Research Group
Applied Functional Analysis
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Functional Analysis I

Tutorial Assignment 12

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Exercise 1: Prove that the shift operator $T : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$, defined by

$$T(x_1, x_2, \dots) := (x_2, x_3, \dots),$$

is a Fredholm operator and compute its index. What changes for the modified operator $\tilde{T} : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$, $\tilde{T}(x_1, x_2, \dots) := (0, x_2, x_3, \dots)$?

Exercise 2: Let E be a Banach space, and let $T \in L(E)$ be a Fredholm operator. Prove that also T^* is a Fredholm operator with $\mathrm{ind}(T^*) = -\mathrm{ind}(T)$.

In Homework 12.4 you need to show, that every non-zero spectral value of a compact operator in a Banach space is already an eigenvalue. This you can use throughout the following tasks.

Exercise 3: We know from the previous tutorial that the operator $T : \ell_2 \to \ell_2$, defined by $T((x_n)_{n\in\mathbb{N}}) := (\frac{1}{n}x_n)_{n\in\mathbb{N}}$, is compact. Compute the spectrum $\sigma(T)$ of T. Moreover, define the operator $S : \ell_2 \to \ell_2$ by $(Sx)_1 := 0$ and $(Sx)_n := n^{-1}x_{n-1}$ for $n \geq 2$. Show that S is compact with $\sigma(S) = \{0\}$.

Exercise 4: Let E be a Banach space and $T \in L(E)$. Show that

$$\lambda \in \sigma(T) \Leftrightarrow \lambda \in \sigma(T^*)$$

where T^* denotes the dual operator of T.

Exercise 5: Let $K \subset \mathbb{C}$ be non-empty and compact. Show that then there exists an operator T whose spectrum is K.