Requirements for the Matrix Decompositions

LDU decomposition

1. $A \in \mathbb{C}^{n \times n}$, $A(1:k, 1:k)$ is nonsingular for all $k = 1, ..., n$.

LDL$^{\mathbf{H}}$ decomposition

1. $A \in \mathbb{C}^{n \times n}$, $A(1:k, 1:k)$ is nonsingular

2. A is Hermitian.

Cholesky decomposition

1. A is HPD.

Schur decomposition

1. $A \in \mathbb{C}^{n \times n}$. There are no requirements (except that A must be square).

Singular value decomposition

1. $A \in \mathbb{C}^{n \times m}$.

Polar decomposition

1. $A \in \mathbb{C}^{n \times n}$.

QR decomposition

1. $A \in \mathbb{C}^{n \times m}$ with $n \geq m$.

Arnoldi decomposition

1. $A \in \mathbb{C}^{n \times n}$.

Chapter 0: Basic Definitions and Matrix Classes

Theorems

1. $\forall$ Hermitian $A \in \mathbb{C}^{n \times n}, \forall x \in \mathbb{C}^n : x^H A x \in \mathbb{R}$

2. The ordering $\geq$ defines a Loewner partial ordering on the set of Hermitian matrices $C \in \mathbb{C}^{n \times n}$.

   a) $A \geq A$ (reflexivity)
   b) $A \geq B \wedge B \geq A \implies A = B$ (anti symmetrie)

   c) $A \geq B, B \geq C \implies A \geq C$ (transitivity)

   What is missing so that it is a total ordering? Can you give an example why the missing property does not hold?

3. Some useful properties of HPSD/ PSD matrices.

   a) If A is HPSD, then $\lambda \geq 0$. If A is SPD, then $\lambda > 0$. $\det A \geq 0$ or $\det A > 0$ respectively.

   b) If A is HPSD, then $X^H A X$ is HPSD. If A is SPD and $X \in \mathbb{C}^{n \times k}$ has full column rank $(rank(X) = k)$, then $X^H A X$ is SPD.

   c) If A is HPSD or SPD, then $A(1:k, 1:k)$ is HPSD or SPD for $\forall k = 1, ..., n$.

4. Inverse of complex matrices (applies *not only* for HPSD or PSD matrices).

   a) If A is nonsingular then the inverse is unique.

   b) If $AB = I$ or $BA = I$ then A is nonsingular and $B = A^{-1}$. We only need to check one equation.

Proofs

1. **Idea:** Show $x^H A x = \overline{x^H A x} \implies x^H A x \in \mathbb{R}$.

   Proof. $x^H A x = x^H A^H x = (x^H A x)^H = \overline{(x^H A x)^T} = \overline{x^T A^T \overline{x}} = \overline{x^H A x}$. $\qquad \square$

2. Statement (a) is easy to show.

   For statement (b) consider the matrix $D = A - B$ and calculate $e_j D e_j$ and $e_j(-D)e_j$. You will see that $d_{jj} \geq 0$ and $d_{jj} \leq 0$. Hence, $d_{jj} = 0$. Then, consider $(e_i + e_j)(\pm D)(e_i + e_j)$ which will proove that $\Re(d_{ij}) = 0$. To show that $\Im(d_{ij}) = 0$ consider $(e_i + e_j)D(e_i - e_j)$.

   For a counterexample of the totality of $\geq$ consider the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

3. To show: $\lambda \geq 0$. Consider an eigenvektor $x \neq 0$. It holds: $x^H A x \geq 0 \iff \lambda x^H x \geq 0 \implies \lambda \geq 0$ since $x^H x > 0$ $(x \neq 0)$.

   Next, $X^H A X$ is SPD. Substitute $z = Xy$. Then $z \neq 0$ because $rank(X) = k$. Thus, $y^H X^H A X y = z^H A z > 0$ for $z \neq 0$.

   Chose $X = [e_1, ..., e_k]$.

4. Let $AB = BA = I = AC = CA$. Then $B = BI = BAC = IC = C$.

   Idea: Use the rank of AB and A (show that A is nonsingular). If $AB = I_n$ then $rank(AB) = n \leq rank(A) \implies A$ is nonsingular. Then $A^{-1} = A^{-1} I = A^{-1} AB = B$.

Chapter 1: Matrix Decompositions

Theorems

1. **LDU decomposition:** The following assertions are equivalent. Let $A \in \mathbb{C}^{n \times n}$.

   a) $A = LDU$ where L and U are *unit* lower/ upper triangular matrices and D is a *nonsingular* diagonal matrix. The decomposition is *unique*.

   b) $A(1:k,1:k)$ is nonsingular for all $k = 1,...,n$ (including n).

