Exercise 2

Sketch of proof

We basically want to prove when equality holds in the Minkowski inequality. For that we reduce the Minkowski inequality to Hoelder's inequality. What we are doing next is just the proof of Minkowski inequality backwards and replacing appropriate inequalities with equalitites by using the given assumptions.

Proof

Let $p > 1$ and $p \neq \infty$. Let $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in \ell_p$ with $||(x_n)|| = ||(y_n)|| = 1$ and $||(x_n) + (y_n)|| = 2$. Define $z_k := |x_k + y_k|$. It follows that

$$||(x_n)|| + ||(y_n)|| = ||(x_n) + (y_n)|| = ||(z_n)||.$$

Thus for all $m \in \mathbb{N}$ it holds

$$(\sum_{k=0}^{m} |z_k|^p)^{\frac{1}{p}} = (\sum_{k=0}^{m} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=0}^{m} |y_k|^p)^{\frac{1}{p}}. \tag{1}$$

Since p is fixed, chose q such that $\frac{1}{p} + \frac{1}{q} = 1$. We that $\frac{1}{p} = 1 - \frac{1}{q}$. So,

$$(\sum_{k=0}^{m} |z_k|^p)^{\frac{1}{p}} = (\sum_{k=0}^{m} |z_k|^p)^{1 - \frac{1}{q}} = \frac{\sum_{k=0}^{m} |z_k|^p}{(\sum_{k=0}^{m} |z_k|^p)^{\frac{1}{q}}}.$$

From this equation, we can multiply (1) with $(\sum_{k=0}^{m} |z_k|^p)^{\frac{1}{q}}$ on both sides, and we obtain

$$\sum_{k=0}^{m} |z_k|^p = (\sum_{k=0}^{m} |x_k|^p)^{\frac{1}{p}} (\sum_{k=0}^{m} |z_k|^p)^{\frac{1}{q}} + (\sum_{k=0}^{m} |y_k|^p)^{\frac{1}{p}} (\sum_{k=0}^{m} |z_k|^p)^{\frac{1}{q}}$$

Now, observe that $p = q(p-1)$. Hence, we can also write former equation as

$$\sum_{k=0}^{m} |z_k|^p = (\sum_{k=0}^{m} |x_k|^p)^{\frac{1}{p}} (\sum_{k=0}^{m} |z_k|^{q(p-1)})^{\frac{1}{q}} + (\sum_{k=0}^{m} |y_k|^p)^{\frac{1}{p}} (\sum_{k=0}^{m} |z_k|^{q(p-1)})^{\frac{1}{q}}$$

$$= [(\sum_{k=0}^{m} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=0}^{m} |y_k|^p)^{\frac{1}{p}}](\sum_{k=0}^{m} |z_k|^{q(p-1)})^{\frac{1}{q}} \tag{2}$$

We also know that

$$\sum_{k=0}^{\infty} |z_k|^p = \sum_{k=0}^{\infty} |z_k||z_k|^{p-1} = \sum_{k=0}^{\infty} |x_k + y_k||z_k|^{p-1}. \tag{3}$$

By triangular inequality we note that

$$\sum_{k=0}^{m} |z_k|^p \overset{(3)}{\leq} \sum_{k=0}^{\infty} (|x_k| + |y_k|) \cdot |z_k|^{p-1}. \tag{4}$$

On the other hand, we can use Hoelder's inequality on (2) to obtain

$$\sum_{k=0}^{\infty} |x_k||z_k|^{p-1} + \sum_{k=0}^{\infty} |y_k||z_k|^{p-1} \leq [(\sum_{k=0}^{m} |x_k|^p)^{\frac{1}{p}} + (\sum_{k=0}^{m} |y_k|^p)^{\frac{1}{p}}](\sum_{k=0}^{m} |z_k|^{q(p-1)})^{\frac{1}{q}}$$

$$= \sum_{k=0}^{\infty} |z_k|^p \tag{5}$$

Using (4) and (5) we find a lower and upper bound for $\sum_{k=0}^{\infty} |z_k|^p$:

$$\sum_{k=0}^{\infty} |x_k||z_k|^{p-1} + \sum_{k=0}^{\infty} |y_k||z_k|^{p-1} \leq \sum_{k=0}^{\infty} |z_k|^p \leq \sum_{k=0}^{\infty} |x_k||z_k|^{p-1} + \sum_{k=0}^{\infty} |y_k||z_k|^{p-1}.$$

Putting two and two together and we get

$$\sum_{k=0}^{\infty} (|x_k| + |y_k|)|z_k|^{p-1} = \sum_{k=0}^{\infty} |z_k|^p \overset{(3)}{=} \sum_{k=0}^{\infty} |x_k + y_k||z_k|^{p-1} \tag{6}$$

> **Lemma 0.1**
>
> Let $(a_n), (b_n)$ be convergent sequences in $\mathbb{C}$. If $\sum_{j=0}^{\infty} |a_j| + |b_j| = \sum_{j=0}^{\infty} |a_j + b_j|$ then $|a_j| + |b_j| = |a_j + b_j|$ for all $j \in \mathbb{N}$.

