7 Complete metric spaces and function spaces

7.1 Completeness

Let (X, d) be a metric space.

Definition 7.1. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is a *Cauchy sequence* if for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that

$$n, m > N \Rightarrow d(x_n, x_m) < \epsilon.$$

It is clear that any convergent sequence is a Cauchy sequence: if $x_n \to x$ then

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x).$$

It is also easy to check that any Cauchy sequence is bounded: take N such that $d(x_n, x_m) < 1$ for $n, m > N$ and $M = \max_{n,m=1,\ldots,N}\{d(x_n, x_m)\}$, then

$$d(x_n, x_m) \leq \max\{M, 1\}, \forall n, m \in \mathbb{N} \;\Rightarrow\; \operatorname{diam}(\{x_n : n \in \mathbb{N}\}) \leq \max\{M, 1\}.$$

Definition 7.2. A metric space is *complete* if any Cauchy sequence is convergent.

A subspace A of a metric space (X, d) is complete if, and only if, it is complete for the induced metric $d_{|A}$.

Examples 7.3.

1. $]0, 1[$ not complete: $x_n = 1/n$ not convergent.
2. $\mathbb{Q}$ not complete.
3. Any discrete topological space X with the metric

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y, \end{cases}$$

is complete, as any Cauchy sequence is eventually constant.

Proposition 7.4. *Let (X, d) be a metric space*

1. *If $A \subset X$ is closed and X is complete then A is complete.*
2. *If $A \subset X$ is complete, then A is closed.*

Proof. (i) If (a_n) is a Cauchy sequence in A, then it is a Cauchy sequence in X, hence it is convergent in X, as X complete. Since A closed: $\lim a_n \in \overline{A} = A$.

(ii) In a metric space[7] A is closed if, and only if, for any sequence $a_n \to x$, $a_n \in A \Rightarrow x \in A$. Let then $a_n \in A$, with $a_n \to x$. Then (a_n) is a Cauchy sequence in X, hence in A, hence it converges in A. Since there is unicity of limits (X Hausdorff), $\lim a_n = x \in A$.

(Or: if $x \in \overline{A}$, then there is $a_n \in A$ with $a_n \to x$ - as X is metric / first countable. If $x \notin A$, then a_n is a Cauchy sequence in A that does not converge in A, so A not complete.) □

[7] In fact, in any first countable space.

Example 7.5. $\mathbb{Q} \cap [0, 2]$ is closed in $\mathbb{Q}$ but not complete.

Lemma 7.6. *X is complete if, and only if, any Cauchy sequence has a convergent subsequence.*

Proof. Let (x_n) be a Cauchy sequence and $x_{n_k} \to x, k \in \mathbb{N}$, ($n_k \in \mathbb{N}$ an increasing sequence) be a convergent subsequence. We prove that $x_n \to x$:

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x).$$

Given $\epsilon > 0$, choose N such that $d(x_n, x_m) < \epsilon/2$ for $n, m > N$ and $k \in \mathbb{N}$ such that $n_k > N$ and $d(x_{n_k}, x) < \epsilon/2$. Then $n > N \Rightarrow d(x_n, x) < \epsilon$. □

Corollary 7.7. *If (X, d) is compact then X is complete.*

Proof. In metric spaces, compact $\Leftrightarrow$ sequentially compact, hence any sequence has a convergent subsequence. □

Now we check completeness of well-known spaces.

Theorem 7.8. *$\mathbb{R}^n$ complete with usual or square metric $\rho(x, y) = \max_{i=1,\ldots,n} |x_i - y_i|$.*

Proof. First note that $\mathbb{R}^n$ with the usual Euclidean metric is complete if, and only if, $(\mathbb{R}^n, \rho)$ is complete: the induced topologies are the same, hence the convergent sequences are the same, and check that (x_n) is Cauchy sequence in the Euclidean metric, if, and only if, it is a Cauchy sequence in $(\mathbb{R}^n, \rho)$.

