# Cheat sheet: Lebesgue theory

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## 0 Preface

A concise overview of the Lebesgue theory for the Analysis III class at the Technical University in the winter term of 2018. Bullet points with a $(\star)$ mean that a proof exists on the auxiliary proof sheet. This sheet is primarily written for me as a learning guide but may be useful for others. Feel free to use it.

# 1 Measure

- Consider $\mathbb{R}^2$, theorems and examples can easily extended to $\mathbb{R}^d$. Let $\mathcal{R}$ be the set that contains all rectangles
- A **measure** $\lambda$ is a function that maps elementary sets to real values, i.e.

$$\lambda : \mathcal{R} \to [0, \infty]$$

- Let $R_i$ be pairwise disjoint intervals and $A = \bigcup_{n \in \mathbb{N}} R_i$:

$$\lambda(A) \coloneqq \sum_{n \in \mathbb{N}} R_i.$$

- The general properties of a measure are: $\lambda : \mathcal{F} \to [0, \infty]$ must map sets from a $\sigma-$algebra to real values including infinity.
  (1) $\lambda(\emptyset) = 0$,
  (2) $\lambda$ is countably additive.
- Some propositions for the measure $\lambda$:
  (1) Let $A, B \subset \mathcal{F}$. $A \subset B \implies \lambda(A) \leq \lambda(B)$ (monotocity)
  (2) $A_n \subset A_{n+1} \implies \lambda(\bigcup A_n) = \lim_{n\to\infty} \lambda(A_n)$ (continuous from below)
  (3) $A_n \supset A_{n+1}, \lambda(A_1) < \infty \implies \lambda(\bigcap A_n) = \lim_{n\to\infty} \lambda(A_n)$ (continuous from above)
- Multiplies of measures and sums of measures are measures.
- A measure only operates on *measurable* sets, i.e. $\lambda(A)$ is defined only for $A \in \mathcal{F}$ (we will later say, $A$ is measurable if $A \in \mathcal{F}$).

# 2 Outer measure

**Motivation:** How can we measure more sets? Until now, we can only measure sets that are the union of rectangles. We introduce a more powerful measure but as it turns out, it *cannot* act on the entire power set of $\mathbb{R}^2$ without preserving the countably additivity.

- The **outer measure** $\lambda^* : \mathcal{P}(\mathbb{R}^2) \to [0, \infty]$ is defined as

$$\lambda^*(A) = \inf\{\sum_{n \in \mathbb{N}} \lambda(R_n) : R_n \in \mathcal{R}, A \subset \bigcup R_n\}.$$

  Note that the outer measure is defined for *every* subset $A \subset \mathbb{R}^2$.

- $\lambda^*(\emptyset) = 0$ and $\lambda$ is only countably *sub*additive. The outer measure is *not* countably additive!
- A subset $A \subset \mathbb{R}^2$ is **Lebesgue measurable** if

$$\forall E \subset \mathbb{R}^2 : m^*(A) = m^*(A \cap E) + m^*(A \setminus E)$$

or if

$$\forall \epsilon > 0, \exists B \in \mathcal{E} : m^*(A \Delta B) \leq \epsilon.$$

- Let $\mathcal{M}_{Leb}$ be the set of all Lebesgue measurable sets of $\mathbb{R}^2$
- Define the **Lebesgue measure** as

$$\lambda : \mathcal{M}_{Leb} \to [0, \infty], A \mapsto \lambda^*(A).$$

Note that the Lebesgue measure is the same map as the outer measure whose domain is restricted on the Lebesgue measurable sets.

- $\lambda$ is countably additive!
- $A \subset \mathbb{R}^2$ is a **null set** if $\lambda^*(A) = 0$.

# 3 Measurable sets

- Every null set is measurable
- Every interval/ rectangle is Lebesgue measurable
- $\mathcal{M}_{\text{Leb}}$ is closed under countable unions, countable intersections and complement.
- All open subsets, and all closed subsets of $\mathbb{R}^2$ are Lebesgue measurable.
- A property is said to hold **almost everywhere** if there exists a null set that contains all element for which the property does not hold true. Let $E$ be the subset where the property does not hold. Then, the property holds almost everywhere if $E \subset N$ for a null set $N$.

