

Discrete Geometrie I - Assignment 02

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Exercise 1

- (a) Let $n \in \mathbb{N}$ be arbitrary, and let e_i denote the i -th unit vector in \mathbb{R}^n . Consider the hyperplanes

$$A_i = \{x \in \mathbb{R}^n : \langle x, e_i \rangle = 0\}, \quad i = 1, \dots, n$$
$$A_{n+1} = \{x \in \mathbb{R}^n : \langle x, \sum_{i=1}^n e_i \rangle = 42\}$$

- It holds $\bigcap_{i=1}^{n+1} A_i = \emptyset$: we see that

$$\bigcap_{i=1}^{n+1} A_i = \left(\bigcap_{i=1}^n A_i \right) \cap A_{n+1} = \{0\} \cap A_{n+1} = \emptyset.$$

- A_i is convex for all $i = 1, \dots, n+1$, since affine linear spaces are convex and A_i is the solution space of a linear equation, which is an affine linear space.
 - $\bigcap_{j=1}^n A_{i_j} \neq \emptyset$: For any A_{i_1}, \dots, A_{i_n} , each A_{i_j} represents a hyperplane, whose normal vector is not colinear to the other normal vectors. Therefore, $\bigcap_{j=1}^n A_{i_j}$ defines a system of n linear equations with n variables that has exactly one solution, for the normal vectors are linear independent. Thus, $\bigcap_{j=1}^n A_{i_j}$ is nonempty.
- (b) Let $d \in \mathbb{N}_{\geq 3}$. Consider the sets $A_i = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_2, \dots, x_d \geq 0, x_1 \geq i\}$. Then, $(A_i)_{i \in \mathbb{N}}$ is a family of convex sets, and $\bigcap_{i \in \mathbb{N}} A_i = \emptyset$. Furthermore, any sets $A_{i_1}, \dots, A_{i_{d+1}}$ share a common point since $(i, 0, \dots, 0) \in \bigcap_{j=1, \dots, d+1} A_{i_j}$ for $i = \max_{j=1, \dots, d+1} \{i_j\}$.

Exercise 2

- **Closed under multiplication:** Let $f, g \in \mathcal{C}(\mathbb{R}^d)$. Then, $f = \sum_{i=1}^n \lambda_i [A_i]$ and $g = \sum_{i=1}^m \mu_i [B_i]$, where A_i and B_i are closed convex sets in \mathbb{R}^d . We would like to prove that $f \cdot g$ is a linear combination of indicator functions of closed convex sets. In that case, $f \cdot g \in \mathcal{C}(\mathbb{R}^d)$. So,

$$\begin{aligned}
f \cdot g &= \left(\sum_{i=1}^n \lambda_i [A_i] \right) \cdot \left(\sum_{i=1}^m \mu_i [B_i] \right) = \sum_{i=1}^n \left(\lambda_i [A_i] \cdot \sum_{j=1}^m \mu_j [B_j] \right) \\
&= \sum_{i=1}^n \sum_{j=1}^m (\lambda_i [A_i] \mu_j [B_j]) \\
&= \sum_{i=1}^n \sum_{j=1}^m \lambda_i \mu_j [A_i \cap B_j].
\end{aligned}$$

For each i, j the set $A_i \cap B_j$ is closed and convex. Thus, $f \cdot g$ is indeed a linear combination of indicator functions of closed convex sets.

- $\mathcal{C}(\mathbb{R}^d)$ is not a finite dimensional real vector space. Assume it would be. In that case, there exists a basis $([A_1], \dots, [A_n])$ of $\mathcal{C}(\mathbb{R}^d)$. Furthermore, we can assume that each A_i and A_j are pairwise disjoint for $i, j = 1, \dots, n$. If there would be A_i and A_j such that $A_i \cap A_j \neq \emptyset$, then define the following sets:

$$B_1 = A_i \cap A_j, \quad B_2 = A_i \setminus A_j, \quad B_3 = A_j \setminus A_i.$$

Note that

$$\text{span}\{[A_1], \dots, [A_n]\} \subset \text{span}\{[A_1], \dots, [A_{i-1}], B_1, B_2, B_3, [A_{i+1}], \dots, [A_n]\}.$$

Therefore, we can assume that A_i and A_j are indeed disjoint. Next, we show that there exists a convex and closed set B that cannot be a linear combination of $[A_1], \dots, [A_n]$. First, there is a set A_i that contains more than one point (otherwise, we would have an infinite basis $[A_1], [A_2], \dots$). Let $B \subset A_i$ with $A_i \setminus B \neq \emptyset$. Now, we can easily see that $[B] \notin \text{span}\{[A_1], \dots, [A_n]\}$, because $[B]$ can only be combined from $\{[A_i]\}$ but B is a true subset of A_i . Thus, $\mathcal{C}(\mathbb{R}^d)$ is not a finite dimensional real vector space.

