Cheat sheet: Linear Algebra and Differential Equations

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Contents

0 Preface

A concise overview of solving linear differential equations with constant coefficients by methods of Eigenvalues and Eigenvectors. It is based on the monography *Introduction to Linear Algebra* by Gilbert Strang. This sheet is primarily written for me as a learning guide but may be useful for others. Feel free to use it.

1 The matrix is diagonalisable

Consider the problem

$$\dot{\mathbf{u}} = A\mathbf{u}. \tag{1}$$

This is a first order differential equation with constant coefficients (*non-autonomous*). The problem becomes easy when $A\mathbf{x} = \lambda\mathbf{x}$ for specific $\mathbf{x}$. If this is the case, we get a solution of (1) with

$$\boxed{\mathbf{u}(t) = e^{\lambda t}\mathbf{x},}$$

for $\dot{\mathbf{u}} = \lambda e^{\lambda t}\mathbf{x}$ and substituting in (1) yields

$$\dot{\mathbf{u}}(t) = \lambda e^{\lambda t}\mathbf{x} = Ae^{\lambda t}\mathbf{x} = \lambda e^{\lambda t}\mathbf{x} = A\mathbf{u}(t).$$

The questions is: <u>how do we find such λ and $\mathbf{x}$?</u> The scalars λ are the Eigenvalues of A with Eigenvectors $\mathbf{x}$. If A with dimension n decomposes into n distinct eigenvalues λ_i, we obtain n distinct solutions $\mathbf{u}_i(t) = e^{\lambda_i t}\mathbf{x}_i$.

The idea is basically a change of coordinates. We try to display the solution $\mathbf{u}$ as a linear combination of Eigenvectors $\mathbf{x}_i$. We get specific solutions by setting the i-th coordinate to $e^{\lambda_i t}$ and everything else to null. As linear combinations of Eigenvectors remain Eigenvectors, $A\mathbf{u}$ is the same as $\lambda\mathbf{u}$, and simple derivation confirms the solution.

By change of coordinates to a basis of Eigenvectors, we are essentially decoupling the system. The matrix A can be displayed as a matrix in *diagonal* form, which means the equations are now independent from each other. We just need to solve n linear differential equations with constant coefficients (without change of coordinates this is not possible because the equations depend on each other).

2 Higher order equations

We increase the dimension of A and as a tradeoff we obtain a system of first order differential equations. For a system of n-th order, we introduce $n-1$ new variables $z_1, ..., z_{n-1}$ that reflect the derivatives. Given a higher order differential equation

$$a_0 y + a_1 y' + ... + a_n y^{(5)} = 0,$$

we convert it to

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 0 \\ & & 0 & 1 & 0 \\ & & & 0 & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & ... & ... & -\frac{a_{n-1}}{a_n} \end{pmatrix} \begin{pmatrix} y \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \frac{d}{dt} \begin{pmatrix} y \\ y' \\ y'' \\ y''' \\ y^{(4)} \end{pmatrix}.$$

It contains the following constraints: $z_1 = y'$,..., $z_4 = \frac{d}{dt}y'''$ and $y^{(5)} = ...$ We can solve $A\mathbf{y} = \dot{\mathbf{y}}$ for y, and obtain the original solution.

3 Stability of 2×2 matrices

If all Eigenvalues have negative real part, then A is stable and solutions $\mathbf{u}(t) \to 0$ for $t \to \infty$. The reason is that every component of $\mathbf{u}$ is of the form $e^{\lambda_i t}$. This part indeed converges to zero for negative λ_i. The matrix A must pass two requirements:

$$\mathrm{tr} A < 0 \quad \text{and} \quad \lambda_1 \lambda_2 > 0.$$

First, we know that the sum of all Eigenvalues equals the trace. Secondly, the determinant of A is the product of all Eigenvalues.

4 Matrix expontial

We cannot work with Eigenvalues and Eigenvectors if A is not diagonalisable. Instead, we can use the matrix exponential, <u>which will always work.</u> The problem consists of finding the exponential of a matrix A, i.e. calculate e^{At}.

The matrix exponential e^A is a matrix defined as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

The problem will be calculating A^k. If we take the matrix At, the formula is

$$\boxed{e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.}$$

Note that $t \in \mathbb{R}$ is a scalar and not a vector. The solution

$$\mathbf{u}(t) = e^{At}\mathbf{u}(0)$$

solves $\dot{\mathbf{u}} = A\mathbf{u}$ with initial value $\mathbf{u}(0)$ for $t = 0$. We can prove it by

$$\frac{d}{dt} e^{At} = A e^{At}.$$

Just put it into the equation and we get the result.

We collect some properties:

- The inverse of e^A is e^{-A}
- For <u>commutative</u> matrices: $e^A e^B = e^B e^A = e^{A+B}$

- $e^{(A^T)} = (e^A)^T$. This means, the exponential matrix maps *symmetric* matrices to *symmetric* matrices.

$$(e^A)^T = e^{(A^T)} = e^A \quad \text{for symmetric } A.$$

The exponential matrix maps *skew-symmetric* matrices ($A^T = -A$) to *orthogonal* matrices ($AA^T = I$).

$$e^A(e^A)^T = e^A e^{(A^T)} = e^A e^{-A} = I \quad \text{for skew-symmetric } A.$$

4.1 A is diagonalisable

If A decomposes into $S\Lambda S^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)$, the matrix exponential e^{At} is easy to compute.

$$e^{At} = S(e^{\Lambda t})S^{-1} = S\left(\mathrm{diag}(e^{\lambda_1 t}, ..., e^{\lambda_n t})\right)S^{-1}.$$

The trick is that e^A becomes really easy if $A = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ is a diagonal matrix. For

$$e^A = \sum_{k=0}^{\infty} \frac{\mathrm{diag}(\lambda_1^k, ..., \lambda_n^k)}{k!} = \mathrm{diag}(e^{\lambda_1}, ..., e^{\lambda_n}).$$

5 Examples

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