Technische Universität Berlin
Institut für Mathematik

Prof. Dr. Yuri B. Suris
Jan Techter

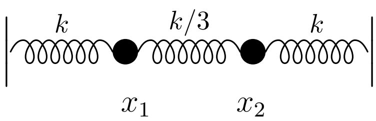
Mathematical Physics I - WS 2018/2019

http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/MP1/

Exercise Sheet 2

Exercise 1: (7 pts)

Consider a one-dimensional system consisting of two masses m and three springs with elasticity constants k, $k/3$ and k, $k > 0$. The springs with elasticity constant k are firmly attached to the wall, while the spring with elasticity constant $k/3$ connects the two masses.



Denote by x_1 and x_2 the positions of the masses. Their equilibrium positions are $x_1 = e_1$ and $x_2 = e_2$. Consider the initial conditions

$$(x_1(0), x_2(0)) = (a, b), \qquad (\dot{x}_1(0), \dot{x}_2(0)) = (c, d),$$

with $b > a > 0$ and $c, d > 0$.

(i) According to Hooke law, express the spring forces acting on each mass (frictional forces are neglected) and write down the coupled system of Newton equations of motion for the two masses.

Hint: Hooke law states that the spring force is $f(x) = -k\,x$, where $k > 0$ is the elasticity constant and x is the displacement of the end of the spring from its equilibrium position.

(ii) Decouple the obtained differential equations by introducing new variables $(\widetilde{x}_1, \widetilde{x}_2) := (x_1 + x_2, x_1 - x_2)$. Solve the IVP in the variables $\widetilde{x}_1$, $\widetilde{x}_2$.

(iii) Determine explicitly the flow of the system in terms of the variables x_1, x_2.

Exercise 2: (5 pts)

Consider the map $\Phi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\Phi_t(x, y, z) \mapsto \left(x + 2ty,\ y,\ e^{-tx-t^2y} z\right).$$

(i) Prove that Φ_t is the flow of a continuous dynamical system.

(ii) Compute the vector field of Φ_t.

(iii) Write down the IVP corresponding to Φ_t.

Exercise 3: (4 pts)

Fix $\alpha > 0$ and consider a discrete dynamical system defined by iterations of the map

$$\Phi : \left\{ (x, y) \in \mathbb{R}^2 \, : \, x \neq \frac{1}{\alpha} \right\} \to \mathbb{R}^2 : (x, y) \mapsto \left(\frac{x}{\alpha\, x - 1}, \frac{y + \alpha\, x(x - y)}{\alpha\, x - 1} \right).$$

(i) Show that the image of Φ is contained in $\left\{ (x, y) \in \mathbb{R}^2 \, : \, x \neq \frac{1}{\alpha} \right\}$, so the iteration can be continued indefinitely.

(ii) Show that each orbit is periodic and determine the period.

Exercise 4: (4 pts)

Let $(\Phi^t)_{t\in\mathbb{R}}$, with $\Phi^t : M \to M \subset \mathbb{R}^n$, be a continuous dynamical system. Consider a point $x_0 \in M$ and consider the orbit $\mathcal{O}(x_0) := \{\Phi^t(x_0) : t \in \mathbb{R}\} \subset M$. Assume that $\Phi^T(x_0) = x_0$ and that T is the smallest positive number with this property. Prove that $\mathcal{O}(x_0)$ is a periodic orbit with period T.

Due Monday, November 05.