Functional Analysis II

Lecture Notes

Prof. Dr. Gitta Kutyniok

Wintersemester 2018/2019

TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Applied Functional Analysis Group
Technische Universität Berlin

1 Spectral theory for bounded operators on complex Hilbert spaces

1.1 Notation

We recall some basic notation and definitions.

Definition 1.1 Let E be a Banach space over $\mathbb{K}$ and let $T \in L(E)$ be a continuous linear operator from E to E.

(1) The *spectrum of* T is the set $\sigma(T) \subset \mathbb{K}$ defined as

$$\sigma(T) := \{\lambda \in \mathbb{K} : \lambda \operatorname{Id} - T \text{ does not have a bounded inverse}\}.$$

(2) $\lambda \in \mathbb{K}$ is called an *eigenvalue*, if $\ker(\lambda \operatorname{Id} - T) \neq \{0\}$, i.e. if there is an $x \neq 0$ with $Tx = \lambda x$.

(3) If λ is an eigenvalue, the elements of $\ker(\lambda \operatorname{Id} - T)$ are called *eigenvectors* of T associated with λ.

(4) $\varrho(T) := \mathbb{K} \setminus \sigma(T)$ is the *resolvent set of* T.

Remark 1.2 We recall the basic properties of $\sigma(T)$:

(1) If $\lambda \in \sigma(T)$, then $|\lambda| \leq \|T\|$.

(2) $\sigma(T)$ is closed, i.e. $\varrho(T)$ is open.

(3) If $\mathbb{K} = \mathbb{C}$, then $\sigma(T) \neq \varnothing$.

$\Diamond$

The following definition recalls a variety of types of operators:

Definition 1.3 Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ and let $T \in L(\mathcal{H})$.

(1) The operator T is called *positive*, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. We sometimes write this as $T \geq 0$.

(2) There is a unique operator $T^* \in L(\mathcal{H})$, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$. This operator is called the *adjoint operator to* T.

(3) If $T = T^*$, T is called *self-adjoint*.

(4) If $TT^* = T^*T$, then T is called a *normal operator*.

The main aim of this section is to generalize the Spectral Theorem for compact self-adjoint operators to bounded self-adjoint (or even normal) operators.

1.2 Functional calculus

From now on, let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$.

Definition 1.4 Let $T \in L(\mathcal{H})$. Then $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$ is called the *numerical range of* T.

Lemma 1.5 *Let $T \in L(\mathcal{H})$. Then $\sigma(T) \subset \overline{W(T)}$.*

Proof. Let $\lambda \notin \overline{W(T)}$ and put

$$d := \operatorname{dist}(\lambda, W(T)) = \inf\{|\lambda - \mu| : \mu \in W(T)\} > 0.$$

Then, for $\|x\| = 1$ we have

$$d \leq |\lambda - \langle Tx, x\rangle| = |\langle \lambda x - Tx, x\rangle| \leq \|(\lambda \operatorname{Id} - T)x\| \cdot \|x\| = \|(\lambda \operatorname{Id} - T)x\|.$$

Therefore, $\lambda \operatorname{Id} - T$ is injective and $(\lambda \operatorname{Id} - T)^{-1} : \operatorname{ran}(\lambda \operatorname{Id} - T) \to \mathcal{H}$ exists and has norm at most $1/d$. Hence, $\operatorname{ran}(\lambda \operatorname{Id} - T)$ and $\mathcal{H}$ are isomorphic and $\operatorname{ran}(\lambda \operatorname{Id} - T)$ is closed.

Now, towards a contradiction assume, that there exists some $x_0 \in \operatorname{ran}(\lambda \operatorname{Id} - T)^\perp$ with $\|x_0\| = 1$. Then we obtain

$$0 = \langle (\lambda \operatorname{Id} - T)x_0, x_0\rangle = \lambda - \langle Tx_0, x_0\rangle$$

and $\lambda \in W(T)$, which is not possible. Hence $\operatorname{ran}(\lambda \operatorname{Id} - T) = \mathcal{H}$ and $\lambda \in \varrho(T)$. ∎

If T is self-adjoint, then $\langle Tx, x\rangle \in \mathbb{R}$ for every $x \in \mathcal{H}$ and the previous lemma gives another proof of $\sigma(T) \subset \mathbb{R}$. We even have:

Corollary 1.6 *If $T \in L(\mathcal{H})$ is self-adjoint, then*

$$\sigma(T) \subset [m(T), M(T)],$$

where

$$m(T) = \inf\{\langle Tx, x\rangle : \|x\| = 1\}, \qquad M(T) = \sup\{\langle Tx, x\rangle : \|x\| = 1\}.$$

If $T \geq 0$ (i.e. T is positive), then

$$\sigma(T) \subset [0, \infty).$$

Remark 1.7 *Functional calculus* is a means to define $f(T)$ for (at least some) functions f and operators T. Surely, if $T \in L(\mathcal{H})$ and $p(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial, we would like to have $p(T) = \sum_{k=0}^{n} a_k T^k$.

Moreover, for $\mathcal{H} = \mathbb{C}^n$ and $T \in L(\mathcal{H})$ is self-adjoint (i.e. a hermitian matrix), we consider the diagonalization

$$T = U^{-1} D U,$$

where U is unitary, i.e. $U^*U = UU^* = \mathrm{Id}$, and D is diagonal. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be holomorphic on $\mathbb{C}$, then (at least formally)

$$\begin{aligned} f(T) &= \sum_{k=0}^{\infty} a_k T^k = \sum_{k=0}^{\infty} a_k (U^{-1}DU)^k = \sum_{k=0}^{\infty} a_k U^{-1} D^k U = U^{-1} \left(\sum_{k=0}^{\infty} a_k D^k \right) U \\ &= U^{-1} f(D) U, \end{aligned}$$

where

$$f(D) = \begin{pmatrix} f(d_1) & 0 & 0 & \dots \\ 0 & f(d_2) & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(d_n) \end{pmatrix}$$

is again a diagonal matrix. We see, that to define $f(T)$, f needs to be defined only on $\sigma(T)$ and "$f(\sigma(T)) = \sigma(f(T))$" holds, which is the so-called *Spectral Mapping Theorem*. ◊

Theorem 1.8 (Continuous Functional Calculus) *Let $T \in L(\mathcal{H})$ be self-adjoint. Then there is a unique mapping*

$$\Phi : C(\sigma(T)) \to L(\mathcal{H}),$$

such that

(i) $\Phi(z) = T, \Phi(1) = \mathrm{Id}$*;*

(ii) Φ *is an involutive homomorphism of algebras, i.e.*

  (a) Φ *is linear;*

  (b) Φ *is multiplicative:* $\Phi(fg) = \Phi(f) \circ \Phi(g)$*;*

(c) Φ *is involutive:* $\Phi(f)^* = \Phi(\overline{f})$;

(iii) Φ *is continuous.*

Φ *is called* continuous functional calculus *of* T *and we denote* $f(T) := \Phi(f)$ *for* $f \in C(\sigma(T))$.

Proof. We first show the uniqueness of Φ. By the multiplicativity, we have that $\Phi(z^n) = T^n$, hence Φ is unique on all polynomials. Furthermore, $\sigma(T) \subset [m(T), M(T)]$ is compact and polynomials are dense in $C(\sigma(T))$. Due to its continuity of, Φ is unique on the whole space $C(\sigma(T))$.

Now we show the existence of Φ. We set $\Phi(f) = \sum_{k=0}^{n} a_k T^k$ for $f(z) = \sum_{k=0}^{n} a_k z^k$ being a polynomial. If we show continuity of Φ on polynomials, then there would be a unique extension to $C(\sigma(T))$, which we denote by Φ again.

We shall use the *Spectral Mapping Theorem* for polynomials, which is left as an exercise, and which states that

$$\sigma(\Phi(f)) = f(\sigma(T)) = \{f(\lambda) : \lambda \in \sigma(T)\}$$

to obtain

$$\begin{aligned}
\|\Phi(f)\|^2 &= \|\Phi(f)^*\Phi(f)\| = \|\Phi(\overline{f}f)\| \\
&= \sup\{|\lambda| : \lambda \in \sigma(\Phi(\overline{f}f))\} \\
&= \sup\{|\overline{f}f(\lambda)| : \lambda \in \sigma(T)\} \\
&= \sup\{|f(\lambda)|^2 : \lambda \in \sigma(T)\} = \|f\|_\infty^2.
\end{aligned}$$

We refer to the exercises for the first equality, the second equality follows from a simple calculation for polynomials, the third one is also based on an exercise and the fact that $\Phi(\overline{f}f)$ is self-adjoint.

All the required properties are then proven by approximation. Let us assume (for example) that $p_n \to f$ and $q_n \to g$ for some polynomials p_n, q_n and $f, g \in C(\sigma(T))$. Then we get

$$\Phi(fg) \leftarrow \Phi(p_n q_n) = \Phi(p_n) \circ \Phi(q_n) \to \Phi(f) \circ \Phi(g).$$

The proof of the other properties is similar. ∎

Theorem 1.9 *Let* $T \in L(\mathcal{H})$ *be self-adjoint and let* $f \to f(T)$ *be the continuous functional calculus for* $f \in C(\sigma(T))$. *Then*

(i) $\|f(T)\| = \|f\|_\infty := \sup_{\lambda \in \sigma(T)} |f(\lambda)|$.

(ii) If $f \geq 0$ *on* $\sigma(T)$, *then* $f(T) \geq 0$, *i.e.* $f(T)$ *is positive.*

(iii) *If $Tx = \lambda x$ for some $x \in \mathcal{H}$, then also $f(T)x = f(\lambda)x$.*

(iv) *$\sigma(f(T)) = f(\sigma(T))$, i.e. the Spectral Mapping Theorem holds for all $f \in C(\sigma(T))$.*

(v) *$\{f(T) : f \in C(\sigma(T))\}$ is a commutative Banach algebra of operators.*

(vi) *All $f(T)$ are normal; if f is real, then $f(T)$ is self-adjoint.*

Proof. (i). was proven for polynomials already before. For general $p_n \to f$ we have $\|\Phi(f)\| \leftarrow \|\Phi(p_n)\| = \|p_n\|_\infty \to \|f\|_\infty$.
To show (ii)., let $f \geq 0$ and take $g \in C(\sigma(T))$ with $g^2 = f$ and $g \geq 0$. Then

$$\langle f(T)x, x\rangle = \langle g(T)x, g(T)^*x\rangle = \langle g(T)x, \overline{g}(T)x\rangle = \langle g(T)x, g(T)x\rangle = \|g(T)x\|^2 \geq 0.$$

(iii). is again clear for polynomials, through approximation it follows for all $f \in C(\sigma(T))$. Also (iv). was already discussed for polynomials. Let $\mu \notin f(\sigma(T))$. Then $g := (f - \mu)^{-1} \in C(\sigma(T))$ and $g(f - \mu) = (f - \mu)g = 1$. Hence, we get

$$g(T)(f(T) - \mu\,\mathrm{Id}) = (f(T) - \mu\,\mathrm{Id})g(T) = \mathrm{Id}\,.$$

Hence $\mu \in \varrho(f(T))$. This shows the "$\subset$" part in (iv).
Let on the other hand $\mu = f(\lambda)$ for some $\lambda \in \sigma(T)$ and choose polynomials p_n with $\|p_n - f\|_\infty \leq 1/n$. Then

$$|f(\lambda) - p_n(\lambda)| \leq 1/n \quad \text{and} \quad \|f(T) - p_n(T)\| \leq 1/n.$$

We know that $p_n(\lambda) \in \sigma(p_n(T))$, i.e. there exists $x_n \in \mathcal{H}$ with $\|x_n\| = 1$ and $\|(p_n(T) - p_n(\lambda)\,\mathrm{Id})x_n\| \leq 1/n$. Finally, from

$$\begin{aligned}\|(f(T) - \mu\,\mathrm{Id})x_n\| &\leq \|(f(T) - p_n(T))x_n\| + \|(p_n(T) - p_n(\lambda)\,\mathrm{Id})x_n\| + \|(p_n(\lambda) - \mu)Ix_n\| \\ &\leq 1/n + 1/n + 1/n = 3/n\end{aligned}$$

we see, that $f(T) - \mu\,\mathrm{Id}$ is not cont. invertible, i.e. $\mu \in \sigma(f(T))$. Finally, (v). is clear now and (vi). follows from $f(T)^* = \overline{f}(T)$ and $f(T)^*f(T) = \overline{f}f(T) = f\overline{f}(T) = f(T)f(T)^*$. ∎

Let us now have a look at the spectral decomposition theorem for compact self-adjoint operators and how it could be generalized to bounded operators. One obvious obstacle would be that the spectrum does not have to be countable any more, i.e. we will have to replace the sums by certain (appropriately interpreted) integrals. Furthermore, we have now spent some time to define $f(T)$ for (at least some) functions f on $\sigma(T)$. We therefore rewrite

$$Tx = \sum_{k=0}^{\infty} \lambda_k \langle x, x_k\rangle x_k = \sum_{k=0}^{\infty} \lambda_k E_k x,$$

where $E_k x = \langle x, x_k\rangle x_k$ is the projection of x onto the linear span of x_k. And if we put $f_k(\lambda_j) = \delta_{j,k}$, we get even

$$f_k(T)x = \sum_{n=0}^{\infty} f_k(\lambda_n)\langle x, x_n\rangle x_n = \langle x, x_k\rangle x_k = E_k x,$$

where (for simplicity) we assumed that all the λ_k's are different.

We will therefore write the spectral decomposition of a bounded self-adjoint operator as an integral with respect to a certain system of projections. Unfortunately, in this case $0-1-$valued functions do not have to be continuous on $\sigma(T)$ anymore. We therefore have to extend the definition of $f(T)$ to bounded measurable functions f. This shall be done through duality. For this sake, we shall need (another) Riesz Representation Theorem, and for this we shall need some notation.

Let (K, τ) be a compact topological space. We denote by $\mathcal{B}$ the smallest σ-algebra containing τ (the so-called *Borel σ-algebra*). As usual, $f : K \to \mathbb{C}$ is called *Borel measurable*, if $f^{-1}(U) \in \mathcal{B}$ for every $U \subset \mathbb{C}$ open.

The interaction between measures on $\mathcal{B}$ and the topology τ is described by the following regularity conditions.

A finite measure μ on $(K, \mathcal{B})$ is said to be a *Radon measure*, if

$$\mu(B) = \inf\{\mu(G) : G \supset B, G \text{ open}\} \text{ for every } B \in \mathcal{B},$$
$$\mu(G) = \sup\{\mu(T) : T \subset G, T \text{ compact}\} \text{ for every open } G \in \tau.$$

A *signed Radon measure* is a σ-additive mapping on $\mathcal{B}$ with $\mu = \mu^+ - \mu^-$, where both μ^+ and μ^- are Radon measures. Unfortunately, this representation is not unique. Indeed, if ν is another bounded positive Radon measure, we get $\mu = \mu^+ - \mu^- = (\mu^+ + \nu) - (\mu^- + \nu)$. Nevertheless, one of these representations is "minimal" (i.e. μ^+ and μ^- are singular to each other).

Using this minimal decomposition, we denote by $|\mu| = \mu^+ + \mu^-$ the *total variation* of μ. Moreover, one can obtain complex-valued measures $\mu = \mu_1 + i \cdot \mu_2$, where μ_1, μ_2 are signed Radon measures. The notion of the total variation can also be generalized to complex measures.

Finally, we denote by $\mathcal{M}(K)$ the space of all complex Radon measures on K with $\|\mu\|_{\mathcal{M}(K)} = |\mu|(K)$.

Theorem 1.10 (Another Riesz Representation Theorem)

$$C(K)^* \approx \mathcal{M}(K),$$

where $\mu(f) = \int_K f d\mu$. Furthermore, positive functionals (i.e. those with $f \geq 0 \Rightarrow \varphi(f) \geq 0$) correspond to non-negative measures.

Lemma 1.11 *Let $\mathcal{H}$ be a complex Hilbert space and let $Q : \mathcal{H} \to \mathbb{C}$ be a function. Then the following are equivalent.*

1. *There exists exactly one operator $A \in L(\mathcal{H})$ such that $Q(x) = \langle Ax, x\rangle$ for all $x \in \mathcal{H}$.*
2. *There is a constant $C > 0$ such that $|Q(x)| \leq C\|x\|^2$, $Q(x+y) + Q(x-y) =$*

$2Q(x) + 2Q(y)$ and $Q(\lambda x) = |\lambda|^2 Q(x)$ for all $x, y \in \mathcal{H}$ and all $\lambda \in \mathbb{C}$.

Proof. It is easy to show that (i). implies (ii). If (ii) is satisfied, we put (motivated by polarization identity)

$$\Psi(x,y) := 1/4(Q(x+y) - Q(x-y) + iQ(x+iy) - iQ(x-iy))$$

and

$$Ax = \sum_{\gamma\in\Gamma} \Psi(x, e_\gamma)e_\gamma,$$

where $(e_\gamma)_{\gamma\in\Gamma}$ is some orthonormal basis of $\mathcal{H}$. The rest follows by simple calculations. Finally, let us observe that $\Psi(x,y) = \langle Ax, y\rangle$. The uniqueness of A follows from the exercises. ∎

Remark 1.12 (The main idea of Borel-measurable functional calculus)
Let $T \in L(\mathcal{H})$ be self-adjoint and let $x \in \mathcal{H}$ be fixed. We consider the mapping

$$f \to \langle f(T)x, x\rangle, \quad C(\sigma(T)) \to \mathbb{C}.$$

This mapping is linear, non-negative (i.e. $f \geq 0$ implies $\langle f(T)x, x\rangle \geq 0$ for all $x \in \mathcal{H}$). Therefore, there exists a non-negative Radon measure E^x on $\sigma(T)$, such that

$$\langle f(T)x, x\rangle = \int_{\sigma(T)} f dE^x \quad \text{for all} \quad f \in C(\sigma(T)).$$

If g is a bounded Borel-measurable function on $\sigma(T)$, the integral $\int_{\sigma(T)} g dE^x$ converges and we shall show (using the Lemma above) that it is equal to some $\langle Gx, x\rangle$ for all $x \in \mathcal{H}$ and some $G \in L(\mathcal{H})$. This G will then be defined to be $g(T)$. ◊

Let us now elaborate on the program just stated.

Theorem 1.13 *Let $T \in L(\mathcal{H})$ be self-adjoint.*

1. *Let $x \in \mathcal{H}$. Then there is a non-negative Radon measure E^x, such that*

$$\langle f(T)x, x\rangle = \int_{\sigma(T)} f dE^x$$

   for every $f \in C(\sigma(T))$.

2. *Let g be a bounded Borel-measurable function on $\sigma(T)$. Then there is a unique $G \in L(\mathcal{H})$ with*

$$\langle Gx, x\rangle = \int_{\sigma(T)} g dE^x$$

   for all $x \in \mathcal{H}$. If g is real, G is self-adjoint, if $g \geq 0$, G is positive.

Proof. The proof is based on Lemma 1.11. Obviously, the operator is unique ($\langle Gx, x\rangle = \langle \tilde{G}x, x\rangle$ for all $x \in \mathcal{H}$ implies $G = \tilde{G}$ by an exercise). Furthermore, $g \geq 0$ implies $\langle Gx, x\rangle \geq 0$, i.e. positivity, and g real implies $\langle Gx, x\rangle$ is real, i.e. G is self-adjoint.
So, we have to prove the existence. We put

$$Q(x) = \int_{\sigma(T)} g dE^x.$$

Then

$$|Q(x)| \leq \int_{\sigma(T)} \|g\|_\infty dE^x = \|g\|_\infty \int_{\sigma(T)} 1 dE^x = \|g\|_\infty \langle 1(T)x, x\rangle = \|g\|_\infty \|x\|^2. \tag{1.1}$$

For every $f \in C(\sigma(T))$, we obtain

$$\underbrace{\int_{\sigma(T)} f dE^{x+y}}_{\langle f(T)(x+y), x+y\rangle} + \underbrace{\int_{\sigma(T)} f dE^{x-y}}_{\langle f(T)(x-y), x-y\rangle} = \underbrace{2\int_{\sigma(T)} f dE^{x}}_{2\langle f(T)x, x\rangle} + \underbrace{2\int_{\sigma(T)} f dE^{y}}_{2\langle f(T)y, y\rangle}$$

and

$$\underbrace{\int_{\sigma(T)} f dE^{\lambda x}}_{\langle f(T)(\lambda x), \lambda x\rangle} = \underbrace{|\lambda|^2 \int_{\sigma(T)} f dE^{x}}_{|\lambda|^2 \cdot \langle f(T)x, x\rangle}$$

Due to the uniqueness of the Riesz Representation Theorem, we get $E^{x+y} + E^{x-y} = 2E^x + 2E^y$ and $E^{\lambda x} = |\lambda|^2 E^x$. This verifies the assumptions of Lemma 1.11 and finishes the proof. ∎

Yet another theorem from measure theory we shall use is that the set of bounded Borel-measurable functions is the smallest set, which includes bounded continuous functions and is closed under pointwise limits of such functions.

Theorem 1.14 *Let $K \subset \mathbb{C}$ be compact. Let $\mathcal{B}^d(K)$ be the Banach space of bounded Borel-measurable functions on K equipped with the supremum norm. Let $C(K) \subset U \subset \mathcal{B}^d(K)$ be a set of functions with the following property*

$$(f_n)_{n\in\mathbb{N}} \subset U \text{ with } f(t) := \lim_{n\to\infty} f_n(t) \text{ existing everywhere and } \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty \Rightarrow f \in U.$$

Then $U = \mathcal{B}^d(K)$.

Proof. The proof is based on measure theory. We refer to Werner, Lemma VII.1.5 for details. ∎

Theorem 1.15 (Borel measurable functional calculus) *Let $T \in L(\mathcal{H})$ be self-adjoint. Then there exists a unique mapping $\Phi : \mathcal{B}^d(\sigma(T)) \to L(\mathcal{H})$ with* *(i)* $\Phi(z) = T, \Phi(1) = \mathrm{Id}$;

(ii) Φ *is an involutive homomorphism of algebras;*

(iii) Φ *is continuous;*

(iv) $f_n \in \mathcal{B}^d(\sigma(T))$ *with* $\sup_n \|f_n\|_\infty < \infty$ *and* $f_n(t) \to f(t)$ *for every* $t \in \sigma(T)$ *implies* $\langle \Phi(f_n)x, y\rangle \to \langle \Phi(f)x, y\rangle$ *for every* $x, y \in \mathcal{H}$.

Proof. Concerning the uniqueness part, observe that (i).,(ii). and (iii). imply uniqueness on $C(\sigma(T))$ and Theorem 1.14 implies uniqueness on all $\mathcal{B}^d(\sigma(T))$.
Now we want to turn our attention to the proof of the existence. We define $\Phi(g) = G$, where G is the operator from Theorem 1.13. We have to verify (i).-(iv). We know that

$$\langle f(T)x, x\rangle = \int_{\sigma(T)} f dE^x$$

for all $f \in C(\sigma(T))$, i.e. $\Phi(f) = f(T)$ for all $f \in C(\sigma(T))$, where $f(T)$ is the continuous functional calculus. This implies (i).

Let g be real valued. Then we obtain

$$\|\Phi(g)\| = \|G\| = \sup\{|\langle Gx, x\rangle| : \|x\| = 1\} \leq \sup\{\|g\|_\infty \cdot \|x\|^2 : \|x\| = 1\} = \|g\|_\infty,$$

where we have used (1.1) and the fact that G is self-adjoint. If g is complex-valued, we split it first into its real and imaginary part.
The proof of (iv) follows from

$$\langle \Phi(f_n)x, x\rangle = \int_{\sigma(T)} f_n dE^x \to \int_{\sigma(T)} f dE^x = \langle \Phi(f)x, x\rangle,$$

which holds by the Lebesgue Dominated Convergence Theorem, and by the polarization identity.
(ii). follows from a limit procedure, which is quite often used in measure theory. For example, to show that $\Phi(fg) = \Phi(f) \circ \Phi(g)$, we proceed as follows. If $f, g \in C(\sigma(T))$, then the result is known. If $g \in C(\sigma(T))$, we set $U := \{f \in \mathcal{B}^d(\sigma(T)) : \Phi(fg) = \Phi(f) \circ \Phi(g)\}$. Then we know that $C(\sigma(T)) \subset U$ and U is closed under pointwise limits of uniformly bounded sequences. According to Theorem 1.14, $U = \mathcal{B}^d(\sigma(T))$.
Finally, if $f \in \mathcal{B}^d(\sigma(T))$, we put $V := \{g \in \mathcal{B}^d(\sigma(T)) : \Phi(fg) = \Phi(f) \circ \Phi(g)\}$. We have just shown that $C(\sigma(T)) \subset V$, furthermore, V is again closed on pointwise limits of uniformly bounded functions, i.e. $V = \mathcal{B}^d(\sigma(T))$ and the statement is true for all $f, g \in \mathcal{B}^d(\sigma(T))$.
The other properties follow in the same way. ∎

Remark 1.16 (iv). in Theorem 1.15 can be improved to $f_n(T)x \to f(T)x$ for every $x \in \mathcal{H}$. This follows from

$$\begin{aligned}\|f_n(T)x\|^2 &= \langle f_n(T)x, f_n(T)x\rangle = \langle f_n(T)^* f_n(T)x, x\rangle \\ &= \langle (\overline{f}_n f_n)(T)x, x\rangle \to \langle (\overline{f} f)(T)x, x\rangle = \|f(T)x\|^2\end{aligned}$$

and $f_n(T)x \xrightarrow{w} f(T)x$. ◊

Remark 1.17 (Analytic functional calculus) We were able to define $f(T)$ for (rather) ugly functions (i.e. all Borel-measurable functions) and very nice operators (i.e. self-adjoint operators on a Hilbert space over $\mathbb{C}$). It comes probably as no surprise that for nicer functions we can define $f(T)$ also for more general classes of operators.

