Chapter 4:
The Simplex Method

(cp. Bertsimas & Tsitsiklis, Chapters 3.1, 3.2)

Linear Program in Standard Form

Throughout this chapter, we consider the following standard form problem:

$$\begin{array}{ll} \text{minimize} & c^T \cdot x \\ \text{subject to} & A \cdot x = b \\ & x \geq 0 \end{array}$$

with $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$.

Basic Directions

Observation 4.1.

Let $B = (A_{B(1)}, \ldots, A_{B(m)})$ be a basis matrix. The values of the basic variables $x_{B(1)}, \ldots, x_{B(m)}$ in the system $A \cdot x = b$ are uniquely determined by the values of the non-basic variables.

Proof:

$$A \cdot x = b \iff B \cdot x_B + \sum_{j \neq B(1),\ldots,B(m)} A_j \cdot x_j = b$$

$$\iff x_B = B^{-1} \cdot b - \sum_{j \neq B(1),\ldots,B(m)} B^{-1} \cdot A_j \cdot x_j$$

□

Definition 4.2.

For fixed $j \neq B(1), \ldots, B(m)$, let $d \in \mathbb{R}^n$ be given by

$$d_j := 1, \quad d_B := -B^{-1} \cdot A_j, \quad \text{and} \quad d_{j'} := 0 \quad \text{for } j' \neq j, B(1), \ldots, B(m).$$

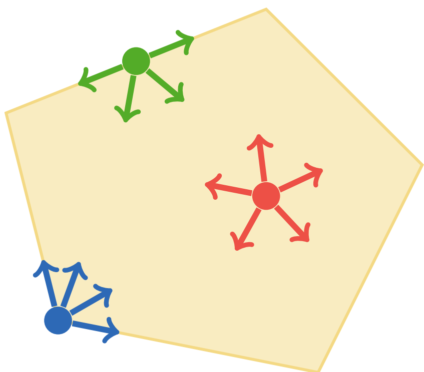
Then $A \cdot (x + \theta\, d) = b$, for all $\theta \in \mathbb{R}$, and d is the jth basic direction.

Feasible Directions

Definition 4.3.

Let $P \subseteq \mathbb{R}^n$ a polyhedron. For $x \in P$ the vector $d \in \mathbb{R}^n \setminus \{0\}$ is a feasible direction at x if there is a $\theta > 0$ with $x + \theta\, d \in P$.

Example: Some feasible directions at several points of a polyhedron.



Basic Directions and Feasible Directions

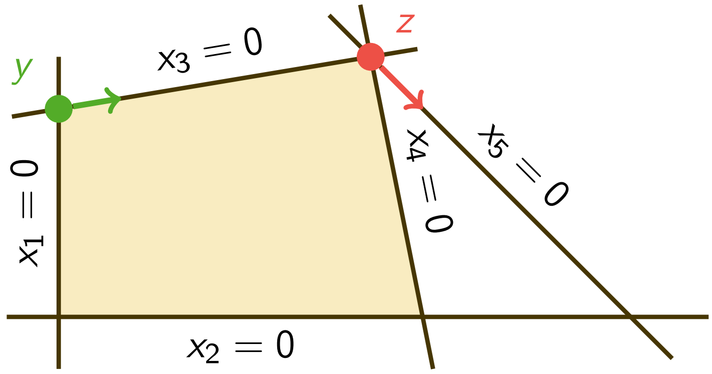
Consider a basic feasible solution x.

Question: Is the jth basic directions d a feasible direction?

Case 1: If x is a non-degenerate feasible solution, then $x_B > 0$ and $x + \theta\, d \geq 0$ for $\theta > 0$ small enough. $\longrightarrow$ answer is yes!

Case 2: If x is degenerate, the answer might be no!
E. g., if $x_{B(i)} = 0$ and $d_{B(i)} < 0$, then $x + \theta\, d \not\geq 0$, for all $\theta > 0$.

Example: $n = 5$, $m = 3$, $n - m = 2$



- 1st basic direction at y is feasible
  (basic variables x_2, x_4, x_5)
- 3rd basic direction at z is infeasible
  (basic variables x_1, x_2, x_4)

Reduced Cost Coefficients

Consider a basic solution x.

Question:
How does the cost change when moving along the jth basic direction d?

$$c^T \cdot (x + \theta\, d) = c^T \cdot x + \theta\, c^T \cdot d = c^T \cdot x + \theta\, \underbrace{(c_j - {c_B}^T \cdot B^{-1} \cdot A_j)}_{\bar{c}_j :=}$$

Definition 4.4.

