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Mathematical Physics I - WS 2018/2019

http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/MP1/

Exercise Sheet 3

Exercise 1: (6 pts)

Consider the following two one-parameter family of maps acting on $\mathbb{R}^2$ ($t \in \mathbb{R}$):

$$\Phi^t : (x_1, x_2) \mapsto \frac{(2\,x_1, 2\,x_2 \cos t + (1 - x_1^2 - x_2^2)\sin t)}{1 + x_1^2 + x_2^2 + (1 - x_1^2 - x_2^2)\cos t - 2\,x_2 \sin t},$$

and

$$\Psi^t : (x_1, x_2) \mapsto \left(x_1 + t, \frac{x_1 x_2}{x_1 + t} \right).$$

(i) Prove that both Φ^t and Ψ^t define dynamical systems.

(ii) Compute the vector fields f and g of Φ^t and Ψ^t respectively.

(iii) Check whether Φ^t and Ψ^t commute.

Exercise 2: (3 pts)

Consider vector field g from Exercise 1 and reconstruct the corresponding flow Ψ^t in terms of its Lie series, i.e.

$$\Psi^t(x_1, x_2)_i = \sum_{k=0}^{\infty} \frac{t^k}{k!} (L_g)^k (x_i), \qquad i = 1, 2.$$

Exercise 3: (5 pts)

Consider the following second-order ODEs in $\mathbb{R}$:

$$\ddot{x} + \alpha^2 x = 0, \qquad \ddot{x} - \alpha^2 x = 0, \qquad \alpha > 0.$$

For both ODEs:

(i) Write down the corresponding IVP with initial condition $(x(0), \dot{x}(0)) \in \mathbb{R}^2$.

(ii) Find an integral of motion, i.e., a function which is invariant under the flow of the IVP.

(iii) Find the fixed points.

(iv) Draw the phase portrait. Is there any periodic orbit?

(v) What can you say about the stability of fixed points (just by looking at the phase portrait)?

Exercise 4: (6 pts)

Let $n \geq 3$, and consider the dynamical system defined by the flow of an n-dimensional IVP with ODEs

$$\dot{x}_i = \prod_{j \neq i}^{n} x_j, \qquad i = 1, \ldots, n. \tag{1}$$

(i) Prove that the flow is volume preserving.

(ii) Prove that the functions

$$F_{ij}(x_i, x_j) := x_i^2 - x_j^2, \qquad i, j = 1, \ldots, n.$$

are integrals of motion. How many of these are functionally independent?

Hint: Functions are independent, if their gradients are linearly independent.

(iii) Define n new coordinates by the transformation $\varphi : x \mapsto y$ defined by

$$y_i := \frac{1}{x_i} \prod_{j \neq i}^{n} x_j, \qquad i = 1, \ldots, n.$$

Prove that the system of ODEs (1) is transformed into the quadratic system

$$\dot{y}_i = y_i \left(-2\, y_i + \sum_{j=1}^{n} y_j \right), \qquad i = 1, \ldots, n. \tag{2}$$

(iv) Find the functions $K_{ij}(y) := F_{ij}(\varphi^{-1}(y))$. Are the functions K_{ij}, $i, j = 1, \ldots, n.$, integrals of motion of the system of ODEs (2)?

Due Monday, November 12.