3236 L 125: Mathematical Physics I

Dynamical Systems and Classical Mechanics

Lecture by Yuri B. Suris
Technical University Berlin
Winter term 2018

Notes by Viet Duc Nguyen

Contents

0 About the notes

These are *personal* lecture notes for the course *Mathematical Physics I: Dynamical Systems and Classical Mechanics*, which is a undergraduate and graduate course of the Berlin Mathematical School held at the Technical University Berlin. The lecture was held by Professor Yuri B. Suris in the winter term 2018. As pointed out, these lecture notes are *personal* notes written by me for the course; thus it may contain mistakes!

Letters in bold typefaces usually refer to vectors in $\mathbb{R}^n$, e.g. $\mathbf{x} = (x_1, ..., x_n)^T$.

Literature recommendations by Professor Suris: V. Arnold, Mathematical Methods of

Classical Mechanics; V. Arnold, Ordinary differential equations; A. Knauf, Mathematische Physik: Klassische Mechanik

1 Introduction: Basic Principles of Dynamical Systems

Week 01
Tue, 16.10

States of a dynamical system are denoted by the *phase space* X. The phase space X is a vector space in $\mathbb{R}^n$ or a manifold. The evolution of states is described either by continuous time or discrete time. The entire future and past of a dynamical system is completely determined by one state $\mathbf{x}_0$ at any fixed moment of time $t_0 = 0$. If the world were purely described by dynamical systems, we would live in a completely deterministic world!

The one parameter family of maps $\Phi^t : X \to X, \mathbf{x}_0 \mapsto \mathbf{x}(t, \mathbf{x}_0)$ describes the evolution of any state $\mathbf{x}_0$ for time t. Φ^t is called a phase flow. It can be continuous $(\Phi^t)_{t\in\mathbb{R}}$ or discrete $(\Phi^t)_{t\in\mathbb{Z}}$.

We collect some properties of phase flows:

- $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$ for all $\mathbf{x}_0 \in X$ if and only if $\Phi^0 = \mathrm{id}_X$
- $\mathbf{x}(t + s, \mathbf{x}_0) = \mathbf{x}(t, \mathbf{x}(s, \mathbf{x}_0))$ for all $\mathbf{x}_0 \in X$ and for all $t, s \in \mathbb{R}$
- $\Phi^t \circ \Phi^{-t} = \Phi^{t-t} = \Phi^0 = \mathrm{id}_X \iff \Phi^{-t} = (\Phi^t)^{-1}$ for all $t \in \mathbb{R}$

A 1-parameter family of maps $(\Phi^t)_{t\in G}$ (G is an abelian group written additively, e.g. $G = \mathbb{R}$ or $G = \mathbb{Z}$) with properties $\Phi^0 = \mathrm{id}_X$ and $\Phi^{t+s} = \Phi^t \circ \Phi^s = \Phi^{s+t}$ is called an action of G on X of a dynamical system with phase space X and time G.

2 Existence and uniqueness of solutions of ordinary differential equations

Week 01
Thu, 18.10

Consider the system of *first-order ordinary*[1] differential equations:

$$\dot{\mathbf{x}} = f(t, \mathbf{x}), \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}.$$

[1]First-order means that only a derivative of rank one occurs. Ordinary means that one derives only after one variable, i.e. there are no partial derivatives.

We can rewrite the system of ODEs (ordinary differential equations) as

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, ..., x_n) \\ \quad \vdots \\ \dot{x}_n = f_n(t, x_1, ..., x_n). \end{cases}.$$

The question is: When is a function $x : I \to U$[2] a solution of the ODE? And especially: <u>When is a solution unique?</u>

Definition 1 (Lipschitz continuous). *Let f be a function. f is locally Lipschitz continuous with respect to x if and only if $\forall (t_0, x_0) \in \Omega, \exists$ neighborhood $\Omega_0 \subset \Omega$ of (t_0, x_0) and $L > 0$ such that:*

$$||f(t, x_1) - f(t, x_2)|| \leq L||x_1 - x_2||, \quad \forall (t, x_1), (t, x_2) \in \Omega_0.$$

Definition 2 (Solution of differential equations). *Let $I \subset \mathbb{R}$ be an <u>interval</u> and $f : \Omega \to \mathbb{R}^n$. A function $x : I \to \mathbb{R}^n$ is called a solution of an ODE $\dot{x} = f(t, x)$ if*

- *x is differentiable on I,*
- *$(t, x(t)) \in \Omega$ for all $t \in I$,*
- *$\dot{x}(t) = f(t, x(t))$ for all $t \in I$.*

2.1 Existence of solutions

Theorem 3 (Local existence after Picard-Lindelöf). *Let $f : \Omega \to \mathbb{R}^n$ be locally Lipschitz continuous. Consider $(t_0, x_0) \in \Omega$. Then, there exists one $\delta > 0$ such that the initial value problem*

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

has a solution on $I = (t_0 - \delta, t_0 + \delta)$.

Proof. We only provide a sketch of proof.

1. Reformulate the IVP (1) as integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

   Note that the integral exists since f is continuous.

[2] $I \subset \mathbb{R}$ and $U \subset \mathbb{R}^n$

2. Choose a compact neighborhood

$$K = \underbrace{[t_0 - a, t_0 + a]}_{:=I \subset \mathbb{R}} \times \underbrace{\{x : ||x - x_0|| \leq b\}}_{:=U \subset \mathbb{R}^n} \subset \Omega.$$

Let $M := \max_{(t,x) \in K} ||f(t,x)||$ (M exists because K is compact) and consider the function space $\mathcal{M}$ with:

$$\mathcal{M} = \mathcal{M}_{a,b} = \{\gamma \in C^0(I, \mathbb{R}^n) \mid \forall t \in I : ||\gamma(t) - x_0|| \leq b\}.$$

$\mathcal{M}$ is a complete, metric space (Banach space) of continuous functions with a metric given by:

$$d(\gamma_1, \gamma_2) = \max_{t \in I} ||\gamma_1(t) - \gamma_2(t)||.$$

Define an operator $P : \mathcal{M}_{a,b} \to \mathcal{M}_{a,b}$ that takes a function y and maps it to another function P_y:

$$(P_y)(t) = x_0 + \int_{t_0}^{t} f(s, y(s)) ds.$$

Is $P_y \in \mathcal{M}_{a,b}$? Indeed:

$$||P_y(t) - x_0|| = ||x_0 + \int_{t_0}^{t} f(s, y(s)) ds - x_0|| = || \int_{t_0}^{t} f(s, y(s)) ds|| \overset{(*)}{\leq} Ma \leq b.$$

We get the inequality $(*)$ by the mean value theorem. If $Ma > b$, we can adjust a such that $a \leq \frac{b}{M}$.

3. We construct the Picard-iterations:

$$y^{(0)}(t) := x_0, \quad y^{(n+1)}(t) := P_{y^{(n)}}(t), \quad t \in I.$$

We claim that $P_{y^{(n)}} \to y^*$ for $n \to \infty$ with $P_{y^*} = y^*$. In other words: y^* is a fixed point of P and we found a solution for the ODE (1).

We will only show that the iterations converge but we do not show that it actually converges to the fixed point of P. We could prove the converge to the fixed point by showing that P is a contraction but instead, observe that:

$$\begin{aligned} y^{(n)}(t) &= y^{(0)}(t) + (y^{(n)}(t) - y^{(n-1)}(t)) + (y^{(n-1)}(t) - y^{(n-2)}(t)) + ... \\ &= y^{(0)}(t) + \sum_{k=1}^{n} y^{(k)}(t) - y^{(k-1)}(t) \end{aligned}$$

So convergence of $\left(y^{(n)}(t)\right)_{n=0}^{\infty}$ is equivalent to convergence of the series $\sum_{k=1}^{\infty} y^{(k)}(t) - y^{(k-1)}(t)$. Let L be the Lipschitz constant of f. We will show that

$$\|y^{(n+1)}(t) - y^{(n)}(t)\| \leq \frac{M}{L} \left(\frac{(L|t - t_0|)^{n+1}}{(n+1)!} \right).$$

This is done by induction. Induction base for $n = 0$:

$$\|y^{(1)}(t) - y^{(0)}(t)\| = \| \int_{t_0}^{t} f(s, x_0) ds\| \le M|t - t_0|.$$

$n \rightsquigarrow n + 1$:

$$\begin{aligned}
\|y^{(n+1)}(t) - y^{(n)}(t)\| &= \| \int_{t_0}^{t} \Big(f\big(s, y^{(n)}(s)\big) - f\big(s, y^{(n-1)}(s)\big)\Big)\, ds\| && \text{Lipschitz continuous} \\
&\le L \int_{t_0}^{t} \|y^{(n)}(s) - y^{(n-1)}(s)\| ds && \text{induction basis} \\
&\le L \int_{t_0}^{t} \frac{M}{L} \left(\frac{(L|s - t_0|)^n}{n!} \right) ds && \text{integrate} \\
&= \frac{M}{L} \frac{(L|t - t_0|)^{n+1}}{(n + 1)!}
\end{aligned}$$

We showed the convergence of the Picard iterations.

□

2.2 Uniqueness of solutions

Week 02
Tue, 23.10

Remember: $\dot{x} = f(t, x), \quad (t, x) \in I \times U = \Omega \subset \mathbb{R}^{n+1}, I \subset \mathbb{R}, U \subset \mathbb{R}^n$, where I is the time space and U is the phase space.

Theorem 4 (Uniqueness theorem). *Let $f : \Omega \to \mathbb{R}^n$ be locally Lipschitz continuous with respect to x. If $x^{(1)} : I \to \mathbb{R}, x^{(2)} : I \to \mathbb{R}$ are two solutions of $\dot{x} = f(t, x)$ with $x^{(1)}(t_0) = x^{(2)}(t_0)$ for some $t_0 \in I$, then*

$$x^{(1)}(t) = x^{(2)}(t), \quad \forall t \in I.$$

This follows from the *Lemma of Gronwall.* So we will show this lemma first, and then use it to prove the theorem of uniqueness.

Theorem 5 (Lemma of Gronwall). *Let $a, b, w : [t_0, t_0 + c] \to \mathbb{R}_{\ge 0}$ be three non-negative functions satisfying*

$$w(t) \le a(t) + \int_{t_0}^{t} b(s)w(s)ds, \quad \forall t \in [t_0, t_0 + c].$$

Then there holds:

$$w(t) \le a(t) + \int_{t_0}^{t} a(s)b(s)e^{\int_s^t b(\tau)d\tau} ds, \quad \forall t \in [t_0, t_0 + c].$$

Proof. Let $y(t) := \int_{t_0}^t b(s)w(s)ds$. If we prove that the following holds

$$y(t) \leq \int_{t_0}^t a(s)b(s)e^{\int_s^t b(\tau)d\tau}ds,$$

we get the claim since $w(t) \leq a + y(t)$. We know that $w(t) \leq a(t) + y(t)$, so it follows $w(t) - y(t) \leq a(t)$. Now, lets multiply this with $b(t)$ and since b is non-negative, we get:

$$\underbrace{w(t)b(t)}_{=y'(t)} - y(t)b(t) \leq a(t)b(t).$$

Multiply this with $e^{-\int_{t_0}^t b(s)ds}$

$$\underbrace{\Big(y'(t) - y(t)b(t)\Big)e^{-\int_{t_0}^t b(s)ds}}_{=\frac{d}{dt}\left(y(t)e^{-\int_{t_0}^t b(s)ds}\right)} \leq a(t)b(t)e^{-\int_{t_0}^t b(s)ds}.$$

So we get

$$\frac{d}{dt}\left(y(t)e^{-\int_{t_0}^t b(s)ds}\right) \leq a(t)b(t)e^{-\int_{t_0}^t b(s)ds}.$$

We integrate both sides with $\int_{t_0}^t$ and since $y(t_0) = 0$ we get:

$$y(t)e^{-\int_{t_0}^t b(s)ds} \leq \int_{t_0}^t a(s)b(s)e^{-\int_{t_0}^s b(\tau)d\tau}ds$$

In the end, we multiply by $e^{\int_{t_0}^t b(s)ds}$ and we get our final result

$$y(t) \leq \int_{t_0}^t a(s)b(s)\underbrace{e^{-\int_{t_0}^s b(\tau)d\tau}}_{=e^{\int_s^{t_0} b(\tau)d\tau}}e^{\int_{t_0}^t b(s)ds}ds = \int_{t_0}^t a(s)b(s)e^{\int_s^t b(\tau)d\tau}ds.$$

□

Remark. This lemma is useful to estimate w with known functions a and b. There are three particular cases:

1. Let $b(s) = b =$ constant. So it holds:

$$w(t) \leq a(t) + b\int_{t_0}^t w(s)ds \implies w(t) \leq a(t) + b\int_{t_0}^t a(s)e^{b(t-s)}ds.$$

2. Let $a(s) = a =$ constant and $b(s) = b =$ constant. One can show that:

$$w(t) \leq a + b\int_{t_0}^t w(s)ds \implies w(t) \leq ae^{b(t-t_0)}$$

3. Let $a = 0$ and let $b = b(s) =$ constant.

$$w(t) \leq b \int_{t_0}^{t} w(s)ds \implies w = 0. \tag{2}$$

Next, we will prove the uniqueness lemma which is not that difficult anymore.

Proof of the uniqueness theorem. Let $J \subset I$ be a closed interval with $t_0 \in J$. Then the solution curves $x^{(1)}, x^{(2)} : J \to \mathbb{R}$ are compact. Set $w(t) := \|x^{(1)}(t) - x^{(2)}(t)\|$. Remember that $\dot{x} = f(s, x)$. Then for $t \in J$ we have:

$$\begin{aligned} w(t) &= \| \int_{t_0}^{t} f(s, x^{(1)}(s)) - f(s, x^{(2)}(s))ds \| \\ &\leq L \int_{t_0}^{t} \|x^{(1)}(s) - x^{(2)}(s)\| ds, \end{aligned}$$

where L is the maximal local Lipschitz constant over J. Note that L exists since J is compact and f is continuous (it is even local Lipschitz continuous).

$$w(t) \leq L \int_{t_0}^{t} \|x^{(1)}(s) - x^{(2)}(s)\| ds = L \int_{t_0}^{t} w(s)ds \overset{(2)}{\implies} w(t) = 0, \quad \forall t \in J.$$

□

Remark. If $f(t, x)$ is not locally Lipschitz continuous, then uniqueness may not hold and as a rule of thumb, there will be no uniqueness of solutions. Murphy's Law in action. The IVP

$$\dot{x} = \sqrt[3]{x^2}, \quad x(0) = 0 \tag{3}$$

has an infinite solutions. For instance, there are solutions with $x(t) = 0$ or for $b > 0$:

$$x_b(t) = \begin{cases} 0, & \text{for } t < b \\ \frac{1}{27}(t - b)^3, & \text{for } t \geq b. \end{cases}$$

2.3 Continuation of solutions

Definition 6. *We say that a solution $x : (t_0, t_1) \to \mathbb{R}^n$ can be* continued *to the right if there exists a solution $\tilde{x} : (t_0, t_2) \to \mathbb{R}^n$ with $t_2 > t_1$ such that:*

$$\tilde{x}(t) = x(t), \quad \forall t \in (t_0, t_1).$$

We say that x is a maximal *solution if it cannot be continued neither to the right nor to the left.*

Theorem 7. *Let $x : (t_0, t_1) \to \mathbb{R}^n$ be a solution for $\dot{x} = f(t, x)$ and $f : \Omega \to \mathbb{R}^n$. A solution x can be continued to the right if and only if there exists the limit $x_1 := \lim_{t \nearrow t_1} x(t)$ and $(t_1, x_1) \in \Omega$.*

Proof. Assume the limit exists $x_1 = \lim_{t \nearrow t_1} x(t)$. We can apply the uniqueness and existence theorem to show that the x can be continued to the right. Consider the IVP with $x(t_1) = x_1$. There exists a solution on $(t_1 - \delta, t_1 + \delta)$ for the IVP which we call y. Define $\tilde{x}$ as follows:

$$\tilde{x}(t) := \begin{cases} x(t), & \text{for } t \in (t_0, t_1) \\ y(t), & \text{for } t \in (t_1, t_1 + \delta) \end{cases}.$$

We expanded the solution on $(t_0, t_1 + \delta)$. This function $\tilde{x}(t)$ is also continuous due to

$$\lim_{t \nearrow t_1} \tilde{x}(t) = \lim_{t \nearrow t_1} x(t) = x_1 = y(t_1) = \tilde{x}(t_1).$$

The other direction is easy(?). I don't know... $\square$

Theorem 8 (Theorem about maximal solutions). *Let $f : \Omega \to \mathbb{R}^n$ be locally Lipschitz continuous with respect to x and let $x : (t_-, t_+) \to \mathbb{R}^n$ be a maximal solution of $\dot{x} = f(t, x)$. Then:*

- *either $t_+ = \infty$,*
- *or $t_+ < \infty$ and for each <u>compact</u> set $K \subset \Omega$ there exists $t' < t_+$ such that*

  $$\forall t \in (t', t_+) : \big(t, x(t)\big) \notin K.$$

  In other words, $x(t)$ leaves any compact subset $K \subset \Omega$, which usually means that $x(t)$ becomes infinite in finite time.

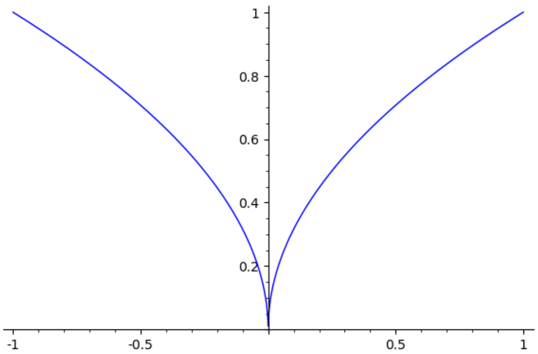
Remark. The only reason for a solution to have finite time is the blow-up within finite time. Consider the ODE

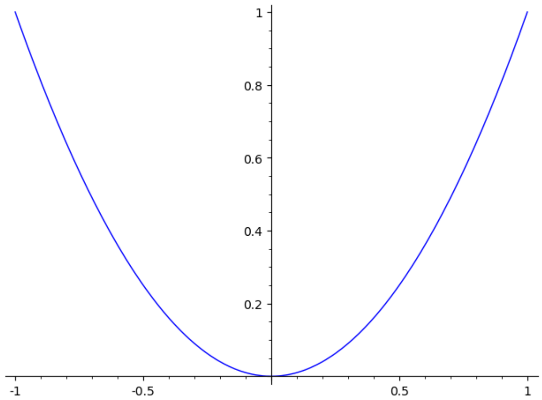
$$\dot{x} = x^2. \tag{4}$$

It has solutions $x(t) = \frac{1}{c-t} \to \infty$ for $t \nearrow c$.

Theorem 9 (No blow-up for linearly growing f). *If $\|f(t, x)\| \leq \alpha(t)\|x\| + \beta(t)$ for all $(t, x) \in I \times \mathbb{R}^n$, then the maximal solutions are defined on the whole intervall I (usually $I = \mathbb{R}$).*

Remark. If $\|f(t,x)\|$ grows as $\|x\|^p$ with $p > 1$, one should expect blow-up. For $p < 1$, the uniqueness of solutions is not granted, see (3). For $p > 1$, it usually blows up, see (4).





2.4 Dependence of solutions on initial data and on parameters

Consider the IVP

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = x_0 \end{cases}$$

with solution $x(t, t_0, x_0)$. We use additional parameters (t_0, x_0) to highlight the dependency of the solution $x(t)$ on the initial value. How does a solution $x(t, t_0, x_0)$ depend on x_0?

Theorem 10. *Let $f : \Omega \to \mathbb{R}^n$ be continuously differentiable with respect to x. Then the solution $x(t, t_0, x_0)$ is continuously differentiable with respect to x_0. The Jacobi matrix*

$$y := \frac{\partial x(t, t_0, x_0)}{\partial x_0} \in \mathrm{Mat_{n\times n}}$$

solves Poincare's variational equation*:*

$$\begin{cases} \dot{y} = A(t)y \\ y(t_0) = I_{n\times n} \end{cases} \quad \textit{with} \quad A(t) := \frac{\partial f(t,x)}{\partial x}\Big|_{x=x(t,t_0,x_0)} = \frac{\partial f}{\partial x}\big(t, x(t, t_0, x_0)\big). \tag{5}$$

Theorem 11. *Let f depend on some parameters $\lambda \in \mathbb{R}^p$, so that*

$$f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p \supset \Omega \to \mathbb{R}^n$$

is $\mathcal{C}^1$ with respect to x and λ. Consider the IVP

$$\begin{cases} \dot{x} = f(t, x, \lambda), \\ x(t_0) = x_0 \end{cases}$$

with solution $x(t, t_0, x_0, \lambda)$. Then the function $x(t, t_0, x_0, \lambda)$ is continuously differentiable with respect to λ and the Jacobi matrix

$$y := \frac{\partial x(t, t_0, x_0, \lambda)}{\partial \lambda} \in \mathrm{Mat}_{n \times p}$$

solves the following IVP:

$$\begin{cases} \dot{y} = Aty + B(t) \\ y(t_0) = 0_{n \times p} \end{cases} \quad \textit{with} \quad \begin{aligned} A &:= \frac{\partial f}{\partial x}(t, x(t), \lambda) \\ B &:= \frac{\partial f}{\partial \lambda}(t, x(t), \lambda). \end{aligned}$$

3 Dynamical systems

Week 02
Thu, 25.10

3.1 Autonomous differential equations

Loosely spoken: autonomous differential equations $\equiv$ dynamical systems with continuous time. Let $f : \mathbb{R}^n \supset U \to \mathbb{R}^n$ be a vector field and locally Lipschitz continuous. Consider the ODE with

$$\dot{x} = f(x).$$

This means, the law of motion does not depend on time. The system is completely deterministic. In other words, the instantaneous velocity $\dot{x}$ only depends on the position x and not on the moment of time t at which the system arrives at x.

Remark. We observe: if $x : I \to U$ is a solution of $\dot{x} = f(x)$, then $y(t) := x(t - \tau)$ is a solution, as well.

Proof. $\dot{y}(t) = \dot{x}(t - \tau) = f(x(t - \tau)) = f(y(t))$. □

Remark. Try the proof for a non-autonomous ODE $\dot{x} = f(t, x)$ with solution $y(t) := x(t - \tau)$.

$$\dot{y}(t) = \dot{x}(t - \tau) = f(t - \tau, x(t - \tau)) = f(t - \tau, y(t)) \neq f(t, y(t)) \quad 21\lightning$$

Definition 12 (Group). *A group $(G, *)$ is a set G together with $* : G \times G \to G$ such that*

- $\forall g, h \in G : g * h \in G$ *(closure),*
- $\forall g, h, k \in G : (g * h) * k = g * (h * k)$ *(associativity),*
- $\forall g \in G, \exists! e \in G : g * e = e * g = g$ *(neutral element),*
- $\forall g \in G, \exists h \in G : g * h = h * g = e$ *(inverse element).*

Definition 13 (Left group action). *Let M be a set and G be a group. The left action of G on M is the map $\Phi_g : G \times M \to M, (g, x) \mapsto \Phi_g(x)$ such that*

1. $\forall g, h \in G, x \in M : \Phi_{g+h}(x) = \Phi_g(\Phi_h(x)) = (\Phi_g \circ \Phi_h)(x)$,
2. $\exists! e \in G, \forall x \in M : \Phi_e(x) = x$.

If these two axioms are satisfied, then Φ_g is a bijection from M to M for fixed g.

3.2 Realisation of dynamical systems

We now know that every initial value problem

$$\begin{cases} \dot{x} = \mathbf{f}(t, x) \\ x(t_0) = x_0 \end{cases} \qquad \text{where } \mathbf{f} \in \mathcal{C}^k(I \times M^{3}, \mathbb{R}^n), k \geq 1$$

has a local solution $\varphi(t, t_0, x_0)$ that is of class $\mathcal{C}^k$ with respect to (t, t_0, x_0). Moreover, the solution can be extended in time until either reaches the homology of M or blows up at infinity.

So, what is a dynamical system? It is a mathematical formalisation of a deterministic process. A dynamical system consists of

- a complete metric space M, which is called *phase space*. It constains all possible states of a region. We assume that $\dim M < \infty$.
- a time variable $\underbrace{t \in \mathbb{R}, \mathbb{R}_+}_{\text{continuous time}}$ or $\underbrace{t \in \mathbb{Z}, \mathbb{Z}_+}_{\text{discrete time}}$.
- and a law of evolution in time (time evolution operator).

Definition 14 (Dynamical system). *Let $M \subset \mathbb{R}^n$ be a phase space. Let G be a number set with a group structure ($G \subset \mathbb{R}$) and $g \in G$ denotes the time variable. A* dynamical system *is a triple $\{\Phi_y, G, M\}$ defined in terms of the G-action:*

$$\Phi_G : G \times M, (g, x) \mapsto \Phi_g(x) \qquad (\textit{time evolution operator})$$

[3] $I \subset \mathbb{R}, M \subset \mathbb{R}^n$

If we consider $G = \mathbb{R}$ and $G * G \to G, g * h := g + h$, we have an abelian group. For every left group action holds: $\Phi_{g+h}(x) = \Phi_g(\Phi_h(x)) = \Phi_h(\Phi_g(x)) = \Phi_{h+g}(x)$.

How do we define dynamical systems in terms of initial value problems?

