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**Adressed to:** Jan Macdonald

# Functional Analysis

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*Bring back Moses!*

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## We need your help!

We the People want Moses back to rate us. We believe this is crossing a moral line and ask that the law concerning the banishment of Moses be overturned.

You can support us on by signing the petition under

***https://petitions.whitehouse.gov/petition/bring-moses-back***

This will help us to bring the issue to the desk of the currently in charged first President of the United States ***Donald Trump***.

Thanks!

## Exercise 1

To prove

If $E$ is separable, then $F := \{x \in E : ||x|| = 1\}$ is separable.

*Proof.* Let $E$ be a normed space, which is separable. This means that there exists a *countable dense* subset $Q \subset E$.

**Claim:** $\tilde{Q} := \{\frac{x}{||x||} : x \in Q\}$ is a countable dense subset of $F$.

1. ***Countable subset:*** For all $x \in \tilde{Q}$ it holds $||x|| = 1$. Thus, $x \in F$.

   We know $Q$ is countable. There exists a bijection $\tau : \tilde{Q} \to Q$ with

   $$\tau(\tilde{q}) = \tilde{q}||\tilde{q}||.$$

   So, $\tilde{Q}$ is countable.

2. ***Dense in*** $F$***:*** Let $x \in F$. We know that $Q$ is dense in $E$. Hence, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset Q$ with $x_n \neq 0$ for all $n \in \mathbb{N}$ such that

   $$\lim_{n\to\infty} x_n = x.$$

   Define a new sequence $(y_n)_{n\in\mathbb{N}}$ with

   $$y_n := \frac{x_n}{||x_n||} \in \tilde{Q} \qquad \forall n \in \mathbb{N}.$$

   We will see that $y_n \to x$:

   $$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{x_n}{||x_n||} = \frac{\lim_{n\to\infty} x_n}{\underbrace{||\lim_{n\to\infty} x_n||}_{||\cdot|| \text{ is continuous}}} = \frac{x}{\underbrace{||x||}_{=1}} = x.$$

   So, $\tilde{Q}$ is dense in $F$.

We have shown the claim. Therefore, $F$ is separable per definition, as we have found a countable dense subset of $F$, namely $Q$. $\square$

To prove

If $F := \{x \in E : ||x|| = 1\}$ is separable, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset E$ such that

$$\overline{\text{span}((x_n)_{n\in\mathbb{N}})} = E.$$

*Proof.* Let $F = \{x \in E : ||x|| = 1\}$ be separable. Hence, we find a countable subset $Q \subset F$, that is dense in $F$ where $Q$ is of the Form $Q = (x_n)_{n\in\mathbb{N}} \subset E$ and $||x_n|| = 1$ for all $n \in \mathbb{N}$. Let now $x \in E$ be arbitrary. We want to find a sequence $(y_n)_{n\in\mathbb{N}}$ in $Q$ with $y_n \to x$ for $n \to \infty$ and every $y_n$ a finite linear combination of our $(x_n)$

Note that in any case where $x \neq 0$ (for $x = 0$ everything is clear) we know $\frac{x}{||x||} = 1$. Thus, $\frac{x}{||x||} \in F$ so by assumption we can find a sequence $(\tilde{y}_n)_{n\in\mathbb{N}} \subset Q$ with $\tilde{y}_n \overset{n\to\infty}{\longrightarrow} \frac{x}{||x||}$ and . This of course means

$$\lim_{n\to\infty} \tilde{y}_n = \frac{x}{||x||} \iff ||x|| \lim_{n\to\infty} \tilde{y}_n = \lim_{n\to\infty} ||x||\tilde{y}_n = x.$$

Since for all natural numbers $n$ we can find an $i_n \in \mathbb{N}$ with $y_n = x_{i_n}$ and hence we know that $||x||y_n \in \text{ span}((x_n)_n)$ and that means

$$x \in \overline{\text{span}((x_n)_n)} \implies E = \overline{\text{span}((x_n)_n)}$$

*To show:* $\overline{\text{span}((x_n)_{n\in\mathbb{N}})} \subset E$. Let $x \in \overline{\text{span}((x_n)_{n\in\mathbb{N}})}$.

