Research Group
Applied Functional Analysis
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Functional Analysis I

Tutorial Assignment 7

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Exercise 1: Prove or disprove (by giving a counterexample) the following statements:

(i) A linear, open map $T : X \to Y$, where X, Y are normed spaces, maps closed sets in X onto closed sets in Y.

(ii) The map $T : \ell_\infty \to c_0, (t_n)_n \mapsto (t_n/n)_n$ is open. Here, c_0 denotes the space of all sequences with elements in $\mathbb{K}$ which converge to 0 with respect to $\|\cdot\|_\infty$.

Exercise 2: Let E and F be normed spaces and $T, T_n \in L(E, F)$, $n \in \mathbb{N}$. Discuss: T_n converges to T pointwise if and only if T_n converges to T in operator norm.

Exercise 3: Let E, F, and G be Banach spaces, let $T : E \to F$ be linear and let $S \in L(F, G)$ be injective. Show that $S \circ T \in L(E, G)$ implies $T \in L(E, F)$.

Exercise 4: Let E and F be Banach spaces and $T_n \in L(E, F)$, $n \in \mathbb{N}$. Moreover, assume $\sup_{n\in\mathbb{N}} \|T_n\| = \infty$. Prove that there exists some $x \in E$ such that $\sup_{n\in\mathbb{N}} \|T_n x\| = \infty$.

Exercise 5: Let E and F be Banach spaces, let $\mathcal{D} \subset E$ be dense, and let $T, T_n \in L(E, F)$, $n \in \mathbb{N}$. Prove that the following conditions are equivalent.

(i) For all $x \in E$ we have $T_n x \to Tx$ as $n \to \infty$.

(ii) $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$ and for all $x \in \mathcal{D}$ we have $T_n x \to Tx$ as $n \to \infty$.

Exercise 6: Let E be a Banach space, F a normed space, and $T_n : E \to F$, $n \in \mathbb{N}$, a sequence of linear bounded operators. Further, assume that for all $(x_n)_n \subseteq E$ with $\|x_n\| \to 0$ as $n \to \infty$ we have $\|T_n x_n\| \to 0$ as $n \to \infty$. Prove that $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$.