Consider a continuous dynamical system $\{\Phi_t, \mathbb{R}, M\}, M \subset \mathbb{R}^n$. The time evolution operator Φ_t is the *global flow* of an autonomous IVP

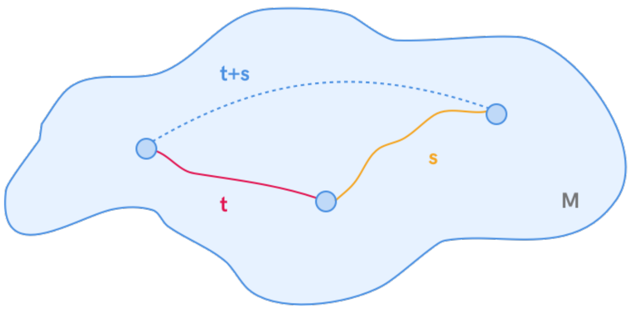
$$\dot{x} = f(x), x(0) = x_0 \in M. \tag{6}$$

That means, that Φ_t solves the IVP above. Let φ be the unique local solution of (6). So Φ_t is defined as

$$\Phi_t : \mathbb{R} \times M \to M, (t, x) \mapsto \Phi_t(x) := \varphi(t, x).$$

The following properties hold

1. $\Phi_{t+s}(x) = (\Phi_t \circ \Phi_s)(x)$ for every $t, s \in \mathbb{R}$ and $x \in M$

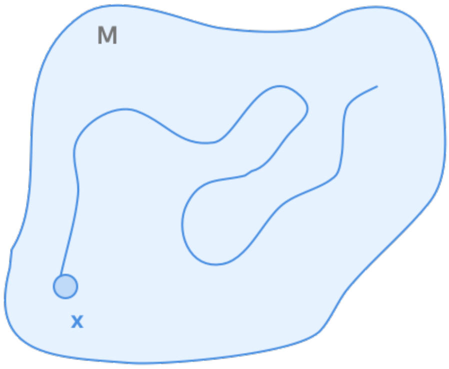


2. $\Phi_0(x) = x$ for every $x \in M$
3. $\Phi_t \circ \Phi_{-t} = \mathrm{id}_{\mathrm{M}}$.

Definition 15 (Orbit). *An* orbit *starting at $x_0 \in M$ is the ordered subset of M defined by*

$$\mathcal{O}(x_0) = \{x \in M : x = \Phi_g(x_0), \forall g \in G\}.$$

It is a parametrised curve in the phase space M.



Remark. If $y_0 = \Phi_g(x_0)$ for some $g \in G$, then the sets $\mathcal{O}(x_0)$ and $\mathcal{O}(y_0)$ are equal. Different orbits are disjoint.

Definition 16 (Fixed point, cycle, phase portrait, invariant set, invariant functions).

1. *A point $\tilde{x} \in M$ is a* fixed point *if $\tilde{x} = \Phi_G(\tilde{x})$ for every $g \in G$.*
2. *A* cycle *is an orbit $\mathcal{O}(x_0)$ such that each point $x \in \mathcal{O}(x_0)$ satisfies*
$$\Phi_{g+T}(x) = \Phi_g(x), \forall g \in G \textit{ with some fixed } T > 0.$$
The minimal T is called the period.
3. *The* phase portrait *is a partitioning of M into orbits (collection of orbits).*
4. *An* invariant set *is a subset $S \subset M$ such that $\Phi_g(S) \subset S$ for every $g \in G$.*
5. *An* invariant function *is a function $F : M \to \mathbb{R}$ such that $F(\Phi_g(x)) = F(x)$ for all $g \in G, x \in M$. This means that the F cannot distinguish between points in an orbit.*

Remark. The IVP in (6) is autonomous (independent of time). This means that the IVP has a time-translational symmetry. We can always set $t_0 = 0$. If $\varphi(t, x_0)$ with $t \in I_{x_0}$ is a solution for the IVP with initial value $t(0) = x_0$, so is $\psi(t, x_0) = \varphi(t + s, x_0)$ with $t + s \in I_{x_0}$. We get
$$\dot{\psi}(t, x_0) = \dot{\varphi}(t + s, x_0) = f(\varphi(t + s, x_0)) = f(\psi(t, x_0)).$$

Example. Not every flow is global. Consider $W : \bigcup_{x_0 \in M} I_{x_0} \times \{x_0\} \subset \mathbb{R} \times M$ where I_{x_0} is the maximal intervall on which a solution φ exists for the IVP with initial value $x(0) = x_0$. Look at the scalar differential equation:
$$\begin{cases} \dot{x} = x^3 \\ x(0) = x_0, x_0 \in M \end{cases}$$

The flow is defined as

$$W = \{(t, x_0) : 2tx_0^2 < 1\},$$
$$I_{x_0} = (-\infty, \frac{1}{2x_0^2}),$$
$$\Phi_t : W \to \mathbb{R}, (t, x_0) \mapsto \varphi(t, x_0) = \frac{x_0}{\sqrt{1 - 2x_0^2 t}}.$$

Graphical interpretations

A solution $t \mapsto \varphi(t, x_0)$ defines for each $x_0 \in M$ two curves:

1. A solution curve (integral curve, trajectory curve):

$$\gamma(x_0) = \{(t, x) : x = \Phi_t(x_0), \forall t \in I_{x_0}\} \subset I_{x_0} \times M.$$

2. An orbit or the projection of $\gamma(x_0)$ on M is

$$\mathcal{O}(x_0) = \{x : x = \Phi_t(x_0), \forall t \in I_{x_0}\} \subset M.$$

An orbit cannot intersect itself. If two orbits have a common point, they coincide.

Remark. The zeros of f of (6) define the fixed points. If $f(\tilde{x}) = 0$, then $\Phi_t(\tilde{x}) = \tilde{x}$ for every $t \in I_{x_0}$.

Week 03
Tue, 30.10

From autonomous ordinary differential equations to dynamical systems: Consider the IVP $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ with solution $x(t, 0, x_0)$. Define the flow as:

$$\Phi^t_{x_0} \coloneqq x(t, 0, x_0) \quad \text{and} \quad \Phi^t : X \to X, x_0 \mapsto \Phi^t_{x_0}$$

It holds:

(1) $\Phi^0 = \mathrm{id_X}$

(2) $\Phi^t \circ \Phi^s = \Phi^s \circ \Phi^t = \Phi^{t+s}$

Proof. We only prove the second statement. Consider the Taylor expansion at $t_0 = 0$:

$$x(t, 0, x_0) = x(0) + \sum_{i=1}^{\infty} \frac{x^{(i)}(0, 0, x_0)}{i!} t = x_0 + \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} t$$

It holds that

$$\begin{aligned}
x(s,0,x(t,0,x_0)) &= x(t,0,x_0) + \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} s \\
&= x_0 + \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} t + \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} s \\
&= x_0 + \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} (t+s) \\
&= x(t+s,0,x_0).
\end{aligned}$$

□

From a dynamical system to an autonomuous ODE: Let $(\Phi_t, \mathbb{R}, M), M \subset \mathbb{R}^n$ be a continuous dynamical system. We obtain the corresponding IVP by:

$$f(x) = \frac{d}{dt} \Phi^t(x) \Big|_{t=0}.$$

3.3 Diffeomorphism, Lie derivative and integral of motion

We will show that maps Φ^t are local diffeomorphisms. Before, we need the following lemma.

Lemma 17 (Liouville's formula). *Let $I \subset \mathbb{R}$ be an intervall and $A : I \to \mathbb{R}^{n \times n}$ continuous. Let $Y : I \to \mathbb{R}^{n \times n}$ be a* general *solution for the ODE*

$$\dot{y} = A(t)y. \tag{7}$$

Then for all $t, t_0 \in I$, it holds:

$$\det Y(t) = \det Y(t_0) e^{\int_{t_0}^{t} \operatorname{tr} A(z) dz}. \tag{8}$$

Proof. We will compute $\frac{d}{dt} \det Y(t)$. Since $\det Y(t)$ is multilinear with respect to all entries Y_{ij} of the matrix Y, it holds for a fixed $i \in \{1, ..., n\}$

$$\det Y = \sum_{j=1}^{n} Y_{ij} \underbrace{(-1)^{i+j} \det \tilde{Y}_{ij}}_{:=u_{ij}} = \sum_{j=1}^{n} Y_{ij} u_{ij}. \tag{9}$$

The u_{ij} are also called *cofactors*. So we have with chain rule and product rule applied on det:

$$\frac{d}{dt}\det Y = \sum_{i,j=1}^{n} \frac{\partial \det Y}{\partial Y_{ij}} \dot{Y}_{ij} \overset{(9)}{=} \sum_{i,j=1}^{n} u_{ij} \dot{Y}_{ij} \overset{(7)}{=} \sum_{i,j=1}^{n} u_{ij} \left(\sum_{k=1}^{n} A_{ik}(t) Y_{kj} \right)$$
$$= \sum_{i,k=1}^{n} A_{ik}(t) \underbrace{\sum_{j=1}^{n} Y_{kj} u_{ij}}_{(*)}$$
$$= \mathrm{tr} A(t) \cdot \det Y.$$

$(*)$ means that if $i = k$, this is $\det Y$ by (9).

If $i \neq k$, $(*)$ is zero since $(*) = \det(\text{matrix } Y \text{ with row } i \text{ replaced by row } k)$.

Transforming the last equation yields

$$\frac{\frac{d}{dt}\det Y(t)}{\det Y(t)} = \frac{d}{dt} \ln(\det Y(t)) = \mathrm{tr} A(t).$$

Now we integrate with arbitrary $t_0 \in I$ and apply the exponential function

$$\ln(\det Y(t)) - \ln(\det Y(t_0)) = \int_{t_0}^{t} \mathrm{tr} A(z) dz \implies \det Y(t) = \det Y(t_0) e^{\int_{t_0}^{t} \mathrm{tr} A(z) dz}.$$

□

Collorary 18.

$$\exists t_0 \in I : \det Y(t_0) \neq 0 \implies \forall t \in I : \det Y(t) \neq 0 \tag{10}$$

Proof. Since $e^c \neq 0$ for all $c \in \mathbb{R}$ and we can chose t_0 such that $\det Y(t_0) \neq 0$, it holds that for all $\det Y(t) \neq 0$ after (8). □

Theorem 19. *Maps Φ^t are local diffeomorphisms.*

Proof. It is clear that Φ^t is differentiable and continuous. By theorem about inverse functions, we just need to show that $\frac{\partial}{\partial x_0}\Phi^t$ exists and is invertible. As said, the derivative exists and we show that the determinant of the invert of the derivative is non-zero (and thus invertible):

$$\det \left(\frac{\partial}{\partial x_0} \Phi^t \right) \neq 0.$$

We know from theorem (10) that $Y(t) := \frac{\partial}{\partial x_0}\Phi^t = \frac{\partial x(t,0,x_0)}{\partial x_0}$ satisfies a linear differential equation

$$\begin{cases} \dot{Y} = A(t) \cdot Y \\ Y(0) = I \end{cases}$$

with $A(t) = \frac{\partial f(x(t,0,x_0))}{\partial x}$. Now we can use the result of collorary (18):

$$\det Y(0) = \det I = 1 \neq 0 \overset{(10)}{\Longrightarrow} \forall |t| < \delta(x_0) : \det Y(t) = \det \left(\frac{\partial}{\partial x_0} \Phi^t \right) \neq 0.$$

□

Remark. If δ can be chosen one and the same for all $x_0 \in X$, then Φ^t is defined for all $t \in \mathbb{R}$ and for all $x_0 \in X$. Then Φ^t is a *global* diffeomorphism.

It is useful to know how some function $F : X \to \mathbb{R}$ evolves along solutions $\Phi^t(x)$. Thus, we consider $F(\Phi^t(x))$. This is studied by considering

$$\frac{d}{dt} F(\Phi^t(x)) = dF(\Phi^t(x)) \cdot \frac{d}{dt}\Phi^t = dF(\Phi^t(x)) \cdot f(\Phi^t(x)).$$

We can omit the arguments:

$$\frac{dF}{dt} = \dot{F} = dF \cdot f = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial x_k} f_k \right).$$

Definition 20 (Lie derivative). *The* Lie derivative *of the function F along the vector field $f(x)$ is defined as:*

$$\mathcal{L}_f F(x) := \frac{d}{dt} F(\Phi^t(x)) \bigg|_{t=0} = \sum_{k=1}^{n} f_k(x) \frac{\partial F}{\partial x_k}(x).$$

One often writes $\mathcal{L}_f$ as an operator acting on functions $F : X \to \mathbb{R}$

$$\mathcal{L}_f = \sum_{k=1}^{n} f_k(x) \frac{\partial}{\partial x_k}.$$

Definition 21 (Conserved quantity, first integral of motion). *A function $F : X \to \mathbb{R}$ is a conserved quantity or first integral of motion or simply an integral of the ODE $\dot{x} = f(x)$ if*

$$\forall t : F(\Phi^t(x)) = F(x).$$

Theorem 22. *$F : X \to \mathbb{R}$ is a conserved quantity iff $\mathcal{L}_f F = 0$.*

Remark. Given a function F and a vector field f, it is straightforward to check, whether F is a conserved quantity for $\dot{x} = f(x)$.

The problem of finding an integral of motion F for a given vector field f is usually very complicated, for it consists of solving a partial differential equation

$$\sum f_k \frac{\partial F}{\partial x_k} = 0.$$

Example. Let $U : \mathbb{R}^n \to \mathbb{R}$ be the potential energy. Consider the ODE:

$$\ddot{X} = -\frac{\partial U(x)}{\partial x}.$$

Written as first order ODE:

$$\begin{cases} \dot{x} = u \\ \dot{u} = -\frac{\partial U(x)}{\partial x}. \end{cases}$$

We claim that $\epsilon(x, u) := \frac{1}{2}\langle u, u\rangle + U(x)$ is an integral of motion. The vector field is

$$f(x, u) = \begin{pmatrix} u \\ -\frac{\partial U(x)}{\partial x}. \end{pmatrix} \text{ with } u \in \mathbb{R}^n, -\frac{\partial U(x)}{\partial x} \in \mathbb{R}^n.$$

Check that this is indeed an integral of motion:

$$\mathcal{L}_f \epsilon = \sum_{k=1}^{n} \frac{\partial \epsilon}{\partial x_k} u_k + \sum_{k=1}^{n} \frac{\partial \epsilon}{\partial x_k} \left(-\frac{\partial U(x)}{\partial x_k} \right) = \sum_{k=1}^{n} \frac{\partial U}{\partial x_k} u_k + \sum_{k=1}^{n} u_k \left(-\frac{\partial U(x)}{\partial x_k} \right) = 0.$$

Note, that $(\frac{1}{2}\langle u, u\rangle)_k = \frac{1}{2}u_k^2$ and therefore $(\frac{d}{2du}\langle u, u\rangle)_k = u_k$. Since we have $\mathcal{L}_f \epsilon = 0$, ϵ is indeed an integral of motion.

Example (Lotka-Volterra system). Let x denote the number of hares, and y denote the number of foxes. Consider the ODE

$$\begin{cases} \dot{x} = ax - by \\ \dot{y} = -cy + dy \end{cases}$$

with $a, b, c, d > 0$. This possess an integral of motion

$$F(x, y) = dx - by - c\ln(x) - a\ln(y).$$

Check:

$$\mathcal{L}_f F = (d - \frac{c}{x}) \underbrace{(ax - by)}_{=f_1} + (-b - \frac{a}{y}) \underbrace{(-cy + dy)}_{=f_2} = 0.$$

Remark. If F is an integral of motion for an ODE $\dot{x} = f(x)$, then the level sets of F are the orbits of the flows Φ^t. For instance, let $Q \coloneqq \{x : F(x) = y\}$ be the level set of y. Then $Q \supset \mathcal{O}(y)$.

Remark. If the level sets are closed curves, then all solutions $\Phi^t(x_0) = x(t, 0, x_0)$ and hence all orbits are periodic.

Definition 23 (Conserved quantity for discrete systems). *A function $F : X \to \mathbb{R}$ is called a conserved quantity or an integral of motion of a discrete dynamical system $(\Phi^n)_{n\in\mathbb{Z}}$ iff*

$$\forall n \in \mathbb{Z}, \forall x : F(\Phi^n(x)) = F(x)$$

Remark. We have no Lie derivatives to check this! But it is sufficient to validate for $n = 1$, i.e.

$$\forall x : F(\Phi(x)) = F(x).$$

3.4 Commutator of vector fields

Week 03
Thu, 01.11

Definition 24 (Lie algebra). *A Lie algebra is a vector space $\mathcal{H}$ supplied with a blinear operation*

$$[\cdot, \cdot] : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$$

with the properties:

- *for all $f, g, h \in H$ and for all $a, b \in \mathbb{R}$: $[af + bg, h] = a[f, h] + b[g, h]$,*
- *for all $f, g, h \in H$ and for all $a, b \in \mathbb{R}$: $[f, ag + bh] = a[f, g] + b[f, h]$,*
- *$[f, g] = [g, f]$ (skew symmetry),*
- *$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$ for all $f, g, h \in \mathcal{H}$ (Jacobi identity).*

Example (Standard example). Let $\mathcal{H} = \mathbb{R}^{n\times n}$ and $[f, g] \coloneqq fg - gf$. Then $(\mathcal{H}, [\cdot, \cdot])$ is a lie algebra. We should check that all the properties hold but this is omitted.

Definition 25 (Lie bracket). *The* Lie bracket *of two vector fields $f, g : X \to X$ is the vector field denoted by $[\mathcal{L}_f, \mathcal{L}_g]$ and defined by*

$$[\mathcal{L}_f, \mathcal{L}_g] \coloneqq \sum_{i,j=1}^{n} \Big(f_j(x) \frac{\partial g_i}{\partial x_j} - g_j(x) \frac{\partial f_i}{\partial x_j} \Big) \frac{\partial}{\partial x_i}$$

Suppose we have two vector fields f, g on the phase space X. Consider the ODEs

$$\dot{x} = f(x) \quad \text{and} \quad \dot{x} = g(x)$$

with the corresponding phase flows $\Phi^t, \Psi^t : X \to X$. Take an arbitrary function $F : X \to \mathbb{R}$ with $F \in \mathcal{C}^2(X, \mathbb{R})$ and compute

$$\Delta(t_1, t_2, x) \coloneqq F(\Phi^{t_1}(\Psi^{t_2}(x))) - F(\Psi^{t_2}(\Phi^{t_1}(x)).$$

We want to estimate Δ for small t_1, t_2. If $t_2 = 0$, then $\Delta(t_1, 0, x) = F(\Phi^{t_1}(x)) - F(\Phi^{t_1}(x)) = 0$. Similarly, if $t_1 = 0$, then $\Delta(0, t_2, x) = 0$.

The Taylor expansion for $\Delta(t_1, t_2, x)$ for small $(t_1, t_2) \in \mathbb{R}^2$ at point $p = (0, 0)$ looks like

$$\begin{aligned}\Delta(t_1, t_2) = \underbrace{\Delta(0,0)}_{=0} + \underbrace{\frac{\partial \Delta(0,0)}{\partial t_1} t_1}_{=0} + \underbrace{\frac{\partial \Delta(0,0)}{\partial t_2} t_2}_{=0} + \underbrace{\frac{1}{2}\frac{\partial^2 \Delta(0,0)}{\partial t_1^2} t_1^2}_{=0} + \underbrace{\frac{1}{2}\frac{\partial^2 \Delta(0,0)}{\partial t_2^2} t_2^2}_{=0} \\ + \underbrace{\frac{\partial^2 \Delta(0,0)}{\partial t_1 \partial t_2}}_{:=\delta} t_1 t_2 + O(t_1^2 t_2, t_1 t_2^2)\end{aligned}$$

Why do some derivates become zero? Because the trick is that $\Psi^0(x) = x$...

$$\begin{aligned}\frac{\partial \Delta(0,0)}{\partial t_1}(x) &= \frac{\partial}{\partial t_1} F(\Phi^{t_1}(\Psi^{t_2}(x))) - F(\Psi^{t_2}(\Phi^{t_1}(x)))\Big|_{(t_1,t_2)=(0,0)} \\ &= \frac{\partial F\left(\Phi^{t_1}(\Psi^{t_2}(x))\right)}{\partial \Phi^{t_1}(\Psi^{t_2}(x))} \cdot \frac{\partial}{\partial t_1}\Phi^{t_1}(\Psi^0(x)) - \frac{\partial F\left(\Psi^{t_2}(\Phi^{t_1}(x))\right)}{\partial \Psi^{t_2}(\Phi^{t_1}(x))} \cdot \frac{\partial}{\partial t_1}\Psi^0(\Phi^{t_1}(x)) \\ &= \frac{\partial F(y)}{\partial y}\Big|_{y=\Phi^0(\Psi^0(x))} \frac{\partial}{\partial t_1}\Phi^{t_1}(x) - \frac{\partial F(y)}{\partial y}\Big|_{y=\Psi^0(\Phi^0(x))} \frac{\partial}{\partial t_1}\Phi^{t_1}(x) \\ &= \frac{\partial F(x)}{\partial x}\frac{\partial}{\partial t_1}\Phi^{t_1}(x) - \frac{\partial F(x)}{\partial x}\frac{\partial}{\partial t_1}\Phi^{t_1}(x) \\ &= 0.\end{aligned}$$

So we get

$$\Delta(t_1, t_2) = \delta ts + \mathcal{O}(t^2 s, ts^2)$$

with $\delta = \frac{\partial^2 \Delta(0,0)}{\partial t \partial s} \in \mathbb{R}$.

Claim: $\delta = (\mathcal{L}_g \mathcal{L}_f - \mathcal{L}_f \mathcal{L}_g) F(x)$ for $F \in \mathcal{C}^2(X, \mathbb{R})$.

Proof. Let $\Phi^t(x)$ be a solution of an ODE with vector field f and $\Psi^s(x)$ be a solution for the corresponding ODE with vector field g. We know that $\mathcal{L}_f F(x) = G(x) \coloneqq \frac{\partial}{\partial t} F(\Phi^t(x))\Big|_{t=0}$ by definition 20. So

$$\mathcal{L}_g \mathcal{L}_f F(x) = \mathcal{L}_g G(x) = \frac{\partial}{\partial s} G(\Psi^s(x))\Big|_{s=0} = \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} F\Big(\Phi^t\big(\Psi^x(x)\big)\Big).$$

Similarly

$$\mathcal{L}_f\mathcal{L}_g F(x) = \frac{\partial}{\partial t}\Big|_{t=0}\frac{\partial}{\partial s}\Big|_{s=0} F(\Psi^s(\Phi^t(x))).$$

Since the partial derivatives of F are conitnuous, we can use Schwarz's theorem to swap the order of the partial derivatives and we get

$$(\mathcal{L}_g\mathcal{L}_f - \mathcal{L}_f\mathcal{L}_g)F = \frac{\partial^2}{\partial t\partial s}\Big|_{t=s=0}\Big(F\big(\Phi^t\circ\Psi^s(x)\big) - F\big(\Psi^s\circ\Phi^t(x)\big)\Big).$$

□

Remark. $\mathcal{L}_g\mathcal{L}_f$ and $\mathcal{L}_f\mathcal{L}_g$ are <u>second order</u> differential operators! For instance,

$$\begin{aligned}\mathcal{L}_g\mathcal{L}_f F = \sum_{k=1}^n g_k\frac{\partial}{\partial x_k}(\mathcal{L}_f F) &= \sum_{k=1}^n g_k\frac{\partial}{\partial x_k}\left(\sum_{j=1}^n f_j\frac{\partial F}{\partial x_j}\right)\\ &= \sum_{k=1}^n\sum_{j=1}^n g_k\frac{\partial f_j}{\partial x_k}\frac{\partial F}{\partial x_j} + \sum_{k=1}^n\sum_{j=1}^n g_k f_j\frac{\partial^2 F}{\partial x_k\partial x_j}\\ \mathcal{L}_f\mathcal{L}_g F &= \sum_{k=1}^n\sum_{j=1}^n f_k\frac{\partial g_j}{\partial x_k}\frac{\partial F}{\partial x_j} + \sum_{k=1}^n\sum_{j=1}^n f_k g_j\frac{\partial^2 F}{\partial x_k\partial x_j}.\end{aligned}$$

However, $\mathcal{L}_g\mathcal{L}_f - \mathcal{L}_f\mathcal{L}_g$ is a <u>first order</u> differential operator, for the second order contributions cancel away.

$$(\mathcal{L}_g\mathcal{L}_f - \mathcal{L}_f\mathcal{L}_g)F = \sum_{j=1}^n\left(\sum_{k=1}^n\big(g_k\frac{\partial f_j}{\partial x_k} - f_k\frac{\partial g_j}{\partial x_k}\big)\frac{\partial F}{\partial x_j}\right)$$

Thus, $(\mathcal{L}_g\mathcal{L}_f - \mathcal{L}_f\mathcal{L}_g)F$ corresponds to a vector field $h = \begin{pmatrix} h_1\\ \vdots\\ h_n\end{pmatrix}$ with components

$$h_j = \sum_{k=1}^n\left(g_k\frac{\partial f_j}{\partial x_k} - f_k\frac{\partial g_j}{\partial x_k}\right).$$

By definition (?)

$$h = [\mathcal{L}_g, \mathcal{L}_f].$$

Thus, we get

$$\mathcal{L}_h = \mathcal{L}_{[g,f]} = \mathcal{L}_g\mathcal{L}_f - \mathcal{L}_g\mathcal{L}_f = [\mathcal{L}_g, \mathcal{L}_f].$$