For a given basis B and corresponding basic solution x, the reduced cost of variable x_j, $j = 1, \dots, n$, is

$$\bar{c}_j := c_j - {c_B}^T \cdot B^{-1} \cdot A_j.$$

Observation 4.5.

The reduced cost of a basic variable $x_{B(i)}$ is zero.

Proof: $\bar{c}_{B(i)} = c_{B(i)} - {c_B}^T \cdot \underbrace{B^{-1} \cdot A_{B(i)}}_{= e_i} = c_{B(i)} - c_{B(i)} = 0$ $\square$

Optimality Criterion

Theorem 4.6.

Let x be a basic feasible solution and $\bar{c}$ the vector of reduced costs.

a) If $\bar{c} \geq 0$, then x is an optimal solution.

b) If x is an optimal solution and non-degenerate, then $\bar{c} \geq 0$.

Definition 4.7.

A basis matrix B is optimal if

a) $B^{-1} \cdot b \geq 0$ and

b) $\bar{c}^T = c^T - {c_B}^T \cdot B^{-1} \cdot A \geq 0$.

Observation 4.8.

If B is an optimal basis, the associated basic solution x is feasible and optimal.

Proof of Theorem 4.6

a) Let B be the basis corresponding to x and let $y \in P$. Then,

$$
\begin{aligned}
c^T \cdot y &= {c_B}^T \cdot y_B + \sum_{j \neq B(1), \ldots, B(m)} c_j \, y_j \\
&= {c_B}^T \cdot \Big(B^{-1} \cdot b - \sum_{j \neq B(1), \ldots, B(m)} B^{-1} \cdot A_j \, y_j \Big) + \sum_{j \neq B(1), \ldots, B(m)} c_j \, y_j \\
&= {c_B}^T \cdot \underbrace{B^{-1} \cdot b}_{= x_B} + \sum_{j \neq B(1), \ldots, B(m)} \underbrace{\big(c_j - {c_B}^T \cdot B^{-1} \cdot A_j \big)}_{= \bar{c}_j} \, y_j \\
&= \underbrace{{c_B}^T \cdot x_B}_{= c^T \cdot x} + \sum_{j \neq B(1), \ldots, B(m)} \bar{c}_j \, y_j \geq c^T \cdot x
\end{aligned}
$$

b) Assume by contradiction that $\bar{c}_j < 0$ for some $j \neq B(1), \ldots, B(m)$.

Since x is non-degenerate, the jth basic direction is a feasible direction and the cost can thus be decreased as $\bar{c}_j < 0$. □

Developement of the Simplex Method

Assumption (for now): only *non-degenerate* basic feasible solutions

Let x be a basic feasible solution with $\bar{c}_j < 0$ for some $j \neq B(1), \dots, B(m)$.

Let d be the jth basic direction:

$$0 > \bar{c}_j = c^T \cdot d$$

It is desirable to go to $y := x + \theta^* \, d$ with $\theta^* := \max\{\theta \mid x + \theta \, d \in P\}$.

Question: How to determine θ^*?

By construction of d, it holds that $A \cdot (x + \theta \, d) = b$ for all $\theta \in \mathbb{R}$, i.e.,

$$x + \theta \, d \in P \qquad \Longleftrightarrow \qquad x + \theta \, d \geq 0.$$

Case 1: $d \geq 0 \quad \Longrightarrow \quad x + \theta \, d \geq 0$ for all $\theta \geq 0 \quad \Longrightarrow \quad \theta^* = \infty$

Thus, the LP is unbounded.

Case 2: $d_k < 0$ for some $k \quad \Longrightarrow \quad \left(x_k + \theta \, d_k \geq 0 \quad \Longleftrightarrow \quad \theta \leq \dfrac{-x_k}{d_k} \right)$

Thus, $\theta^* = \min\limits_{k\colon d_k < 0} \dfrac{-x_k}{d_k} = \min\limits_{\substack{i=1,\dots,m \\ d_{B(i)}<0}} \dfrac{-x_{B(i)}}{d_{B(i)}} > 0.$

Developement of the Simplex Method (Cont.)

Assumption (for now): only *non-degenerate* basic feasible solutions

Let x be a basic feasible solution with $\bar{c}_j < 0$ for some $j \neq B(1), \dots, B(m)$.