We know that $\mathbb{C}_{\mathbb{Q}} := \{z = a + ib : a, b \in \mathbb{Q}\}$ is countable and dense in $\mathbb{C}$. Thus, $\text{span}_{\mathbb{Q}} := \left\{x \in E \mid \exists m_x \in \mathbb{N} : x = \sum_{i=1}^{m_x} \lambda_i x_{\tau_x(i)} \in \text{span}((x_n)_n) \text{ and } \lambda_i \in \mathbb{C}_{\mathbb{Q}}\right\}$ - where $\tau_x$ is some permutation of $\mathbb{N}$ - is also dense in $\text{span}((x_n)_n)$

Now we can see that

$$\begin{aligned}\text{span}_{\mathbb{Q}} &= \bigcup_{m\in\mathbb{N}} \left\{x = \sum_{i=1}^{m} \lambda_i x_{\tau_x(i)} \in E \mid \lambda_i \in \mathbb{C}_{\mathbb{Q}} \text{ for all } i \in \{1, ..., m\}\right\} \\ &= \bigcup_{m\in\mathbb{N}} \bigcup_{(n_1,...,n_m)\in\mathbb{N}^m} \bigcup_{(\lambda_1,...,\lambda_m)\in(\mathbb{C}_{\mathbb{Q}})^m} \left\{\sum_{i=1}^{m} \lambda_i x_{n_i}\right\}\end{aligned}$$

which is countable. Thus we have a countable set for which

$$\overline{\text{span}_{\mathbb{Q}}} = \overline{\text{span}((x_n)_n)} = E.$$

Hence $E$ is separable.

Now for the second part of the task: We show that $E^*$ being separable implies $E$ being separable by proving that $\text{span}((x_n)_n)$ is dense in E for a sequence $(x_n)$ as discussed in the task. We will, however, assume that $||x_n|| = 1$ for all natural $n$. We can do that because $1 = ||\ell_n|| = \sup_{||x||=1} |\ell(x)|$ implies that for all positive $\epsilon$ we can find an $x$ with $||x|| = 1$ and still $|\ell_n(x)| > 1 - \epsilon$.

Now towards a contradiction assume there exists a $x \in E$ with $x \notin \text{span}((x_n)_n)$. This means we have

$$\inf_{y\in\text{span}((x_n)_n)} ||x - y|| > 0.$$

Then Corollary 4.8 in the script yields us an $\ell \in E^*$ with $\ell|_{\text{span}((x_n)_n)} = 0$ and $||\ell|| = 1$, so $\ell \in S$.

This leads us for any $n \in \mathbb{N}$ to

$$||\ell - \ell_n|| = \sup_{x\neq 0} \frac{|\ell(x) - \ell_n(x)|}{||x||} \geq \frac{|\ell(x_n) - \ell_n(x_n)|}{||x_n||} = \frac{|\ell_n(x_n)|}{||x_n||} = \frac{1}{2||x_n||} > 0.$$

Thus, $\ell \notin \overline{(\ell_n)_n}$ and hence $(\ell_n)_n$ is not dense in $S$. That is a contradiction!

□

## Exercise 2

Let $E$ be a $\mathbb{C}$ vector space with a seminorm $\rho : E \to \mathbb{R}$ on $E$, and let $F$ be a linear subspace $F \subset E$. Let $x : F \to \mathbb{K}$ with

$$|x(y)| \leq \rho(y), \quad \forall y \in F.$$

To prove

There exists an extension $\ell : E \to \mathbb{K}$ such that

$$\ell_{|F} = x \text{ and } |\ell(y)| \leq \rho(y) \quad \forall y \in F.$$

Lemma 0.1

Let $l \in E'$. Define $l_{re} : E \to \mathbb{R}, z \mapsto \text{Re}(l(z))$. The map $l_{re}$ is $\mathbb{R}$-***linear*** and it holds

$$||l_{re}|| = ||l||.$$

*Proof.* Let $l \in E'$. We show the $\mathbb{R}$-linearity of $l_{re}$.