Theorem 26. *If $\mathcal{L}_{[g,f]}F(x) = 0$ for all $F : X \to X$ and for all $x \in X$, then $\Phi^t\circ\Psi^s = \Psi^s\circ\Phi^t$ for all $s, t \in \mathbb{R}$, that is, the flows commute globally. This means, the flows commute if*

$$[\mathcal{L}_f, \mathcal{L}_g] = 0.$$

3.5 Phase volume and its conversation

Let $\Phi^t : X \to X$ be a dynamical system with the corresponding ODE $\dot{x} = f(x)$. The volume of a arbitrary set U is defined as

$$\mathrm{vol}(U) = \int_U \mathrm{d}x_1 ... \mathrm{d}x_n.$$

So it follows from the change of formula or transformation theorem that

$$\mathrm{vol}(\Phi^t(U)) = \int_{\Phi^t(U)} \mathrm{d}x_1 ... \mathrm{d}x_n = \int_U 1 \cdot \left| \det \frac{\partial \Phi^t}{\partial x} \right| \mathrm{d}x_1 ... \mathrm{d}x_n. \tag{11}$$

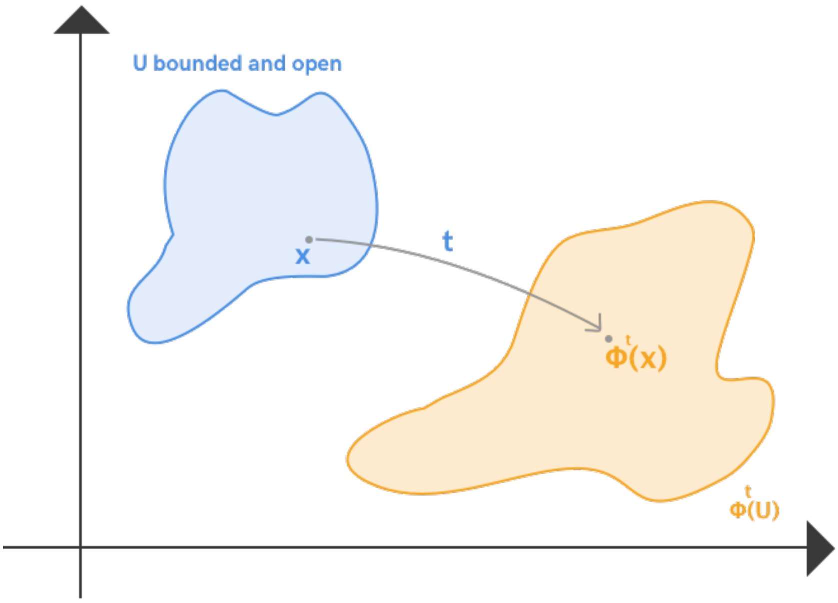
where $\frac{\partial \Phi^t}{\partial x}$ denotes the Jacobi matrix. Remember that for $\varphi : U \to V$

$$\int_V f(x) dx = \int_U f(\varphi(x)) |\det D\varphi(x)| dx. \tag{Transformation formula}$$

Definition 27 (Phase volume preserving). Φ^t *is* phase volume preserving *if*

$$\mathrm{vol}(\Phi^t(U)) = \mathrm{vol}(U) \quad \textit{for all open } U \subset X.$$

One says that the phase volume is an integral invariant *of* Φ^t *(see definition 21).*



Week 04
Tue, 06.11

Recall from (5) that $\frac{\partial \Phi^t}{\partial x}$ is a solution of Poincare's variational equation with $f(t, x) = \Phi^t(x)$:

$$\dot{Y} = A(t)Y \quad \text{with} \quad A(t) = \frac{\partial f}{\partial \Phi}(\Phi^t(x))$$

From Liouville's formula (8) we know the solution $\frac{\partial \Phi^t}{\partial x}$ satisfies

$$\det Y(t) = \det Y(0) e^{\int_0^t \mathrm{tr} A(\tau) \mathrm{d}\tau}.$$

Since $Y(0) = I$ we get

$$\det Y(t) = \det \frac{\Phi^t}{\partial x} = \exp \int_0^t \mathrm{tr} \left(\frac{\partial f}{\partial x} (\Phi^\tau(x)) \right) d\tau.$$

Now remember that $f : X \to X$ with $X \subset \mathbb{R}^n$:

$$\mathrm{tr} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial f_1}{\partial x_1} + ... + \frac{\partial f_n}{\partial x_n} =: \mathrm{div} f.$$

In words, the trace of a Jacobian matrix of f is the divergence of f. Thus, from (11) it follows:

$$\boxed{\mathrm{vol}(\Phi^t(U)) = \int_U \exp \left(\int_0^t \mathrm{div} f(\Phi^\tau(x)) d\tau \right) dx_1 ... dx_n.}$$

Theorem 28. *It holds*

$$\forall t \in \mathbb{R} : \mathrm{vol}(U) = \mathrm{vol}(\Phi^t(U)) \iff \mathrm{div} f \equiv 0$$

for any open set $U \subset X$. In other words the vector field is divergence free.

Proof. If $\mathrm{div} f \equiv 0$, then $\mathrm{vol}(\Phi^t(U)) = \int_U \exp(0) dx_1 ... dx_n = \int_U dx_1 ... dx_n = \mathrm{vol}(U)$. Otherwise... (proof to be completed) □

Example.

$$\ddot{x} = F(x) \iff \begin{cases} \dot{x} = v \\ \dot{v} = F(x) \end{cases} \qquad \text{with vector field } f(x, v) = \begin{pmatrix} v \\ F(x) \end{pmatrix}.$$

We obtain a linear differential equation in $\mathbb{R}^{2n}$. It follows that

$$\frac{\partial f_i}{\partial x_i} = 0 \implies \mathrm{div} f = 0.$$

Therefore, all flows Φ^t for ODEs that have been transformed to a system of first order differential equations are phase volume preserving:

$$\int_U dx_1 ... dx_n dv_1 ... dv_n = \int_{\Phi^t(U)} dx_1 ... dx_n dv_1 ... dv_n.$$

For volume preserving maps there holds the Poincare recurrence theorem.

Theorem 29 (Recurrence by Poincare (toy version)). *Let Φ be a volume preserving diffeomorphism of a* <u>*bounded*</u> *domain $D \subset \mathbb{R}^n$. Then for any small neighbourhood $U \subset D$ there exists a point $x \in U$ such that $\Phi^n(x) \in U$ for a certain $n \in \mathbb{N}$.*

Proof. It holds due the volume preserving property of Φ that

$$\forall n \in \mathbb{N} : \mathrm{vol}(U) = \mathrm{vol}(\Phi(U)) = \mathrm{vol}(\Phi^2(U)) = ... = \mathrm{vol}(\Phi^n(U)).$$

If all $U, \Phi(U), \Phi^2(U), ..., \Phi^n(U) \subset D$ would be disjoint, we would have $\mathrm{vol}(D) = \infty$ which contradicts to the boundness of D. Therefore, not all $\Phi^n(U)$ are disjoint:

$$\exists n_1, n_2 \in \mathbb{N} : \Phi^{n_1}(U) \cap \Phi^{n_2}(U) \neq \emptyset \iff U \cap \Phi^{n_2 - n_1}(U) \neq \emptyset.$$

So, the image of U under $\Phi^{n_2 - n_1}$ intersects U again.

I dont get it... p. 51: unicity of solutions.... $\square$

Theorem 30 (General version of the Poincare recurrence theorem). *Let (X, μ) be a probability space and $f : X \to X$ a map preserving measure of μ (i.e. $\mu(f^{-1}(U)) = \mu(U)$) for all measurable $U \subset X$). Then for any measurable $E \subset X$, it holds*

$$\mu(\{x \in E \,|\, \exists N \in \mathbb{N} : f^n(x) \notin E,\ \forall n \geq N\}) = 0.$$

In words: Almost all points in E return to E infinitely often.

Proof. It holds that $f^n(x) \in E \iff x \in f^{-n}(E)$. Therefore for arbritary $N \in \mathbb{N}$:

$$x \in E \wedge \forall n \geq N : f^n(x) \notin E \iff x \in \bigcap_{n \geq N} \big(E \setminus f^{-n}(E)\big) = E \setminus \bigcup_{n \geq N} f^{-n}(E).$$

We set $A_N \coloneqq \bigcup_{n \geq N} f^{-n}(E)$. Now, we get

$$\begin{aligned} A_{N+1} = \bigcup_{n \geq N+1} f^{-n}(E) = f^{-1}(A_N) &\implies \mu(A_{N+1}) = \mu(f^{-1}(A_N)) = \mu(A_N) \\ &\implies \forall i, j \in \mathbb{N} : \mu(A_i) = \mu(A_j). \end{aligned}$$

Then due to $E \subset A_0$ ($A_0 \supset f^0(E)$) and $A_i \subset A_j$ for $i \geq j$ (A_i is decreasing):

$$E \setminus A_N \subset A_0 \setminus A_N \implies \mu(E \setminus A_N) \leq \mu(A_0 \setminus A_N) = \mu(A_0) - \mu(A_N) = 0.$$

Note that $\mu(A \setminus B) = \mu(A) - \mu(B)$ holds only if $B \subset A$. $\square$

3.6 Fixed points (equilibria) of dynamical systems

Remember from Definition 16 that x_0 is called a fixed point if $\Phi^t(x_0) = x_0$ for all $t \in \mathbb{R}$ or $t \in \mathbb{Z}$. We also know that x_0 is a fixed point iff $f(x_0) = 0$ in case of a continuous dynamical system.

Theorem 31. *Let $x : (t_-, t_+) \to X \subset \mathbb{R}^n$ be a maximal solution of $\dot{x} = f(x)$ with a continuous vector field $f : X \to \mathbb{R}^n$. If $\lim\limits_{t \nearrow t_+} x(t) = x_0 \in X$ exists, then x_0 is a fixed point and $t_+ = \infty$.*

Proof. First, we show that $t_+ = \infty$ by proof by contradiction. If $t_+ < \infty$, then $x(t)$ should leave any compact set $K \subset X$ as $t \nearrow t_+$ (see Theorem 8). However, that is a contradiction to $x(t) \in B_\epsilon(x_0)$ for all $t_+ - \delta < t < t_+$ due to the existence of the limit $\lim\limits_{t \nearrow t_+} x(t) = x_0$. Therefore, $t_+ = \infty$.

From here, I do not understand what happens next From $\dot{x}(t) = f(x(t))$ follows that $x_i(t+1) - x_i(t) = x_i(\xi_i) = f_i(x(\xi_i))$ with $x_i \in [t, t+1]$. Send $t \to \infty$, then $\xi \to \infty$ and using the continuity of f

$$0 = \lim_{t \to \infty} (x_i(t+1) - x_i(t)) = f_i(x_0), \quad i = 1, ..., n.$$

$\square$

Example. Consider the ODE

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -2x_2 \end{cases} \quad \text{with solutions} \quad \begin{cases} x_1(t) &= x_{10}e^{-t} \\ x_2(t) &= x_{20}e^{-2t}. \end{cases}$$

We consider the limits for $t \to \infty$ and see that

$$x_1(t) \to 0 \quad \text{and} \quad x_2(t) \to 0.$$

All solutions have

$$\lim_{t \to \infty} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as a fixed point.

Example.

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2 \end{cases}$$

with solution

$$\begin{cases} x_1(t) = x_{10}e^{-t} \\ x_2(t) = x_{20}e^{t} \end{cases}.$$

The limit $\lim_{t\to\infty}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$ exists if and only if $x_{20}=0$. Incomplete!!!

4 Stability of fixed points

Consider the IVP $\dot{x}=f(x)$ with initial value $x(0)=x_0$. Remember that a point $\tilde{x}\in\mathbb{R}^n$ is called a fixed point if $f(\tilde{x})=0$. That means, $\Phi^t(\tilde{x})=\tilde{x}$ for all $t\in I\subset\mathbb{R}$.

Definition 32. *A fixed point x_0 of a dynamical system $\Phi^t: X\to X$ is called*

1. ***stable*** *(Lyapunov stable) if*

$$\forall\epsilon>0,\exists\delta>0: y\in B_\delta(x_0) \implies \forall t\geq 0:\Phi^t(y)\in B_\epsilon(x_0)$$

In words "start near, stay near".

2. ***asymptotically stable*** *if it is stable and*

$$\forall y\in B_\delta(x_0):\lim_{t\to\infty}\Phi^t(y)=x_0.$$

3. ***unstable*** *if it is not stable, i.e.*

$$\exists\epsilon>0,\exists\{y_n\}_{n\in\mathbb{N}}\quad\text{with}\quad\lim_{n\to\infty}y_n=x_0\text{ and }\forall n\in\mathbb{N},\exists t_n>0:\Phi^{t_n}(y_n)\notin B_\epsilon(x_0).$$

What can you say about the *stability* of the particular fixed point $\tilde{x}=0$? If $f(x)=Ax$ (thus f is a linear vector field) we have essentially a linear algebra problem. Solutions and stability properties of the fixed point $\tilde{x}=0$ are completely determined by the eigenstructure of A. Moreover, the fixed point $\tilde{x}=0$ is the only fixed point if $\det A\neq 0$.

If f is nonlinear, there are two ways to assess the stability properties. We can try to find *Lyapunov functions* or try to *linearise* the vector field f. The method of Lyapunov functions is always applicable but to find a Lyapunov function is not always easy. The method of linearisation only gives information about the stability of a fixed point $\tilde{x}$ if $\tilde{x}$ is a *hyperbolic* fixed point.

4.1 Linearisation of nonlinear systems

Week 04
Thu, 08.11

For continuous time dynamical systems with $\dot{x}=f(x)$ and solution $x_0(t)$, the linearised equation below is solved by $x_0(t)$:

$$\dot{y}=A(t)y,\quad A(t):=\frac{\partial f}{\partial x}(x_0(t)).\qquad\text{(Poincaré's variational equation)}$$

Poincaré's variational equation describes the behaviour of the solution $x_0(t)$. We would like to know how the solution, which depends on the initial value x_0, behaves if one pertuabates the initial value x_0 by small values. Is this new solution with the small pertuabation nearby the original solution $x_0(t)$ or far away? Basically, we are interested in the stability properties of the dynamical system with solution $x_0(t)$.

Therefore, we consider nearby solutions $x(t) = x_0(t) + y(t)$ of the ODE $\dot{x} = f(x)$ with $y(t)$ being small. For $x(t)$ it holds that

$$\dot{x}_0(t) + \dot{y}(t) = f(x_0(t) + y(t)).$$

So we get

$$\begin{aligned}\dot{y}(t) = f(x_0(t) + y(t)) - \dot{x}_0(t) &= f(x_0(t) + y(t)) - f(x_0(t)) \\ &= \frac{\partial f}{\partial x}(x_0(t))y(t) + O(||y(t)||^2)\end{aligned}$$

Thus the behaviour of $y(t)$ as approximately described by

$$\dot{y} = A(t)y = \frac{\partial f}{\partial x}(x_0(t))y$$

If $x_0(t) \equiv x_0$, that is, x_0 is a fixed point with $f(x_0) = 0$, then the Poincare's variational equation reads $\dot{y} = Ay$ with $A = \frac{\partial f}{\partial x}(x_0)$ being a constant $n \times n$ Matrix.

Theorem 33 (Poincaré-Lyapunov). *The following statements hold true:*

1. *If <u>all</u> eigenvalues λ_i of $A = \frac{\partial f}{\partial x_i}(x_0)$ have $\mathrm{Re}(\lambda_i) < 0$ then the fixed point x_0 of $\dot{x} = f(x)$ is asymptotically stable.*
2. *If <u>at least one</u> eigenvalue λ_i of A has $\mathrm{Re}(\lambda_i) > 0$ then the fixed point is unstable.*

Remark. If all λ_i have $\mathrm{Re}(\lambda_i) \leq 0$ with some $\mathrm{Re}(\lambda_i) = 0$ then one cannot decide about the stability of x_0 based on the linearised equation only. One has to take into account the higher order terms.

To prove this theorem, we will discuss the theory of solutions for linear systems of differential equations.

4.2 Linear Systems of Differential Equations

Consider the homogenous differential equation

$$\dot{y} = A(t)y \quad \text{with} \quad A : \mathbb{R} \supset I \to \mathbb{R}^{n\times n} \text{ or } \mathbb{C}^{n\times n}.$$

The non-homogenous ODE has the form

$$\dot{y} = A(t)y + b(t) \quad \text{with} \quad b : \mathbb{R} \supset I \to \mathbb{R}^n \text{ or } \mathbb{C}^n.$$

Note that solutions exist and are unique for a given initial value due to Picard-Lindelöf.

4.2.1 Homogenous systems

Theorem 34. *Let y_1 and y_2 be solutions of the ODE $\dot{y} = A(t)y$. Linear combinations of y_1 and y_2 are solutions, as well.*

$$\forall c_1, c_2 \in \mathbb{R} : \frac{d}{dt}(c_1 y_1 + c_2 y_2) = A(t)(c_1 y_1 + c_2 y_2).$$

In other words, solutions of $\dot{y} = A(t)y$ build a vector space.

Theorem 35. *Let $x_1, ..., x_m : I \to \mathbb{R}^n$ be m solutions of $\dot{y} = A(t)y$. If for some $t_0 \in I$, the vectors $x_1(t_0), ..., x_m(t_0)$ are linearly dependant, then the functions $x_1, ..., x_m : I \to \mathbb{R}^n$ are linearly dependant, as well.*

Proof. Let $c_1 x_1(t_0) + ... + c_m x_m(t_0) = 0$ for some $c_1, ..., c_m \in \mathbb{R}$. Build the solution

$$y(t) \coloneqq c_1 x_1(t) + ... + c_m x_m(t),$$

which is indeed a solution for $\dot{y} = A(t)y$ due to Theorem 34. It yields

$$y(t_0) = 0.$$

However, $x(t) = 0$ and $y(t)$ solve the IVP $\begin{cases} \dot{y} = A(t)y, \\ y(0) = 0 \end{cases}$. Due to uniqueness, $y(t) \equiv x(t) = 0$. Thus, $x_1, ..., x_m$ are linearly dependent. ☐

If the solutions $x_1, ..., x_m$ are linearly independent, then $x_1(t), ..., x_m(t) \in \mathbb{R}^n$ are linearly independent for all $t \in I$. The other direction is obvious. If $x_1(t), ..., x_m(t)$ are linearly independent for all $t \in \mathbb{I}$, then $x_1, ..., x_m$ are linearly independent solutions. So we get: $x_1, ..., x_m$ are linearly independent solutions if and only if $x_1(t_0), ..., x_m(t_0)$ are linearly independent for all $t_0 \in I$.

Theorem 36. *Any n linearly inpendent solutions build a basis in the space of solutions. Thus, the solution space is n-dimensional.*

Proof. Let $x_1(t), ..., x_n(t)$ be n linearly independent solutions and let $x(t)$ be any further solution. Then $x_1(t_0), ..., x_n(t_0)$ are linearly independent for all $t_0 \in I$ and are therefore a basis of $\mathbb{R}^n$:

$$x(t_0) = c_1 x_1(t_0) + ... + c_n x_n(t_0) \implies x(t) = c_1 x_1(t) + ... + c_n x_n(t).$$

$\square$

Example. Let $x_1(t), .., x_n(t)$ be solutions of $\dot{y} = A(t)y$ with initial values

$$x_1(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad x_2(t_0) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad ..., \quad x_n(t_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

We can write the solutions as a matrix

$$X(t) \coloneqq \begin{pmatrix} | & | & ... & | \\ x_1(t) & x_2(t) & ... & x_n(t) \\ | & | & ... & | \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad \forall t \in I.$$

This matrix $X(t)$ solves the matrix initial value problem

$$\begin{cases} \dot{X} = A(t)X \\ X(t_0) = I_{n\times n} \end{cases}.$$

How can we find a solution for $\dot{y} = A(t)y$ with arbitrary initial value $y(t_0) = a \in \mathbb{R}^n$? The solution is given by

$$x(t) = X(t)a.$$

Any matrix solution of $\dot{X} = A(t)X$ with $\det X \neq 0$ (the columns are linearly independent) is called a **fundamental matrix** of the ODE $\dot{y} = A(t)y$.

Theorem 37. *If $X : I \to \mathrm{GL}(n,\mathbb{R})$[4] is a fundamental matrix of $\dot{y} = A(t)y$, i.e. it holds $\dot{X} = A(t)X$ and if $Y : I \to Mat_{n\times m}$ is any other solution of $\dot{Y} = A(t)Y$, then*

$$Y(t) = X(t) \cdot C, \quad \textit{with} \quad C \in \mathrm{Mat_{n\times m}}.$$

In particular, if Y is another fundamental matrix, then $C \in \mathrm{GL}(n,\mathbb{R})$.

4.2.2 Linear Systems with constant coefficients

Consider the system with constant matrix A, i.e. $\dot{x} = Ax$ for $A \in \mathbb{R}^{n\times n}$. This is the only case where it can solved explicitly. The fundamental matrix $X(t)$ is given by

$$X(t) = e^{At} \quad \text{with} \quad I(0) = I_{n\times n}.$$

[4] $\mathrm{GL}(n,\mathbb{R})$ is the space of invertible real matrices of dimension $n \times n$

We obtain a solution for the ODE with initial value $x(0) = x_0 \in \mathbb{R}^n$ by

$$x(t) = X(t)x_0.$$

The real problem is <u>how to compute the matrix exponential e^{At}</u>. As a reminder, the **matrix exponential** is defined as

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \in \mathrm{Mat_{n\times n}}.$$

If A is diagonalisable, we can write A as

$$A = U \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} U^{-1}.$$

Thus, the matrix exponential is really easy to compute

$$e^{At} = U \begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix} U^{-1}.$$

Let $\tilde{a}_{ij}$ denote the entry in row i and column j of the exponential matrix e^{At}, i.e. $e^{At} = (\tilde{a}_{ij})_{i,j=1}^n$. Each entry $\tilde{a}_{ij}$ is a linear combination $\sum_{k=1}^n c_k e^{\lambda_k t}$. If the real part of all eigenvalues λ_k are negative ($\operatorname{Re}\lambda_k < 0$), then the matrix A converges to the null matrix:

$$\lim_{t\to\infty} e^{\lambda_k t} = 0, \quad \forall k \in \{1, ..., n\}.$$

That means, the fixed point $\mathbf{x} = 0$ is *stable* for matrix A. Systems with small perturbations of the starting point $\mathbf{x} = 0$ still end up with $\mathbf{x} = 0$.

If there exists an Eigenvalue with $\mathrm{Re}(\lambda_k) > 0$, we get

$$\lim_{t\to\infty} e^{\lambda_k t} = \infty.$$

The fixed point $\mathbf{x} = 0$ is therefore *unstable*, as the system moves away from zero.

If there is an eigenvalue λ_k with null real part $\operatorname{Re}\lambda_k = 0$, every solution $x(t)$ that is a combination of $e^{\lambda_k t}$ oscillates, and stays bounded. For if $\operatorname{Re}\lambda_k = 0$, it holds due to Euler's Formula[5]:

$$e^{\lambda_k t} = e^{\mathrm{Re}(\lambda_k)t}\Big(\cos(\mathrm{Im}(\lambda_k)t) + i\sin(\mathrm{Im}(\lambda_k)t)\Big) = \cos(\mathrm{Im}(\lambda_k)t) + i\sin(\mathrm{Im}(\lambda_k)t).$$

[5] $e^{a+bi} = e^a(\cos b + i \sin b)$

How do we compute e^{At} if A is not diagonalisable. A has at least a Jordan normal form. Let A have n Jordan blocks:

$$A = U \begin{pmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_n \end{pmatrix} U^{-1} \quad \text{with blocks} \quad J_k = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, k = 1, ..., n.$$

Then the matrix exponential is as follows

$$e^{At} = U \begin{pmatrix} e^{J_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{J_n t} \end{pmatrix} U^{-1} \quad \text{with blocks} \quad e^{J_k t} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \dots & \frac{t^{p-1}}{(p-1)!} \\ 0 & 1 & t & \dots & \frac{t^{p-1}}{(p-2)!} \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Where does the formula $e^{J_k t}$ come from? Note that $J_k = \lambda I + N$ and N is the matrix with only ones above the main diagonal.

$$J_k = \begin{pmatrix} \lambda & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

For matrix exponential, it holds

$$e^X e^Y = e^{X+Y} \quad \text{only for } \underline{\text{commutative}} \text{ matrices!}$$

We see that $IN = NI$, i.e. the matrices I and N commutate. Therefore,

$$e^{J_k} = e^{\lambda I + N} = e^{\lambda I} e^N.$$

Now, $e^{\lambda I} = \operatorname{diag}(\lambda, ..., \lambda)$. But what is e^N? The matrix N is nilpotent, that means, there exists l such that N^l is the null matrix. Thus with the matrix exponential series, we obtain

$$e^{\lambda I} e^N = e^{\lambda} \left[\begin{pmatrix} 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & 0 \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{t^2}{2} & 0 \\ & \ddots & \ddots & \frac{t^2}{2} \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix} + \dots + \begin{pmatrix} 0 & 0 & 0 & 0 \\ & \ddots & \ddots & 0 \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix} \right].$$

Any entry of e^{At} takes the form $\sum c_k(t) e^{\lambda_k t}$, where $c_k(t)$ is a polynomial. To compare, if A is diagonalisable, each entry has the form $\sum c_k e^{\lambda_k t}$, where $c_k \in \mathbb{R}$ is constant.

If all $\operatorname{Re}\lambda_k < 0$, every entry $\sum c_k(t)e^{\lambda_k t}$ converges to zero for $t \to \infty$, since the exponential function grows "faster". The fixed point 0 is therefore *stable*.

