Chapter 2: Linear Programming Basics

(cp. Bertsimas & Tsitsiklis, Chapter 1)

Example of a Linear Program

$$
\begin{array}{lrcrcrcrl}
\text{minimize} & 2x_1 & - & x_2 & + & 4x_3 & & & \\
\text{subject to} & x_1 & + & x_2 & & & + & x_4 & \leq 2 \\
 & & & 3x_2 & - & x_3 & & & = 5 \\
 & & & & & x_3 & + & x_4 & \geq 3 \\
 & x_1 & & & & & & & \geq 0 \\
 & & & & & x_3 & & & \leq 0
\end{array}
$$

Remarks.

- objective function is linear in vector of variables $x = (x_1, x_2, x_3, x_4)^T$
- constraints are linear inequalities and linear equations
- in this example, the last two constraints are special: non-negativity and non-positivity constraint, respectively

General Linear Program

$$\begin{aligned}
\text{minimize} \quad & c^T \cdot x \\
\text{subject to} \quad & a_i{}^T \cdot x \geq b_i && \text{for } i \in M_1, \\
& a_i{}^T \cdot x = b_i && \text{for } i \in M_2, \\
& a_i{}^T \cdot x \leq b_i && \text{for } i \in M_3, \\
& x_j \geq 0 && \text{for } j \in N_1, \\
& x_j \leq 0 && \text{for } j \in N_2,
\end{aligned}$$

with $c \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$ for $i \in M_1 \dot\cup M_2 \dot\cup M_3$ (finite index sets), and $N_1, N_2 \subseteq \{1, \dots, n\}$ given.

- $x \in \mathbb{R}^n$ satisfying all constraints is a feasible solution;
- feasible solution x^* is optimal solution if
$$c^T \cdot x^* \leq c^T \cdot x \qquad \text{for all feasible solutions } x;$$
- linear program is unbounded if, for all $k \in \mathbb{R}$, there is a feasible solution $x \in \mathbb{R}^n$ with $c^T \cdot x \leq k$.

Special Forms of Linear Programs

- Maximizing $c^T \cdot x$ is equivalent to minimizing $(-c)^T \cdot x$.
- Any linear program can be written in the form
$$\begin{aligned}
\text{minimize} \quad & c^T \cdot x \\
\text{subject to} \quad & A \cdot x \geq b
\end{aligned}$$
for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:
  - rewrite $a_i{}^T \cdot x = b_i$ as: $a_i{}^T \cdot x \geq b_i \ \wedge \ a_i{}^T \cdot x \leq b_i$,
  - rewrite $a_i{}^T \cdot x \leq b_i$ as: $(-a_i)^T \cdot x \geq -b_i$.
- Any linear program is equivalent to a linear program in standard form:
$$\begin{aligned}
\min \quad & c^T \cdot x \\
\text{s.t.} \quad & A \cdot x = b \\
& x \geq 0
\end{aligned}$$
with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ (see Slide 34 for details).

Example: Diet Problem

Given:

- n different foods, m different nutrients
- a_{ij} := amount of nutrient i in one unit of food j
- b_i := requirement of nutrient i in some ideal diet
- c_j := cost of one unit of food j

Task: find a cheapest ideal diet consisting of foods $1, \ldots, n$.

LP formulation: Let x_j := number of units of food j in the diet:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x \geq b \\ & x \geq 0 \end{aligned}$$

with $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, $b = (b_i) \in \mathbb{R}^m$, $c = (c_j) \in \mathbb{R}^n$.

Example: LP Relaxation of Mixed-Integer Linear Program

Definition 2.1.

Let $X \subseteq Y$ and $f : Y \to \mathbb{R}$. Then, the optimization problem

$$\text{minimize} \quad f(x) \qquad \text{subject to} \quad x \in Y$$

is a relaxation of the optimization problem

$$\text{minimize} \quad f(x) \qquad \text{subject to} \quad x \in X.$$

The second problem is a tightening of the first problem.

