Differential Equations I (Week 10)

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Contents

1 Lecture on 20th December 2018

These notes were written in the class *Differential Equations I*, which is an undergraduate course at the Technical University of Berlin. The lecture was held by Hans-Christian Kreusler. These notes are not endorsed by the lecturers, and all mistakes were certainly made by me.

Fundamental Theorem of Calculus for Lebesgue Integrals

Today's lecture is all about the fundamental theorem of calculus. The theorem should be known for real valued functions: Let $f : [a, b] \to \mathbb{R}$ be continuous, then F with

$$F(x) = F(x_0) + \int_{x_0}^{x} f(t)dt$$

is differentiable, and it holds

$$F'(x) = f(x), \quad \forall x \in [a, b].$$

Note that the integral $\int f(t)dt$ exists because f is continuous. However, if f is not continuous, we cannot apply this thereom. Yet, there exists functions which have an antiderivative and are not continuous.

We will introduce a more generalised version of the fundamental theorem of calculus which applies for Lebesgue integrals and just requires absolute continuity.

Theorem 1 (Fundamental theorem of calculus). *Let $f : [a, b] \to \mathbb{R}$ be continuous and monotonically increasing. Then, the following statements are equivalent:*

1. *f is absolutely continuous,*
2. *f maps null sets onto null sets,*
3. *f is differentiable almost everywhere, f' is integrable, and it holds*

$$f(x) = f(a) + \int_a^x f'(t)dt.$$

Proof. First, we will show that

$$f \text{ is absolutely continuous} \implies f \text{ maps null sets onto null sets.}$$

Let $\mathcal{H}$ be the $\sigma-$algebra. Let f be absolutely continuous and let $A \in \mathcal{H}$ be a null set with $\lambda(A) = 0$. W.l.o.g. $A \subset (a, c)$.

Let $\epsilon > 0$ and δ chosen as defined by the absolute contuinity of f. □

2 Lecture on 21th December 2018

Differential Equations after Carathéodory

Today, we want to introduce a new notion of solution for initial value problems. This notion is based on the theory of Carathéodory and requires weaker conditions. We will see that a solution need not be differentiable everywhere, instead the solution must only be absolutely continuous, which is a weaker condition. As usual, consider the initial value problem given by

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases} \tag{1}$$

where $f : [0, T] \times \overline{B(u_0, r)} \to \mathbb{R}^d$ and $u_0 \in \mathbb{R}^d$.

Definition. *A function $u : [0, T] \to \mathbb{R}^d$ is a called a* ***solution*** *of the equation (1) if*

- *u solves the differential equation almost everywhere,*
- *u suffices the initial value constraint,*
- *and u is absolutely continuous.*

Definition 2. *Eine absolut stetige Funktion* $[0.T] \to \mathbb{R}^d$ *die das AWP*

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$$

fast überall löst, d.h. u erfüllt die DGL fast überall und u erfüllt die AB, heißt Lösung Sinne von Carathédoroy.

Remark. *Geht auch für* $\dim X = \infty \rightsquigarrow$ *Bochner-Integral (DGL III). Im mehrdimensionalen nimmt man das Integral komponentenweise.*

Definition 3. *Eine Funktion* $f : [0, T] \times \overline{B(u_0, t)} \mapsto \mathbb{R}^d$ *erfüllt die Carathéodory-Bedingung, falls*

(1) *Für alle* $v \in \bar{B}(u_0, t), i = 1, ..., d$ *ist* $t \mapsto f_i(t, v)$ *messbar auf* $[0, T]$*;*

(2) *Für fast alle* $t \in [0, T]$*, alle* $i = 1, ..., d$ *ist* $v \mapsto f_i(t, v)$ *stetig auf* $\bar{B}(u_0, t)$*.*

Remark. *Ersetzt die Stetigkeitsbedingung.*

Definition 4. *Eine Funktion f erfüllt eine Majoranten-Bedingung, falls es eine inte-Unktion* $m : [0, T] \to \mathbb{R}$ *mit*

$$|f_(t, v)| \leq m(t), \quad \textit{für alle } v \in \bar{B}(u_0, r), i = 1, ..., d \textit{ und fast alle } t \in [0, T].$$

Remark. *Ersetzt die Beschränktheitsbedingung.*

Der Integrand sollte integrierbar sein. Dafür brauchen wir das folgende Lemma.

