Research Group
Applied Functional Analysis
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Functional Analysis I

Tutorial Assignment 11

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Exercise 1: Prove that the operator $T : \ell_2 \to \ell_2$, defined by $T((x_n)_{n\in\mathbb{N}}) := (\frac{1}{n}x_n)_{n\in\mathbb{N}}$, is compact. *Hint:* Note that $Tx = \sum_{k=1}^{\infty} \frac{1}{k}\langle x, e_k\rangle e_k$, $x \in \ell_2$.

Exercise 2: Let E and F be Banach spaces and $T_n \in L(E, F)$ such that $\operatorname{ran} T_n$ is finite-dimensional for each $n \in \mathbb{N}$. Assume for $T \in L(E, F)$ that $T_n x \to Tx$ for every $x \in E$. Is T compact? What can we say if $T = \lim_{n\to\infty} T_n$ in operator norm?

Exercise 3: Let E be a normed space and $S, T \in L(E)$.

(a) Does $ST \in \mathcal{K}(E)$ imply that $S \in \mathcal{K}(E)$ and $T \in \mathcal{K}(E)$?

(b) Does $ST \in \mathcal{K}(E)$ imply that $S \in \mathcal{K}(E)$ or $T \in \mathcal{K}(E)$?

Exercise 4: Show that a bounded linear projection in a Banach space is compact if and only if its range is finite-dimensional.

Exercise 5: Let E and F be Banach spaces. By $\mathbb{F}(E, F)$ we denote the subspace of finite-dimensional operators in $L(E, F)$. Assume that there exists a bounded sequence $(S_n)_{n\in\mathbb{N}} \subset \mathbb{F}(F, F)$ such that

$$\forall y \in F : \lim_{n\to\infty} \|S_n y - y\| = 0.$$

Prove that $\overline{\mathbb{F}(E, F)} = \mathcal{K}(E, F)$. *Hint:* For $T \in \mathcal{K}(E, F)$ consider the operators $S_n T$.