Functional Analysis II - Exercise 02

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Orthogonal Projections

Definition 0.1

Let E be a vector space. A mapping $P : E \to E$ is called a **projection** if $P^2 = P$.

From now on we only consider *linear* projections.

Definition 0.2

Let $Y \subset E$ be a subspace. A mapping $P : E \to E$ is called a **projection onto** Y if $P^2 = P$ and $\mathrm{ran}(P) = Y$.

Proposition

1. P is a projection onto Y if and only if
$$\forall x \in E : Px \in Y \quad \text{and} \quad P|_Y = \mathrm{Id}|_Y.$$
2. If P is a projection, then $E = \ker P \dotplus \operatorname{ran} P$, where $\dotplus$ denotes the direct sum.
3. If P is a projection, so is $\mathrm{Id} - P$. Additionally, $\ker \mathrm{Id} - \mathrm{P} = \operatorname{ran} P$ and $\operatorname{ran} \mathrm{Id} - P = \ker P$.
4. For every subspace $Y \subset E$ there exists a projection onto Y. The proof uses Zorn's lemma.

Now, let E be a normed space. Under which circumstances is P continuous?

Proposition

Let $P \in L(E)$ be a projection. Then,

1. $\ker P$ and $\operatorname{ran} P$ are closed
2. $||P|| \geq 1$ or $||P|| = 0$.

Proof. 1. Let $P \in L(E)$. Because of continuity we immediately get that $\ker P = P^{-1}(\{0\})$ is closed *(preimage of closed sets of continuous functions are closed)*. To show that the range of P is closed note that $\operatorname{ran} P = \ker \mathrm{Id} - P$ and $\mathrm{Id} - P \in L(E)$. We just showed that the $\ker \mathrm{Id} - P$ is closed, and so is $\operatorname{ran} P$.

2. It holds $||P|| = ||P^2|| \leq ||P||^2$. Therefore, $Px \geq 1$ or $Px = 0$. Otherwise, $||P|| \leq ||P||^2$ cannot hold.

□

There exist normed spaces E and ***closed*** subspaces $Y \subset E$ such that there is no continuous projection onto Y.

In Hilbert spaces the situation is different. We showed in Functional Analysis I that

$$E = Y \oplus Y^\perp, \quad E \text{ is Hilbert and } Y \text{ is closed.}$$

For $x = y + z$ where $y \in Y$ and $z \in Y^\perp$ define

$$P : E \to Y, x \mapsto y$$

Definition 0.3

P is called **orthogonal projection** if $\ker P \perp \operatorname{ran} P$.

Proposition

Let H be a Hilbert space. Let $P \in L(H), P \neq 0$ be a projection. Then the following are equivalent

1. P is an orthogonal projection
2. $||P|| = 1$
3. $P = P^*$
4. $P^*P = PP^*$
5. P is positive, i.e. $\langle Tx, x\rangle \geq 0$ for all $x \in H$.

Proof. • $(i) \implies (ii)$: Let P be an orthogonal projection. We want to show that $||P|| = 1$. Let $x \in H$ with $x = y + z = Py + z$ where $y \in \operatorname{ran} P$ and $x \in \ker P$. Then,

$$||Px||^2 = ||Py||^2 \leq ||Py||^2 + ||z||^2 \overset{(*)}{=} ||x||^2$$

where $(*)$ is true because $Py \perp z$. To see this, note that

$$||Py||^2 + ||z||^2 = ||Py||^2 + 2\langle Py, z\rangle + ||z||^2 = ||Py + z||^2.$$

- $(ii) \implies (i)$: Let $x \in \ker P$ and $y \in \operatorname{ran} P$. To show that P is an orthogonal projection we need to prove that $\ker P \perp \operatorname{ran} P$. For any λ we have

  $$||\lambda y||^2 = ||P(\lambda y + x)||^2 \leq ||\lambda y + x||^2 = ||x||^2 + 2\Re(\bar{\lambda}\langle x, y\rangle) + \lambda||y||^2$$

  Hence, $-2\Re(\bar{\lambda}\langle x, y\rangle) \leq ||x||^2$. Since this is true for all $\lambda \in \mathbb{C}$ we have $\langle x, y\rangle = 0$.

- $(iii) \implies (iv)$: This is clear.

- $(i) \implies (iii)$: Homework exercise.

- $(i) \implies (v)$: Let P be an orthogonal projection. We want to show that P is positive. Let $x = y + z \in H$ with $y \in \operatorname{ran} P$ and $z \in \ker P$. Then,

  $$\langle Px, x\rangle = \langle y, y + z\rangle = ||y||^2 \geq 0.$$

  Note that $\langle Px, x\rangle \in \mathbb{R}$ because of $||y||$.

- $(iii) \implies (i)$: Let P be self-adjoint. We want to show that P is an orthogonal projection. Let $y \in H$ and $z \in \ker P$. Then, $\langle Py, z\rangle = \langle y, Pz\rangle = 0$ as $Pz = 0$.

- $(v) \implies (iii)$: Let P be positive. We want to show that $P = P^*$. $P \geq 0 \implies \langle Px, x\rangle \in \mathbb{R} \implies P = P^*$ (by exactly the same homework)

- $(iv) \implies (i)$: From homework sheet 1 #1 we know

  $$T \in L(H) \text{ is normal} \iff ||Tx|| = ||T^*x|| \quad \forall x \in H.$$

  Thus, $\ker P = \ker P^*$. We proved earlier that the range of P is closed. By the closed range theorem we have

  $$\ker P^* = (\operatorname{ran} P)^\perp \implies \ker P = (\operatorname{ran} P)^\perp.$$

$\square$