ADM I Problem Sheet 01

Team 42 Linear and Combinatorial Optimization Workgroup

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Exercise 1

Part (a)

To prove

Every convex function $f : \mathbb{R} \to \mathbb{R}$ is continuous.

This is a true statement, that we want to prove in the following. First, we need a lemma.

Lemma 0.1

Let $f : [a, b] \to \mathbb{R}$ be convex. Let $a \leq x < y < z \leq b$. Then, it holds

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Proof. Let $x < y < z$. It holds

$$y = \frac{y - z}{x - z}x + \left(1 - \frac{y - z}{x - z}\right)z \quad \text{and} \quad 0 < \frac{y - z}{x - z} < 1.$$

With convexity of f it follows

$$f\left(\frac{y - z}{x - z}x + (1 - \frac{y - z}{x - z})z\right) = f(y) \leq \frac{y - z}{x - z}f(x) + (1 - \frac{y - z}{x - z})f(z)$$
$$\Downarrow$$
$$(x - z)(f(y) - f(z)) \leq (y - z)(f(x) - f(z))$$
$$\Downarrow$$
$$\frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

Similarly, we see that

$$y = \frac{y - x}{z - x}z + \left(1 - \frac{y - x}{z - x}\right)x \quad \text{and} \quad 0 < \frac{y - x}{z - x} < 1.$$

So,

$$f\left(\frac{y-x}{z-x}z + (1-\frac{y-x}{z-x})x\right) = f(y) \leq \frac{y-x}{z-x}f(z) + (1-\frac{y-x}{z-x})f(x)$$

$$\Downarrow$$

$$(z-x)(f(y)-f(x)) \leq (y-x)(f(z)-f(x))$$

$$\Downarrow$$

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x}.$$

Combining these two results, we obtain our final result

$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}.$$

□

Next, we want to prove that every convex function $f : (a,b) \to \mathbb{R}$ is continuous. Especially, it follows that $f : \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. Let $[\alpha,\beta] \subset (a,b)$. Our aim is to prove that f is continuous for all $x \in [\alpha,\beta]$. Let $a_0, b_0 \in \mathbb{R}$ such that $\beta < b_0 < b$ and $a < a_0 < \alpha$.

Applying the lemma above on f yields

$$\frac{f(x)-f(a_0)}{x-a_0} \leq \frac{f(y)-f(x)}{y-x} \leq \frac{f(b_0)-f(y)}{b_0-y} \quad \forall x,y \in [\alpha,\beta], y > x.$$

This implies that there exists a real value $M > 0$ such that

$$|f(y)-f(x)| \leq M|y-x| \quad \forall x,y \in [\alpha,\beta].$$

Thus, f is Lipschitz continuous on $[\alpha,\beta]$, and therefore also continuous on $[\alpha,\beta]$. Since $[\alpha,\beta]$ was arbitrary and (a,b) can be written as the union of closed intervals,it follows that f is continuous on (a,b). □

Part (b)

> To prove
>
> Every convex function $f : X \to \mathbb{R}$ is continuous for any $X \subset \mathbb{R}$.

This statement is wrong. To see that, we would like to find a function $f : [a,b] \to \mathbb{R}$ that is convex but not continuous.

Consider $f : [0,1] \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & x > 0 \end{cases}.$$

This function is not continuous (that is clear), but it is convex, which is shown below.

- Let $x = 0$ and $y > 0$. Let $z = \alpha x + (1-\alpha)y$ for $0 \leq \alpha \leq 1$. If $\alpha = 1$, then $f(z) = 1 \leq 1 = f(x) + (1-1)f(y)$. If $\alpha < 1$, then $f(z) = 0 < \alpha = \alpha f(x) + (1-\alpha)f(y)$.
- Let $0 < x < y$. Then, $f(z) = 0 \leq 0 = \alpha f(x) + (1-\alpha)f(y)$.

We have shown that f is not convex.

Exercise 2

Consider the Maximum Set Packing problem

Given: a finite set U and a family $\mathcal{S} \subseteq 2^U$ of subsets of U.

Task: find a subfamily $\mathcal{P} \subseteq \mathcal{S}$ of maximum cardinality such that all sets in $\mathcal{P}$ are pairewise disjoint; such a set $\mathcal{P}$ is called *set packing*

(a) Formulate the general probelm as an integer linear program (IP)

