Problem Sheet 01

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Exercise 1

Given a chart $\varphi : U \to V$ and $\tilde{\varphi} : \tilde{U} \to V$. Let $\tau : \varphi^{-1} \circ \tilde{\varphi}$ be the coordinate transformation between the charts. We want to show that

$$\det \tilde{g}(y) = |\det D(\tau(y))|^2 \det g(\tau(y))$$

where $g, \tilde{g}$ denotes the Gramian matrix of $\varphi, \tilde{\varphi}$ respectively.

Proof.

$$\begin{aligned} \det \tilde{g}(y) = \det(D\tilde{\varphi}(y)^T D\varphi(y)) = \det(D(\varphi \circ \tau)(y))^2 &= \det(D\varphi(\tau(y))D\tau(y))^2 \\ &= \det g(\tau(y))|\det D\tau(y)|^2 \end{aligned}$$

$\square$

Next we want to show that the integral over a manifold is independent of the chart φ. Define:

$$I(f) = \int_U f(\varphi(x))\sqrt{\det g(x)}dx.$$

Proof.

$$\begin{aligned} I(f) = \int_{\tilde{U}} f(\tilde{\varphi}(x))\sqrt{\det \tilde{g}(x)}dx &= \int_{\tilde{U}} f(\varphi(\tau(x)))\sqrt{\det D(\tau(x))^2 \det g(\tau(x))}dx \\ &= \int_U f(\varphi(y))\sqrt{\det g(y)}dy. \end{aligned}$$

$\square$

Exercise 2

To show: Let $U \subset \mathbb{R}^d$ and $f \in C^0(U)$, $u \in C^2(U)$. For every $\varphi \in C_c^\infty(U)$ it holds

$$\int_U u\Delta\varphi dx = \int_U f\varphi dx$$

iff $\Delta u = f$.

Proof. First, notice that for $f \in C^2(U)$ and $g \in C^2_c(U)$:

$$\int_U D_i^2 f g dx = -\int_U D_i f D_i g dx = \int_U f D_i^2 g dx$$

Summing over i yields

$$\int_U \Delta f g dx = -\int_U \langle \nabla f, \nabla g \rangle dx = \int_U f \Delta g dx.$$

Let $\delta u = f$, then

$$\int_U u \Delta \varphi dx = \int_U \Delta u \varphi dx = \int_U f \varphi dx.$$

Now, let $\int_U u \Delta \varphi dx = \int_U f \varphi dx$. We know that

$$\int_U u \Delta \varphi dx = \int_U \Delta u \varphi dx = \int_U f \varphi dx$$

Thus, $\int_U (\Delta u - f)\varphi dx = 0$ for every $\varphi \in C_c^\infty(U)$. So, $\Delta u = f$ after a well known lemma. For completeness, we will state this lemma.

Lemma 0.1

Let $h \in C(U)$. If $\int_U h(x)g(x)dx = 0$ for every $g \in C_c^\infty(U)$ then $h(x) = 0$ for all $x \in U$.

$\square$

Exercise 3

Consider the sequences

$$g_n(x) = \frac{2}{\pi}\arctan(nx) \quad \text{and} \quad f_n(x) = \int_0^x \sqrt{g_n(t)}dt$$

Goal: $(f_n)_n$ is a Cauchy sequence in $(C_0^1[0,1], ||\cdot||_\nabla)$ but does not converge in this metric space.

To show that $(f_n)_n$ is indeed a Cauchy sequence observe that

$$\begin{aligned}||f_n - f_m||_\nabla^2 = \int_0^1 (f_n'(x) - f_m'(x))^2 dx &\leq \int_0^1 (f_n'(x))^2 + (f_m'(x))^2 dx \\ &\leq \int_0^1 |g_n(x) - g_m(x)| dx\end{aligned}$$

We show that $(g_n)_n$ is a Cauchy sequence with standard norm $|\cdot|$. Let $\epsilon > 0$ and wlog $n \geq m$:

$$\forall x > \epsilon : |g_n(x) - g_m(x)| \leq |1 - g_m(x)| \leq |1 - g_m(\epsilon)| \to 0 \quad \text{für } m \to \infty.$$

Thus,

$$||f_n - f_m||_\nabla^2 \leq \int_0^1 |g_n(x) - g_m(x)| dx \leq \epsilon + |1 - g_m(\epsilon)|$$

So, $(f_n)_n$ is a Cauchy sequence with respect to $||\cdot||_\nabla$.

Assume $f_n \to f \in C_0^1[0,1]$. Then, $f(0) = 0$ because $f(0) = \int_0^0 f'(x)dx = 0$ for a $f' \in C^0[0,1]$. Nun ist f stetig und somit gibt es ein $\delta > 0$, sodass $f(x) < 1$ für alle $0 \leq x < \delta$.

`https://math.stackexchange.com/questions/1932023/is-c1a-b-with-the-norm-left-f-right-1-1935026?noredirect=1#comment3975378_1935026`

Fail.

1.

2. Consider $f_n(x) = 2\sqrt{x + \frac{1}{n}}$. It is differentiable with $f_n'(x) = \frac{1}{\sqrt{x+\frac{1}{n}}}$. Thus, $f_n \in C_0^1[0,1]$.

