Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 9

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Exercise 1: **6 Points**

Let $\mathcal{H}$ be a Hilbert space and let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ be a sesquilinear form. Prove that the following conditions are equivalent:

(a) B is continuous.

(b) For each $x \in \mathcal{H}$ the mapping $y \mapsto B(x, y)$ is continuous, and for each $y \in \mathcal{H}$ the mapping $x \mapsto B(x, y)$ is continuous.

(c) B is bounded, i.e. there exists $c > 0$ such that $|B(x, y)| \leq c\|x\|\|y\|$ for all $x, y \in \mathcal{H}$.

Now assume, that B is continuous with c as in (c). Then it was shown in the lecture that there exists exactly one linear mapping $T : \mathcal{H} \to \mathcal{H}$ with the property that $B(x, y) = \langle x, Ty \rangle$ for all $x, y \in \mathcal{H}$ and $\|T\| \leq c$. Assume in addition, that there exists some $0 < c_0 \leq c$ such that for all $x \in X$ there holds

$$\mathrm{Re}(B(x, x)) \geq c_0\|x\|^2.$$

Prove that T is boundedly invertible with $\|T^{-1}\| \leq 1/c_0$.

Exercise 2: **4 Points**

Let $\mathcal{H}$ be a vector space over $\mathbb{K}$ and $\langle \cdot, \cdot \rangle$ a scalar product on $\mathcal{H}$. Further, let $\| \cdot \|$ denote the norm induced by the scalar product. Prove for $f, g \in \mathcal{H}$ the equivalence of the following two statements.

(a) $\|f + g\| = \|f\| + \|g\|$.

(b) There exists $\lambda \geq 0$ such that $f = \lambda g$ or $g = \lambda f$.

Exercise 3: **5 Points**

Let $\mathcal{H}$ be a Hilbert space, $\dim \mathcal{H} \geq 2$. We know that $L(\mathcal{H})$ equipped with the operator norm $\| \cdot \|$ is a Banach space. Prove that the operator norm on $L(\mathcal{H})$ is not induced by a scalar product.

Exercise 4: **5 Points**

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{K}$ whose elements are functions $f : S \to \mathbb{K}$ defined on a fixed set S. The addition and scalar multiplication in $\mathcal{H}$ shall thereby coincide with the respective point-wise operations on the functions. A function $K : S \times S \to \mathbb{K}$ is called a *reproducing kernel* for $\mathcal{H}$ if the following holds:

(i) For all $t \in S$ we have $k_t \in \mathcal{H}$, where $k_t(s) := k(s, t)$, $s, t \in S$.

(ii) If $f \in \mathcal{H}$, then $f(t) = \langle f, k_t \rangle$ for all $t \in S$.

Prove the following statements:

(a) A reproducing kernel is unique.

(b) A reproducing kernel exists if and only if for every $t \in S$ the functional ϕ_t given by $\phi_t(f) := f(t)$, $f \in \mathcal{H}$, is continuous.

(c) If k is a reproducing kernel, then $\mathrm{span}\{k_t : t \in S\}$ is dense in $\mathcal{H}$.

(d) Let $\mathcal{H}$ be the two-dimensional subspace of $L_2(0, 1)$ consisting of the functions $f(t) = a + bt$, $a, b \in \mathbb{K}$. Find the reproducing kernel for $\mathcal{H}$.

Please hand the homework exercises in the BEGINNING of the big exercise on June, 17.