Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 10

Martin Genzel, Mones Raslan Summer Term 2019

Exercise 1: **6 Points**

Let $\mathcal{H}$ be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{H}$. Show that the following statements are equivalent:

(a) There exists some constant $B > 0$ such that $\sum_{k \in \mathbb{N}} |\langle f, f_k \rangle|^2 \leq B \|f\|^2$ for all $f \in \mathcal{H}$.

(b) The operator $T : \ell_2(\mathbb{N}) \to \mathcal{H}$ given by $T((c_k)_{k \in \mathbb{N}}) := \sum_{k \in \mathbb{N}} c_k f_k$ is well-defined and bounded from $\ell_2(\mathbb{N})$ into $\mathcal{H}$ and $\|T\| \leq \sqrt{B}$.

Exercise 2: **4 Points**

Construct a functional f on some non-complete inner product space $\mathcal{H}$ such that f is not in the image of the Riesz map.

Exercise 3: **4 Points**

Let $\mathcal{M}$ be a closed subspace of a Hilbert space $\mathcal{H}$. Then for each $x \in \mathcal{H}$ there exist uniquely determined $u \in \mathcal{M}, v \in \mathcal{M}^\perp$ with $x = u + v$. We set $\mathcal{P}_\mathcal{M} x := u$. Then $\mathcal{P}_\mathcal{M}$ is the orthogonal projection onto $\mathcal{M}$. Let $\mathcal{N}$ be another closed subspace of $\mathcal{H}$. Show that

$$\mathcal{M} \subset \mathcal{N} \iff \mathcal{P}_\mathcal{M} = \mathcal{P}_\mathcal{N} \mathcal{P}_\mathcal{M} = \mathcal{P}_\mathcal{M} \mathcal{P}_\mathcal{N}$$

Exercise 4: **6 Points**

In this exercise we prove that the Fourier transform can be seen as an isometry in $L_2(\mathbb{R})$. We use the following definition of the Fourier transform $\mathcal{F} f$ of a function $f \in L_1(\mathbb{R})$:

$$(\mathcal{F} f)(y) := \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} f(x) e^{-ixy} \, dx, \quad y \in \mathbb{R}.$$

Everything you need to know for our aim is the following:

(i) For $y \in \mathbb{R}$ define the function $e_y : (-\pi, \pi) \to \mathbb{C}$, $e_y(x) := \frac{1}{\sqrt{2\pi}} e^{ixy}$. Then $\{e_k : k \in \mathbb{Z}\}$ is a complete orthonormal system in $L_2(-\pi, \pi)$.

(ii) Define the functions $s_k : \mathbb{R} \to \mathbb{R}$, $s_k(x) := \frac{\sin((k-x)\pi)}{(k-x)\pi}$. Then $\{s_k : k \in \mathbb{Z}\}$ is an orthonormal system in $L_2(\mathbb{R})$.

You do not need to prove (i) or (ii)! You can now proceed as follows:

(a) Consider the space $L_2(-\pi,\pi)$ as a closed subspace of $L_2(\mathbb{R})$ (by setting $f \in L_2(\mathbb{R})$ to zero on $\mathbb{R}\backslash(-\pi,\pi)$) and prove $\|\mathcal{F}f\|_2 = \|f\|_2$ for all $f \in L_2(-\pi,\pi)$. For this, first show for $f \in L_2(-\pi,\pi)$ that

$$\|\mathcal{F}f\|_2^2 = \int_{\mathbb{R}} |\langle f, e_y\rangle|^2 \, dy \,,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L_2(-\pi,\pi)$. Now, use (i) in order to decompose f with respect to the e_k's, $k \in \mathbb{Z}$. Prove that $\langle e_k, e_y\rangle = s_k(y)$. Next, make use of (ii) and (i) again to see that indeed $\|\mathcal{F}f\|_2^2 = \|f\|_2^2$ holds.

(b) For each $n \in \mathbb{N}$ consider $L_2(-n\pi, n\pi)$ as a closed subspace of $L_2(\mathbb{R})$ and prove that $\|\mathcal{F}f\|_2 = \|f\|_2$ for all $f \in L_2(-n\pi, n\pi)$.

(c) Prove that $\bigcup_{n\in\mathbb{N}} L_2(-n\pi, n\pi)$ is dense in $L_2(\mathbb{R})$ and use this to extend $\mathcal{F}$ by continuity to an isometric operator on $L_2(\mathbb{R})$.

The following bonus exercise introduces you to frames. Frame theory is nowadays a fundamental research area in (applied) functional analysis with links to other areas in both pure and applied mathematics as well as to computer science and engineering. Frames are a means to provide redundant, yet stable decompositions of functions/signals and are a generalization of orthonormal bases (to be more precise, a generalization of the Parseval identity is assumed to hold).

Bonus Exercise* **+5 points**

Let $\mathcal{H}$ be a separable Hilbert space, and let $J \subset \mathbb{N}$ be an index set. A system $(\phi_j)_{j\in J} \subset \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist $A, B > 0$ such that for all $x \in \mathcal{H}$

$$A\|x\|^2 \leq \sum_{j\in J} |\langle x, \phi_j\rangle|^2 \leq B\|x\|^2.$$

Let $\Phi := (\phi_j)_{j\in J}$ be a frame for $\mathcal{H}$. The associated *analysis operator* $T_\Phi : \mathcal{H} \to \ell_2(J)$ is defined by $T_\Phi x := (\langle x, \phi_j\rangle)_{j\in J}$, $x \in \mathcal{H}$. Prove that T_Φ is well-defined, bounded, injective, and that its range is closed. The *synthesis operator* of Φ is defined as the *Hilbert space adjoint* T_Φ^* of T_Φ, i.e., the unique linear map $T_\Phi^* : \ell_2(J) \to \mathcal{H}$ which satisfies $\langle x, T_\Phi y\rangle_{\ell_2(J)} = \langle T_\Phi^* x, y\rangle_{\mathcal{H}}$ for all $x \in \ell_2(J)$ and $y \in \mathcal{H}$. Confirm that T_Φ^* indeed exists, is well-defined and uniquely determined. Further, show that T_Φ^* is bounded, surjective, and that it has the form

$$T_\Phi^*((c_j)_{j\in J}) = \sum_{j\in J} c_j\phi_j, \quad (c_j)_{j\in J} \in \ell_2(J).$$

Please submit your homework in the BEGINNING of the big exercise on Monday, June 24.