If there exists an eigenvalue with $\operatorname{Re}\lambda_k > 0$, there is an entry $\sum c_k(t)e^{\lambda_k t}$ that grows to infinity for $t \to \infty$. The fixed point 0 is *unstable*.

These results justify the Theorem of Poincaré-Lyapunov, see Theorem 33.

The fixed point $x_0 = 0$ of $\dot{x} = Ax$ is asympotically stable if all $\operatorname{Re}\lambda_k < 0$ and unstable if there is at least one λ_k with $\operatorname{Re}\lambda_k > 0$.

4.2.3 Nonhomogenous systems

Consider the ODE

$$\dot{x} = A(t)x + b(t).$$

The solution space is a n-dimensional affine space. Take any particular solution $x_0(t)$ of $\dot{x}_0 = A(t)x_0 + b(t)$, then for any other solution $x(t)$ we have $y(t) := x(t) - x_0(t)$ is a solution of the homogenous system $\dot{y} = A(t)y$.

Let $X(t)$ be the fundamental matrix of $\dot{x} = A(t)x$ with $X(t_0) = I$. The solution of the IVP

$$\begin{cases} \dot{x} = A(t)x + b(t) \\ x(t_0) = x_0 \end{cases}$$

is given by the **variation of constants** method

$$x(t) = X(t)x_0 + \int_{t_0}^{t} X(t) \cdot X^{-1}(s)b(s)ds. \qquad \text{(Variation of constants)}$$

Here, $X(t)x_0$ is the solution of the homogenous system. $X(t)\int_{t_0}^{t} X^{-1}(s)b(s)ds$ is a particular solution of the non-homogenous system.

Proof. The solution $x(t) = X(t)c$ solves the homogenous system if $c = x_0 \in \mathbb{R}^n$; thus $\dot{X} = A(t)X$. The idea is to change the constant vector c to a function $c(t) : \mathbb{R} \to \mathbb{R}^n$. So, we consider the solution given by

$$x(t) = X(t)c(t) \quad \text{with} \quad x(t_0) = c(t_0) = x_0.$$

and we are looking for $c(t)$. We substitute $x(t)$ in the inhomogenous system, and we get

$$\begin{aligned}
& \frac{d}{dt}(X(t)c(t)) = A(t)X(t)c(t) + b(t) \\
\iff \quad & \dot{X}(t)c(t) + X(t)\dot{c}(t) = A(t)X(t)c(t) + b(t) \\
\iff \quad & A(t)X(t)c(t) + X(t)\dot{c}(t) = A(t)X(t)c(t) + b(t) \\
\iff \quad & X(t)\dot{c}(t) = b(t) \\
\iff \quad & \dot{c}(t) = X^{-1}(t)b(t).
\end{aligned}$$

Integrating yields

$$c(t) = \int_{t_0}^{t} X^{-1}(s)b(s)ds + c(t_0) = x_0 + \int_{t_0}^{t} X^{-1}(s)b(s)ds.$$

Thus,

$$x(t) = X(t)\left(x_0 + \int_{t_0}^{t} X^{-1}(s)b(s)ds\right).$$

□

In particular, if we have a inhomogenous ODE with initial value $x(0) = x_0$ and if $A(t) = A$ is constant, then we set $X(t) = e^{At}$ and we get

$$\boxed{x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}b(s)ds.}$$

Note that the inverse of a matrix exponential is just $X^{-1}(s) = (e^{As})^{-1} = e^{-As}$.

4.3 Classification of fixed points

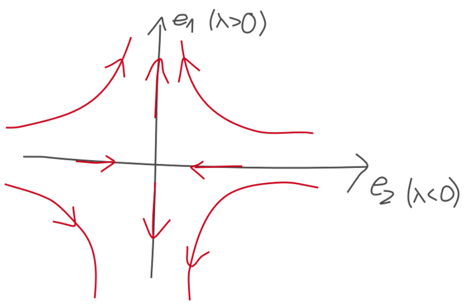
Consider a genereal linear system $\dot{\mathbf{x}} = A\mathbf{x}$, where $A \in \mathrm{Mat}_{2\times 2}$. We can classify the fixed point $\mathbf{x} = 0$ based on the eigenstructure of A. The characteristic polynomial is given by

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det A = 0.$$

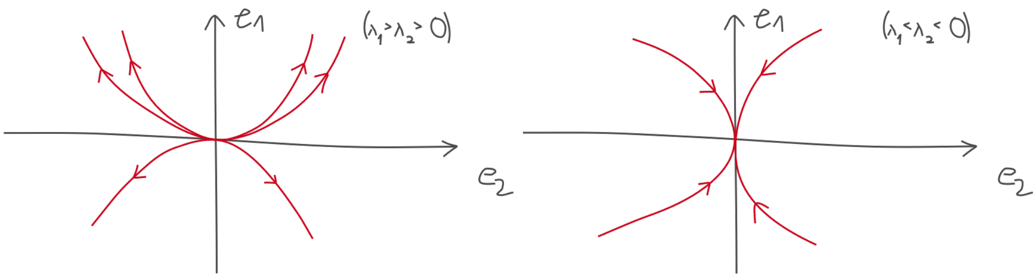
So the eigenvalues are

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}A \pm \sqrt{\frac{1}{4}(\mathrm{tr}A)^2 - \det A}.$$

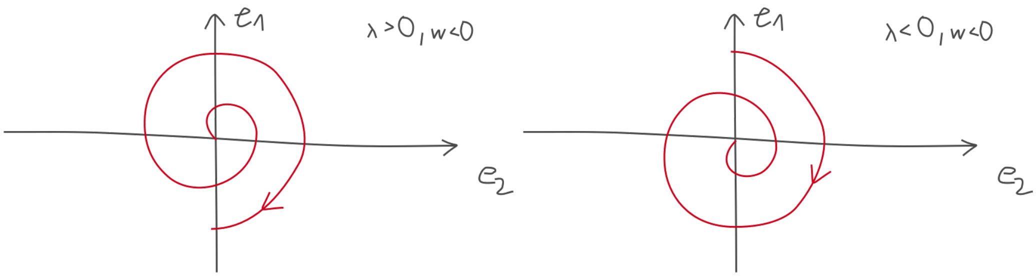
- **Saddle point** ($\det A < 0$): The eigenvalues λ_1 and λ_2 have opposite signs, i.e. $\lambda_1\lambda_2 < 0$. E.g. $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}$, canonical form: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$



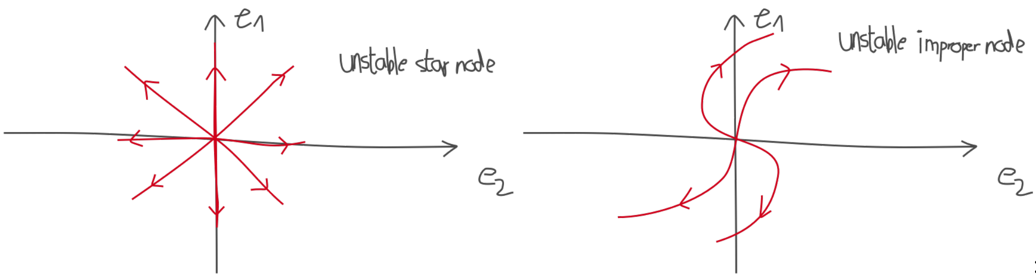
- **Node** ($(\operatorname{tr} A)^2 > 4 \det A > 0$): The eigenvalues have the same sign $\lambda_1 \lambda_2 > 0$. If $\operatorname{tr} A > 0$, then it is an unstable/ repelling node. If $\operatorname{tr} A < 0$, then it is a stable/ attracting node. Remember that $\operatorname{tr} A = \sum \lambda_i$. Canonical form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$



- **Focus (Spiral)** ($(\operatorname{tr} A)^2 < 4 \det A$): Eigenvalues $\lambda \pm iw$ are complex. Both eigenvalues have either positive real part ($\operatorname{tr} A > 0$: unstable, repelling focus) or both have negative real part ($\operatorname{tr} A < 0$: stable, attracting focus). Canonical form: $\begin{pmatrix} \lambda & w \\ -w & \lambda \end{pmatrix}$



- **Star/ Stellar nodes, Improper nodes** ($(\operatorname{tr} A)^2 = 4 \det A \neq 0$) E.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$



:

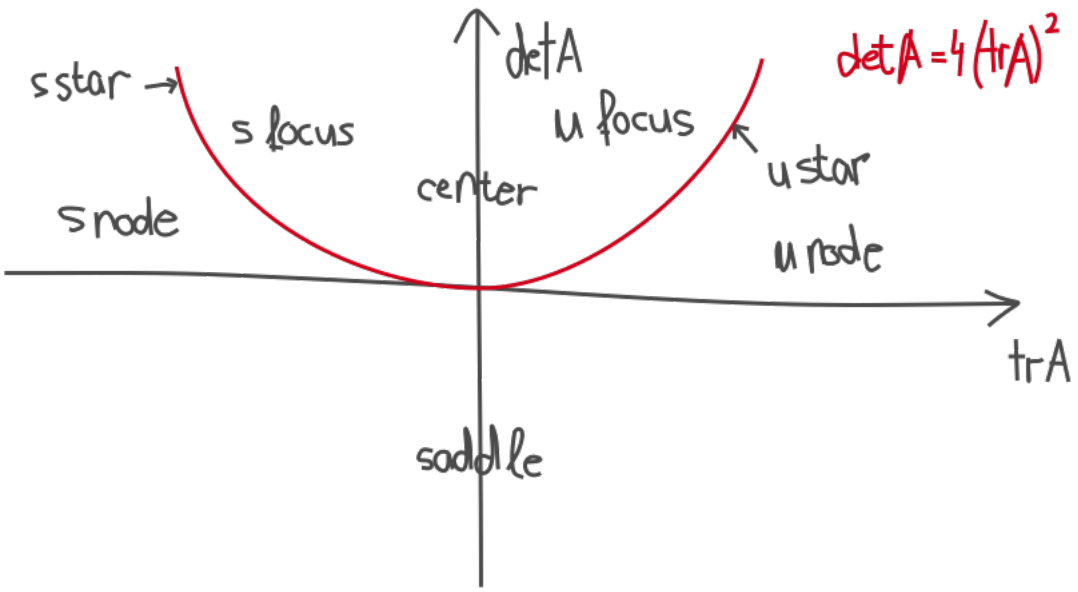
In all these cases the fixed point is *hyperbolic*.

Definition 38 (Hyperbolic fixed point). *A fixed point* $\mathbf{x}$ *is called* ***hyperbolic*** *if all eigenvalues of the linearisation* A *of the system in* $\mathbf{x}$ *have non-zero real part.*

Thus the nonhyperbolic cases are those fixed points whose linearised systems have eigenvalues with zero real part. These are of three kinds:

- A is the null matrix, which means $\det A = \operatorname{tr} A = 0$.
- $\det A = 0$, one eigenvalue is zero and we have line of fixed points, e.g. $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$
- $\det A > 0$ and $\operatorname{tr} A = 0$. This is called a **centre**. The eigenvalues are $\pm iw$ for the canonical form $\begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}$ and trajectories are closed curves. E.g. $\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$.

To find the canonical forms, just diagonalise the matrix. In case of complex eigenvalues, we have two complex eigenvalues $e, \bar{e}$; take $(\operatorname{Re} e, \operatorname{Im} e)$ as a basis. This helps in drawing trajectories. Note that classification is independent of coordinates.



4.4 Method of Lyapunov functions

Week 05
Tue, 13.11

The idea of Lyapunov functions comes from physical systems that involve friction. Such systems do not converse energy, so that the total enery is a decreasing function of time. If we regard a pendulum, we see that the equilibrium point of such system lies at angle $0°$ and $180°$. The stability properties of both fixed point yield completely different behaviours. Linear stability analysis would linearise around the point corresponding to the angle $0°$, and examine small perturbations of the angle. Lyapunov recognised that the reason for the stability of this fixed point has not much to do with small perturbations of the angle, but instead the sytem depends on its finite energy that keeps being dissipated. He developed a mathematical formalisation for scalar energly like functions that can be used to examine the behaviour of trajectories of the system.

Video Link: General idea of Lyapunov functions by Professor Slotine at the MIT

Consider a scalar function $F : \mathbb{R}^n \to \mathbb{R}$ that is zero at some point $\tilde{\mathbf{x}}$ and positive elsewhere for a ball around zero. Such a function F might be thought of an energy function.

Let $\dot{F}$ denote the time derivative of F along a trajectory $\Phi^t(\mathbf{x})$ of the dynamical system. That means $\dot{F}$ describes the current rate of change of energy for a given moment t:

$$\dot{F}(t) := \dot{F}(\Phi^t(\mathbf{x})) := \frac{d}{dt}F(\Phi^t(\mathbf{x})), \quad \forall t \in \mathbb{R} \text{ and } \mathbf{x} \text{ is fixed.}$$

If $\dot{F}$ is negative throughout the system trajectory (except at $\tilde{\mathbf{x}}$), the energy function F is strictly decreasing over time. Since $F(\tilde{\mathbf{x}}) = \mathbf{0}$ is a lower bound, the energy of the system approaches zero, and any trajectory converges to this zero state. As we will see, we will call a function F with these properties a *Lyapunov function*.

Let's further examine the time derivative $\dot{F}$. We will see that the rate at which energy varies along any system trajectory does not depend on time but only on the current position of the trajectory; that means we can write $F(t)$ as $F(\mathbf{x})$. To see this, we derive $F(\Phi^t(\mathbf{x}))$ with respect to t by the chain rule:

$$\dot{F}(\Phi^t(x)) = \frac{d}{dt}F(\Phi^t(\mathbf{x})) = \frac{d}{d\Phi^t(\mathbf{x})}F(\Phi^t(\mathbf{x}))\cdot\frac{d}{dt}\Phi^t(\mathbf{x}) \overset{(!)}{=} \frac{d}{d\Phi^t}F(\Phi^t(\mathbf{x}))\cdot f(\Phi^t(\mathbf{x})), \quad \forall t \in \mathbb{R}.$$

The important step happens at (!). For the system is autonomous, the derivative of $\Phi^t(\mathbf{x})$ is $f(\Phi^t(\mathbf{x}))$; the derivative does not depend on t. Intuitively, the system knows where to go next, based on the information given by the current position. For any t, $\Phi^t(\mathbf{x})$ identifies a particular position of the system, namely the position that the system arrives at when $\mathbf{x}$ is the starting point and time t has passed. Let $\mathbf{x}$ denote this position, $\mathbf{x} := \Phi^t(\mathbf{x})$. Substitution for the time derivative $\dot{F}(\Phi^t(\mathbf{x}))$ yields:

$$\dot{F}(\mathbf{x}) = \frac{d}{d\mathbf{x}}F(\mathbf{x}) \cdot f(\mathbf{x}).$$

To compare, if the system were nonautonomous: $\dot{F}(\Phi^t(\mathbf{x})) = \frac{d}{d\Phi^t}F(\Phi^t(\mathbf{x})) \cdot f(\Phi^t(\mathbf{x}), t)$. The rate of change of energy for such a system hinges on its current position $\Phi^t(\mathbf{x})$ and on how much time t has passed since moving away from the starting point of the trajectory.

The derivative $\frac{d}{d\mathbf{x}}F(\mathbf{x})$ is a row vector

$$\frac{d}{d\mathbf{x}}F(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1}F(\mathbf{x}) & \dots & \frac{\partial}{\partial x_n}F(\mathbf{x}) \end{pmatrix} = \boldsymbol{\nabla} F(\mathbf{x})^T.$$

Thus, we obtain

$$\dot{F}(\mathbf{x}) = \boldsymbol{\nabla} F(\mathbf{x})^T \cdot f(\mathbf{x}) = \sum_{k=1}^{n} \frac{\partial F}{\partial x_k}(\mathbf{x}) \cdot f_k(\mathbf{x}) = \mathcal{L}_f F(\mathbf{x}).$$

Definition 39 (Lyapunov function). *Let $\tilde{\mathbf{x}}$ be a fixed point of $\Phi^t(\mathbf{x})$. Let $\tilde{M} \subset \mathbb{R}^n$ be a neighborhood of $\tilde{x}$. We call a scalar function $F \in \mathcal{C}^1(\tilde{M}, \mathbb{R})$ a **Lyapunov function** of Φ^t if*

- *F is* positive definite *around $\tilde{\mathbf{x}}$, i.e.*

$$F(\tilde{\mathbf{x}}) = 0 \quad \textit{and} \quad F(\mathbf{x}) > 0, \ \textit{for all } \mathbf{x} \in \tilde{M} \setminus \{\tilde{\mathbf{x}}\},$$

- *the time derivative of F along any trajectory $\Phi^t(x)$ is* negative semidefinite*, i.e.*

$$(\mathcal{L}_f F)(\mathbf{x}) \leq 0, \quad \textit{for all } \mathbf{x} \in \tilde{M}.$$

*F is called a **strict Lyapunov function** if the derivative of F along the system's trajectories is* strictly negative definite*, i.e. $(\mathcal{L}_f F)(\mathbf{x}) < 0$ for all $\mathbf{x} \in \tilde{M} \setminus \{\tilde{\mathbf{x}}\}$.*

Theorem 40 (Lyapunov theorem). *Let $\tilde{\mathbf{x}}$ be a fixed point of Φ^t.*

1. *If there exists a Lyapunov function F, then $\tilde{\mathbf{x}}$ is stable.*
2. *If there exists a strict Lyapunov function F, then $\tilde{\mathbf{x}}$ is asymptotically stable.*

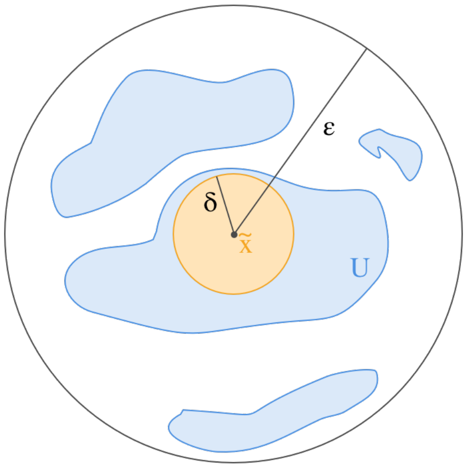
Proof of part one. We want to show that $\tilde{\mathbf{x}}$ is stable if there is a Lyapunov function F.

Let $\epsilon > 0$. We need to find a $\delta > 0$ such that $\Phi^t(B_\delta(\tilde{\mathbf{x}})) \subset B_\epsilon(\tilde{\mathbf{x}})$ for all $t \geq 0$. Define m as the minimum of F in the boundary of $B_\epsilon(\tilde{\mathbf{x}})$:

$$m := \min_{\mathbf{x} \in \partial B_\epsilon(\tilde{\mathbf{x}})} F(\mathbf{x}). \tag{12}$$

Note that the minimum m is well defined because F is continuous. That means, the boundary $\partial B_\epsilon(\tilde{\mathbf{x}})$ is not empty. Additionally, $m > 0$ because $F(\mathbf{x}) > 0$ for all $\mathbf{x} \in \tilde{M} \setminus \{\tilde{\mathbf{x}}\}$.

Define the level set $U = \{\mathbf{x} \in B_\epsilon(\tilde{\mathbf{x}}) : F(\mathbf{x}) < m\}$, which is not empty (due to $m > 0$ and $F(\tilde{\mathbf{x}}) = 0$) and open (due to the continuity of F). As U is open, we find a ball with radius δ around $\tilde{\mathbf{x}}$ such that $B_\delta(\tilde{\mathbf{x}}) \subset U$. Note that U may not be connected, yet we still manage to find such ball around $\tilde{\mathbf{x}}$. To conclude, every $\mathbf{x} \in B_\delta(\tilde{\mathbf{x}})$ satisfies: $F(\mathbf{x}) < m$.



To prove stability of $\tilde{x}$, we show $\Phi^t(B_\delta(\tilde{x})) \subset B_\epsilon(\tilde{x})$ for all $t \geq 0$.

Assume, there exists a point $\mathbf{x} \in B_\delta(\tilde{\mathbf{x}})$ with t_1 such that $\Phi^{t_1}(\mathbf{x}) \notin B_\epsilon(\tilde{\mathbf{x}})$. By continuity of the vector field f we must have that at an earlier time t_2, it holds: $\Phi^{t_2}(\mathbf{x}) \in \partial B_\epsilon(\tilde{\mathbf{x}})$. Due to (12), we get

$$F(\Phi^{t_2}(\mathbf{x})) \geq m.$$

However, $\mathbf{x} \in B_\delta(\tilde{\mathbf{x}})$ and thus $F(\mathbf{x}) < m$. We know that F is nonincreasing because of $\dot{F}(\Phi^t(\mathbf{x})) \leq 0$ for all t and therefore:

$$F(\Phi^{t_2}(\mathbf{x})) - F(\mathbf{x}) = \int_0^{t_2} \frac{d}{ds} F(\Phi^s(\mathbf{x}))ds \leq 0$$
$$\Downarrow$$
$$F(\Phi^{t_2}(\mathbf{x})) \leq F(\mathbf{x}) < m.$$

This is a contradiction, and thus it must hold: $\Phi^t(\mathbf{x}) \in B_\epsilon(\tilde{\mathbf{x}})$ for all t and $\mathbf{x} \in B_\delta(\tilde{\mathbf{x}})$. □

Proof of part two. We want to show that $\tilde{\mathbf{x}}$ is asymptotically stable if F is a strict Lyapunov function. The idea is to prove that $\lim_{t\to\infty} F(\Phi^t(\mathbf{x})) = 0$ for all $\mathbf{x} \in B_\delta(\tilde{\mathbf{x}})$. If that holds, it follows from the continuity of F

$$F(\lim_{t\to\infty} \Phi^t(\mathbf{x})) = 0 \implies \lim_{t\to\infty} \Phi^t(\mathbf{x}) = \mathbf{0},$$

because $F(\mathbf{x}) = 0 \iff \mathbf{x} = \mathbf{0}$. Thus, $\tilde{\mathbf{x}}$ is asymptotically stable, q.e.d.

What remains is the proof of $\lim_{t\to\infty} F(\Phi^t(\mathbf{x})) = 0$. Since $\dot{F}(\Phi^t(\mathbf{x}))$ is strictly decreasing and $F(\Phi^t(\mathbf{x})) \geq 0$ for all t, we know that $\lim_{t\to\infty} F(\Phi^t(\mathbf{x})) = c$ for some $c \geq 0$. We show that $c = 0$ by contradiction.

Suppose $c > 0$. Let S be the level set with $S := \{\mathbf{x} \in \mathbb{R}^n : F(\mathbf{x}) \leq c\}$. Let $B_\alpha(\tilde{\mathbf{x}}) \subset S$ and let $\Phi^t(\mathbf{x})$ be a trajectory with $\mathbf{x} \in B_\delta(\tilde{\mathbf{x}})$. $F(\Phi^t(\mathbf{x}))$ strictly decreases monotonically to c and $F(\Phi^t(\mathbf{x})) > c$. Thus, $\Phi^t(\mathbf{x}) \notin B_\alpha(\tilde{x})$ for all t. Additionally, $\Phi^t(\mathbf{x})$ stays in $B_\epsilon(\tilde{\mathbf{x}})$ for any t as we have shown in part one. Therefore, the trajectory moves within $\|\mathbf{x}-\tilde{\mathbf{x}}\|$ and we can define

$$\gamma := -\max_{\alpha \leq \|\mathbf{x}-\tilde{\mathbf{x}}\| \leq \epsilon} \dot{F}(\mathbf{x}).$$

Clearly, $\gamma > 0$ because $\dot{F}(\mathbf{x}) < 0$. Observe that,

$$F(\Phi^t(\mathbf{x})) = F(\mathbf{x}) + \int_0^t \dot{F}(\Phi^\tau(\mathbf{x}))d\tau \leq F(\mathbf{x}) - t\gamma.$$

For suffieciently large t, it would follow that $F(\Phi^t(\mathbf{x})) < 0$ but that is a contradiction because F is positive definite. Therefore, $c = 0$. □

4.5 Lyapunov functions and linear systems

Week 05
Thu, 15.11

Consider a linear, homogenous system

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}.$$

We will provide a characterisation of a Lyapunov functions for linear systems. This result will later be needed to prove the Theorem of Poincaré-Lyapunov.

Theorem 41. *The fixed point $\tilde{\mathbf{x}} = 0$ of a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ is asympototically stable if and only if for any positive definite symmetric matrix $Q \in \mathrm{GL}(n,\mathbb{R})$[6], there exists a positive definite symmetric matrix $P \in \mathrm{GL}(n,\mathbb{R})$ that satisfies the matrix equation*

$$PA + A^T P = -Q.$$

Proof. " $\Longleftarrow$ " We show if for any Q there exists P such that the matrix equation is satisfied, then $\tilde{\mathbf{x}} = 0$ is a fixed point. Consider the candidate Lyapunov function $F(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$.

Prove that F is indeed a Lyapunov function: F is positive definite, i.e. $F(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \tilde{\mathbf{x}}$ and $F(\tilde{\mathbf{x}}) = 0$. Note that F is a dot product because P is positive definite and symmetric.

