Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 11

Martin Genzel, Mones Raslan Summer Term 2019

Exercise 1: **5 Points**

For $t \in \mathbb{R}$, let $e_t : \mathbb{R} \to \mathbb{C}$ be the function defined by $e_t(x) := e^{itx}$, $x \in \mathbb{R}$, and set $\mathcal{L} := \operatorname{span}\{e_t : t \in \mathbb{R}\}$. Prove that $\langle \cdot, \cdot \rangle$, where

$$\langle f, g \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x)\overline{g(x)}\, dx, \quad f, g \in \mathcal{L},$$

is a well-defined scalar product on $\mathcal{L}$. Show that the system $\{e_t : t \in \mathbb{R}\}$ is an ONS in $\mathcal{L}$. Further, show that $\mathcal{L}$ is not complete.

Exercise 2: **5 Points**

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $T \in L(\mathcal{H}_1, \mathcal{H}_2)$. Prove that the following statements are equivalent:

1. T is compact.
2. T^*T is compact.
3. $\lim_{n\to\infty} \|Tx_n\| = 0$ for every sequence $(x_n)_n$ which converges weakly to zero.

Bonus Exercise 1 **+4 Points**

Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $T \in L(\mathcal{H}_1, \mathcal{H}_2)$ be compact. Show that $\operatorname{ran}(T)$ and $\ker(T)^\perp$ are separable.

Bonus Exercise 2 **+6 Points**

Let $f \in L_1([a, b]^2)$. Recall (Fubini) that $F(x) := \int_a^b |f(x, y)|\, dy$ exists for almost every $x \in [a, b]$ and that $\int_a^b F(x)\, dx = \|f\|_1$. Furthermore, we recall that for each $\varepsilon > 0$ there exist mutually disjoint open rectangles $Q_j = (\alpha_j, \beta_j) \times (\gamma_j, \delta_j)$ and $c_j \in \mathbb{K}$, $j \in \{1, \dots, n\}$, such that

$$\Big\| f - \sum_{j=1}^{n} c_j \chi_{Q_j} \Big\|_1 < \varepsilon.$$

Now let $k \in L_2([a, b]^2)$. For $f \in L_2([a, b])$ define

$$(Kf)(x) := \int_a^b k(x, y) f(y)\, dy, \quad x \in [a, b].$$

Prove the following statements:

(a) Kf is well-defined for $f \in L_2([a,b])$.

(b) $\|Kf\|_2 \leq \|k\|_2 \|f\|_2$ for $f \in L_2([a,b])$. In particular, K is a bounded linear operator from $L_2([a,b])$ to itself.

(c) K (as an operator from $L_2([a,b])$ to itself) is compact. (*Hint:* Approximate k.)

Please submit your homework in the BEGINNING of the big exercise on Monday, July 1st.