Let then (x_n) be a Cauchy sequence in $(\mathbb{R}^n, \rho)$, then (x_n) is bounded in $\mathbb{R}^n$, hence it is contained in some ρ-ball:

$$\{x_n : n \in \mathbb{N}\} \subset B_\rho(0, r) = [-r, r]^n.$$

Since $[-r, r]^n$. is compact, hence sequentially compact, (x_n) has a convergent subsequence, and completeness follows from the previous lemma. □

Remark 7.9. Completeness is not a topological property: $\mathbb{R} \simeq]0, 1[$, and $\mathbb{R}$ complete, $]0, 1[$ not complete.

In fact, being a Cauchy sequence is not a topological property: $f :]0, 1] \to [1, +\infty[, x \mapsto 1/x$ is a homeomorphism but $x_n = 1/n$ is Cauchy in $]0, 1]$, $f(x_n) = n$ not Cauchy in $[1, +\infty[$. It is true however that if $f : X \to Y$ is *uniformly continuous*[8] and (x_n) is a Cauchy sequence in X then $(f(x_n))$ is a Cauchy sequence in Y.

[8] f is uniformly continuous in A if given $\epsilon > 0$, there is $\delta > 0$ such that for all $x, y \in A$: $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$

The finite product of complete metric spaces is always complete in the square metric, which induces the product topology. The countable product of complete metric spaces (Y_i, d_i) is also complete, with the metric

$$D(x,y) = \sup_{i \in \mathbb{N}} \{\overline{d_i}(x_i, y_i)/i\}$$

where $\overline{d_i}$ is the standard bounded metric with respect to d_i. In particular, if (Y,d) is complete, the space of sequences $(Y^{\mathbb{N}}, D)$ is complete, where D induces the product topology.

Example 7.10. l^p is complete

When we take uncountable products, then the product topology on Y^X, X uncountable, is not even metrizable. But if we take the uniform metric

$$\overline{\rho}(f,g) := \sup\{\overline{d}(f(x), g(y)) : x \in X\}$$

where $\overline{d}$ is the standard bounded metric with respect to d on Y, i.e., $\overline{d}(f(x), g(x)) = \min\{d(f(x), g(x)), 1\}$ then we do obtain a complete metric space.

Theorem 7.11. *If (Y,d) is complete then $(Y^X, \overline{\rho})$ is complete.*

Proof. First note that if (Y,d) is complete, then $(Y, \overline{d})$ is also complete. Let (f_n) be a Cauchy sequence in $(Y^X, \overline{\rho})$. Then, for each $x \in X$,

$$\overline{d}(f_n(x), f_m(x)) \leq \overline{\rho}(f_n(x), f_m(x)),$$

hence $(f_n(x)$ is a Cauchy sequence in Y, and therefore it is convergent, as Y is complete.

Define $f : X \to Y$ such that $f(x) := \lim f_n(x)$. Then, by definition, $f_n \to f$ pointwise (ie in the product topology). We now prove that $f_n \to f$ in the $\overline{\rho}$ metric.

Given $\epsilon > 0$, let N be such that $n, m > N$ then

$$\overline{\rho}(f_n(x), f_m(x)) < \epsilon \;\Rightarrow\; \overline{d}(f_n(x), f_m(x)) < \epsilon, \forall x \in X.$$

Fix $n > N$, $x \in X$ and let $m \to \infty$, then it follows that also $\overline{d}(f_n(x), f(x)) < \epsilon$, hence

$$\overline{\rho}(f_n(x), f(x)) = \sup_{x \in X} \overline{d}(f_n(x), f(x)) < \epsilon,\ n > N \;\Rightarrow\; f_n \to f \text{ in } (Y^X, \overline{\rho}).$$

□

(Note that we proved in particular that if (f_n) is a Cauchy sequence in the uniform metric and $f_n \to f$ pointwise, then $f_n \to f$ uniformly.)

Now assume that X is also a topological space and consider the subspace of Y^X given by *continuous* functions:

$$\mathbb{C}(X,Y) := \{f : X \to Y : f \text{ is continuous}\}.$$

Recall the uniform limit theorem: if (Y, d) is a metric space and $(f_n) \in \mathbb{C}(X, Y)$ is such that, for some $f \in Y^X$, given $\epsilon > 0$ one can find N such that $d(f_n(x), f(x)) < \epsilon$, for all $x \in X$, then $f \in \mathbb{C}(X, Y)$.