Let X be a Banach space and let $T \in L(X)$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a potential series with convergence radius bigger then $r(T)$. Then f is analytic on $\sigma(T)$ and we can define

$$f(T) := \sum_{n=0}^{\infty} a_n T^n.$$

One can show:

(1) The series for $f(T)$ converges in $L(X)$ and $f(T)$ is therefore well-defined.

(2) If $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is another potential series with radius of convergence bigger then $r(T)$, then $(fg)(T) = f(T)g(T)$.

(3) The Spectral mapping theorem $\sigma(f(T)) = f(\sigma(T))$ holds.

(4) If T is a self-adjoint operator on a Hilbert space over $\mathbb{C}$, then this definition coincides with the continuous (and also with the Borel-measurable) functional calculus.

◊

Remark 1.18 (Dunford's analytic functional calculus) The analytic functional calculus described above allows to study spectral theory also in the frame of Banach spaces. Nevertheless, the basis for such a theory is different. If f is holomorphic on a neighborhood Ω of $\sigma(T)$ (i.e. on an open set containing $\sigma(T)$), then we may define

$$f(T) := \frac{1}{2\pi i} \oint_{\Gamma} f(\lambda) R_\lambda(T) d\lambda,$$

where $\Gamma \subset \Omega$ is a suitable curve, which goes around $\sigma(T)$ exactly once, and $R_\lambda(T)$ is the resolvent mapping of T. Of course, this idea is heavily inspired by the Cauchy integral known from complex analysis. ◊

1.3 Spectral theorem for bounded self-adjoint operators

In view of the spectral theorem, which we will show in this section, $f(T)$ is especially important for $f = \chi_A$, where $A \subset \mathbb{R}$ is a Borel measurable set.

Lemma 1.19 *Let $T \in L(\mathcal{H})$ be self-adjoint and let $A \subset \sigma(T)$ be a Borel set. Then $E_A := \chi_A(T)$ is an orthogonal projection.*

Proof. Due to $\chi_A^2 = \chi_A$, we have $E_A = E_A^2$ and $E_A^* = E_A$ follows from $\overline{\chi_A} = \chi_A$ and an exercise. ∎

Lemma 1.20 *Let $T \in L(\mathcal{H})$ be self-adjoint. Then*

(i) $E_\varnothing = \chi_\varnothing(T) = 0, E_{\sigma(T)} := \chi_{\sigma(T)}(T) = \mathrm{Id}$;

(ii) For pairwise disjoint sets $A_1, A_2, \dots \subset \sigma(T)$, and for $x \in \mathcal{H}$, we have

$$\sum_{i=1}^{\infty} \chi_{A_i}(T)x = \chi_{\bigcup_{i=1}^{\infty} A_i}(T)x.$$

(iii) $\chi_A(T)\chi_B(T) = \chi_{A\cap B}(T)$ *for Borel sets $A, B \subset \sigma(T)$.*

Proof. The proof follows from the properties of the Borel-measurable functional calculus. ∎

Remark 1.21 Let us note, that

$$\sum_{i=1}^{\infty} \chi_{A_i}(T) \neq \chi_{\bigcup_{i=1}^{\infty} A_i}(T),$$

i.e. the identity must be interpreted in the "weak sense". ◊

Definition 1.22 Let Σ be the σ-algebra of Borel sets on $\mathbb{R}$. A mapping

$$E : \Sigma \to L(\mathcal{H}), \quad E(A) = E_A$$

is called a *spectral measure* if all E_A are orthogonal projections and

(1) $E_\varnothing = 0, E_\mathbb{R} = \mathrm{Id}$;

(2) For pairwise disjoint sets $A_1, A_2, \dots \in \Sigma$, and for $x \in \mathcal{H}$, we have

$$\sum_{i=1}^{\infty} E_{A_i}(x) = E_{\bigcup_{i=1}^{\infty} A_i}(x).$$

A spectral measure E has *compact support*, if there is a compact set K with $E_K = \mathrm{Id}$.

Let us state some simple properties of spectral measures.

(1) $E_A + E_B = E_{A\cap B} + E_{A\setminus B} + E_B = E_{A\cap B} + E_{A\cup B}$ holds for every Borel sets $A, B \subset \mathbb{R}$.

(2) If $A \subset B \subset \mathbb{R}$, then $E_{B\setminus A} = E_B - E_A$ is itself a projection, i.e. $E_B \geq E_A$ and (with the help of Exercises) $E_B E_A = E_A E_B = E_A$.

(3) If A and B are disjoint, we get $E_A + E_B = E_{A\cup B}$, i.e. $E_A^2 + E_A E_B = E_A E_{A\cup B} = E_A$ due to the previous point; hence $E_A E_B = 0$.

(4) $E_A E_B = E_A(E_{A\cap B} + E_{B\setminus A}) = E_A E_{A\cap B} + E_A E_{B\setminus A} = E_{A\cap B}$ holds (due to the previous points) for all Borel sets $A, B \subset \mathbb{R}$.

(5) Finally, if $A, B \subset \mathbb{R}$ are disjoint, then $\langle E_A x, E_B y\rangle = \langle E_B^* E_A x, y\rangle = \langle E_B E_A x, y\rangle = \langle 0, y\rangle = 0$, i.e. E_A and E_B project onto mutually orthogonal subspaces.

Interpreting the aforementioned properties, "in some sense, E is a measure with values in $L(\mathcal{H})$ instead of $[0,\infty)$."

For the sake of spectral decomposition theorem, we need to find a way, how to calculate $\int f dE \in L(\mathcal{H})$ for a spectral measure E and a complex-valued function f. Of course, for $f = \chi_A$, we define $\int f dE := E(A) = E_A \in L(\mathcal{H})$. For $f = \sum\limits_{i=1}^{n} \alpha_i \chi_{A_i}$ a simple function, we put $\int f dE := \sum\limits_{i=1}^{n} \alpha_i E_{A_i} \in L(\mathcal{H})$. Finally, for a general bounded Borel-measurable function f we consider a uniformly convergent sequence $f_n \to f$ of simple functions and define

$$\int f dE := \lim_{n\to\infty} \int f_n dE.$$

We have to check, that this definition is consistent. It is quite easy to see that the integral of a simple function f does not depend on the form in which f was written. The limit procedure is then justified by the following Lemma:

Lemma 1.23 *For a simple function f, we have*

$$\left\| \int f dE \right\| \leq \|f\|_\infty.$$

Proof. Let $\|x\| = 1$ and let $f = \sum\limits_{i=1}^{n} \alpha_i \chi_{A_i}$ with mutually disjoint $A_1, \dots, A_n$. Then

$$\begin{aligned} \left\| \left(\int f dE \right)(x) \right\|^2 &= \left\| \sum_{i=1}^{n} \alpha_i E_{A_i}(x) \right\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 \cdot \|E_{A_i}(x)\|^2 \\ &\leq \max_{i=1,\dots,n} |\alpha_i|^2 \cdot \sum_{i=1}^{n} \|E_{A_i}(x)\|^2 \leq \|f\|_\infty^2 \cdot \|x\|^2 = \|f\|_\infty^2. \end{aligned}$$

∎

This lemma shows that if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in $\|\cdot\|_\infty$ (i.e. $f_n \to f$), then $(\int f_n dE)_n$ is a Cauchy-sequence in $L(\mathcal{H})$, its limit exists and it does not depend on the choice of the sequence $(f_n)_{n\in\mathbb{N}}$. Therefore, $\int f dE$ is well defined.

We summarize our findings in the following theorem:

Theorem 1.24 (Properties of integration with respect to the spectral measure) *Let E be a spectral measure. Then $\int f dE \in L(\mathcal{H})$ exists for every bounded Borel measurable function $f \in \mathcal{B}^d(\mathbb{R})$. The mapping $f \to \int f dE$ is linear and continuous, i.e. $\left\|\int f dE\right\| \leq \|f\|_\infty$. If f is real, then $\int f dE$ is self-adjoint. If $K \subset \mathbb{R}$ is compact with $E_K = \mathrm{Id}$, then it is enough if f is defined and bounded on K.*

Remark 1.25 On the one hand, we have for each $T \in L(\mathcal{H})$ self-adjoint the spectral measure $E_A := \chi_A(T)$. On the other hand, we can define for every spectral measure with compact support the operator $T := \int \lambda dE_\lambda$, i.e. we integrate the identity function $\lambda \mapsto \lambda$ with respect to E. We shall show that these two operations are actually inverse to each other. $\Diamond$

Theorem 1.26 *Let E be a spectral measure on $\mathbb{R}$ with compact support. Let $T \in L(\mathcal{H})$ be the self-adjoint operator defined by $T = \int \lambda dE_\lambda$. Then the mapping*

$$\Psi : f \to \int_{\sigma(T)} f dE, \quad \Psi : \mathcal{B}^d(\sigma(T)) \to L(\mathcal{H})$$

is the Borel-measurable functional calculus from Theorem 1.15. Especially, we have $E_{\sigma(T)} = \mathrm{Id}$.

Proof. We first sketch the proof that $E_{\sigma(T)} = \mathrm{Id}$. Let $f(\lambda) := \lambda$ and let $f_\delta \to f$ be a collection of simple functions $f_\delta = \sum_{i=1}^{n(\delta)} \alpha_{i,\delta}\chi_{A_{i,\delta}}$ with $|f_\delta(\lambda) - \lambda| \leq \delta$ on the support of E and $A_{1,\delta}, \ldots, A_{n(\delta),\delta}$ mutually disjoint. Let $\mu \in \varrho(T)$. As $\varrho(T)$ is open and

$$\left\|\int_{\mathbb{R}} f_\delta dE - T\right\| = \left\|\int_{\mathbb{R}} (f_\delta(\lambda) - \lambda) dE\right\| \leq \delta,$$

we conclude, that $\int_{\mathbb{R}} f_\delta dE - \mu\,\mathrm{Id} = \sum_{i=1}^{n(\delta)} \alpha_{i,\delta} E_{A_{i,\delta}} - \mu\,\mathrm{Id}$ is invertible for $0 < \delta \leq \delta_0$ (for $\delta_0 > 0$ small enough). On the other hand, the inverse of such an operator has a norm $\max\{|\alpha_{i,\delta} - \mu|^{-1} : 1 \leq i \leq n(\delta), E_{A_{i,\delta}} \neq 0\} \to \|(T - \mu\,\mathrm{Id})^{-1}\|$. Hence, $E_{A_{i,\delta}} = 0$ for $|\alpha_{i,\delta} - \mu|$ small and we conclude that $E_U = 0$ for some small neighborhood of $\mu \in U$. Hence $E_{\varrho(T)} = 0$ and $E_{\sigma(T)} = \mathrm{Id}$.

To finish the proof, we have to verify that the mapping Ψ has all the properties described in Theorem 1.15. We know already, that it is linear, continuous, multiplicative (start with characteristic functions, then simple functions and then limits) and involutive. The condition (iv) from Theorem 1.15 reads in our case as

$$\left\langle \left(\int f_n dE \right) x, y \right\rangle \to \left\langle \left(\int f dE \right) x, y \right\rangle$$

and follows from the fact, that the mapping $g \to \langle (\int g dE) x, y \rangle$ is in $C(\sigma(T))^*$ and may be therefore represented by a measure $\nu_{x,y}$, i.e. $\langle (\int g dE) x, y \rangle = \int g d\nu_{x,y}$. The proof is then finished by applying the Lebesgue Dominated Convergence Theorem. ∎

Hence, integration with respect to a spectral measure E is a functional calculus of the self-adjoint operator $\int \lambda dE_\lambda$. The spectral theorem says, that actually every self-adjoint operator may be written in this way - i.e. there is a one-to-one correspondence between bounded self-adjoint operators and spectral measures with compact support.

Theorem 1.27 (Spectral theorem for bounded self-adjoint operators) *Let $T \in L(\mathcal{H})$ be self-adjoint. Then there exists a unique spectral measure E on $\mathbb{R}$ with compact support with*

$$T = \int_{\sigma(T)} \lambda dE_\lambda.$$

The mapping $f \to f(T) = \int f(\lambda) dE_\lambda$ coincides with the Borel-measurable functional calculus and

$$\langle f(T)x, y \rangle = \int_{\sigma(T)} f(\lambda) d\langle E_\lambda x, y \rangle$$

holds for any $x, y \in \mathcal{H}$. By $\langle E_\lambda x, y \rangle$ we denote the (complexed-valued) measure $A \to \langle E_A x, y \rangle$.

Proof. Let E be the spectral measure associated to T as described above, i.e. $E : A \to \chi_A(T)$. We have to show, that $S := \int_{\sigma(T)} \lambda dE_\lambda$ is equal to T. Let f be a simple function with $\|f(\lambda) - \lambda\|_\infty = \sup_{\lambda \in \sigma(T)} |f(\lambda) - \lambda| \le \delta$. Then

$$\|T - S\| \le \|T - f(T)\| + \|f(T) - f(S)\| + \|f(S) - S\|.$$

We estimate all the three summands as follows. Let Ψ be the Borel-measurable functional calculus of T. Then $\Psi(\lambda) = T$ and $\Psi(f) = f(T)$. Hence

$$\|T - f(T)\| = \|\Psi(\lambda) - \Psi(f)\| = \|\Psi(\lambda - f(\lambda))\| \le \|f(\lambda) - \lambda\|_\infty \le \delta,$$

where we have used the boundedness of Borel-measurable functional calculus.

Next we estimate $\|f(T) - f(S)\|$. We write $f(t) = \sum_i \alpha_i \chi_{A_i}(t)$ with $A_i \subset \mathbb{R}$ disjoint. Then, by the definition of the E_{A_i}, we get

$$f(T) = \sum_i \alpha_i \chi_{A_i}(T) = \sum_i \alpha_i E_{A_i}.$$

Furthermore, according to Theorem 1.26 applied to S, we know that

$$f(S) = \int_{\mathbb{R}} f(\lambda) dE_\lambda = \int_{\mathbb{R}} \sum_i \alpha_i \chi_{A_i}(\lambda) dE_\lambda = \sum_i \alpha_i E_{A_i}. \tag{1.2}$$

Hence, $\|f(T) - f(S)\| = 0$.

Finally, we estimate $\|f(S) - S\|$. We use again (1.2) combined with the norm estimate from Theorem 1.24 and obtain

$$\|f(S) - S\| = \left\| \int (f(\lambda) - \lambda) dE_\lambda \right\| \leq \|f(\lambda) - \lambda\|_\infty \leq \delta$$

Hence, $\|S - T\| \leq \delta$ for $\delta > 0$ arbitrary small, i.e. $S = T$.

The properties of the mapping $f \to f(T) := \int_{\sigma(T)} \lambda dE_\lambda$ follow from Theorem 1.26, once the identity $S = T$ was proven. Finally, the identity for $\langle f(T)x, y\rangle$ is a matter of definition for $f = \chi_A$, and also for simple functions. Finally, taking a limit, it follows for every bounded Borel measurable f. ■

Let us summarize the structure of this section into the following diagram.

$$\begin{tikzcd}
\overset{T \in L(\mathcal{H})}{\text{self-adjoint}} \arrow[r, "\text{Weierstraß Theorem}", "\text{density of polynomials}"'] \arrow[d, equal] & \text{Cont. F.C. (Thm. 1.8)} \arrow[r, "\text{Riesz}"] & \text{Borel FC (Thm. 1.15)} \arrow[d, "E_A := \chi_A(T)"'] \\
\overset{T \in L(\mathcal{H})}{\text{self-adjoint}} & \int f dE \arrow[l, "T := \int \lambda dE_\lambda"', "\text{Thm. 1.26}"] & \text{Spectral measure} \arrow[l, "\text{Thm. 1.24}"']
\end{tikzcd}$$

1.4 Some applications of the spectral theorem

Theorem 1.28 *Let $S \in L(\mathcal{H})$ be self-adjoint, $g : \sigma(S) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be Borel-measurable and bounded. Then*

$$(f \circ g)(S) = f(g(S)).$$

Proof. First, note that $f \circ g$ is again Borel-measurable. Furthermore, $g(S)$ is self-adjoint and, therefore, $f(g(S))$ can be defined.

It is enough to consider $f = \chi_A$, the rest follows in the usual way. Then

$$\chi_A \circ g = \chi_{g^{-1}(A)}$$

and we have to show that $\chi_{g^{-1}(A)}(S) = \chi_A(g(S))$. Let F be the spectral measure of S and E the spectral measure of $g(S)$; then we have to show that $F_{g^{-1}(A)} = E_A$ for all Borel sets A, or, equivalently,

$$\langle F_{g^{-1}(A)} x, y\rangle = \langle E_A x, y\rangle$$

for all $x, y \in \mathcal{H}$ and all Borel sets A.

Let $x, y \in \mathcal{H}$ be fixed and let us consider the measures

$$\begin{aligned}\nu^1_{x,y} &: A \to \langle F_{g^{-1}(A)}x, y\rangle,\\ \nu^2_{x,y} &: A \to \langle E_A x, y\rangle,\\ \mu_{x,y} &: A \to \langle F_A x, y\rangle.\end{aligned}$$

Hence, $\nu^1_{x,y}(A) = \mu_{x,y}(g^{-1}(A))$. We use the transformation law for the (complex-valued) measures $\nu^1_{x,y}$ and $\mu_{x,y}$ for

$$\int h d\nu^1_{x,y} = \int (h \circ g) d\mu_{x,y},$$

written as

$$\int h(\lambda) d\nu^1_{x,y}(\lambda) = \int (h \circ g) d\mu_{x,y} = \int h(g(\lambda)) d\langle F_\lambda x, y\rangle$$

and obtain

$$\int \lambda^n d\nu^1_{x,y}(\lambda) = \int g(\lambda)^n d\langle F_\lambda x, y\rangle = \langle g(S)^n x, y\rangle = \int \lambda^n d\langle E_\lambda x, y\rangle = \int \lambda^n d\nu^2_{x,y}(\lambda).$$

Due to Riesz and Weierstrass, $\nu^1_{x,y} = \nu^2_{x,y}$. ∎

Remark 1.29 We recall the notion of the *push-forward measure*. Let μ be a measure on X_1 and $f : X_1 \to X_2$ a mapping. Then $f_*(\mu)(B) = \mu(f^{-1}(B))$ for $B \subset X_2$ is called the push-forward measure of μ under f. To develop the complete theory (which includes the transformation law we have used just now) one has to take care also about the corresponding σ-algebras. ◊

Corollary 1.30 *Let $T \in L(\mathcal{H})$ be positive and let $n \in \mathbb{N}$. Then there exists a unique positive $S \in L(\mathcal{H})$ with $S^n = T$.*

Proof. The functions $f_n : t \to t^{1/n}$ are continuous, bounded and non-negative on $\sigma(T) \subset [0, \infty)$. We set $S := f_n(T)$. Then $S \geq 0$ and from $t^{1/n} \cdot \dots \cdot t^{1/n} = t$, it follows $S^n = T$. Let $g_n(t) = t^n$, then $(f_n \circ g_n)(t) = t$ for $t \in [0, \infty) \supset \sigma(S)$ and

$$S = (f_n \circ g_n)(S) = f_n(g_n(S)) = f_n(S^n) = f_n(T)$$

shows the uniqueness of S. ∎

Corollary 1.31 (Polar decomposition) *For an operator $T \in L(\mathcal{H})$, we define $|T| = (T^*T)^{1/2}$. Furthermore, there is a partial isometry U with $T = U|T|$.*

Proof. The proof is the same as for compact operators. We see that

$$\||T|x\|^2 = \langle x, T^*Tx \rangle = \langle Tx, Tx \rangle = \|Tx\|^2.$$

Therefore, $U(|T|x) := Tx$ is an isometry of $\operatorname{ran}|T|$ to $\operatorname{ran} T$. This can be extended to $U : \overline{\operatorname{ran}|T|} \to \overline{\operatorname{ran} T}$. Finally, we put $U = 0$ on $(\operatorname{ran}|T|)^\perp = \ker|T| = \ker T$. ∎

2 Spectral Theorem for Unbounded Operators on Complex Hilbert Spaces

Also in this section, $\mathcal{H}$ denotes a Hilbert space over $\mathbb{C}$.

2.1 Motivation and Basic Definitions

Remark 2.1 Many important operators on Hilbert spaces appearing for example in physics are not bounded. This is especially the case for many differential operators. As an example, consider $\mathcal{H} = L^2([0,1])$, $T = \frac{d}{dt}$, which is only defined on $C([0,1])$ and which is not continuous with respect to $\|\cdot\|_2$. Therefore, one is interested in the analysis of operators defined on a subspace $\mathrm{dom}(T)$ of $\mathcal{H}$. $\Diamond$

Definition 2.2 (1) A linear mapping $T : \mathrm{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ is called an *operator*, if $\mathrm{dom}(T)$ is a linear subspace of $\mathcal{H}$.

(2) We say, that T is *densely defined* if $\overline{\mathrm{dom}(T)} = \mathcal{H}$.

(3) The *graph* of an operator T is the set $G(T) := \{(Tx, x) : x \in \mathrm{dom}(T)\} \subset \mathcal{H} \times \mathcal{H}$, where $\mathcal{H} \times \mathcal{H} = \{(x,y) : x, y \in \mathcal{H}\}$ is a Hilbert space with the scalar product $\langle (u,v),(x,y)\rangle_{\mathcal{H}\times\mathcal{H}} = \langle u, x\rangle + \langle v, y\rangle$. We say that T is a *closed operator* if $G(T)$ is closed in $\mathcal{H} \times \mathcal{H}$.

(4) T is called *closable* if $\overline{G(T)}$ is a graph of some linear operator T_0. This operator is then called *closure* of T. We shall denote this by $T_0 = \overline{T}$.

(5) An operator $S : \mathrm{dom}(S) \subset \mathcal{H} \to \mathcal{H}$ is called *extension* of T if $G(T) \subset G(S)$ or, equivalently, if $\mathrm{dom}(T) \subset \mathrm{dom}(S)$ and $Sx = Tx$ for all $x \in \mathrm{dom}(T)$. We shall denote this by $T \subset S$.

(6) $T = S$ if $T \subset S$ and $S \subset T$, i.e. if $\mathrm{dom}(T) = \mathrm{dom}(S)$ and $Sx = Tx$ for all $x \in \mathrm{dom}(T)$.

The domain of T, i.e. $\mathrm{dom}(T)$, is an essential part of the definition of every operator T. For example, if S and T are two (unbounded) operators on $\mathcal{H}$, we set

$$\mathrm{dom}(S+T) := \mathrm{dom}(S) \cap \mathrm{dom}(T), \quad (S+T)(x) = Sx + Tx,$$
$$\mathrm{dom}(ST) := \{x \in \mathrm{dom}(T) : Tx \in \mathrm{dom}(S)\}, \quad (ST)(x) = S(T(x)).$$

Definition 2.3 An operator $T : \mathrm{dom}(T) \to \mathcal{H}$ is called *symmetric* if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in \mathrm{dom}(T).$$

Theorem 2.4 (Hellinger-Toeplitz) *Let T be a symmetric operator defined on* $\mathrm{dom}(T) = \mathcal{H}$. *Then T is bounded, i.e. $T \in L(\mathcal{H})$.*

Proof. Let $x_n \to 0$ and $Tx_n \to z$. We show that $z = 0$. The claim then follows from the Closed Graph theorem. But we have

$$\langle z, z\rangle = \left\langle \lim_{n\to\infty} Tx_n, z \right\rangle = \lim_{n\to\infty} \langle Tx_n, z\rangle = \lim_{n\to\infty} \langle x_n, Tz\rangle = 0.$$

Let us mention that in this case $T = T^*$. ∎

Next we define the adjoint of a densely defined operator T.

Definition 2.5 Let T be a densely defined operator. We define

$$\mathrm{dom}(T^*) := \{y \in \mathcal{H} : x \to \langle Tx, y\rangle \text{ is continuous on } \mathrm{dom}(T)\}.$$

For $y \in \mathrm{dom}(T^*)$, the mapping $x \to \langle Tx, y\rangle$ may be extended to a continuous mapping on all $\mathcal{H}$ and (according to the Riesz Theorem) be represented as $x \to \langle x, z\rangle$, for some $z \in \mathcal{H}$. We denote this z by T^*y. Due to the density of $\mathrm{dom}(T)$, z is unique. T^* is called the *adjoint operator of* T. If $T = T^*$, then T is called self-adjoint.

Lemma 2.6 *We have $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in$* $\mathrm{dom}(T)$ *and all $y \in$* $\mathrm{dom}(T^*)$.

If T and T^* are densely defined, we can also have a look at T^{**}. If T is densely defined and symmetric, then $T \subset T^*$.

Theorem 2.7 *Let $T :$* $\mathrm{dom}(T)$ *$\to \mathcal{H}$ be densely defined. Then*

(i) T^* is closed.

(ii) If T^* is densely defined, then $T \subset T^{**}$.

(iii) If T^* is densely defined, then T^{**} is the closure of T.

