Exercise 11.1

(i) The Hamilton equations are given by

$$\dot{x}_j = \frac{\partial H}{\partial p_j} \text{ and } \dot{p}_j = -\frac{H}{\partial x_j}.$$

So for $j = 1, 2, 3$ we get

$$\dot{x}_j = \frac{\partial}{\partial p_j}\left(\sqrt{\alpha^2 + p_1^2 + p_2^2 + p_3^2} + a_1x_1 + a_2x_2 + a_3x_3\right) = \frac{p_j}{\sqrt{\alpha^2 + p_1^2 + p_2^2 + p_3^2}}$$

and

$$\dot{p}_j = -\frac{\partial}{\partial x_j}(\sqrt{\alpha^2 + p_1^2 + p_2^2 + p_3^2} + a_1x_1 + a_2x_2 + a_3x_3) = -a_j.$$

We get a first order system of linear differential equations

$$\begin{cases} \dot{x}_j = \frac{p_j}{\sqrt{\alpha^2 + \sum_{i=1}^3 p_i^2}} \\ \dot{p}_j = -a_j \end{cases}, j = 1, 2, 3.$$

(ii) Idea: Legendre transformation of the Hamilton function gives the Lagrangian function. We know that $\dot{x}_j = \frac{p_j}{\sqrt{\alpha^2 + \sum_{i=1}^3 p_i^2}} \iff \dot{x}_j^2(\alpha^2 + \sum p_i^2) = p_j^2$. Let $v_j := \dot{x}_j$ for $j = 1, 2, 3$. So, we use SageMath to solve that equation for p_1 and we get

$$p_1^2 = \frac{-(\alpha^2 + p_2^2 + p_3^2)v_1^2}{v_1^2 - 1}$$

Then we use this information to solve for (again with SageMath)

$$p_2^2 = -\frac{(\alpha^2 + p_3^2)v_2^2}{v_1^2 + v_2^2 - 1}.$$

Plugging everything into p_3 yields

$$p_3^2 = -\frac{\alpha^2 v_3^2}{v_1^2 + v_2^2 + v_3^2 - 1}$$

and substituting back into the first two equations finally gives

$$p_2^2 = -\frac{\alpha^2 v_2^2}{v_1^2 + v_2^2 + v_3^2 - 1} \quad \text{and} \quad p_1^2 = -\frac{\alpha^2 v_1^2}{v_1^2 + v_2^2 + v_3^2 - 1}.$$

The Lagrangian is then given by

$$\mathcal{L}(x, v) = \sum_{j=1}^{3} p_j \frac{\partial H}{\partial p_j} - \mathcal{H}(x, p)\bigg|_{p=p(x,v)} = \alpha\sqrt{1 - \sum_{j=1}^{3} v_j^2} - \langle a, x\rangle.$$

Everything we have computed was done by SageMath. The code can be found on `https://github.com/geniegeist/Mathematical-Physics-I/tree/master/UE11`.

(iii) The Euler-Lagrange equation is given by

$$\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial v_i}) = 0.$$

We have

$$\frac{\partial \mathcal{L}}{\partial x_i} = -a_i$$

and

$$\frac{\partial \mathcal{L}}{\partial v_i} = \frac{\alpha \cdot (-2v_i)}{2\sqrt{1-\sum v_j^2}} = \frac{-\alpha v_i}{\sqrt{1-\sum v_j^2}}.$$

Hence,

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v_i} &= \frac{-\alpha \dot{v}_i \sqrt{1-\sum v_j^2} + \alpha v_i \frac{1}{2\sqrt{1-\sum v_j^2}} \cdot (-\sum 2 v_j \dot{v}_j)}{1-\sum v_j^2} \\
&= \frac{-\alpha \dot{v}_i \sqrt{1-\sum v_j^2} - \frac{\alpha v_i}{\sqrt{1-\sum v_j^2}} \cdot \sum v_j \dot{v}_j}{1-\sum v_j^2} \\
&= \frac{-\alpha \dot{v}_i}{\sqrt{1-\sum v_j^2}} - \frac{\alpha v_i \cdot \sum v_j \dot{v}_j}{(1-\sum v_j^2)\sqrt{1-\sum v_j^2}} \\
&= \frac{-\alpha \dot{v}_i}{\sqrt{1-\sum v_j^2}}(1 + \frac{v_i \sum v_j \dot{v}_j}{\dot{v}_i \sqrt{1-\sum v_j^2}})
\end{aligned}$$