Since F is a Lyapunov function, we use the theorem of Lyapunov to show that $\tilde{\mathbf{x}}$ is

[6]space of all real invertible $n \times n$ matrices

asympotically stable. We need to show that $\mathcal{L}_f F(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \tilde{\mathbf{x}}$.

$$\begin{aligned}
\mathcal{L}_f F(\mathbf{x}) &= \langle f(\mathbf{x}), \boldsymbol{\nabla} F(\mathbf{x})\rangle \\
&\Downarrow && \text{(note: } \boldsymbol{\nabla} F(\mathbf{x}) = 2P\mathbf{x}) \\
&= 2\langle A\mathbf{x}, P\mathbf{x}\rangle \\
&= \langle \mathbf{x}, A^T P\mathbf{x}\rangle + \langle \mathbf{x}, PA\mathbf{x}\rangle \\
&= \langle \mathbf{x}, (A^T P + PA)\mathbf{x}\rangle \\
&= -\langle \mathbf{x}, Q\mathbf{x}\rangle < 0 && (Q \text{ is positive definite.})
\end{aligned}$$

" $\implies$ " The fixed point $\tilde{\mathbf{x}} = 0$ is asymptotically stable. Thus all eigenvalues λ_i have negative real part. We define the matrix P as follows:

$$P := \int_0^\infty e^{A^T t} Q e^{At} dt.$$

The matrix P is positive definite because Q is positive definite and symmetric, and thus $e^{A^T t} Q e^{At}$ defines a dot product in the vector space $\mathrm{Mat}_{n\times n}$. Additionally, the integral is converges because A is the matrix corresponding to an asymptotically stable IVP, i.e. $e^{At} \to \mathbf{0}_{n\times n}$ for $t \to \infty$.

We substitute this into $PA + A^T P = -Q$ and see that the matrix equation is indeed satisfied. This step is omitted. $\square$

Note that this proof explicitly gives a plan how to construct a Lyapunov function for linear systems if there are symmetric and positive definite matrices given with $PA + A^T P = -Q$.

... Proof of Lyapunov is missing, see raw notes from 15 November.

4.6 Topological equivalence of dynamical systems

We try to find some kind of equivalence classes between dynamical systems. For that we need to specify when two dynamical systems are equivalent. Naturally, we expect that the phase portrait look qualitatively similar so that the number of fixed points or cycles are the same. We could also say that the phase portrait can be transformed into each other by a continuous map.

Definition 42. *Two dynamical systems* $(\Phi^t, \mathbb{R}, M_1)$ *and* $(\Psi^t, \mathbb{R}, M_2)$ *are* ***topologically equivalent*** *if there exists a homeomorphism*[7] $\Lambda : M_1 \to M_2$ *and a time-increasing function* $\tau(t)$[8] *such that*

$$(\Lambda \circ \Phi^t)(\mathbf{x}) = (\Psi^{\tau(t)} \circ \Lambda)(\mathbf{x}), \quad \forall \mathbf{x} \in M_1, t \in \mathbb{R}.$$

[7] A homeomorphism is a bijection that is continuous and whose inverse is continuous.

[8] $\tau : \mathbb{R} \to \mathbb{R}, t \mapsto \tau(t)$ is continuous and monotically increasing. Such a map preserves the direction of time.

*In other words, it is possible to find a map Λ from one phase space to the other, and a map τ from one time in a phase space to time in the other, in such a way that the evolution is the same. Clearly topological equivalence maps fixed points to fixed points and periodic orbits to periodic orbits (though not necessarily of the same period). A stronger condition is **topologically conjugate**, that means time is preserved: $\tau = \mathrm{id}_{\mathbb{R}}$.*

$$(\Lambda \circ \Phi^t)(\mathbf{x}) = (\Psi^t \circ \Lambda)(\mathbf{x}), \quad \forall \mathbf{x} \in M_1, t \in \mathbb{R}.$$

If Λ is a diffeomorphism[9] *of class $\mathcal{C}^k$, then $(\Phi^t, \mathbb{R}, M_1)$ and $(\Psi^t, \mathbb{R}, M_2)$ are called **$\mathcal{C}^k$-diffeomorphic**.*

Assume that $(\Phi^t, \mathbb{R}, M)$ with vector field f and $(\Psi^t, \mathbb{R}, M)$ with vector field g are $\mathcal{C}^k$-diffeomorphic with $\tau = \mathrm{id}_{\mathbb{R}}$. Then there exists a diffeomorphism $\Lambda \in \mathcal{C}^k(M, M)$ such that

$$\Lambda(\Phi^t(\mathbf{x})) = \Psi^t(\Lambda(\mathbf{x})), \quad \forall \mathbf{x} \in M.$$

Differentiating with respect to t at $t = 0$ yields

$$D\Lambda(\mathbf{x}) \cdot f(\mathbf{x}) = g(\Lambda(\mathbf{x}))$$
$$\Downarrow$$
$$f(\mathbf{x}) = (D\Lambda(\mathbf{x}))^{-1} g(\Lambda(\mathbf{x})).$$

Note that $D\Lambda(\mathbf{x})$ is the Jacobian matrix of Λ. Two diffeomorphic systems are practically identical and can be viewed as the same system using different coordinates.

Example. Consider the three dynamical systems

$$\begin{cases} \dot{x}_1 = -2x_1 \\ \dot{x}_2 = -2x_2 \end{cases} \quad \begin{cases} \dot{x}_1 = -2x_1 + x_2 \\ \dot{x}_2 = -x_1 - 2x_2 \end{cases} \quad \begin{cases} \dot{x}_1 = -2x_1 \\ \dot{x}_2 = x_1 - 2x_2 \end{cases}$$

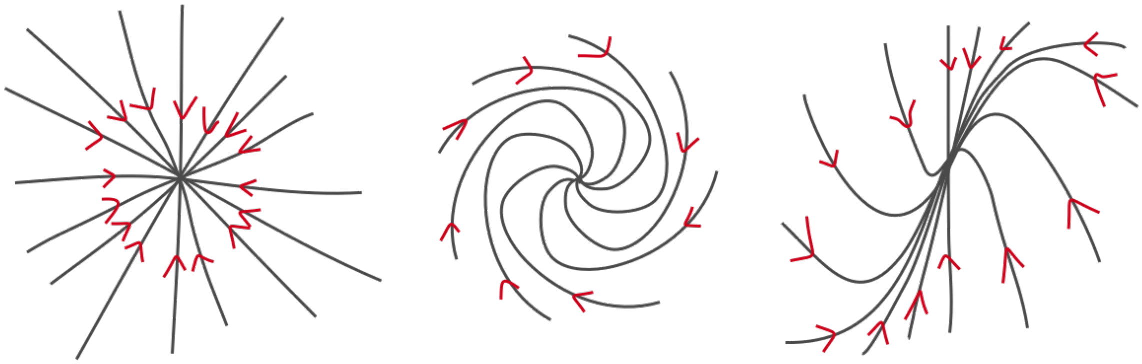
The eigenvalues of the corresponding matrices are

$$\lambda_{1,2} = -2 \quad \lambda_{1,2} = -2 \pm i, \quad \lambda_{1,2} = -2.$$

The fixed point $(0, 0)$ is asymptotically stable in all three cases because all eigenvalues have positive real part.

The phase portraits of the system are depicted by the graphics below:

[9] A diffeomorphism a is continuously differentiable functions and its inverse is continuously differentiable as well.



We rewrite the systems in polar coordinates. The first two systems expressed in polar coordinates are

$$\begin{cases} \dot{r} = -2r \\ \dot{\theta} = 0 \end{cases} \quad \text{and} \quad \begin{cases} \dot{s} = -2s \\ \dot{\varphi} = -1 \end{cases}$$

The behaviour of the systems expressed in polar coordinates match the pictures. The radius is decreasing. For the first system the angle remains constant while for the second the angle constantly decreases, which results in rotation to the right.

System one and two are topologically conjugate for the homeomorphism $\Lambda : (s, \varphi) \mapsto (r, \theta)$ maps the phase spaces from system one to two and preserves time:

$$\Lambda(s, \varphi) = (s, \varphi - \frac{1}{2} \ln s).$$

To show that this is indeed the corresponding homeomorphism integrate both system to obtain the solution of the ODEs

$$\Phi_1^t(r, \theta) = (re^{-2t}, \theta) \quad \text{and} \quad \Phi_2^t(s, \varphi) = (se^{-2t}, \varphi - t)$$

and check

$$\begin{aligned} \Lambda(\Phi_2^t(s, \varphi)) &= \Lambda(se^{-2t}, \varphi - t) \\ &= (se^{-2t}, \varphi - t - \frac{1}{2} \ln\big(se^{-2t}\big)) \\ &= (se^{-2t}, \varphi - t - \frac{1}{2} \ln(s) + t) \\ &= (se^{-2t}, \varphi - \frac{1}{2} \ln(s)) \\ &= \Phi_1^t(s, \varphi - \frac{1}{2} \ln(s)) \\ &= \Phi_1^t(\Lambda(s, \varphi)). \end{aligned}$$

Therefore, system one and two are topologically conjugate.

The first and third system are topologically conjugate. The homeomorphism is

$$\Lambda : (x_1, x_2) \mapsto (x_1, x_2 - \frac{x_1}{2} \ln(|x_1|).$$

First, obtain the explicit solutions for system one and system three. The solutions are

$$\Phi_1^t(x_1, x_2) = \begin{pmatrix} x_1 e^{-2t} \\ x_2 e^{-2t} \end{pmatrix} \quad \text{and} \quad \Phi_3^t(x_1, x_2) = \begin{pmatrix} x_1 e^{-2t} \\ x_1 t e^{-2t} + x_2 e^{-2t} \end{pmatrix}.$$

For the solution of the third system, we calculated the Jordan normalform which is

$$A_3 = \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then compute the matrix exponential $e^{A_3 t} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} e^{-2t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e^{-2t} & 0 \\ te^{-2t} & e^{-2t} \end{pmatrix}$.

Finally, the solution is given by $\Phi_3^t(x_1, x_2) = e^{At} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Now when we have both solutions, we just need to validate the formula given by Definition 42, which we will omit.

The following theorem justifies the approach to examine a linearised version of a hyperbolic nonlinear system in order to analyse the behaviour of the nonlinear system around the equilibrium.

Theorem 43 (Hartman-Grobman). *Consider a hyperbolic nonlinear IVP in $M \subset \mathbb{R}^n$:*

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + h(\mathbf{x}) \\ \mathbf{x}(0) = x_0 \end{cases} \qquad \text{with} \quad h \in \mathcal{C}^1(M, \mathbb{R}^n), h(\mathbf{0}) = \mathbf{0} \text{ and } \left.\frac{dh}{d\mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}} = \mathbf{0}.$$

For a neighborhood $\tilde{M}$ of the hyperbolic fixed point $\tilde{\mathbf{x}} = \mathbf{0}$, the nonlinear system is locally topologically conjugate to the linearised system.

Precisely, if Φ^t is the flow of the nonlinear system, then there exists a homeomorphism Λ such that

$$\Lambda(e^{At}\mathbf{x}) = \Phi^t(\Lambda(\mathbf{x})), \quad \forall \mathbf{x} \in \tilde{M}, t \in \mathbb{R}.$$

Example (Hyperbolic fixed point). Consider the nonlinear IVP in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = 2x_1 + x_2^2 \\ \dot{x}_2 = x_2 \\ x_1(0), x_2(0) \in \mathbb{R} \end{cases}.$$

The orbits of the nonlinear system are given by the curves

$$x_1 = x_2^2(\alpha + \log|x_2|), \quad \alpha \in \mathbb{R}.$$

The parameter α depends on the initial conditions $(x_1(0), x_2(0))$. The linearisation of the nonlinear system around the fixed point $(0,0)$ reads as

$$\frac{\partial f}{\partial x_i}(0,0) = \begin{pmatrix} 2 - x_2^2 & 2x_1 - 2x_2 \\ 0 & 1 \end{pmatrix}\Bigg|_{\mathbf{x}=(0,0)}$$

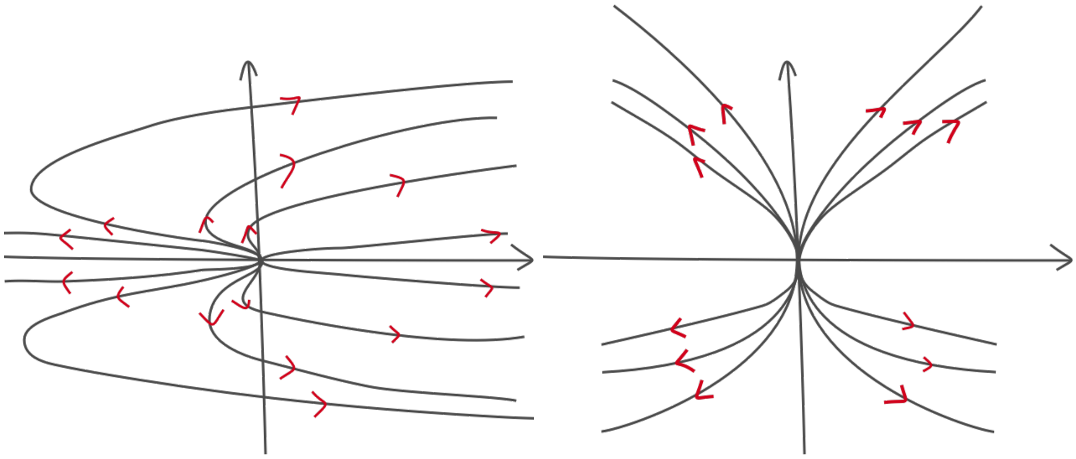
$$\Downarrow$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We see that the system is hyperbolic since all eigenvalues have positive real part (the eigenvalues are 1 and 2). Additionally, the linear system is unstable and after Hartman-Grobman the nonlinear system is unstable, too. The orbits of the linearised system are given by

$$x_1 = \alpha x_2^2, \quad \alpha \in \mathbb{R}.$$

Phase portrait of the nonlinear (left) and linear (right) system:



Week 06
Tue, 20.11

one example missing...

Example (Nonhyperbolic fixed point). Consider the ODE

$$\begin{cases} \dot{x}_1 = x_2 + \lambda x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = -x_1 + \lambda x_2(x_1^2 + x_2^2) \end{cases}.$$

$(0,0)$ is a fixed point. The linearisation $\dot{\mathbf{y}} = A\mathbf{y}$ at $(0,0)$ is given by

$$A = Df(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with eigenvalues} \quad \lambda_{1,2} = \pm i.$$

Since at least one eigenvalue has zero real part, the fixed point $(0,0)$ is not hyperbolic and the theorem of Grobman-Hartman is *not* applicable. To examine the stability behaviour of the nonlinear system we need to find a Lyapunov function instead. Therefore, we write system in polar coordinates where r denotes the radius: $r^2(x_1,x_2) = x_1^2 + x_2^2$. Clearly, the system converges to zero if the radius vanishes with the time. The angle θ has no influence on the stability behaviour.

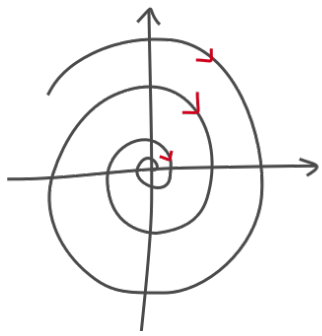
To be a Lyapunov function r must be positive definite. This is true since $r(0,0) = 0$ and $r(x_1,x_2) > 0$ for all $(x_1,x_2) \neq \mathbf{0}$. Now we have to validate that the time derivative of r along any system trajectory $x_1^2 + x_2^2$ is negative, i.e. $\frac{d}{dt} r^2(x_1,x_2) < 0$.

$$
\begin{aligned}
(r^2(x_1,x_2))^{\cdot} \overset{(*)}{=} 2r(x_1,x_2)\dot{r}(x_1,x_2) &= (x_1^2 + x_2^2)^{\cdot} \\
&= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \\
&= 2x_1(x_2 + \lambda x_1 r^2(x_1,x_2)) + 2x_2(-x_1 + \lambda x_2 r^2(x_1,x_2)) \\
&\overset{(**)}{=} 2\lambda r^4(x_1,x_2)
\end{aligned}
$$

Compare $(*)$ and $(**)$ and rearrange the equation by dividing with $2r(x_1,x_2)$:

$$
\begin{aligned}
\dot{r}(x_1,x_2) = \lambda r^3(x_1,x_2) &= \lambda\sqrt{(x_1^2 + x_2^2)^3} \\
&\Downarrow \text{ if } \lambda \text{ is negative} \\
&< 0.
\end{aligned}
$$

Therefore, if $\lambda < 0$ we have found a strict Lyapunov function for the trivial fixed point which means that the $(0,0)$ is asymptotically stable. If $\lambda > 0$ the radius grows with the time and thus the system is unstable in $(0,0)$.



4.7 Fixed points in dynamical systems with discrete time

Consider a map $\Phi : X \to X$ generating a discrete dynamical system $(\Phi)_{n\in\mathbb{Z}}$.

$$\mathbf{x}_0 \text{ is a fixed point if and only if } \mathbf{x}_0 = \Phi(\mathbf{x}_0).$$

Stability is defined exactly as for continuous time system.

We linearise the discrete system $\mathbf{x}_{n+1} = \Phi(\mathbf{x}_n)$ around the fixed point $\mathbf{x}_0$

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{y} \quad \text{with a linear map } \mathbf{y}.$$

A step is described by $\mathbf{x}_{n+1} = \Phi(\mathbf{x}_n)$ leading to

$$\begin{aligned}
\mathbf{x}_0 + \mathbf{y}_{n+1} &= \Phi(\mathbf{x}_0 + \mathbf{y}_n) \\
&\Downarrow \text{Taylor expansion up to the first order} \\
&\approx \Phi(\mathbf{x}_0) + d\Phi(\mathbf{x}_0)\mathbf{y}_n \\
&= \mathbf{x}_0 + d\Phi(\mathbf{x}_0)\mathbf{y}_n \\
&\Downarrow \\
\mathbf{y}_{n+1} &= d\Phi(\mathbf{x}_0)\mathbf{y}_n.
\end{aligned}$$

The linearised discrete system is obtained by

$$\mathbf{y}_{n+1} = B\mathbf{y}_n, \quad B = d\Phi(\mathbf{x}_0), \quad B \text{ is the Jacobi matrix of } \Phi.$$

Compare this with a linearised continuous system

$$\dot{\mathbf{y}} = A\mathbf{y}, \quad A = df(\mathbf{x}_0).$$

Note that a solution $\mathbf{y}_n$ of the linear discrete system is given by $\mathbf{y}_n = B^n\mathbf{y}_0$. If B is a diagonalisable, i.e. $B = U\text{diag}(\mu_1, ..., \mu_d)U^{-1}$, then $B^n = U\text{diag}(\mu_1^n, ..., \mu_d^n)U^{-1}$.

Observe

- if $|\mu_i| < 1$ then $\lim_{n\to\infty} \mu_i^n = 0$
- if $|\mu_i| > 1$ then $\lim_{n\to\infty} \mu_i^n = \infty$
- if $|\mu_i| = 1$ then $|\mu_i|^n = 1$.

	Continuous time	Discrete time
Dynamical system	$\dot{\mathbf{x}} = f(\mathbf{x})$	$\mathbf{x}_{n+1} = \Phi(\mathbf{x}_n)$
Fixed point	$f(\mathbf{x}_0) = \mathbf{0}$	$\mathbf{x}_0 = \Phi(\mathbf{x}_0)$
Linearisation around a fixed point	$\begin{cases} \dot{\mathbf{y}} = A\mathbf{y} \\ A = df(\mathbf{x}_0) \end{cases}$	$\begin{cases} \mathbf{y}_{n+1} = B\mathbf{y}_n \\ B = d\Phi(\mathbf{x}_0) \end{cases}$
Solution of the linearised system	$\mathbf{y} = e^{At}\mathbf{y}_0$	$\mathbf{y}_n = B^n\mathbf{y}_0$
Asymptotical stability	All eigenvalues have negative real part	All eigenvalues have absolute value smaller one
Instablity	At least one eigenvalue has positive real part	At least one eigenvalue has absolute value bigger than one
Contributions of eigenvalues to solutions of the linearised system	$e^{\lambda_i t}$	μ_i^n

5 Stability of periodic solutions

Periodic solutions are flows whose orbits are closed trajectories. While pariticular points on periodic orbits $\mathbf{x}(t_0)$ are no fixed points and therefore it does not make sense to talk about stability of single points, we can consider the stability of the whole orbit as an invariant set.

5.1 Definition and examples

Before we introduce the notion of stability of periodic orbits, let's examine one example to get an intuitive grasping of the concept.

Example. Consider the continuous system:

$$\begin{cases} \dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2) \end{cases} .$$

Transform into polar coordinates.

$$\begin{cases} x_1 = r\cos(\varphi) \\ x_2 = r\sin(\varphi) \end{cases}$$

Set $\rho := r^2 = x_1^2 + x_2^2$ and try to find the constraint of ρ.

$$\begin{aligned} \dot\rho = (x_1^2 + x_2^2)^{\cdot} = 2x_1\dot x_1 + 2x_2\dot x_2 &= 2x_1(-x_2 + x_1(1-\rho)) + 2x_2(x_1 + x_2(1-\rho)) \\ &= 2(x_1^2 + x_2^2)(1-\rho) \\ &= 2\rho(1-\rho). \end{aligned}$$

Separation of variables results in

$$\frac{d\rho}{\rho(1-\rho)} = 2dt \iff \frac{d\rho}{\rho} + \frac{d\rho}{1-\rho} = 2dt \iff \ln\left(\frac{\rho}{1-\rho}\right) = 2t + \text{const} \iff \frac{\rho}{1-\rho} = \text{const}\cdot e^{2t}.$$

Depending on the sign of the constant we get

$$\frac{\rho}{1-\rho} = \pm e^{2(t-t_0)}.$$

t_0 is a parameter which is to determined such that $\rho(0) = \rho_0$. The angle of the coordinates is given by $\varphi = \arctan\left(\frac{x_2}{x_1}\right)$. Remember that $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$. Thus

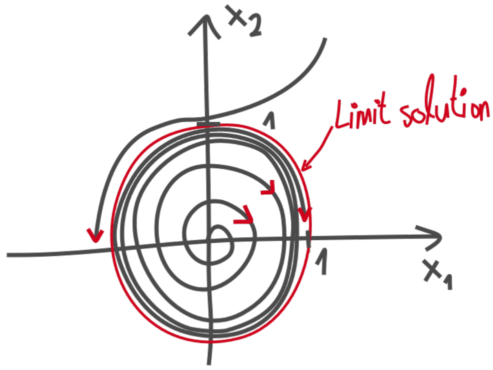
$$\begin{aligned} \dot\varphi = (\arctan\left(\frac{x_2}{x_1}\right))^{\cdot} = \frac{\frac{\dot x_2 x_1 - \dot x_1 x_2}{x_1^2}}{1 + \frac{x_2^2}{x_1^2}} &= \frac{\dot x_2 x_1 - \dot x_1 x_2}{x_1^2 + x_2^2} \\ &= \frac{x_1(x_1 + x_2(1-\rho)) - x_2(-x_2 + x_1(1-\rho))}{x_1^2 + x_2^2} \\ &= \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} \\ &= 1 \\ &\Downarrow \\ \varphi(t) &= t + \varphi_0. \end{aligned}$$

For the evolution of ρ: If the initial value $\rho(0)$ lies between $0 < \rho(0) < 1$ then $\frac{\rho(0)}{1-\rho(0)} > 0$ so that we must take the positive sign of $\frac{\rho(t)}{1-\rho(t)} = e^{2(t-t_0)}$. Thus

$$\rho(t) = \frac{e^{2(t-t_0)}}{e^{2(t-t_0)} + 1} \to \begin{cases} 1, & t \to \infty \\ 0, & t \to -\infty \end{cases}$$

If $\rho > 1$ then $\frac{\rho(0)}{1-\rho(0)} < 0$ so that $\frac{\rho(t)}{1-\rho(t)} = -e^{2(t-t_0)}$.

$$\rho(t) = \frac{e^{2(t-t_0)}}{e^{2(t-t_0)} - 1} \to \begin{cases} 1, & t \to \infty \\ \infty, & t \searrow t_0 \end{cases}$$



The limit solution is a circle with radius one. It is asymptotically stable. Therefore the isolated periodic solution is $\rho(t) \equiv 1, \varphi(t) = t + \varphi_0$. This is equivalent to

$$\begin{cases} x(t) = \cos(t + \varphi_0) \\ y(t) = \sin(t + \varphi_0) \end{cases}$$

All other solutions approach this solution as t goes to infinity.

Definition 44 (Stability of periodic solutions). *Let $\mathbf{x}_0(t)$ be a T-periodic solution of $\dot{\mathbf{x}} = f(\mathbf{x})$ with trajectory $\gamma = \{\mathbf{x} : \mathbf{x} = \mathbf{x}_0(t), t \in \mathbb{R}\}$. It is called* ***asymptotically stable*** *if the following requirements hold true*

- $\forall \epsilon > 0, \exists \delta > 0, \forall \mathbf{y} \in B_\delta(\gamma) : \Phi^t(\mathbf{y}) \in B_\epsilon(\gamma), \quad \forall t \in \mathbb{R}$,
- $\exists \delta > 0, \forall \mathbf{y} \in B_\delta(\gamma) : \operatorname{dist}(\Phi^t(\mathbf{y}), \gamma) = \min_{\mathbf{x}_0 \in \gamma} |\Phi^t(\mathbf{y}) - \mathbf{x}_0| \to 0$ *for* $t \to \infty$.