Proof. (i). We have to show that for $(y_n)_n \subset \mathrm{dom}(T^*), y_n \to y \in \mathcal{H}$ and $T^* y_n \to z$ implies $y \in \mathrm{dom}(T^*)$ and $z = T^* y$. We have

$$\langle Tx, y \rangle = \lim_{n\to\infty} \langle Tx, y_n \rangle = \lim_{n\to\infty} \langle x, T^* y_n \rangle = \langle x, z \rangle, \quad \text{for every} \quad x \in \mathrm{dom}(T).$$

Hence $x \mapsto \langle Tx, y \rangle$ is continuous, $y \in \mathrm{dom}(T^*)$ and $T^* y = z$.
(ii). If $x \in \mathrm{dom}(T)$ and $y \in \mathrm{dom}(T^*)$, then $\langle Tx, y \rangle = \langle x, T^* y \rangle$. Hence $y \mapsto \langle x, T^* y \rangle$ is continuous and $x \in \mathrm{dom}(T^{**})$. Furthermore,

$$\langle x, T^* y \rangle = \langle T^{**} x, y \rangle, \quad \text{for all } y \in \mathrm{dom}(T^*).$$

We conclude that $Tx = T^{**} x$ for all $x \in \mathrm{dom}(T)$, i.e. $T \subset T^{**}$.
(iii). We show that

$$\overline{G(T)} = G(T^{**}).$$

The inclusion "$\subset$" follows from (i) and (ii). Let $(v, u) \in G(T)^\perp$. Then $\langle x, u \rangle + \langle Tx, v \rangle = 0$ for all $x \in \mathrm{dom}(T)$. Then $v \in \mathrm{dom}(T^*)$ and $T^* v = -u$.
For $(T^{**} z, z) \in G(T^{**})$, we have

$$\langle (T^{**} z, z), (v, u) \rangle = \langle z, u \rangle + \langle T^{**} z, v \rangle = \langle z, u \rangle + \langle z, T^* v \rangle = \langle z, u + T^* v \rangle = 0,$$

hence $(v, u) \in G(T^{**})^\perp$ and the second inclusion follows.

■

Corollary 2.8 *Let $T : \mathrm{dom}(T) \to \mathcal{H}$ be densely defined. Then the following holds:*

(i) T is symmetric if and only if $T \subset T^*$. Then we have $T \subset T^{**} \subset T^* = T^{***}$ and also T^{**} is symmetric.

(ii) T is closed and symmetric if and only if $T = T^{**} \subset T^*$.

(iii) T is self-adjoint if and only if $T = T^* = T^{**}$.

2.2 Spectral properties of unbounded operators

Definition 2.9 Let $T : \mathrm{dom}(T) \subset \mathcal{H} \to \mathcal{H}$ be densely defined.

(1) The set

$$\varrho(T) := \{\lambda \in \mathbb{C} : \lambda \operatorname{Id} - T : \mathrm{dom}(T) \to \mathcal{H} \text{ has a bounded inverse in } \mathcal{H}\}$$

is called the *resolvent set* of T.

(2) The mapping $R : \varrho(T) \to L(\mathcal{H})$, $R_\lambda = R(\lambda) := (\lambda \operatorname{Id} - T)^{-1}$ is called the *resolvent mapping*.

(3) $\sigma(T) := \mathbb{C} \setminus \varrho(T)$ is called the *spectrum* of T.

Theorem 2.10 *Let $T : \operatorname{dom}(T) \to \mathcal{H}$ be densely defined. Then*

(i) $\varrho(T)$ is open.

(ii) The resolvent mapping is analytic and the resolvent identity

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$$

holds.

(iii) $\sigma(T)$ is closed.

Proof. The proof works in a similar way as for bounded operators.

(i). Let $\lambda_0 \in \varrho(T)$ and $|\lambda - \lambda_0| < \|(\lambda_0 \operatorname{Id} - T)^{-1}\|^{-1}$. Then

$$\lambda \operatorname{Id} - T = (\lambda_0 \operatorname{Id} - T) + (\lambda - \lambda_0) \operatorname{Id} = (\lambda_0 \operatorname{Id} - T) \underbrace{[\operatorname{Id} - (\lambda - \lambda_0)(\lambda_0 \operatorname{Id} - T)^{-1}]}_{[\dots]^{-1} = \sum_{n=0}^{\infty} ((\lambda - \lambda_0)(\lambda_0 \operatorname{Id} - T)^{-1})^n}.$$

Hence, $\lambda \operatorname{Id} - T$ is invertible. (ii). A formal calculation immediately yields

$$(\lambda \operatorname{Id} - T)(\mu \operatorname{Id} - T)[(\lambda \operatorname{Id} - T)^{-1} - (\mu \operatorname{Id} - T)^{-1}] = (\mu - \lambda) \operatorname{Id}$$

and a similar identity can be derived by multiplying from the right side. So, only the (easy) inspection of domains is missing. (iii). follows from (i).

■

Theorem 2.11 *Let T be a densely defined operator. Then* $\ker(T^*) = \operatorname{ran}(T)^\perp$, *where*

$$\ker(T^*) = \{x \in \operatorname{dom}(T^*) : T^* x = 0\},$$
$$\operatorname{ran}(T) = \{Tx : x \in \operatorname{dom}(T)\}.$$

Proof. We have that $y \in \ker(T^*)$ if and only if, $\langle Tx, y \rangle = 0$ for all $x \in \operatorname{dom}(T)$ which holds if and only if $y \perp \operatorname{ran}(T)$. ■

Theorem 2.12 *Let T be a densely defined symmetric operator and $z \in \mathbb{C} \setminus \mathbb{R}$. Then the following statements are equivalent:*

(i) T is self-adjoint.

(ii) *T is closed and* $\ker(T^* - z\,\mathrm{Id}) = \ker(T^* - \bar{z}\,\mathrm{Id}) = \{0\}$.

(iii) $\operatorname{ran}(T - z\,\mathrm{Id}) = \operatorname{ran}(T - \bar{z}\,\mathrm{Id}) = \mathcal{H}$.

Proof. We shall use the fact proven in the exercises, that if T is densely defined and $z \in \mathbb{C}$, then $(T - z\,\mathrm{Id})^* = T^* - \bar{z}\,\mathrm{Id}$.

(i) $\Rightarrow$ (ii). Since $T = T^*$ and T^* is closed, also T is closed. Let $x \in \ker(T^* - z\,\mathrm{Id}) = \ker(T - z\,\mathrm{Id})$, which holds, since $T = T^*$. Then

$$z\langle x, x\rangle = \langle zx, x\rangle = \langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, zx\rangle = \bar{z}\langle x, x\rangle,$$

i.e. $x = 0$. The same holds for $\ker(T^* - \bar{z}\,\mathrm{Id})$.

(ii) $\Rightarrow$ (iii). By Theorem 2.11, we have

$$\operatorname{ran}(T - z\,\mathrm{Id})^{\perp} = \ker(T^* - \bar{z}I) = \{0\}.$$

So, it is enough to show, that $\operatorname{ran}(T - zI)$ is closed.
Let $y_n \in \operatorname{ran}(T - zI)$, i.e. $y_n = (T - zI)x_n$ for some $x_n \in \operatorname{dom}(T)$ and we suppose that $y_n \to y$. Let $z = a + ib$ with $b \neq 0$ and let $u \in \operatorname{dom}(T)$. Then we have

$$\|(T - z\,\mathrm{Id})u\|^2 = \langle (T - a - ib)u, (T - a - ib)u\rangle = \|(T - a\,\mathrm{Id})u\|^2 + b^2\|u\|^2 \geq b^2\|u\|^2,$$

as $\langle -ibu, (T - aI)u\rangle + \langle (T - aI)u, -ibu\rangle = 0$, since $u \in \operatorname{dom}(T) \subset \operatorname{dom}(T^*)$. Hence

$$\|u\| \leq \frac{1}{|b|} \cdot \|(T - zI)u\|.$$

We apply this inequality to $u = x_m - x_n$ and observe, that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence and $x := \lim_{n\to\infty} x_n$ exists. Then $Tx_n = y_n + zx_n$ converges also to $y + zx$. As T is closed, we get $x \in \operatorname{dom}(T)$, $Tx = y + zx$, or $y = (T - zI)x \in \operatorname{ran}(T - zI)$. This shows the claim.

(iii) $\Rightarrow$(i). Let $x \in \operatorname{dom}(T^*)$. By Assumption (iii), we can find some $y \in \operatorname{dom}(T)$, such that

$$(T - zI)y = (T^* - zI)x = (T^* - zI)y,$$

as $T \subset T^*$. Hence, $x - y \in \ker(T^* - zI)$, which, by Theorem 2.11, is equal to $\operatorname{ran}(T - \bar{z}I)^{\perp} = \{0\}$. Therefore, $x = y \in \operatorname{dom}(T)$, $\operatorname{dom}(T^*) \subset \operatorname{dom}(T)$ and $T = T^*$. ∎

Corollary 2.13 *Let T be a densely defined, self-adjoint operator. Then $\sigma(T) \subset \mathbb{R}$.*

Proof. Let $z \in \mathbb{C} \setminus \mathbb{R}$. By Theorem 2.12 we have that $\ker(T - zI) = 0$ and $\operatorname{ran}(T - zI) = \mathcal{H}$. ∎

2.3 Spectral Theorem for unbounded operators

Theorem 2.14 (Spectral Theorem for unbounded operators) *Let $T : \mathrm{dom}(T) \to \mathcal{H}$ be a self-adjoint operator. Then there exists a unique spectral measure E on the Borel sets of $\sigma(T)$, such that*

$$T = \int_{\sigma(T)} \lambda dE_\lambda,$$

in the sense that

$$\langle Tx, y \rangle = \int_{\sigma(T)} \lambda d\langle E_\lambda x, y \rangle \quad \textit{for all } x \in \mathrm{dom}(T) \textit{ and all } y \in \mathcal{H}.$$

Proof. See [Werner]. ∎

Remark 2.15 (a) Formally, the result looks very similar to the case of bounded operators.

(b) $\sigma(T)$ might be an unbounded subset of $\mathbb{R}$.

◊

3 Fourier Analysis, Distributions and Sobolev Spaces

3.1 The Fourier Transforms on $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$

Definition 3.1 Let $f \in L^1(\mathbb{R}^n)$. Then the *Fourier Transform of* f is defined by

$$\mathcal{F}f(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-i\langle x,\xi\rangle} dx,$$

where $\xi \in \mathbb{R}^n$.

Remark 3.2 (1) There are various adaptations to the definition of $\hat{f}$.

(2) One can interpret the value $|\hat{f}(\xi)|$ as the "strength" with which the "frequency" ξ appears in f.

$\Diamond$

Lemma 3.3 $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ *is linear and bounded.*

Proof. The linearity is clear. For $\xi \in \mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$ we have

$$|\mathcal{F}f(\xi)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(x)| dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \|f\|_1 < \infty$$

Hence $\mathcal{F}f \in L^\infty(\mathbb{R}^n)$ and $\mathcal{F}$ is bounded. $\blacksquare$

Definition 3.4 For $y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}_+$, let

$$T_y : L^1(\mathbb{R}^n)/L^\infty(\mathbb{R}^n) \to L^1(\mathbb{R}^n)/L^\infty(\mathbb{R}^n),\ f(\cdot) \mapsto f(\cdot - y)$$

be the *translation operator*,

$$M_y : L^1(\mathbb{R}^n)/L^\infty(\mathbb{R}^n) \to L^1(\mathbb{R}^n)/L^\infty(\mathbb{R}^n),\ f(\cdot) \mapsto f(\cdot)e^{i\langle y,\cdot\rangle}$$

be the *modulation operator*, and

$$D_\lambda : L^1(\mathbb{R}^n)/L^\infty(\mathbb{R}^n) \to L^1(\mathbb{R}^n)/L^\infty(\mathbb{R}^n),\ f(\cdot) \mapsto f\left(\frac{\cdot}{\lambda}\right)$$

the *dilation operator*.

Theorem 3.5 *Let $f \in L^1(\mathbb{R}^n)$, $\xi, y \in \mathbb{R}^n$ and $\lambda > 0$. Then*

(i) $\mathcal{F}(T_y f) = M_{-y}\mathcal{F}(f)$

(ii) $\mathcal{F}(M_y f) = T_y\mathcal{F}(f)$

(iii) $\mathcal{F}(D_\lambda f) = \lambda^n D_{\lambda^{-1}}\mathcal{F}(f)$

(iv) If $g(x) = \overline{f(-x)}$, $x \in \mathbb{R}^n$, then $\mathcal{F}g = \overline{\mathcal{F}f}$

(v) If $g(x) = -ix_j f(x)$, $g \in L^1(\mathbb{R}^n)$, then $\mathcal{F}f$ is differentiable and $\partial_j(\mathcal{F}f) = \mathcal{F}g$.

Proof. (i). We have

$$\begin{aligned}
\mathcal{F}(T_y f)(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} T_y f(x) e^{-i\langle x,\xi\rangle} dx \\
&= e^{-i\langle y,\xi\rangle}\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x) e^{-i\langle x,\xi\rangle} dx \\
&= e^{-i\langle y,\xi\rangle}\mathcal{F}(f)(\xi) \\
&= M_{-y}\mathcal{F}(f)(\xi)
\end{aligned}$$

(ii). There holds

$$\begin{aligned}
\mathcal{F}(M_y f)(\xi) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i\langle y,x\rangle} f(x) e^{-i\langle x,\xi\rangle} dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x) e^{-i\langle x,\xi-y\rangle} dx \\
&= T_y(\mathcal{F}f)(\xi)
\end{aligned}$$

(iii). and (iv). can be proven in a similar way.

(v). We have

$$\begin{aligned}
\frac{\hat{f}(\xi + he_j) - \hat{f}(\xi)}{h} &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x)\frac{e^{-i\langle \xi+he_j,x\rangle} - e^{-i\langle \xi,x\rangle}}{h} dx \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x) e^{-i\langle \xi,x\rangle}\frac{e^{-ihx_j} - 1}{h} dx
\end{aligned}$$

Hence, by the Lebesgue Dominated Convergence Theorem and the fact, that

$$\lim_{h\to 0}\frac{e^{-ihx_j}-1}{h}=\lim_{h\to 0}(-ix_j)e^{-ihx_j}=-ix_j,$$

the claim follows. ■

Definition 3.6 Let $f,g\in L^1(\mathbb{R}^n)$. Then the *convolution*

$$f*g$$

is defined as

$$(f*g)(x)=\int_{\mathbb{R}^n}f(y)g(x-y)dy,\qquad \text{for } x\in\mathbb{R}^n.$$

Theorem 3.7 *Let $f,g\in L^1(\mathbb{R}^n)$. Then $f*g\in L^1(\mathbb{R}^n)$ and $\mathcal{F}(f*g)=(2\pi)^{\frac{n}{2}}\mathcal{F}(f)\mathcal{F}(g)$.*

Proof. We have

$$\begin{aligned}\|f*g\|_1&=\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}f(y)g(x-y)dy\right|dx\\&\leq\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(y)||g(x-y)|dydx\\&=\int_{\mathbb{R}^n\times\mathbb{R}^n}|f(y)||g(x)|d(x,y)=\|f\|_1\|g\|_1.\end{aligned}$$

Moreover,

$$\begin{aligned}\mathcal{F}(f*g)(\xi)&=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(y)g(x-y)dye^{-i\langle x,\xi\rangle}dx\\&=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}f(y)\int_{\mathbb{R}^n}g(x-y)e^{-i\langle x-y,\xi\rangle}dxe^{-i\langle y,\xi\rangle}dy\\&=(2\pi)^{\frac{n}{2}}\mathcal{F}f(\xi)\mathcal{F}g(\xi)\end{aligned}$$

■

Theorem 3.8 (Riemann-Lebesgue Lemma) *Let $f\in L^1(\mathbb{R}^n)$. Then $\mathcal{F}f\in C_0(\mathbb{R}^n)$, i.e., $\mathcal{F}f\in C(\mathbb{R}^n,\mathbb{R}^n)$, $\lim_{\xi\to\infty}|\mathcal{F}f(\xi)|=0$.*

Proof. To show continuity, observe that by the Lebesgue Dominated Convergence Theorem there holds

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi+h\rangle}dx \to \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} f(x)e^{-i\langle x,\xi\rangle}dx, \qquad \text{as } h\to 0.$$

Now we want to analyze the behavior of the Fourier Transform at ∞. First of all, consider the case $n=1$. Then we have for $f := \chi_{[a,b]}$ that

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_a^b e^{-i\langle x,\xi\rangle}dx = \frac{1}{(2\pi)^{\frac{n}{2}}}\frac{e^{-i\langle b,\xi\rangle}-e^{-i\langle a,\xi\rangle}}{\xi} \to 0, \qquad \text{as } |\xi|\to\infty$$

Now, let $n>1$ and consider $f := \chi_{\prod_{i=1}^n [a_i,b_i]}$. Then there holds

$$\mathcal{F}f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\prod_{i=1}^n \frac{e^{-i\langle b_i,\xi\rangle}-e^{-i\langle a_i,\xi\rangle}}{i\xi_i} \to 0, \qquad \text{as } |\xi|\to\infty$$

The same conclusion can be drawn for finite sums of such functions. Let $f \in L^1(\mathbb{R}^n)$, then for $\varepsilon>0$, there exists a step function h on $\mathbb{R}^n$ with $\|f-h\|_1 \le \varepsilon$. Thus, by Lemma 3.3

$$\begin{aligned}|\mathcal{F}f(\xi)| &\le |\mathcal{F}(f-h)(\xi)| + |\mathcal{F}h(\xi)| \\ &\le \frac{\|f-h\|_1}{(2\pi)^{\frac{n}{2}}} + |\mathcal{F}h(\xi)| \\ &\le \frac{\varepsilon}{(2\pi)^{\frac{n}{2}}} + |\mathcal{F}h(\xi)| \to \frac{\varepsilon}{(2\pi)^{\frac{n}{2}}},\end{aligned}$$

as $|\xi|\to\infty$. Since $\varepsilon>0$ can be chosen arbitrarily small, this shows the claim.

∎

Theorem 3.9 (Fourier Inversion Formula) *Let $f\in L^1(\mathbb{R}^n)$, $\mathcal{F}f \in L^1(\mathbb{R}^n)$. Then, for almost every $x\in\mathbb{R}^n$*

$$f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\hat{f}(\xi)e^{i\langle x,\xi\rangle}d\xi = \mathcal{F}^{-1}(\mathcal{F}f)(x).$$

Proof. We will prove the statement only for $n=1$. Towards this goal, we will use the Fejér-Kernels which are given by

$$F_\lambda := \lambda D_{\lambda^{-1}}F,$$

for $\lambda>0$ and with

$$F(x) = \frac{1}{2\pi}\int_{-1}^1 (1-|t|)e^{ixt}dt$$

for $x \in \mathbb{R}$.

These kernels satisfy $\lim_{\lambda\to\infty} \|f - f * F_\lambda\|_1 = 0$ and $f * F_\lambda \to f$ pointwise almost everywhere. Then, for $x \in \mathbb{R}$, we have

$$\begin{aligned}
f * F_\lambda(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} f(t)[\lambda\int_{-1}^{1}(1-|\theta|)e^{i(x-t)\theta\lambda}d\theta]dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} f(t)[\int_{-\lambda}^{\lambda}(1-|\frac{\theta}{\lambda}|)e^{i(x-t)\theta}d\theta]dt \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\lambda}^{\lambda}(1-|\frac{\theta}{\lambda}|)[\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)e^{-it\theta}dt]e^{-ix\theta}d\theta \\
&= \frac{1}{\sqrt{2\pi}}\int_{|\theta|\le\sqrt{\lambda}}(1-|\frac{\theta}{\lambda}|)\hat{f}(\theta)e^{ix\theta}d\theta + \frac{1}{\sqrt{2\pi}}\int_{\sqrt{\lambda}\le|\theta|\le\lambda}(1-|\frac{\theta}{\lambda}|)\hat{f}(\theta)e^{ix\theta}d\theta \\
&\to \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{f}(\theta)e^{ix\theta}d\theta, \qquad \text{as } \lambda\to\infty.
\end{aligned}$$

■

Theorem 3.10 (Plancherel Theorem) *Let $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then*

$$\langle f, g\rangle = \langle \mathcal{F}f, \mathcal{F}g\rangle.$$

In particular, $\|f\|_2 = \|\mathcal{F}f\|_2$, hence $\mathcal{F}$ is an isometry, if restricted to $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Proof. Define $h(x) := \int_{\mathbb{R}^n} f(t)g(t-x)dt = (f * g(-\cdot))(x)$ for $x \in \mathbb{R}^n$. Then $h(0) = \langle f, g\rangle$ and by Theorem 3.7 and Theorem 3.5.(iv) there holds

$$\hat{h}(\xi) = (2\pi)^{\frac{n}{2}}\hat{f}(\xi)\mathcal{F}(g(-\cdot))(\xi) = (2\pi)^{\frac{n}{2}}\hat{f}(\xi)\overline{\hat{g}(\xi)}$$

Assume now that $\hat{f}, \hat{g} \in L^2(\mathbb{R}^n)$ (We will see later that the functions in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ whose Fourier transforms are in $L^2(\mathbb{R}^n)$ are dense in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with respect to the $\|\cdot\|_1$- as well as the $\|\cdot\|_2$-norm.) Then $\hat{h} \in L^1(\mathbb{R}^n)$ and by employing the Fourier Inversion Formula we obtain

$$\begin{aligned}
\langle f, g\rangle = h(0) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\hat{h}(\xi)e^{i\langle\xi,0\rangle}d\xi \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\hat{h}(\xi)d\xi = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(2\pi)^{\frac{n}{2}}\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi = \langle\hat{f}, \hat{g}\rangle.
\end{aligned}$$

A density argument leads to the proof of the claim for all $f, g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. ■

Remark 3.11 Ultimatively, we would like to define $\mathcal{F}f$ for $f \in L^2(\mathbb{R}^n)$. Let $f \in L^2(\mathbb{R}^n)$ and define $f_k := f\chi_{[-k,k]^n} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ for $k \in \mathbb{N}$. This implies $\|f - f_k\| \to_{k\to\infty} 0$. Hence, by the Plancherel Theorem,

$$\|\hat{f}_k - \hat{f}_l\|_2 = \|\mathcal{F}(f_k - f_l)\|_2 = \|f_k - f_l\|_2$$

Thus $(\hat{f}_k)_k$ is a Cauchy-sequence in $L^2(\mathbb{R}^n)$. Hence there exists a limit $g \in L^2(\mathbb{R}^n)$ of $(\hat{f}_k)_{k\in\mathbb{N}}$. Finally, we set $\mathcal{F}f := g$. $\lozenge$

3.2 Uncertainty Principles

Theorem 3.12 *Let $f \in L^2(\mathbb{R})$ and $a, b \in \mathbb{R}$. Then*

$$\left(\int_{\mathbb{R}} (x-a)^2 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} (\xi - b)^2 |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \geq \frac{1}{2}\|f\|_2^2$$

Proof. Without loss of generality, let $a = b = 0$, $\|f\|_2^2 = 1$. Then

$$\begin{aligned} 1 = \int_{\mathbb{R}} |f(x)|^2 dx &= -\int_{\mathbb{R}} x \frac{d}{dx} |f(x)|^2 dx \\ &= -\int_{\mathbb{R}} x f'(x) \bar{f(x)} + x \bar{f'(x)} f(x) dx, \end{aligned}$$

where partial integration and $|f|^2 = f\overline{f}$ was used.

Hence

$$1 \leq 2\int_{\mathbb{R}} |x||f'(x)||f(x)| dx \leq 2\left(\int_{\mathbb{R}} x^2 |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f'(x)|^2 dx\right)^{\frac{1}{2}}$$

By Plancherel, we finally obtain

$$\|f'\|_2 = \|\mathcal{F}f'\|_2 = \|\xi \mathcal{F}f\|_2$$

■

Theorem 3.13 *Let S, T be (possibly unbounded) self-adjoint operators on $\mathcal{H}$, $a, b \in \mathbb{R}$. Then we have*

$$\|(S - aI)f\| \cdot \|(T - bI)f\| \geq \frac{1}{2}|\langle [S, T]f, f\rangle|$$

for all $f \in \mathrm{dom}(ST) \cap \mathrm{dom}(TS)$ where $[S, T] := ST - TS$ is the commutator of S and T.

Proof. We have

$$\begin{aligned}\langle [S,T]f,f\rangle &= \langle ((S-aI)(T-bI)-(T-bI)(S-aI))f,f\rangle \\ &= \langle (T-bI)f,(S-aI)f\rangle - \langle (S-aI)f,(T-bI)f\rangle \\ &= 2i\cdot \operatorname{Im}\langle (T-bI)f,(S-aI)f\rangle.\end{aligned}$$

The statement then follows by an application of the Cauchy-Schwarz inequality. ∎

Remark 3.14 Theorem 3.12 / 3.13 are called Heisenberg's Uncertainty Principle. ◊

We now give a second version of the proof of Theorem 3.12

Proof. Theorem 3.12

Choose $(Sf)(x) := xf(x)$ for $f \in L^2(\mathbb{R}^n)$ and $(Tf)(x) := if'(x)$ for any differentiable $f \in L^2(\mathbb{R}^n)$. Then for $f \in \operatorname{dom}(ST)\cap\operatorname{dom}(TS)$ we have

$$([S,T]f)(x) = ixf'(x) - i(xf(x))' = -if(x)$$

and

$$\frac{\|f\|_2^2}{2} = \frac{1}{2}|\langle -if(x),f(x)\rangle| \le \|(S-aI)f\|_2\cdot\|(T-bI)\|_2.$$

Then

$$\|(T-bI)f\|_2 = \|\mathcal{F}((T-bI)f)\|_2 = \|(\xi-b)\mathcal{F}f\|_2$$

holds by the Plancherel Theorem, which yields the claim. ∎

3.3 Fourier Transforms on $L^1(G)$, $L^2(G)$ (Excursion)

Remark 3.15 The theory on Fourier Transforms can be generalized to functions on so-called *locally-compact (abelian) groups*. This leads automatically to Fourier Transforms on $L^2(\mathbb{T})$, $\ell_2(\mathbb{Z})$,... such that analagous results like the Plancherel Theorem hold. ◊

Definition 3.16 A *topological group* is a group G equipped with a topology such that the group operations $(x,y)\mapsto xy$ and $x\mapsto x^{-1}$ are continuous.