   Let's see if this is a Cauchy sequence. Consider

$$\begin{aligned}
||f_n - f_m||_\nabla^2 &= \int_0^1 \Big(\frac{1}{\sqrt{x + \frac{1}{n}}} - \frac{1}{\sqrt{x + \frac{1}{m}}}\Big)^2 \\
&= \int_0^1 \frac{1}{x + \frac{1}{n}} - \frac{2}{\sqrt{(x + \frac{1}{n})(x + \frac{1}{m})}} + \frac{1}{x + \frac{1}{m}} dx \\
&= \int_0^1 \frac{1}{x + \frac{1}{n}} - \frac{2}{\underbrace{\sqrt{x^2 + x\frac{m+n}{mn} + \frac{1}{nm}}}_{sad}} + \frac{1}{x + \frac{1}{m}} dx \\
&\leq [\ln(x + \frac{1}{n})]_0^1 + [\ln(x + \frac{1}{m})]_0^1
\end{aligned}$$

   This is no good. Overestimated too generously.

   Assume that $f_n \to f$ with $f \in C_0^1[0,1]$. For every $\epsilon > 0$ we find $N \geq \mathbb{N}$ such that for all $n \geq N$:

$$||f_n - f||_\nabla^2 = \int_0^1 \left(\frac{1}{\sqrt{x + \frac{1}{n}}} - f'(x)\right)^2 dx = \int_0^1 \frac{1}{x + \frac{1}{n}} - \frac{2f'(x)}{\sqrt{x + \frac{1}{n}}} + f'(x)^2 dx < \epsilon$$

   Because f' is continuous on $[0,1]$ we can estimate the left hand side term with

$$\int_0^1 \frac{1}{x + \frac{1}{n}} - \frac{2c}{\sqrt{x + \frac{1}{n}}} < \epsilon,$$

where $c = \inf_{x\in[0,1]} f'(x)$. Integrating then yields

$$[\ln(x+\frac{1}{n})]_0^1 - 4c[\sqrt{x+\frac{1}{n}}]_0^1 = \underbrace{\ln(1+\frac{1}{n})}_{\to 0} - \underbrace{\ln(\frac{1}{n})}_{\to -\infty} - 4c\underbrace{\sqrt{1+\frac{1}{n}}}_{\to 1} - 4c\underbrace{\sqrt{\frac{1}{n}}}_{\to 0}$$

As we see, increasing n will lead to a value bigger than ϵ. Thus, f is not in $C_0^1[0,1]$.

3.

Exercise 4

- It holds $\mu(\emptyset) = 0$. To see this, consider the sequence $A_n = \emptyset$. It holds $A_{n+1} \subset A_n$ and $\bigcap A_n = \emptyset$. Thus, $\mu(A_n) = \mu(\emptyset) \to 0$ implies $\mu(\emptyset) = 0$.
- We show the ***monotonicity*** of μ. Let $A \subset B$. We know that

$$B = (A \cap B)\dot{\cup}(B \setminus A).$$

  So, $\mu(B) = \mu(A \cap B) + \mu(B \setminus A) \geq \mu(A)$. We only needed *finite additivty* and the *positivity* of μ to show the monotonicity.
- Now to the σ-additivity. Let $A_i \in \mathcal{A}$, A_i and A_j be pairwise disjoint and $A = \bigcup_{i=1}^{\infty} A_i$. Define the sequence

$$B_n = \bigcup_{i=n}^{\infty} A_i.$$

  It holds the following properties:

  - $B_{n+1} \subset B_n$
  - $\bigcap B_n = \emptyset$
  - $B_1 = A$

  From the first two properties it follows that

$$\lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(\bigcup_{i=n}^{\infty} A_n) = 0.$$

We see that $A = B_n \dot\cup \bigcup_{i=1}^{n-1} A_i$.

$$\begin{aligned}
\mu(A) &= \mu(B_n) + \mu(\bigcup_{i=1}^{n-1} A_i), \quad \forall n \in \mathbb{N} \\
&\Downarrow \text{limit} \\
&= \lim_{n\to\infty} \mu(B_n) + \mu(\bigcup_{i=1}^{n} A_i) \\
&= \lim_{n\to\infty} \mu(\bigcup_{i=1}^{n} A_i) \\
&\Downarrow \text{finite additivity} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(A_i) \\
&= \sum_{i=1}^{\infty} \mu(A_i) < \infty \quad \text{due to monotonicity}
\end{aligned}$$

Exercise 5

(a) **To show:** $\mathcal{F}^\mu$ is a σ-algebra.

Proof. – $\Omega = \Omega \cup \emptyset$ and $\Omega \in \mathcal{F}$. Thus, $\Omega \in \mathcal{F}^\mu$.

– Next, let $B \in \mathcal{F}^\mu$. We want to show that $B^c \in \mathcal{F}^\mu$. We can write

$$B = F \cup A \subset F \cup N \implies B^c = F^c \cap A^c \supset F^c \cap N^c, \quad F \in \mathcal{F}, A \in \mathcal{N}, A \subset N.$$

Goal: Write $B^c = \tilde{F} \cup \tilde{A}$ with $\tilde{F} \in \mathcal{F}$ and $\tilde{A} \in \mathcal{N}$.
Let $N = A \dot\cup X$ for a subset $X \subset \Omega$. So $N^c = A^c \cap X^c$.

$$\begin{aligned}
B^c \supset F^c \cap N^c = F^c \cap A^c \cap X^c \implies B^c &= F^c \cap A^c \cap X^c \cup (F^c \cap A^c \setminus X^c) \\
&= F^c \cap \underbrace{A^c \cap X^c}_{(A\cup X)^c = N^c} \cup \underbrace{(F^c \cap A^c \cap X)}_{=\tilde{A}}
\end{aligned}$$

– Let $B_1, B_2, \ldots \in F^\mu$. We want to show that $B = \bigcup B_i \in F^\mu$. Every B_i can be written as $B_i = F_i \cup A_i$ with $N_i \supset A_i$ and $\mu(N_i) = 0$. So,

$$B = \underbrace{\bigcup F_i}_{\in \mathcal{F}} \cup \underbrace{\bigcup A_i}_{\in \mathcal{N}}$$

□