Exercise 12.1

(i) Given $f(q,p) = \begin{pmatrix} p^\alpha q^\beta \\ -p^{\alpha+1} q^\delta \end{pmatrix}$ we need to find $\alpha, \beta, \delta \in \mathbb{R}$ such that f is a Hamiltonian vector field. We use a theorem from the lecture which states that

Theorem. *Let $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a continuous differentiable vector field. It is a Hamiltonian vector field if and only if $\frac{\partial f}{\partial x} \in \mathfrak{Sp}_{2n}$.*

$\mathfrak{Sp}_{2n}$ is the symplectic Lie algebra, i.e. matrices D that satisfy $JD^TJ = D$. If we have 2×2 matrices, then $JD^TJ = D$ is the same as

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus we get

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & a \\ -d & b \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} -d & b \\ c & -a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So, $-a = d \iff 0 = a + d \iff \mathrm{tr}A = 0$. In the end, we must show that $\mathrm{tr}(\frac{\partial f}{\partial x}) = 0$. First, let's compute the Jacobi matrix

$$\frac{\partial f}{\partial(q,p)} = \begin{pmatrix} \beta p^\alpha q^{\beta-1} & \alpha p^{\alpha-1} q^\beta \\ -\delta p^{\alpha+1} q^{\delta-1} & -(\alpha+1) p^\alpha q^\delta \end{pmatrix}.$$

The trace is then given by $\beta p^\alpha q^{\beta-1} - (\alpha+1) p^\alpha q^\delta$, which should equal to zero for all $p, q \in \mathbb{R}$. If $p = 0$ or $q = 0$, the trace is trivially zero for all α, β, δ. So, assume $p \neq 0$ and $q \neq 0$. We must find α, β, δ such that $\beta p^\alpha q^{\beta-1} = (\alpha+1) p^\alpha q^\delta$. We divide by $p^\alpha \neq 0$, and obtain

$$\beta q^{\beta-1} = (\alpha+1) q^\delta \implies \alpha = \beta q^{\beta-1-\delta} - 1, \qquad q^\delta \neq 0.$$

Now, if $\alpha = \beta q^{\beta-1-\delta} - 1$ should hold for fixed α, β, δ and for all $p, q \in \mathbb{R}$, then $q^{\beta-1-\delta}$ must evaluate to a constant for all q, which is only the case if $\beta - 1 - \delta$ equals zero. Hence, we get for any $\delta \in \mathbb{R}$

$$\beta = 1 + \delta.$$

And then,

$$\alpha = \delta.$$

The Hamiltonian vector field has the form

$$f_{Ham}(q,p) = \begin{pmatrix} p^\delta q^{1+\delta} \\ -p^{\delta+1} q^\delta \end{pmatrix}.$$

(ii) The Hamilton function can be obtained by integrating f_{Ham} with respect to p and q. Both yield the same result

$$H(q,p) = \frac{1}{\delta+1} p^{\delta+1} q^{\delta+1} + \text{const.}$$