Lemma 3.17 *Let G be a topological group. Then the topology is invariant under translation and inversion, i.e. if U is open, $xU := \{xy : y\in U\}$, $Ux := \{yx : y \in U\}$ and $U^{-1} := \{y^{-1} : y\in U\}$ are open for all $x\in G$.*

Proof. Follows from Definition 3.16. ∎

Definition 3.18 Let G be a topological group.

(1) G is a *locally compact group*, if it is a topological group whose topology is locally compact and Hausdorff. A topology is called locally compact if every point has a compact neighbourhood. A Hausdorff space is a topological space where you can separate any two points with open sets.

(2) If G is a locally compact group, which is abelian, then G is called *locally compact abelian (LCA) group*.

Definition 3.19 A *left (respectively right) Haar-measure* on a locally compact group G is a non-zero Radon-measure μ which satisfies $\mu(xE) = \mu(E)$ (respectively $\mu(Ex) = \mu(E)$) for all Borel-sets $E \subset G$ and $x \in G$.

Theorem 3.20 *Every locally compact group G possesses a left (respectively right) Haar-measure μ. μ is unique up to rescaling by a positive number.*

Proof. Similar to existence of the Lebesgue measure. See Folland - A course in harmonic analysis. ■

Remark 3.21 From now on only consider locally-compact abelian groups (abbreviated LCAG). However most of the following can be generalized to locally compact groups. ♢

Definition 3.22 Let G be a LCAG.

(1) A continuous homomorphism $\gamma : G \to \mathbb{T}$ is called a *character* of G.

(2) The set of all characters is denoted by $\hat{G}$, which is the *dual group* of G.

The torus $\mathbb{T}$ is isomorphic to, can hence also be viewed as, $\mathbb{R}/\mathbb{Z}$ and $S^1 = \{z \in \mathbb{C} : \ |z| = 1\}$.

Remark 3.23 Notice that $\hat{G}$ is a group with group operation

$$(\gamma\omega)(x) := \gamma(x)\omega(x) \in \mathbb{T},$$

unit element $1(x) = 1 \in \mathbb{T}$ and inverse $\gamma^{-1}(x) = \overline{\gamma(x)} = \gamma(x^{-1})$ (where $\mathbb{T}$ is interpreted as $\{z \in \mathbb{C} : \ |z| = 1\}$. Moreover, we equip $\hat{G}$ with the topology of compact convergence under which the group operations are continuous. Then $\hat{G}$ is a LCAG. ♢

Theorem 3.24 *We have*

(i) $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{T}$ *and* $\mathbb{Z}_k$ *are LCAG.*

(ii) $\hat{\mathbb{R}} \cong \mathbb{R}$ *via* $x \mapsto \gamma_x$, $\gamma_x(y) = e^{ixy}$, $y \in \mathbb{R}$

(iii) $\hat{\mathbb{T}} \cong \mathbb{Z}$ *via* $m \mapsto \gamma_m$, $\gamma_m(\theta) = \theta^m$

(iv) $\hat{\mathbb{Z}} \cong \mathbb{T}$ *via* $\theta \mapsto \gamma_\theta$, $\gamma_\theta(m) = \theta^m$

(v) $\hat{\mathbb{Z}}_k \cong \mathbb{Z}_k$ *via* $m \mapsto \gamma_m$, $\gamma_m(n) = e^{2\pi i \frac{mn}{k}}$

Proof. (i). is clear.

(ii). Since $\gamma \in \hat{\mathbb{R}}$, we have $\gamma(0) = 1$. Hence there exists some $a > 0$ such that $\int_0^a \gamma(y)dy \neq 0$. Let $A := \int_0^a \gamma(y)dy$. Then, for $x \in \mathbb{R}$,

$$A\gamma(x) - \int_0^a \gamma(y+x)dy = \int_x^{a+x} \gamma(y)dy$$

Hence, γ is differentiable and

$$\gamma'(x) = A^{-1}(\gamma(a+x) - \gamma(x)) =: c\gamma(x)$$

Thus $\gamma(x) = e^{cx}$. But since $|e^{cx}| = |\gamma(x)| = 1$, $c = iy$ for some $y \in \mathbb{R}$.

(iii). We identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$ via $x \mapsto e^{2\pi i x}$. Thus the characters of $\mathbb{T}$ are the characters of $\mathbb{R}$ which are 1 on $\mathbb{Z}$. The rest follows from (i).

(iv). Let $\gamma \in \hat{\mathbb{Z}}$, then $\alpha := \gamma(1) \in \mathbb{T}$. Thus, for any $m \in \mathbb{Z}$, $\gamma(m) = \gamma(1)^m = \alpha^m$.

(v). The characters of $\mathbb{Z}_k$ are the characters of $\mathbb{Z}$ which are trivial on $k\mathbb{Z}$. Thus, they are of the form $\gamma(m) = \alpha^m$, where α is the k-th root of 1. ∎

Proposition 3.25 *If $G_1, \ldots, G_n$ are LCAG, then*

$$(G_1 \times \cdots \times G_n)\hat{} \cong \hat{G}_1 \times \ldots \hat{G}_n.$$

In particular, $\hat{\mathbb{R}}^n \cong \mathbb{R}^n$, $\hat{\mathbb{T}}^n \cong \mathbb{Z}^n$, $\hat{\mathbb{Z}}^n \cong \mathbb{T}^n$, $\hat{G} \cong G$ for any finite LCAG group G.

Definition 3.26 Let $f \in L^1(G)$, where G is a LCAG and μ a left-invariant Haar-measure on G. Then, for $\gamma \in \hat{G}$

$$\mathcal{F}f(\gamma) := \hat{f}(\gamma) := \int_G f(x)\overline{\gamma(x)}d\mu(x)$$

is the *Fourier Transform of f*.

Theorem 3.27 *For a LCAG G, we have $\mathcal{F} : L^1(G) \to C_0(\hat{G})$, where $C_0(\hat{G})$ denotes the set of continuous and bounded functions on $\hat{G}$.*

Theorem 3.28 (Plancherel Theorem) *The Fourier Transform on $L^1(G) \cap L^2(G)$ uniquely extends to a unitary isomorphism from $L^2(G)$ to $L^2(\hat{G})$.*

Theorem 3.29 (Pontryagin Duality Theorem) *Let G be a LCAG. Then the map $\Phi : G \mapsto \hat{\hat{G}}$, $(\Phi(x))(\gamma) := \gamma(x)$ is an isomorphism of topological groups.*

Theorem 3.30 (Fourier Inversion Formula) *For a LCAG G and $f \in L^1(G)$ such that also $\hat{f} \in L^1(\hat{G})$ we have*

$$f(x) = \hat{\hat{f}}(x^{-1}) \qquad \text{for a.e. } x \in G,$$

i.e.

$$f(x) = \int_{\hat{G}} \hat{f}(\gamma)\gamma(x)d\mu(\gamma) \qquad \text{for a.e. } x \in G,$$

where μ is the appropriately normalized left-invariant Haar-measure on $\hat{G}$. If $f \in C(G)$, these identities hold for every *$x \in G$.*

Definition 3.31 Let $f, g \in L^1(G)$, where G is a LCAG. Then the *convolution of f and g* is defined as

$$(f * g)(x) := \int_G f(y)g(y^{-1}x)d\mu(y),$$

for a.e. $x \in G$.

Proposition 3.32 *Let G be a LCAG.*

(i) For $f, g \in L^1(G)$, there holds $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

(ii) For $f, g \in L^2(G)$, we have $(fg)\hat{} = \hat{f} * \hat{g}$

Example 3.33 Let $G = \mathbb{Z}$ equipped with the standard topology $\mathcal{P}(\mathbb{Z})$, and the the counting measure μ. Then, for some $c = (c_n)_{n \in \mathbb{Z}}$ such that $c \in \ell^1(\mathbb{Z})$, the Fourier Transform is defined as

$$\hat{c}(\theta) = \sum_{n \in \mathbb{Z}} c_n \theta^{-n}, \qquad \theta \in \mathbb{T}.$$

By Theorem 3.27, $\hat{c} \in C(\mathbb{T})$ for $c \in \ell^1(\mathbb{Z})$. By Theorem 3.30, we have for $c \in \ell^1(\mathbb{Z})$ such that $\hat{c} \in \ell^1(\mathbb{T})$ (which always holds since $\hat{c}$ is continuous and $\mathbb{T}$ is compact)

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \hat{c}(\theta)\theta^n d\theta$$

Also by Theorem 3.28, there holds

$$\|c\|_{\ell^2(\mathbb{Z})} = \|\hat{c}\|_{L^2(\mathbb{T})}$$

since $\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z})$ (which has been shown in a homework in FA 1).

Thus, this theory is powerful enough to deliver properties of the usually seperately-treated Fourier Transform of a sequence $(c_n)_{n\in\mathbb{Z}}$. This is just a glimpse into this theory which belongs to abstract harmonic analysis (theory of locally compact groups). $\Diamond$

3.4 The Fourier Transform on $\mathcal{S}(\mathbb{R}^n)$ (Schwartz Functions)

Remark 3.34 We use the usual multiindex notation for $\alpha \in \mathbb{N}_0^n$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$
$$|\alpha| := \alpha_1 + \cdots + \alpha_N$$
$$D^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$
$$\alpha! := \alpha_1! \cdots \alpha_n!$$
$$|x| := \|x\|_2.$$

$\Diamond$

Definition 3.35 Let $n \in \mathbb{N}$. Then the *Schwartz-Space* $\mathcal{S}(\mathbb{R}^n)$ is defined as

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) : \|\varphi\|_{(k,l)} < \infty \text{ for all } k, l \in \mathbb{N}_0\}$$

where $\|\varphi\|_{(k,l)} := \sup_{x\in\mathbb{R}^n}(1 + |x|^2)^{\frac{k}{2}} \sum_{|\alpha|\leq l}|D^\alpha \varphi(x)|$. This means that all derivatives vanish very quickly (faster than any polynomial). A sequence $(\varphi_k)_k \subset \mathcal{S}(\mathbb{R}^n)$ is said to converge to $\varphi \in \mathcal{S}(\mathbb{R}^n)$, if

$$\lim_{j\to\infty}\|\varphi_j - \varphi\|_{(k,l)} = 0 \text{ for all } k, l \in \mathbb{N}_0.$$

We write $\varphi_j \xrightarrow{\mathcal{S}} \varphi$.

Remark 3.36 $\mathcal{S}(\mathbb{R}^n)$ is a vector space. But it is not a normed space in such a way that it is not possible to define a norm on $\mathcal{S}(\mathbb{R}^n)$ such that convergence with respect to this norm is equivalent to the convergence we just introduced. But $\mathcal{S}(\mathbb{R}^n)$ can be equipped with a topology such that convergence in this topology equals the convergence we defined. This turns $\mathcal{S}(\mathbb{R}^n)$ into a topological vector space. $\Diamond$

Theorem 3.37 *The following properties hold:*

(i) Let $(\varphi_j)_j \subset \mathcal{S}(\mathbb{R}^n)$ be a sequence with $\varphi_j \xrightarrow{\mathcal{S}} \varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $\varphi_j \to \varphi$ in $L^p(\mathbb{R}^n)$ for all $0 < p < \infty$.

(ii) Let $(\varphi_j)_j \subset \mathcal{S}(\mathbb{R}^n)$ such that $\varphi_j \xrightarrow{\mathcal{S}} \varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $D^\alpha \varphi \xrightarrow{\mathcal{S}} D^\alpha \varphi$ for all $\alpha \in \mathbb{N}_0^n$.

(iii) Let $(\varphi_j)_j$ such that $\varphi_j \xrightarrow{\mathcal{S}} \varphi$. Then also $(x \mapsto x^\alpha \varphi_j(x)) \xrightarrow{\mathcal{S}} (x \mapsto x^\alpha \varphi(x))$ for all $\alpha \in \mathbb{N}_0^n$.

(iv) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $T_h \varphi \xrightarrow{\mathcal{S}} \varphi$ as $h \to 0$, where T_h denotes the translation operator.

(v) Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Then $\varphi\psi$, $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$ and $D^\alpha(\varphi * \psi) = (D^\alpha \varphi) * \psi = \varphi * (D^\alpha \psi)$ for all $\alpha \in \mathbb{N}_0^n$.

Proof. (i). We have

$$|\varphi_j(x) - \varphi(x)| \leq (1 + |x|^2)^{-\frac{k}{2}} \|\varphi_j - \varphi\|_{(k,0)}.$$

Choosing k large enough, integrating over $\mathbb{R}^n$ and using that $\|\varphi_j - \varphi\|_{(k,0)} \xrightarrow{j\to\infty} 0$ yields the claim.

(ii).,(iii). follow immediately from the definition of $\mathcal{S}(\mathbb{R}^n)$.

(iv). By Taylor's Theorem, there holds

$$|D^\alpha \varphi(x+h) - D^\alpha \varphi(x)| \leq C_\alpha |h| \sup_{|\beta|=1} \sup_{y \in B_{|h|}(x)} |D^{\alpha+\beta} \varphi(y)|$$

where C_α is a constant. Hence we obtain

$$\|T_h \varphi - \varphi\|_{(k,l)} \leq C_{k,l} |h| \|\varphi\|_{(k,l+1)} \to 0, \qquad \text{as } h \to 0,$$

where $C_{k,l}$ is another constant.

(v). First of all, $\varphi * \psi$ is well-defined, since the integral exists because of the decay properties of Schwartz functions. Hence, for $j = 1, ..., n$, there holds

$$\begin{aligned} D^{e_j}(\varphi * \psi)(x) &= \lim_{h\to 0} \frac{(\varphi * \psi)(x + he_j) - (\varphi * \psi)(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \int_{\mathbb{R}^n} (\varphi(x + he_j - y) - \varphi(x - y))\psi(y)dy \\ &= \lim_{h\to 0} \int_{\mathbb{R}^n} \frac{\varphi(z + he_j) - \varphi(z)}{h} \psi(x - z)dz \end{aligned}$$

We have $\lim_{h\to 0} \frac{\varphi(z+he_j)-\varphi(z)}{h} = (D^{e_j}\varphi)(z)$. Again, using the properties of Schwartz functions, we obtain that $D^{e_j}(\varphi * \psi) = (D^{e_j}\varphi) * \psi$. By an iterative argument and exploiting the commutativity of the convolution, we obtain

$$D^\alpha(\varphi * \psi) = (D^\alpha \varphi) * \psi = \varphi * D^\alpha \psi \qquad \text{for all } \alpha \in \mathbb{N}_0^n.$$

We now show that $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$. We have for any $N \in \mathbb{N}$

$$\begin{aligned}\left|\int \varphi(x-y)\psi(y)dy\right| &\le c_1 \int_{|y|\le \frac{|x|}{2}} \frac{1}{|x-y|^N}|\psi(y)|dy + c_2 \int_{|y|>\frac{|x|}{2}} |\psi(y)|dy \\ &\le \frac{c_1'}{|x|^N}\int_{|y|\le \frac{|x|}{2}} |\psi(y)|dy + c_2\int_{|y|>\frac{|x|}{2}} |y|^{-N}|y|^N|\psi(y)|dy \\ &\le \frac{c_1''}{|x|^N} + \frac{c_2''}{|x|^N} = c|x|^{-N}\end{aligned}$$

for suitably chosen constants $c_1, c_2, c_1', c_1'', c_2''$.

Hence, for all $N \in \mathbb{N}_0$, we have $|(\varphi * \psi)(x)| \le \tilde{C}_N(1+|x|^2)^{-\frac{N}{2}}$. Now, by using $D^\alpha(\varphi * \psi) = (D^\alpha\varphi) * \psi$ and $D^\alpha\varphi \in \mathcal{S}(\mathbb{R}^n)$ one can invoke the same argument to show $|D^\alpha(\varphi * \psi)(x)| \le \tilde{C}_N^\alpha(1+|x|^2)^{-\frac{N}{2}}$ for all $N \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$ as well as some constants $\tilde{C}_N^\alpha > 0$. It follows that $\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$.

Finally, also $\varphi\psi \in \mathcal{S}(\mathbb{R}^n)$ by the decay properties of Schwartz functions.

■

Theorem 3.38 *The following properties of the Fourier Transform on $\mathcal{S}(\mathbb{R}^n)$ hold.*

(i) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}\varphi, \mathcal{F}^{-1}\varphi \in \mathcal{S}(\mathbb{R}^n)$.

(ii) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then for all $\alpha \in \mathbb{N}_0^n, x \in \mathbb{R}^n$.

(a) $D^\alpha(\mathcal{F}\varphi)(\xi) = (-i)^{|\alpha|}\mathcal{F}(x^\alpha\varphi)(\xi)$.

(b) $\xi^\alpha(\mathcal{F}\varphi)(\xi) = (-i)^{|\alpha|}\mathcal{F}(D^\alpha\varphi)\xi$.

(iii) Let $(\varphi_j)_j \subset \mathcal{S}(\mathbb{R}^n)$ such that $\varphi_j \xrightarrow{\mathcal{S}} \varphi$. Then

$$\mathcal{F}\varphi_j \xrightarrow{\mathcal{S}} \mathcal{F}\varphi, \qquad \mathcal{F}^{-1}\varphi_j \xrightarrow{\mathcal{S}} \mathcal{F}^{-1}\varphi.$$

Proof. (ii). We first show that (a) holds. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$, then one gets from Theorem 3.37 that $x^\alpha\varphi \in \S(\mathbb{R}^n)$ and $D^\alpha\varphi \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, we get by Lebesgue's Dominated Convergence Theorem

$$\frac{\partial}{\partial\xi_l}(\mathcal{F}\varphi)(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}(-i)x_l e^{-i\langle x,\xi\rangle}\varphi(x)dx = (-i)\mathcal{F}(x_l\varphi(x))(\xi).$$

By iteration, we get (a). To prove (b), we use partial integration in x_l-direction to obtain

$$\begin{aligned}\xi_l(\mathcal{F}\varphi)(\xi) &= \frac{i}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\frac{\partial}{\partial x_l}(e^{-i\langle x,\xi\rangle})\varphi(x)dx = \frac{-i}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\frac{\partial\varphi}{\partial x_l}(x)e^{-i\langle x,\xi\rangle}dx\\ &= (-i)\mathcal{F}\left(\frac{\partial\varphi}{\partial x_l}\right)(\xi).\end{aligned}$$

These formulas can be easily iterated, for instance we obtain

$$\frac{\partial^2}{\partial\xi_m\partial\xi_l}(\mathcal{F}\varphi)(\xi) = (-i)\frac{\partial}{\partial\xi_m}(\mathcal{F}(x_l\varphi(x))(\xi)) = (-i)^2\mathcal{F}(x_m x_l\varphi(x))(\xi).$$

By induction, the claim follows.

To show (i)., note that by the Fourier Inversion Formula we obtain

$$\begin{aligned}\|\mathcal{F}\varphi\|_{(k,l)} &\le c\sup_{\xi\in\mathbb{R}^n}\max_{|\beta|\le k}\max_{|\alpha|\le l}|\xi^\beta|\cdot|D^\alpha(\mathcal{F}\varphi)(\xi)|\\ &\le c\max_{|\beta|\le k}\max_{|\alpha|\le l}\|\mathcal{F}(D^\beta(x^\alpha\varphi(x)))\|_\infty\\ &\le c\max_{|\beta|\le k}\max_{|\alpha|\le l}\|D^\beta(x^\alpha\varphi(x))\|_1\\ &\le c\max_{|\beta|\le k}\max_{|\alpha|\le l}\|x^\alpha(D^\beta\varphi)(x)\|_1\\ &\le c\,\|\varphi\|_{(l+n+1,k)}, \qquad \text{for all} \quad \varphi\in\mathcal{S}(\mathbb{R}^n).\end{aligned}$$

(iii). follows immediately from the considerations above. ■

Remark 3.39 Theorem 3.5.(i)-(iv) also hold trivially for all $\varphi\in\mathcal{S}(\mathbb{R}^n)$. ◊

Theorem 3.40 *The following properties hold.*

(i) If $\varphi\in\mathcal{S}(\mathbb{R}^n))$ and $\psi\in\mathcal{S}(\mathbb{R}^m)$, then $(\varphi\otimes\psi)(x,y)=\varphi(x)\psi(y)\in\mathcal{S}(\mathbb{R}^{n+m})$ is the tensor product *of φ and ψ. And the formula $\mathcal{F}(\varphi\otimes\psi)(\xi,\eta)=(\mathcal{F}\varphi)(\xi)(\mathcal{F}\psi)(\eta)$ holds for every $\xi\in\mathbb{R}^n$ and every $\eta\in\mathbb{R}^m$.*

(ii) Let $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}(\varphi\psi)=(2\pi)^{-n/2}\mathcal{F}(\varphi)*\mathcal{F}(\psi)$.

(iii) It holds

$$\mathcal{F}(e^{-|x|^2/2})(\xi)=e^{-|\xi|^2/2}, \quad \xi\in\mathbb{R}^n.$$

Proof. (i). is clear.

(ii). follows from Theorem 3.7 and the Fourier Inversion Formula.

It remains to show (iii). Due to the product structure, it is enough to show (iii). for $n = 1$. Let

$$h(s) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-t^2/2} e^{-its} dt, \quad s \in \mathbb{R}.$$

Then, by using polar coordinates,

$$h(0)^2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-(t^2+u^2)/2} d(t,u) = \frac{1}{2\pi} \int_0^{\infty} (2\pi r)\, e^{-r^2/2} dr = 1,$$

i.e. $h(0) = 1$. On the other hand, by integration by parts we get

$$\begin{aligned} h'(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} e^{-its}(-it)dt = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dt}\left(e^{-t^2/2}\right) e^{-its} dt \\ &= \frac{-s}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} e^{-its} dt = -sh(s). \end{aligned}$$

Solving this equation, we observe that $h(s) = h(0)e^{-s^2/2} = e^{-s^2/2}$, which finishes the proof.

■

Theorem 3.41 (Fourier Inversion Formula) *We have $\mathcal{F}(\mathcal{F}^{-1}(\varphi)) = \varphi = \mathcal{F}^{-1}(\mathcal{F}(\varphi))$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Also, $\mathcal{F}, \mathcal{F}^{-1}$ are bijective on $\mathcal{S}(\mathbb{R}^n)$.*

Proof. We will give a more elegant proof than before. Consider

$$\mathrm{I} := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \mathcal{F}(\varphi)(\xi) e^{i\langle x,\xi\rangle} e^{-\varepsilon^2 \frac{|\xi|^2}{2}} d\xi$$

As $\varepsilon \to 0$, $\mathrm{I} \to \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} (\mathcal{F}\varphi)(\xi) e^{i\langle x,\xi\rangle} d\xi = \mathcal{F}^{-1}(\mathcal{F}(\varphi))$.

Set $\psi(x) = e^{-\varepsilon^2 \frac{|x|^2}{2}}$. Then $(\mathcal{F}\psi)(\xi) = \varepsilon^{-n} e^{-\frac{|\xi|^2}{2\varepsilon^2}}$ for all $\xi \in \mathbb{R}^n$ by Theorem 3.40.(ii). As $\varepsilon \to 0$, we have, by a change of variables,

$$\begin{aligned} \mathrm{I} &= \varepsilon^{-n} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(x+y) e^{-\frac{|y|^2}{2\varepsilon^2}} dy \\ &\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \varphi(x+\varepsilon z) e^{-\frac{|z|^2}{2}} dz \to \varphi(x), \end{aligned}$$

where we have employed the Lebesgue Dominated Convergence Theorem in the last step. ■

3.5 Tempered Distributions

Definition 3.42 Let $\mathcal{S}'(\mathbb{R}^n)$ be the set of all continuous complex, linear functions on $\mathcal{S}(\mathbb{R}^n)$, i.e. all $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ such that

(1) $T(\lambda\varphi + \mu\psi) = \lambda T(\varphi) + \mu T(\psi)$ for all $\lambda, \mu \in \mathbb{C}$, $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$

(2) $T\varphi_k \to T\varphi$ for all $\varphi_k \xrightarrow{\mathcal{S}} \varphi$.

The elements of $\mathcal{S}'(\mathbb{R}^n)$ are called *tempered distributions*.

Remark 3.43 By setting

$$(\lambda T + \mu S)(\varphi) = \lambda T\varphi + \mu S\varphi \text{ for all } \lambda, \mu \in \mathbb{C},\, T, S \in \mathcal{S}'(\mathbb{R}^n),\, \varphi \in \mathcal{S}(\mathbb{R}^n)$$

$\mathcal{S}'(\mathbb{R}^n)$ becomes a linear space. We further equip $\mathcal{S}'(\mathbb{R}^n)$ with a weak convergence by $T_k \xrightarrow{\mathcal{S}} T$ if and only if : $T_k\varphi \to T\varphi$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$ holds. $\Diamond$

Theorem 3.44 *Let T be a linear functional on $\mathcal{S}(\mathbb{R}^n)$. Then the following are equivalent:*

(i) $T \in \mathcal{S}'(\mathbb{R}^n)$

(ii) $|T\varphi| \le c\|\varphi\|_{(k,l)}$ *for some* $c < 0$, $k, l \in \mathbb{N}_0$ *and all* $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. (i).⇒(ii). Towards a contradiction, assume that for all $c > 0$, $k, l \in \mathbb{N}_0$ there exists $\varphi_{c,k,l} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$1 = |T\varphi_{c,k,l}| > c\|\varphi_{c,k,l}\|_{(k,l)}$$

Next, consider $\varphi_k := \varphi_{k,k,k}$. We have $\varphi_k \xrightarrow{\mathcal{S}} 0$ as $k \to \infty$ and $T\varphi_k \to 0$ since T is continuous. This is a contradiction to $|T\varphi_k| = 1$.