It is easy to check that the condition above is equivalent to $f_n \to f$ in $(Y^X, \overline{\rho})$, hence the uniform limit theorem basically says that $\mathbb{C}(X, Y)$ is closed in Y^X.

Theorem 7.12. *Let (Y, d) be complete and X be a topological space. Then $\mathbb{C}(X, Y)$ is complete in the uniform metric.*

Proof. We have just seen that $\mathbb{C}(X, Y)$ is closed in the complete metric space $(Y^X, \overline{\rho})$. □

In particular, $\mathbb{C}(X, \mathbb{R}^n)$ is complete.

Another closed subspace, as we shall see next, is the space of *bounded* functions:

$$\mathcal{B}(X, Y) := \{f : X \to Y : f \text{ is bounded}\}.$$

Here, we can take the equivalent metric, usually called the *sup metric* given by

$$\rho(f, g) = \sup_{x \in X}\{d(f(x), g(x)\},$$

that is, $\overline{\rho} = \min\{\rho, 1\}$ is the standard bounded metric with respect to ρ. Hence it is clear that a sequence is Cauchy with respect to ρ if, and only if, it is Cauchy with respect to $\overline{\rho}$. Note that when X is compact,

$$\mathbb{C}(X, Y) \subset \mathcal{B}(X, Y)$$

as we have seen in Theorem 3.13 that any compact set in a metric space is bounded (and a continuous function maps compacts to compacts).

Theorem 7.13. *Let (Y, d) be complete and X be a topological space. Then $\mathcal{B}(X, Y)$ is complete in the uniform / sup metric.*

Proof. We show that $\mathcal{B}(X, Y)$ is closed. Let $f_n \in \mathcal{B}(X, Y)$ be such that $f_n \to f$ uniformly, or equivalently, in the sup metric. Then, given $\epsilon > 0$,

$$\sup_{x \in X} d(f_n(x), f(x)) < \epsilon, n > N.$$

Take N such that $d(f_N(x), f(x)) < 1/2$, for all $x \in X$ and let $M > 0$ be such that $d(f_N(x), f_N(y)) < M$, for all $x, y \in$ X, which exists since $f_N(X)$ is bounded. Then

$$d(f(x), f(y)) \leq d(f_N(x), f(x)) + d(f_N(y), f(y)) + d(f_N(x), f_N(y)) < M + 1, \ \forall x, y \in X$$

hence $f \in \mathcal{B}(X, Y)$ and completeness follows. □

We now see that any metric space can be embedded in a complete metric space.

Theorem 7.14. *(X, d) be a metric space. Then there is an isometric embedding of X in a complete metric space.*

Proof. To prove this result, we embed X in $\mathcal{B}(X, \mathbb{R})$.

Let $x_0 \in X$ be fixed. Given $a \in X$, let

$$\phi_a(x) := d(x, a) - d(x, x_0), \quad \phi_a : X \to \mathbb{R}.$$

Always have $|d(x, a) - d(x, b)| \leq d(a, b)$, hence $|\phi_a(x)| \leq d(a, x_0)$ and $\phi_a \in \mathcal{B}(X, \mathbb{R})$.

Show now that the map $\Phi : X \to \mathcal{B}(X, \mathbb{R})$, $a \mapsto \phi_a$ is an isometric embedding:

$$\rho(\phi_a, \phi_b) = \sup_{x \in X} |\phi_a(x) - \phi_b(x)| = \sup_{x \in X} |d(x, a) - d(x, b)| = d(a, b)$$

since $|d(x, a) - d(x, b)| \leq d(a, b)$, for all $x \in X$ and for $x = a$, $|d(x, a) - d(x, b)| = d(a, b)$. □

Definition 7.15. Let (X, d) be a metric space and $h : (X, d) \to (Y, D)$ be an isometric embedding, with (Y, D) complete. The *completion* of X is defined as $h(\overline{X})$,

The completion is unique, up to an isometry.

