Research Group
Applied Functional Analysis
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Functional Analysis I

Tutorial Assignment 10

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Exercise 1: Let $(E, \|\cdot\|)$ be a Banach space which is the direct sum of two closed subspaces $U, V \subset E$, i.e. $E = U \oplus V$. Let P_U and P_V be the canonical projections onto U and V, respectively. Prove that $|||\cdot|||$, defined by

$$|||x||| := \sqrt{\|P_U x\|^2 + \|P_V x\|^2}, \quad x \in E,$$

is a norm on E which is equivalent to $\|\cdot\|$. With this, show that E is isomorphic to a Hilbert space provided that $(U, \|\cdot\|)$ and $(V, \|\cdot\|)$ are Hilbert spaces.

Exercise 2: Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $T \in L(\mathcal{H}, \mathcal{K})$. Prove that the following statements are equivalent:

(i) T is isometric.

(ii) $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in \mathcal{H}$.

(iii) $T^*T = I_{\mathcal{H}}$.

Exercise 3: Let $\mathcal{H}$ be a Hilbert space, and let P be a bounded linear projection in $\mathcal{H}$ (i.e. $P^2 = P$). Prove that P is an orthogonal projection if and only if $P^* = P$ (that is, P is *selfadjoint*).

Exercise 4: Prove that the functions f_k, $k \in \mathbb{Z}$, defined by

$$f_k(t) := \frac{1}{\sqrt{2\pi}} e^{ikt}, \quad t \in (-\pi, \pi),$$

form an orthonormal system in $L_2(-\pi, \pi)$.

Exercise 5: For $f \in L_1(\mathbb{R})$ the *Fourier transform* of f is defined by

$$(\mathcal{F}_1 f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx, \quad \xi \in \mathbb{R}.$$

Prove that $\hat{f}$ is well-defined and that $\hat{f} \in C_b(\mathbb{R})$, which is the set of all bounded continuous functions on $\mathbb{R}$. Moreover, prove that $\mathcal{F}_1 \in L(L_1(\mathbb{R}), C_b(\mathbb{R}))$ with $\|\mathcal{F}_1\| \leq 1$.