Note: it is not required that $E$ is measurable.

# 4 Measure spaces

- Let $\Omega$ be any set. $\mathcal{F} \subset \mathcal{P}(\Omega)$ is a $\sigma-$**algebra** if
  (1) $\emptyset \in \mathcal{F}$
  (2) $E \in \mathcal{F} \implies E^c \in \mathcal{F}$

(3) $\forall n \in \mathbb{N} : E_n \in \mathcal{F} \implies \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{F}$

- $\forall n \in \mathbb{N} : E_n \in \mathcal{F} \implies \bigcap_{n\in\mathbb{N}} E_i \in \mathcal{F}$
- The tupel $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is a $\sigma-$algebra is called **measure space**. Any set $A \in \mathcal{F}$ is called $\mathcal{F}-$**measurable**.
- $(\mathbb{R}, \mathcal{M}_{\text{Leb}})$ is a measure space. $\mathcal{M}_{\text{Leb}}$ is a $\sigma-$algebra. Any set $A \in \mathcal{M}_{\text{Leb}}$ is called lebesgue measurable.

# 5 Borel $\sigma-$algebra

- For every $\mathcal{B} \subset \mathcal{P}(\Omega)$, there exists a *unique* $\sigma-$algebra $\mathcal{F}_{\mathcal{B}}$ that is generated by $\mathcal{B}$ in a sense that
  1. $\mathcal{B} \subset \mathcal{F}_{\mathcal{B}}$
  2. if $\mathcal{F}$ is a $\sigma-$algebra and contains $\mathcal{B}$, then $\mathcal{F}_{\mathcal{B}} \subset \mathcal{F}$

  $\mathcal{F}_{\mathcal{B}}$ is the smallest $\sigma-$algebra that contains $\mathcal{B}$.
- The **Borel $\sigma-$algebra**, denoted by $\mathcal{M}_{\text{Bor}}$, is the smallest $\sigma-$algebra that contains <u>all open sets</u> in $\mathbb{R}$. In other words, $\mathcal{M}_{\text{Bor}}$ is the $\sigma-$algebra generated by the open sets of $\mathbb{R}$. A **Borel set** is a set $B \in \mathcal{M}_{\text{Bor}}$.
- Here are some other equivalent definitions of the Borel $\sigma-$algebra:
  - $\mathcal{M}_{\text{Bor}} = \sigma(\{(a, b) : -\infty \leq a < b \leq \infty\})$... the smallest $\sigma$-algebra containing <u>all open intervals</u>
  - $\mathcal{M}_{\text{Bor}} = \sigma(\{[a, b] : -\infty < a < b < \infty\})$... the smallest $\sigma$-algebra containing <u>all closed intervals</u>
  - the smallest $\sigma$-algebra containing <u>all intervals</u>
  - the smallest $\sigma$-algebra containing <u>intervals of the form $(a, \infty), (-\infty, a)$</u>
  - the smallest $\sigma$-algebra containing <u>intervals of the form $[a, \infty), (-\infty, a]$</u>
- Loosely speaken, the Borel $\sigma-$algebra contains subsets that can be obtained from intervals through countable union, intersection or complement operations.

  *"However this has to be treated with caution, because it is not necessarily possible to obtain a given Borel set by performing the countable number of steps in a single sequence." (Lecture notes, Prof. Charles Batty)*

  The Borel $\sigma-$algebra includes
  - all open, closed and compact sets
  - all intervals $(a, b), [a, b], (a, b], [a, b)$ and $(a, \infty), (-\infty, a), [a, \infty), (-\infty, a]$

  - all countable sets
  - all sets of single points

- Due to the definition of $\sigma-$algebra, the countable unio and intersection, complement and difference of Borel sets are Borel sets, too
- $\mathcal{M}_{\text{Bor}}$ is smaller than $\mathcal{M}_{\text{Leb}}$. There are two possible ways to describe the relation between these two sets:

$$\mathcal{M}_{\text{Leb}} = \{B \setminus N : B \in \mathcal{M}_{\text{Bor}}, N \text{ null}\} = \{B \cup N : B \in \mathcal{M}_{\text{Bor}}, N \text{ null}\}.$$

