Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Tutorial Assignment 1

Martin Genzel, Mones Raslan Summer Term 2019

Exercise 1: Let (X,d) be a metric space. Prove the following assertions.

(i) We have

 (a) $\varnothing, X$ are open.

 (b) $U_1, \dots U_r \subseteq X$ open $\Rightarrow \bigcap_{i=1}^{r} U_i$ is open.

 (c) $U_i \subseteq X, i \in I$ open $\Rightarrow \bigcup_{i\in I} U_i$ is open.

(ii) We have

 (a) $\varnothing, X$ are closed.

 (b) $A_i \subseteq X, i \in I$ closed $\Rightarrow \bigcap_{i\in I} A_i$ is closed.

 (c) $A_1, \dots A_r \subseteq X$ closed $\Rightarrow \bigcup_{i=1}^{r} A_i$ is closed.

(iii) For each $x \in X$, $r > 0$, the set $K_r(x)$ is closed.

(iv) For $E \subseteq X$, $\overline{E}$ is the smallest closed set containing E.

(v) For $E \subseteq X$, $\mathring{E}$ is the biggest open set contained in E.

Background: A *topology* on a set X is a collection $\mathfrak{U} := \{U_j\}_{j\in J}$ of subsets such that $\varnothing \in \mathfrak{U}$, $X \in \mathfrak{U}$, and each union and each finite(!) intersection of sets in $\mathfrak{U}$ is again contained in $\mathfrak{U}$. The pair $(X, \mathfrak{U})$ is then called a *topological space* and the elements of $\mathfrak{U}$ *open sets*. Hence, Exercise 1 (i) states that the open sets of a metric space (X,d) form a topology on X .

Exercise 2: Let (X, d) be a metric space. Prove the following.

(i) A sequence can have at most one limit.

(ii) Let $E \subset X$. Then $x \in \overline{E}$ if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset E$ with $x_n \to x$ as $n \to \infty$.

(iii) If $(x_n)_{n\in\mathbb{N}} \subset X$ is convergent, then $(x_n)_{n\in\mathbb{N}} \subset X$ is a Cauchy-sequence. The converse is not always true[1]. A Cauchy-sequence is convergent, if it contains a convergent subsequence.

(iv) If X is complete and $E \subset X$ closed, then E is complete. If $E \subset X$ is complete, then E is closed in X.

Exercise 3: Let (X, d) and (X', d') be metric spaces and $f : X \to X'$.

(i) Prove that f is continuous at $x \in X$ if and only if for each sequence $(x_n) \subset X$ with $d(x_n, x) \to 0$ we have $d'(f(x_n), f(x)) \to 0$ as $n \to \infty$.

(ii) Show that f is continuous (on X) if and only if for every open set $V \subset X'$ the set $f^{-1}(V) \subset X$ is open.

(iii) Find an example where f is continuous, U open, but $f(U)$ not open in X'.

(iv) If X is compact, show that f is continuous if and only if f is uniformly continuous.

Exercise 4: Prove that every norm induces a metric. Does every metric come from a norm?

[1]For example, consider $X = (0, 1]$, $x_n = \frac{1}{n}$.