TECHNISCHE UNIVERSITÄT BERLIN
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Mathematical Physics I - WS 2018/2019

http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/MP1/

Exercise Sheet 8

Exercise 1: (7 pts)

Consider the following system of ODEs in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = x_1^4 + x_1 x_2, \\ \dot{x}_2 = -2\, x_2 - x_1^2 + x_1 x_2^2. \end{cases} \tag{1}$$

(i) Linearize (1) around the fixed point $(0,0)$. What can you say about the stability of $(0,0)$ on the basis of the Poincaré-Lyapunov Theorem?

(ii) Find the center (linear) space $E^c(0,0)$. Construct an approximation of the center manifold $W^c(0,0)$.

(iii) Find the first three nonzero terms of the series expansion of the system obtained by reducing (1) on $W^c(0,0)$. What can you say now about the stability of the fixed point $(0,0)$ for system (1)?

Hint: The center manifold is parametrized, in a neighborhood of $(0,0)$, by $x_2 = h(x_1)$ for some function h. An approximation of $W^c(0,0)$ is given by the series expansion – say up to $O(x_1^5)$ – of the function h around $x_1 = 0$.

Exercise 2: (8 pts)

Consider the following system of ODEs in $\mathbb{R}^3$:

$$\begin{cases} \dot{x}_1 = \frac{1}{2}(x_3 - x_1) - x_2 + 2x_2^3, \\ \dot{x}_2 = \frac{1}{2}(x_3 - x_1), \\ \dot{x}_3 = x_3 - x_2^2(x_1 + x_3). \end{cases} \tag{2}$$

(i) Find all fixed points.

(ii) Linearize system (2) around the fixed point $(0,0,0)$ and discuss the stability of $(0,0,0)$. Write the general solution of the linearized system.

(iii) Prove that $M := \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 - 2x_2 + x_3 = 0\}$ is an invariant manifold for (2).

(iv) Reduce system (2) to M by eliminating the variable x_1, thus getting a first-order ODE for x_2 and x_3. Then eliminate the variable x_3, thus obtaining a second-order ODE for x_2. Find an integral of motion of the resulting ODE.

Is this an integral of motion of the full system (2)?

Exercise 3: (5 pts)

Consider the planar system of ODEs

$$\begin{cases} \dot{x}_1 = x_2 - 2\,x_1, \\ \dot{x}_2 = \mu + x_1^2 - x_2, \end{cases}$$

with $\mu \geq 0$. Study the bifurcations that occur as μ is varied.

Due Monday, December 17.