Remark 7.16. There is an alternative way of proving the existence of a completion: take equivalence classes of Cauchy sequences (so proceed in a similar way as to define $\mathbb{R}$ as the completion of $\mathbb{Q}$): let

$$\tilde{X} := \{x = (x_n) \in X^{\mathbb{N}} : (x_n) \text{ is Cauchy } \}$$

and define an equivalence relation on $\tilde{X}$ such that $x \sim y$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Let $Y = X/\sim$ and $D : Y^2 \to \mathbb{R}$, $D([x], [y]) := \lim_{n\to\infty} d(x_n, y_n)$. Show that D is well-defined (let $z_n = d(x_n, y_n)$, then show (z_n) is Cauchy sequence in $\mathbb{R}$ hence it converges) and it is a metric such that (Y, D) is complete.

Now let $h : X \to Y$ given by $h(x) = [(x, x, ..., x, ...)]$: it is injective, and $D(h(x), h(y)) = d(x, y)$ hence it is an isometric embedding.

The space $h(X)$ is actually dense in $Y = X/\sim$, that is Y is the completion of X.

Examples 7.17. $\mathbb{R}$ is the completion of $\mathbb{Q}$.

7.2 Compactness

We have seen already that in a metric space (X, d)

$$\text{compact} \Leftrightarrow \text{sequentially compact} \Leftrightarrow \text{limit point compact}$$

Moreover, in a metric space, any compact set is closed and bounded, but the converse is not true in general (it is true in $\mathbb{R}^n$, according to the Heine-Borel theorem).

We have seen already that any compact metric space is complete. Our aim now is to give a generalization of Heine-Borel's theorem to metric spaces, using the notion of completeness. To do this, we need a richer notion of 'boundedness'.

Definition 7.18. A metric space (X, d) is *totally bounded* if for any $\epsilon > 0$ there is a finite covering of X by ϵ-balls:

$$X = B(x_1, \epsilon) \cup ... \cup B(x_p, \epsilon).$$

It is clear that X totally bounded yields X bounded: take $\epsilon = 1, x_1, ..., x_p$ as above and $M = \max_{i,j=1,...,p} d(x_i, x_j)$, then for $x, y \in X$, $x \in B(x_i, 1)$ and $y \in B(x_j, 1)$ for some x_i, x_j and

$$d(x, y) \leq d(x, x_i) + d(y, x_j) + d(x_j, x_i) < M + 2, \ \forall x, y \in X.$$

The converse is not true, for instance, with the standard bounded metric any set is bounded, but not necessarily totally bounded ($\mathbb{R}$ not totally bounded with $\overline{d}(x, y) = \min\{1, |x - y|\}$).

Note that for subspaces, $A \subset X$ is totally bounded if, and only if, given $\epsilon > 0$, there is a covering of A by ϵ-balls in X, that we can assume, centered in $a_i \in A$:

$$A \subset B(a_1, \epsilon) \cup ... \cup B(a_p, \epsilon), \ a_i \in A.$$

In particular, if A totally bounded and $B \subset A$ then B is totally bounded. Also, A totally bounded if, and only if, $\overline{A}$ is totally bounded (if $\epsilon' < \epsilon$ then $\overline{B(a, \epsilon')} \subset B(a, \epsilon)$[9]).

Examples 7.19.

1. In $\mathbb{R}^n$ with the usual Euclidean metric (or square), bounded $\Leftrightarrow$ totally bounded.

2. X infinite with the discrete metric, then any set is bounded but no infinite set is totally bounded: if $\epsilon < 1$ then an ϵ-ball only covers a one point set.

If X is a compact metric space, then X is always totally bounded: just take a covering of X by ϵ-balls $B(x, \epsilon)$, $x \in X$, then there is a finite subcover. Now we have:

Theorem 7.20. *A metric space (X, d) is compact if, and only if, it is complete and totally bounded.*

Proof. We need only show that if X is complete and totally bounded then it is compact. We show it is sequentially compact.

Let (a_n) be a sequence in X. We use total boundedness to construct a Cauchy subsequence of (a_n), which will then converge, as X is complete.

Since X can be covered by finite number of 1-balls, there is a 1-ball B_1 that contains an infinite number of a_n's, that is, such that $B_1 \cap \{a_n : n \in \mathbb{N}\}$ is infinite. Let $J_1 = \{n \in \mathbb{N} : x_n \in B_1\}$ is infinite.

