Research Group
Applied Functional Analysis
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Functional Analysis I

Tutorial Assignment 4

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Exercise 1: Let X and Y be compact metric spaces. We call an operator $T : C(X) \longrightarrow C(Y)$ *positive* if $f \geq 0 \Rightarrow T(f) \geq 0$. Prove that if $T : C(X) \longrightarrow C(Y)$ is a linear positive operator, then T is continuous and $\|T\| = \|T(\mathbf{1})\|$ where $\mathbf{1} \in C(X)$ is the constant function equal to 1.

Exercise 2: Let $S_1, S_2 \in L(E)$ and assume that S_1S_2 is boundedly invertible.

(i) Show that $\operatorname{ran} S_1 = E$, $\ker S_2 = \{0\}$, and that $\operatorname{ran} S_2$ is closed.

(ii) Show that $\operatorname{ran} S_2S_1 = \operatorname{ran} S_2$, $\ker S_2S_1 = \ker S_1$.

(iii) Show that $E = \operatorname{ran} S_2 \oplus \ker S_1$ (direct sum).
Hint: $x = S_2(S_1S_2)^{-1}S_1x + \ldots$

Exercise 3: Let E and F be normed spaces and let $L \subset E$ be a linear subspace. Suppose that $T : L \to F$ is bijective. Prove that T us closed if and only if T^{-1} is closed. Now, let $E = F = \ell_2$ and and set

$$L := \{(x_n)_n \in \ell_2 : (nx_n)_n \in \ell_2\}.$$

Moreover, define the operator $T : L \to \ell_2$ by $T((x_n)_n) := (nx_n)_n$ and prove the following statements:

(i) L is a linear subspace.

(ii) T is bijective.

(iii) T^{-1} is closed (what is T^{-1}?)

(iv) T is closed.

(v) L is *not* closed.

Exercise 4: Let X be a normed space, $Y \subseteq X$ a linear subspace, and $i : Y \to X$ the inclusion operator. Prove that $i^* : X^* \to Y^*$ is the restriction operator, i.e., $i^*(x^*) = x^*|_Y$ for all $x^* \in X^*$.

Exercise 5: Let E, F, and G be normed spaces. Show $(ST)^* = T^*S^*$ for $T \in L(E, F)$ and $S \in L(F, G)$.