We say that the asymptotic stability takes place with an ***asymptotic phase*** $c \in \mathbb{R}$ *if*

$$\exists \delta > 0, \forall \mathbf{y} \in B_\delta(\gamma) : |\Phi^t(\mathbf{y}) - \mathbf{x}_0(t + c)| \to 0 \quad \textit{for} \quad t \to \infty.$$

Week 06
Thu, 22.11

Example. Consider the dynamical system examined from last week

$$\begin{cases} \dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2) \end{cases} . \tag{13}$$

We know that $\mathbf{x}(t) = (\cos t, \sin t)$ is a solution of this ODE. The period is $T = 2\pi$ as $\mathbf{x}(t + T) = \mathbf{x}(t)$ for all $t \in \mathbb{R}$. We want to linearise the nonlinear system along the periodic solution and see how the periodicity is reflected in the linear system. The linear system is $\dot{\mathbf{y}} = A(t)\mathbf{y}$ with $A(t) = \frac{\partial f}{\partial x} f(\mathbf{x}(t))$.

$$\begin{pmatrix} \dot{y}_1 \\ \\ \dot{y}_2 \end{pmatrix} = \left.\begin{pmatrix} 1-9x_1^2 & -1-2x_1x_2 \\ \\ 1-2x_1x_2 & 1-x_1^2-3x_2^2 \end{pmatrix}\right|_{(x_1,x_2)=(\cos t,\sin t)} \begin{pmatrix} y_1 \\ \\ y_2 \end{pmatrix}$$
$$= \underbrace{\begin{pmatrix} -2\cos^2 t & -1-2\cos t\sin t \\ \\ 1-2\cos t\sin t & -2\sin^2 t \end{pmatrix}}_{=A(t)} \begin{pmatrix} y_1 \\ \\ y_2 \end{pmatrix}.$$

The matrix $A(t)$ is 2π periodic (accidentically, it is even π-periodic but we will ignore this). Thus $A(t+2\pi) = A(t)$ for all real t.

Imagine we do not know that the solution $\mathbf{x}(t)$ is 2π periodic but we do know that the linearised system is 2π periodic. Can we interfere periodicity properties for the nonlinear system? We could if we know that the linearised system is hyperbolic for at least one point along the solution. This is great because we only need to find one hyperbolic solution along $A(t)$, i.e. find t_0 such that all eigenvalues of $A(t_0)$ have nonnegative real part. If we would find such t_0, we can conclude that the nonlinear system and linear system are topologically conjugate after the theorem of Hartman-Grobman. That means the nonlinear system is periodic, for $x(t_0+T) = x_{(t_0)}$ implies $x(t+T) = x(t)$ for all $t \in \mathbb{R}$ (we have shown in an exercise sheet that if a solution is at one particular point periodic, it is periodic everywhere). How can find a t_0 such that all eigenvalues of $A(t_0)$ have nonzero real parts? We use the mean value theorem for integrals

$$\frac{1}{2\pi}\int_0^{2\pi} A(t)dt = \begin{pmatrix} -1 & -1 \\ \\ 1 & -1 \end{pmatrix} = A(t_0), \quad t_0 \in (0, 2\pi).$$

$A(t_0)$ is hyperbolic for the characteristic polynomial is $(\lambda+1)^2 = 0$ and the eigenvalues are $\lambda_{1,2} = -1 \pm i$. Thus the nonlinear system is 2π-periodic.

5.2 Floquet theory of linear differential equations with periodic coefficients

5.2.1 Homogeneous systems

Consider a periodic homogeneous linear ODE in $\mathbb{R}^n$:

$$\dot{\mathbf{x}} = A(t)\mathbf{x}, \quad A : \mathbb{R} \to \mathrm{Mat}_{n\times n} \text{ is continuous and } T\text{-periodic}^{10}.$$

Theorem 45 (Theorem of Floquet). *Every fundamental matrix $X(t)$ of $\dot{X} = A(t)X$ can be represented in the form*

$$X(t) = P(t)e^{tR}$$

where $P(t)$ is a T-periodic matrix-valued function and $R \in \mathrm{Mat}_{n\times n}$.

Proof. Take any fundamental matrix $X(t)$ and consider the time-shifted matrix solution $Z(t) := X(t+T)$ with period $T \in \mathbb{R}$. We show that $Z(t)$ is also a fundamental matrix:

$$\dot{Z}(t) = \dot{X}(t+T) = A(t+T)X(t+T) = A(t)X(t+T) = A(t)Z(t).$$

By Theorem 37 we get that $X(t+T) = X(t)C$ for an invertible matrix C.

Claim: Any non-degenerate $C \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ has a **matrix logartihm** $S \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ such that $e^S = C$. We prove the claim: If C is non-degenerate and diagonalisable, then $C = U\Lambda U^{-1}$ with $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n)$ and $\lambda_i \neq 0$ for all $i = 1, ..., n$. Set

$$S := U\mathrm{diag}(\ln(\lambda_1), ..., \ln(\lambda_n))U^{-1}.$$

If C is non-degenerate and not diagonalisable, then it has at least a Jordan normal form: $C = U\mathrm{diag}(J_1, ..., J_k)U^{-1}$ where $J_i = \lambda I + N$ are Jordan blocks. Set

$$S := U\mathrm{diag}(\ln(J_1), ..., \ln(J_k))U^{-1}$$

where

$$\ln(\lambda I + N) = \ln(\lambda)I + \ln\left(I + \frac{1}{\lambda}N\right) = \ln(\lambda)I + \underbrace{\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j}(\frac{1}{\lambda}N)^j}_{<\infty \text{ because } N \text{ is nilpotent}} .$$

This proves the claim of the existence of a matrix logarithm.

Having proved the claim, we return back to the proof of the theorem of Floquet. We set

[10] The matrix function $A(t)$ is called T-periodic if $A(t+T) = A(t)$ for all $t \in \mathbb{R}$

$$\boxed{R \coloneqq \frac{1}{T}\ln C, \quad P(t) \coloneqq X(t)e^{-tR}.}$$

Check that P is a T-periodic matrix function, i.e. $P(t+T) = P(t)$ holds true for all $t \in \mathbb{R}$. Indeed

$$\begin{aligned}
P(t+T) = X(t+T)e^{(-t+T)R} = X(t+T)e^{-TR-tR} &= X(t+T)e^{-TR}e^{-tR} \\
&\Downarrow \text{note: } e^{-TR} = e^{-\ln C} = C^{-1} \\
&= X(t+T)C^{-1}e^{-tR} \\
&\Downarrow \text{note: } X(t+T) = X(t)C \\
&= X(t)e^{-tR} \\
&= P(t).
\end{aligned}$$

Check that $P(t)e^{tR} = X(t)e^{-tR}e^{tR} = X(t)$. The theorem of Floquet is proved. □

Note that the proof gives us a formula on how to chose R and $P(t)$. Besides, we showed that $X(t+T)$ is a fundamental solution.

Definition 46. *A matrix C is called a* ***monodromy matrix*** *for a fundamental matrix $X(t)$ if $X(t+T) = X(t)C$ for all $t \in \mathbb{R}$. Eigenvalues of C are called* ***Floquet multipliers****.*

Theorem 47. *Any other monodromy matrix $\tilde{C}$ for any other fundamental matrix $\tilde{X}(t)$ is conjugate to the monodromy matrix C of $X(t)$. Thus, Floquet multipliers do not depend on the choice of solution.*

Proof. Let $\tilde{X}$ be any other fundamental matrix and let $\tilde{C}$ be its monodromy matrix, i.e. $\tilde{X}(t+T) = \tilde{X}(t)\tilde{C}$. After Theorem 37, there exists D such that $\tilde{X}(t) = X(t)D$. We get

$$\tilde{X}(t+T) = X(t+T)D = X(t)CD = \tilde{X}(t)D^{-1}CD \implies \tilde{C} = D^{-1}CD.$$

The eigenvalues of C and $\tilde{C}$ are equal. □

Theorem 48. *A complex value $\mu \in \mathbb{C}$ is a Floquet multiplier for $\dot{\mathbf{x}} = A(t)\mathbf{x}$ if and only if there exists a nontrivial solution $\mathbf{x}(t)$ such that $\mathbf{x}(t+T) = \mu\mathbf{x}(t)$. Such solutions are called* ***Floquet solutions****.*

Proof. μ is a multiplier if and only if $C\mathbf{u} = \mu\mathbf{u}$ for eigenvectors $\mathbf{u} \neq \mathbf{0}$. Set $\mathbf{x}(t) \coloneqq X(t)\mathbf{u}$. We have

$$\mathbf{x}(t+T) = X(t+T)\mathbf{u} = X(t)C\mathbf{u} = \mu X(t)\mathbf{u} = \mu\mathbf{x}(t).$$

$\square$

Collorary 49. *The system* $\dot{\mathbf{x}} = A(t)\mathbf{x}$ *with a* T*-periodic matrix function* $A(t)$ *has a* T*-periodic solution if and only if* 1 *is a Floquet multiplier.*

It is very difficult to analytically find multipliers for a given system; it is an easy task to find it numerically. The formula by Liouville gives additional information about the Floquet multipliers:

$$\begin{aligned} \det X(T) &= \det X(0) e^{\int_0^T \operatorname{tr}(A(z))dz} \\ X(T+t) = X(t)C \Downarrow & \\ \det(X(0) \cdot C) &= \det X(0) e^{\int_0^T \operatorname{tr}(A(z))dz} \\ &\Downarrow \text{divide by } \det X(0) \\ \mu_1 \cdot \ldots \cdot \mu_n = \det C &= e^{\int_0^T \operatorname{tr} A(z)dz}. \end{aligned}$$

5.2.2 Nonhomogenous systems

Consider a non-homogeneous system

$$\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{b}(t), \quad A(t) \text{ and } \mathbf{b}(t) \text{ are } T\text{-periodic.}$$

Theorem 50. *A solution* $\mathbf{x}(t)$ *of this system is* T*-periodic if and only if* $\mathbf{x}(0) = \mathbf{x}(T)$*.*

Proof. The necessity is obvious: if a solution $\mathbf{x}(t)$ is periodic with period T, then $\mathbf{x}(0) = \mathbf{x}(0+T) = \mathbf{x}(T)$. To show sufficiency: Let $\mathbf{x}(t)$ be a solution of the non-homogeneous system with $\mathbf{x}(0) = \mathbf{x}(T)$. We need to show that $\mathbf{x}(t)$ is T-periodic. Consider $\mathbf{y}(t) := \mathbf{x}(t+T)$ which solves the non-homogeneous system:

$$\begin{aligned} \dot{\mathbf{y}}(\mathbf{t}) = \dot{\mathbf{x}}(t+T) &= A(t+T)\mathbf{x}(t+T) + \mathbf{b}(t+T) \\ &= A(t)\mathbf{x}(t+T) + \mathbf{b}(t) \\ &= A(t)\mathbf{y}(t) + \mathbf{b}(t). \end{aligned}$$

Additionally, it holds $\mathbf{y}(0) = \mathbf{x}(T) = \mathbf{x}(0)$; solutions $\mathbf{x}$ and $\mathbf{y}$ intersect at $t = 0$. By uniqueness it follows that $\mathbf{y}(t) = \mathbf{x}(t)$ and therefore $\mathbf{x}(t+T) = \mathbf{x}(t)$. The solution $\mathbf{x}$ is T-periodic. $\square$

Theorem 51. *If* 1 *is not a Floquet multiplier of* $\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{b}(t)$ *with* T*-periodic* A*, then the system has exactly one* T*-periodic solution for any* T*-periodic* $\mathbf{b}(t)$*.*

Note that in this case $\dot{\mathbf{x}} = A(t)\mathbf{x}$ *has no nontrivial* T*-periodic solutions!*

Proof. Let $X(t)$ be a fundamental matrix of $\dot{X} = A(t)X$ with $X(0) = I$ and A is T-periodic. Then $X(T)$ is its monodromy matrix because $X(T) = X(0+T) = X(0) \cdot C = I \cdot C = C$. We get the solution $\mathbf{x}(t)$ of the non-homogeneous system

$$\begin{cases} \dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{b}(t) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

by variation of constants:

$$\mathbf{x}(t) = X(t)\mathbf{x}_0 + \int_0^t X(t) \cdot X^{-1}(s)\mathbf{b}(s)ds.$$

The solution $\mathbf{x}(t)$ is T-periodic iff $\mathbf{x}(T) = \mathbf{x}_0$ (see Theorem 50). Thus we get

$$\mathbf{x}_0 = X(T)\mathbf{x}_0 + \int_0^T X(T) \cdot X^{-1}(s)\mathbf{b}(s)ds \iff (I - X(T))\mathbf{x}_0 = \int_0^T X(T) \cdot X^{-1}(s)\mathbf{b}(s)ds.$$

If $I - X(T)$ is non-degenerate, i.e. $\det(I - X(T)) \neq 0$, then the equation above admits a unique solution for any given T-periodic $\mathbf{b}(t)$. The condition $\det(I - X(T)) \neq 0$ is equivalent to the condition that 1 is not an eigenvalue of $X(t)$ which is the monodromy matrix. The value 1 cannot be a Floquet multiplier. □

5.2.3 Examples

Consider the example from the beginning

$$\begin{cases} \dot{x}_1 = -x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 = x_1 + x_2(1 - x_1^2 - x_2^2) \end{cases} \quad \text{with } A(t) = \begin{pmatrix} -2\cos^2 t & -1 - 2\cos t \sin t \\ 1 - 2\cos t \sin t & -2\sin^2 t \end{pmatrix}$$

We will see that the linearised system $\dot{\mathbf{y}} = A(t)\mathbf{y}$ with period 2π has a multiplier 1. To see this, we must find a nontrivial 2π-periodic solution $\mathbf{y}(t)$ due to Collorary 49. Luckily,

$\mathbf{y}(t) = (\sin t, -\cos t)$ is a solution:

$$
\begin{aligned}
A(t)\mathbf{y}(t) &= \begin{pmatrix} -2\cos^2 t & -1 - 2\cos t \sin t \\ \\ 1 - 2\cos t \sin t & -2\sin^2 t \end{pmatrix} \begin{pmatrix} \sin t \\ \\ -\cos t \end{pmatrix} \\
&= \begin{pmatrix} -2\cos^2(t)\sin(t) + \cos(t) + 2\cos^2(t)\sin(t) \\ \\ \sin(t) - 2\cos(t)\sin^2(t) + 2\sin^2(t)\cos(t) \end{pmatrix} \\
&= \begin{pmatrix} \cos t \\ \\ \sin(t) \end{pmatrix} \\
&= \dot{y}(t).
\end{aligned}
$$

Obviously, $\mathbf{y}(t)$ is 2π periodic. Thus, 1 is indeed a Floquet multiplier.

In general: Differentiating $\dot{\mathbf{x}} = f(\mathbf{x}(t))$ yields

$$
\ddot{\mathbf{x}}(t) = \frac{\partial f}{\partial x}(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) \iff \mathbf{y} = \dot{\mathbf{x}}(t) \text{ is a solution of } \dot{\mathbf{y}} = A(t)\mathbf{y}(t),
$$

where $A(t) = \frac{\partial f}{\partial x}(\mathbf{x}(t))$. If $\mathbf{x}(t)$ is a periodic solution, then $\mathbf{y}(t)$ is T-periodic. *The value* 1 *is always a multiplier of* $\dot{\mathbf{y}} = A(t)\mathbf{y}$; remember that in the proof of the theorem of Floquet we have shown that $\mathbf{y}(t+T)$ is a solution if $\mathbf{y}(t)$ is a solution. For 1 is a multiplier, we have $\mu_1 \cdot \ldots \cdot \mu_n = e^{\int_0^T \operatorname{tr} A(z) dz}$ and $\mu_1 = 1$. This gives us both multipliers if $n = 2$.

Example. Consider

$$
\begin{cases} \mu_1 \mu_2 = e^{\int_0^{2\pi} -2\cos^2 t - 2\sin^2 t dt} = e^{-4n} \\ \mu_1 = 1 \end{cases}.
$$

It follows that $\mu_2 = e^{-4n}$. The fact that $|\mu_2| < 1$ makes for the asymptotically stability of the limit cycle.

Week 07
Tue, 27.11

5.3 Stability of limit cycles

Consider any continuous system $\dot{\mathbf{x}} = f(\mathbf{x})$. The vector field f may be nonlinear and nonperiodic. If this system admits a periodic solution $\mathbf{x}_0(t)$ with period T, what can we say about the stability of this limit cycle? Again, linearisation may give an answer. If we linearise this system along the orbit $\Gamma := \{\mathbf{x}_0(t) : t \in \mathbb{R}\}$, the linearised system

$$\dot{\mathbf{y}} = A(t)\mathbf{y} \quad \text{with} \quad A(t) = \frac{\partial f}{\partial x}(\mathbf{x}_0(t)) \qquad \text{(Poincare variational equation)}$$

has periodic coefficients with period T, i.e. $A(t+T) = A(t)$. Thus, we obtain a *periodic homogeneous linear* ODE, and for such kinds of systems we have developed the **Floquet theory**. Remember that one multiplier of $\dot{\mathbf{y}}(t) = A(t)\mathbf{y}$ is always unity; we will denote this multiplier by μ_1.

Theorem 52. *Consider a periodic solution $\mathbf{x}_0(t)$ with period T of $\dot{\mathbf{x}} = f(\mathbf{x})$. If the Poincare variatonal equation $\dot{\mathbf{y}} = A(t)\mathbf{y}$ only has multipliers $|\mu_j| < 1$ (except for the unity multiplier), then the periodic orbit $\Gamma = \{\mathbf{x}_0(t) : t \in \mathbb{R}\}$ is* ***asymptotically stable****, i.e. it holds*

- *for any $\epsilon > 0, \exists \delta > 0$ such that for any $\mathbf{z} \in B_\delta(\Gamma)$ we have: $\Phi^t(\mathbf{z}) \in B_\epsilon(\Gamma)$ for all $t \geq 0$*
- *$\exists \delta > 0$ such that for any $\mathbf{z} \in B_\delta(\Gamma)$ we have : $\lim_{t\to\infty} \operatorname{dist}(\Phi^t(\mathbf{z}), \Gamma) = 0$*

Moreover, the periodic orbit Γ is asymptotically stable with an ***asymptotic phase****, i.e. for any $\mathbf{z} \in B_\delta(\Gamma)$ there exists $c \in \mathbb{R}$ such that $\lim_{t\to\infty} \|\Phi^t(\mathbf{z}) - \mathbf{x}_0(t+c)\| = 0$.*

In order to utilise this theorem we need all multipliers. This is not an easy task! However, we can define the **Poincaré section** which lowers the dimension of the problem by one and discretises the system. The idea is to contruct a local hypersurface S, i.e. a manifold of dimension $n-1$, which is transversal to the orbit Γ at point $\mathbf{p} = \mathbf{x}_0(0)$. This hypersurface is called the Poincaré section. The transversality of S means that orbits starting on the surface flow through the surface and not parallel to it. If all nearby trajectories around $\mathbf{p}$ intersect the surface, we may conclude stability properties of the periodic orbit Γ. Actually, this is the way we will prove Theorem 52 by showing some kind of equivalence between fixed points of the Poincaré map and periodic orbits (see Lemma 55).

I recommend watching the video **Intro to Poincaré maps** *found on* `https://scholar.harvard.edu/siams/class-17-coupled-oscillators-and-poincare-maps` *to get a good overview of Poincaré maps. It helps understanding the following definition.*

Theorem 53 (Poincaré map). *If f is a $\mathcal{C}^k$ vector field, then there exists open neighbourhoods $W_0, W_1 \subset S$ of $\mathbf{p}$ and a $\mathcal{C}^k$ diffeomorphism $P : W_0 \to W_1$ such that for all $\mathbf{q} \in W_0$ it holds*

- *$\exists \tau(\mathbf{q}) \in \mathbb{R} : P(\mathbf{q}) = \Phi^{T+\tau(\mathbf{q})}(\mathbf{q}) \in W_1$,*
- *and $\Phi^t(\mathbf{q}) \notin W_1$ for $0 < t < T + \tau(\mathbf{q})$.*

The diffeomorphism P is called ***Poincaré map*** *(also known as* ***first return map****) on the Poincaré section S. The Poincaré map $P(\mathbf{q})$ gives the point on the surface S where the trajectory starting at $\mathbf{q}$ first returns to S.*

This is a proof I do not quite understand yet...

Proof. Chose any Poincaré section S. Then, $\Phi^t : U_0 \to U_1$ is a diffeomorphism, where U_0 and U_1 are small open neighbourhoods of $\mathbf{p}$ in $\mathbb{R}^n$. For $\frac{d}{dt}\Phi^{T+t}(\mathbf{p})\Big|_{t=0} = \dot{\mathbf{x}}_\mathbf{0}(t)$, the surface $\frac{d}{dt}\Phi^{T+t}(\mathbf{q})\Big|_{t=0}$ is transversal to S. Since $\Phi^{T+t}(\mathbf{q})$ is differentiable with respect to t, $\Phi^T(\mathbf{p}) = \mathbf{p}$, and the transaversality of $\frac{d}{dt}\Phi^{T+t}(\mathbf{q})\Big|_{t=0}$, the surface $\frac{d}{dt}\Phi^{T}(\mathbf{q})\Big|_{t=0}$ is also transversal to S in some neighbourhood of $\mathbf{p}$. Therefore, there exists a unique $\tau(\mathbf{q})$ close to 0 such that $P(\mathbf{q}) = \Phi^{T+\tau(\mathbf{q})}(\mathbf{q}) \in S$ by the implicit function theorem.

To show that $P : \mathbf{q} \mapsto \Phi^{T+\tau(\mathbf{q})}(\mathbf{q})$ is a local diffeomorphism, it is sufficient to show that $\det(dP(\mathbf{p})) \neq 0$. In other words, $dP(\mathbf{p})$ is non-degenerate. □

Lemma 54. *Let $\frac{d}{dt}\Phi^t(\mathbf{p})\Big|_{t=0} = \mathbf{u}$ and let $\mathbb{R}^n = \text{span}(\mathbf{u}) \oplus W$ such that W is a $(n-1)$-dimensional vector space and $\mathbf{u} \notin W$. Let (u, w) be some coordinates in $\mathbb{R}^n$ corresponding to this splitting, and let*

$$\Phi^t(u, w) := \begin{pmatrix} \Phi_1^t(u, w) \\ \\ \Phi_2^t(u, w) \end{pmatrix},$$

where $\Phi_1^t(u, w) \in \text{span}(\mathbf{u})$ describes the first coordinate and $\Phi_2^t(u, w) \in W$ describes the $(n-1)$ coordinates.

Then, the monodromy matrix $d\Phi^t(0, 0)$ of the Poincaré equation $\dot{\mathbf{y}} = A(t)\mathbf{y}$ is given by

$$d\Phi^T(0, 0) = \begin{pmatrix} 1 & * \\ \\ 0 & d_2\Phi_2^T(0, 0) \end{pmatrix} = \begin{pmatrix} 1 & * \\ \\ 0 & dP(\mathbf{p}) \end{pmatrix}.$$

In particular, eigenvalues of $dP(\mathbf{p})$ are exactly the multipliers $\mu_2, ..., \mu_n$ of the periodic solution $\mathbf{x}_0(t)$. Thus, the eigenvalues are independent of $\mathbf{p}$ and of the transversal Poincaré section.

Proof. We have coordinates (u, w). The coordiante $\mathbf{w} = (0, w)$ refers to points $\mathbf{w} \in S$.

For any $\mathbf{w} \in S$ it holds:

$$\begin{aligned} P(\mathbf{w}) &= \Phi_2^{T+\tau(\mathbf{w})}(0, \mathbf{w}) \\ &\Downarrow \\ dP(\mathbf{w}) &= \frac{d}{dt}\Phi_2^t(0, \mathbf{w})\Big|_{t=T+\tau(\mathbf{w})} \cdot \tau'(\mathbf{w}) + d_2\Phi_2^{T+\tau(\mathbf{w})}(0, \mathbf{w}) \\ &\Downarrow \\ dP(\mathbf{0}) &= \frac{d}{dt}\Phi_2^t(0, 0)\Big|_{t=T} \cdot \tau'(0) + d_2\Phi_2^T(0, 0). \end{aligned}$$

On the other hand,

$$\frac{d}{dt}\Phi^t(0, 0)\Big|_{t=T} = \begin{pmatrix} \frac{d}{dt}\Phi_1^T(0, 0)\Big|_{t=T} \\ \\ \frac{d}{dt}\Phi_2^T(0, 0)\Big|_{t=T} \end{pmatrix} = \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix},$$

as it corresponds to $\frac{d}{dt}\Phi^t(\mathbf{p})\Big|_{t=T} := \mathbf{u}$, and the vector $\mathbf{u}$ has a zero w-coordinate in $\mathbb{R}^n = \operatorname{span}(\mathbf{u}) \oplus W$. Thus, it holds

$$dP(\mathbf{0}) = d_2\Phi_2^T(0, 0).$$

Next, we show that $d\Phi^T(0, 0)\mathbf{u} = \mathbf{u}$, i.e. 1 is an eigenvalue of $d\Phi^T(\mathbf{p})$ with eigenvector $\mathbf{u}$. It holds

$$\frac{d}{ds}\Phi^{T+s}(0, 0)\Big|_{s=0} = \frac{d}{dt}\Phi^t(0, 0)\Big|_{t=T} = \frac{d}{dt}\Phi^t(0, 0)\Big|_{t=0} = \mathbf{u}.$$

On the other hand, we have

$$\frac{d}{ds}\Phi^{T+s}(0, 0)\Big|_{s=0} = \frac{d}{ds}\Phi^T \circ \Phi^s(0, 0)\Big|_{s=0} = d\Phi^T(0, 0)\frac{d}{ds}\Phi^s(0, 0)\Big|_{s=0} = d\Phi^T(0, 0)\mathbf{u}.$$

Thus, $\mathbf{u}$ is an eigenvector of $d\Phi^T(\mathbf{p})$ with eigenvalue 1. In our local coordinates, $\mathbf{u} \sim (1, 0, ..., 0)^T$, which implies that $d\Phi^T(0, 0)$ must have the block structure

$$d\Phi^T(0, 0) = \begin{pmatrix} 1 & * \\ & \\ 0 & d_2\Phi_2^T(0, 0) = dP(\mathbf{p}) \end{pmatrix}.$$

□

From this lemma we know that the eigenvalues of $dP(\mathbf{p})$ are exactly the floquet multipliers $\mu_2, ..., \mu_n$ of the periodic solution $\mathbf{x}_0(t)$. By the discrete time analogy of the Poincaré-Lyapunov theorem the fixed point of P is asympotically stable if all eigenvalues of $dP(\mathbf{p})$ have absolute value smaller than one (see section 4.7).

