Problem Sheet 01

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Exercise 1

Given a chart $\varphi : U \to V$ and $\tilde{\varphi} : \tilde{U} \to V$. Let $\tau : \varphi^{-1} \circ \tilde{\varphi}$ be the coordinate transformation between the charts. We want to show that

$$\det \tilde{g}(y) = |\det D(\tau(y))|^2 \det g(\tau(y))$$

where $g, \tilde{g}$ denotes the Gramian matrix of $\varphi, \tilde{\varphi}$ respectively.

Proof.

$$\begin{aligned} \det \tilde{g}(y) = \det(D\tilde{\varphi}(y)^T D\varphi(y)) = \det(D(\varphi \circ \tau)(y))^2 &= \det(D\varphi(\tau(y))D\tau(y))^2 \\ &= \det g(\tau(y))|\det D\tau(y)|^2 \end{aligned}$$

$\square$

Next we want to show that the integral over a manifold is independent of the chart φ. Define:

$$I(f) = \int_U f(\varphi(x))\sqrt{\det g(x)}dx.$$

Proof.

$$\begin{aligned} I(f) = \int_{\tilde{U}} f(\tilde{\varphi}(x))\sqrt{\det \tilde{g}(x)}dx &= \int_{\tilde{U}} f(\varphi(\tau(x)))\sqrt{\det D(\tau(x))^2 \det g(\tau(x))}dx \\ &= \int_U f(\varphi(y))\sqrt{\det g(y)}dy \end{aligned}$$

$\square$