Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 3

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Definition:
A subset M of a linear space X is called *convex*, if for all $\lambda \in [0, 1]$ and $x_1, x_2 \in M$ there holds

$$\lambda x_1 + (1 - \lambda)x_2 \in M.$$

Furthermore, a normed space $(X, \|\cdot\|)$ is called *strictly convex*, if for all $x_1, x_2 \in X$ such that $\|x_1\| = \|x_2\| = 1$ there holds

$$\left\| \frac{x_1 + x_2}{2} \right\| = 1 \implies x_1 = x_2$$

Exercise 1: **4 Points**
Let X be a linear space and $p : X \to [0, \infty)$ such that

(i) $p(x) = 0$ if and only if $x = 0$,

(ii) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.

Show that p is a norm on X if and only if $\{x \in X : p(x) \leq 1\}$ is convex.

Exercise 2: **5 Points**
Show, that for $p \in (1, \infty)$, the spaces ℓ_p are strictly convex, but ℓ_1, ℓ_∞ are not.

Hint: You may use that for all $a, b \geq 0$ and $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$ there holds

$$ab = \frac{a^p}{p} + \frac{b^q}{q} \iff a^p = b^q.$$

Moreover, you may use that for two complex numbers x, y there holds

$$|x + y| = |x| + |y| \iff (x = 0 \text{ or } y = 0 \text{ or there exists some } \lambda > 0 \text{ such that } \lambda x = y).$$

Exercise 3: **4 Points**

Let $(X, \|\cdot\|)$ be a normed space and $Y \subseteq X$ be a finite-dimensional subspace. Prove that for any element of X here exists a projection on Y, i.e. for all $x \in X$ there exists $y_x \in Y$ such that $\|x - y_x\| = \inf\{\|x - y\| : \; y \in Y\}$. Is this projection unique ? Give a proof or counterexample.

The existence of y_x does not hold in general, if Y is just assumed to be a closed subspace. However, it remains true, if X is reflexive.

Exercise 4: **4 Points**

In the second tutorial, you showed that in a normed space in general the sum of two closed sets is not closed anymore. Prove, that for two closed linear subspaces F, G of a normed space E the sum $F + G$ is closed, if G is finite-dimensional.

Exercise 5: **3 Points**

Let $\mathcal{P}$ be the space of all real-valued polynomials, defined on $\mathbb{R}$. For a polynomial $p \in \mathcal{P}$, such that $p(t) = \sum_{k=0}^{n} a_k t^k$, set $\|p\| := \sum_{k=0}^{n} |a_k|$. Then (this you do not need to show) the set $(\mathcal{P}, \|\cdot\|)$ is a normed space. Consider the linear mappings $f_i : \mathcal{P} \to \mathbb{R},\; i = 1, ..., 3$, which are given by

$$f_1(p) = \int_0^1 p(t)dt, \quad f_2(p) = p'(0), \quad f_3(p) = p'(1).$$

Examine for $i = 1, ..., 3$ whether f_i is continuous and determine the operator norm $\|f_i\|$.

Please submit your homework in the BEGINNING of the big exercise on Monday, May 6.