Proof. Let $A \in \mathbb{R}^{m \times n}$. Assume every basic feasible solutions is non degenerate. Since we know that $\text{rank}(A) = m$ it follows from observation 3.42. that any basic feasible solution must have exactly m positive components.

By contradiction, we will show that every feasible solution $x \in \mathbb{R}$ with m positive components must be a basic feasible solution.

Let x be feasible solution (i.e. $Ax = b$ and $x \geq 0$) with m positive components (i.e. $x_{B(i)} > 0$ for $i = 1, ..., m$ where $B(i) \in \{1, ..., n\}$ and $B(i) \neq B(j)$ for $i \neq j$) and $n - m$ components that are zero (i.e. $x_j = 0$ for all $j \notin \{B(i)\}_{i=1,...,m}$). In order to show that x is a basic feasible solution, we need to show that there exist exactly n linearly independent constraints which satisfy x. We know that x satisfies $Ax = b$, which means that m constraints are already fulfilled. Furthermore, $n - m$ inequalities of kind $x_i \geq 0$ are tight due to $x_j = 0$ for all $j \notin \{B(i)\}_{i=1,...,m}$. So, x satisfies n constraints, but it remains unclear if these constraints are linearly independent. This is what we want to show next.

Let $a_i \in \mathbb{R}^n$ denote the i-th row of A. We know $a_i^T x = b_i$ for all $i = 1, ..., m$ because of $Ax = b$. Additionally, let e_i be the i-th unit vector in $\mathbb{R}^n$. It holds $e_j^T x = 0$ for all $j \notin \{B(i)\}_{i=1,...,m}$ because exactly n components of x are non-zero, namely the components which belong to $\{B(i)\}_{i=1,...,m}$. Let's denote the constraints $\{e_j\}_{j \neq B(i), i=1,...,m}$ by $\{e_{C(i)}\}_{i=1,...,n-m}$.

The constraints above are satisfied by the feasible solution x. Now, assume the constraints $\{a_1, ..., a_m, e_{C(1)}, ..., e_{C(n-m)}\}$ are linearly dependent. By theorem 3.18, this is equivalent to the fact that the linear system of equations given by

$$\begin{pmatrix} a_1^T \\ \vdots \\ a_m^T \\ e_{C(1)}^T \\ \vdots \\ e_{C(n-m)}^T \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad y \in \mathbb{R}^n$$

has infinitely many solutions (one solution is given by x). The introduced system of linear equations can also be written as

$$\begin{pmatrix} \hat{a}_1^T \\ \vdots \\ \hat{a}_m^T \end{pmatrix} \begin{pmatrix} y_{B(1)} \\ \vdots \\ y_{B(m)} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, \quad y_j = 0, \quad \forall j \notin \{B(i)\}_{i=1,...,m} \tag{*}$$

where $\hat{a}_i \in \mathbb{R}^m$ is the vector $a_i = \begin{pmatrix} a_{i,1} & ... & a_{i,n} \end{pmatrix}$ which only contains components $a_{i,j}$ such that $j \in \{B(i)\}_{i=1,...,m}$, i.e. $\hat{a}_i = \begin{pmatrix} a_{i,B(1)} & ... & a_{i,B(m)} \end{pmatrix}^T$. By assumption, the system (*) has infinitely many solutions. Equivalently, it holds

$$\text{rank}(\begin{pmatrix} \hat{a}_1^T \\ \vdots \\ \hat{a}_m^T \end{pmatrix}) = \text{rank}(\begin{pmatrix} \hat{a}_1^T & b_1 \\ \vdots & \vdots \\ \hat{a}_m^T & b_m \end{pmatrix}) < m.$$

Hence, there exists a column of $(\hat{a}_i^T)_{i=1,...,m}$ which is linearly dependent on the other remaning columns. W.l.o.g. let this column be the j-th column of $(\hat{a}_i^T)_{i=1,...,m}$, and denote this column by $c = (\hat{a}_{1,B(j)}^T, ..., \hat{a}_{m,B(j)}^T)^T$.

Next, replace the column c of the matrix $(\hat{a}_i^T)_{i=1,...,m}$ by another column $d = (\hat{a}_{1,k}^T, ..., \hat{a}_{m,k}^T)^T$ for $k \notin \{B(i)\}_{i=1,...,m}$ such that (1) the resultant matrix (denoted by B) has full rank, and (2) $x_B = B^{-1}b \geq 0$. (1) is possible due to $\text{rank}(A) = m$, and (2) holds because x is a feasible solution of the system (*).

Since all basic feasible solutions are non degenerate, it holds $x_B > 0$, where the vector $\begin{pmatrix} x_B & 0 & ... & 0 \end{pmatrix}^T \in \mathbb{R}^n$ is the unique solution of $By = b, y \in \mathbb{R}^m$. However, we know that $c = \sum_{j \neq i, j=1}^{m} \lambda_i \hat{a}_i^T$. Thus, a solution of $By = b$ is given by

$$(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_m) + x_j(\lambda_1, ..., \lambda_{j-1}, 0, \lambda_{j+1}, ..., \lambda_m)$$

For the solution of $By = b$ is unique, it holds $x_B = x$, and we found a contradiction due to $x_B > 0$. $\square$