Remark: For a minimization (maximization) problem, the optimal value of a relaxation yields a lower (upper) bound on the optimum.

Example: Relaxing integrality conditions of a MIP yields its LP relaxation:

$$\begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x \geq b \\ & x_i \in \mathbb{Z} \qquad \forall\, i \in N_1 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & c^T \cdot x \\ \text{s.t.} \quad & A \cdot x \geq b \end{aligned}$$

MIP LP relaxation

LP Relaxation of Node Cover Problem

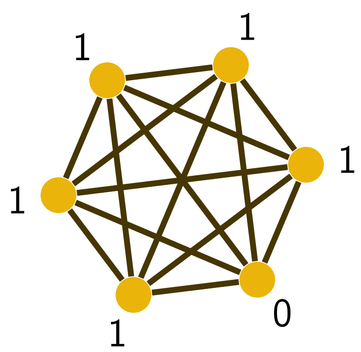
$$\begin{aligned} \min \quad & \sum_{v \in V} w_v \cdot x_v \\ \text{s.t.} \quad & x_v + x_{v'} \geq 1 \quad \forall \{v, v'\} \in E \\ & x_v \in \{0,1\} \qquad \forall\, v \in V \end{aligned}$$

Node Cover IP

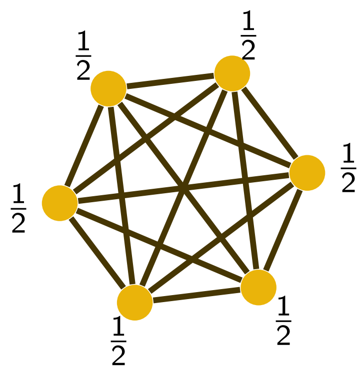
$$\begin{aligned} \min \quad & \sum_{v \in V} w_v \cdot x_v \\ \text{s.t.} \quad & x_v + x_{v'} \geq 1 \quad \forall \{v, v'\} \in E \\ & x_v \in [0,1] \qquad \forall\, v \in V \end{aligned}$$

Node Cover LP relaxation

Example: 'integrality gap' between IP and LP relaxation (for unit weights)



optimal IP solution of value 5



optimal LP solution of value 3

Reduction to Standard Form

Any linear program can be brought into standard form:

- elimination of free (unbounded) variables x_j:

  replace x_j with $x_j^+, x_j^- \geq 0$: $\quad x_j = x_j^+ - x_j^-$

- elimination of non-positive variables x_j:

  replace $x_j \leq 0$ with $(-x_j) \geq 0$

- elimination of inequality constraint ${a_i}^T \cdot x \leq b_i$:

  introduce slack variable $s_i \geq 0$ and rewrite: ${a_i}^T \cdot x + s_i = b_i$

- elimination of inequality constraint ${a_i}^T \cdot x \geq b_i$:

  introduce slack variable $s_i \geq 0$ and rewrite: ${a_i}^T \cdot x - s_i = b_i$

Example

The linear program

$$
\begin{array}{lrcrl}
\min & 2\,x_1 & + & 4\,x_2 & \\
\text{s.t.} & x_1 & + & x_2 & \geq 3 \\
& 3\,x_1 & + & 2\,x_2 & = 14 \\
& x_1 & & & \geq 0
\end{array}
$$

is equivalent to the following standard form problem:

$$
\begin{array}{lrcrcrcrl}
\min & 2\,x_1 & + & 4\,x_2^+ & - & 4\,x_2^- & & & \\
\text{s.t.} & x_1 & + & x_2^+ & - & x_2^- & - & x_3 & = 3 \\
& 3\,x_1 & + & 2\,x_2^+ & - & 2\,x_2^- & & & = 14 \\
& & & & & \multicolumn{3}{r}{x_1,\,x_2^+,\,x_2^-,\,x_3} & \geq 0
\end{array}
$$

Affine Linear Functions

Lemma 2.2.

a) An affine linear function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = c^T x + d$ with $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, is both convex and concave.

b) If $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}$ are convex functions, then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \max_{i=1,\dots,k} f_i(x)$ is also convex.