Let d be the jth basic direction:

$$0 > \bar{c}_j = c^T \cdot d$$

It is desirable to go to $y := x + \theta^* \cdot d$ with $\theta^* := \max\{\theta \mid x + \theta \cdot d \in P\}$.

$$\theta^* = \min_{k\colon d_k < 0} \frac{-x_k}{d_k} = \min_{\substack{i=1,\dots,m \\ d_{B(i)}<0}} \frac{-x_{B(i)}}{d_{B(i)}}$$

Let $\ell \in \{1, \dots, m\}$ with $\theta^* = \dfrac{-x_{B(\ell)}}{d_{B(\ell)}}$, then $y_j = \theta^*$ and $y_{B(\ell)} = 0$.

$\Longrightarrow \quad x_j$ replaces $x_{B(\ell)}$ as a basic variable and we get a new basis matrix

$$\bar{B} = \Big(A_{B(1)}, \dots, A_{B(\ell-1)}, A_j, A_{B(\ell+1)}, \dots, A_{B(m)}\Big) = \Big(A_{\bar{B}(1)}, \dots, A_{\bar{B}(m)}\Big)$$

with $\qquad \bar{B}(i) = \begin{cases} B(i) & \text{if } i \neq \ell, \\ j & \text{if } i = \ell. \end{cases}$

Core of the Simplex Method

Theorem 4.9.

Let x be a non-degenerate basic feasible solution, $j \neq B(1), \dots, B(m)$ with $\bar{c}_j < 0$, d the jth basic direction, and $\theta^* := \max\{\theta \mid x + \theta\, d \in P\} < \infty$.

a $\theta^* = \min\limits_{\substack{i=1,\dots,m\\ d_{B(i)}<0}} \dfrac{-x_{B(i)}}{d_{B(i)}} = \dfrac{-x_{B(\ell)}}{d_{B(\ell)}}$ for some $\ell \in \{1, \dots, m\}$.

Let $\bar{B}(i) := B(i)$ for $i \neq \ell$ and $\bar{B}(\ell) := j$.

b $A_{\bar{B}(1)}, \dots, A_{\bar{B}(m)}$ are linearly independent and $\bar{B}$ is a basis matrix.

c $y := x + \theta^*\, d$ is a basic feasible solution associated with $\bar{B}$ and $c^T \cdot y < c^T \cdot x$.

Proof: ... □

An Iteration of the Simplex Method

Given: basis $B = (A_{B(1)}, \dots, A_{B(m)})$, correspond. basic feasible solution x

i Let $\bar{c}^T := c^T - c_B{}^T \cdot B^{-1} \cdot A$. If $\bar{c} \geq 0$, then STOP; else choose j with $\bar{c}_j < 0$.

ii Let $u := B^{-1} \cdot A_j$. If $u \leq 0$, then STOP (optimal cost is $-\infty$).

iii Let $\theta^* := \min\limits_{i:u_i>0} \dfrac{x_{B(i)}}{u_i} = \dfrac{x_{B(\ell)}}{u_\ell}$ for some $\ell \in \{1, \dots, m\}$.

iv Form new basis by replacing $A_{B(\ell)}$ with A_j; corresponding basic feasible solution y is given by

$$y_j := \theta^* \qquad \text{and} \qquad y_{B(i)} = x_{B(i)} - \theta^* u_i \quad \text{for } i \neq \ell.$$

Remark: We say that the nonbasic variable x_j enters the basis and the basic variable $x_{B(\ell)}$ leaves the basis.

Correctness of the Simplex Method

Theorem 4.10.

If every basic feasible solution is non-degenerate, the simplex method terminates after finitely many iterations in one of the following two states:

a) we have an optimal basis B and an associated basic feasible solution x which is optimal;

b) we have a vector d satisfying $A \cdot d = 0$, $d \geq 0$, and $c^T \cdot d < 0$; the optimal cost is $-\infty$.