   Give an example where the LDU decomposition does not hold but the LU decomposition.

2. **LDL$^\mathbf{H}$ decomposition:** Given *Hermitian* $A \in \mathbb{C}^{n \times n}$ with $A(1:k,1:k)$ is nonsingular for all $k = 1,...,n$. We can find *real, nonsingular, unique, diagonal* $D \in \mathbb{R}^{n \times n}$ and *unique* unit lower triangular L such that $A = LDL^H$.

3. **Cholesky decomposition:** Given a HPD matrix $A \in \mathbb{C}^{n \times n}$ we can write it as $A = LL^H$ where L is a lower, unique, triangular matrix. L has positive entries on its diagonal.

4. **Schur decomposition:** Let $A \in \mathbb{C}^{n \times n}$. Then $A = URU^H$ where U is unitary and R is upper triangular matrix.

5. **Spectral decompositions of normal matrices**:

   (a) A is normal iff A can unitarily diagonalized, i.e. $A = UDU^H$ with U is unitary and D is diagonal.

   (b) A is Hermitian iff A can be unitarily diagonalized and all the eigenvalues of A are real.

   (c) A is unitary iff A can unitarily diagonalized and all $|\lambda| = 1$.

6. **Singular Value decomposition:** Let $A \in \mathbb{C}^{n \times m}$ with $rank(A) = r$. Then $A = U\Sigma_+ V^H$ with $U \in \mathbb{C}^{n \times n}$ and $V \in \mathbb{C}^{m \times m}$ unitary and $\Sigma_+ = diag(\sigma_1,...,\sigma_r), \sigma_1 \geq ... \geq \sigma_r > 0$, $\Sigma = \begin{pmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{pmatrix}$. Alternatively, $A = \sum_{i=1}^{r} \sigma_i u_i v_i^H$.

Proofs

1. $\Longrightarrow$: Given $A = LDU$ show that $A(1:k,1:k)$ is nonsingular. Consider

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} = LDU$$

with $A_{11} = A(1:k,1:k)$. Since L_{11} and U_{11} are lower/ upper triangular matrices we obtain $\det(A_{11}) = \det(D_{11}) \neq 0$ because D and thus D_{11} is nonsingular.

$\Longleftarrow$: Show that there exists such a decomposition for $A \in \mathbb{C}^{n \times n}$ whose leading principal submatrix are all nonsingular. **Idea:** Proof by induction over n.

(Beginning:) For $A = (a_{11}) = 1 * a_{11} * 1$ and $a_{11} \neq 0$ because A is nonsingular.

(Assumption:) Assume for every $A \in \mathbb{C}^{(n-1) \times (n-1)}$ with nonsingular $A(1:k,1:k), k = 1,...,n-1$ there exists a decomposition L,D,U.

(Step:) Consider a matrix $A \in \mathbb{C}^{n \times n}$ with nonsingular leading principle submatrix:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I_{n-1} & 0 \\ A_{21}A_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} I_{n-1} & A_{11}^{-1}A_{12} \\ 0 & 1 \end{pmatrix}$$

and nonsingular $A_{11} = A(n-1,n-1)$, $s = A_{22} - A_{21}A_{11}^{-1}A_{12}$. Thus, $s \neq 0$. Since A_{11} is nonsingular, we can use the induction hypothesis to show that $A_{11} = L_{n-1}D_{n-1}U_{n-1}$. So

$$A = \begin{pmatrix} I_{n-1} & 0 \\ A_{21}A_{11}^{-1} & 1 \end{pmatrix} \begin{pmatrix} L_{n-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D_{n-1} & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} U_{n-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_{n-1} & A_{11}^{-1}A_{12} \\ 0 & 1 \end{pmatrix}.$$

Show the unicity of $A = L_1 D_1 U_1 = L_2 D_2 U_2 \implies L_2^{-1} L_1 D_1 = D_2 U_2 U_1^{-1}$. This is a diagonal matrix. The diagonal entries of $L_2^{-1}, L_1, U_2, U_1^{-1}$ are ones.

Counter example: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2. Let $A = LDU$. Since $A = LDU = (LDU)^H = U^H D^H L^H = A^H$ it follows that $L = U^H$ and $D = D^H$ (LU decomposition is unique and D is nonsingular). Thus, D is real and $A = LDL^H$.

3. Write $A = LDL^H$ since A is Hermitian and $A(1:k,1:k)$ is nonsingular. Then, $D = L^H AL \in \mathbb{R}^{n \times n}$. A is HPD and L has full rank (it is a lower unit triangular). Thus, D is also HPD and has positive diagonal entries. Set $\tilde{L} = L \cdot D^{\frac{1}{2}} = L \cdot diag(d_1^{\frac{1}{2}},...,d_n^{\frac{1}{2}})$.

