Research Group
Applied Functional Analysis
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# Functional Analysis I

Tutorial Assignment 8

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**Exercise 1:** Let $X = \{1, 2, 3, 4\}$. Which of the following systems $\mathcal{T}_i$, $i = 1, 2, 3$, define a topology on $X$,

(i) $\mathcal{T}_1 = \{\{1, 2, 3\}, \{1\}, \{2\}, \{3\}\}$,

(ii) $\mathcal{T}_2 = \{\{\varnothing, \{1\}, X\}$,

(iii) $\mathcal{T}_3 = \{\varnothing, X, \{1, 2, 3\}, \{1\}, \{2\}, \{3\}\}$?

In case $(X, \mathcal{T}_i)$ is a topological space, find the corresponding closed sets.

**Exercise 2:** Prove Lemma 6.18:
Let $X \neq \varnothing$, and let $\gamma$ be a set of subsets of $X$, i.e., $\gamma \subset \mathcal{P}(X)$. By $\mathcal{L}$ denote the set consisting of all finite intersections of sets in $\gamma$, the empty set $\varnothing$, and $X$ itself. Further, let $\mathcal{T}$ be the set which consists of all unions of sets in $\mathcal{L}$. Then $\mathcal{T}$ is a topology on $X$, and $\mathcal{L}$ is a basis for $\mathcal{T}$.

**Exercise 3:** Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The closure $\overline{A}$ of $A$ is defined as the intersection of all closed sets containing $A$. Prove that $\overline{A}$ is indeed closed and that $x \in \overline{A}$ if and only if for all $U \in \mathcal{T}$ with $x \in U$ we have that $U \cap A \neq \varnothing$.

**Exercise 4:** Let $(X_1, \mathcal{T}_1)$, $(X_2, \mathcal{T}_2)$ be topological spaces and $\gamma$ be a subbasis for $\mathcal{T}_2$. Show that a mapping $f : X_1 \to X_2$ is continuous if and only if $f^{-1}(V) \in \mathcal{T}_1$ for all $V \in \gamma$.

**Exercise 5:** Let $E$ be a normed space and let $\mathcal{T}$ be the topology of $E$. Show that $\sigma(E, E^*) \subset \mathcal{T}$.

**Exercise 6:** Let $E$ be a normed space. Show that if $E$ is reflexive, then $\sigma(E^*, E) = \sigma(E^*, E^{**})$.

**Exercise 7:** Prove, that sequences in Hausdorff spaces can have at most one limit.

**Exercise 8:** Prove that every compact subset $C \subset X$ of a Hausdorff space $(X, \mathcal{T})$ is closed.

**Exercise 9:** A topological space $(X, \mathcal{T})$ is called *Hausdorff space* if for each $x, y \in X$, $x \neq y$, there exist neighborhoods $U_x, U_y \in \mathcal{T}$ of $x$ and $y$, respectively, such that $U_x \cap U_y = \varnothing$. Let $E$ be a normed space. Prove that $(E, \sigma(E, E^*))$ and $(E^*, \sigma(E^*, E))$ are Hausdorff spaces.