Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 12

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Exercise 1: **5 Points**
Let $T : C[0,1] \to C[0,1]$ be defined by $(Tf)(t) = tf(t), t \in [0,1]$. Compute the spectrum of T.

Exercise 2: **5 Points**
Let E be a Banach space and $T \in L(E)$ a Fredholm operator with $\operatorname{ind} E = 0$. Prove that $0 \in \sigma(T)$ if and only if $\ker T \neq \{0\}$, i.e., zero is an eigenvalue of T. Conclude from this that the non-zero spectral points of a compact operator are eigenvalues with finite-dimensional eigenspaces.

The aim of Bonus Exercise 2 is to obtain alternative characterizations of Fredholm operators. The statement of Bonus Exercise 1 is an auxiliary tool for Bonus Exercise 2.

Bonus Exercise 1 **+5**
Let E be a Banach space, $A \in L(E)$. Show that $\dim(\ker(A)) < \infty$ and $\operatorname{ran}(A)$ is closed in E if and only if each sequence $(x_n)_n \subset E$ such that $\|x_n\| \leq 1$ and $Ax_n \to 0$ has a convergent subsequence.

Bonus Exercise 2 **+5**
Let E be a Banach space, $A \in L(E)$. Use Bonus Exercise 1 to show that the following statements are equivalent:

1. A is a Fredholm operator.
2. There exists $B \in L(E)$ such that $\mathrm{Id} - AB$ and $\mathrm{Id} - BA$ are finite-dimensional.
3. There exists $B \in L(E)$ such that $\mathrm{Id} - AB$ and $\mathrm{Id} - BA$ are compact.
4. There exist $B, C \in L(E)$ such that $\mathrm{Id} - BA$ and $\mathrm{Id} - AC$ are compact.

Please submit your homework in the beginning of the LECTURE (!!!) on Tuesday, July 9.