Proof. Let $(a_n), (b_n)$ be convergent sequences in $\mathbb{C}$ with

$$\sum_{j=0}^{\infty} |a_j| + |b_j| = \sum_{j=0}^{\infty} |a_j + b_j|.$$

For all $n \in \mathbb{N}$ it holds

$$|a_0| + |b_0| + ... + |a_n| + |b_n| \leq |a_0 + b_0| + ... + |a_n + b_n| \leq |a_0| + |b_0| + ... + |a_n| + |b_n|.$$

Assume that there exists $j \in \{1, ..., n\}$ such that $|a_j + b_j| \neq |a_j| + |b_j|$ Thus, it must follow by triangluar inequality $|a_j + b_j| < |a_j| + |b_j|$. So, we get

$$|a_j| + |b_j| - |a_j + b_j| + \sum_{k \in \{1,...,n\} \setminus \{j\}} |a_k| + |b_k| \leq \sum_{k \in \{1,...,n\} \setminus \{j\}} |a_k + b_k|,$$

which implies that $\sum_{k \in \{1,...,n\} \setminus \{j\}}^{n} < \sum_{k \in \{1,...,n\} \setminus \{j\}}^{n} |a_k + b_k|$. *Contradiction!* Thus, there cannot be such j and $|a_j| + |b_j| = |a_j + b_j|$ for all $j \in \{1, ..., n\}$. For $n \in \mathbb{N}$ was arbitrary it holds for all $j \in \mathbb{N}$ that

$$|a_j| + |b_j| = |a_j + b_j|.$$

$\square$

Using the hint

We apply the Lemma 0.1 on (6), and we get

$$|x_k| + |y_k| = |x_k + y_k|, \qquad \forall k \in \mathbb{N}$$

Let $k \in \mathbb{N}$. Using the second hint given on the exercise sheet, we get two cases:

(1) Either it holds: $x_k = \alpha y_k$ for $\alpha > 0$, and since $||(y_k)|| = \alpha ||(x_k)|| = 1$ it follows $\alpha = 1$. So $x_k = y_k$.

(2) Consider sequences $(x_k), (y_k) \in \ell_p$ with $||(x_k)|| = ||(y_k)|| = 1$ and $x_k = 0$ or $y_k = 0$ for an index $k \in \mathbb{N}$. W.l.o.g. let $x_k = 0$. For now we assume that there is exactly one member x_i, namely x_k, which is zero, i.e. we assume that $y_i \neq 0$ and $x_i \neq 0$ *(the proof can be easily generalized to the case if* $x_i = 0$ *for more than one* $i \in \mathbb{N}$*)*. Due to case (1) it holds $x_i = y_i$ for all $i \neq k$. We want to prove the contraposition of

$$\frac{||(x_k) + (y_k)||}{2} = 1 \implies (x_k) = (y_k).$$

Assume $(x_k) \neq (y_k)$. Thus, there must exist an index $l \in \mathbb{N}$ such that $x_l \neq y_l$. Because of $x_i = y_i$ for all $i \neq k$ we have $x_k = 0$ and $y_k \neq 0$. Additionally, there must exist an index $j \neq k$ such that $x_j \neq 0$ and $y_j = 0$ due to $||(x_k)|| = ||(y_k)|| = 1$ and $|x_i| = |y_i|$ for all $i \neq k$.

It holds

$$\frac{\left(\sum_{i \in \mathbb{N}} |x_i + y_i|^p \right)}{2} = \frac{1}{2} \Big(\sum_{i \neq k, i \neq j}^{\infty} |x_i + y_i|^p + |x_j|^p + |y_k|^p \Big)$$

$$\Downarrow \textit{due to } |x_k| + |y_k| = |x_k + y_k| \textit{ for all } k \in \mathbb{N}$$

$$= \frac{1}{2} \Big(\sum_{i \neq k, i \neq j}^{\infty} (|x_i| + |y_i|)^p + |x_j|^p + |y_k|^p \Big)$$

$$\Downarrow \textit{due to } x_i \neq 0 \textit{ and } y_i \neq 0 \textit{ for all } i \in \mathbb{N} \setminus \{j, k\}$$

$$> \frac{1}{2} \Big(\sum_{i \neq k, i \neq j}^{\infty} |x_i|^p + |y_i|^p + |x_j|^p + |y_k|^p \Big)$$

$$= \frac{1}{2} \Big(\sum_{i \in \mathbb{N}}^{\infty} |x_i|^p + \sum_{i \in \mathbb{N}}^{\infty} |y_i|^p \Big) = 1.$$

Thus by contraposition it holds that $x_k = y_k = 0$.

Conclusion

Finally, it holds $(y_k)_{k \in \mathbb{N}} = (x_k)_{k \in \mathbb{N}}$. ℓ_p is strictly convex for $p \in (1, \infty)$.

Counterexamples for ℓ_1 and ℓ_∞

Consider $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \in \ell_1$ with

$$(x_k)_{k\in\mathbb{N}} = (\frac{1}{2}, \frac{1}{2}, 0, 0, ...) \quad \text{and} \quad (y_k)_{k\in\mathbb{N}} = (0, \frac{1}{2}, \frac{1}{2}, 0, ...).$$

We see that $||(x_k)|| = \sum_{k\in\mathbb{N}} |x_k| = 1 = \sum_{k\in\mathbb{N}} |y_k| = ||(y_k)||$ but $(x_k)_{k\in\mathbb{N}} \neq (y_k)_{k\in\mathbb{N}}$. So, ℓ_1 is not strictly convex.

Consider $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}} \in \ell_\infty$ with

$$(x_k)_{k\in\mathbb{N}} = (1, 0, 0, 0, ...) \quad \text{and} \quad (y_k)_{k\in\mathbb{N}} = (0, 1, 0, 0, ...).$$

We see that $||(x_k)|| = \sup_{k\in\mathbb{N}} |x_k| = 1 = \sup_{k\in\mathbb{N}} |y_k| = ||(y_k)||$ but $(x_k)_{k\in\mathbb{N}} \neq (y_k)_{k\in\mathbb{N}}$. So, ℓ_∞ is not strictly convex.