Now, X is a finite union of 1/2-balls, hence there is a 1/2-ball B_2 such that $B_2 \cap \{x_n : n \in J_1\}$ is infinite. Let $J_2 = \{n \in J_1 : x_n \in B_2\}$.

By induction, we construct a sequence of infinite index sets

$$J_1 \supset J_2 \supset ... \supset J_n \supset ...$$

[9] It is not always true that $\overline{B(a, \epsilon')} = \{x \in X : d(x, a) \leq \epsilon'\}$ - discrete topology.

such that $J_{k+1} = \{n \in J_k : x_n \in B_{k+1}$, with B_{k+1} a $\frac{1}{k+1}$-ball. Now we define our Cauchy subsequence: just pick $n_1 \in J_1$, and $n_2 \in J_2$ with $n_2 > n_1$ (possible since J_2 infinite) and in general, given $n_k \in J_k$, pick $n_{k+1} > n_k$, with $n_{k+1} \in J_{k+1}$. Then, for any $k \in \mathbb{N}$:

$$\begin{aligned} i, j, > k &\Rightarrow n_i, n_j \in J_k \\ &\Rightarrow a_{n_i}, a_{n_j} \in \frac{1}{k} - \text{ball} \\ &\Rightarrow d(a_{n_i}, a_{n_j}) < \frac{2}{k}. \end{aligned}$$

Hence (a_{n_k}) is a Cauchy subsequence of (a_n), hence it converges, as X is complete.

□

In fact, we saw that if X is totally bounded, then any sequence in X has a Cauchy subsequence. (The converse is also true.)

Corollary 7.21. *Let (X, d) be complete. Then $A \subset X$ is compact if, and only if, it is closed and totally bounded.*

A subset $A \subset X$, X a topological space, is said to be *relatively compact* if $\overline{A}$ is compact.

Corollary 7.22. *Let (X, d) be complete. Then $A \subset X$ is totally bounded if, and only if, A is relatively compact.*

Example 7.23. In $\mathbb{R}^{\mathbb{N}}$ with the uniform topology: it is a complete space, and $B = \{x \in \mathbb{R}^{\mathbb{N}} : \overline{\rho}(x) \leq 1\}$ is closed. However, it is not totally bounded as it contains an infinite dense subset. Hence it is not compact.

We shall use the characterization above to study compact subsets of spaces of *continuous funtions*. We make use of the following notion, often easier to check than total boundedness:

Definition 7.24. Let X be a topologcal space, (Y, d) be a metric space. A subset $\mathcal{F} \subset \mathbb{C}(X, Y)$ is said to be *equicontinuous* if for $x_0 \in X$, there is an open set U, $x_0 \in U$, such that for *all* $f \in \mathcal{F}$,

$$x \in U \Rightarrow d(f(x), f(y)) < \epsilon.$$

The point is that the same neighborhood U can be chosen, independently of $f \in \mathcal{F}$. We now see what is the relation between equicontinuous families and total boundedness in $\mathbb{C}(X, Y)$.

Lemma 7.25. *Let X be a topological space, (Y, d) be a metric space, and $\mathcal{F} \subset \mathbb{C}(X, Y)$ endowed with the uniform metric*

(i) $\mathcal{F}$ is totally bounded then is equicontinuous

(ii) If X, Y are compact: $\mathcal{F}$ equicontinuous then totally bounded (with respect to uniform or sup metric).

Proof. (i) Let $x_0 \in X$ and $\epsilon > 0$ be given, can assume $\epsilon < 1$. Since $\mathcal{F}$ is totally bounded, cover $\mathcal{F}$ by finite union of δ-balls: $\mathcal{F} \subset B(f_1, \delta) \cup ... \cup B(f_p, \delta)$. Given $f \in \mathcal{F}$, take i such that

$$f \in B(f_i, \delta) \Leftrightarrow \bar{d}(f(x), f_i(x)) < \delta, \forall x \in X \Leftrightarrow d(f(x), f_i(x)) < \delta, \forall x \in X$$

(assuming $\delta < 1$). Then

$$d(f(x), f(x_0)) \le d(f(x), f_i(x)) + d(f_i(x), f_i(x_0)) + d(f_i(x_0), f(x_0).$$

Now let $\delta < \epsilon/3$ and note that since each f_k is continuous, we can choose U_k such that $d(f_k(x), f_k(x_0)) < \epsilon/3$, for $x \in U_k$. Setting $U = U_1 \cap ... \cap U_p$, we have $d(f_k(x), f_k(x_0)) < \epsilon/3$, for any $k = 1, ..., p$, $x \in U$, and

$$x \in U \Rightarrow d(f(x), f(x_0)) < \epsilon.$$

Since $f \in \mathcal{F}$ was arbitrary and U is independent of f, $\mathcal{F}$ is equicontinuous.

