Proofs: Multiple Riemann integral

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1 One dimensional integral

- One may differentiate under the integral sign and the derivative is continuous.

  Proof. First, we will show the following theorem: Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continuous and continuously partially differentiable in $[c, d]$. Consider a sequence $(y_k) \subset [c, d]$ with $y_k \to y$ and $y_k \neq y$. Then

  $$x \mapsto \frac{f(x, y_k) - f(x, y)}{y_k - y} \quad \text{uniformly converges to} \quad x \mapsto \frac{\partial}{\partial y} f(x, y).$$

  It holds

  $$\lim_{k\to\infty} \frac{\int_a^b f(x, u_k)dx - \int_a^b f(x, u)dx}{u_k - u} = \int_a^b \lim_{k\to\infty} \frac{f(x, u_k) - f(x, u)}{u_k - u} dx = \int_a^b \frac{\partial}{\partial u} f(x, u)dx.$$

  After (1), the integral is continuous, since $\frac{\partial}{\partial u} f(x, u)$ is continuous. □

2 Multiple integral on compact cuboids

- If $f : [a, b] \times U \to \mathbb{R}$ is continuous, then

$$(u_1, ..., u_n) \mapsto \int_b^a f(x, u_1, ..., u_n)dx \text{ is continuous} \tag{1}$$

Proof. First, we need a theorem that states: Let $f : [a, b] \times U \to \mathbb{R}$ be continuous. Consider a sequence $(u_k) \subset U$ with $u_k \to u$. It holds:

$$x \mapsto f(x, u_k) \quad \text{uniformly converges for } k \to \infty \text{ to} \quad x \mapsto f(x, u). \tag{2}$$

To show that $F(u_1, ..., u_n) = \int_b^a f(x, u_1, ..., u_n)dx$ is continuous, we will prove for any sequence $(u_k) \subset U$ with $u_k \to u$ that $\lim_{k\to\infty} F(u_k) = F(u)$. It holds

$$\lim_{k\to\infty} F(u_k) = \lim_{k\to\infty} \int_b^a f(x, u_k)dx \overset{(*)}{=} \int_b^a \lim_{k\to\infty} f(x, u_k)dx = \int_b^a f(x, u)dx = F(u).$$

In the step $(*)$, we used that $f(x, u_k)$ uniformly converges to $f(x, u)$ due to (2). Therefore, we can exchange limit and integral. □

- **Theorem of Fubini:** For any continuous function $f : [a, b] \times [c, d] \to \mathbb{R}$, it holds:

$$\int_c^d \left(\int_a^b f(x, y)dx \right) dy = \int_a^b \left(\int_c^d f(x, y)dx \right) dy.$$

Proof. Consider the derivative of $y \mapsto \int_a^b \left(\int_c^y f(x, u)du \right) dx$. It is

$$\int_a^b \left(\frac{d}{dy} \int_c^y f(x, y)dy \right) dx = \int_a^b f(x, y)dx.$$

Now, we will immediately see the result

$$\int_c^d \left(\int_a^b f(x, y)dx \right) dy = \int_c^d \frac{d}{dy} \left(\int_a^b \left(\int_c^y f(x, y)dy \right) dx \right) dy = \int_a^b \left(\int_c^d f(x, y)dy \right) dx$$

Theorems we used to show the proofs: $\int f$ is continuous $\implies$ the double integral is well defined and one may differentiate under the integral sign. □

Proof. Alternatively, the permutation is given by an orthogonal matrix A. Then $\int f(Ax)dx = \int f(x)dx$. □

3 Linear, monotonic and translation-invariant functionals

- Let $f_k \to f$ uniformly and $f_k, f \in \mathcal{C}_c(\mathbb{R}^d)$. The support of all f_k is contained in a compact cuboid Q. Then $J(f_k) \to J(f)$.

Proof. There is continuous $\Phi : \mathbb{R}^d \to [0,1]$ with $\Phi|_Q = 1$ and compact support Q. Then $-\|f_k - f\|\Phi \le f_k - f \le \|f_k - f\|\Phi$ due to $\operatorname{supp}(f_k - f) \subset Q$. Applying the functional yields due to monotocity of J

$$J(-\|f_k - f\|\Phi) \le J(f_k - f) \le J(\|f_k - f\|\Phi).$$

Here we need that there exists such a Φ that is continuous and compact. We need it here otherwise we cannot apply the functional on the term $\|f_k - f\|$ since it is not compact. We use linearity:

$$-\|f_k - f\|J(\Phi) \le J(f_k) - J(f) \le \|f_k - f\|J(\Phi) \iff |J(f_k) - J(f)| \le \|f_k - f\|J(\Phi).$$

Thus $J(f_k) \to J$ since $\|f_k - f\| \to 0$. $\square$

- Every functional on $\mathcal{C}_c(\mathbb{R}^d)$ that is linear, monotonic and translation-invariant is unique up to a constant.

  Proof. Let $c := J(\Psi)$, It holds $J(\Psi_{2^{-n}}) = cI(\Psi_{2^{-n}})$. We approximate f by $f_n = \sum f(k2^{-n})\theta_{k2^{-n}}\Psi_{2^{-n}}$. Since it is translation-invariant and linear, it follows $J(f_n) = cI(f_n)$. All f_n, f are contained in a compact cuboid Q. Additionally, $f_n \to f$ uniformly. Thus, $J(f) = cI(f)$. $\square$

4 Integration by substitution

- $\int_{\mathbb{R}^d} f(Ax)dx = \int_{\mathbb{R}^d} f(x)dx$ if A is orthogonal.

  Proof. There exists c such that $\int f(Ax)dx = c\int f(x)dx$ because $\int \circ A$ is a linear, monotonic and translation-invariant functional. Then $f_0(x) = (1 - \|x\|)_+$ implies $c = 1$ because A is orthogonal. $\square$

5 Volumes

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