Exercise 8.1

(a) Linearise around $(0,0)$:

$$A := \frac{df}{dx}\Big|_{x=(0,0)} = \begin{pmatrix} 4x_1^3 + x_2 & x_1 \\ -2x_1 + x_2^2 & -2 + 2x_1x_2 \end{pmatrix}\Bigg|_{x=(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}.$$

We see that the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -2$. As $\Re(\lambda_1) = 0$, the fixed point $(0,0)$ is not hyperbolic and thus the theorem of Poincaré-Lyapunov is not applicable.

(b) The center subspace $E^C(0,0)$ can be obtained by

$$(A - \lambda_1 E)x = 0 \iff Ax = 0 \iff -2x_2 = 0 \iff x_2 = 0.$$

Therefore $E^C(0,0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\} = \text{span}((1,0))$.
Idea: $W^C(0,0) = \{(x_1, h(x_1)) =: H(x_1) : |x_1| \le \epsilon\}$. Due to $\Phi^t(W^C(0,0)) \subset W^C(0,0)$, it follows

$$f(H(x_1)) \in T_{H(x_1)}W^C(0,0) \implies \langle f(H(x_1)), N(x_1)\rangle = 0$$

with $N(x_1) = \begin{pmatrix} 1 \\ h'(x_1) \end{pmatrix}^\perp = \begin{pmatrix} h'(x_1) \\ -1 \end{pmatrix}$. So we obtain

$$\langle \begin{pmatrix} x_1^4 + x_1h(x_1) \\ -2h(x_1) - x_1^2 + x_1h^2(x_1) \end{pmatrix}, \begin{pmatrix} h'(x_1) \\ -1 \end{pmatrix}\rangle = 0. \tag{1}$$

Now we expand $h(x_1)$ by Taylor Expansion up to degree 5:

$$\begin{aligned} h(x_1) &= h(0) + h'(0)x_1 + \frac{1}{2}h''(0)x_1^2 + \frac{1}{6}h^{(3)}x_1^3 + \frac{1}{24}h^{(4)}x_1^4 + O(x_1^5) \\ &= \frac{1}{2}ax_1^2 + \frac{1}{6}bx_1^3 + \frac{1}{24}cx_1^4 + O(x_1^5), \end{aligned}$$

where a, b, c are yet to be determined. Derivating the local approximation of $h(x_1)$ yields

$$h'(x_1) = ax_1 + \frac{1}{2}bx_1^2 + \frac{1}{6}cx_1^3 + O(x^4)$$

Substituting in (1) gives

$$\begin{pmatrix} x_1^4 + x_1h(x_1) \\ -2h(x_1) - x_1^2 + x_1h^2(x_1) \end{pmatrix}, \begin{pmatrix} ax_1 + \frac{1}{2}bx_1^2 + \frac{1}{6}cx_1^3 + O(x^4) \\ -1 \end{pmatrix}\rangle = 0$$

The first row gives a term $O(x^5)$ and therefore

$$2h(x_1) + x_1^2 - \underbrace{x_1h^2(x_1)}_{=O(x^5)} + O(x^5) = 0 \iff ax_1^2 + \frac{1}{3}bx_1^3 + \frac{1}{12}cx_1^4 + x_1^2 + O(x^5) = 0.$$

Now we get

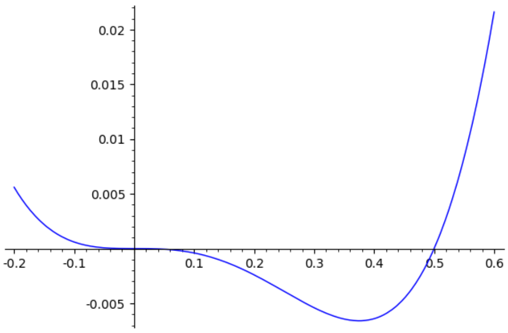
$$0 = (a+1)x_1^2 + \frac{1}{3}bx_1^3 + \frac{1}{12}cx_1^4 \implies a = -1, b = c = 0.$$

Therefore, $h(x_1) = -\frac{1}{2}x_1^2$ and

$$W^C(0,0) = \{\begin{pmatrix} 2x_1 \\ -x_1^2 \end{pmatrix} : |x_1| \le \epsilon\}.$$

(c) Restricting the dynamical system locally on $W^C(0,0)$ results in

$$\dot{x}_1 = x_1^4 + x_1(-0.5x_1^2) \iff \dot{x}_1 = x_1^4 - 0.5x_1^3.$$



We that the fixed point $(0,0)$ is stable. For positive x_1 values and $x_1 < 0.5$, $\dot{x}_1$ is negative, which results in convergence to 0. Similarly, for negative x_1 values, $\dot{x}_1$ is positive, which results in convergence to 0.

Exercise 8.2