# Differential Equations I (Week 10)

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## Appendix: Osgood's Uniqueness Theorem

Consider the following initial value problem stated as

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(0) = u_0 \end{cases} \tag{1}$$

with $U \subset \mathbb{R}^d$ open and $f : [0, T] \times U \to \mathbb{R}^d$ and $u_0 \in \mathbb{R}^d$. We know that there exists a unique solution if the Cauchy-Lipschitz condition is satisfied. However, Osgood could prove that there exists a weaker condition that asserts the uniqueness of a solution for a local neighbourhood around 0. For that, we need to define functions which satisfy the *Osgood condition*.

**Definition 1.** *A function $\omega : [0, \infty) \to [0, \infty)$ is said to satisfy* ***Osgood's condition*** *if $\omega(0) = 0$, $\omega(z) > 0$ for all $z > 0$ and for any $\delta > 0$ it holds*

$$\lim_{\epsilon \searrow 0} \int_\epsilon^\delta \frac{1}{\omega(r)} dr = \infty.$$

**Theorem 2.** *Let $U \subset \mathbb{R}^d$ be open. Let $f : [0, T] \times U \to \mathbb{R}^d$ be continuous and let $\omega : [0, \infty) \to [0, \infty)$ satisfy Osgood's condition. If*

$$|f(t, v) - f(t, u)| \le \omega(|v - u|) \quad \forall t \in [0, T], \forall u, v \in U$$

*then for any $u_0 \in U$ there is a $\delta > 0$ for which there exists a unique solution $u : [0, \delta] \to U$ of the initial value problem defined in (1).*