Lemma 5. 1. *Erfüllt f die Carathéodory Bedingiung, so bildet der zugehäroge Nemyzki-Operator messbare Funktionen auf messbare Funktionen ab.*

2. *Erfüllt f zusätzlich eine Majoranten-Bedingung, so bildet der Nemyzki-Operator messbare Funktionen auf integrierbare Funktionen ab.*

Proof. 1. Sei $u : [0, T] \to \mathbb{R}^d$ messbar. Wir zeigen, dass $t \mapsto f_i(t, u(t)), i = 1, ..., d$ messbar ist. Sei (u_n) eine Folge einfacher Funktionen mit $u_n \to u$ fast überall.

$$u_n = \sum_{i=1}^{N_n} v_i^{(n)} \chi_{A_i^{(n)}}, \quad \text{wobei } A_i^{(n)} \cap A_j^{(n)} = \emptyset, i \neq j, \bigcup_{i=1}^{N_n} A_i^{(n)} = [0, T], v_i \in \mathbb{R}^d.$$

Die Abbildung $t \mapsto f_i(t, u(t))$ ist messbar:

$$f_i(t, u_n(t)) = f_i(t, \sum_{i=1}^{N_n} v_i^{(n)} \chi_{A_i^{(n)}}(t)) \overset{(*)}{=} \sum_{i=1}^{N_n} f_i(t, v_i^{(n)}) \chi_{A_i^{(n)}}(t)$$

(∗) es gilt $t \in A_\tau : f_i(t, u_n(t)) = f_i(t, v_\tau)$. Da A_i messbar ist und f die Carathéodory-Bedingung erfüllt, sind alle Summanden messbar. Es gilt $f_i(\cdot, u_n) \to f_i(\cdot, u)$ fast überall: Es gelte $u_n(t) \to u(t)$. Da f die Carathéodory Bedingung erfüllt, folgt die Stetigkeit im zweiten Argument:

$$f_i(t, u_n(t)) \to f_i(t, u(t)).$$

Somit ist $f(\cdot, u)$ als gast überall punktweise Grenzwert einer Folge messbarer Funktionen messbar.

2. Wegen f messbar existiert $\int_0^T |f_i(t, u(t))| dt$. Also $\int_0^T |f_i(t, u(t))| dt \leq \int_0^T m(t) dt < \infty$.

□

Remark. *Es sind äquivalent (falls f die Carathßeodory Bedingung und Majoranten Bedingung erfüllt):*

1. *u ist absolut stetig und eine Carathéodory Lösung des AWP.*
2. *u ist stetig und löst*

$$u(t) = \int_0^t f(s, u(s)) ds + u_0 \quad \textit{für fast alle } t \in I, I \textit{ ist kompakt.}$$

3. *(u ist integrierbar und $u(t) = \int_0^t f(s, u(s)) ds + u_0$ für fast alle $t \in I$.)*
4. *()u ist messbar und $u(t) = \int_0^t f(s, u(s)) ds + u_0$ für fast alle $t \in I$.)*

Proof. (1) $\implies$ (2): Da $u'(t) = f(t, u(t))$ fast überall gilt und $f(\cdot, u)$ integrierbar ist, folgt dies wieder durch Integration. (2) $\implies$ (3) $\implies$ (4) easy. (4) $\implies$ (1), ist u messbar , so ist wieder $f(\cdot, u)$ integrierbar und die Behauptung folgt aus dem Hauptsatz für absolut stetige Funktionen. □

Theorem 6 (lokale Lösbarkeit, Carathéodory, 1918). *Es sei $f : [0, T] \times \bar{B}(u_0, r) \to \mathbb{R}^d, u_0 \in \mathbb{R}^d, t_0 \in [0, T]$. Falls f eine Carathéodory Bedingung und Majorantenbedingung, so besitzt das AWP*

$$\begin{cases} u'(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$$

mindestends eine Lösung im SInne von Carathéodory auf dem Intervall $I = [0, T] \cap [t_0 - a, t_0 + a]$, wobei a so gewählt ist, dass $\int_I m(t) dt \leq r$.

Remark. *Falls f beschränkt ist, das heißt $m \equiv M > 0$ gewählt werden kann, so kann $a = \frac{r}{2M}$ gewählt werden.*

Proof. sd □

3 Big Tutorial on 19th December 2018

Differential equations after Carathéodory

Our goal in this tutorial is to weaken the notion of solution for a differential equation. Consider the ODE of the form

$$\begin{cases} u'(t) = f'(t, u(t)) \\ u(t_0) = u_0 \end{cases} \tag{$\star$}$$

with $f : J \times D \to \mathbb{R}^d, (t, u) \mapsto f(t, u)$ and $J \subset \mathbb{R}, D \subset \mathbb{R}^d$. We shall say that $u(t) : J \to D$ is a *solution after Carathéodory* if

- u is absolute continuous on J (notation: $u \in \mathrm{AC}(J, D)$),
- u solves the ODE ($\star$) almost everywhere,
- and u satisfies the initial values.