Here comes the new insight: Fixed points of the Poincaré maps directly correspond to periodic orbits of the invesgitaed system. Even more, the stability of fixed points of the Poincaré map translates to asymptotic stability of periodic orbits!

Lemma 55. *The periodic orbit* $\Gamma = \{\mathbf{x}_0(t) : t \in \mathbb{R}\}$ *is asymptotically stable if and only if* $\mathbf{p} = \mathbf{x}_0(0)$ *is an asymptotically stable fixed point of the Poincaré map* P.

Proof. $\Longleftarrow$: Let $\mathbf{p}$ be an asymptotically stable fixed point of the Poincaré map P. The Poincaré map P in coordinates of the Poincaré section S around $\mathbf{p}$ is given by the map $\mathbf{w} \mapsto \tilde{\mathbf{w}} = A\mathbf{w} + W(\mathbf{w})$ with $A = dP(\mathbf{p})$ and $W(\mathbf{w})$ representing higher order terms of the Taylor expansion, i.e. $W(\mathbf{0}) = \mathbf{0}$ and $\frac{\partial W}{\partial \mathbf{w}}(\mathbf{0}) = \mathbf{0}$. Since $\mathbf{p}$ is asymptotically stable A has eigenvalues $\mu_2, ..., \mu_n$ with $|\mu_j| < 1$.

If $\|\mathbf{w}\|$ is sufficiently small, then $P(\mathbf{w})$ is a contraction, i.e. $\|\tilde{\mathbf{w}}\| \leq \mu\|\mathbf{w}\|$ for some $\mu > \max(|\mu_2|, ..., |\mu_n|)$ and $0 < \mu < 1$ (note that $\mu < 1$ is possible since all eigenvalues lie inside the unit circle). Let $\mathbf{w}_0 \in S$ be sufficiently small such that P is a contraction as described. There follows that $\mathbf{w}_n := P^n(\mathbf{w}_0)$ is defined for all $n = 1, 2, ...$ and

$$\lim_{n\to\infty} \|\mathbf{w}_n\| \leq \lim_{n\to\infty} \mu^n \|\mathbf{w}_0\| \leq 0.$$

Remember that $P(\mathbf{q}) = \Phi^{\tau(\mathbf{q})}(\mathbf{q})$[11]. Thus, $\mathbf{w}_n = \Phi^{t_n}(\mathbf{w}_0)$ with $t_{n+1} = t_n + \tau(\mathbf{w}_n)$ and $t_0 = 0$. Due to $\tau(\mathbf{0}) = T$[12] and the continuity of τ, we have $\lim_{n\to\infty} \tau(\mathbf{w}_n) = T$. Thus,

$$\lim_{n\to\infty} \frac{t_n}{nT} = 1.$$

Acutally, an even stronger statement holds true

$$\exists c := \lim_{n\to\infty} (t_n - nT) < \infty \quad \text{and} \quad \exists L_1 : |(t_n - nT) - c| \leq L_1 \mu^n \|\mathbf{w}_0\|.$$

To show that such L_1 exists, we observe

$$\begin{aligned}
|(t_n - nT) - (t_{n-1} - (n-1)T)| &= |\tau(\mathbf{w}_{n-1}) - \underbrace{T}_{=\tau(\mathbf{0})}| \\
&\Downarrow \text{ use mean value theorem as } \tau \in \mathcal{C}^1 \\
&\leq L_0 \|\mathbf{w}_{n-1},\| \qquad (L_0 \in \mathbb{R} \text{ is some constant}) \\
&= L_0 \mu^{n-1} \|\mathbf{w}_0\|.
\end{aligned}$$

From this follows the existence of $\lim_{n\to\infty}(t_n - nT) = \sum_{k=1}^{\infty}(t_k - kT) + t_0$; this series is absolutely convergent with $L_1 := \frac{L_0}{1-\mu}$.

[11] Professor Suris used a slightly other definition than at the beginning where $P(\mathbf{q}) = \Phi^{T+\tau(\mathbf{q})}(\mathbf{q})$. The definitions are equivalent however.

[12] note that the coordinate vector $\mathbf{0}$ refers to $\mathbf{p}$ which has a return time of T. Hence $\tau(\mathbf{0}) = 0$.

Since $\Phi^t \in \mathcal{C}^1$ we have

$$\|\Phi^{t+t_n}(\mathbf{w}_0) - \mathbf{x}_0(t)\| = \|\Phi^t(\mathbf{w}_n) - \Phi^t(\mathbf{0})\| \leq L_2\|\mathbf{w}_n\| \leq L_2\mu^n\|\mathbf{w}_0\|.$$

Furthermore,

$$\|\Phi^{t+t_n}(\mathbf{w}_0) - \Phi^{t+nT+c}(\mathbf{w}_0)\| \leq L_3|t_n - (nT+c)| \leq L_4\mu^n\|\mathbf{w}_0\|.$$

Hence,

$$\|\Phi^{t+nT+c}(\mathbf{w}_0) - \mathbf{x}_0(t)\| \leq (L_2 + L_4)\mu^n\|\mathbf{w}_0\|$$

Replace t by $t - nT$:

$$\|\Phi^{t+c}(\mathbf{w}_0) - \mathbf{x}_0(t)^{13}\| \leq (L_2 + L_4)\mu^{\frac{t}{T}}\|\mathbf{w}_0\| \quad \text{for} \quad nT \leq t \leq (n+1)T.$$

Thus, the periodic orbit Γ is by definition asymptotically stable.

$\Longrightarrow$: This direction is obvious. □

Collorary 56. *If all eigenvalues of $dP(\mathbf{p})$ lie inside the unit circle, i.e. $|\mu_j| < 1$ for $j = 2, ..., n$, then the periodic orbit Γ is stable.*

All the hard work is done, and the collorary proves theorem 52.

The book "Ordinary Differential Equations and Dynamical Systems" by Gerald Teschl has an extensive chapter about the stability of limit cycles. It wraps up a lot of concepts that were introduced in this chapter. A free digital copy can be obtained on `https://www.mat.univie.ac.at/~gerald/ftp/book-ode/`.

6 Invariant manifolds of fixed points

6.1 Local manifolds

Week 07
Thu, 29.11

Let $\mathbf{x}_0$ be a fixed point of $\dot{\mathbf{x}} = f(\mathbf{x})$ with a $\mathcal{C}^r$-smooth vector field f. We define the following sets

- **local *stable* manifold of $\mathbf{x}_0$:**

$$W^s_\epsilon(\mathbf{x}_0) := \{\mathbf{x} \in B_\epsilon(\mathbf{x}_0) : \Phi^t(\mathbf{x}) \in B_\epsilon(\mathbf{x}_0) \text{ for all } t \geq 0 \text{ and } \lim_{t\to\infty} \Phi^t(\mathbf{x}) = \mathbf{x}_0\}$$

- **local *unstable* manifold of $\mathbf{x}_0$:**

$$W^u_\epsilon(\mathbf{x}_0) := \{\mathbf{x} \in B_\epsilon(\mathbf{x}_0) : \Phi^t(\mathbf{x}) \in B_\epsilon(\mathbf{x}_0) \text{ for all } t \leq 0 \text{ and } \lim_{t\to-\infty} \Phi^t(\mathbf{x}) = \mathbf{x}_0\}$$

[13] $\mathbf{x}_0(t) = \mathbf{x}_0(t - nT)$ for all $n \in \mathbb{N}$ since $\mathbf{x}_0$ is T-periodic.

Both sets are invariant sets, i.e.

$$\Phi^t(W^s_\epsilon(\mathbf{x}_0)) \subset W^s_\epsilon(\mathbf{x}_0) \quad \forall t \geq 0 \quad \text{and} \quad \Phi^t(W^u_\epsilon(\mathbf{x}_0)) \subset W^u_\epsilon(\mathbf{x}_0) \quad \forall t \leq 0.$$

If $\mathbf{x}_0$ is asymptotically stable, then for sufficiently small ϵ it holds: $W^s_\epsilon(\mathbf{x}_0) = B_\epsilon(\mathbf{x}_0)$ and $W^u_\epsilon(\mathbf{x}_0) = \{\mathbf{x}_0\}$. The more interesting case occurs when $\mathbf{x}_0$ is *hyperbolic* and $A = df(\mathbf{x}_0)$ has eigenvalues with $\operatorname{Re}\lambda_i > 0$ and with $\operatorname{Re}\lambda_j < 0$. We define the **stable subspace** $\mathcal{E}^s$ and **unstable subspace** $\mathcal{E}^u$ as

$$\begin{aligned}
\mathcal{E}^s(\mathbf{x}_0) &\coloneqq \operatorname{span}\{\text{generalised eigenvectors of } A \text{ for eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda < 0\} \\
\mathcal{E}^u(\mathbf{x}_0) &\coloneqq \operatorname{span}\{\text{generalised eigenvectors of } A \text{ for eigenvalues } \lambda \text{ with } \operatorname{Re}\lambda > 0\}
\end{aligned}$$

As a reminder vectors $\mathbf{u}$ *are generalised eigenvectors if* $(A - \lambda I)^k\mathbf{u} = \mathbf{0}$ *for some* $k \in \mathbb{N}$. *If* $k = 1$, $\mathbf{u}$ *is an eigenvector.*

If $\mathbf{x}_0$ is hyperbolic, then $\mathbb{R}^n = \mathcal{E}^s(\mathbf{x}_0) \oplus \mathcal{E}^u(\mathbf{x}_0)$. Therefore, we can choose an coordinate system $\{(y, z)\}$ in $\mathbb{R}^n$ such that $\mathcal{E}^s(\mathbf{x}_0) = \{(y, 0) \in \mathbb{R}^n\}$ and $\mathcal{E}^u(\mathbf{x}_0) = \{(0, z) \in \mathbb{R}^d\}$.

Theorem 57 (Local invariant manifolds). *Let* $\mathbf{x}_0$ *be a hyperbolic fixed point. For* $\epsilon > 0$ *sufficiently small, the local stable manifold* $W^s_\epsilon(\mathbf{x}_0)$ *is an embedded submanifold of* $\mathbb{R}^n$, *which can be represented as a graph of a function* h^s *on* $\mathcal{E}^s(\mathbf{x}_0)$ *such that*

$$W^s_\epsilon(\mathbf{x}_0) = \{(y, h^s(y)) \in \mathbb{R}^n\} \quad \textit{and} \quad h^s \textit{ is tangent to } \mathcal{E}^s(\mathbf{x}_0)^{14}.$$

Additionally, if the vector field $f \in \mathcal{C}^r$, *then* $h^s \in \mathcal{C}^r$.

Example. Consider

$$\ddot{x} = \sin(-x) = \Longleftrightarrow \begin{cases} \dot{x} = u \\ \dot{u} = -\sin(x) \end{cases} .$$

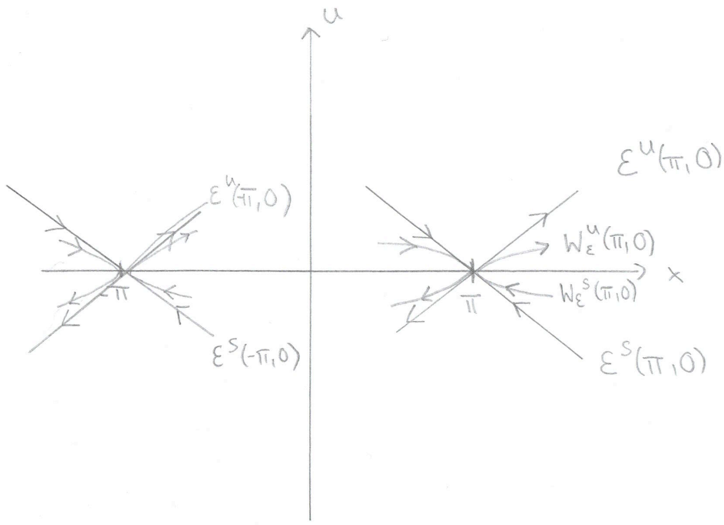
The fixed points are given by $(x, u) = (n\pi, 0)$ for all $n \in \mathbb{Z}$. We will compute for which n the fixed points becomes hyperbolic or nonhyperbolic. For that we need the eigenvalues of $df(n\pi, 0)$.

$$A = df(0, n\pi) = \left.\begin{pmatrix} 0 & 1 \\ -\cos(x) & 0 \end{pmatrix}\right|_{x=n\pi} = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1} & 0 \end{pmatrix}$$

If n is even, $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The eigenvalues are $\lambda_{1,2} = \pm i$. Thus, the fixed point is nonhyperbolic.

[14] h^s is tangent to $\mathcal{E}^s(\mathbf{x}_0)$ if $h^s(0) = 0$ and $dh^s(0) = 0$

If n is odd, $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The eigenvalues are $\lambda_{1,2} = \pm 1$. Therefore, the fixed point is hyperbolic. The eigenraum of $\lambda = -1$ is $\mathcal{E}^s(\pi, 0) = (1, -1)$ and of $\lambda = 1$ it is $\mathcal{E}^u(\pi, 0) = (1, 1)$.



6.2 Gloabal invariant manifolds

The global invariant manifolds are defined as

$$W^s(\mathbf{x}_0) = \bigcup_{t \leq 0} \Phi^t(W^s_\epsilon(\mathbf{x}_0)) \quad \text{and} \quad W^u(\mathbf{x}_0) = \bigcup_{t \geq 0} \Phi^t(W^u_\epsilon(\mathbf{x}_0)).$$

6.3 Discrete time

The theory of absolutely parallel in the discrete time situation.

Theorem 58. *Consider a $\mathcal{C}^1$-map...*

incomplete

6.4 Center Subspaces

Week 08
Tue, 04.12

7 Bifurcations

incomplete...

Week 08
Thu, 06.12

8 Lagrangian Mechanics

8.1 Basic principles

- States point in the configuration space X. X is a vector space in $\mathbb{R}^n$ or a manifold.
- The evolution of the mechanical system is a **curve** $\mathbf{x} : [t_0, t_1] \mapsto X, t \mapsto \mathbf{x}(t)$ in this space.
- Admissible curves[15] satisfy a minimisation property. To formulate it, we need a function $\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}})$. If $X = \mathbb{R}^n$
$$\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$
or if X is a manifold
$$\mathcal{L} : T_{\mathbf{x}}X \to \mathbb{R}.$$
$\mathcal{L}$ is called the **Lagrange function**.
- The **action** along the curce $\mathbf{x}$ is defined as
$$S[\mathbf{x}] = \int_{t_0}^{t_1} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(\mathbf{t}))dt.$$

8.2 Principle of least action

Week 09
Tue, 11.12

The motion of a mechanical system with the Lagrange function $\mathcal{L}$ are those curves $\mathbf{x} : [t_0, t_1] \to X$ which deliver minimal critical points of the action functional $S[\mathbf{x}]$ under variations of the curve fixing its endpoints.

Consider nearby curves $\mathbf{y} : [t_0, t_1] \to \mathbb{R}^n$ with $\mathbf{y}(t_0) = \mathbf{x}(t_0)$ and $\mathbf{y}(t_1) = \mathbf{x}(t_0)$. The variation of the curve is described by $\delta_{\mathbf{x}} : [t_0, t_1| \to \mathbb{R}^n, \delta_{\mathbf{x}} \coloneqq \mathbf{y}(t) - \mathbf{x}(t)$. For $\delta_{\mathbf{x}}(t_0) = \delta_{\mathbf{x}}(t_1) = 0$ the variation is null .

In the old classical formulation, the **principle of least action** is
$$S[\mathbf{x}] \leq S[\mathbf{x} + \delta_{\mathbf{x}}]$$
for all $\delta_{\mathbf{x}}$ which are sufficiently small.

[15] those which describe a real motion of the mechanical system

Example. Consider a natural mechanical system with the Lagrange function

$$\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) = K(\dot{\mathbf{x}}) - U(\mathbf{x})$$

where K is the kinetic energy and U is the potential energy.

- Two body gravitational problem

$$\begin{aligned} K(\dot{\mathbf{x}}) &= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) \\ U(\mathbf{x}) &= -\frac{\gamma m_1 m_2}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}} \end{aligned}$$

- Mathematical pendulum

$$\begin{aligned} K(\dot{x}) &= \frac{1}{2}m\dot{x}^2 \\ U(x) &= -m \cdot g \cdot l \cdot \cos(x) \end{aligned}$$

Now, we will introduce some notations. We define the vector space

$$\mathcal{X} = \mathcal{X}_{\mathbf{x}_0, \mathbf{x}_1} = \{\mathbf{x} \in \mathcal{C}^1([t_0, t_1], \mathbb{R}^n) : \mathbf{x}(t_0) = \mathbf{x}_0, (t_1) = \mathbf{x}_1\}.$$

Note that $\mathcal{X}$ is *not* a vector space but rather an affine space. The space of differences between elements in $\mathcal{X}$ is a vector space

$$\mathcal{X}_{\mathbf{0}} = \mathcal{X}_{\mathbf{0},\mathbf{0}} = \{\mathbf{h} \in \mathcal{C}^1([t_0, t_1], \mathbb{R}^n) : \mathbf{h}(t_0) = \mathbf{0}, \mathbf{h}(t_1) = \mathbf{0}\}$$

with norm $\|\mathbf{h}\|_1 = \sup\limits_{t \in [t_0, t_1]} (\|\mathbf{h}(t)\| + \|\dot{\mathbf{h}}(\mathbf{t})\|)$.

Definition 59. *A functional $S : \mathcal{X} \to \mathbb{R}$ is called **differentiable** if for any curve $\mathbf{x} \in \mathcal{X}$ there exists a linear map $\delta S[\mathbf{x}] : \mathcal{X}_{\mathbf{0}} \to \mathbb{R}$ such that*

$$S[\mathbf{x} + \mathbf{h}] - S[\mathbf{x}] = \delta S[\mathbf{x}]\mathbf{h} + o(\|\mathbf{h}\|_1), \quad \forall \mathbf{h} \in \mathcal{X}_{\mathbf{0}}.$$

*We say that $\mathbf{x} \in \mathcal{X}$ delivers a **critical point** of S if $\delta S[\mathbf{x}] = 0$.*

*We say that $\mathbf{x} \in \mathcal{X}$ delivers a **local minimum** of S if $S[\mathbf{x} + \mathbf{h}] \geq S[\mathbf{x}]$ for all $h \in \mathcal{X}_{\mathbf{0}}$ with $\|\mathbf{h}\|_1$ sufficiently small.*

Theorem 60. *The local minimum of a differentiable functional is its critical point.*

Theorem 61. *If $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^2$, then the action functional $S[\mathbf{x}] = \int_{t_0}^{t_1} \mathcal{L}(\mathbf{x}(t), \dot{\mathbf{x}}(\mathbf{t}))dt$ is differentiable with*

$$\delta S[\mathbf{x}]h = \int_{t_0}^{t_1} \sum_{i=1}^{n} \underbrace{\left[\frac{\partial \mathcal{L}}{\partial x_i} (\mathbf{x}(t), \dot{\mathbf{x}}(t)) - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i} (\mathbf{x}(t), \dot{\mathbf{x}}(t)) \right) \right]}_{\substack{:= E\mathcal{L}_i \text{ ... } \textbf{\textit{Euler-Lagrange expression}} \\ \textit{for the Lagrange function } \mathcal{L}}} h_i(t)dt.$$

Proof. ... ☐

Theorem 62. *A curve $\mathbf{x} \in \mathcal{X}$ is critical for the action S if and only if it satisfies the* ***Euler-Lagrange differential equation***

$$E\mathcal{L}_i = \frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) = 0, \quad \textit{for all } i = 1, ..., n.$$

In order to prove this theore we will need the following lemma.

Lemma 63. *Let $f : [t_0, t : 1] \to \mathbb{R}$ be continuous. If for all...*

Proof of Lemma. Inhalt... ☐

Proof of Theorem. The lemma proves the theorem for $n = 1$. To prove it for $n > 1$ use the lemma componentwise. ☐

Example. The Euler-Lagrange equations for natural mechanical systems are given by

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) &= \frac{1}{2} \langle \dot{\mathbf{x}}, M \dot{\mathbf{x}} \rangle - U(\mathbf{x}), \quad M \in \mathrm{Symm}(n) \\ &= \frac{1}{2} \sum_{i,j=1}^{n} m_{ij} \dot{x}_i \dot{x}_j - U(x_1, ..., x_n) \end{aligned}$$

The Euler-Lagrange expression is

$$E\mathcal{L}_i = \frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i} \right) = -\frac{\partial U}{\partial x_i} - \frac{d}{dt} \sum_{j=1}^{n} m_{ij} \dot{x}_j = -\frac{\partial U}{\partial x_i} - \sum_{j=1}^{n} m_{ij} \ddot{x}_j.$$

So the Euler-Lagrange differential equation $E\mathcal{L}_i = 0$ yields

$$\boxed{\sum_{j=1}^{n} m_{ij} \ddot{x}_j = -\frac{\partial U}{\partial x_i}, \quad i = 1, ..., n.}$$

Those are implicit second order differential equations

$$\boxed{M\ddot{\mathbf{x}} = -\Delta U.}$$

Usually, one would like to the Euler Lagrange equations for $\ddot{x}_i = g_i(\mathbf{x}, \dot{\mathbf{x}})$. For natural mechnical system, one needs $\det M \neq 0$ for this so that

$$\ddot{\mathbf{x}} = \underbrace{-M^{-1}\Delta U(\mathbf{x})}_{=g(\mathbf{x})}.$$

In general

$$E\mathcal{L}_i = \frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i} - \underbrace{\sum_{j=1}^{n} \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_i \partial x_j}\dot{x}_j}_{\text{complicated}} - \underbrace{\sum_{j=1}^{n} \frac{\partial^2 \mathcal{L}}{\partial \dot{x}_i \partial \dot{x}_j}\ddot{x}_j}_{\text{easy}}.$$

To solve the system of Euler Lagrange equations for $\ddot{x}_j = g_j(\mathbf{x}, \dot{\mathbf{x}})$ one needs the **non-degeneracy condition**

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{x}_i \partial \dot{x}_j}\right)_{i,j=1,\ldots,n} \neq 0.$$

Theorem 64. *Euler Lagrange equations admit a conserved quantity*

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \sum_{i=1}^{n} \dot{x}_i \frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}).$$

Proof. dwdw □

Example. For natural mechanical systems

$$E(\mathbf{x}, \dot{\mathbf{x}}) = \sum_{i=1}^{n}$$

Week 09
Thu, 13.12

8.3 Submanifolds in $\mathbb{R}^n$

We will give a quick introduction into submanifolds in $\mathbb{R}^n$ before we move to Lagrangian mechanics on manifolds.

Definition 65. *$M \subset \mathbb{R}^n$ is called a d-dimensional differentiable submanifold in $\mathbb{R}^n$ if any point $\mathbf{p} \in M$ has an open neighbourhood $U \subset \mathbb{R}^n$ and a diffeomorphism*[16] *$H : U \to U'$ onto an open neighbourhood $U' \subset \mathbb{R}^n$ such that*

$$H(M \cap U) = (\mathbb{R}^d \times \mathbf{0}^{n-d}) \cap U'.$$

In any neighbourhood of $\mathbf{p} \in M$ the manifold M looks like a small piece of $\mathbb{R}^d$.

8.4 Lagrangian mechanics on manifolds under holonomic constraints

There are two kinds of constraints

- holonomic $f_i(\mathbf{x}) = \mathbf{0}$ for $i = 1, ..., n$. This is the case which we will treat in our course.
- non-holonomic $f_i(\mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}$. We won't discuss this.

Let M be the configuration manifold and $\mathcal{L} : TM \to \mathbb{R}$. For any curve $\mathbf{x} : [t_0, t_1] \to M$ the action along $\mathbf{x}$ is given by

$$S[\mathbf{x}] = \int_{t_0}^{t_1} \mathcal{L}(\mathbf{x}(t), \underbrace{\dot{\mathbf{x}}(t))}_{\in T_{\mathbf{x}(t)}}\, dt.$$

Motions of a mechanical systems are critical points of the action S, i.e. curves $\mathbf{x}$ with fixed endpoints at $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$ and which are stationary values of the action S (least action principle). We are interested in finding stationary values of S since these are the admissable curves of the system. The remarkable insight is (due to Euler and Lagrange) that there is a single method which deals with all kinds of questions concerning the analysis of stationary values of an functional of the form

$$I[y] = \int_a^b F(y, \dot{y})dt$$

for general F. This approach has many applications in physics due to the fact that the concept of stationary values is independent of coordinates.

