Problem Sheet 01

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Exercise 1

> To prove
>
> (X, d_1) and (X, d_2) are equivalent $\iff$ the convergent sequences concide

Proof. $\implies$ Let (X, d_1) and (X, d_2) be equivalent metric spaces. Let $(x_n)_{n \in \mathbb{N}}$ be any convergent sequence in (X, d_1), i.e. $x_n \to x \in X$ in (X, d_1).

Goal: We will show that $(x_n)_n$ converges in (X, d_2), i.e. $x_n \to x$ in (X, d_2).

Let $\epsilon > 0$. Due to the equivalence of U_1 and U_2, we find $r > 0$ such that $U_1(x, r) \subset U_2(x, \epsilon)$. Then, choose $N \in \mathbb{N}$ such that $x_n \in U_1(x, r)$ for all $n \geq N$; finding such N is possible since $(x_n)_n$ is convergent in (X, d) after the assumption. Thus, it follows that $x_n \in U_2(x, \epsilon)$ for all $n \geq N$. Per definition, $(x_n)_n$ converges in (X, d_2). Same argument applies to show that $(x_n)_n$ converges in (X, d_1) if $(x_n)_n$ is convergent in (X, d_2).

$\impliedby$ Assume that the convergent sequences in (X, d_1) and (X, d_2) are the same. We will prove the equivalence of the metric spaces by contradiction.

Goal: We will construct a sequence $(x_n)_n$ that converges in (X, d_1) but not in (X, d_2) if we assume that the metric spaces are *not* equivalent.

Let (X, d_1) and (X, d_2) be not equivalent. Hence, there is a $x \in X$ and $\epsilon > 0$ such that for all $r > 0$ it holds: $U_1(x, r) \not\subset U_2(x, \epsilon)$ or $U_2(x, r) \not\subset U_1(x, \epsilon)$. Consider the first case (the proof for the other case is identical). Thus, there exists x_r for each $r > 0$ such that $d_1(x_r, x) < r$ and $d_2(x_r, x) \geq \epsilon$. Construct the sequence $(y_n)_n$ with $y_n := x_{\frac{1}{n}}$. Finally, $y_n \to x$ in (X, d_1) but $y_n \not\to x$ in (X, d_2). Contradiction, for the sequences in both metric spaces conincide per assumption. The metric spaces must be therefore equivalent. $\square$

> To prove
>
> Show that $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric for a metric space (X, d).

Proof. Let $x, y, z \in X$. δ is zero if and only if the numerator is zero. The numerator is $d(x, y)$; thus, $\delta(x, y) = 0 \iff d(x, y) = 0 \iff x = y$. The ***symmetry*** follows from the symmetry of d. Observe that $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \delta(y, x)$. Concerning the ***triangle inequality***:

$$\delta(x,z) \leq \frac{d(x,z)}{1+d(x,z)} \overset{(*)}{\leq} \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$
$$= \delta(x,y) + \delta(y,z),$$

where $(*)$ follows from the monotony of $x \mapsto \frac{x}{1+x}$ (remains to be shown). δ is therefore a norm. $\square$

Lemma 0.1

$f(x) = \frac{x}{1+x}$ is monotonically increasing.

Proof. The derivative reads $\frac{1}{(1+x)^2}$. Since the derivative is strictly positive and f is continuous, f is monotonically increasing. $\square$

To prove

Show that (X, d) and (X, δ) are equivalent metric spaces.

Proof. To show the claim we prove that the convergent sequences coincide. Let $(x_n)_n$ be a convergent sequence in (X, d) with limit $x \in X$. The convergence of $(x_n)_n$ in (X, δ) follows from $\delta(x, y) = \frac{d(x,y)}{\underbrace{1 + d(x,y)}_{\geq 1}} \leq d(x, y)$ for all $x, y \in X$. Thus, for any $\epsilon > 0$ we will find a number $N \in \mathbb{N}$ (due to the convergence of $(x_n)_n$ in (X, d)) such that

$$\forall n \geq N : \delta(x_n, x) \leq d(x_n, y) < \epsilon,$$

which implies the convergence of $(x_n)_n$ in (X, δ).

Consider a convergent sequence $(x_n)_n$ in (X, δ) with limit $x \in X$. We want to prove that $(x_n)_n$ also converges to the same limit x in (X, d). Again, take any arbitrary positive ϵ, and we need to find a number $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

Define $f : X \to \mathbb{R}, u \mapsto \frac{u}{1+u}$. This function is continuous *(make sure this is clear to you!!!!!!)*, and injective (due to the strict monotonicity as stated in the previous lemma). We know that $f((x_n, x)) \to 0$ from the convergence of $(x_n)_n$ in (X, δ). Observe that $f(u) = 0 \iff u = 0$. By injectivity and monotonicity of f, we deduct

$$f(u_n) \to 0 \implies u_n \to 0, \quad \forall (u_n)_n \subset X.$$

Thus, $d(x_n, x) \to 0$ impliying the convergence of $(x_n)_n$ in (X, d). $\square$

Aufgabe 2

Aufgabe 3

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Aufgabe 4

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