In comparision, the "classical" notion of solution requires that u is differentiable on J, and that u solves the ODE everywhere, i.e. on entire J.

u is absolute continuous	vs	u is differentiable on J
u solves the ODE almost everywhere	vs	u solves the ODE everywhere

As we see, the requirements have been fairly weakened. This means, even if we cannot find classical solutions, we can still hope for a solution after Carathéodory. For instance, all theorems we have known so far do not apply for an ODE $u'(t) = f(t, u(t))$ if f is not continuous in some way. However, we learn a new theorem that asserts the existence of a solution even if f is *not* continuous!

Before we move on, we will refresh some notions, in particular the notion of *absolute continuity.*

Definition 7 (Absolute continuity). *A function $u : J \to \mathbb{R}^d$ is* absolutely continuous *if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every finite family of pairwise disjoint intervalls $\{(a_i, b_i) \subset J\}_{i=1,\dots,n}$ with total length $\sum_{i=1}^n |b_i - a_i| < \delta$ holds:*

$$\sum_{i=1}^{n} \|u(b_i) - u(a_i)\| < \epsilon.$$

We see that every function u that is uniformly continuous is also absolutely continuous.

Lipschitz-continuous $\implies$ absolutely continuous $\implies$ uniformly continuous $\implies$ continuous.

Moreover, f being absolutely continuous is equivalent to f having a derivative f' almost everywhere, f' being Lebesgue integrable and

$$\forall x \in [a, b] : f(x) = f(a) + \int_a^x f'(t)dt.$$

Local existence of solutions

Definition 8 (Carathéodory condition). *A function f suffices the* Carathéodory condition *if for all $i = 1, ..., d$ holds:*

- *$t \mapsto f_i(t, u)$ is measurable on J for all $u \in D$ (u is fixed);*
- *$u \mapsto f_i(t, u)$ is continuous on D for almost all $t \in J$ (t is fixed).*

Definition 9 (Majorant condition). *A function suffices the* majorant condition *if there exists a Lebesgue integrable function $m : J \to \mathbb{R}$ such that for all $i = 1, ..., n$ holds*

$$\forall (t, v) \in J \times D : |f_i(t, v)| \leq m(t)$$

In the lecture, we will see that if f satisfies the Carathéodory and majorant condition and u is measurable, then the map $t \mapsto f(t, u(t))$ is measurable, as well. Moreover, if u is integrable, then $t \mapsto f(t, u(t))$ is integrable, too.

We use the two conditions to state a theorem that yields the existence of solutions by a more general case.

Theorem 10 (Local existence of solutions after Carathéodory, 1918). *Let $X = \mathbb{R}^d, t \in [0, T]$. Let $f : [0, T] \times \overline{B(u_0, r)} \to \mathbb{R}^d$ be a function that suffices the Carathéodory and majorant condition (with majorant $m \in L^1([0, T])$). Then, the integral equation*

$$u(t) = u_0 + \int_{t_0}^t f(s, u(s))ds, \quad u_0 \in X$$

has at least one solution u after Carathéodory on the interval $I_a := [t_0 - a, t_0 + a] \cap [0, T]$. The solution $u : I_a \to \overline{B(u_0, r)}$ is absolutely continuous.

How must we chose $a > 0$ for the interval I_a? Chose it as follows

$$\max_{t \in I_a} |\int_{t_0}^t m(s)ds| \leq r.$$

Proof. Analogue to the proof of Peano by the Schauder fixed-point theorem. $\square$

Example. *Consider the IVP*

$$\begin{cases} u'(t) = \chi_{\mathbb{R}\setminus\mathbb{Q}}(t) \\ u(0) = 0 \end{cases}$$

with $t \in I = [0,1]$ *and* $u(t) \in D = \mathbb{R}$. *So,* $f(t,u) = \chi_{\mathbb{R}\setminus\mathbb{Q}}(t)$ *(note that* f *is independent of* u*). In case,* $\chi_A(t)$ *is the characteristic function which gives* 1 *if* $t \in A$ *and otherwise* 0. *We check that the Carathéodory and majorant conditions hold:*

- $t \mapsto f(t,u) = \chi_{\mathbb{R}\setminus\mathbb{Q}}(t)$ *is measurable since* χ *is a simple function and simple functions are measurable.*
- $u \mapsto f(t,u) = \chi_{\mathbb{R}\setminus\mathbb{Q}}(t)$ *is constant and therefore continuous on* D.
- f *suffices the majorant condition due to* $|f(t,u)| \leq 1 \in L^1([0,1])$ *for all* $(t,u) \in I \times D$.