Since the set U is finite, we assume that we are able to enumerate the elements in U such that $\{V_j : j \in \{1,..,m\}\}, m \in \mathbb{N}$ represents all the elements in U and since 2^U is also finite we can enumerate the elements in $\mathcal{S}$, such that $\{S_j : j \in \{1,...,n\}\}, n \in \mathbb{N}$ represents the sets in $\mathcal{S}$. To formulate the problem as an integer linear program we will introduce the variables $c_k \in \{0,1\}$ for $k \in \{1,..,n\}$ and $x_{i,k} \in \{0,1\}$ for $i \in \{1,..,m\}, k \in \{1,..,n\}$. We will set $x_{i,k} = 1$ if $V_i \in S_k$ and $x_{i,k} = 0$ otherwise. The resulting formulation of the linear program is

$$\max \sum_{j=1}^{n} c_j$$

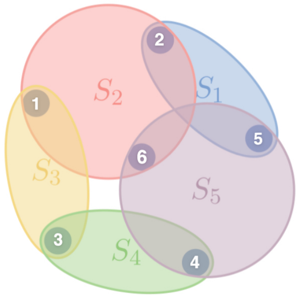
$$\text{s.t.} \sum_{k=1}^{n} x_{i,k} c_k \leq 1 \text{ for } i \in \{1,...,m\}$$

The constrain ensures that all the subsets of $\mathcal{S}$ that are choosen for the set packing will be disjoint.

(b) Specify the IP for the instance on the right side and, if possible, give feasible, an optimal, and an infeasible solution by assigning values to your variables

For the example on the right side we will set $n = 5$ and $m = 6$. The matrix $X = (x_{i,k})_{i=1,...,6,k=1,...,5}$ will then be

$$X = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$



A feasible solution would be $c_2 = 1$ and $c_j = 0$ for $j \in \{1, 3, 4, 5\}$, in matrix notation we obtain with $c = (c_i)_{i=1,\dots,5}$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The constraining function would be fulfilled. An infeasible solution would be $c_2, c_5 = 1$ and $c_j = 0$ for $j \in \{1, 3, 4\}$, since then follows

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

The constraining function is not fulfilled, therefore this is an infeasible solution. A optimal solution would be $c_2, c_4 = 1$ and $c_j = 0$ for $j \in \{1, 3, 5\}$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(c) Relax the integrality constrains on your variables, and give an optimal solution for the resultingu linear program

By relaxing the integrality constrains we obtain the problem

$$\max \sum_{j=1}^{5} c_j$$
$$\text{s.t.} \sum_{k=1}^{5} x_{i,k} c_k \leq 1 \text{ for } i \in \{1, ..., 6\}$$

and $c_i \in [0, 1]$ for $i \in \{1, .., 5\}$. The optimal solution would then be to choose $c_i = \frac{1}{2}$ for all $i \in \{1, .., 5\}$. That this is the optimal solution can be seen, by assuming that one of the c_i for $i \in \{1, .., 5\}$ is choosen as $c_i = \frac{1}{2} + \varepsilon$ for $\varepsilon > 0, \varepsilon \in (0, \frac{1}{2}]$. Since the graph of this problem is symmetric we only need to consider the cases $i = 1, 2, 3$

Case 1: By setting $c_2 = \frac{1}{2} + \varepsilon$ we would also have to set $c_3, c_5, c_1 = \frac{1}{2} - \varepsilon$ in order to fulfill the constraining function and $c_4 = \frac{1}{2} + \varepsilon$, so overall we would lose $-\varepsilon$.

Case 2: Now we chose $c_1 = \frac{1}{2} + \varepsilon$. This would lead to $c_2, c_5 = \frac{1}{2} - \varepsilon$ and $c_3 = \frac{1}{2} + \varepsilon$ and $c_4 = \frac{1}{2} - \varepsilon$ or $c_3 = \frac{1}{2} - \varepsilon$ and $c_4 = \frac{1}{2} + \varepsilon$. Overall there would be no improvement, we would also lose $-\varepsilon$.

Case 3: Let $c_3 = \frac{1}{2} + \varepsilon$. We then obtain $c_2, c_4 = \frac{1}{2} - \varepsilon$ and we can chose one of c_1, c_5 to be $\frac{1}{2} + \varepsilon$ and the other one to be $\frac{1}{2} - \varepsilon$. Hence in this case there would also be a loss of $-\varepsilon$.

Exercise 3

To prove

Let a set N of possible locations for the establishment of warehouses and a set $I = \{1, ..., m\}$ of customers to be served by these warehouses be given. To build a warehouse at location $j \in N$ costs $c_j > 0$. To serve the total needs of customer i from a warehouse in location j costs $h_{ij} \geq 0$; if you serve p% of your needs from the warehouse in j, it costs $\frac{p}{100} h_{ij}$ accordingly. The task is to decide where to set up warehouses and by which amount each warehouse serves each customer so that the costs are minimal. **Formulate this problem as a mixed integer linear program.**

First, let's introduce some variables.

- $w_j \in \{0, 1\}$: If $w_j = 1$, then build a warehouse at location j. If $w_j = 0$, do not build a warehouse at location j.
- $p_{ij} \in [0, 1]$: p_{ij}% of customer i's total need is supplied by warehouse j

The mixed integer linear program looks as follows

$$\begin{aligned}
\min \quad & \sum_{j \in N} w_j \left(c_j + \sum_{i \in I} p_{ij} h_{ij} \right) \\
\text{s.t.} \quad & \sum_{j \in N} w_j p_{ij} = 1 \quad \forall i \in I
\end{aligned}$$