Measure- and Integrationtheory - Assignment 02

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Exercise 1

- **From (i), *(ii)*, (iii) to (i), *(ii')*, (iii):** Consider sets A and B with the latter being a superset of the first from a system of sets $\mathcal{D}$, where the properties (i), (ii), and (iii) hold. Then, B without A is included in $\mathcal{D}$, for one concludes $B \setminus A = B \cap A^c = (B^c \cup A)^c$. Since the complement of B and A share no common element, the claim (ii') follows.

  From (i), *(ii')*, (iii) to (i), *(ii)*, (iii): Consider a set $A \in \mathcal{D}$. Its complement is also contained in $\mathcal{D}$, following from $A^c = \Omega \setminus A$. Since Ω and A are in $\mathcal{D}$ with Ω being a superset of A, the claim follows.

- **From (i), (ii'), *(iii)* to (i), (ii'), *(iii')*:** Consider sets (A_i) from $\mathcal{D}$ with $A_i \uparrow A$. Define pairwise disjoint sets $A'_i = A_i \setminus A_{i-1}$, which yield the set A by union. All A'_i are in $\mathcal{D}$ because of (ii'), and A_i being a superset of A_{i-1}. From property (iii) it follows (iii)'.

  From (i), (ii'), *(iii')* to (i), (ii'), *(iii)*: Consider sets (A_i) from $\mathcal{D}$ that are pairwise disjoint. First, for sake of simplicity, we will derive property (ii). This follows directly from (ii') by setting $B = \Omega$. With this in mind, one sees that the union of any disjoint sets A_i and A_{i+1} is the same as $(A_i^c \setminus A_{i+1})^c$. Being disjoint, A_i^c is a superset of A_{i+1}. From (ii') and (ii), it follows $(A_i^c \setminus A_{i+1})^c \in \mathcal{D}$.

  Next, one defines $B_n = \bigcup_{i=1}^n A_i = (A_n^c \setminus B_{n-1})^c$ and $B_1 = A_1$. All sets B_n are in $\mathcal{D}$, as shown before. Finally, $B_n \uparrow A \in \mathcal{D}$ due to property (iii').

- **From (i), (ii), *(iii)* to (i), (ii), *(iii')*:** Consider sets $A_i \in \mathcal{D}$ with $A_i \subset A_{i+1}$. We showed that (ii) is equivalent to (ii') if (i) and (iii) hold. Thus, we can make use of the difference operator. It holds $\bigcup A_i = \bigcup (A_{i+1} \setminus A_i)$; the latter being a union of disjoint sets, where $A_{i+1} \setminus A_i$ is in $\mathcal{D}$ due to (ii'). Property (iii) then implies (iii').

- **Counterexample:** Let $\Omega = \{1, 2, 3\}$ and

  $$\mathcal{D} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{2, 3\}, \{3\}, \{\}\}.$$

  The set $\{1\} \cup \{3\}$ is not included in $\mathcal{D}$.

Exercise 2

Let Ω be an universal set, and let $\mathcal{S}$ be a semiring over Ω with a σ-additive function $\mu : \mathcal{S} \to [0, \infty]$. Additionally, let $\mu(\emptyset) = 0$.

(i) **Show that for any set $A \subset \Omega$ there exists a set $B \in \sigma(\mathcal{S})$ such that both sets have the same outer measure.**

Let $A \subset \Omega$. By definition of the outer measure, for all $n \in \mathbb{N}$ there is a sequence of sets $(A_{nk})_{k\in\mathbb{N}} \subset \mathcal{S}$ with

$$\bigcup_{k=1}^{\infty} A_{nk} \supset A \quad \text{and} \quad \sum_{k=1}^{\infty} A_{nk} \le \mu^*(A) + \frac{1}{n}.$$

Define $B_n = \bigcup_{k=1}^{\infty} A_{nk} \in \sigma(\mathcal{S})$. It holds $\bigcap_{n=1}^{N} B_n \supset A$ for any $N \in \mathbb{N}$. Thus, the following inequality holds

$$\mu^*(A) \le \mu^*(\cap_{n=1}^{N} B_n) \le \mu^*(A) + \frac{1}{N}, \quad \forall N \in \mathbb{N}.$$

Taking the limit $N \to \infty$ yields

$$\mu^*(A) = \mu^*(\underbrace{\bigcap_{n=1}^{\infty} B_n}_{\in\sigma(\mathcal{S})}).$$

Thus, define $B = \bigcap_{n=1}^{\infty} B_n$.

(ii)

- **Show that for any sets A and B in Ω it holds that $\mu^*(A) + \mu^*(B) \ge \mu^*(A \cap B) + \mu^*(A \cup B)$.**

  We make use of the previous result. Let $\lambda = \mu^*|_{\sigma(\mathcal{S})}$. Let A and B be subsets in Ω. Let A' and B' be from $\sigma(\mathcal{S})$ such that $\mu^*(C') = \mu^*(C)$ and $C' \supset C$ for all $C \in \{A, B\}$. Due to monotonicity of the outer measure, it holds

  $$\mu^*(A \cup B) + \mu^*(A \cap B) \le \mu^*(A' \cup B') + \mu^*(A' \cap B').$$

  The right hand side is equal to

  $$\lambda(A' \cup B') + \lambda(A' \cap B') = \lambda(A') + \lambda(B') = \mu^*(A) + \mu^*(B).$$

  This gives the desired inequality.

- **If A or B is measurable, then equality holds.**

  Let A be measurable. We obtain

  $$\begin{aligned}\mu^*(A) + \mu^*(B) &= \mu^*(A \cap B) + \mu^*(B \cap A^c) + \mu^*(A) \\ &= \mu^*(A \cap B) + \mu^*((A \cup B) \cap A^c) + \mu^*(A) \\ &= \mu^*(A \cap B) + \mu^*((A \cup B) \cap A^c) + \mu^*((A \cup B) \cap A) \\ &= \mu^*(A \cap B) + \mu^*(A \cup B),\end{aligned}$$

  where for the first and last equality we made use of the fact that A is measurable.