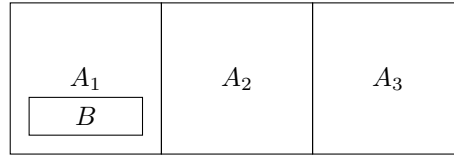


Figure 1: Example case for \mathbb{R}^2 . We can see that the set B cannot be approximated by the sets A_1, A_2 and A_3 .

Exercise 3

1. I_2 is an open square in \mathbb{R}^2 , i.e. a square without its edges.

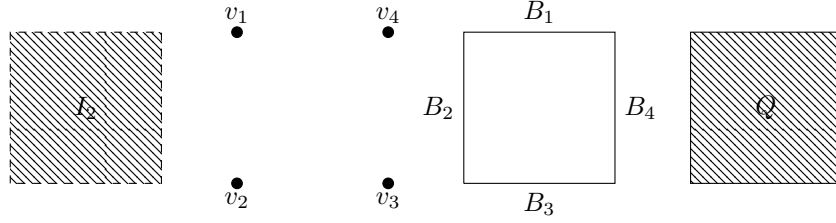


Figure 2: The set I_2 , the edges of the unit square, the four vertices and $Q = [0, 1]^2$.

First,

$$Q = [0, 1]^2 = I_2 \cup \bigcup_{i=1}^4 v_i \cup \bigcup_{i=1}^4 b_i,$$

where $v_i \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ are the four vertices of the unit square, and $B_i \in \{\{0\} \times]0, 1[, \{1\} \times]0, 1[,]0, 1[\times \{0\},]0, 1[\times \{1\}\}$ are the four edges of the unit square. Now, it holds $\chi(\{v_i\}) = 1$ since $\{v_i\}$ is closed and convex. Additionally,

$$1 = \chi(\{0\} \times [0, 1]) = \chi(B_1) + \chi(\{(0, 0)\}) + \chi(\{(0, 1)\}) = \chi(B_1) + 1 + 1$$

and from this follows $\chi(B_1) = -1$. The same for B_2, B_3 and B_4 .

So,

$$1 = \chi(Q) = \chi(I_2) + \underbrace{\sum_{i=1}^4 \chi(\{v_i\})}_{=4} + \underbrace{\sum_{i=1}^4 \chi(B_i)}_{=-4}$$

Therefore, $\chi(I_2) = 1$.

2. Let's compute $\chi(I_3)$, where I_3 is the open unit cube in \mathbb{R}^3 . The unit cube $Q = [0, 1]^3$ has 8 vertices, and 12 edges. Each vertex has an Euler characteristic of 1 and each edge has an Euler characteristic of -1 . So, $1 = \chi(Q) = \chi(I_3) + 8 - 12 \implies \chi(I_3) = 5$.

Exercise 4

Let $A_1, \dots, A_n \subset \mathbb{R}^d$ be closed and convex such that $\bigcap_{i=1}^n A_i \neq \emptyset$. Consider $n = 2$. Then, $\chi(A_1 \cup A_2) = \chi(A_1) + \chi(A_2) - \chi(A_1 \cap A_2) = 1 + 1 - 1 = 1$. Note that $A_1 \cap A_2$ is convex and closed as well as not empty.

$n \rightsquigarrow n + 1$. Assume $\chi(A_1 \cup \dots \cup A_n) = 1$ for some $n \in \mathbb{N}$ if A_i are closed and

convex such that $A_1 \cap \dots \cap A_n \neq \emptyset$. Then,

$$\begin{aligned}
\chi(A_1 \cup \dots \cup A_n \cup A_{n+1}) &= \chi\left(\left(\bigcup_{i=1}^n A_i\right) \cup A_{n+1}\right) \\
&= \chi\left(\bigcup_{i=1}^n A_i\right) + \chi(A_{n+1}) - \chi\left(\bigcup_{i=1}^n A_i \cap A_{n+1}\right) \\
&= 1 + 1 - \chi\left(\bigcup_{i=1}^n A_i \cap A_{n+1}\right)
\end{aligned}$$

Consider the set $B_i = A_i \cap A_{n+1}$. It holds $B_1 \cap \dots \cap B_n \neq \emptyset$ and B_i is convex and closed. Therefore, $\chi(B_1 \cup \dots \cup B_n) = 1$. Thus,

$$\chi(A_1 \cup \dots \cup A_n \cup A_{n+1}) = 1 + 1 - \underbrace{\chi\left(\bigcup_{i=1}^n A_i \cap A_{n+1}\right)}_{=B_1 \cup \dots \cup B_n} = 1 + 1 - 1 = 1.$$