Discrete Geometry I Summer term 2020 Viet Duc Nguyen (395220)
Tutor: Ole or Holger
Homework group 9

Exercise sheet 2

Exercise 1 **6 Points**

In this exercise we examine the assumptions of some theorem that we saw in the lectures.

a) Let $n \in \mathbb{N}$. Prove that there exists a finite family of convex sets $A_1, \ldots, A_m \subset \mathbb{R}^n$, $m \geq n$, such that any intersection of n distinct $A_i\,'s$ is non-empty while $\cap_{i=1}^m A_i = \emptyset$ holds.

b) Give an example of a countably infinite familiy of convex sets $\{A_i\}$ in some $\mathbb{R}^d, d > 2$, such that every $d+1$ of the A_i share a common point while $\cap_{i \in \mathbb{N}} A_i = \emptyset$ holds.

our solution:

(a) Let $n \in \mathbb{N}$ be arbitrary, and let e_i denote the i-th unit vector in $\mathbb{R}^n$. Consider the hyperplanes

$$A_i = \{x \in \mathbb{R}^n : \langle x, e_i \rangle = 0\}, \quad i = 1, ..., n$$
$$A_{n+1} = \{x \in \mathbb{R}^n : \langle x, \sum_{i=1}^n e_i \rangle = 42\}$$

- It holds $\bigcap_{i=1}^{n+1} A_i = \emptyset$: we see that
$$\bigcap_{i=1}^{n+1} A_i = \left(\bigcap_{i=1}^{n} A_i \right) \cap A_{n+1} = \{0\} \cap A_{n+1} = \emptyset.$$
- A_i is convex for all $i = 1, ..., n+1$, since affine linear spaces are convex and A_i is the solution space of a linear equation, which is an affine linear space.
- $\bigcap_{j=1}^n A_{i_j} \neq \emptyset$: For any $A_{i_1}, ..., A_{i_n}$, each A_{i_j} represents a hyperplane, whose normal vector is not colinear to the other normal vectors. Therefore, $\bigcap_{j=1}^n A_{i_j}$ defines a system of n linear equations with n variables that has exactly one solution, for the normal vectors are linear independent. Thus, $\bigcap_{j=1}^n A_{i_j}$ is nonempty.

(b) Let $d \in \mathbb{N}_{\geq 3}$. Consider the sets $A_i = \{(x_1, ..., x_d) \in \mathbb{R}^d : x_2, ..., x_d \geq 0, x_1 \geq i\}$. Then, $(A_i)_{i \in \mathbb{N}}$ is a family of convex sets, and $\bigcap_{i \in \mathbb{N}} A_i = \emptyset$. Furthermore, any sets $A_{i_1}, ..., A_{i_{d+1}}$ share a common point since $(i, 0, ..., 0) \in \bigcap_{j=1,...,d+1} A_{i_j}$ for $i = \max_{j=1,...,d+1}\{i_j\}$.

grading: $\boxed{/6}$

Exercise 2 **2 Points**

Recall the function space $\mathcal{C}(\mathbb{R}^d)$ and show that the (ring) multiplication of the underlying $\mathbb{R}$-algebra structure is well-defined. Is $\mathcal{C}(\mathbb{R}^d)$ a finite dimensional real vector space?

our solution:

- **Closed under multiplication:** Let $f, g \in \mathcal{C}(\mathbb{R}^d)$. Then, $f = \sum_{i=1}^{n} \lambda_i[A_i]$ and $g = \sum_{i=1}^{m} \mu_i[B_i]$, where A_i and B_i are closed convex sets in $\mathbb{R}^d$. We would like to prove that $f \cdot g$ is a linear combination of indicator functions of closed convex sets. In that case, $f \cdot g \in \mathcal{C}(\mathbb{R}^d)$. So,

$$\begin{aligned} f \cdot g = \left(\sum_{i=1}^{n} \lambda_i[A_i]\right) \cdot \left(\sum_{i=1}^{m} \mu_i[B_i]\right) &= \sum_{i=1}^{n} \left(\lambda_i[A_i] \cdot \sum_{j=1}^{m} \mu_j[B_j]\right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} (\lambda_i[A_i]\mu_j[B_j]) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i\mu_j[A_i \cap B_j]. \end{aligned}$$

  For each i, j the set $A_i \cap B_j$ is closed and convex. Thus, $f \cdot g$ is indeed a linear combination of indicator functions of closed convex sets.

- $\mathcal{C}(\mathbb{R}^d)$ is not a finite dimensional real vector space. Assume it would be. In that case, there exists a basis $([A_1], ..., [A_n])$ of $\mathcal{C}(\mathbb{R}^d)$. Furthermore, we can assume that each A_i and A_j are pairwise disjoint for $i, j = 1, ..., n$. If there would be A_i and A_j such that $A_i \cap A_j \neq \emptyset$, then define the following sets:

$$B_1 = A_i \cap A_j, \quad B_2 = A_i \setminus A_j, \quad B_3 = A_j \setminus A_i.$$

  Note that

$$\operatorname{span}\{[A_1], ..., [A_n]\} \subset \operatorname{span}\{[A_1], ..., [A_{i-1}], B_1, B_2, B_3, [A_{i+1}], ..., [A_n]\}.$$

  Therefore, we can assume that A_i and A_j are indeed disjoint. Next, we show that there exists a convex and closed set B that cannot be a linear combination of $[A_1], ..., [A_n]$. First, there is a set A_i that contains more than one point (otherwise, we would have an infinite basis $[A_1], [A_2], ...$). Let $B \subset A_i$ with $A_i \setminus B \neq \emptyset$. Now, we can easily see that $[B] \notin \operatorname{span}\{[A_1], ..., [A_n]\}$, because $[B]$ can only be combined from $\{[A_i]\}$ but B is a true subset of A_i. Thus, $\mathcal{C}(\mathbb{R}^d)$ is not a finite dimensional real vector space.

grading: /2

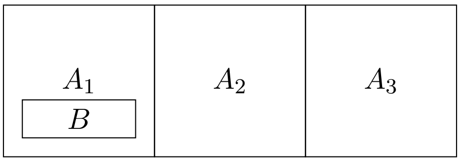

Abbildung 1: Example case for $\mathbb{R}^2$. We can see that the set B cannot be approximated by the sets A_1, A_2 and A_3.

