Lemma

Let $f : [a, b] \to \mathbb{R}$ be convex. Let $a \leq x < y < z \leq b$. Then, it holds

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Proof. Let $x < y < z$. It holds

$$y = \frac{y - z}{x - z} x + \left(1 - \frac{y - z}{x - z}\right) z \quad \text{and} \quad 0 < \frac{y - z}{x - z} < 1.$$

With convexity of f it follows

$$f\left(\frac{y - z}{x - z} x + (1 - \frac{y - z}{x - z}) z\right) = f(y) \leq \frac{y - z}{x - z} f(x) + (1 - \frac{y - z}{x - z}) f(z)$$
$$\Downarrow$$
$$(x - z)(f(y) - f(z)) \leq (y - z)(f(x) - f(z))$$
$$\Downarrow$$
$$\frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Similarly, we see that

$$y = \frac{y - x}{z - x} z + \left(1 - \frac{y - x}{z - x}\right) x \quad \text{and} \quad 0 < \frac{y - x}{z - x} < 1.$$

So,

$$f\left(\frac{y - x}{z - x} z + (1 - \frac{y - x}{z - x}) x\right) = f(y) \leq \frac{y - x}{z - x} f(z) + (1 - \frac{y - x}{z - x}) f(x)$$
$$\Downarrow$$
$$(z - x)(f(y) - f(x)) \leq (y - x)(f(z) - f(x))$$
$$\Downarrow$$
$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x}.$$

Combining these two results, we obtain our final result

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

$\square$

Proof

Now, we want to prove that every convex function $f : (a, b) \to \mathbb{R}$ is continuous. Especially, it follows that $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. Let $[\alpha, \beta] \subset (a, b)$. Our aim is to prove that f is continuous for all $x \in [\alpha, \beta]$. Let $a_0, b_0 \in \mathbb{R}$ such that $\beta < b_0 < b$ and $a < a_0 < \alpha$.

Applying the lemma above on f yields

$$\frac{f(x) - f(a_0)}{x - a_0} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(b_0) - f(y)}{b_0 - y} \quad \forall x, y \in [\alpha, \beta], y > x.$$

This implies that there exists a real value $M > 0$ such that

$$|f(y) - f(x)| \leq M|y - x| \quad \forall x, y \in [\alpha, \beta].$$

Thus, f is Lipschitz continuous on $[\alpha, \beta]$, and therefore also continuous on $[\alpha, \beta]$. Since $[\alpha, \beta]$ was arbitrary and (a, b) can be written as the union of closed intervals,it follows that f is continuous on (a, b). $\square$

Counter example

We would like to find a function $f : [a, b] \to \mathbb{R}$ that is convex but not continuous.

Consider $f : [0, 1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & x > 0 \end{cases}.$$

This function is not continuous (that is clear), but it is convex, which is shown below.

- Let $x = 0$ and $y > 0$. Let $z = \alpha x + (1 - \alpha)y$ for $0 \leq \alpha \leq 1$. If $\alpha = 1$, then $f(z) = 1 \leq 1 = f(x) + (1 - 1)f(y)$. If $\alpha < 1$, then $f(z) = 0 < \alpha = \alpha f(x) + (1 - \alpha)f(y)$.
- Let $0 < x < y$. Then, $f(z) = 0 \leq 0 = \alpha f(x) + (1 - \alpha)f(y)$.

We have shown that f is not convex.