Functional Analysis Sheet 5

Featuring Diddle

TO-DO

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Wichtig: *https://www.youtube.com/watch?v=0-gR9KPCVbo*

Exercise 1

> To prove
>
> If E is separable, then $F := \{x \in E : ||x|| = 1\}$ is separable.

Proof. Let E be a normed space, which is separable. This means that there exists a *countable dense* subset $Q \subset E$.

Claim: $\tilde{Q} := \{\frac{x}{||x||} : x \in Q\}$ is a countable dense subset of F.

1. ***Countable subset:*** For all $x \in \tilde{Q}$ it holds $||x|| = 1$. Thus, $x \in F$.

   We know Q is countable. There exists a bijection $\tau : \tilde{Q} \to Q$ with

   $$\tau(\tilde{q}) = \tilde{q}||\tilde{q}||.$$

   So, $\tilde{Q}$ is countable.

2. ***Dense in F:*** Let $x \in F$. We know that Q is dense in E. Hence, there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset Q$ with $x_n \neq 0$ for all $n \in \mathbb{N}$ such that

   $$\lim_{n \to \infty} x_n = x.$$

Define a new sequence $(y_n)_{n\in\mathbb{N}}$ with

$$y_n := \frac{x_n}{||x_n||} \in \tilde{Q} \qquad \forall n \in \mathbb{N}.$$

We will see that $y_n \to x$:

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{x_n}{||x_n||} = \frac{\lim_{n\to\infty} x_n}{\underbrace{||\lim_{n\to\infty} x_n||}_{||\cdot|| \text{ is continuous}}} = \frac{x}{\underbrace{||x||}_{=1}} = x.$$

So, $\tilde{Q}$ is dense in F.

We have shown the claim. Therefore, F is separable per definition, as we have found a countable dense subset of F, namely Q. □

To prove

If $F := \{x \in E : ||x|| = 1\}$ is separable, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset E$ such that

$$\overline{\text{span}((x_n)_{n\in\mathbb{N}})} = E.$$

Proof. Let $F = \{x \in E : ||x|| = 1\}$ be separable. Hence, we find a countable subset $Q \subset F$, that is dense in F where Q is of the Form $Q = (x_n)_{n\in\mathbb{N}} \subset E$ and $||x_n|| = 1$ for all $n \in \mathbb{N}$. Let now $x \in E$ be arbitrary. We want to find a sequence $(y_n)_{n\in\mathbb{N}}$ in Q with $y_n \to x$ for $n \to \infty$ and every y_n a finite linear combination of our (x_n)

Note that in any case where $x \neq 0$ (for $x = 0$ everything is clear) we know $\frac{x}{||x||} = 1$. Thus, $\frac{x}{||x||} \in F$ so by assumption we can find a sequence $(\tilde{y}_n)_{n\in\mathbb{N}} \subset Q$ with $\tilde{y}_n \overset{n\to\infty}{\longrightarrow} \frac{x}{||x||}$ and . This of course means

$$\lim_{n\to\infty} \tilde{y}_n = \frac{x}{||x||} \iff ||x|| \lim_{n\to\infty} \tilde{y}_n = \lim_{n\to\infty} ||x||\tilde{y}_n = x.$$

Since for all natural numbers n we can find an $i_n \in \mathbb{N}$ with $y_n = x_{i_n}$ and hence we know that $||x||y_n \in \text{ span}((x_n)_n)$ and that means

$$x \in \overline{\text{span}((x_n)_n)} \implies E = \overline{\text{span}((x_n)_n)}$$

To show: $\overline{\text{span}((x_n)_{n\in\mathbb{N}})} \subset E$. Let $x \in \overline{\text{span}((x_n)_{n\in\mathbb{N}})}$.

We know that $\mathbb{C}_\mathbb{Q} := \{z = a + ib : a, b \in \mathbb{Q}\}$ is countable and dense in $\mathbb{C}$. Thus, $\text{span}_\mathbb{Q} := \left\{x \in E \mid \exists m_x \in \mathbb{N} : x = \sum_{i=1}^{m_x} \lambda_i x_{\tau_x(i)} \in \text{span}((x_n)_n) \text{ and } \lambda_i \in \mathbb{C}_\mathbb{Q}\right\}$ - where τ_x is some permutation of $\mathbb{N}$ - is also dense in $\text{span}((x_n)_n)$

Now we can see that

$$\begin{aligned}\text{span}_\mathbb{Q} &= \bigcup_{m\in\mathbb{N}} \left\{x = \sum_{i=1}^{m} \lambda_i x_{\tau_x(i)} \in E \mid \lambda_i \in \mathbb{C}_\mathbb{Q} \text{ for all } i \in \{1, ..., m\}\right\} \\ &= \bigcup_{m\in\mathbb{N}} \bigcup_{(n_1,\ldots,n_m)\in\mathbb{N}^m} \bigcup_{(\lambda_1,\ldots,\lambda_m)\in(\mathbb{C}_\mathbb{Q})^m} \left\{\sum_{i=1}^{m} \lambda_i x_{n_i}\right\}\end{aligned}$$

which is countable. Thus we have a countable set for which

$$\overline{\text{span}_\mathbb{Q}} = \overline{\text{span}((x_n)_n)} = E.$$

Hence E is separable.