(ii) Let $\mathcal{F}$ be equicontinuous and $\epsilon > 0$. Show that $\mathcal{F}$ can be covered by finite ϵ-balls. For each $a \in X$, let $U_a \subset X$ open such that

$$x \in U_a \Rightarrow d(f(x), f(a)) < \epsilon/3, \forall f \in \mathcal{F}.$$

Then cover X by finitely many such U_a's: $X = U_{a_1} \cup ... \cup \mathcal{U}_{a_p}$.

Now cover Y by finitely many $\epsilon/6$-balls: $Y = B(b_1, \epsilon/6) \cup ... \cup B(b_k, \epsilon/6)$, $b_i \in Y$, $i = 1, ..., k$.

Let $A = \{a_1, ..., a_p\}$, $B = \{b_1, ..., b_k\}$. Then the set B^A of maps $\alpha : A \to B$ is finite. To each such map α, let $f_\alpha \in \mathcal{F}$ be such that $f_\alpha(a_i) \in B(\alpha(a_i), \epsilon/6)$, if it exists. Then $\mathcal{F}$ is covered by ϵ-balls centered in f_α: given $f \in \mathcal{F}$, for each $i = 1, ..., p$, $f(a_i) \in B(b_j, \epsilon/6)$, for some b_j and we let $\alpha \in B^A$ be such that $\alpha(a_i) = b_j$. In this case, for $x \in U_{a_i}$,

$$d(f(x), f_\alpha(x)) \le d(f(x), f(a_i)) + d(f(a_i), f_\alpha(a_i)) + d(f_\alpha(a_i), f_\alpha(x)) < \epsilon,$$

by definition of U_{a_i} and since $f(a_i), f_\alpha(a_i) \in B(\alpha(a_i), \epsilon/6)$. Hence,

$$\rho(f, f_\alpha) = \sup_{x \in X} d(f(x), f_\alpha(x)) < \epsilon \;\Rightarrow f \in B_\rho(f_\alpha, \epsilon),$$

and therefore $\mathcal{F} \subset \cup_{\alpha \in B^A} B_\rho(f_\alpha, \epsilon)$. □

We now characterize compact subsets of $\mathbb{C}(X, \mathbb{R}^n)$, X compact, with the sup metric, which, as we have seen is a complete metric space (since $\mathbb{R}^n$ is complete).

A family $\mathcal{F} \subset \mathbb{C}(X, Y)$ is said to be *pointwise bounded* if for any $a \in X$ the set

$$\mathcal{F}_a := \{f(a) : f \in \mathcal{F}\} \text{ is bounded in } Y.$$

Theorem 7.26 (Ascoli)**.** *Let X be compact and $\mathbb{R}^n$ have the usual / square metric. A subspace $\mathcal{F} \subset \mathbb{C}(X, \mathbb{R}^n)$ is relatively compact if, and only if, it is equicontinuous and pointwise bounded.*

Proof. We have seen that, since $\mathbb{C}(X, \mathbb{R}^n)$ is complete

$$\text{relatively compact} \Leftrightarrow \text{totally bounded.}$$

Assume $\mathcal{F}$ is totally bounded, then it is equicontinuous by the previous lemma (i), and bounded with the sup norm, hence also pointwise bounded.

Conversely, we show that equicontinuous and pointwise bounded yields totally continuous. By the previous lemma (ii), it suffices to check that there is Y compact such that $f(X) \subset Y$ for all $f \in \mathcal{F}$.

