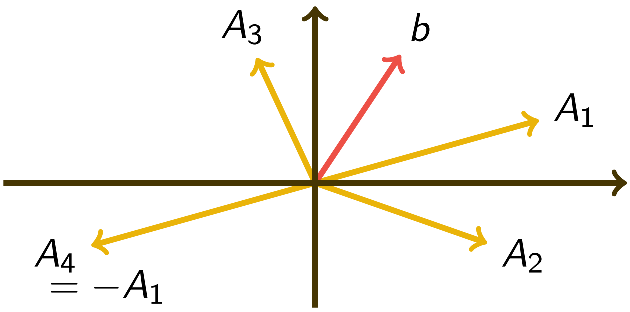
Basic Columns and Basic Solutions

Observation 3.37.

Let $x \in \mathbb{R}^n$ be a basic solution, then:

- $B \cdot x_B = b$ and thus $x_B = B^{-1} \cdot b$;
- x is a basic feasible solution if and only if $x_B = B^{-1} \cdot b \geq 0$.

Example: $m = 2$



- A_1, A_3 or A_2, A_3 form bases with corresp. basic feasible solutions.
- A_1, A_4 do not form a basis.
- A_1, A_2 and A_2, A_4 and A_3, A_4 form bases with infeasible basic solution.

Bases and Basic Solutions

Corollary 3.38.

- Every basis $A_{B(1)}, \dots, A_{B(m)}$ determines a unique basic solution.
- Thus, different basic solutions correspond to different bases.
- But: two different bases might yield the same basic solution.

Example: If $b = 0$, then $x = 0$ is the only basic solution.

Adjacent Bases

Definition 3.39.

Two bases $A_{B(1)}, \dots, A_{B(m)}$ and $A_{B'(1)}, \dots, A_{B'(m)}$ are adjacent if they share all but one column.

Observation 3.40.

- a Two adjacent basic solutions can always be obtained from two adjacent bases.
- b If two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

Degeneracy

Definition 3.41.

A basic solution x of a polyhedron P $\subseteq \mathbb{R}^n$ is degenerate if more than n constraints are active at x.

Observation 3.42.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ be a polyhedron in standard form with $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$, and $b \in \mathbb{R}^m$.

- a A basic solution $x \in P$ is degenerate if and only if more than $n - m$ components of x are zero.
- b For a non-degenerate basic solution $x \in P$, there is a unique basis.

Three Different Reasons for Degeneracy

i. redundant variables

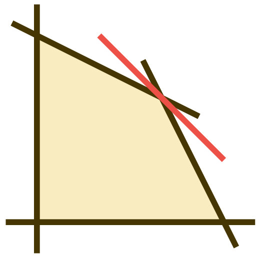
Example:

$$\begin{aligned} x_1 + x_2 \quad &= 1 \\ x_3 &= 0 \\ x_1, x_2, x_3 &\geq 0 \end{aligned} \qquad \longleftrightarrow \qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii. redundant constraints

Example:

$$\begin{aligned} x_1 + 2\,x_2 &\leq 3 \\ 2\,x_1 + x_2 &\leq 3 \\ x_1 + x_2 &\leq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$



iii. geometric reasons (non-simple polyhedra)

Example: Octahedron

Observation 3.43.

Perturbing the right hand side vector b may remove degeneracy.

Chapter 4: The Simplex Method

(cp. Bertsimas & Tsitsiklis, Chapters 3.1, 3.2)

Linear Program in Standard Form

Throughout this chapter, we consider the following standard form problem:

$$\begin{aligned} \text{minimize} \quad & c^T \cdot x \\ \text{subject to} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned}$$

with $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Basic Directions

Observation 4.1.

Let $B = (A_{B(1)}, \dots, A_{B(m)})$ be a basis matrix. The values of the basic variables $x_{B(1)}, \dots, x_{B(m)}$ in the system $A \cdot x = b$ are uniquely determined by the values of the non-basic variables.

Proof:
$$\begin{aligned} A \cdot x = b \quad & \Longleftrightarrow \quad B \cdot x_B + \sum_{j \neq B(1),\dots,B(m)} A_j \cdot x_j = b \\ & \Longleftrightarrow \quad x_B = B^{-1} \cdot b - \sum_{j \neq B(1),\dots,B(m)} B^{-1} \cdot A_j \cdot x_j \end{aligned}$$
$\square$

Definition 4.2.

For fixed $j \neq B(1), \dots, B(m)$, let $d \in \mathbb{R}^n$ be given by

$$d_j := 1, \quad d_B := -B^{-1} \cdot A_j, \quad \text{and} \quad d_{j'} := 0 \quad \text{for } j' \neq j, B(1), \dots, B(m).$$

Then $A \cdot (x + \theta \cdot d) = b$, for all $\theta \in \mathbb{R}$, and d is the jth basic direction.