A_{21}A_{11}^{-1} & 1\end{pmatrix}\begin{pmatrix}L_{n-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}D_{n-1} & 0\\ 0 & s\end{pmatrix}\begin{pmatrix}U_{n-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}I_{n-1} & A_{11}^{-1}A_{12}\\ 0 & 1\end{pmatrix}.$$

   Show the unicity of $A = L_1D_1U_1 = L_2D_2U_2 \implies L_2^{-1}L_1D_1 = D_2U_2U_1^{-1}$. This is a diagonal matrix. The diagonal entries of $L_2^{-1}, L_1, U_2, U_1^{-1}$ are ones.

   Counter example: $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$

2. Let $A = LDU$. Since $A = LDU = (LDU)^H = U^HD^HL^H = A^H$ it follows that $L = U^H$ and $D = D^H$ (LU decomposition is unique and D is nonsingular). Thus, D is real and $A = LDL^H$.

3. Write $A = LDL^H$ since A is Hermitian and $A(1:k,1:k)$ is nonsingular. Then, $D = L^HAL \in \mathbb{R}^{n\times n}$. A is HPD and L has full rank (it is a lower unit triangular). Thus, D is also HPD and has positive diagonal entries. Set $\tilde{L} = L\cdot D^{\frac{1}{2}} = L\cdot \mathrm{diag}(d_1^{\frac{1}{2}},...,d_n^{\frac{1}{2}})$.

4. Proof by induction. For $n = 1$ chose $U = 1$ and $R = a_{11}$. $n-1 \rightsquigarrow n$: Chose any eigenvalue of A with its unit eigenvector x. Find any matrix $Y \in \mathbb{C}^{n\times(n-1)}$ such that $X = [x, Y]$ is unitary. This means that

$$\begin{pmatrix}x^H\\ Y^H\end{pmatrix}(x, Y) = E_n$$

   It holds that $x^HY = (0,...,0) \in \mathbb{C}^{1\times(n-1)}$ und $Y^Hx = (0,...,0)^T \in \mathbb{C}^{n-1}$. Consider

$$X^HAX = \begin{pmatrix}\lambda & x^HAY\\ 0 & Y^HAY\end{pmatrix}.$$

   By induction hypothesis, $Z^H(Y^HAY)Z = \tilde{R}$ where R is upper triangular. Then, $U = [x, YZ]$. To show: U is unitary and $U^HAU = \begin{pmatrix}x^HAx & x^HAYZ\\ Z^HY^HAx & Z^HY^HAYZ\end{pmatrix} = \begin{pmatrix}\lambda & x^HAYZ\\ 0 & \tilde{R}\end{pmatrix}$ which is relatively easy.

5. Consider A^HA. It is HPSD, for $x^HA^HAx = ||Ax||_2^2 \geq 0$. Thus, it is unitarily diagonalizable: $V^HA^HAV = diag(\sigma_1^2,...,\sigma_r^2,0,...,0)$ with $\underline{r = rank(A) = rank(A^HA)}$. Partition $V = [V_1, V_2]$ and $V_1 \in \mathbb{C}^{m\times r}$. Then, $V_2^HA^HAV_2 = 0 \implies AV_2 = 0$. Define $U_1 = AV\Sigma_+^{-1}$ where $\Sigma_+ = diag(\sigma_1,...,\sigma_r)$. We see that it U_1 is unitary which means $U_1^HU_1 = \Sigma_+^{-H}V^HA^HAV\Sigma_+^{-1} = \Sigma^{-H}\Sigma_+^2\Sigma^{-1} = I_r$. So, we find $U = [U_1, U_2]$ such that U is unitary. Finally,

$$U^HAV = \begin{pmatrix}U_1^HAV_1 & U_1^HAV_2\\ U_2^HAV_1 & U_2^HAV_2\end{pmatrix} = \begin{pmatrix}\Sigma_+ & 0\\ U_2^HU_1\Sigma_+ & 0\end{pmatrix} = \begin{pmatrix}\Sigma_+ & 0\\ 0 & 0\end{pmatrix}.$$

General questions

1. Why is $rank(A) = rank(A^H A)$ for any matrix $A \in \mathbb{C}^{n \times m}$?