For each $a \in X$, let $U_a \subset X$ open such that

$$x \in U_a \Rightarrow d(f(x), f(a)) < 1, \forall f \in \mathcal{F}.$$

Since X is compact, we have that

$$X = U_{a_1} \cup ... \cup U_{a_p}.$$

Moreover, since $\mathcal{F}_{a_i}$ is bounded in $\mathbb{R}^n$, the finite union $\mathcal{F}_{a_1} \cup ... \cup \mathcal{F}_{a_p}$ is also bounded, hence it is contained in some ball $B(0, R)$, for some $R > 0$. Then, for any $f \in \mathcal{F}$, if $x \in U_{a_i}$,

$$d(f(x), 0) \leq d(f(x), f(a_i)) + d(f(a_i, 0) < 1 + R.$$

We have then $f(X) \subset B(0, R + 1)$, for all $f \in \mathcal{F}$. Let $Y = \overline{B(0, R + 1)}$, closed and bounded in $\mathbb{R}^n$, hence compact. □

Corollary 7.27. *Let X be compact and $\mathbb{R}^n$ have the usual / square metric. A subspace $\mathcal{F} \subset \mathbb{C}(X, \mathbb{R}^n)$ is compact if, and only if, it is closed, equicontinuous and bounded (under sup norm).*

Proof. $\mathcal{F}$ compact, then always closed and bounded, and equicontinuous by previous result.

If $\mathcal{F}$ is closed then $\mathcal{F}$ compact iff relatively compact. □

Example 7.28. In $\mathbb{C}([0, 1])$ with the sup norm, $B = \{f \in C([0, 1]) : \|f\| \leq 1\}$ is closed and bounded but not compact: take $f_n(t) = t^n$. Then f_n does not have a convergent subsequence, since a convergent subsequence would converge pointwise and for each $t \in [0, 1]$, $f_n(t) \to f(t)$ with $f(t) = 0, 0 \leq t < 1$, $f(1) = 1$, f not continuous. Check that $\{f_n\}$ not equicontinuous at $t = 1$.

The following result is often used, e.g., to prove existence of solutions of differential equations:

Theorem 7.29 (Ascoli-Arzela)**.** *Let X be compact and $f_n \in \mathbb{C}(X, \mathbb{R}^n)$. If $\{f_n\}$ is pointwise bounded and equicontinuous then (f_n) has a convergent subsequence.*

Proof. Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$. By Ascoli's theorem, $\overline{\mathcal{F}}$ is compact, hence sequentally compact. Then any sequence in $\overline{\mathcal{F}}$, in particular (f_n), has a convergent subsequence. □

Theorem 7.30. (X, d) *is complete if and only if, any nested sequence of closed, non-empty sets* $F_1 \supset \ldots \supset F_n \supset \ldots$ *such that* $\mathrm{diam}(F_n) \to 0$ *has non-empty intersection* $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$.

(Note that if (X, d) compact then the previous is clear as the family $\{F_n\}$ has the finite intersection property.)

Proof. Suppose (X, d) is complete and let F_n be a family of closed sets as above. Pick $x_n \in F_n$, for each $n \in \mathbb{N}$ and $X_n = \{x_p : p \geq n\}$. Then $X_n \subset F_n$ hence $\mathrm{diam}(X_n) \to 0$ and therefore (x_n) is Cauchy, so it converges to some $x = \lim x_n$. We have $x \in \cap_{n\in\mathbb{N}} \overline{X_n} \subset \cap_{n\in\mathbb{N}} F_n \neq \emptyset$.

Conversely: let (x_n) be a Cauchy sequence and again $X_n = \{x_p : p \geq n\}$. Define $F_n = \overline{X_n}$. Then, since (x_n) Cauchy, $\mathrm{diam}\, F_n = \mathrm{diam}\, X_n \to 0$ and clearly F_n closed such that $F_{n+1} \subset F_n$, hence $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$. But if $x \in \cap_{n\in\mathbb{N}} F_n$ then any neighborhood of x intersects X_n so x is a sublimit of (x_n). We conclude that X is complete. □

Theorem 7.31 (Baire)**.** *Let* (X, d) *be a complete metric space. If* X *is given by a countable union of closed sets then at least one has non-empty interior.*

Hence $\mathbb{Q}$ not complete. In fact, a complete metric space with no isolated points is necessarily uncountable.