- Let $x, y \in E$. It holds

$$\begin{aligned} l_{re}(x+y) = \text{Re}(l(x+y)) = \text{Re}(l(x)+l(y)) &= \text{Re}(l(x)) + \text{Re}(l(y)) \\ &= l_{re}(x) + l_{re}(y). \end{aligned}$$

- Let $\lambda \in \mathbb{R}$ and $x \in E$. It holds

$$l_{re}(\lambda x) = \text{Re}(l(\lambda x)) = \lambda \, \text{Re}(l(x)) = \lambda l_{re}(x).$$

We now show $||l_{re}|| = ||l||$. For every $x \in E$ it holds

$$|l_{re}(x)| = |\,\text{Re}(l(x))| \leq ||l(x)||,$$

which implies $||l_{re}|| \leq ||l||$.

Let $x \in E$ with $||x|| = 1$. By using the $\mathbb{C}$-linearity of $l_{re}$ we obtain

$$0 \leq |l(x)| = \underbrace{\frac{|l(x)|}{l(x)}}_{\in \mathbb{C}} l(x) = \underbrace{l(\frac{|l(x)|}{l(x)} x)}_{\in \mathbb{R}} = l_{re}(\frac{|l(x)|}{l(x)} x).$$

Since $||\frac{|l(x)|}{l(x)} x|| = 1$ for all $||x|| = 1$ it follows $|l(x)| \leq ||l_{re}||$, implying $||l|| \leq ||l_{re}||$. Thus, $||l|| = ||l_{re}||$. $\square$

Lemma 0.2

Let $l_{re} : E \to \mathbb{R}$ be $\mathbb{R}$-linear. Set $l(x) := l_{re}(x) - i l_{re}(ix)$ for all $x \in E$. Then $l \in E'$, $||l|| = ||l_{re}||$ and $\text{Re}(l) = l_{re}$.

*Proof.* Let $x \in E$. Then

$$\text{Re}(l(x)) = \text{Re}(l_{re}(x)) - \text{Re}(i\,\underbrace{l_{re}(ix)}_{\in\mathbb{R}}) = \text{Re}(l_{re}(x)) = \text{Re}(l_{re}(x)) = l_{re}(x).$$

We use the $\mathbb{R}$ linearity of $l_{re}$ to show that $l$ is linear with complex scalars. Let $x \in E$.

$$\begin{aligned} l((a+bi)x) &= l_{re}(ax+ibx) - il_{re}(iax-bx) \\ &= al_{re}(x) + bl_{re}(ix) - ial_{re}(ix) + ibl_{re}(x) \\ &= (a+bi)l_{re}(x) - (ai-b)l_{re}(ix) \\ &= (a+bi)l(x). \end{aligned}$$

Let $x, y \in E$. It holds

$$l(x+y) = l_{re}(x) + l_{re}(y) - il_{re}(x) - il_{re}(y) = l(x) + l(y).$$

The proof for $||l|| = ||l_{re}||$ is the same as in the first lemma. We use that $\text{Re} \circ l = l_{re}$ which implies $||l_{re}|| \leq ||l||$, and then we show the converse way $||l_{re}|| \geq ||l||$ in the exact same way as in lemma 0.1 by using the $\mathbb{C}$-linearity of $l$. $\square$

Now to the actual proof.

