Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 2

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Exercise 1: **6 Points**

Prove the missing direction in the proof of the Arzelà-Ascoli theorem: *Let X be a compact metric space and $F \subset C(X)$. If $\overline{F}$ is compact, then F is equicontinuous and pointwise bounded.* As an application, show that the closure of a set $M \subseteq C([a,b],\mathbb{R})$ for which there exist $m, L > 0$ as well as $x_0 \in [a,b]$ such that

$$|f(x_0)| \leq m, \qquad \text{for all } f \in M,$$

and

$$|f(x) - f(y)| \leq L|x-y|, \qquad \text{for all } f \in M, x, y \in [a,b],$$

is compact in the space $C([a,b],\mathbb{R})$ of real-valued continuous functions on $[a,b]$.

Exercise 2: **5 Points**

Let (X,d) and (X',d') be metric spaces and $f : X \to X'$ be a continuous, surjective function such that $d(x,y) \leq d'(f(x),f(y))$ for all $x,y \in X$. Prove or disprove (by giving a counterexample) whether the following statements are true:

(a) If X is complete, then X' is complete.

(b) If X' is complete, then X is complete.

Exercise 3: **5 Points**

A metric space is called *separable* if it contains a countable dense subset. Examine, for which $p \in [1,\infty]$ the space ℓ_p is separable.
Hint: You may use, that the set $H := \{(x_n)_{n\in\mathbb{N}} : x_n \in \{0,1\} \text{ for all } n \in \mathbb{N}\}$ of $0-1-$sequences is uncountable.

Exercise 4: **5 Points**

Show, that the space $C([0,1])$, equipped with

$$\|f\|_1 := \int_0^1 |f(x)|dx,$$

is a normed space, but not a Banach space.

From time to time you will find Bonus Exercises on the homework sheets. Some of these exercises are designed to help you gain more points and are similarily difficult as the regular homework exercises. The others will be more challenging and will ONLY be corrected, if you have already gained at least 15 points in the regular homework exercises of the sheet at hand. We will label the more challenging tasks by a "*". The following task is of a similar difficulty as the other homework, so it will be corrected even if you did not reach 15 points in the regular tasks.

Bonus Exercise: **+5 Points**

Find all continuous functions $\phi : [0,1] \to [0,\infty)$ such that the closure of the set

$$M_\phi := \{f \in C[0,1] : |f(x)| \leq \phi(x) \text{ for all } x \in [0,1]\}$$

is compact in $C[0,1]$.

Please submit your homework in the BEGINNING of the big exercise on Monday, April 29.