
Exercise sheet 1

Exercise 1**4 Points**

As we saw in the lecture, a hyperplane is an affine subspace of codimension one.

- a) Let $H_1, H_2 \subset \mathbb{R}^n$ be hyperplanes. Determine all possible values for $\dim(H_1 \cap H_2)$.
- b) A hyperplane may be described by a (translated) normal vector. Consider hyperplanes $H_1, \dots, H_7 \subset \mathbb{R}^9$ with the following normal vectors (all starting in the origin, of course):

$$\begin{aligned} n_1 &= (1, 0, 1, 0, 0, 0, 0, 0, 1) \\ n_2 &= (0, 1, 0, 0, 0, 0, 0, 0, 0) \\ n_3 &= (0, 0, 1, 0, 1, 1, 0, 0, 0) \\ n_4 &= (1, 1, 1, 1, 0, 1, 1, 1, 1) \\ n_5 &= (1, 0, 0, 0, 0, 0, 1, 1, 1) \\ n_6 &= (1, 0, 0, 0, 1, 0, 1, 0, 1) \\ n_7 &= (0, 1, 1, 0, 0, 0, 0, 0, 0) . \end{aligned}$$

Calculate $\dim(H_1 \cap \dots \cap H_7)$.

our solution:

- (a) Consider two hyperplanes that have no point in common. The dimension of the intersection of both hyperplanes is zero.

For the case that both hyperplanes H_1 and H_2 have a point in common in \mathbb{R}^n , note that one can shift any linear subspace to contain the origin without affecting its dimension. This is known as $\dim(a + V) = \dim(V)$ for any linear subspace V . The intersection of two hyperplanes results in an affine linear subspace that can be shifted to include the origin. For the sake of simplicity, let H_1 and H_2 denote the shifted hyperplanes whose intersection holds the origin. Then, the dimension formula which reads $\dim(H_1 \cap H_2) = \dim H_1 + \dim H_2 - \dim(H_1 + H_2)$ holds. The Minkowski sum of H_1 and H_2 either results in H_1 if both hyperplanes are identical, or \mathbb{R}^n otherwise. Thus, the dimension of $H_1 \cap H_2$ takes the value $2n - 2 - (n - 1) = n - 1$ or $2n - 2 - n = n - 2$.

All in all, possible values for $\dim(H_1 \cap H_2)$ are zero, $n - 1$ or $n - 2$.

- (b) **Find** $\dim(H_1 \cap \dots \cap H_7)$.

The idea is to apply the previous result inductively on all the hyperplanes. First, all hyperplanes contain the origin, yet are distinct due to the linear independence of the normal vectors. Thus, each intersection of two hyperplanes shrinks the dimension by one as seen in the previous statement. Consequently, the dimension is $9 - 7 = 2$.

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Exercise 2

2 Points

Let $X, Y \subset \mathbb{R}^n$. Show that $\text{conv}(X + Y) = \text{conv}(X) + \text{conv}(Y)$ holds.

our solution:

- $\text{conv}(X+Y) \subset \text{conv}(X)+\text{conv}(Y)$: First, $X+Y \subset \text{conv}(X)+\text{conv}(Y)$. In addition, $\text{conv}(X)+\text{conv}(Y)$ is convex as the Minkowski sum of convex sets is again convex. Since $\text{conv}(X+Y)$ is the smallest convex set containing $X+Y$, the inclusion stated above is proven.
- $\text{conv}(X+Y) \supset \text{conv}(X)+\text{conv}(Y)$: Any point $x \in \text{conv}(X)$ translated by a vector $y \in Y$ is in $\text{conv}(X+Y)$ as it is shown below:

$$x + y = \sum_{i=1}^k \alpha_i (x_i + y), \quad x_i \in X, \sum_{i=1}^k \alpha_i = 1.$$

If one translates x not by y but instead by a convex combination of vectors y_1, \dots, y_k from Y , one obtains

$$x + \sum_{i=1}^k \alpha_i y_i = \sum_{i=1}^k \alpha_i (x + y_i) \in \text{conv}(\text{conv}(X + Y)) = \text{conv}(X + Y).$$

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Exercise 3

6 Points

Let $X \subset \mathbb{R}^n$. Prove or disprove the following:

- If X is closed then $\text{conv}(X)$ is closed.
- If X is convex then \overline{X} is convex.
- If X is open then $\text{conv}(X)$ is open.

our solution:

- (a) **Statement:** If a set is closed, then its convex hull is also closed.