Proof: a) For $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$:

$$
\begin{aligned}
\lambda \cdot f(x) + (1-\lambda)\cdot f(y) &= \big(\lambda \cdot c^T x + \lambda \cdot d\big) + \big((1-\lambda)\cdot c^T y + (1-\lambda)\cdot d\big) \\
&= c^T\big(\lambda \cdot x + (1-\lambda)\cdot y\big) + \big(\lambda + (1-\lambda)\big)\cdot d = f\big(\lambda\cdot x + (1-\lambda)\cdot y\big)
\end{aligned}
$$

b) For $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$:

$$
\begin{aligned}
\lambda \cdot f(x) + (1-\lambda)\cdot f(y) &= \lambda \cdot \max_{i=1,\dots,k} f_i(x) + (1-\lambda)\cdot \max_{i=1,\dots,k} f_i(y) \\
&\geq \max_{i=1,\dots,k} \big\{\lambda \cdot f_i(x) + (1-\lambda)\cdot f_i(y)\big\} \\
&\geq \max_{i=1,\dots,k} f_i\big(\lambda\cdot x + (1-\lambda)\cdot y\big) = f\big(\lambda \cdot x + (1-\lambda)\cdot y\big) \qquad \square
\end{aligned}
$$

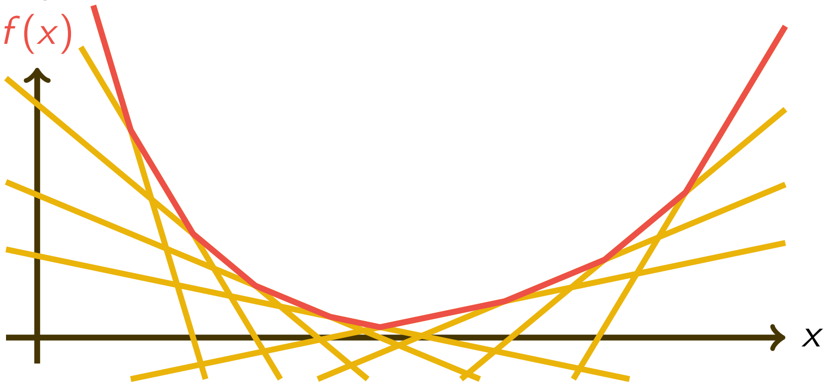
Affine Linear Functions

Lemma 2.2.

a. An affine linear function $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = c^T x + d$ with $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, is both convex and concave.
b. If $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}$ are convex functions, then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \max_{i=1,\dots,k} f_i(x)$ is also convex.

Proof: . . . □

Example: The point-wise maximum of affine linear functions is convex.



Piecewise Linear Convex Objective Functions

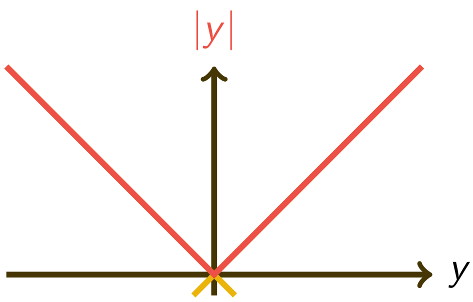
Let $c_1, \dots, c_k \in \mathbb{R}^n$ and $d_1, \dots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\dots,k} {c_i}^T \cdot x + d_i$:

$$\begin{array}{ll} \min & \max\limits_{i=1,\dots,k} {c_i}^T \cdot x + d_i \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{lll} \min & z & \\ \text{s.t.} & z \geq {c_i}^T \cdot x + d_i & \text{for all } i \\ & A \cdot x \geq b & \end{array}$$

Example: let $c_1, \dots, c_n \geq 0$

$$\begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \leftrightarrow \quad \begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot z_i \\ \text{s.t.} & z_i \geq x_i \\ & z_i \geq -x_i \\ & A \cdot x \geq b \end{array}$$



Piecewise Linear Convex Objective Functions

Let $c_1, \ldots, c_k \in \mathbb{R}^n$ and $d_1, \ldots, d_k \in \mathbb{R}$.