(ii).⇒(i). This is immediate. ■

Remark 3.45 Let $f \in L^1(\mathbb{R}^n)$. Define

$$T_f : \varphi \to T_f(\varphi) := \int_{R^n} f(x)\varphi(x)dx \quad \text{ for } \varphi \in \mathcal{S}(\mathbb{R}^n).$$

This is well-defined since $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and linear. Also,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} f(x)\varphi(x)dx\right| &\le \int_{\mathbb{R}^n} |f(x)||\varphi(x)|dx \\ &\le \|\varphi\|_\infty \|f\|_1 = \|\varphi\|_{(0,0)}\|f\|_1\end{aligned}$$

Theorem 3.44 now implies $T_f \in \mathcal{S}'(\mathbb{R}^n)$. This can be extended to $f \in L^2(\mathbb{R}^n)$, even to all $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty]$, but not to $p < 1$. $\Diamond$

Definition 3.46 Let T be a tempered distribution. If there exists $f \in L^1(\mathbb{R}^n)$ with $T = T_f$, then T is called *regular distribution*. All other tempered distributions are referred to as *singular distributions*.

Lemma 3.47 *Let $f, g \in L^1(\mathbb{R}^n)$ such that $T_f(\varphi) = T_g(\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then $f = g$ almost everywhere.*

Proof. We shall present one proof which uses the technique of *mollification*. Obviously, it is enough to show that if $T_h(\varphi) = 0$ for some $h \in L_1(\mathbb{R}^n)$ and all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $h = 0$ a.e. Let $\omega \in \mathcal{S}(\mathbb{R}^n)$ be a smooth non-negative symmetric function with support in $B := \{y : |y| \leq 1\}$ and $\int_{\mathbb{R}^n} \omega(x)dx = 1$. We define $\omega_\varepsilon(x) := \varepsilon^{-n}\omega(x/\varepsilon)$. Observe that also $\int_{\mathbb{R}^n} \omega_\varepsilon(x)dx = 1$. Next, we notice that $h * \omega_\varepsilon \in C(\mathbb{R}^n)$ for each $\varepsilon > 0$. Using Fubini's Theorem (and the symmetry of ω) we get quickly (as $\omega_\varepsilon * \varphi \in \mathcal{S}(\mathbb{R}^n)$) that

$$0 = \int_{\mathbb{R}^n} h(x)(\omega_\varepsilon * \varphi)(x)dx = \int_{\mathbb{R}^n} \varphi(y)(h * \omega_\varepsilon)(y)dy. \tag{3.1}$$

As $\omega_\varepsilon * h$ is a continuous function, for each $x \in \mathbb{R}^n$ with $(\omega_\varepsilon * h)(x) > 0$ there is some neighborhood U_x of x such that

$$(\omega_\varepsilon * h)(y) > 0 \quad \text{for all } y \in U_x.$$

For each such x, consider a non-negative, non-zero function $\varphi_x \in \mathcal{S}(\mathbb{R}^n)$ with support in U_x. By (3.1), we obtain

$$(\omega_\varepsilon * h)(x) = 0 \quad \text{for all } x \in \mathbb{R}^n.$$

Finally, we use that, for every $t > 0$, $h \in L_1(\mathbb{R}^n)$ may be written as

$$h = h_1 + h_2,$$

where h_1 is continuous with compact support and $\|h_2\|_1 \leq t$.

Then

$$\begin{aligned} \|h\|_1 &= \|h - \omega_\varepsilon * h\|_1 = \|h_1 + h_2 - \omega_\varepsilon * h_1 - \omega_\varepsilon * h_2\|_1 \\ &\leq \underbrace{\|h_1 - \omega_\varepsilon * h_1\|_1}_{(*)} + \underbrace{\|h_2\|_1 + \|\omega_\varepsilon * h_2\|_1}_{\leq 2t}, \end{aligned}$$

where

$$\begin{aligned} (*) &= \int_{\mathbb{R}^n} |h_1(x) - \omega_\varepsilon * h_1(x)|dx \\ &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} [h_1(x) - h_1(x-y)]\omega_\varepsilon(y)dy \right| dx \end{aligned}$$

$$\begin{aligned}&\leq \int_{\mathbb{R}^n} \omega_\varepsilon(y) \int_{\mathbb{R}^n} |h_1(x) - h_1(x-y)| dx dy \\ &\leq \sup_{\{y:|y|\leq\varepsilon\}} \|h_1(\cdot) - h_1(\cdot - y)\|_1.\end{aligned}$$

Due to the bounded support of h_1 and its uniform continuity, the last expression goes to zero as $\varepsilon \to 0$. Hence, choosing $\varepsilon, t > 0$ small enough, $\|h\|_1$ is arbitrary small and, therefore, equal to zero, i.e. $h = 0$ a.e. ■

Remark 3.48 (1) Any finite Borel measure μ on $\mathbb{R}^n$ is a tempered distribution via

$$\mu(\varphi) = \int_{\mathbb{R}^n} \varphi(x) d\mu(x).$$

(2) We show how that one can extend the convolution operation also to measures. Let first $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}(\mathbb{R}^n)$ as well. We denote by μ the measure on $\mathbb{R}^n$, which has density φ with respect to the Lebesgue measure. Furthermore, λ corresponds to ψ in the same way. Then

$$\begin{aligned}(\varphi * \psi)(f) &= \int_{\mathbb{R}^n} (\varphi * \psi)(x) f(x) dx = \int_{\mathbb{R}^n} f(x) \int_{\mathbb{R}^n} \varphi(x-y)\psi(y) dy dx \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x)\varphi(x-y)\psi(y) dy dx = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(z+y)\varphi(z)\psi(y) dz dy \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} f(z+y) d\mu(z) d\lambda(y).\end{aligned}$$

We observe that the last expression makes sense for all Borel measures μ and λ and $f \in C_0(\mathbb{R}^n)$, i.e. continuous functions which tend to zero at infinity. Hence, for two finite Borel measures μ and λ on $\mathbb{R}^n$, we define $\mu * \lambda$ to be the unique measure on $\mathbb{R}^n$, such that (using the Riesz representation theorem)

$$\int_{\mathbb{R}^n} f d(\mu * \lambda) = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x+y) d\mu(x) d\lambda(y) \tag{3.2}$$

for all $f \in C_0(\mathbb{R}^n)$. Using standard arguments, this holds also for all bounded Borel-measurable functions, in particular for the functions $f_\xi(x) = e^{i\langle x,\xi\rangle}$. Therefore, we obtain immediately the formula

$$\mathcal{F}(\mu * \lambda)(\xi) = (2\pi)^{n/2} (\mathcal{F}\mu)(\xi)(\mathcal{F}\lambda)(\xi), \quad \xi \in \mathbb{R}^n.$$

Furthermore, $\|\mu * \lambda\| \leq \|\mu\| \cdot \|\lambda\|$, where the norms denote total variation of measures and $\mathcal{F}(\mu)(\xi) := \int e^{-i\langle x,\xi\rangle} d\mu(x)$

Finally, (3.2) implies that

$$(\mu * \lambda)(E) = (\mu \otimes \lambda)(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x + y \in E\})$$

for every Borel-measurable set $E \subset \mathbb{R}^n$.

◊

Definition 3.49 Let $T \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha \in \mathbb{N}_0^n$. Then we put

$$\begin{aligned}(D^\alpha T)(\varphi) &= (-1)^{|\alpha|} T(D^\alpha \varphi),\\ (\mathcal{F}T)(\varphi) &= T(\mathcal{F}\varphi),\\ (\mathcal{F}^{-1}T)(\varphi) &= T(\mathcal{F}^{-1}\varphi),\\ (\varphi T)(\psi) &= T(\varphi\psi).\end{aligned}$$

Corollary 3.50 *We have that $D^\alpha T, (\varphi T), \mathcal{F}T, \mathcal{F}^{-1}T \in \mathcal{S}'(\mathbb{R}^n)$.*

Proof. This follows immediately from Theorem 3.44. ∎

Theorem 3.51 *Let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then the following properties hold.*

(i) Then $\mathcal{F}(\mathcal{F}^{-1}T) = \mathcal{F}^{-1}(\mathcal{F}T) = T$. Furthermore, both $\mathcal{F}$ and $\mathcal{F}^{-1}$ map $\mathcal{S}'(\mathbb{R}^n)$ bijectively and continuously onto itself.

(ii) Let $\alpha \in \mathbb{N}_0^n$. Then $x^\alpha T \in \mathcal{S}'(\mathbb{R}^n)$ and $D^\alpha T \in \mathcal{S}'(\mathbb{R}^n)$. Furthermore,

$$\mathcal{F}(D^\alpha T) = i^{|\alpha|} x^\alpha (\mathcal{F}T) \quad \textit{and} \quad \mathcal{F}(x^\alpha T) = i^{|\alpha|} D^\alpha (\mathcal{F}T).$$

(iii) Let $\varepsilon > 0$ and let $T \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$T_\varepsilon(\varphi) := T(\varepsilon^{-n}\varphi(\cdot/\varepsilon)), \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$$

is the dilation *of T and*

$$\mathcal{F}(T_\varepsilon) = \varepsilon^{-n}\mathcal{F}(T)(\cdot/\varepsilon).$$

(iv) Let $h \in \mathbb{R}^n$ and $T \in \mathcal{S}'(\mathbb{R}^n)$. Then we denote by $(\tau_h T)(\varphi) = T(\varphi(\cdot + h))$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the translation operator. *The formula $\mathcal{F}(\tau_h T) = e^{-i\langle h,\xi\rangle}\mathcal{F}T$ holds for all $T \in \mathcal{S}'(\mathbb{R}^n)$.*

(v) Let $h \in \mathbb{R}^n$ and $T \in \mathcal{S}'(\mathbb{R}^n)$. Then we denote by $(M_h T)(\varphi) = T(e^{i\langle h,x\rangle}\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the modulation *of T and the formula $\mathcal{F}(M_h T) = \tau_h(\mathcal{F}T)$ holds for every $T \in \mathcal{S}'(\mathbb{R}^n)$.*

Proof. We obtain

$$\mathcal{F}(\mathcal{F}^{-1}T)(\varphi) = (\mathcal{F}^{-1}T)(\mathcal{F}\varphi) = T(F^{-1}(\mathcal{F}\varphi)) = T(\varphi)$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, i.e. $\mathcal{F}(\mathcal{F}^{-1}T) = T$. The other identities follow in the same manner. ∎

3.6 The spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$

Remark 3.52 $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are not well-suited for partial differential equations (PDEs) on domains. This gave rise to the following definitions, which are historically older than those of the Schwartz spaces and tempered distributions. ◊

Definition 3.53 Let $\Omega \subset \mathbb{R}^n$ be open.

(1) The space of *test functions* is definded by

$$\mathcal{D}(\Omega) := \{\varphi \in C^\infty(\Omega) : \mathrm{supp}(\varphi) \text{ is a compact subset of } \Omega\}.$$

(2) A sequence $(\varphi_j)_j \subset \mathcal{D}(\Omega)$ is said to converge to $\varphi \in \mathcal{D}(\Omega)$ in $\mathcal{D}(\Omega)$, if there exists a compact set $K \subset \Omega$ with $\mathrm{supp}(\varphi_j) \subset K$ for all j and $D^\alpha \varphi_j \to D^\alpha \varphi$ holds for all $\alpha \in \mathbb{N}_0$. We write this as $\varphi_j \xrightarrow{\mathcal{D}} \varphi$.

(3) $\mathcal{D}'(\Omega)$ is the collection of all complex-valued, linear, continuous functionals on $\mathcal{D}(\Omega)$, i.e. $T : \mathcal{D}(\Omega) \to \mathbb{C}$ belongs to $\mathcal{D}'(\Omega)$, if

   (a) $T(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 T\varphi_1 + \lambda_2 T\varphi_2$ holds for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega)$.

   (b) $T(\varphi_j) \to T(\varphi)$ for all $\varphi_j \xrightarrow{\mathcal{D}} \varphi$.

   The elements of $\mathcal{D}'(\Omega)$ are called *distributions*.

(4) The space $\mathcal{D}'(\Omega)$ is turned into a vector space by setting

$$(\lambda_1 T_1 + \lambda_2 T_2)(\varphi) = \lambda_1 T_1\varphi_1 + \lambda_2 T_2\varphi_2 \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{C},\ T_1, T_2 \in \mathcal{D}'(\Omega),\ \varphi \in \mathcal{D}(\Omega).$$

(5) We equip $\mathcal{D}'(\Omega)$ with the following convergence. T_j converges to T in $\mathcal{D}(\Omega)$, if

$$T_j(\varphi) \to T(\varphi) \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Remark 3.54 The spaces $\mathcal{D}(\Omega)$ are much more flexible due to the inclusion of the compact support in their definition. However, a lot of algebraic structure is lost, e.g. Fourier Transforms or convolutions do not have an easy counterpart on $\mathcal{D}(\Omega)$. ◊

3.7 Sobolev Spaces

In this chapter, we would like to extend the definition of the C^k-spaces to functions which are differentiable in a weaker sense.

Definition 3.55 Let $k \in \mathbb{N}_0$ and $1 \le p < \infty$. Then

$$W_p^k(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : \; D^\alpha f \in L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \; \text{ for all } \; \alpha \in \mathbb{N}_0^n, |\alpha| \le k\}$$

are the *classical Sobolev spaces*. We write $L^p(\mathbb{R}^n) = W_p^0(\mathbb{R}^n)$.

Theorem 3.56 *Let $k \in \mathbb{N}_0$, $1 \le p < \infty$. Then $W_p^k(\mathbb{R}^n)$, equipped with norm*

$$\|f\|_{W_k^p(\mathbb{R}^n)} := \left(\sum_{|\alpha| \le k} \|D^\alpha f\|_p^p \right)^{1/p},$$

is a Banach space and

$$\mathcal{S}(\mathbb{R}^n) \subset W_k^p(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).$$

Moreover, $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are dense in $W_k^p(\mathbb{R}^n)$.

Proof. (i). $W_k^p(\mathbb{R}^n)$ is a linear subspace of $\mathcal{S}'(\mathbb{R}^n)$, if the elements of $W_k^p(\mathbb{R}^n)$ are interpreted as tempered distributions. Furthermore, the norm properties follow from the norm properties on $L^p(\mathbb{R}^n)$.

(ii). To show, that $W_k^p(\mathbb{R}^n)$ is complete, let $(f_j)_j \subset W_k^p(\mathbb{R}^n)$ be a Cauchy sequence. Then, by the completeness of $L^p(\mathbb{R}^n)$, for every $|\alpha| \le k$ there exists some $f^\alpha \in L^p(\mathbb{R}^n)$ such that $D^\alpha f_j \to f^\alpha$ in $L^p(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} D^\alpha f_j(x)\varphi(x)dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f_j(x)(D^\alpha \varphi)(x)dx, \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n).$$

By applying Hölder's inequality to every $D^\alpha f_j - f^\alpha, f_j - f^0 \in L^p(\mathbb{R}^n)$, as well as $\varphi, D^\alpha \varphi \in L^{p'}(\mathbb{R}^n)$ for conjugate exponent p' and $\varphi \in C_c^\infty$, we obtain

$$\int_{\mathbb{R}^n} f^\alpha(x)\varphi(x)dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f^0(x) D^\alpha \varphi(x)dx$$

and $f^\alpha = D^\alpha f$ for every $|\alpha| \le k$. Thus, $f_j \to f$ in $W_k^p(\mathbb{R}^n)$ as $j \to \infty$.

Concerning the embeddings, let us note that we only need to show that $\mathcal{D}(\mathbb{R}^n)$ is dense in $W_k^p(\mathbb{R}^n)$. The rest is clear. We use again the technique of mollification. Choose $\omega \in \mathcal{D}(\mathbb{R}^n), \omega \neq 0$, such that $\int_{\mathbb{R}^n} \omega(x)dx = 1$. For $\varepsilon > 0$, let $\omega_\varepsilon(x) := \frac{1}{\varepsilon^n}\omega\left(\frac{x}{\varepsilon}\right)$. Let $f \in W_k^p(\mathbb{R}^n)$ and f_ε be given by $f_\varepsilon := w_\varepsilon * f$. We have

$$(D^\alpha f_\varepsilon)(x) = \int_{\mathbb{R}^n} D_x^\alpha \omega_\varepsilon(x-y)f(y)dy$$

$$\begin{aligned}&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} D_y^\alpha \omega_\varepsilon(x-y) f(y) dy \\ &= \int_{\mathbb{R}^n} \omega_\varepsilon(x-y)(D^\alpha f)(y) dy = (D^\alpha f)_\varepsilon(x).\end{aligned}$$

Hence, $\|D^\alpha f - D^\alpha f_\varepsilon\| \to 0$, as $\varepsilon \to 0$ holds for all $|\alpha| \leq k$. Thus, $f_\varepsilon \in C^\infty(\mathbb{R}^n) \cap W_k^p(\mathbb{R}^n)$ and $f_\varepsilon \to f$ in $W_k^p(\mathbb{R}^n)$ as $\varepsilon \to 0$ holds. This implies, that it suffices to approximate functions $g \in C^\infty(\mathbb{R}^n) \cap W_k^p(\mathbb{R}^n)$ in $W_k^p(\mathbb{R}^n)$ by functions in $\mathcal{D}(\mathbb{R}^n)$. But this follows from the fact that for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\varphi(x) = 1$ for all $|x| \leq 1$ there holds

$$\varphi(2^{-j}\cdot)g(\cdot) \to g(\cdot)$$

holds in $W_k^p(\mathbb{R}^n)$ as $j \to \infty$.

■

Remark 3.57 For $k \in \mathbb{N}_0$, the spaces $W_2^k(\mathbb{R}^n)$, equipped with

$$\langle f, g \rangle_{W_2^k(\mathbb{R}^n)} := \sum_{|\alpha| \leq k} \int_{\mathbb{R}^n} D^\alpha f(x) \overline{D^\alpha g(x)} dx,$$

are Hilbert spaces. ◊

Our next goal is to describe Sobolev spaces in term of the Fourier Transform.

Definition 3.58 Let $n \in \mathbb{N}$ and w a continuous, positive function on $\mathbb{R}^n$. Then the *weighted L^2-space* is defined by

$$L^2(\mathbb{R}^n, w) := \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : wf \in L^2(\mathbb{R}^n)\}.$$

Remark 3.59 (1) $L^2(\mathbb{R}^n, w)$ is a Hilbert space, when equipped with

$$\langle f, g \rangle_{L^2(\mathbb{R}^n, w)} := \int_{\mathbb{R}^n} w(x) f(x) \overline{w(x) f(x)} dx = \langle wf, wg \rangle_{L^2(\mathbb{R}^n)}.$$

Also,

$$f \mapsto wf,\ L^2(\mathbb{R}^n, w) \to L^2(\mathbb{R}^n)$$

is unitary.

(2) An important instance of weights are

$$w_s(x) := \langle x \rangle := (1 + |x|^2)^{s/2}, \quad s \in \mathbb{R},\ x \in \mathbb{R}^n.$$

◊

Proposition 3.60 *For all $s \in \mathbb{R}$, we have*

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n, w_s) \subset \mathcal{S}'(\mathbb{R}^n).$$

Both $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are dense in $L^2(\mathbb{R}^2, w_s)$.

Proof. The first inclusion follows from the definition of $\mathcal{S}(\mathbb{R}^n)$, the second inclusion is also clear. The proof of the denseness follows in a similar way as the proof of the denseness in Theorem 3.56. ■

Theorem 3.61 *Let $k \in \mathbb{N}_0$. The Fourier Transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ generate unitary maps of $W_2^k(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n, w_k)$ and of $L^2(\mathbb{R}^n, w_k)$ onto $W_2^k(\mathbb{R}^n)$, i.e.*

$$\mathcal{F}W_2^k(\mathbb{R}^n) = \mathcal{F}^{-1}W_2^k(\mathbb{R}^n) = L^2(\mathbb{R}^n, w_k).$$

Proof. We only show the claim for $\mathcal{F}$, the claim for $\mathcal{F}^{-1}$ follows in a similar way. Let $f \in W_2^k(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|f\|_{W_2^k}^2 &= \sum_{|\alpha|\le k} \|D^\alpha f\|_2^2 \\
&= \sum_{|\alpha|\le k} \|\mathcal{F}(D^\alpha f)\|_2^2 \\
&= \int_{\mathbb{R}^n} \left(\sum_{|\alpha|\le k} |x^\alpha|^2 \right) |\mathcal{F}f(x)|^2 dx \\
&\sim \int_{\mathbb{R}^n} w_k^2(x) |\mathcal{F}f(x)|^2 dx.
\end{aligned}$$

Hence, $\mathcal{F} : W_2^k(\mathbb{R}^n) \to L^2(\mathbb{R}^n, w_k)$ is isometric. Conversely, for $g \in L^2(\mathbb{R}^n, w_k)$ and $f = \mathcal{F}^{-1}g$, we have by assumption for all $|\alpha| \le k$ that

$$D^\alpha f = i^{|\alpha|}\mathcal{F}^{-1}(x^\alpha g) \in L^2(\mathbb{R}^n),$$

hence $f \in W_2^k(\mathbb{R}^n)$. This shows the claim. ■

Remark 3.62 One can rewrite Theorem 3.61 as

$$W_2^k(\mathbb{R}^n) = \mathcal{F}L^2(\mathbb{R}^n, w_k) = \mathcal{F}^{-1}L^2(\mathbb{R}^n, w_k),$$

hence $W_2^k(\mathbb{R}^n)$ can also be defined as the Fourier image of $L^2(\mathbb{R}^n, w_k)$.

◊

Definition 3.63 Let $s \in \mathbb{R}$. Then

$$H^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \ w_s \mathcal{F} f \in L^2(\mathbb{R}^n)\}$$

is the *Sobolev space of order s*.

Remark 3.64 (1) By Theorem 3.61,

$$H^k(\mathbb{R}^n) = W_2^k(\mathbb{R}^n), \quad k \in \mathbb{N}_0.$$

(2) Also, in the definition of $H^s(\mathbb{R}^n)$, $\mathcal{F}$ can be replaced by $\mathcal{F}^{-1}$.

(3) The Sobolev spaces $H^s(\mathbb{R}^n)$ are a natural extension of the classical Sobolev spaces $W_2^k(\mathbb{R}^n)$.

(4) It is also possible to define the Sobolev spaces $H_p^s(\mathbb{R}^n)$, which extend $W_p^k(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$ to $s \in \mathbb{R}$ for all $1 \le p < \infty$.

$\lozenge$

Proposition 3.65 *Let $s \in \mathbb{R}$. The spaces $H^s(\mathbb{R}^n)$, equipped with*

$$\langle f, g \rangle_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} w_s(x) \mathcal{F} f(x) \overline{w_s(x) \mathcal{F} g(x)} dx$$

are Hilbert spaces. Moreover,

$$\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n),$$

and $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

Proof. By definition,

$$f \mapsto w_s \mathcal{F} f, \ H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

is isometric. Given some $g \in L^2(\mathbb{R}^n)$, we can choose $f = \mathcal{F}^{-1}(w_{-s} g) \in \mathcal{S}'(\mathbb{R}^n)$, hence this map is unitary. Thus, $H^s(\mathbb{R}^n)$ is a Hilbert space. We can argue, that this map also maps $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ both onto themselves. Hence, the density follows from Theorem 3.56. ■

Remark 3.66 By definition,

$$H^{s_1}(\mathbb{R}^n) \subset H^{s_2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n), \quad 0 \le s_2 \le s_1 < \infty.$$

Thus, the gaps between the smoothnes parameters $k \in \mathbb{N}_0$ of $W_2^k(\mathbb{R}^n) = H^k(\mathbb{R}^n)$ are filled by the continuous smoothness parameters s. $\lozenge$

Remark 3.67 To conclude this section, let us state some additional results.

(1) Interpreting fractional smoothness in terms of differences

$$(\Delta_h f)(x) = f(x+h) - f(x), \quad h \in \mathbb{R}^n,\ x \in \mathbb{R}^n,$$

we can define the spaces

$$W_2^s(\mathbb{R}^n)$$

for $s = k + \sigma$, where $k \in \mathbb{N}_0$ and $0 < \sigma < 1$ without usage of the Fourier Transform. It can be shown that this leads to the same spaces as $H^s(\mathbb{R}^n)$, i.e.

$$H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n).$$

(2) This then leads to one of the *Sobolev embedding theorems.* There holds

$$W_2^s(\mathbb{R}^n) \hookrightarrow C^l(\mathbb{R}^n)$$

for $l \in \mathbb{N}_0$ and $s > l + n/2$.

$\Diamond$

4 Linear Ill-Posed Inverse Problems

4.1 Introduction

Remark 4.1 We consider operator equations of the form

$$Fx = y,$$

where $F : X \to Y$ is a linear/non-linear/etc. operator and $y \in Y$ is given. We are interested in methods to solve such equations. In the most general case, X and Y are Banach spaces and $F : X \to Y$ is a (linear) operator. Given an element $y \in Y$, we aim to find a "suitable" $x \in X$ with $Fx = y$. $\Diamond$

Definition 4.2 (Hadamard) Let X, Y be Banach spaces, $F : X \to Y$. Then the operator equation $Fx = y$ is *well-posed* if, for all $y \in Y$:

(1) There exists some $x \in X$ such that $Fx = y$.

(2) The solution is unique.

(3) The solution depends continuously on y.

If at least one of these conditions is not fulfilled, the equation is *ill-posed*.