*Proof.* Let $X_{\mathbb{R}} \subset X$ be the vector space by restricting $X$ to real scalars. Let $x \in X'$. From the first lemma we know that the functional $y = \text{Re} \circ x$ lies in $Y_{\mathbb{R}}$ with $||y|| = ||\text{Re} \circ x||$. We know from theorem 4.5 (for $\mathbb{K} = \mathbb{R}$) that there exists an extension $\tilde{\ell} \in E'_{\mathbb{R}}$ for $y$ such that $||\tilde{\ell}|| = ||y|| = ||x||$. From the second lemma we obtain a linear functional $\ell \in E'$ with $||\ell|| = ||\tilde{\ell}|| = ||x||$ that extends $x$. To see this we use the $\mathbb{C}$-linearity of $x$:

$$\forall a \in E : \ell(a) = \tilde{\ell}(a) - i\ell(ai) = \text{Re}(x(a)) - i\,\text{Re}(i\,\underbrace{x(a)}_{\in\mathbb{R}}) = x(a).$$

$\square$

## Exercise 3

(i) Let $X$ be a normed space and $C \subset X$ be open and convex with $0 \in C$. We show that $p_C$ is a sublinear functional.

To prove

For all $x, y \in X$ it holds

$$p_C(x+y) \leq p_C(x) + p_C(y).$$

*Proof.* Let $x, y \in X$. From 3(ii) we know that $C$ is absorbing. Thus, $p_C(x) \neq \infty, p_C(y) \neq \infty$ and $p_C(x+y) \neq \infty$. Let $\epsilon > 0$ be arbitrary. For $x$ there exists $\mu_x \in \mathbb{R}_{>0}$ and $z_y \in C$ such that $x = \mu_x z_x$ and $\mu_x < p_C(x) + \frac{\epsilon}{2}$. Likewise for $y$: there is $\mu_y \in \mathbb{R}_{>0}$ such that $y = \mu_y z_y$ with $\mu_y < p_C(y) + \frac{\epsilon}{2}$. Adding $\mu_x$ and $\mu_y$ together yields

$$\mu_x + \mu_y < p_C(x) + p_C(y) + \epsilon.$$

Goal

Next, we try to find a vector $z_{x+y} \in C$ and scalar $\mu_{x+y} > 0$ such that

$$\mu_{x+y} z_{x+y} = x + y.$$

Consider the vector

$$z_{x+y} := \frac{\mu_x}{\mu_x + \mu_y} z_x + \frac{\mu_y}{\mu_x + \mu_y} z_y,$$

which luckily lies in $C$. To see this, remember that $C$ is a convex set, and note that $z_{x+y}$ is a convex combination of $z_x \in C$ and $z_y \in C$ since

$$\frac{\mu_x}{\mu_x + \mu_y} + \frac{\mu_y}{\mu_x + \mu_y} = 1.$$

Define the scalar $\mu_{x+y}$ as $\mu_{x+y} := \mu_x + \mu_y > 0$. We see that

$$\mu_{x+y} z_{x+y} = \mu_x z_x + \mu_y z_y = x + y.$$

Putting everything together

$$p_C(x+y) \overset{(*)}{\leq} \mu_{x+y} = \mu_x + \mu_y < p_C(x) + p_C(y) + \epsilon.$$

Explanation

$(*)$ holds by definition of $p_C(x+y)$.

As $\epsilon$ was chosen arbitrarily, it holds

$$p_C(x+y) \leq p_C(x) + p_C(y).$$

$\square$

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To prove

For any $x \in X$ and $\lambda \in \mathbb{R}_{>0}$ it holds

$$p_C(\lambda x) = \lambda p_C(x).$$

*Proof.* Let $x \in X$ and $\lambda > 0$. Then,

$$\begin{aligned} p_C(\lambda x) = \inf\{\alpha > 0 : \lambda x \in \alpha C\} &= \inf\{\alpha > 0 : x \in \frac{\alpha}{\lambda} C\} \\ &= \inf\{\lambda\alpha > 0 : x \in \alpha C\} \\ &= \lambda \inf\{\alpha > 0 : x \in \alpha C\} \\ &= \lambda p_C(x). \end{aligned}$$

$\square$

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**Conclusion:** $p_C : X \to [0, \infty)$ is a sublinear functional.

