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Mathematical Physics I - WS 2018/2019

http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/MP1/

Exercise Sheet 13

Exercise 1: (10 pts)

In Exercise 1, Sheet 11, we considered the following Lagrangian ($\alpha = 1$)

$$\mathscr{L}(q, \dot{q}) = -\sqrt{1 - \langle \dot{q}, \dot{q} \rangle} - \langle a, q \rangle,$$

where $(q, \dot{q}) \in \mathbb{R}^6$. The Euler-Lagrange equations are

$$\frac{\mathsf{d}}{\mathsf{d}t}\left(\frac{\dot{q}}{\sqrt{1 - \langle \dot{q}, \dot{q} \rangle}}\right) + a = 0, \tag{1}$$

and the corresponding canonical Hamilton equations are

$$\begin{cases} \dot{q} = \dfrac{p}{\sqrt{1 + \langle p, p \rangle}}, \\ \dot{p} = -a. \end{cases} \tag{2}$$

Now consider the following discretization of the Lagrangian $\mathscr{L}$:

$$\mathbf{L}(q, \widetilde{q}) = \epsilon \left(-\sqrt{1 - \frac{\langle \widetilde{q} - q, \widetilde{q} - q \rangle}{\epsilon^2}} - \frac{1}{2}\langle a, q + \widetilde{q} \rangle \right), \qquad 0 < \epsilon \ll 1,$$

where $q = q(n)$, $\widetilde{q} = q(n+1)$, $n \in \mathbb{N}$. Note that first derivative $\dot{q}$ is discretized by the finite difference $(\widetilde{q} - q)/\epsilon$, that is, we assume $\widetilde{q} = q + \epsilon \dot{q} + O(\epsilon^2)$.

(i) Show that $\mathbf{L}(q, \widetilde{q}) = \epsilon \mathscr{L}(q, \dot{q}) + O(\epsilon^2)$.

(ii) Compute the explicit form of the symplectic map defined by the equations

$$p = -\mathsf{grad}_q \mathbf{L}(q, \widetilde{q}), \qquad \widetilde{p} = \mathsf{grad}_{\widetilde{q}} \mathbf{L}(q, \widetilde{q}).$$

(iii) Use (ii) to find the discrete analog of equations (2). In other words, compute the quantities $(\widetilde{q} - q)/\epsilon$ and $(\widetilde{p} - p)/\epsilon$. Check that they converge to (2) in the continuous limit.

(iv) Construct the discrete Euler-Lagrange equations

$$\mathsf{grad}_q \mathbf{L}(q, \widetilde{q}) + \mathsf{grad}_q \mathbf{L}(\underset{\sim}{q}, q) = 0,$$

where $\underset{\sim}{q} = q(n-1)$. Check that they converge to (1) in the continuous limit.

Exercise 2: (10 pts)

In the canonical Hamiltonian phase space $\mathbb{R}^4$ consider the discrete dynamical system defined by iterations of the map $\Phi : \mathbb{R}^4 \setminus \{q_1^2 = q_2^2\} \to \mathbb{R}^4$, $(q_1, q_2, p_1, p_2) \mapsto (\widetilde{q}_1, \widetilde{q}_2, \widetilde{p}_1, \widetilde{p}_2)$ given by:

$$\begin{cases} \widetilde{q}_1 = p_1(q_1^2 + q_2^2) + 2\, q_1\, q_2\, p_2, \\ \widetilde{q}_2 = p_2(q_1^2 + q_2^2) + 2\, q_1\, q_2\, p_1, \\ \widetilde{p}_1 = \dfrac{q_1}{q_1^2 - q_2^2}, \\ \widetilde{p}_2 = -\dfrac{q_2}{q_1^2 - q_2^2}. \end{cases}$$

(i) Prove that the map Φ preserves the canonical Poisson brackets.

(ii) Prove that the functions

$$F_1(q_1, q_2, p_1, p_2) := q_1\, p_1 + q_2\, p_2,$$
$$F_2(q_1, q_2, p_1, p_2) := q_1\, p_2 + q_2\, p_1,$$

are two functionally independent integrals of motion of Φ.

(iii) Prove that F_1 and F_2 are in Poisson involution, i.e., $\{\, F_1, F_2 \,\} = 0$

Bonus: (8 pts)

Consider the change of coordinates $\Psi : \mathbb{R}^4 \setminus \{q_1^2 = q_2^2\} \to \mathbb{R}^4$, $(q_1, q_2, p_1, p_2) \mapsto (Q_1, Q_2, P_1, P_2)$ defined by:

$$\begin{cases} Q_1 = \dfrac{1}{2}\big(\log(q_1 + q_2) + \log(q_1 - q_2) \big), \\ Q_2 = \dfrac{1}{2}\big(\log(q_1 + q_2) - \log(q_1 - q_2) \big), \\ P_1 = q_1\, p_1 + q_2\, p_2, \\ P_2 = q_1\, p_2 + q_2\, p_1. \end{cases}$$

(i) Prove that the map Ψ preserves the canonical Poisson brackets.

(ii) Prove that Ψ allows to *linearize* the map Φ:

$$\begin{cases} \widetilde{Q}_1 = Q_1 + \nu_1(P_1, P_2), \\ \widetilde{Q}_2 = Q_2 + \nu_2(P_1, P_2), \\ \widetilde{P}_1 = P_1, \\ \widetilde{P}_2 = P_2. \end{cases}$$

Here ν_1 and ν_2 are two functions of the integrals of motion to be determined.

Due Monday, February 04.