Now for the second part of the task: We show that E^* being separable implies E being separable by proving that $\text{span}((x_n)_n)$ is dense in E for a sequence (x_n) as discussed in the task. We will, however, assume that $||x_n|| = 1$ for all natural n. We can do that because $1 = ||\ell_n|| = \sup_{||x||=1} |\ell(x)|$ implies that for all positive ϵ we can find an x with $||x|| = 1$ and still $|\ell_n(x)| > 1 - \epsilon$.

Now towards a contradiction assume there exists a $x \in E$ with $x \notin \text{span}((x_n)_n)$. This means we have

$$\inf_{y\in\text{span}((x_n)_n)} ||x - y|| > 0.$$

Then Corollary 4.8 in the script yields us an $\ell \in E^*$ with $\ell|_{\text{span}((x_n)_n)} = 0$ and $||\ell|| = 1$, so $\ell \in S$.

This leads us for any $n \in \mathbb{N}$ to

$$||\ell - \ell_n|| = \sup_{x\neq0} \frac{|\ell(x) - \ell_n(x)|}{||x||} \geq \frac{|\ell(x_n) - \ell_n(x_n)|}{||x_n||} = \frac{|\ell_n(x_n)|}{||x_n||} = \frac{1}{2||x_n||} > 0.$$

Thus, $\ell \notin \overline{(\ell_n)_n}$ and hence $(\ell_n)_n$ is not dense in S. That is a contradiction!

□

Exercise 2

Let E be a $\mathbb{C}$ vector space with a seminorm $\rho : E \to \mathbb{R}$ on E, and let F be a linear subspace $F \subset E$. Let $f : F \to \mathbb{K}$ with

$$|f(x)| \leq \rho(x), \quad \forall x \in F.$$

To prove

There exists an extension $\ell : E \to \mathbb{K}$ such that

$$\ell_{|F} = f \text{ and } |\ell(x)| \leq \rho(x) \quad \forall x \in F.$$

Exercise 3

(i) Let X be a normed space and $C \subset X$ be open and convex with $0 \in C$. We show that $p_C : X \to [0, \infty)$ (note that $p_C(x) < \infty$ for all $x \in X$, which we will show later) is a sublinear functional.

To prove

For all $x, y \in X$ it holds

$$p_C(x + y) \leq p_C(x) + p_C(y).$$

Proof. Let $x, y \in X$. From 3(ii) we know that C is absorbing. Thus, $p_C(x) \neq \infty, p_C(y) \neq \infty$ and $p_C(x + y) \neq \infty$. Let $\epsilon > 0$ be arbitrary. For x there exists $\mu_x \in \mathbb{R}_{>0}$ and $z_y \in C$ such that $x = \mu_x z_x$ and $\mu_x < p_C(x) + \frac{\epsilon}{2}$. Likewise for y: there is $\mu_y \in \mathbb{R}_{>0}$ such that $y = \mu_y z_y$ with $\mu_y < p_C(y) + \frac{\epsilon}{2}$. Adding μ_x and μ_y together yields

$$\mu_x + \mu_y < p_C(x) + p_C(y) + \epsilon.$$

Goal

Next, we try to find a vector $z_{x+y} \in C$ and scalar $\mu_{x+y} > 0$ such that

$$\mu_{x+y} z_{x+y} = x + y.$$

Consider the vector

$$z_{x+y} := \frac{\mu_x}{\mu_x + \mu_y} z_x + \frac{\mu_y}{\mu_x + \mu_y} z_y,$$

which luckily lies in C. To see this, remember that C is a convex set, and note that z_{x+y} is a convex combination of $z_x \in C$ and $z_y \in C$ since

$$\frac{\mu_x}{\mu_x + \mu_y} + \frac{\mu_y}{\mu_x + \mu_y} = 1.$$

Define the scalar μ_{x+y} as $\mu_{x+y} \coloneqq \mu_x + \mu_y > 0$. We see that

$$\mu_{x+y} z_{x+y} = \mu_x z_x + \mu_y z_y = x + y.$$

Putting everything together

$$p_C(x+y) \overset{(*)}{\leq} \mu_{x+y} = \mu_x + \mu_y < p_C(x) + p_C(y) + \epsilon.$$

Explanation

$(*)$ holds by definition of $p_C(x+y)$.

As ϵ was chosen arbitrarily, it holds

$$p_C(x+y) \leq p_C(x) + p_C(y).$$

$\square$

To prove

For any $x \in X$ and $\lambda \in \mathbb{R}_{>0}$ it holds

$$p_C(\lambda x) = \lambda p_C(x).$$

Proof. Let $x \in X$ and $\lambda > 0$. Then,

$$\begin{aligned} p_C(\lambda x) = \inf\{\alpha > 0 : \lambda x \in \alpha C\} &= \inf\{\alpha > 0 : x \in \frac{\alpha}{\lambda} C\} \\ &= \inf\{\lambda\alpha > 0 : x \in \alpha C\} \\ &= \lambda \inf\{\alpha > 0 : x \in \alpha C\} \\ &= \lambda p_C(x). \end{aligned}$$

$\square$

Conclusion: $p_C : X \to [0, \infty)$ is a sublinear functional.

(ii)

(iii) We want to show:

To prove

$$C = \{x \in X : p_C(x) < 1\}$$

Proof. sadasd ☐