Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 7

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The aim of Exercise 1 is to obtain alternative proofs for the fundamental theorems of functional analysis which were presented in the lectures. As a preparation, consider the following definition.

Definition:
Let E be a normed space. We say that a series $\sum_{k=1}^{\infty} x_k$ with $x_k \in E$, $k \in \mathbb{N}$, is *convergent in* E, if there exists $x \in E$ with

$$\lim_{N\to\infty} \left\| x - \sum_{k=1}^{N} x_k \right\| = 0.$$

Exercise 1: **8 Points**
Let E be a Banach space and p a seminorm on E such that

$$p\left(\sum_{k=1}^{\infty} x_k\right) \leq \sum_{k=1}^{\infty} p(x_k).$$

Then it can be shown (you do not need to do this) that there exists some $M \geq 0$ with $p(x) \leq M\|x\|$ for all $x \in X$. You are required to use this statement in the sequel. More precisely, you are obligated to use it to show the following three assertions:

(i) Show the Uniform Boundedness Prinicple, Theorem 5.8. *Hint:* $p(x) = \sup_i \|T_i x\|$.

(ii) Show the Inverse Mapping Theorem, Corollary 5.5(i).

(iii) Show the Closed Graph Theorem, Theorem 5.7.

Exercise 2: **4 Points**
Let E and F be Banach spaces, and assume that $T \in L(E, F)$ is injective.

(i) Show that $T \in L(E, \operatorname{ran} T)$ is boundedly invertible if and only if $\operatorname{ran} T$ is closed in F.

(ii) Assume further, that $T \in L(E, F)$ is open. Does T map closed sets in E onto closed sets in F? Give a proof or counterexample.

Exercise 3: **4 Points**

Let E be a Banach space and suppose there exist two closed linear subspaces F and G of E such that any $x \in E$ has a unique representation $x = y + z$ with $y \in F$ and $z \in G$. Use the Closed Graph Theorem to prove that there exists a $C \geq 0$ such that

$$\|y\| \leq C\|x\| \quad \text{and} \quad \|z\| \leq C\|x\|$$

for all $x = y + z \in E$.

Exercise 4: **4 Points**

Let $(a_n)_n \subset \mathbb{K}$ be a sequence such that for all $(x_n)_n \in c_0$ the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k x_k$$

exists. Use the Uniform Boundedness Principle to prove $(a_n)_n \in \ell_1$.

This task will only be corrected if you have gained at least 15 points in Exercise 1-4.

Bonus*: **+5 Points**

Prove, that the Open Mapping Theorem, the Inverse Mapping Theorem and the Closed Graph Theorem are equivalent.

Please submit your homework in the BEGINNING of the big exercise on Monday, June 3.