4. Proof by induction. For $n = 1$ chose $U = 1$ and $R = a_{11}$. $n-1 \rightsquigarrow n$: Chose any eigenvalue of A with its unit eigenvector x. Find any matrix $Y \in \mathbb{C}^{n \times (n-1)}$ such that $X = [x,Y]$ is unitary. This means that

$$\begin{pmatrix} x^H \\ Y^H \end{pmatrix} (x,Y) = E_n$$

It holds that $x^H Y = (0,...,0) \in \mathbb{C}^{1 \times (n-1)}$ und $Y^H x = (0,...,0)^T \in \mathbb{C}^{n-1}$. Consider

$$X^H AX = \begin{pmatrix} \lambda & x^H AY \\ 0 & Y^H AY \end{pmatrix}.$$

By induction hypothesis, $Z^H(Y^H AY)Z = \tilde{R}$ where R is upper triangular. Then, $U = [x,YZ]$. To show: U is unitary and $U^H AU = \begin{pmatrix} x^H Ax & x^H AYZ \\ Z^H Y^H Ax & Z^H Y^H AYZ \end{pmatrix} = \begin{pmatrix} \lambda & x^H AYZ \\ 0 & \tilde{R} \end{pmatrix}$ which is relatively easy.

5. a) Let A be normal. We have $A = URU^H$ by Schur decomposition and $A^H A = UR^H U^H URU^H = AA^H$ implies that $R^H R = RR^H$. Write $R = \begin{pmatrix} R_1 & r \\ 0 & p \end{pmatrix}$. This gives

$$R^H R = \begin{pmatrix} R_1^H R_1 & R_1^H r \\ r^H R_1 & ||r||_2^2 + |p|^2 \end{pmatrix} = RR^H = \begin{pmatrix} R_1 R_1^H + rr^H & r\bar{p} \\ pr^H & |p|^2 \end{pmatrix}$$

and we see that $r = 0$. Because R is normal we can apply the same argument for R_1 inductively. Now let A be unitarily diagonalizable. We want to show that A is normal. $A^H A = UR^H U^H URU^H = UR^H RU^H = URR^H U^H = AA^H$.

   b) Let A be Hermitian. Then $A = UDU^H \implies A^H = A \iff UD^H U^H = UDU^H$. We see that $D^H = D$ and the eigenvalues are real. Now let A be unitarily diagonalizable with real eigenvalues. Then $A = UDU^H = UD^H U^H = A^H$.

6. Consider $A^H A$. It is HPSD, for $x^H A^H A x = ||Ax||_2^2 \geq 0$. Thus, it is unitarily diagonalizable: $V^H A^H A V = diag(\sigma_1^2, ..., \sigma_r^2, 0, ..., 0)$ with $r = rank(A) = rank(A^H A)$. Partition $V = [V_1, V_2]$ and $V_1 \in \mathbb{C}^{m \times r}$. Then, $V_2^H A^H A V_2 = 0 \implies AV_2 = 0$. Define $U_1 = AV\Sigma_+^{-1}$ where $\Sigma_+ = diag(\sigma_1, ..., \sigma_r)$. We see that it U_1 is unitary which means $U_1^H U_1 = \Sigma_+^{-H} V^H A^H A V \Sigma_+^{-1} = \Sigma^{-H} \Sigma_+^2 \Sigma^{-1} = I_r$. So, we find $U = [U_1, U_2]$ such that U is unitary. Finally,

$$U^H A V = \begin{pmatrix} U_1^H A V_1 & U_1^H A V_2 \\ U_2^H A V_1 & U_2^H A V_2 \end{pmatrix} = \begin{pmatrix} \Sigma_+ & 0 \\ U_2^H U_1 \Sigma_+ & 0 \end{pmatrix} = \begin{pmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{pmatrix}.$$

General questions

1. Why is $rank(A) = rank(A^H A)$ for any matrix $A \in \mathbb{C}^{n \times m}$?

    Answer: Show that the null spaces are equal. If $x \in N(A)$, then $Ax = 0 \implies A^H Ax = 0$. Now regard $A^H Ax = 0$. Multiply by x^H from the left and we get $x^H A^H Ax = ||Ax||_2^2 = 0$ which implies that $Ax = 0$.

2. Schur decomposition: Let $U = [u_1, ..., u_n]$. Why do $u_1, ..., u_k$ build an invariant subspace for A? That means: $Au_k = span(u_1, ..., u_k)$. Why is the first column u_1 a unit eigenvector of the first eigenvalue of r_{11}?