Exercise 3 **4 Points**

Let $I_n : \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_1, \ldots, x_n < 1\}$. Determine $\chi(I_2)$ and $\chi(I_3)$.

our solution:

1. I_2 is an open square in $\mathbb{R}^2$, i.e. a square without its edges.

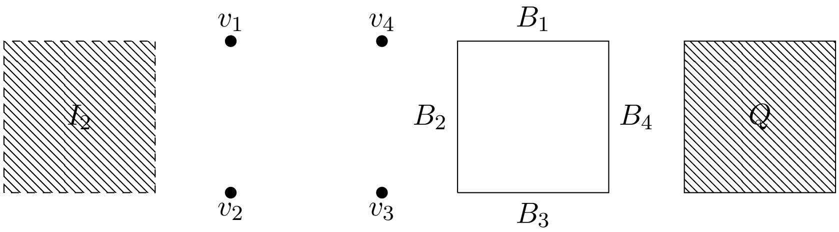

Abbildung 2: The set I_2, the edges of the unit square, the four vertices and $Q = [0,1]^2$.

First,

$$Q = [0,1]^2 = I_2 \cup \bigcup_{i=1}^{4} v_i \cup \bigcup_{i=1}^{4} b_i,$$

where $v_i \in \{(0,0), (0,1), (1,0), (1,1)\}$ are the four vertices of the unit square, and $B_i \in \{\{0\} \times]0,1[, \{1\} \times]0,1[,]0,1[\times \{0\},]0,1[\times \{1\}\}$ are the four edges of the unit square. Now, it holds $\chi(\{v_i\}) = 1$ since $\{v_i\}$ is closed and convex. Additionally,

$$1 = \chi(\{0\} \times [0,1]) = \chi(B_1) + \chi(\{(0,0)\}) + \chi(\{(0,1)\}) = \chi(B_1) + 1 + 1$$

and from this follows $\chi(B_1) = -1$. The same for B_2, B_3 and B_4.

So,

$$1 = \chi(Q) = \chi(I_2) + \underbrace{\sum_{i=1}^{4} \chi(\{v_i\})}_{=4} + \underbrace{\sum_{i=1}^{4} \chi(B_i)}_{=-4}$$

Therfore, $\chi(I_2) = 1$.

2. Let's compute $\chi(I_3)$, where I_3 is the open unit cube in $\mathbb{R}^3$. The unit cube $Q = [0,1]^3$ has 8 vertices, and 12 edges. Each vertex has an Euler characteristic of 1 and each edge has an Euler characteristic of -1. So, $1 = \chi(Q) = \chi(I_3) + 8 - 12 \implies \chi(I_3) = 5$.

grading: $\boxed{/4}$

Exercise 4 **4 Points**

Let $A_1, \ldots, A_m \subset \mathbb{R}^d$ be closed convex sets with $\cap_{i=1}^m A_i \neq \emptyset$. Prove that $\chi(\cup_{i=1}^m A_i) = 1$.

our solution:

Let $A_1, ..., A_n \subset \mathbb{R}^d$ be closed and convex such that $\bigcap_{i=1}^n A_i \neq \emptyset$. Consider $n = 2$. Then, $\chi(A_1 \cup A_2) = \chi(A_1) + \chi(A_2) - \chi(A_1 \cap A_2) = 1 + 1 - 1 = 1$. Note that $A_1 \cap A_2$ is convex and closed as well as not empty.

$n \rightsquigarrow n+1$. Assume $\chi(A_1 \cup ... \cup A_n) = 1$ for some $n \in \mathbb{N}$ if A_i are closed and convex such that $A_1 \cap ... \cap A_n \neq \emptyset$. Then,

$$\begin{aligned}
\chi(A_1 \cup ... \cup A_n \cup A_{n+1}) &= \chi(\left(\bigcup_{i=1}^n A_i\right) \cup A_{n+1}) \\
&= \chi\left(\bigcup_{i=1}^n A_i\right) + \chi(A_{n+1}) - \chi(\bigcup_{i=1}^n A_i \cap A_{n+1}) \\
&= 1 + 1 - \chi(\bigcup_{i=1}^n A_i \cap A_{n+1})
\end{aligned}$$

Consider the set $B_i = A_i \cap A_{n+1}$. It holds $B_1 \cap ... \cap B_n \neq \emptyset$ and B_i is convex and closed. Therefore, $\chi(B_1 \cup ... \cup B_n) = 1$. Thus,

$$\chi(A_1 \cup ... \cup A_n \cup A_{n+1}) = 1 + 1 - \chi(\underbrace{\bigcup_{i=1}^n A_i \cap A_{n+1}}_{=B_1 \cup ... \cup B_n}) = 1 + 1 - 1 = 1.$$

grading: $\boxed{/4}$

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total grade:

$\boxed{/16}$