Remark 4.3 In finite dimensions and for F linear, 4.2, (3) is always satisfied. Operator equations are then much easier to handle. $\Diamond$

Example 4.4 Consider $L^\infty([0,1]) = Y$. We aim to find a derivative $x := y'$. If $y(0) = 0$, then $x = y'$ if and only if

$$y(t) = \int_0^t x(s)ds = \int_0^1 k(s,t)x(s)ds$$

where

$$k(s,t) = \begin{cases} 1 & \text{if } s \leq t \\ 0 & \text{else} \end{cases}$$

This leads to an equation of the form $y = Kx$. Assume we only know that $y = \tilde{y} + \eta$, $\tilde{y} \in C^1([0,1])$, $\eta \in L^\infty([0,1])$, where η is interpreted as noise or some other perturbation of $\tilde{y}$. The solution only exists if η is differentiable, but even then the problem is ill-posed. Consider $(\delta_n)_n \subset \mathbb{R}_{>0}$ with $\delta_n \to 0$. Then, for $k \in \mathbb{N}$, choose $\eta_n(t) = \delta_n \sin\left(\frac{kt}{\delta_n}\right)$.

Then $\eta_n \in C^1([0,1]) \hookrightarrow L^\infty([0,1])$ and $\|\eta_n\|_\infty = \delta_n \to 0$. We have $x_n(t) = y_n'(t) = \tilde{y}(t) + k\cos(\frac{kt}{\delta_n})$ for $n \in \mathbb{N}$. Set $\tilde{x}(t) := \tilde{y}'(t)$. Then $\|\tilde{x} - x_n\|_{L^\infty} = \|k\cos(\frac{kt}{\delta_n})\|_{L^\infty} = k$ which does not converge to 0. However, the problem becomes well-posed, if $Y = C^1([0,1])$ with an appropriate norm. $\lozenge$

Example 4.5 One large classes of possible operators are (partial) differential operators. They exhaust the whole range of difficulties: unbounded, non-linear, etc. $\lozenge$

Example 4.6 Some examples from imaging science:

(1) Inpainting/Recovery of missing data:

Let $F : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $M \subset \mathbb{R}^2$, and let $P_{L^2(\mathbb{R}^2 \backslash M)}$ be the orthogonal projection onto $L^2(\mathbb{R}^2 \backslash M)$. Then for $F = P_{L^2(\mathbb{R}^2 \backslash M)}$, the problem is ill-posed, violating 4.2.(3).

(2) Magnetic Resonance Imaging (MRI)

Let $F : L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \to \ell^2(\Delta)$, $f \mapsto (\hat{f}(\lambda))_{\lambda \in \Delta}$, $\Delta \subset \mathbb{R}^2$ discrete. Here, 4.2.(1).-(3). can be violated.

$\lozenge$

4.2 The Moore-Penrose Pseudoinverse

Remark 4.7 We introduce some new notation. For $T : X \to Y$ we write from now on

(1) $\mathcal{R}(T) := \operatorname{ran}(T) = T(X)$ (the range of T)

(2) $\mathcal{N}(T) := \ker(T) = T^{-1}(0_Y)$ (the kernel or nullspace of T)

(3) $\mathcal{D}(T) := \operatorname{dom}(T)$ (the domain of T)

Let X, Y be Hilbert spaces. We now consider $T \in L(X, Y)$ and the equation $Tx = y$. If $y \notin \mathcal{R}(T)$, then there does not exist a solution.

Question 1: How to define a solution? $\rightsquigarrow$ Choose $x \in X$ with $\|Tx - y\|$ minimal. However, if $\mathcal{N}(T) \neq 0$, there exist infinitely many solutions.

Question 2: Which one to pick? $\rightsquigarrow$ Choose $x \in X$ with $\|Tx - y\|$ and $\|x\|$ minimal. $\lozenge$

Definition 4.8 Let X, Y be Hilbert spaces, $T \in L(X, Y)$ and $y \in Y$.

(1) $x \in X$ is called *least-square solution* of $Tx = y$, if $\|Tx - y\| = \min_{z \in X} \|Tz - y\|$.

(2) $x \in X$ is called *minimal norm solution* (or *best approximate solution*) of $Tx = y$, if x is a least-square solution and has minimal norm among all least-square solutions.

Definition 4.9 Let $T \in L(X,Y)$, where X,Y are Hilbert spaces. Set $\tilde{T} : \mathcal{N}(T)^{\perp} \to \mathcal{R}(T)$, $\tilde{T} := T|_{\mathcal{N}(T)^{\perp}}$. Then the *Moore-Penrose Pseudoinverse* T^{+} is the unique linear extension of $\tilde{T}^{-1}$ with $\mathcal{D}(T^{+}) = \mathcal{R}(T) \bigoplus \mathcal{R}(T)^{\perp}, \mathcal{N}(T^{+}) = \mathcal{R}(T)^{\perp}$.

Lemma 4.10 *T^{+} is well-defined on $\mathcal{D}(T^{+})$.*

Proof. $\tilde{T}$ is injective and surjective, hence $\tilde{T}^{-1}$ exists. Thus T^{+} is well-defined on $\mathcal{R}(T)$. For $y \in \mathcal{D}(T^{+})$, we can write $y = y_1 + y_2$, where $y_1 \in \mathcal{R}(T)$ and $y_2 \in \mathcal{R}(T)^{\perp}$ are uniquely determined. Hence, since $\mathcal{N}(T^{+}) = \mathcal{R}(T)^{\perp}$, there holds

$$T^{+}y = T^{+}y_1 + T^{+}y_2 = T^{+}y_1 = \tilde{T}^{-1}y_1. \tag{4.1}$$

Thus, T^{+} is well-defined on $\mathcal{D}(T^{+})$. ∎

Lemma 4.11 *Let X,Y and T be as before. Then T^{+} satisfies that $\mathcal{R}(T^{+}) = \mathcal{N}(T)^{\perp}$ and the* Moore-Penrose equations *hold:*

(i) $TT^{+}T = T$

(ii) $T^{+}TT^{+} = T^{+}$

(iii) $T^{+}T = \mathrm{Id} - P_{\mathcal{N}(T)} = P_{\mathcal{N}(T)^{\perp}}$

(iv) $TT^{+} = (P_{\overline{\mathcal{R}(T)}})|_{\mathcal{D}(T^{+})}$

These already characterize T^{+} uniquely (we will not show this).

Proof. By definition and Equation (4.1), for all $y \in \mathcal{D}(T^{+})$ there holds

$$T^{+}y = \tilde{T}^{-1}P_{\overline{\mathcal{R}(T)}}y = T^{+}P_{\overline{\mathcal{R}(T)}}y \tag{4.2}$$

since in this case already $P_{\overline{\mathcal{R}(T)}} \in \mathcal{R}(T)$. Hence, $T^{+}y \in \mathcal{R}(\tilde{T}^{-1}) = \mathcal{N}(T)^{\perp}$. Thus, $\mathcal{R}(T^{+}) \subset \mathcal{N}(T)^{\perp}$. Conversely, for all $x \in \mathcal{N}(T)^{\perp}$,

$$T^{+}Tx = \tilde{T}^{-1}\tilde{T}x = x$$

hence $x \in \mathcal{R}(T^{+})$. Thus, there holds that

$$\mathcal{R}(T^{+}) = \mathcal{N}(T)^{\perp}. \tag{4.3}$$

We will now prove the four equations.

(iv): For $y \in \mathcal{D}(T^+)$, by Equation (4.2) and Equation (4.3), we have

$$TT^+y = TT^+P_{\overline{\mathcal{R}(T)}}y = T\tilde{T}^{-1}P_{\overline{\mathcal{R}(T)}}y = \tilde{T}\tilde{T}^{-1}P_{\overline{\mathcal{R}(T)}}y = P_{\overline{\mathcal{R}(T)}}y$$

where we used that $P_{\overline{\mathcal{R}(T)}}y \in \mathcal{R}(T)$ and $\tilde{T}^{-1}P_{\overline{\mathcal{R}(T)}}y \in \mathcal{N}(T)^\perp$.

(iii): By definition of T^+, for all $x \in X$,

$$T^+Tx = \tilde{T}^{-1}Tx$$

Hence

$$\begin{aligned} T^+Tx &= \tilde{T}^{-1}T(P_{\mathcal{N}(T)}x + (\mathrm{Id} - P_{\mathcal{N}(T)})x) \\ &= \tilde{T}^{-1}TP_{\mathcal{N}(T)}x + \tilde{T}^{-1}\tilde{T}(\mathrm{Id} - P_{\mathcal{N}(T)})x \\ &= (\mathrm{Id} - P_{\mathcal{N}(T)})x \\ &= P_{\mathcal{N}(T)^\perp} \end{aligned}$$

(ii): Applying (4) to Equation (4.2) yields for all $y \in \mathcal{D}(T^+)$ that

$$T^+y = T^+P_{\overline{\mathcal{R}(T)}}y = T^+TT^+y$$

(i): Using (3), there holds

$$TT^+T - T(\mathrm{Id} - P_{\mathcal{N}(T)}) = 0$$

∎

Theorem 4.12 *Let X, Y, and T be as before and $y \in \mathcal{D}(T^+)$. Then $Tx = y$ has a unique minimal-norm solution x^+, which is given by $x^+ = T^+y$. The set of all least-square solutions is given by $x^+ + \mathcal{N}(T)$.*

Proof. We first prove the existence of least-square solutions. Consider the set

$$S := \{z \in X : Tz = P_{\overline{\mathcal{R}(T)}}y\}.$$

Since, by assumption, $y \in \mathcal{D}(T^+)$, we obtain $P_{\overline{\mathcal{R}(T)}}y \in \mathcal{R}(T)$ and thus $S \neq \varnothing$. By using the properties of orthogonal projections, for all $z \in S$ there holds

$$\|Tz - y\| = \|P_{\overline{\mathcal{R}(T)}}y - y\| = \min_{w \in \overline{\mathcal{R}(T)}} \|w - y\| \leq \|Tx - y\|$$

for all $x \in X$. Hence every $z \in S$ is a least-square solution of $Tz = y$. Conversely, for all least-square solutions $z \in X$ we have, employing the same argument as before,

$$\|P_{\overline{\mathcal{R}(T)}}y - y\| \leq \|Tz - y\| = \min_{x \in X}\|Tx - y\| = \min_{w \in \mathcal{R}(T)}\|w - y\| \leq \|P_{\overline{\mathcal{R}(T)}}y - y\|$$

Thus,

$$Tz = P_{\overline{\mathcal{R}(T)}}y. \tag{4.4}$$

We have just shown that

$$S = \{x \in X : x \text{ is a least-square solution of } Tx = y\} \neq \varnothing.$$

Now we turn our attention to the existence of a unique minimal norm solution. By Equation (4.4), each least-square solution of $Tx = y$ is a solution of $Tx = P_{\overline{\mathcal{R}(T)}}y$. Let $x \in X$ be such a solution and write $x = \bar{x} + x_0$ for $\bar{x} \in \mathcal{N}(T)^\perp$, $x_0 \in \mathcal{N}(T)$. Then

$$\|x\|^2 = \|\bar{x} + x_0\|^2 = \|\bar{x}\|^2 + 2\langle \bar{x}, x_0\rangle + \|x_0\|^2 = \|\bar{x}\|^2 + \|x_0\| \geq \|\bar{x}\|^2.$$

Moreover, T is injective on $\mathcal{N}(T)^\perp$, hence $\bar{x}$ is independent of x because, informally:

If $x_2 \in S$ such that $x_2 = \bar{x}_2 + x_1$ where $\bar{x}_2 \neq \bar{x}$ and again $\bar{x}_2 \in \mathcal{N}(T)^\perp$, $x_1 \in \mathcal{N}(T)$, then $Tx_2 = T\bar{x}_2 \neq T\bar{x} = P_{\overline{\mathcal{R}(T)}}y$ which is a contradiction.

It follows that $x^+ = \bar{x}$ is the unique minimal-norm solution of $Tx = y$. Finally, by Lemma 4.11, we have

$$x^+ = P_{\mathcal{N}(T)^\perp}x^+ = (\mathrm{Id} - P_{\mathcal{N}(T)})x^+ = T^+Tx^+ \overset{(4.4)}{=} T^+P_{\overline{\mathcal{R}(T)}}y \overset{(iv)}{=} T^+TT^+y \overset{(i)}{=} T^+y$$

This finishes the proof. ∎

Theorem 4.13 *Let X, Y, and T be as before, $y \in \mathcal{D}(T^+) = \mathcal{R}(T) \bigoplus \mathcal{R}(T)^\perp$. Then the following are equivalent:*

(i) $x \in X$ is a least-square solution of $Tx = y$

(ii) $x \in X$ satisfies the normal equation $T^*Tx = T^*y$ where T^* denotes the usual Hilbert space adjoint operator.

If in addition $x \in \mathcal{N}(T)^\perp$, then $x = x^+$.

Proof. By the proof of theorem 4.12, (i) is equivalent to (ii'), where (ii') denotes

$$Tx = P_{\overline{\mathcal{R}(T)}}y.$$

But, by properties of orthogonal projections, (ii') is equivalent to $Tx \in \overline{\mathcal{R}(T)}$ and $Tx - y \in \overline{\mathcal{R}(T)}^\perp = \mathcal{N}(T^*)$ which is equivalent to $T^*(Tx - y) = 0$.

By the proof of Theorem 4.12, the minimal-norm solution x^+ is the only least-square solution in $\mathcal{N}(T)^\perp$. ∎

Corollary 4.14 *The minimal-norm solution also satisfies $x^+ = (T^*T)^+T^*y$. This can be obtained by multiplying $(T^*T)^+$ on both sides of $T^*Tx = T^*y$.*

Remark 4.15 (1) By construction, $\mathcal{D}(T^+) = \mathcal{R}(T) \bigoplus \mathcal{R}(T)^\perp$. Hence,

$$\overline{\mathcal{D}(T^+)} = \overline{\mathcal{R}(T)} \bigoplus \mathcal{R}(T)^\perp = Y (= \mathcal{N}((T^*)^\perp) \bigoplus \mathcal{N}(T^*))$$

meaning $\mathcal{D}(T^+)$ is dense in Y.

(2) If $\mathcal{R}(T)$ is closed, then $\mathcal{D}(T^+) = Y$ and the converse is true as well.

(3) For $y \in \mathcal{R}(T)^\perp = \mathcal{N}(T^+)$, the minimal-norm solution is $x^+ = 0$.

◊

Theorem 4.16 *Let X, Y be Hilbert spaces, $T \in L(X, Y)$. Then the following are equivalent:*

(i) $T^+ \in L(\mathcal{D}(T^+), X)$.

(ii) $\mathcal{R}(T)$ is closed or, in other words, $\mathcal{D}(T^+) = Y$.

Proof. (ii) $\Rightarrow$ (i):

Claim 1: T^+ is closed.

Let $(y_n) \subset \mathcal{D}(T^+)$ with $y_n \to y \in Y$ and $T^+y_n \to x \in X$. By Lemma 4.11.(4),

$$TT^+y_n = P_{\overline{\mathcal{R}(T)}}y_n \to P_{\overline{\mathcal{R}(T)}}y$$

by the continuity of orthogonal projections. Since T is continuous, we obtain

$$P_{\overline{\mathcal{R}(T)}}y = \lim_{n\to\infty} P_{\overline{\mathcal{R}(T)}}y_n = \lim_{n\to\infty} TT^+y_n = Tx,$$

implying that x is a least-square solution of $Tx = y$. Furthermore,

$$T^+y_n \in \mathcal{R}(T^+) = \mathcal{N}(T)^\perp \Rightarrow T^+y_n \to x \in \mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}.$$

Similarly to the proof of Theorem 4.12, x is a minimal-norm solution of $Tx = y$. Hence, $x = T^+y$. Claim (i) follows.

By Remark 4.15.(2), (ii) implies $\mathcal{D}(T^+) = Y$. The closed graph theorem now implies (i).

(i) $\Rightarrow$ (ii):

By Remark 4.15.(1), $\mathcal{D}(T^+)$ is dense in Y. Hence, T^+ can be uniquely and continuously extended to Y by $\overline{T^+} \in L(Y, X)$ defined by $\overline{T^+}y := \lim_{n\to\infty} T^+y_n$ for some sequence

$(y_n) \subset \mathcal{D}(T^+)$ with $y_n \to y \in Y$. That this procedure is well-defined follows from results showed in Functional Analysis 1. Now, let $y \in \overline{\mathcal{R}(T)}$, $(y_n) \subset \mathcal{R}(T)$ such that $y_n \to y$. By Lemma 4.11.(iv) and the continuity of T, we have

$$y = P_{\overline{\mathcal{R}(T)}} y = \lim_{n\to\infty} P_{\overline{\mathcal{R}(T)}} y_n = \lim_{n\to\infty} TT^+ y_n = TT^+ y \in \mathcal{R}(T)$$

Hence $\overline{\mathcal{R}(T)} = \mathcal{R}(T)$. ∎

Corollary 4.17 *Let $K \in \mathcal{K}(X,Y)$ with $\dim(\mathcal{R}(K)) = \infty$. Then K^+ is not continuous.*

Proof. Assume towards a contradiction that K^+ is continuous. By Theorem 4.16, $\mathcal{R}(K)$ is closed. Let $\tilde{K} := K|_{\mathcal{N}(K)^\perp} : \mathcal{N}(K)^\perp \to \mathcal{R}(K)$. Then the bijective operator $\tilde{K}$ has a continuous inverse, $\tilde{K}^{-1} \in L(\mathcal{R}(K), \mathcal{N}(K)^\perp)$. Since K is compact, $K \circ \tilde{K}^{-1}$ is compact. Since for $x \in \mathcal{R}(K)$ we have that $K\tilde{K}^{-1}x = x$, the operator $\mathrm{Id} : \mathcal{R}(K) \to \mathcal{R}(K)$ is compact. Hence (using results from Functional Analysis 1) $\dim(\mathcal{R}(K)) < \infty$. This is the contradiction we were working towards. ∎

4.3 Singular Value Decomposition for Compact Operators

Theorem 4.18 *Let X, Y be Hilbert spaces, $K \in \mathcal{K}(X,Y)$. Then there exists a decreasing sequence $(\sigma_n)_n \subset \mathbb{R}^+$ such that $\sigma_n \to 0$, an orthonormal basis $(u_n)_n \subset Y$ of $\overline{\mathcal{R}(K)}$, and an orthonormal basis $(v_n)_n \subset X$ of $\overline{\mathcal{R}(K^*)}$ with*

(i) $Kv_n = \sigma_n u_n$, $K^*u_n = \sigma_n v_n$ for all $n \in \mathbb{N}$, and

(ii) $Kx = \sum_{n\in\mathbb{N}} \sigma_n \langle x, v_n \rangle u_n$ for all $x \in X$.

The sequence $((\sigma_n, u_n, v_n))_{n\in\mathbb{N}}$ is called singular value decomposition of *K.*

Proof. This proof relies on the spectral theorem for compact and self-adjoint operators from Functional Analysis 1. Since $K^*K : X \to X$ is compact and self-adjoint, there exists a (with respect to the absolute value) decreasing sequence $(\lambda_n) \subset \mathbb{R}\backslash\{0\}$ such that $\lambda_n \to 0$ and a corresponding orthonormal system $(v_n)_n \subset X$ with the property that for all $x \in X$ there holds

$$K^*Kx = \sum_{n\in\mathbb{N}} \lambda_n \langle x, v_n \rangle v_n$$

We have

$$\lambda_n = \lambda_n \|v_n\|^2 = \langle \lambda_n v_n, v_n \rangle = \langle K^*Kv_n, v_n \rangle = \langle Kv_n, Kv_n \rangle = \|Kv_n\|^2 > 0,$$

hence also $\|Kv_n\|^2 > 0$, $\lambda_n > 0$. This implies, that for all $n \in \mathbb{N}$, we can set $\sigma_n := \sqrt{\lambda_n} > 0$ and $u_n := \sigma_n^{-1} K v_n \in Y$. $(u_n)_{n\in\mathbb{N}}$ is an orthonormal system in Y, since

$$\langle u_i, u_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle K v_i, K v_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle K^* K v_i, v_j \rangle = \frac{\lambda_i}{\sigma_i \sigma_j} \langle v_i, v_j \rangle = \delta_{i,j}$$

Moreover,

$$K^* u_n = \sigma_n^{-1} K^* K v_n = \sigma_n^{-1} \lambda_n v_n = \sigma_n v_n \tag{4.5}$$

By the spectral theorem, $(v_n)_{n\in\mathbb{N}}$ is an orthonormal basis for $\overline{\mathcal{R}(K^*K)}$. But $\overline{\mathcal{R}(K^*K)} = \overline{\mathcal{R}(K^*)}$ since, due to the following informal argument: Let $x \in \overline{\mathcal{R}(K^*)} \setminus \{0\}$. Then there exists a sequence $(y_n) \subset Y$ such that $K^* y_n \to x$. Without loss of generality, $(y_n) \subset \mathcal{N}(K^*)^\perp = \overline{\mathcal{R}(K)}$. Using a diagonal argument, $x \in \overline{\mathcal{R}(K^*K)}$. This implies $\overline{\mathcal{R}(K^*K)} \supseteq \overline{\mathcal{R}(K^*)}$. The other inclusion is clear.

Hence, $(v_n)_{n\in\mathbb{N}}$ is an orthonormal basis for $\overline{\mathcal{R}(K^*)}$. $(v_n)_{n\in\mathbb{N}}$ can be extended to an orthonormal basis V for X. For this, note that the remaining elements must be in $\mathcal{N}(K)$ since $\overline{\mathcal{R}(K^*)}^\perp = \mathcal{N}(K)$. Thus, for all $x \in X$,

$$Kx = \sum_{v\in V} \langle x, v \rangle K v = \sum_{n\in\mathbb{N}} \langle x, v_n \rangle K v_n \overset{(i)}{=} \sum_{n\in\mathbb{N}} \langle x, v_n \rangle \sigma_n u_n \overset{(4.5)}{=} \sum_{n\in\mathbb{N}} \langle x, K^* u_n \rangle u_n = \sum_{n\in\mathbb{N}} \langle Kx, u_n \rangle u_n$$

The last calculation now implies that (u_n) is an orthonormal basis for $\overline{\mathcal{R}(K)}$. ■

Theorem 4.19 *Let $K \in \mathcal{K}(X, Y)$, $(\sigma_n, u_n, v_n)_{n\in\mathbb{N}}$ be a singular system for K and $y \in \overline{\mathcal{R}(K)}$. Then the following are equivalent:*

(i) $y \in \mathcal{R}(K)$.

(ii) The Picard condition $\sum_{n\in\mathbb{N}} \sigma_n^{-2} |\langle y, u_n \rangle|^2 < \infty$ *is satisfied.*

In this case, we have $K^+ y = \sum_{n\in\mathbb{N}} \sigma_n^{-1} \langle y, u_n \rangle v_n$.

Proof. (i) ⇒ (ii):

Let $y \in \mathcal{R}(K)$. Then there exists an $x \in X$ such that $Kx = y$. Hence, for all $n \in \mathbb{N}$

$$\langle y, u_n \rangle = \langle Kx, u_n \rangle = \langle x, K^* u_n \rangle = \sigma_n \langle x, v_n \rangle$$

By the Bessel inequality, this implies

$$\sum_{n\in\mathbb{N}} \sigma_n^{-2} |\langle y, u_n \rangle|^2 = \sum_{n\in\mathbb{N}} |\langle x, v_n \rangle|^2 \leq \|x\|^2 < \infty$$

(ii) ⇒ (i):

Let $y \in \overline{\mathcal{R}(K)}$ and assume that (ii) holds. Then $\left(\sum_{n=1}^{N} \sigma_n^{-2} |\langle y, u_n \rangle|^2\right)_{N\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ and also $X_N := \sum_{n=1}^{N} \sigma_n^{-1} \langle y, u_n \rangle v_n$ is a Cauchy sequence in X since, for $N, M \to \infty$

$$\|X_N - X_M\|^2 = \left\| \sum_{n=N}^{M} \sigma_n^{-1} \langle y, u^n \rangle v_n \right\|^2 = \sum_{n=N}^{M} \sigma_n^{-2} |\langle y, u_n \rangle|^2 \to 0$$

Also, we have that $(v_n)_n \subset \overline{\mathcal{R}(K^*)}$. Hence, $(X_N)_N \subset \overline{\mathcal{R}(K^*)}$ converges to some $x := \sum_{n\in\mathbb{N}} \sigma_n^{-1} \langle y, u_n \rangle v_n \in X$. Since $\overline{\mathcal{R}(K^*)}$ is closed, $x \in \overline{\mathcal{R}(K^*)} = \mathcal{N}(K)^{\perp}$. Observe that

$$Kx = \sum_{n\in\mathbb{N}} \sigma_n^{-1} \langle y, u_n \rangle K v_n = \sum_{n\in\mathbb{N}} \langle y, u_n \rangle u_n = P_{\overline{\mathcal{R}(K)}}(y) = y,$$

which implies that $y \in \mathcal{R}(K)$. By Theorem 4.13, $Kx = P_{\overline{\mathcal{R}(K)}} y$ and $x \in \mathcal{N}(K)^{\perp}$ is equivalent to $K^+ y = x$. ∎

Remark 4.20 (1) By the Picard condition, a minimal norm solution can only exist, if the "Fourier Coefficients" $(\langle y, u_n \rangle)_{n\in\mathbb{N}}$ of y decay rapidly compared to the singular values $(\sigma_n)_{n\in\mathbb{N}}$.

(2) We know that $K^+ y = \sum_{n\in\mathbb{N}} \sigma_n^{-1} \langle y, u_n \rangle v_n$. If $y^\delta := y + \delta u_n$, for some $\delta > 0$, we have

$$\|K^+ y^\delta - K^+ y\| = \delta \|K^+ u_n\| = \delta \sigma_n^{-1} \to \infty.$$

This shows how perturbations of y affect perturbations of x^+, and the faster $(\sigma_n)_n$ decays, the worse the amplification of the error is as $n \to \infty$.