(ii) $C$ is absorbing: More precisely, there is some $M \geq 0$ such that for all $x \in X$ there holds $p_c(x) \leq M$.

By assumption we know $C$ is open and that $0 \in C$. Therefore there exists $\varepsilon > 0$ such that $U_\varepsilon(0) \subset C$. Since $||\frac{\varepsilon}{2}\frac{x}{||x||}|| = \frac{\varepsilon}{2}$, $\frac{x\frac{\varepsilon}{2}}{||x||} \in U_\varepsilon(0)$ and therefore $\frac{x\frac{\varepsilon}{2}}{||x||} \in C$. Setting $M := \frac{2}{\varepsilon}$ we obtain $x = M||x||\frac{x\frac{\varepsilon}{2}}{||x||}$, thus

$$p_c(x) = \inf\{\alpha > 0 : x \in \alpha C\} \leq M||x||$$

(iii) Let $C \subset X$ be open and convex. We want to show:

To prove

$$C = \{x \in X : p_C(x) < 1\}$$

*Proof.* First, we show $C \subset \{x \in X : p_C(x) < 1\}$. Let $x \in C$.

Goal

We want to find $z \in C$ such that $x = \alpha z$ with $\alpha < 1$.

The subset $C$ is **open**. Thus, there exists a ball with radius $\epsilon > 0$ around $x$ such that $U_\epsilon(x) \subset C$. Therefore, $z := (||x|| + \frac{\epsilon}{2})\frac{x}{||x||} \in U_\epsilon(x)$ because

$$||x - z|| = ||x - (||x|| + \frac{\epsilon}{2})\frac{x}{||x||})|| = \frac{\epsilon}{2}\frac{||x||}{||x||} < \epsilon.$$

Hence, $z \in U_\epsilon(x) \implies z \in C$.

We use the **convexity** of $C$ to show that there exists $\alpha < 1$ such that $x = \alpha z$. Define the scalar value $\alpha \in \mathbb{R}$ as

$$\alpha := \frac{1}{1 + \frac{\epsilon}{2||x||}}.$$

It holds $\alpha < 1$ since $\frac{\epsilon}{2||x||} > 0$. We compute $\alpha z$, and we obtain

$$\alpha z = \frac{1}{1 + \frac{\epsilon}{2||x||}}(||x|| + \frac{\epsilon}{2})\frac{x}{||x||} = \frac{1 + \frac{\epsilon}{2||x||}}{1 + \frac{\epsilon}{2||x||}}x = x.$$

□

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*Proof.* Next, we show $\{x \in X : p_C(x) < 1\} \subset C$. Let $y \in \{x \in X : p_C(x) < 1\}$.

Goal

We want to show that $y \in C$.

There exists $z \in C$ with $\alpha < 1$ such that

$$y = \alpha z.$$

Note that $y \in \mathrm{conv}(0, z)$ and in particular $0 \in C$ by assumption. Since $C$ is convex, it holds $\mathrm{conv}(0, z) \subset C$, which implies $y \in C$. □

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**Conclusion:** This proves the claim that $C = \{x \in X : p_C(x) < 1\}$.

## Exercise 4

"$\Longrightarrow$": Let $F$ be dense in $X$ and let $f \in X^*$ with $f|_F = 0$. For all $x \in X$ we can then find a sequence $(x_n)_{n\in\mathbb{N}}$ in $F$ with $x_n \to x$ for $n \to \infty$ because $F$ is dense in $X$. This in turn implies that

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = 0$$

as $f$ is continuous.

"$\Longleftarrow$": Let $f \in X^*$ with $f|_F = 0$. Then, $f$ is continuous. If for some $x \in X$ we had $fx \neq 0$ we would also find an $\epsilon > 0$ with $fx \neq 0$ for all $x \in U_\epsilon(x)$. But then, since $f|_F = 0$ we would have $x \notin \overline{F}$ which would imply $\overline{F} \neq X$. Thus $F$ in that case could not be dense in $X$.

$\square$