Answer: This statement is *wrong*. Consider the set that contains all members of the sequence $(\frac{1}{n})_{n \in \mathbb{N}}$. This set is closed. Its convex hull yields the interval from zero to one, including one but excluding zero. Here is the reason why it does not contain zero: every member of the sequence is strictly positive, and thus the convex combination must be strictly positive, as well.

- (b) **Statement:** If a set is convex, then its closure is convex, too.

Answer: This is *true*. Consider two points a and b from the closure of X . Then, both a and b are the limits of two sequences (a_n) and (b_n) in X , respectively. We will see that any point $c = a + (b - a)\lambda$, where $\lambda \in (0, 1)$, is in the closure of X , too. This follows from:

$$\underbrace{a_n + (b_n - a_n)\lambda}_{\in X} \rightarrow a + (b - a)\lambda = c \in \bar{X}, \quad n \rightarrow \infty.$$

- (c) **Statement:** The convex hull of an open set is open.

Answer: This statement is *true*. Let $A \subset \mathbb{R}^n$ be open and $x \in \text{conv}(A)$. We want to find an open neighborhood of x that is a subset of A . Let $x = \sum_{i=1}^n \alpha_i x_i$ be a convex combination of x . Without loss of generality, let $\alpha_1 \neq 0$. Define the continuous function $\gamma_x : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as

$$\gamma_x(y) : \frac{y - \sum_{i=2}^n \alpha_i x_i}{\alpha_1}.$$

Its inverse function reads

$$\gamma_x^{-1}(\tilde{x}) = \alpha_1 \tilde{x} + \sum_{i=2}^n \alpha_i x_i.$$

We see that $\gamma_x(x) = x_1 \in A$. Therefore, $x \in \gamma_x^{-1}(A)$. So, we found an open neighborhood for x , namely $\gamma_x^{-1}(A)$.

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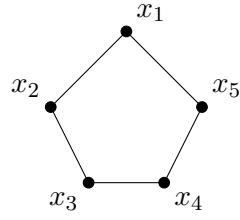
Exercise 4

4 Points

Let $X \subset \mathbb{R}^2$ with cardinality $\#X \geq 5$ such that no three pairwise different points lie on a common line. Show that there are $x_1, \dots, x_4 \in X$ such that $\text{conv}(x_1, \dots, x_4)$ is quadrilateral.

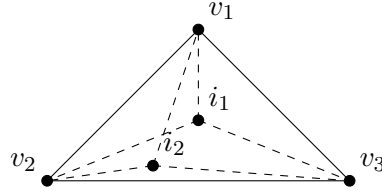
our solution:

- If the convex hull of the given points has four or more vertices, then select any four vertices, and we are done.



- In the other case, the convex hull has three vertices v_1, v_2, v_3 , and two interior points i_1, i_2 . The interior point i_1 partitions the triangle $\overline{v_1 v_2 v_3}$ into smaller subtriangles $\triangle_1, \triangle_2, \triangle_3$. The same for i_2 ; there are three smaller subtriangles $\tilde{\triangle}_1, \tilde{\triangle}_2, \tilde{\triangle}_3$. Then, i_2 must lie in one triangle \triangle_i , and i_1 must lie in one triangle $\tilde{\triangle}_j$ for some $i, j \in \{1, 2, 3\}$. Therefore, there are two vertices v_k and v_l left such that $\overline{v_k v_l i_1} \neq \triangle_i$ and $\overline{v_k v_l i_2} \neq \tilde{\triangle}_j$.

The claim is that $\text{conv}\{v_k, v_l, i_1, i_2\}$ is a convex set, i.e. a quadrilateral. If it were not a convex set, then i_1 or i_2 must be a convex combination of the other three points. But by construction and because of the fact that no three points lie on a line, this is not possible. Thus, $\text{conv}\{v_k, v_l, i_1, i_2\}$ is a convex set.



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total grade:

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