◊

Definition 4.21 (1) $Kx = y$ is *moderately ill-conditioned* if there exist $c, r > 0$ with $\sigma_n \geq c n^{-r}$ for all $n \in \mathbb{N}$, i.e. the decay of the σ_n is at most polynomial.

(2) If (1) is not the case, $Kx = y$ is called *strongly ill-conditioned*.

(3) $Kx = y$ is called *exponentially ill-conditioned*, if there exist $c, r > 0$ such that $\sigma_n \leq c e^{-nr}$ for all $n \in \mathbb{N}$.

Example 4.22 Let $X = L^2([0,1])$, $K \in \mathcal{K}(X, X)$ given as in Example 4.4, that is $(Kx)(t) = \int_0^t x(s) ds$. Then $Kx = y$ is moderately ill-conditioned (this is a homework exercise). ◊

Remark 4.23 The Singular Value Decomposition allows us to define functions of compact operators. Let $\varphi : [0, \infty) \to \mathbb{R}$ be a piecewise continuous locally bounded function. Then for $K \in K(X, Y)$ with singular system $((\sigma_n, u_n, v_n))_{n\in\mathbb{N}}$ and $x \in X$ we define the operator $\varphi(K^* K) : X \to X$ by

$$\varphi(K^* K) x := \sum_{n\in\mathbb{N}} \varphi(\sigma_n^2) \langle x, v_n \rangle v_n + \varphi(0) P_{\mathcal{N}(K)} x$$

This series converges in X, since φ is evaluated on the closed and bounded interval $[0, \sigma_1] = [0, \|K\|^2]$. Moreoever,

$$\|\varphi(K^*K)\| = \sup_{n\in\mathbb{N}} |\varphi(\sigma_n^2)| \leq \sup_{\lambda\in[0,\|K\|^2]} |\varphi(\lambda)| < \infty$$

This implies, that $\varphi(K^*K) \in L(X)$. $\Diamond$

Example 4.24 $\varphi(t) := \sqrt{t}$, then $|K| := \varphi(K^*K)$ is the *absolute value of* K:

$$|K|x = \sum_{n\in\mathbb{N}} \sigma_n \langle x, v_n \rangle v_n, \quad \forall x \in X.$$

$\Diamond$

5 Regularization of linear ill-posed problems

5.1 Regularization and operators

Remark 5.1 Let $Tx = y$ be an ill-posed operator equation. Then, for $y \in \mathcal{D}(T^+)$, there exists a unique minimal norm solution $x^+ := T^+y$. Often, one only knows a perturbation $y^\delta \in Y$ with $\|y - y^\delta\| \leq \delta$. Since, in general, T^+ does not need to be continuous, we cannot expect T^+y^δ to be close to x^+ in all cases. If even $y^\delta \notin \mathcal{D}(T^+)$, it is impossible to calculate T^+y^δ in the first place. $\diamondsuit$

Idea:

(1) Aim for an approximation x_α^δ which depends continuously on y^δ (and hence on δ).

(2) There exist *regularization parameters* $\alpha = \alpha(\delta) > 0$ with $x_{\alpha(\delta)}^\delta \to x^+$ as $\delta \to 0$.

Definition 5.2 Let $T \in L(X, Y)$, where X, Y are Hilbert spaces. A family $(R_\alpha)_{\alpha>0}$ of linear operators $R_\alpha : Y \to X$ is called *regularization* (*family of regularization operators*) for T^+, if

(1) $R_\alpha \in L(Y, X)$ for all $\alpha > 0$, and

(2) $R_\alpha y \to T^+y$ for all $y \in \mathcal{D}(T^+)$.

Remark 5.3 A regularization is a pointwise approximation of the pseudoinverse by continuous operators. $\diamondsuit$

Theorem 5.4 *Let $T \in L(X, Y)$, $(R_\alpha)_{\alpha>0} \subset L(Y, X)$ a regularization of T^+. If T^+ is not continuous, then $(R_\alpha)_{\alpha>0}$ is not uniformly bounded. In particular, there exists a $y \in Y$ such that $\|R_\alpha y\| \to \infty$ as $\alpha \to 0$.*

Proof. Assume that $(R_\alpha)_{\alpha>0}$ is uniformly bounded, i.e. there exists $M > 0$ with $\|R_\alpha\| \leq M$ for all $\alpha > 0$. Using the Banach-Steinhaus theorem, since $R_\alpha \to T^+$ pointwise on $\mathcal{D}(T^+)$, this convergence also holds on $\overline{\mathcal{D}(T^+)} = Y$. Thus, $T^+ \in L(Y, X)$, which is a contradiction. Assume now that there does not exist a $y \in Y$ with $\|R_\alpha y\| \to \infty$ for $\alpha \to 0$, that is $(R_\alpha)_{\alpha>0}$ is pointwise bounded. By the Uniform Boundedness Principle, $(R_\alpha)_{\alpha>0}$ would also have to be uniformly bounded which, again, is a contradiction. ∎

Theorem 5.5 *Let $T \in L(X,Y)$ with T^+ discontinuous and let $(R_\alpha)_{\alpha>0} \subset L(Y,X)$ be a regularization. If*

$$\sup_{\alpha>0} \|TR_\alpha\| < \infty, \tag{5.1}$$

then $\|R_\alpha y\| \to \infty$ for all $y \notin \mathcal{D}(T^+)$ as $\alpha \to 0$.

Proof. Let $y \in Y$ be arbitrary. Assume that there exists a sequence $(\alpha_n)Nn \in \mathbb{N}$ with $\alpha_n \to 0$ such that $(R_{\alpha_n} y)_{n\in\mathbb{N}}$ is bounded. Since X is reflexive, there exists a weakly convergent subsequence $(x_k)_k$ where $x_k := R_{\alpha_{n_k}} y$ with $x_k \overset{w}{\to} x \in X$. Since elements of $L(X,Y)$ are weakly continuous, $Tx_k \overset{w}{\to} Tx$. By the continuity of T and the pointwise convergence of R_α to T^+ on $\mathcal{D}(T^+)$, we have $TR_\alpha z \to TT^+z = P_{\overline{\mathcal{R}(T)}} z$ for all $z \in \mathcal{D}(T^+)$ as $\alpha \to 0$. By (5.1), $TR_\alpha z$ converges for all $z \in Y$. Since $Tx_k \to P_{\overline{\mathcal{R}(T)}} y$ and $Tx_k \overset{w}{\to} Tx$, we conclude that $Tx = P_{\overline{\mathcal{R}(T)}} y$ which implies $y \in \mathcal{D}(T^+)$. ∎

5.2 Parameter choice rules

Remark 5.6 Often, for $y^\delta \in Y$ with $\|y^\delta - y\| \le \delta$ we do not have $y^\delta \in \mathcal{D}(T^+)$. Thus, we are interested in the total error

$$\|R_\alpha y^\delta - T^+y\| \le \|R_\alpha y^\delta - R_\alpha y\| + \|R_\alpha y - T^+y\| \le \|R_\alpha\|\delta + \|R_\alpha y - T^+y\|$$

The second term, the *error of the regularization method,* converges to 0 as $\alpha \to 0$ but the first term, the *data error,* is not bounded for $\delta > 0$ if T^+ is discontinuous. ◊

Definition 5.7 A function $\alpha : \mathbb{R}_{>0} \times Y \to \mathbb{R}_{>0}$ is called *parameter choice rule*. We distinguish between

(1) *A priori parameter choice rules (a priori strategies)*, for which α depends only on δ.

(2) *A posteriori parameter choice rules (a priori strategies)*, for which α depends on both δ and y^δ.

(3) *Heuristic parameter choice rules (heuristic strategies)*, for which α depends only on y^δ.

If $(R_\alpha)_{\alpha>0}$ is a regularization of T^+ and α is a parameter choice rule, then (R_α, α) is called (convergent) *regularization method* if for all $y \in \mathcal{D}(T^+)$ we have

$$\lim_{\delta\to 0} \Big(\sup\{ \|R_{\alpha(\delta,y^\delta)} y^\delta - T^+y\| : y^\delta \in Y, \|y^\delta - y\| \le \delta\} \Big) = 0.$$

Moreover, if (R_α, α) is a convergent regularization method, we assume that for all

$y \in Y$ there holds

$$\lim_{\delta\to 0} \sup\{\alpha(\delta, y^\delta) | y^\delta \in Y, \|y^\delta - y\| \leq \delta\} = 0$$

Theorem 5.8 *Let $(R_\alpha)_{\alpha>0}$ be a regularization of T^+, then there exists an a priori parameter choice rule α such that (R_α, α) is a regularization method.*

Proof. Let $y \in \mathcal{D}(T^+)$. By assumption, $R_\alpha y \to T^+ y$ as $\alpha \to 0$. Hence, for all $\varepsilon > 0$ there exists $\sigma(\varepsilon) > 0$ with $\|R_{\sigma(\varepsilon)} y - T^+ y\| \leq \frac{\varepsilon}{2}$. Notice that σ can be chosen to grow monotonically in ε with $\lim_{\varepsilon\to 0} \sigma(\varepsilon) = 0$. Moreover, for all $\varepsilon > 0$, $R_{\sigma(\varepsilon)}$ is continuous. Hence for all $\varepsilon > 0$ there exists $\varrho(\varepsilon) > 0$ with

$$\|R_{\sigma(\varepsilon)} z - R_{\sigma(\varepsilon)} y\| \leq \frac{\varepsilon}{2}$$

for all $z \in Y$ with $\|z - y\| \leq \varrho(\varepsilon)$. Again, $\varrho : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ can be chosen to be strictly monotonically increasing in ε and $\lim_{\varepsilon\to 0} \varrho(\varepsilon) = 0$ with ϱ being continuous. Hence, there exists a strictly montone and continuous inverse function ϱ^{-1} of ϱ on $\mathcal{R}(\varrho)$ with $\lim_{\delta\to 0} \varrho^{-1}(\delta) = 0$. We now extend ϱ^{-1} monotonically and continuously to $\tilde{\varrho}^{-1} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. Our a priori parameter choice rule is now defined as

$$\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0} : \delta \mapsto \sigma(\tilde{\varrho}^{-1}(\delta))$$

Then, by definition, $\lim_{\delta\to 0} \alpha(\delta) = 0$. Moreover, for all $\varepsilon > 0$ there exists $\delta := \varrho(\varepsilon) > 0$ with the property that for all $y^\delta \in Y$ with $\|y^\delta - y\| \leq \delta$ we have

$$\|R_{\alpha(\delta)} y^\delta - T^+ y\| = \|R_{\sigma(\varepsilon)} y^\delta - R_{\sigma(\varepsilon)} y\| + \|R_{\sigma(\varepsilon)} y - T^+ y\| \leq \varepsilon$$

This implies that for $\delta \to 0$ and each family $(y^\delta)_{\delta>0}$ with $\|y^\delta - y\| \leq \delta$ we have $\mathbb{R}_{\alpha(\delta)} y^\delta - T^+ y\| \to 0$. ∎

Theorem 5.9 *Let $(R_\alpha)_{\alpha>0}$ be a regularization and $\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ an a priori parameter choice rule. Then the following are equivalent:*

(i) (R_α, α) *is a regularization method.*

(ii) We have the following two properties:

(a) $\lim_{\delta\to 0} \alpha(\delta) = 0$, *and*

(b) $\lim_{\delta\to 0} \delta \|R_{\alpha(\delta)}\| = 0$.

Proof. (ii) $\Rightarrow$ (i):

By Remark 5.6, for all $y^\delta \in Y$ such that $\|y^\delta - y\| \le \delta$, as $\delta \to 0$

$$\|R_{\alpha(\delta)} y^\delta - T^+ y\| \le \delta \|R_{\alpha(\delta)}\| + \|R_{\alpha(\delta)} y - T^+ y\| \to 0$$

where the terms on the right side converge to 0 by (a) and (b).

(i) $\Rightarrow$ (ii):

Towards a contradiction, we assume that (ii) is not fulfilled. Since, by assumption, (R_α, α) is a convergent regularization method, (a) is true. Now assume that (b) is not true. Then there exists a sequence $(\delta_n)_n$ such that $\delta_n \to 0$ and $\delta_n \|R_{\alpha(\delta_n)}\| \ge C > 0$ for all $n \in \mathbb{N}$. This implies that there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset Y$ with $\|z_n\| = 1$ and $\delta_n \|R_{\alpha(\delta_n)} z_n\| \ge C$ for all $n \in \mathbb{N}$. Now let $y \in \mathcal{D}(T^+)$ be arbitrary and set $y_n := y + \delta_n z_n$. Then $\|y_n - y\| \le \delta$, but

$$R_{\alpha(\delta_n)} y_n - T^+ y = (R_{\alpha(\delta_n)} y - T^+ y) + \delta_n R_{\alpha(\delta_n)} z_n,$$

which cannot converge to 0 in norm since the first term converges to 0 and the second term is unbounded by construction. Hence, (i) is not true. ■

Remark 5.10 If $\|R_\alpha\| \to \infty$ as $\alpha \to 0$, then α should not converge to 0 too fast to still allow for the condition in Theorem 5.9.(ii).(b) to be satisfied. One can choose $\alpha(\delta) = \delta^r$ for $r < 1$ for instance to achieve "slow" convergence. Choosing r "correctly" requires knowledge of x^+. ♢

Remark 5.11 A posteriori parameter choice rules: Let $y \in \mathcal{D}(T^+)$ and y^δ with $\|y - y^\delta\| = \delta$. Set $R_\alpha y^\delta = x_\alpha^\delta$ and consider the *residuum* $\|Tx_\alpha^\delta - y^\delta\|$. If we even have $y \in \mathcal{R}(T)$, then $\|Tx^+ - y^\delta\| = \|y - y^\delta\| = \delta$. Hence it does not make a lot of sense to aim for a smaller residuum in choosing $\alpha(\delta, y^\delta)$.

This leads to the *Discrepancy principle of Morozov*. For given $\delta > 0$, $y^\delta \in Y$, choose $\alpha = \alpha(\delta, y^\delta)$ with $\|Tx_\alpha^\delta - y^\delta\| \le \tau\delta$ for some $\tau > 1$. This guideline might be not fulfillable: If $y \in \mathcal{R}(T)^\perp$, then $\|Tx^+ - y\| = \|TT^+ y - y\| = \|P_{\overline{\mathcal{R}(T)}} y\| = \|y\| > \delta$ for δ small enough. This problem does not occur if $\mathcal{R}(T)$ is dense. ♢

Theorem 5.12 *Let $(R_\alpha)_\alpha$ be a regularization of T^+. If there exists a heuristic parameter choice rule α such that (R_α, α) is a regularization method, then T^+ is already continuous.*

Proof. Let $\alpha : Y \to \mathbb{R}_{>0}$ be such a (heuristic) parameter choice method. Then define the operator $R : Y \to X : y \mapsto R_{\alpha(y)} y$. Let $y \in \mathcal{D}(T^+)$. Since (R_α, α) is a regularization method, we have, for $y^\delta = y$ and $\delta = 0$, that $Ry = T^+ y$. Now let $(y_n)_n \subset \mathcal{D}(T^+)$ with $y_n \to y$ as $n \to \infty$. Again by the definition of a regularization method and choosing $y^\delta = y_n$ as well as $\delta = \|y^\delta - y\|$, we have

$$T^+ y_n = R y_n = R_{\alpha(y_n)} y_n \overset{n\to\infty}{\to} T^+ y$$

Hence, T^+ is continuous. ■

Remark 5.13 The aforementioned theorem shows that heuristic parameter choice methods are not applicable in general. In practical applications however, they are often used anyway since most of the time only y^δ is known, not δ itself. $\Diamond$

5.3 Convergence Rates

Remark 5.14 Given a regularization $(R_\alpha)_{\alpha>0}$ for T^+, we are interested in error estimates $\|R_\alpha y^\delta - T^+ y\| \leq \varphi(\delta)$ for some function $\varphi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\lim_{t\to 0} \varphi(t) = 0$. $\Diamond$

Definition 5.15 We define the worst case error

$$\varepsilon(y, \delta) := \sup\{\|R_{\alpha(\delta, y^\delta)} y^\delta - T^+ y\| : y^\delta \in Y, \|y^\delta - y\| \leq \delta\}$$

Theorem 5.16 *Let (R_α, α) be a regularization method. If there exists a function $\varphi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\lim_{t\to 0} \varphi(t) = 0$ and $\sup_{y\in\mathcal{D}(T^+)\cap B_1(0_Y)} \varepsilon(y, \delta) \leq \varphi(\delta)$, then T^+ is already continuous.*

Proof. Let $y \in \mathcal{D}(T^+) \cap B_1(0_Y)$ and $(y_n)_n \subset \mathcal{D}(T^+) \cap B_1(0_Y)$ with $y_n \to y$. Then set $\delta_n = \|y_n - y\|$. Assume that $\sup_{y\in\mathcal{D}(T^+)\cap B_1(0_Y)} \varepsilon(y, \delta) \leq \varphi(\delta)$ holds for some $\varphi : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\lim_{t\to 0} \varphi(t) = 0$. This implies

$$\|T^+ y_n - T^+ y\| \leq \|T^+ y_n - R_{\alpha(\delta_n, y_n)} y_n\| + \|R_{\alpha(\delta_n, y_n)} y_n - T^+ y\| \leq 2\varphi(\delta_n) \overset{n\to\infty}{\to} 0$$

This means that $T^+ y_n \to T^+ y$ and hence that T^+ is continuous on $\mathcal{D}(T^+) \cap B_1(0_Y)$, implying the continuity on $\mathcal{D}(T)^+$. $\blacksquare$

Remark 5.17 Various error estimates can be determined, using for instance the modulus of continuity, for which we refer to the exercises. $\Diamond$

6 Spectral Regularization

6.1 General Approach

Remark 6.1 For compact operators $K \in \mathcal{K}(X, Y)$, regularizations can be constructed using the singular value decomposition for K. For this, recall that $K^+ y = (K^*K)^+ K^* y$ for $y \in \mathcal{D}(K^+)$. Let $((\sigma_n, u_n, v_n))_n$ be a singular system for K. By construction, $((\sigma_n^2, v_n, v_n))_n$ is a singular system for K^*K. By the singular value decomposition, we can write (using $Kv_n = \sigma_n u_n$)

$$\begin{aligned}(K^*K)^+ K^* y &= \sum_{n\in\mathbb{N}} \sigma_n^{-2} \langle K^* y, v_n\rangle v_n \\ &= \sum_{n\in\mathbb{N}} \sigma_n^{-2} \langle y, Kv_n\rangle v_n \\ &= \sum_{n\in\mathbb{N}} \sigma_n^{-2} \sigma_n \langle y, u_n\rangle v_n \\ &= \sum_{n\in\mathbb{N}} \varphi(\sigma_n^2) \sigma_n \langle y, u_n\rangle v_n\end{aligned}$$

where $\lambda \overset{\varphi}{\mapsto} \lambda^{-1}$. The discontinuity of K^+ is related to the fact that φ is unbounded on $(0, \|K^*K\|]$, since $\sigma_n \to 0$. To regularize, we replace φ by a family $(\varphi_\alpha)_{\alpha>0}$ of bounded functions, converging pointwise to φ. In the sequel, let $\kappa = \|K\|^2 = \|K^*K\|^2 = \sigma_1^2$. $\Diamond$

Definition 6.2 Let $(\varphi_\alpha)_{\alpha>0}$ be a family of piecewise continuous and bounded functions $\varphi_\alpha : (0, \kappa] \to \mathbb{R}$. If $(\varphi_\alpha)_{\alpha>0}$ satisfies

(1) $\lim_{\alpha\to 0} \varphi_\alpha(\lambda) = \varphi(\lambda)$ $(= \lambda^{-1})$ for all $\lambda \in (0, \kappa]$,

(2) $\lambda|\varphi_\alpha| \le C_\varphi$ for some constant $C_\varphi > 0$, $\lambda \in (0, \kappa]$, $\alpha > 0$,

then the family is called (regularizing) filter.

Lemma 6.3 *Let $(\varphi_\alpha)_{\alpha>0}$ be a regularizing filter. Then, for all $\alpha > 0$*

$$\|K\varphi_\alpha(K^*K)K^*\| \le \sup_{n\in\mathbb{N}} |\varphi_\alpha(\sigma_n^2)|\sigma_n^2 \le C_\varphi$$

Proof. For all $y \in Y$, $\alpha > 0$, we have

$$\begin{aligned} K\varphi_\alpha(K^*K)K^*y &= K[\sum_{n\in\mathbb{N}} \varphi_\alpha(\sigma_n^2)\langle K^*y, v_n\rangle v_n] \\ &= K[\sum_{n\in\mathbb{N}} \varphi_\alpha(\sigma_n^2)\sigma_n\langle y, u_n\rangle v_n] \\ &= \sum_{n\in\mathbb{N}} \varphi_\alpha(\sigma_n^2)\sigma_n\langle y, u_n\rangle Kv_n \\ &= \sum_{n\in\mathbb{N}} \varphi_\alpha(\sigma_n^2)\sigma_n^2\langle y, u_n\rangle v_n \end{aligned}$$

and hence

$$\begin{aligned} \|K\varphi_\alpha(K^*K)K^*y\|^2 &= \|\sum_{n\in\mathbb{N}} \varphi_\alpha(\sigma_n^2)\sigma_n^2\langle y, u_n\rangle u_n\|^2 \\ &\le \sup_{n\in\mathbb{N}}(|\varphi_\alpha(\sigma_n^2)|\sigma_n^2)^2 \sum_{n\in\mathbb{N}} |\langle y, u_n\rangle|^2 \\ &\overset{\text{Bessel}}{\le} (\sup_{n\in\mathbb{N}}(|\varphi_\alpha(\sigma_n^2)|\sigma_n^2)^2)\|y\|^2 \end{aligned}$$

Taking square roots yields the claim using $0 < \sigma_n^2 \le \sigma_1^2 = \|K^*K\| = \kappa$ and the fact that $(\varphi_\alpha)_{\alpha>0}$ is a regularizing filter. ∎

Lemma 6.4 *Let $(\varphi_\alpha)_{\alpha>0}$ be a regularizing filter. Then, for all $\alpha > 0$,*

$$\|\varphi_\alpha(K^*K)K^*\| \le \sqrt{C_\varphi} \cdot \sup_{\lambda\in(0,\kappa]} \sqrt{|\varphi_\alpha(\lambda)|}.$$

In particular, $\|\varphi_\alpha(K^*K)K^*\| < \infty$.

Proof. For all $y \in Y$ and $\alpha > 0$ we have by the proof of Lemma 6.3,

$$\begin{aligned} \|\varphi_\alpha(K^*K)K^*y\|^2 &= \langle\varphi_\alpha(K^*K)K^*y, \varphi_\alpha(K^*K)K^*y\rangle \\ &= \sum_n \varphi_\alpha(\sigma_n^2)\sigma_n\langle y, u_n\rangle\langle\varphi_\alpha(K^*K)K^*y, v_n\rangle \\ &= \sum_n \varphi_\alpha(\sigma_n^2)\langle y, u_n\rangle\langle K\varphi_\alpha(K^*K)K^*y, v_n\rangle \\ &\le \sup |\varphi_\alpha(\sigma_n^2)||\langle K\varphi_\alpha(K^*K)K^*y, \sum_n\langle y, u_n\rangle u_n\rangle \\ &\le \sup |\varphi_\alpha(\sigma_n^2)|\|K\varphi_\alpha(K^*K)K^*y)\|\|P_{\overline{\mathcal{R}(K^*)}}y\| \\ &\le \sup_n |\varphi_\alpha(\sigma_n^2)|C_\varphi\|y\|^2. \end{aligned}$$

The claim now follows from the boundedness of the φ_α. ∎

Theorem 6.5 *Let $(\varphi_\alpha)_{\alpha>0}$ be a regularizing filter. Then the following hold.*

(i) For all $y \in \mathcal{D}(K^+)$ we have $\lim_{\alpha\to 0} \varphi_\alpha(K^*K)K^*y = K^+y$.

(ii) For $y \notin \mathcal{D}(K^+)$, we have that $\lim_{\alpha\to 0}\|\varphi_\alpha(K^*K)K^*y\| = \infty$, if K^+ is discontinuous.

Proof. (i) Let $y \in \mathcal{D}(K^+)$. Set $x^+ = K^+y$, $x_\alpha = \varphi_\alpha(K^*K)K^*y$. Since we have $K^*Kx^+ = K^*y$, there holds

$$x_\alpha = \varphi_\alpha(K^*K)K^*Kx^+$$

Now, let $r_\alpha(\lambda) = 1 - \lambda\varphi_\alpha(\lambda)$. Then we have

$$x^+ - x_\alpha = (\mathrm{Id} - \varphi_\alpha(K^*K)K^*K)x^+ = r_\alpha(K^*K)x^+ = \sum_{n\in\mathbb{N}} r_\alpha(\sigma_n^2)\langle x^+, v_n\rangle v_n.$$

Hence,

$$\|x^+ - x_\alpha\|^2 \overset{(*)}{=} \sum_{n\in\mathbb{N}} r_\alpha(\sigma_n^2)^2|\langle x^+, v_n\rangle|^2 \tag{6.1}$$

Since $(\varphi_\alpha)_{\alpha>0}$ is a regularizing filter, we have $\lim_{\alpha\to 0} r_\alpha(\lambda) = 0$ for all $\lambda \in (0, \kappa]$, $|r_\alpha(\lambda)| \le 1 + C_\varphi$ for all $\alpha > 0$, $\lambda \in (0,\kappa]$ and $\alpha > 0$. By the Bessel-inequality, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ with $\sum_{n=N+1}^\infty |\langle x^+, v_n\rangle|^2 < \frac{\varepsilon^2}{2(1+C_\varphi)^2}$. By the pointwise convergence of $(r_\alpha)_{\alpha>0}$, there exists some $\alpha_0 > 0$ with $r_\alpha(\sigma_n^2)^2 < \frac{\varepsilon^2}{2\|x^+\|^2}$ for all $n \le N$, $\alpha < \alpha_0$. This implies, for $\alpha < \alpha_0$ by Equation (6.1), that

$$\begin{aligned}\|x^+ - x_\alpha\|^2 &= \sum_{n=1}^N r_\alpha(\sigma_n^2)^2|\langle x^+, v_n\rangle|^2 + \sum_{n=N+1}^\infty r_\alpha(\sigma_n^2)^2|\langle x^+, v_n\rangle|^2 \\ &\le \frac{\varepsilon^2}{2\|x^+\|^2}\sum_{n=1}^N |\langle x^+, v_n\rangle|^2 + (1 + C_\varphi)^2\frac{\varepsilon^2}{2(1 + C_\varphi)^2} = \varepsilon^2\end{aligned}$$

This implies that $\|x^+ - x_\alpha\| \overset{\alpha\to 0}{\to} 0$.