  We see from these formulae that $\mathcal{M}_{\text{Bor}} \subset \mathcal{M}_{\text{Leb}}$, as we can set $N = \emptyset$. This characterisation of the Lebesgue measurable sets follows from the next statement

- For every $E \in \mathcal{M}_{\text{Leb}}$, there exists $A, B \in \mathcal{M}_{\text{Bor}}$ such that $A \subset E \subset B$ and $B \setminus A$ is a null set (and so are $E \setminus A$ and $B \setminus E$)
- We can generalise Borel sets to any topological room $\Omega$: the Borel $\sigma$-algebra is the $\sigma$-algebra generated by the open sets of the topological room $\Omega$.

# 6 Borel measurable functions

We know what it means if a set is measurable. Now, we will see what it means when a function is measurable. It builds on the concept of measurable sets.

- Let $(\Omega, \mathcal{F})$ be a measurable space. A function $f : \Omega \to \mathbb{R}$ is **$\mathcal{F}$-measurable** (also denoted by **Borel measurable**) if

$$f^{-1}(I) \in \mathcal{F} \quad \text{for each interval } I \subset \mathbb{R}.$$

  In words, the preimage of each interval under $f$ is measurable.

- $f$ is $\mathcal{F}$-measurbale if and only if $f^{-1}(G) \in \mathcal{F}$ for every $G \in \mathcal{M}_{\text{Bor}}$ or for every open sets, closed intervals, ...

  Alternative definition: $f$ is $\mathcal{F}$-measurbale if and only if

$$f^{-1}(\mathcal{M}_{\text{Bor}}) \subset \mathcal{F}.$$

- $f$ is $\mathcal{F}$-measurable if

$$\forall t \in \mathbb{R} : \{f \leq t\} \in F.$$

- Any continuous function between measure spaces $(\Omega, \mathcal{M}_{\text{Bor}})$ and $(\Omega', \mathcal{M}'_{\text{Bor}})$ is Borel measurable. This follows from the fact that the preimage of open sets under a continuous function is open. Recall that $\mathcal{M}_{\text{Bor}}$ is the $\sigma$-algebra generated by the open sets of $\Omega$.

- Until now, we have never used the notion of measure to define measurability of functions!

# 7 Measurable and Lebesgue measurable functions

Borel measurable functions are special in a way that they map the measure space $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{M}_{\text{Bor}})$. We will introduce a more general notion by allowing functions that map the measure space $(\Omega, \mathcal{F})$ to any measure space $(\Omega', \mathcal{F}')$.

- Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measure spaces. $f$ is called **$\mathcal{F}$-$\mathcal{F}'$-measurable** or just **measurable** if
$$\forall A \in F' : f^{-1}(A) \in \mathcal{F}.$$
- Borel measurable functions ($\mathcal{F}$-measurable functions) are $\mathcal{F}$-$\mathcal{M}_{\text{Bor}}$-measurable functions.
- $\mathcal{M}_{\text{Leb}}$-measurable functions are called **Lebesgue measurable** or $\mathcal{M}_{\text{Leb}}$-$\mathcal{M}_{\text{Bor}}$-measurable. In words: $f$ is Lebesgue measurable if the preimage of every real intervall is Lebesgue measurable.
- Lebesgue measurable functions are a *special type* of Borel measurable functions:
$$\text{Lebesgue measurable function} \implies \text{Borel measurable function.}$$
- $\mathcal{F}$-$\overline{\mathcal{M}_{\text{Bor}}}$-measurable functions are called **measurable numerical** functions. $\overline{\mathcal{M}_{\text{Bor}}}$ is the $\sigma$-algebra that is generated by $\mathcal{M}_{\text{Bor}}$, $\{\infty\}$ and $\{-\infty\}$.

