Research Group
Applied Functional Analysis
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Functional Analysis I

Tutorial Assignment 2

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Exercise 1: Show, that it is not possible to omit the property that the $D_n, n \in \mathbb{N}$ are open in the formulation of Baire's Theorem. For the construction of an example, you may choose $X = \mathbb{R}$.

Exercise 2: Let E be a normed space. For subsets A and B of E one defines

$$A + B := \{a + b : a \in A,\, b \in B\}.$$

Prove the following statements:

(i) If A or B is open then $A + B$ is open.

(ii) If A and B are compact then $A + B$ is compact.

(iii) If A is compact and B is closed then $A + B$ is closed.

Moreover, find an example in $E = \mathbb{R}$ where both A and B are closed, but $A + B$ is not closed.
Hint: It might help to use $U_\varepsilon(x + y) = U_\varepsilon(x) + y$ in (i)–(iii).

Exercise 3: Construct sequences $(f_n)_{n\in\mathbb{N}}$ of real-valued, continuous functions defined on a metric space $X \subset \mathbb{R}$ such that

(a) X is compact and $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$, but the sequence is not equicontinuous.

(b) X is compact and $(f_n)_{n\in\mathbb{N}}$ is equicontinuous, but $\sup_{n\in\mathbb{N}} \|f_n\| = \infty$

Exercise 4: If a subspace F of a normed space E contains a non-empty open set, show that $F = E$.