The idea of finding stationary values is rooted in the idea that small variations $\mathbf{x}(t)+\epsilon\mathbf{v}(t)$ of a curve $\mathbf{x}(t)$ are close to the tangent space $T_{\mathbf{x}(t)}M$. Infinitesimal variations of $\mathbf{x}(t)$ are given by $\mathbf{x}(t) + \epsilon\mathbf{v}(t)$ where $\mathbf{v} : [t_0, t_1] \to TM$ with $\mathbf{v}(t) \in T_{\mathbf{x}(t)}M$ and $\mathbf{v}(t_0) = \mathbf{0} = \mathbf{v}(t_1)$.

$$\delta S[\mathbf{x}]\mathbf{v} = 0 \iff \frac{d}{d\epsilon} S[\mathbf{x} + \epsilon\mathbf{v}]\Big|_{\epsilon=0} = 0.$$

Week 10
Tue, 18.12

[16] H is a diffeomorphism if it is bijective and differentiable, and its inverse is differentiable.

Example. The mathematical pendulum is described by

$$\mathcal{L}(\mathbf{x}, \mathbf{y}, \dot{\mathbf{x}}, \dot{\mathbf{y}}) = \frac{1}{2}m(\dot{\mathbf{x}}^2 + \dot{\mathbf{y}}^2) - mg\mathbf{y}.$$

Week 10
Thu, 20.12

8.5 Noether's theorem

We do not prove the full version. Nevertheless, everything is based on integration by parts (as usual in variatoinal calculus).

9 Hamiltonian mechanics

9.1 Basic principles

In textbooks the Hamiltonian mechanics is described as the more fundamental formulation of classical mechanics in comparison to the Lagrangian Mechanics. To obtain a Hamiltonian system, we transform the Lagrangian systen with Lagrange function $\mathcal{L}$ by a Legrende transformation. Given a point $(\mathbf{x}, \dot{\mathbf{x}})$ in the configuration space, we introduce a new variable $\mathbf{p} \in \mathbb{R}^n$, that we call **conjugate momentum**. The momentum is defined as

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}})$$

and in components for $i = 1, ..., n$

$$\mathbf{p}_i = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}_\mathbf{i}}(\mathbf{x}, \dot{\mathbf{x}}).$$

The conjugate momentum $\mathbf{p} \in T^*_\mathbf{x}X$ is a **covector**, and $T^*_\mathbf{x}X$ is called the **cotangent space** at $\mathbf{x}$. The cotangent space is the dual space of $T_\mathbf{x}X$, i.e. every $a \in T^*_\mathbf{x}X$ is linear functional with $a : T_\mathbf{x}X \to \mathbb{R}$. So, $\mathbf{p}$ is linear functional that takes points in the configuration space T_xX, and maps them to the derivative of $\mathcal{L}$ with respect to the velocity $\dot{\mathbf{x}}$.

If we have a Lagrangian system, we use the **Legrende transformation**

$$(\mathbf{x}, \dot{\mathbf{x}}) \mapsto (\mathbf{x}, \mathbf{p}) = (\mathbf{x}, \frac{\partial L}{\partial \dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}))$$

to obtain the Hamiltonian system. The configuration space which was previously in TX is now in T^*X. The second component is not the velocity $\dot{\mathbf{x}} \in T_\mathbf{x}X$ but the conjugate momentum $\mathbf{p} \in T^*_\mathbf{x}X$. We will later see why this transformation is useful for physical applications.

There is just one problem: the transformation may not be invertible. Thus, we do only consider systems where the Legrende transformation $(\mathbf{x}, \dot{\mathbf{x}}) \mapsto \mathbf{p}$ is invertible, i.e. we can reconstruct the velocity given the position $\mathbf{x}$ and the conjugate momentum $\mathbf{p}$ by

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, \mathbf{p}).$$

We also provide a condition when the Legrende transformation is non-degenerate: the determinant of the Hessian matrix of $\mathcal{L}$ with respect to $\dot{\mathbf{x}}$ must be non-zero[17]

$$\left| \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{x}_i \partial \dot{x}_j} \right)_{i,j=1}^{n} \right| \neq 0.$$

We introduce the **Hamilton function**, which will replace the Lagrange function in Hamiltonian systems. We start by defining the Hamilton function H as the total energy function, where the velocity depends on the conjugate momentum

$$\begin{aligned} H(\mathbf{x}, \mathbf{p}) &= E(\mathbf{x}, \dot{\mathbf{x}}) \Big|_{\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, \mathbf{p})} \\ &= \left(\sum_{k=1}^{n} \dot{x}_k \frac{\partial \mathcal{L}}{\partial \dot{x}_k} - \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) \right) \Big|_{\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, \mathbf{p})}. \end{aligned}$$

Using the identity $\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}}$, we obtain

The Legrende transformation of $\mathcal{L}(\mathbf{x}, \dot{\mathbf{x}})$ is given by

$$H(\mathbf{x}, \mathbf{p}) = \left(\sum_{k=1}^{n} \dot{x}_k p_k - \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) \right) \Big|_{\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, \mathbf{p})} = \Big(\langle \mathbf{x}, \mathbf{p} \rangle - \mathcal{L}(\mathbf{x}, \dot{\mathbf{x}}) \Big) \Big|_{\dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, \mathbf{p})}$$

The Hamilton function maps from T^*X to $\mathbb{R}$. In comparision, the Lagrange function maps from TX to $\mathbb{R}$. Interestingly, the partial derivatives of H are easy to compute:

$$\frac{\partial H}{\partial p_j} = \dot{x}_j + \sum_{k=1}^{n} \frac{\dot{x}_k}{\partial p_j} p_k - \sum_{k=1}^{n} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}_k}}_{=p_k} \frac{\partial \dot{x}_k}{\partial p_j} = \dot{x}_j$$

[17]Note that the same non-degeneracy condition was used to ensure that the Euler-Lagrange equations are 2nd order differential equations and invariant under changes of coordinates.

and

$$
\begin{aligned}
\frac{\partial H}{\partial x_j} &= \sum_{k=1}^{n} \frac{\partial \dot{x}_k}{\partial x_j} p_k - \frac{\mathcal{L}}{\partial x_j}(\mathbf{x}, \dot{\mathbf{x}}(\mathbf{x}, \mathbf{p})) \\
&\Downarrow \text{By chain rule: } \frac{\partial u(v(t), w(t))}{\partial t} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\frac{\partial w}{\partial t} \\
&= \sum_{k=1}^{n} \frac{\partial \dot{x}_k}{\partial x_j} p_k - \frac{\partial \mathcal{L}}{\partial x_j} - \sum_{k=1}^{n} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}_k}}_{=p_k} \frac{\partial \dot{x}_k}{\partial x_j} \\
&= -\frac{\partial \mathcal{L}}{\partial x_j} \\
&\Downarrow \text{Euler-Lagrange equations} \\
&= -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_j}\right) \\
&= -\frac{d}{dt} p_j \\
&= -\dot{p}_j .
\end{aligned}
$$

Using the coordinates $\mathbf{x}$ and $\mathbf{p}$, the motions of the system are given by **Hamilton's equations of motions**

$$
\boxed{\dot{x}_j = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = -\frac{\partial H}{\partial x_j} \quad j = 1, ..., n.}
$$

This is good news for we get a system of $2n$ first order ordinary differential equations. Before, we had a system of second order differential equations in the form of Euler-Lagrange equations. We increase the dimensions, but as a tradeoff we obtain a first order system of ODEs. This allows us to directly solve the linear system of differential equations; no more variational principles or least action principles are needed to obtain solutions!

Generally, we consider Hamiltonian systems opearting on $H : T^*X \to \mathbb{R}$ or $H : \mathbb{R}^{2n} \to \mathbb{R}$. Hence, Hamilton functions take vectors of dimension $2n$ with $(x_1, ..., x_n, p_1, ..., p_n)$. The dimension of the configuration space - also known as *degree of freedom* - is n. The system is called n degree-of-freedom Hamiltonian system. The configuration space T^*X and $\mathbb{R}^{2n}$ are called the *canonical phase space* of dimension $2n$.

Note that the system of linear differential equations is autonomous, i.e. independent of t. Thus, everything we have learned so far about dynamical systems is applicable in our theory. Let us examine some examples.

Example. Consider a natural mechanical system with Lagrange function

$$\mathcal{L}(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}\langle \dot{\mathbf{x}}, M\dot{\mathbf{x}}\rangle - U(\mathbf{x}).$$

The Legrende transformation of the Lagrangian system is given by

$$\mathbf{p} = M\dot{\mathbf{x}}$$

and the inverse of the transformation is

$$\dot{\mathbf{x}} = M^{-1}\mathbf{p}.$$

Thus, the Hamilton function reads

$$\begin{aligned} H(\mathbf{x},\mathbf{p}) &= \left(\langle \dot{\mathbf{x}}, \mathbf{p}\rangle - \mathcal{L}(\mathbf{x},\dot{\mathbf{x}})\right)\Big|_{\dot{\mathbf{x}}=M^{-1}\mathbf{p}} \\ &= \langle M^{-1}\mathbf{p},\mathbf{p}\rangle - \frac{1}{2}\langle \dot{\mathbf{x}}, M\dot{\mathbf{x}}\rangle\Big|_{\dot{\mathbf{x}}=M^{-1}\mathbf{p}} + U(\mathbf{x}) \\ &= \langle M^{-1}\mathbf{p},\mathbf{p}\rangle - \frac{1}{2}\langle M^{-1}\mathbf{p},\mathbf{p}\rangle + U(\mathbf{x}) \\ &= \underbrace{\frac{1}{2}\langle M^{-1}\mathbf{p},\mathbf{p}\rangle}_{\text{kinetic energy}} + \underbrace{U(\mathbf{x})}_{\text{potential energy}}. \end{aligned}$$

Thus, the Hamilton function of a natural mechanical system is the total energy function. As we will later see, the Hamilton function is always an integral of motion of the Hamiltonian system!

The **Hamiltonian vector field** X_H of the Hamiltonian system

$$\begin{cases} \dot{x}_k = \frac{\partial H}{\partial p_k} \\ \dot{p}_k = -\frac{\partial H}{\partial x_k} \end{cases}$$

is given by

$$X_H = \begin{pmatrix} (\frac{\partial H}{\partial p_k})_{k=1}^n \\ \\ -(\frac{\partial H}{\partial x_k})_{k=1}^n \end{pmatrix} = \begin{pmatrix} \boldsymbol{\nabla}_{\mathbf{p}} H \\ \\ -\boldsymbol{\nabla}_{\mathbf{x}} H \end{pmatrix}.$$

Thus, we can write the Hamiltonian system as

$$(\dot{\mathbf{x}},\dot{\mathbf{p}}) = X_H(\mathbf{x},\mathbf{p}).$$

A good overview of Hamiltonian mechanics can be found on `http://www.scholarpedia.org/article/Hamiltonian_systems`.

9.2 General properties of Hamiltonian systems

- $H(\mathbf{x}, \mathbf{p})$ is an integral of motion for the system

$$\begin{cases} \dot{x}_k = \frac{\partial H}{\partial p_k} \\ \dot{p}_k = -\frac{\partial H}{\partial x_k} \end{cases}$$

  Proof. To see this, we compute the Lie-derivative of the Hamilton function H under the vector field X_H, i.e.

$$\mathcal{L}_{X_H} H = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial x_k} \dot{x}_k + \frac{\partial H}{\partial p_k} \dot{p}_k \right) = \sum_{k=1}^{n} \left(\frac{\partial H}{\partial x_k} \frac{\partial H}{\partial p_k} + \frac{\partial H}{\partial p_k} (-\frac{\partial H}{\partial x_k}) = 0. \right)$$

$\square$ **Week 11** Tue, 08.01

- asdasd

9.3 Poisson bracket

9.4 Quick introduction: Alternating forms and differential forms

not quite complete, one example missing **Week 12** Tue, 15.01

Theorem 66. *If a flow of a vector field X on $\mathbb{R}^{2n}(x, p)$ preserves the canonical two form $\omega = \sum_{k=1}^{n} dx_k \wedge dp_k$, then (locally) there exists $H : \mathbb{R}^{2n} \to \mathbb{R}$ such that $X = X_H =$* $\begin{pmatrix} \mathrm{grad}_p H \\ -\mathrm{grad}_x H \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathrm{grad} H$.

Proof. Let

$$X = \begin{pmatrix} f_1(x,p) \\ \vdots \\ f_n(x,p) \\ g_1(x,p) \\ g_n(x,p) \end{pmatrix}$$

so that $\begin{cases} \dot{x}_k = f_k(x,p) \\ \dot{p}_k) g_k(x,p) \end{cases}$. Assume that

$$\begin{aligned} L_x\omega = 0 &\iff \sum_{k=1}^{n}(df_k(x,p)\wedge dp_k + dx_k \wedge dg_k(x,p)) = 0 \\ &\iff \sum_{k=1}^{n}(\sum_{j=1}^{n}(\frac{\partial f_k}{\partial x_j}dx_j + \frac{\partial f_k}{\partial p_j}dp_j)\wedge dp_k + dx_k \wedge (\sum_{j=1}^{n}\frac{\partial g_k}{\partial x_j}dx_j + \frac{\partial g_k}{\partial p_j}\wedge dp_j)) = 0. \end{aligned}$$

Coefficients by $dx_k \wedge dx_j$; $\frac{\partial g_k}{\partial x_j} - \frac{\partial g_j}{\partial x_k} = 0$. By $dp_j \wedge dp_k$: $\frac{\partial f_k}{\partial p_j} - \frac{\partial f_j}{\partial p_k} = 0$. By $dx_j \wedge dp_k$: $\frac{\partial f_k}{\partial x_j} - \frac{\partial g_j}{\partial p_k} = 0$. The first condition is that there exists a function H_1 (defined up to a function of p) such that locally it holds $g_k = \frac{\partial H_1}{\partial x_k}$. The second condition is that there exists a function H_2 (defined up to a function of x) such that locally it holds $f_k = \frac{\partial H_2}{\partial p_k}$. Additionally

$$\begin{aligned} \frac{\partial}{\partial x_j}(\frac{\partial H_2}{\partial p_k}) + \frac{\partial}{\partial p_k}(\frac{\partial H_1}{\partial x_j}) = 0 &\iff \frac{\partial^2}{\partial x_j \partial p_k}(H_1 + H_") = 0 \\ &\iff H_1(x,p) + H_2(x,p) = a(x) + b(p) \\ &\iff H_1(x,p) - b(p) = -H_"(x,p) + a(x) \coloneqq -H(x,p). \end{aligned}$$

Then $f_k = \frac{\partial H}{\partial p_k}, g_k = -\frac{\partial H}{\partial x_k}$. □

An example of a locally Hamiltonian vector field which is not globally Hamiltonian. Consider a vector field on a non-simply-connected manifold - a torus[1]. A torus is analytically described by $(\mathbb{R}\backslash\mathbb{Z})^2 = S^1 \times S^1 = \{(x(\mod 1), p(\mod 1))\}$. The canonical two form of the torus is

$$\omega = dx \wedge dp \quad \text{it measures the area of the torus.}$$

The simples vector field is $\begin{cases} \dot{x} = 1 \\ \dot{p} = 0 \end{cases}$. It just describes a translation. We claim that this vector field is locally Hamiltonian $\begin{pmatrix} \frac{\partial H}{\partial p} = 1 \\ \\ \frac{\partial H}{\partial x} = 0 \end{pmatrix}$, but $H(x,p) = p$ is not a smooth function on the torus since it gets an increment $\neq 0$ along closed non-null-homotopic curves $x = \text{const}$.

Definition 67. *A map $\varphi : \mathbb{R}^{2n}_{(x,p)} \to \mathbb{R}^{2n}_{(x,p)}$ preserving $\omega = \sum_{k=1}^{n} dx_k \wedge dp_k$ (that is $\varphi^*\omega = \omega$) is called a* ***symplectic map*** *(so that the flow of X_H consists of symplectic maps).*

Definition 68. *A matrix $D \in \mathrm{mat}_{2n\times 2n}(\mathbb{R})$ is symplectic, i.e. $D \in \mathrm{Sp}_{2n}(\mathbb{R})$, if it satisfies*

$$D^T J D = J \qquad \textit{with } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Theorem 69. *A map φ is symplectic if and only if $d\varphi \in \mathrm{mat}_{2n\times 2n}(\mathbb{R})$ is symplectic.*

Proof. Let $(\tilde{x}, \tilde{p}) = \varphi(x, p)$.

$$\sum_{k=1}^{n} d\tilde{x}_k \wedge d\tilde{p}_k = \sum_{k=1}^{n}\sum_{j=1}^{n}(\frac{\partial \tilde{x}_k}{\partial x_j}dx_j + \frac{\partial \tilde{x}_k}{\partial p_j}dp_j) \wedge \sum_{i=1}^{n}(\frac{\partial \tilde{p}_k}{\partial x_i}dx_i + \frac{\partial \tilde{p}_k}{\partial p_i}dp_i)$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n}(\sum_{k=1}^{n}\frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{p}_k}{\partial x_i}dx_j \wedge dx_i) + \sum_{j=1}^{n}\sum_{i=1}^{n}(\sum_{k=1}^{n}\frac{\partial \tilde{x}_k}{\partial p_j}\frac{\partial \tilde{p}_k}{\partial p_i}dp_j \wedge dp_i) + \sum_{j=1}^{n}\sum_{i=1}^{n}(\frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{p}_k}{\partial p_i} - \frac{\partial \tilde{x}_k}{\partial p_i}\frac{\partial \tilde{p}_k}{\partial x_j})dx_j \wedge dx_i$$

This should be equal to $\sum_{j=1}^{n} dx_j \wedge dp_j$.

Coefficients by $dx_j \wedge dx_i : \sum_{k=1}^{n} \frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{p}_k}{\partial x_i} - \frac{\partial \tilde{x}_k}{\partial x_i}\frac{\partial \tilde{p}_j}{\partial x_k} = 0 \iff (\frac{\partial \tilde{x}}{\partial x})^T\frac{\partial \tilde{p}}{\partial x} - (\frac{\partial \tilde{p}}{\partial x})^T(\frac{\partial \tilde{x}}{\partial x}) = 0_{n\times n}$.

Coefficients by $dp_j \wedge dp_i : \sum_{k=1}^{n} \frac{\partial \tilde{x}_k}{\partial p_j}\frac{\partial \tilde{p}_k}{\partial p_i} - \frac{\partial \tilde{x}_k}{\partial p_i}\frac{\partial \tilde{p}_k}{\partial p_j} = 0 \iff (\frac{\partial \tilde{x}}{\partial p})^T\frac{\partial \tilde{p}}{\partial p} - (\frac{\partial \tilde{p}}{\partial p})^T(\frac{\partial \tilde{x}}{\partial p}) = 0_{n\times n}$.

Coefficients by $dx_j \wedge dp_i : \sum_{k=1}^{n} \frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{p}_k}{\partial p_i} - \frac{\partial \tilde{x}_k}{\partial p_i}\frac{\partial \tilde{p}_k}{\partial x_j} = \delta_{ij} \iff (\frac{\partial \tilde{x}}{\partial x})^T\frac{\partial \tilde{p}}{\partial p} - (\frac{\partial \tilde{p}}{\partial x})^T\frac{\partial \tilde{x}}{\partial p} = I_{n\times n}$

This is is equivalent to

$$\begin{pmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial p} \\ \frac{\partial \tilde{p}}{\partial x} & \frac{\partial \tilde{p}}{\partial p} \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial p} \\ \frac{\partial \tilde{p}}{\partial x} & \frac{\partial \tilde{p}}{\partial p} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

□

Definition 70. *$Sp_{2n} \coloneqq \{D \in mat_{2n\times 2n}(\mathbb{R}) : D^T J D = J\}$ is called the symplectic group.*

Based on

- $D_1^T J D_1 = J, D_2^T J D_2 = J \implies (D_1 D_2)^T J (D_1 D_2) = J$
- $\det D \neq 0 : \det\big(D^T J D\big) = \det(D)\det(J)\det(D) = \det(J) \implies (\det(D))^2 = 1 \implies \det(D) = \pm 1$.

- $D^T J D = J \iff (D^{-1})^T J (D^{-1}) = J$.

Thus, $\mathrm{Sp}_{2n}(\mathbb{R})$ is a group, indeed.

Some side remarks: Classical matrix Lie groups

$$\begin{aligned}
\mathrm{GL}_n(\mathbb{R}) &= \{A \in \mathrm{mat}_{n\times n}(\mathbb{R}) : \det(A) \neq 0\} \\
O_n(\mathbb{R}) &= \{A \in \mathrm{mat}_{n\times n}(\mathbb{R}) : A^T A = I\} \\
\mathrm{Sp}_{2n}(\mathbb{R}) &= \{A \in \mathrm{mat}_{2n\times 2n}(\mathbb{R}) : A^T J A = J\}
\end{aligned}$$

The group $\mathrm{Sp}_{2n}(\mathbb{R})$ preserves skew symmetric two forms.

We know for othrogonal matrices if the row form an orthonormal basis so do the columns, i.e. $A^T A = I$ and $AA^T = I$. Something similar also holds for the symplectic matrices.

Theorem 71. *It holds: $D \in \mathrm{Sp}_{2n}(\mathbb{R}) \iff DJD^T = J$.*

Proof.

$$\begin{aligned}
D^T J D = J \iff JD = (D^T)^{-1} J \overset{J^{-1} = -J}{\iff} D = -J(D^T)^{-1} J &\iff DJ = J(D^T)^{-1} \\
&\iff DJD^T = J.
\end{aligned}$$

□

Collorary 72. *Symplectic maps preserve Poisson brackets.*

Proof. For $D = d\varphi = \begin{pmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial p} \\ \frac{\partial \tilde{p}}{\partial x} & \frac{\partial \tilde{p}}{\partial p} \end{pmatrix}$ the identitiy $DJD^T = J$ reads

$$\begin{aligned}
\frac{\partial \tilde{x}}{\partial x}\left(\frac{\partial \tilde{x}}{\partial p}\right)^T - \frac{\partial \tilde{x}}{\partial p}\left(\frac{\partial \tilde{x}}{\partial x}\right)^T = 0 &\iff \sum_{k=1}^{n}\left(\frac{\partial \tilde{x}_i}{\partial x_k}\frac{\partial \tilde{x}_j}{\partial p_k} - \frac{\partial \tilde{x}_i}{\partial p_k}\frac{\partial \tilde{x}_j}{\partial x_k}\right) = 0 \\
\frac{\partial \tilde{p}}{\partial p}\left(\frac{\partial \tilde{x}}{\partial p}\right)^T - \frac{\partial \tilde{p}}{\partial p}\left(\frac{\partial \tilde{x}}{\partial x}\right)^T = 0 &\iff \sum_{k=1}^{n}\left(\frac{\partial \tilde{p}_i}{\partial x_k}\frac{\partial \tilde{p}_j}{\partial p_k} - \frac{\partial \tilde{p}_i}{\partial p_k}\frac{\partial \tilde{p}_j}{\partial x_k}\right) = 0 \\
\frac{\partial \tilde{x}}{\partial x}\left(\frac{\partial \tilde{p}}{\partial p}\right)^T - \frac{\partial \tilde{x}}{\partial p}\left(\frac{\partial \tilde{p}}{\partial x}\right)^T = I &\iff \sum_{k=1}^{n}\left(\frac{\partial \tilde{x}_i}{\partial x_k}\frac{\partial \tilde{p}_j}{\partial p_k} - \frac{\partial \tilde{x}_i}{\partial p_k}\frac{\partial \tilde{p}_j}{\partial x_k}\right) = \delta_{i,j}.
\end{aligned}$$

This is equivalent to $\{\tilde{x}_i \tilde{x}_j\} = 0 = \{x_i, x_j\}, \{\tilde{p}_i \tilde{p}_j\} = 0 = \{p_i, p_j\}, \{\tilde{x}_i \tilde{p}_j\} = \delta_{i,j} = \{x_i, p_j\}$. □

This means that the map $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a **Poisson map**.

Definition 73. *A map $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called a* ***Poisson map*** *if it preserves the Poisson brackets: in coordinates,*

$$\begin{aligned} \{\tilde{x}_i, \tilde{x}_j\} &= \{x_i, x_j\} = 0 \\ \{\tilde{p}_i, \tilde{p}_j\} &= \{p_i, p_j\} = 0 \\ \{\tilde{x}_i, \tilde{p}_j\} &= \{x_i, p_j\} = \delta_{i,j} \end{aligned}$$

or intrinsically, $\{F \circ \varphi, G \circ \varphi\} = \{F, G\} \circ \varphi \forall F, G : \mathbb{R}^{2n} \to \mathbb{R}$.

Why is coordinate formulation equivalent to the intrinsic one? Suppose that there is a coordinate system $\{y_i\}$ such that the Poisson bracket of coordinate functions is given by $\{y_i, y_i\} = \omega_{ij} = -\omega_{ji}$. Suppose that $\varphi : y \mapsto \tilde{y}$ and $\{\tilde{y}_i, \tilde{y}_j\} = \omega_{ij}$. Then the poisson bracket is $\{F, G\} = \sum_{i=1}^{2n} \sum_{j=1}^{2n} \frac{\partial F}{\partial y_i} \frac{\partial G}{\partial y_j} \omega_{ij}$, in our case $(\omega_{ij}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. We have

$$\{F \circ \varphi, G \circ \varphi\} = \{F(\tilde{y}), G(\tilde{y})\} = \sum_{k,l=1}^{2n} \frac{\partial F(\tilde{y})}{\partial y_k} \frac{G(\tilde{y})}{\partial y_k} \omega_{kl} = \sum_{k,l=1}^{2n} \sum_{i=1}^{2n} \frac{\partial F(\tilde{y})}{\partial \tilde{y}_i} \frac{\partial \tilde{y}_i}{\partial y_k} \sum_{j=1}^{2n} \frac{\partial G(\tilde{y})}{\partial \tilde{y}_j} \frac{\partial \tilde{y}_j}{\partial y_l} \omega_{kl}$$

Week 12
Thu, 17.01