# 8 Properties of measurable functions

As noted before, measurable functions refer to $\mathcal{F}$-$\mathcal{F}'$- measurable functions. If a function $f$ operates in the real numbers, i.e. $f : \Omega \to \mathbb{R}$ (where $\Omega$ is any set), we say $f$ is measurable if it is Borel measurable.

- Let's characterise some measurable functions. The following functions are measurable:
  - constant functions
  - consider the measure space $(\Omega, \mathcal{F})$. The indicator function (or characteristic function) $\chi_A : \Omega \to \mathbb{R}$ is measurable if and only if $A \subset \Omega$ is a measurable set, i.e. $A \in \mathcal{F}$.

- continuous functions $f : \mathbb{R} \to \mathbb{R}$
- monotone functions $f : \mathbb{R} \to \mathbb{R}$
- functions that are continuous almost everywhere
- if $f = g$ almost everywhere and $f$ is Lebesgue measurable, then $g$ is Lebesgue measurable
- $f_1 \circ f_2$ is measurable if $f_1, f_2$ are measurable
- $f + g$ and $fg$ are measurable for $f, g : \Omega \to \mathbb{R}$ measurable real functions ($\star$)
- $\frac{f}{g}$ is measurable if $g(t) \neq 0$ for all $t \in \Omega$, and $f, g$ are measurable
- $\sup_{n\in\mathbb{N}} f_n$, $\inf_{n\in\mathbb{N}} f_n$, $\limsup f_n$ and $\liminf f_n$ are measurable, where $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions $f_n : \Omega \to \mathbb{R}$; note that the supremum is taken pointwise: $(\sup f_n)(x) = \sup\limits_{n\geq 1}\{f_n(x)\}$ for a fixed $x$; similarly inf ($\star$)
- $\max\{f, g\}$, $f^+ = \max\{f, 0\}$, $f^- = \max(-f, 0)$ and $|f|$ are measurable ($\star$)

- The existnence of a non-measurable functions is equivalent to the existence of a non-measurable set.
- **Fact of life:** ALL FUNCTIONS $f : \mathbb{R} \to \mathbb{R}$ THAT CAN BE EXPLICITLY DEFINED ARE LEBESGUE MEASURABLE.

  *"[...] a non-measurable function involves some non-explicit choice process. Priestley compares the existence of non-measurable functions to the existence of yetis." (Lecture notes 2018, Prof. Charles Batty)*

# 9 Simple functions

The genereal notion of a simple function is as follows: Every measurable function that takes finitely many real values is called a simple function. We will use a (seemingly) more specific definition.

- A simple function $\phi$ is linear combination of characteristic functions on measurable sets.
$$\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i} \quad \alpha_i \in \mathbb{R}, A_i \in \mathcal{F} \text{ for all } i = 1, ..., n.$$
- Let $f : \Omega \to [0, \infty]$ be measurable. There exists an increasing sequence $(\phi_n)$ of non-negative simple functions $\phi_n$ such that
$$\lim_{n\to\infty} \phi_n(x) = f(x) \quad \text{for all } x \in \Omega.$$

Note that the limit is a pointwise limit and $f$ is a non-negative, numerical measurable function (it may take $\infty$).

There are a lot of ways to construct such $(\phi_n)$. For instance:

$$\phi_n = \sum_{k=1}^{n2^n} (k-1)2^{-n} \chi_{\{(k-1)2^{-n} \le f < k2^{-n}\}} + n\chi_{\{f \ge n\}}$$

or

$$\phi_n = \sum_{k=1}^{2^{2n}} (k-1)2^{-n} \chi_{\{(k-1)2^{-n} \le f < k2^{-n}\}} + 2^n \chi_{\{f \ge 2^n\}}.$$

For the latter, we see that $\phi_n \le \phi_{n+1}$, $\phi_n(x) \le f$, $f(x) - \phi_n(x) < 2^{-n}$ for all sufficiently large $n$ and fixed $x$, and $\phi_n(x) = 2^n$ for all $n \in \mathbb{N}$ if $f(x) = \infty$. Thus, $\phi_n$ converges pointwise to $f$ for all $x \in \Omega$.