(ii) If $y \notin \mathcal{D}(K^+)$, then the claim follows from Theorem 5.5 and Lemma 6.3.

∎

Example 6.6 (1) Truncated singular value decomposition

Choose

$$\varphi_\alpha(\lambda) = \begin{cases} \frac{1}{\lambda} & \lambda \ge \alpha \\ 0 & \lambda < \alpha \end{cases}.$$

This turns $(\varphi_\alpha)_{\alpha>0}$ into a regularizing filter, since $|\varphi_\alpha(\lambda)| \le \frac{1}{\alpha}$ and φ_α is piecewise continuous for $\alpha > 0$. Also, $\lim_{\alpha\to 0} \varphi_\alpha(\lambda) = \frac{1}{\lambda}$ for all $\lambda > 0$ and $\lambda|\varphi_\alpha(\lambda)| \le 1$. The associated regularization operator is given by

$$\varphi_\alpha(K^*K)K^*y = \sum_{n\in\mathbb{N}} \varphi_\alpha(\sigma_n^2)\sigma_n\langle y, u_n\rangle v_n = \sum_{n:\,\sigma_n\ge\sqrt{\alpha}} \frac{1}{\sigma_n}\langle y, u_n\rangle v_n$$

(2) The Tychonov regularization. Choose

$$\varphi_\alpha(\lambda) = \frac{1}{\lambda + \alpha}.$$

Then, again, $(\varphi_\alpha)_{\alpha>0}$ is a regularizing filter, since $|\varphi_\alpha(\lambda)| \leq \frac{1}{\alpha}$, φ_α is continuous for $\alpha > 0$, $\lim_{\alpha\to 0} \varphi_\alpha(\lambda) = \frac{1}{\lambda}$ and $\lambda|\varphi_\alpha(\lambda)| \leq 1$. The associated regularization operator is given by

$$\varphi_\alpha(K^*K)K^*y = \sum_{n\in\mathbb{N}} \frac{\sigma_n}{\sigma_n^2 + \alpha} \langle y, u_n\rangle v_n.$$

We will see, that $x_\alpha = \varphi_\alpha(K^*K)y$ can be computed without knowledge of a singular system of K. This is one of the main reasons why the Tychonov regularization is so popular in practical applications.

◊

We now fix the regularizing filter φ_α, given by the Tikhonov regularization. The following holds.

Theorem 6.7 *For $y \in Y$, $\alpha > 0$, $x_\alpha = R_\alpha y = \varphi_\alpha(K^*K)K^*y$ is uniquely determined as the solution of the equation*

$$(K^*K + \alpha\mathrm{Id})x = K^*y$$

Proof. Let $(\sigma_n, u_n, v_n)_{n\in\mathbb{N}}$ be a singular system for K. There holds that

$$\alpha x_\alpha = \sum_{n\in\mathbb{N}} \frac{\alpha\sigma_n}{\sigma_n^2 + \alpha} \langle y, u_n\rangle v_n$$

and

$$K^*Kx_\alpha = \sum_{n\in\mathbb{N}} \frac{\sigma_n}{\sigma_n^2 + \alpha} \langle y, u_n\rangle K^*Kv_n = \sum_{n\in\mathbb{N}} \frac{\sigma_n^3}{\sigma_n^2 + \alpha} \langle y, u_n\rangle v_n$$

Hence,

$$(K^*K + \alpha \cdot \mathrm{Id})\ x_\alpha = \sum_{n\in\mathbb{N}} \sigma_n \langle y, u_n\rangle v_n = K^*y.$$

The uniqueness of the solution follows from the fact that $K^*K + \alpha\mathrm{Id}$ is injective for $\alpha > 0$, since

$$\langle K^*Kx + \alpha\mathrm{Id}\ x, x\rangle \geq \alpha\langle x, x\rangle + \langle Kx, Kx\rangle \geq \alpha\|x\|^2$$

∎

Theorem 6.8 *Let $y \in Y$. Then $x_\alpha = R_\alpha y$ (R_α is the Tikhonov regularization) is the unique minimizer of the* Tikhonov functional

$$J_\alpha(x) = \frac{1}{2}\|Kx - y\|^2 + \frac{\alpha}{2}\|x\|^2.$$

Proof. This proof uses $(K^*K + \alpha \mathrm{Id})x_\alpha = K^*y$ which has been proven before. Let $x \in X$ be arbitrary. Then, we have

$$\begin{aligned} 2(J_\alpha(x) - J_\alpha(x_\alpha)) &= \langle Kx - y, Kx - y\rangle + \alpha\langle x, x\rangle - \langle Kx_\alpha - y, Kx_\alpha - y\rangle - \alpha\langle x_\alpha, x_\alpha\rangle \\ &= \|Kx - Kx_\alpha\|^2 + \alpha\|x - x_\alpha\|^2 + \langle K^*(Kx_\alpha - y) + \alpha x_\alpha, x - x_\alpha\rangle \\ &= \|Kx - Kx_\alpha\|^2 + \|x - x_\alpha\|^2 \geq 0, \end{aligned}$$

and hence x_α is minimal. x_α is also the only minimizer, since for any other minimizer $\bar{x}$

$$0 = 2(J_\alpha(\bar{x}) - J_\alpha(x_\alpha)) \geq \|\bar{x} - x_\alpha\|^2 \geq 0$$

with equality holding if and only if $x_\alpha = \bar{x}$. ■

Remark 6.9 The Tihonov functional $J_\alpha(x) = \frac{1}{2}\|Kx - y\|^2 + \frac{\alpha}{2}\|x\|^2$ can be interpreted as the sum of the so-called *residual* or *data fidelity term* $\frac{1}{2}\|Kx - y\|^2$ (measuring how well x "relates" to y) and the *penalty* or *regularization term* $\frac{\alpha}{2}\|x\|^2$. α is the regularization parameter which balances the terms. The smaller the noise δ is, the more emphasis can be put on minimizing the residual term, hence the smaller α can be chosen. A too large α puts more emphasis on minimizing $\|x\|^2$ without appropriately minimizing the residual term. ♢

Remark 6.10 (1) There exist extensions of the Tichonov functional which are more data adapted. The generalization could be of the form $\tilde{J}_\alpha(x) = \|Kx - y\|^2 + \alpha P(x)$ where is a penalty term to

(a) ensure the continuous dependence on the data,

(b) incorporate properties of the solution.

(2) One such approach has been introduced in (Daubechies,Defrise,DeMol;2008):

Let $\Phi = (\varphi_i)_{i\in\mathcal{I}} \subset X$ be an orthonormal basis for X and $w = (w_i)_{i\in\mathcal{I}} \subset \mathbb{R}_{>0}$ be a sequence of weights. Then consider the functional

$$\tilde{J}_{\alpha,\Phi,w}(x) = \|Kx - y\|^2 + \alpha \sum_{i\in\mathcal{I}} w_i |\langle x, \varphi_i\rangle|^p$$

for $1 \leq p \leq 2$. Notice that α is not really necessary here since w replaces the role of α. Assume that the solution x_w satisfies that $\sum_{i\in\mathcal{I}} w_i|\langle x, \varphi_i\rangle|^p$ is small, for instance if $(\langle x, \varphi_i\rangle)_{i\in\mathcal{I}}$ have rapid decay. Then, for instance, choose $w_i = 1$ for all $i \in \mathcal{I}$ and $p = 1$.

(3) An extension is the notion of *frames* which generalizes the concept of orthonormal bases. For this, see Functional Analysis 3.

♢

6.2 Landweber Regularization

Recall that for $y \in \mathcal{D}(K^+)$ (for $K \in \mathcal{K}(X,Y)$), $x \in X$ is a least-square solution if and only if $K^*Kx = K^*y$. If additionally $x \in \mathcal{N}(K)^\perp$, then $x = x^+$. This can be reformulated as a fixpoint equation

$$x = x - \omega(K^*Kx - K^*y)$$

for any $\omega > 0$. The associated fixed point iteration, called *Richardson iteration*, is given by

$$x_{n+1} = x_n + \omega(K^*y - K^*Kx_n)$$

for $x_0 \in X$, e.g. $x_0 = 0$. The Banach fixed point theorem implies that $(x_n)_n$ converges to a solution of the normal equation if $\|\mathrm{Id} - \omega K^*K\| < 1$ (and $y \in \mathcal{R}(K)$). If $x_0 \in \mathcal{R}(K^*) \subseteq (\mathcal{N}(K))^\perp$, then $x_n \to x^+$. But: If $y^\delta \notin \mathcal{R}(K)$, then convergence does not need to hold. The idea is then to stop after a certain number of iterations.

Lemma 6.11 *For $m \in \mathbb{N}$, we have, when choosing $x_0 = 0$,*

$$x_m = \omega \sum_{n=0}^{m-1} (\mathrm{Id} - \omega K^*K)^n K^*y$$

Proof. To show the statement, we use induction over m. The case for $m = 1$ is clear since $x_1 = \omega K^*y$. Now assume that the statement is true for $m \in \mathbb{N}$, then

$$\begin{aligned} x_{m+1} &= x_m + \omega K^*(y - Kx_m) \\ &= (\mathrm{Id} - K^*K)x_m + \omega K^*y = (\mathrm{Id} - \omega K^*K)\omega \sum_{n=0}^{m-1} (\mathrm{Id} - \omega K^*K)^n K^*y + \omega K^*y \\ &= \omega \sum_{n=0}^{m-1} (\mathrm{Id} - \omega K^*K)^{n+1} K^*y + \omega K^*y = \omega \sum_{n=0}^{m} (\mathrm{Id} - \omega K^*K)^n K^*y \end{aligned}$$

which is what we wanted to show. ∎

Theorem 6.12 *Let $\omega \in (0, 2\kappa^{-1})$. Then we have*

$$x_m = \varphi_m(K^*K)K^*y$$

with $\varphi_m(\lambda) = \frac{1-(1-\omega\lambda)^m}{\lambda}$. Moreover, this determines a regularization $(R_m)_{m\in\mathbb{N}}$ with $R_m := \varphi_m(K^*K)K^*$.

Proof. Observe that

$$\varphi_m(\lambda) = \frac{1-(1-\omega\lambda)^m}{\lambda} = \omega\frac{1-(1-\omega\lambda)^m}{1-(1-\omega\lambda)} = \omega\sum_{n=0}^{m-1}(1-\omega\lambda)^n$$

But this implies that

$$\varphi_m(K^*K)K^*y = \left(\omega\sum_{n=0}^{m-1}(\mathrm{Id}-\omega K^*K)^n\right)K^*y \overset{6.11}{=} x_m$$

Now notice that $\omega \in (0, 2\kappa^{-1})$ implies that

$$-1 < (1-\omega\lambda) < 1 \text{ for all } \lambda \in (0,\kappa]$$

Thus $(1-\omega\lambda)^m \overset{m\to\infty}{\to} 0$, hence $\varphi_m(\lambda) = \frac{1-(1-\omega\lambda)^m}{\lambda} \to \frac{1}{\lambda}$. Also,

$$\lambda|\varphi_m(\lambda)| = |1-(1-\omega\lambda)^m| \leq 2$$

for all $m \in \mathbb{N}$ and $\lambda \in (0,\kappa]$. Thus, $(\varphi_m)_m$ is a regularizing filter. Also, by Theorem 6.5, $(R_m)_m$ is a regularization method. ∎

7 Discretization as Regularization

Remark 7.1 Recall that for given $K \in \mathcal{K}(X,Y)$ with infinite-dimensional range $\mathcal{R}(K)$, the pseudoinverse K^+ is discontinuous. The idea we will use in this chapter is to construct a sequence $(K_n)_n \subset \mathcal{K}(X,Y)$ with $\dim(\mathcal{R}(K)) < \infty$ and such that the (continuous) pseudoinverses $(K_n^+)_n$ approximate K^+y. We will consider the following two approaches:

(1) Least-squares projection: Let $X_n \subseteq X$ be finite-dimensional subspaces and consider $K_n := K|_{X_n}$.

(2) Dual least-squares projection: Let $Y_n \subseteq Y$ be finite-dimensional subspaces and consider $K_n : X \to Y_n$ (for example by $K_n := P_{Y_n}K$).

$\Diamond$

7.1 Least-squares projection

Definition 7.2 Let $T \in L(X,Y)$ and $(X_n)_n$, $X_n \subseteq X$ be subspaces with $X_1 \subseteq X_2 \subseteq ...$, such that $\dim(X_n) = n$ and $\overline{\bigcup_n X_n} = X$. Then set $P_n := P_{X_n}$ (which is the orthogonal projection onto X_n), $T_n := TP_n \in L(X,Y)$. Then $\mathcal{R}(T_n)$ is finite-dimensional, hence T_n^+ is continuous. Further, for some $y \in Y$, we set $x_n := T_n^+ y$, which is the minimum-norm solution of $TP_n x = T_n x = y$. This procedure is called *least-squares projection*.

Lemma 7.3 *Let $(x_n)_n$ be defined as previously. Then the following are equivalent:*

(i) $x_n \xrightarrow{w} x^+$.

(ii) $(\|x_n\|)_n$ *is bounded.*

Proof. (i) $\Rightarrow$ (ii):

This follows immediately from the fact that weakly convergent sequences are bounded.

(ii) $\Rightarrow$ (i):

Since $(\|x_n\|)_n$ is bounded, there exists a weakly convergent subsequence $(x_{n_k})_k$ and a weak limit $\bar{x} \in X$ with $x_{n_k} \xrightarrow{w} \bar{x}$ and $Tx_{n_k} \xrightarrow{w} T\bar{x}$. We write $x_k := x_{n_k}$ as an abbreviation from now on. By definition 7.2, for all $x \in X$, since the x_k are the minimal-norm solutions of $T_k x_k = y$

$$\|T_k x_k - y\| \le \|T_k x - y\| \tag{7.1}$$

Hence, since $x_k \in X_k$,

$$\begin{aligned}\|Tx_k - Tx^+\| = \|T_k x_k - Tx^+\| &\overset{(7.1)}{\le} \|T_k x^+ - Tx^+\| \\ &= \|TP_k x^+ - Tx^+\| \\ &\le \|T\|\|P_k x^+ - x^+\| = \|T\|\|(P_k - \mathrm{Id})x^+\|\end{aligned}$$

Since $X_1 \subseteq X_2 \subseteq ...$ and $\overline{\bigcup_n X_n} = X$, it follows that $\|(P_k - \mathrm{Id})x^+\| \to 0$. Thus, by the previous calculation, we conclude that $Tx_k \to Tx^+$. This implies that $x^+ - \bar{x} \in \mathcal{N}(T)$. By Theorem 4.13, $x^+ \in \mathcal{N}(T)^\perp$. We now claim that $\bar{x} \in \mathcal{N}(T)^\perp$ as well. This can be seen as follows:

First, observe that

$$\mathcal{N}(T_k)^\perp = \mathcal{N}(TP_k)^\perp = X_k \cap (\mathcal{N}(T) \cap X_k)^\perp,$$

since $\mathcal{N}(T_k)^\perp = \overline{\mathcal{R}(T_k^+)} = \overline{\mathcal{R}(P_k T^+)} \subseteq X_k$. We have that

$$(\mathcal{N}(T) \cap X_k)^\perp \supseteq (\mathcal{N}(T) \cap X_{k+1})^\perp \supseteq ... \supseteq \mathcal{N}(T)^\perp.$$

Since orthogonal complements are closed, this implies that

$$\bar{x} \in (\mathcal{N}(T) \cap X_m)^\perp \tag{7.2}$$

for all m. We know that $\overline{\bigcup_n X_n} = X$, hence $\overline{\bigcup_m \mathcal{N}(T) \cap X_m} = \mathcal{N}(T)$. Then (7.2) implies that $\bar{x} \in \mathcal{N}(T)^\perp$. This proves the claim. To conclude, we showed that $\bar{x} - x^+ \in \mathcal{N}(T) \cap \mathcal{N}(T)^\perp = \{0\}$, hence $\bar{x} = x^+$. Since the limit is unique, we proved (i). ■

Corollary 7.4 *Using the same notation and hypothesis as before, the following are equivalent:*

(i) $x_n \to x^+$.

(ii) $\limsup_{n\to\infty} \|x_n\| \le \|x^+\|$.

Proof. (i) ⇒ (ii): We have

$$\|x_n\| \le \|x_n - x^+\| + \|x^+\| \to \|x^+\|.$$

(ii) ⇒ (i):

(ii) implies the boundedness of $(\|x_n\|)_n$. Then the previous Lemma yields $x_n \xrightarrow{w} x^+$. Thus,

$$\|x^+\|^2 = \langle x^+, x^+\rangle = \lim_{n\to\infty} \langle x_n, x^+\rangle \le \liminf_{n\to\infty} \|x_n\|\|x^+\|$$

$$\Rightarrow \|x^+\| \leq \liminf_{n\to\infty} \|x_n\| \leq \limsup_{n\to\infty} \|x_n\| \overset{(ii)}{\leq} \|x^+\|$$

Hence, $\|x_n\| \to \|x^+\|$. Using $x_n \xrightarrow{w} x^+$ implies $x_n \to x^+$. ■

Theorem 7.5 *Let $y \in \mathcal{D}(T^+)$ and*

$$\limsup_{n\to\infty} \|(T_n^*)^+ x_n\| = \limsup_{n\to\infty} \|(T_n^+)^* x_n\| < \infty.$$

Then $x_n \to x^+$.

Proof. First, note that

$$\|x_n\|^2 = \langle x_n - x^+, x_n\rangle + \langle x^+, x_n\rangle \leq |\langle x_n - x^+, x_n\rangle| + \|x^+\|\|x_n\|$$

By Corollary 7.4 it remains to prove that $|\langle x_n - x^+, x_n\rangle| \to 0$. Then $x_n \to x^+$. For this, set $w_n := (T_n^+)^* x_n$. We know that $T_n^* w_n = x_n$. Next, note that

$$|\langle x_n - x_+, x_n\rangle| = |\langle x_n - x^+, T_n^* w_n\rangle| = |\langle T_n x_n - Tx^+, w_n\rangle + \langle Tx^+ - T_n x^+, w_n\rangle|$$
$$\leq \|w_n\|(\|T_n x_n - Tx^+\| + \|T(\mathrm{Id} - P_n)x^+\|) \overset{\text{Proof Lemma 7.3}}{\leq} \|w_n\| 2\|T\|\|(\mathrm{Id} - P_n)x^+\|$$

By the hypothesis, $\limsup_{n\to\infty} \|w_n\| < \infty$, hence $\|w_n\| \leq C$ for all $n \in \mathbb{N}$ and some $C > 0$.This implies that

$$|\langle x_n - x^+, x_n\rangle| \leq 2C\|T\|\|(\mathrm{Id} - P_n)x^+\|$$

By choice of $(X_n)_n$ and thus $(P_n)_n$, we have $\|(\mathrm{Id} - P_n)x^+\| \to 0$. This proves $(*)$. ■

Theorem 7.6 *Let $K \in \mathcal{K}(X, Y)$ and again*

$$\limsup_{n\to\infty} \|(K_n^*)^+ x_n\| = \limsup_{n\to\infty} \|(K_n^+)^* x_n\| < \infty.$$

Then $x^+ \in \mathcal{R}(K^*)$.

Proof. Again, set $w_n := (K_n^+)^* x_n$. By the hypothesis, $(\|w_{n\|})_n$ is bounded and hence there exists a weakly convergent subsequence $w_k \xrightarrow{w} \bar{w}$. Since K and K^* are compact, $K^* w_k \to K^* \bar{w}$. Next, observe that $(K_n^+)^* = (K_n^*)^+ = (P_n K^*)^+$. This implies that

$$K^* w_k = P_k K^* w_k + (\mathrm{Id} - P_k) K^* w_k$$

and using that $K^* w_k \to K^* \bar{w}$, $P_k K^* w_k = x_k \overset{\text{Theorem 7.5}}{\to} x^+$ and $\|(\mathrm{Id} - P_k) K^* w_k\| \to 0$, we obtain $K^* \bar{w} = x^+$ which implies $x^+ \in \mathcal{R}(K^*)$. ■

Theorem 7.7 *Let $K \in \mathcal{K}(X, Y)$ and yet again*

$$\limsup_{n\to\infty} \|(K_n^*)^+ x_n\| = \limsup_{n\to\infty} \|(K_n^+)^* x_n\| < \infty.$$

Then there exists a constant $C > 0$ with

$$\|x_n - x^+\| \leq C\|(\mathrm{Id} - P_n)K^*\|$$

Proof. By the previous theorem, there exists $w \in Y$ with $K^* w = x^+$. Similarly to the proof of Theorem 7.5,

$$|\langle x_n - x_+, x^+\rangle| \leq \|K(\mathrm{Id} - P_n)x^+\|\|w\|$$

This implies, with w_n defined as before, that

$$\begin{aligned}
\|x_n - x^+\|^2 &= \langle x_n - x_+, x_n\rangle - \langle x_n - x^+, x^+\rangle \\
&\overset{\text{Theorem 7.5}}{\leq} \|K(\mathrm{Id} - P_n)x^+\|\|w_n\| + \|K(\mathrm{Id} - P_n)x^+\|\|w\| \leq C\|K(\mathrm{Id} - P_n)x^+\| \\
&= C\|K(\mathrm{Id} - P_n)(\mathrm{Id} - P_n)K^* w\| \leq C\|(Id - P_n)K^*\|^2\|w\|^2 \leq C'\|(\mathrm{Id} - P_n)K^*\|^2.
\end{aligned}$$

■

7.2 Dual Least-Squares Projection

Definition 7.8 Let $T \in L(X, Y)$ and $Y_n \subseteq Y$ be subspaces with $\dim Y_n = n$, $Y_1 \subseteq Y_2 \subseteq ... \subseteq \overline{\mathcal{R}(T)} = \mathcal{N}(T^*)^\perp \subseteq Y$ and $\overline{\cup_n Y_n} = \mathcal{N}(T^*)^\perp$. Further, let $Q_n := P_{Y_n}$ be the orthogonal projection onto Y_n and $T_n := Q_n T \in L(X, Y_n)$. Then T_n^*, T_n^+ and $T_n^+ Q_n$ are continuous and $x_n = T_n^+ Q_n y$ is the minimum norm solution of $Q_n T x = Q_n y$. This procedure is called *dual least-squares projection*.

Remark 7.9 The question is under which circumstances $x_n \to x^+$. ◊

Lemma 7.10 *Set $X_n := T^* Y_n$ and $P_n := P_{X_n}$. Let $y \in \mathcal{D}(T^+)$. Then $x_n = P_n x^+$.*

Proof. By definition of T_n, we have

$$\mathcal{R}(T_n^+) = \mathcal{N}(T_n)^\perp = \mathcal{R}(T_n^*) = \mathcal{R}(T^* Q_n) = T^* Y_n = X_n.$$

This shows that $x_n \in X_n$ and $X_n^\perp = \mathcal{N}(T_n)$. Moreoever, $T_n(\mathrm{Id} - P_n) = 0$, thus

$$T_n = T_n P_n. \tag{7.3}$$

Also,

$$Q_n y = Q_n P_{\overline{\mathcal{R}(T)}} y = Q_n T T^+ y = Q_n T x^+ = T_n x^+ \tag{7.4}$$

Now

$$\|T_n x - Q_n y\| \stackrel{(7.4)}{=} \|T_n x - T_n x^+\| \stackrel{(7.3)}{=} \|T_n x - T_n P_n x^+\| \stackrel{(7.3)}{=} \|T_n P_n (x - P_n x^+)\|$$

and x_n is the minimum-norm solution of $T_n x = Q_n y$. That means, since $P_n x^+$ minimizes the expression and by the uniqueness of the minimum-norm solution, that $x_n = P_n x^+$. ■

Theorem 7.11 *Let X_n, P_n be defined as before and let $y \in \mathcal{D}(T^+)$. Then $x_n \to x_+$.*

Proof. By choice of $(Y_n)_n$, we have $X_1 \subseteq X_2 \subseteq ...$ and $\overline{\cup_n X_n} = \overline{\cup_n T^* Y_n} = T^* \overline{\cup_n Y_n} = T^* \mathcal{N}(T^*)^\perp = \mathcal{N}(T)^\perp$. We have $x^+ \in \mathcal{R}(T^+) = \mathcal{N}(T)^\perp$. Hence, by the previous Lemma, we conclude that $x_n = P_n x^+ \to x^+$. ■

References

[1] John B. Conway, *A course in functional analysis*, Springer, 1990.

[2] Javier Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics, vol. 29, American Mathematical Society, Providence, RI, 2001, Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

[3] Loukas Grafakos, *Classical fourier analysis*, Springer, 2014 (Third Edition).

[4] Walter Rudin, *Functional analysis*, McGraw-Hill, 1991.

[5] Dirk Werner, *Funktionalanalysis*, Springer, 2011.