Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 5

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Exercise 1: **7 Points**

Prove that for a normed space E the following conditions are equivalent:

(i) E is separable.

(ii) $\{x \in E : \|x\| = 1\}$ is separable.

(iii) There exists a sequence $(x_n)_n \subset E$ such that $\overline{\operatorname{span}((x_n)_n)} = E$.

The aim of this task is to prove

$$E^* \text{ is separable} \implies E \text{ is separable.}$$

Towards this aim, assume that E^* is separable. Then $S := \{\ell \in E^* : \|\ell\| = 1\}$ is separable. Let $(\ell_n)_n$ be dense in S. Since $\|\ell_n\| = 1$ for all $n \in \mathbb{N}$, there exists some $x_n \in E$ with $\ell_n(x_n) \geq 1/2$ and $\|x_n\| = 1$. Prove that $\operatorname{span}((x_n)_n)$ is dense in E.

Exercise 2: **5 Points**

Give the proof of Theorem 4.5 for the case $\mathbb{K} = \mathbb{C}$, i.e., prove the following statement. Given a $\mathbb{C}$-vector space E, a seminorm $\rho : E \to \mathbb{R}$ on E, and a linear subspace $F \subset E$, every $f \in F'$, where F' denotes the algebraic dual space of F, with the property

$$|f(x)| \leq \rho(x) \quad \text{for all } x \in F,$$

has an extension $\ell \in E'$ such that $\ell|_F = f$ and $|\ell(x)| \leq \rho(x)$ for all $x \in E$.

Definition:

Let X be a vector space, $C \subset X$. Then the *Minkowski functional of* C is defined as

$$p_C : X \to [0, \infty], \; x \mapsto \inf\{\alpha > 0 : \; x \in \alpha C\}.$$

If $p_C(x) < \infty$, then C *absorbs* x.

Exercise 3: **5 Points**

Let X be a normed space and $C \subset X$ be an open, convex subset of X with $0 \in C$. Show that

(i) p_C is sublinear.

(ii) C is absorbing: More precisely, there is some $M \geq 0$ such that for all $x \in X$ there holds $p_C(x) \leq M\|x\|$.

(iii) $C = \{x \in X : p_c(x) < 1\}$.

Remark: The Minkowski functional plays an important role for instance in the geometric interpretation of the Hahn-Banach Theorem, which will be presented in the big exercise and which uses the statement of this task.

Exercise 4: **3 Points**

Let X ne a normed space and $F \subset X$ be a subspace. Show

$$F \text{ is dense in } X \iff (\forall f \in X^* : \; f|_F = 0 \implies f = 0)$$

Please submit your homework in the BEGINNING of the big exercise on Monday, May 20.