Consider piecewise linear convex function: $x \mapsto \max_{i=1,\ldots,k} {c_i}^T \cdot x + d_i$:

$$\begin{array}{ll} \min & \max\limits_{i=1,\ldots,k} {c_i}^T \cdot x + d_i \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \longleftrightarrow \quad \begin{array}{lll} \min & z & \\ \text{s.t.} & z \geq {c_i}^T \cdot x + d_i & \text{for all } i \\ & A \cdot x \geq b & \end{array}$$

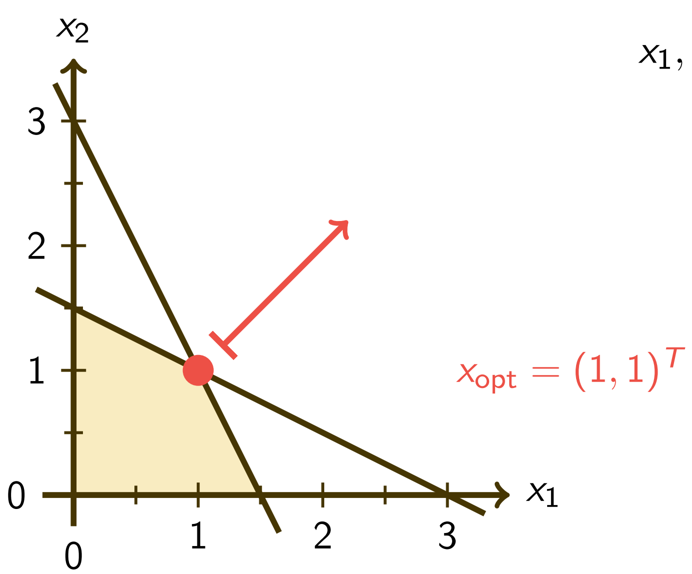
Example: let $c_1, \ldots, c_n \geq 0$

$$\begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.} & A \cdot x \geq b \end{array} \quad \leftrightarrow \quad \begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot z_i \\ \text{s.t.} & z_i \geq x_i \\ & z_i \geq -x_i \\ & A \cdot x \geq b \end{array} \quad \leftrightarrow \quad \begin{array}{ll} \min & \sum\limits_{i=1}^{n} c_i \cdot (x_i^+ + x_i^-) \\ \text{s.t.} & A \cdot (x^+ - x^-) \geq b \\ & x^+, x^- \geq 0 \end{array}$$

Graphical Representation and Solution

2D example:

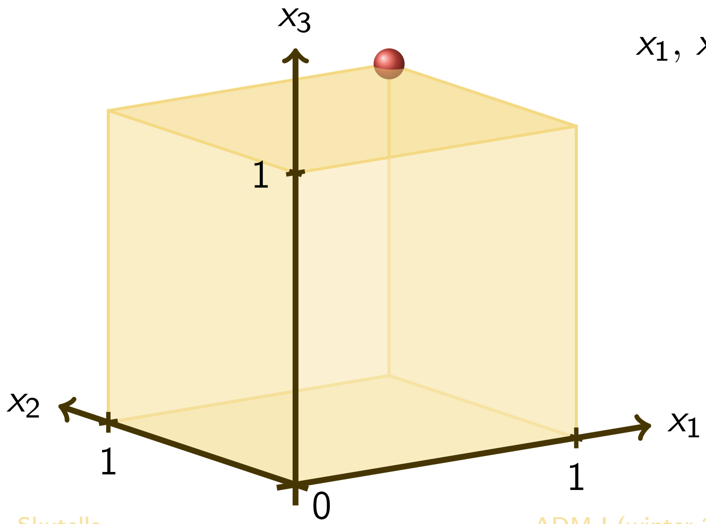
$$\begin{array}{lrcrl} \min & -x_1 & - & x_2 & \\ \text{s.t.} & x_1 & + & 2\,x_2 & \leq 3 \\ & 2\,x_1 & + & x_2 & \leq 3 \\ & & & x_1,\, x_2 & \geq 0 \end{array}$$



Graphical Representation and Solution (Cont.)