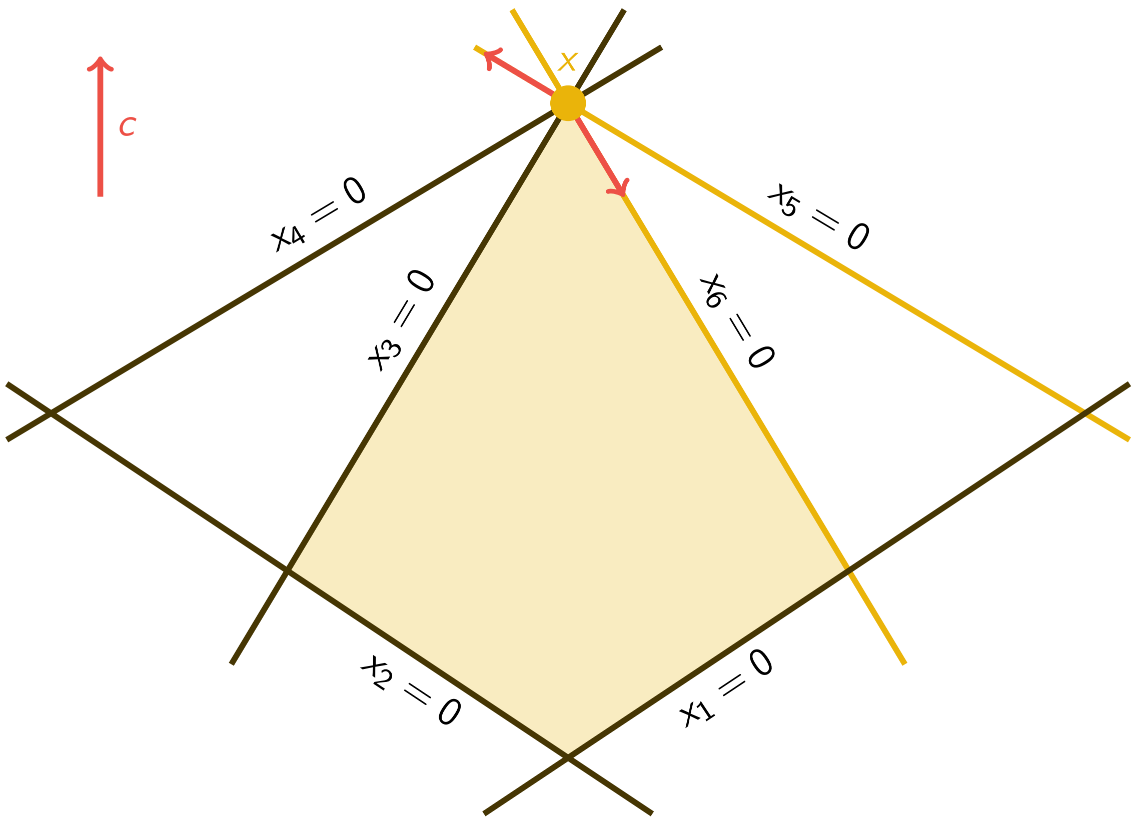
Proof sketch: The simplex method makes progress in every iteration. Since there are only finitely many different basic feasible solutions, it stops after a finite number of iterations. □

Simplex Method for Degenerate Problems

- An iteration of the simplex method can also be applied if x is a degenerate basic feasible solution.
- In this case it might happen that $\theta^* := \min_{i: u_i > 0} \frac{x_{B(i)}}{u_i} = \frac{x_{B(\ell)}}{u_\ell} = 0$ if some basic variable $x_{B(\ell)}$ is zero and $d_{B(\ell)} = -u_\ell < 0$.
- Thus, $y = x + \theta^* \, d = x$ and the current basic feasible solution does not change.
- But replacing $A_{B(\ell)}$ with A_j still yields a new basis with associated basic feasible solution $y = x$.

Remark: Even if θ^* is positive, more than one of the original basic variables may become zero at the new point $x + \theta^* \, d$. Since only one of them leaves the basis, the new basic feasible solution y may be degenerate.

Example



Pivot Selection

Question: How to choose j with $\bar{c}_j < 0$ and ℓ with $\dfrac{x_{B(\ell)}}{u_\ell} = \min\limits_{i:u_i>0} \dfrac{x_{B(i)}}{u_i}$ if several possible choices exist?

Attention: Choice of j is critical for overall behavior of simplex method.

Three popular choices are:

- smallest subscript rule: choose smallest j with $\bar{c}_j < 0$.
  (very simple; no need to compute entire vector $\bar{c}$; usually leads to many iterations)
- steepest descent rule: choose j such that $\bar{c}_j < 0$ is minimal.
  (relatively simple; commonly used for mid-size problems; does not necessarily yield the best neighboring solution)
- best improvement rule: choose j such that $\theta^* \bar{c}_j$ is minimal.
  (computationally expensive; used for large problems; usually leads to very few iterations)