3D example:

$$
\begin{array}{lrcrcrl}
\min & -x_1 & - & x_2 & - & x_3 & \\
\text{s.t.} & x_1 & & & & & \leq 1 \\
& & & x_2 & & & \leq 1 \\
& & & & & x_3 & \leq 1 \\
& & & & x_1, & x_2, x_3 & \geq 0
\end{array}
$$

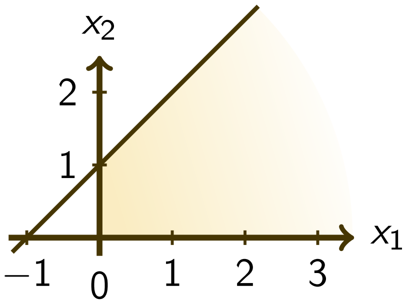


$x_{\text{opt}} = (1, 1, 1)^T$

Graphical Representation and Solution (Cont.)

another 2D example:

$$
\begin{array}{lrcrl}
\min & c_1 x_1 & + & c_2 x_2 & \\
\text{s.t.} & -x_1 & + & x_2 & \leq 1 \\
& & & x_1, x_2 & \geq 0
\end{array}
$$



- for $c = (1, 1)^T$, the unique optimal solution is $x = (0, 0)^T$
- for $c = (1, 0)^T$, the optimal solutions are exactly the points
$$x = (0, x_2)^T \qquad \text{with } 0 \leq x_2 \leq 1$$
- for $c = (0, 1)^T$, the optimal solutions are exactly the points
$$x = (x_1, 0)^T \qquad \text{with } x_1 \geq 0$$
- for $c = (-1, -1)^T$, the problem is unbounded, optimal cost is $-\infty$
- if we add the constraint $x_1 + x_2 \leq -1$, the problem is infeasible

Properties of the Set of Optimal Solutions

In the last example, the following 5 cases occurred:

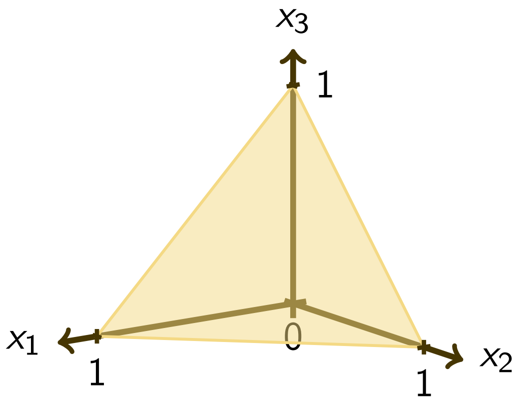
i. there is a unique optimal solution
ii. there exist infinitely many optimal solutions, but the set of optimal solutions is bounded
iii. there exist infinitely many optimal solutions and the set of optimal solutions is unbounded
iv. the problem is unbounded, i.e., the optimal cost is $-\infty$ and no feasible solution is optimal
v. the problem is infeasible, i.e., the set of feasible solutions is empty

These are indeed all cases that can occur in general (see also later).

Visualizing LPs in Standard Form

Example:
Let $A = (1, 1, 1) \in \mathbb{R}^{1\times 3}$, $b = (1) \in \mathbb{R}^1$ and consider the set of feasible solutions

$$P = \{x \in \mathbb{R}^3 \mid A \cdot x = b,\ x \geq 0\} \ .$$



More general:

- if $A \in \mathbb{R}^{m\times n}$ with $m \leq n$ and the rows of A are linearly independent, then
$$\{x \in \mathbb{R}^n \mid A \cdot x = b\}$$
is an $(n-m)$-dimensional affine subspace of $\mathbb{R}^n$.
- set of feasible solutions lies in this affine subspace and is only constrained by non-negativity constraints $x \geq 0$.