After the theorem of the local existence of solutions, a solution after Carathéodory exists for the IVP. Note that no classical solution exists since the derivatie u' *is nowhere continuous. A solution after Carathéodory is* $u(t) = t$. *This solution is absolutely continuous and solves the ODE almost everywhere, i.e. it holds* $u' = 1 = \chi_{\mathbb{R}\setminus\mathbb{Q}}$ *almost everywhere.*

Example. *Consider the IVP*

$$\begin{cases} 0 & \text{if } u(t) = 0, t = 0 \\ \frac{t^2 u(t)^2}{t^4 + u(t)^4} & \text{otherwise} \\ u(t_0) = u_0 \end{cases}$$

with $(t_0, u_0) \in [-1,1] \times \mathbb{R} = I \times D$. *We see that* $f = \begin{cases} 0, & \text{if } t = u = 0 \\ \frac{t^2u^2}{t^4+u^4} & \text{otherwise} \end{cases}$.

- $t \mapsto f(t,u)$ *is measurable and* $u \mapsto f(t,u)$ *is continuous. Thus, the Carathéodory condition holds.*
- *The majorant conditions holds, too:* $|f(t,u)| \leq \frac{t^2u^2}{t^4+u^4} \leq \frac{\frac{1}{2}(t^4+u^4)}{t^4+u^4} = \frac{1}{2} \in L^1([-1,1])$.

Therefore, the IVP has at least one solution after Carathéodory. Note that f *is not continuous on* $[-1,1] \times \mathbb{R}$, *for instance chose* $t = u$ *and let* $t \to 0$; $f(t,t) \to \frac{1}{2} \neq 0$.

Local unicity of solutions

The theorem of Carathéodory does not give any information about the uniqueness of a solution. However, if f additionally suffices a general local Lipschitz-continuity condition, the uniqueness of solutions can indeed be guaranteed! The following theorem will state this.

Theorem 11 (Local uniqueness of solutions). *Let $f : [0, T] \times D \to \mathbb{R}^d$ with $D \subset \mathbb{R}^d$ be open such that f suffices the Carathéodory and majorant condition. If f additionally satisfies the general local Lipschitz continuous condition, i.e. for every compact set $K \subset D$ it holds that*

$$\exists l_K \in L^1([0,T]), \forall u, v \in K, \forall t \in [0,T] : \|f(t,u) - f(t,v)\| \leq l_k(t)\|u - v\|,$$

then there exists <u>exactly one</u> solution.

Proof. Let $\alpha, \beta > 0$ such that the ball around (t_0, u_0) with radius (α, β) is a subset of $[0, T] \times D$:

$$B(\alpha, \beta) = \{(t, u) : |t - t_0| \leq \alpha, |u - u_0| \leq \beta\} \subset [0, T] \times D.$$

Let $I_\alpha \coloneqq [t_0 - \alpha, t_0 + \alpha]$ and let

$$M(t) \coloneqq \int_{t_0}^{t} m(s)ds. \tag{2}$$

Chose $\bar{\alpha}, \bar{\beta}$ such that $0 \leq \bar{\alpha} \leq \alpha$ and $0 \leq \bar{\beta} \leq \beta$ with

$$M(t) \leq \bar{\beta}, \quad \forall t \in I_{\bar{\alpha}}. \tag{3}$$

Let $\Omega = \{\Phi \in \mathcal{C}(I_\alpha, D) \,|\, \Phi(t_0) = u_0, \forall t \in I_{\bar{\alpha}} : |\Phi(t) - u_0| \leq \bar{\beta}\}$. The set Ω is closed, non-empty, bounded and convex in $\mathcal{C}(I_\alpha, D)$. For $\Phi \in \Omega$, define $T\Phi$ as

$$(T\Phi)(t) \coloneqq u_0 + \int_{t_0}^{t} f(s, \Phi(s))ds.$$

We need to check that f is integrable, otherwise $T\Phi$ is not well defined. The map $s \mapsto f(s, \Phi(s))$ is Lebesgue integrable on I_α because f suffices the Carathéodory and majorant condition (see remark of the Definition 9). Therefore, we obtain that T is a function that maps continuous functions from $\mathcal{C}(I_\alpha, \mathbb{R}^d)$ to $\mathcal{C}(I_\alpha, \mathbb{R}^d)$.

Now, we want to prove that T is a contraction on Ω. So, T must be a map from Ω to Ω:

$$\|(T\Phi)(t) - u_0\| \leq |\int_{t_0}^{t} f(s, \Phi(s))ds| \leq \int_{t_0}^{t} m(s)ds \overset{(2)}{=} M(t) \overset{(3)}{\leq} \bar{\beta}, \quad \forall t \in I_{\bar{\alpha}}.$$

To prove the contraction property: Let $\Phi, \Psi \in \Omega$. It holds

$$\begin{aligned} \|(T\Phi)(t) - (T\Psi)(t)\| &= \| \int_{t_0}^{t} f(s, \Phi(s)) - f(s, \Psi(s))ds\| \\ &\overset{(\triangle)}{\leq} \int_{t_0}^{t} \|f(s, \Phi(s)) - f(s, \Psi(s))\|ds. \end{aligned} \tag{4}$$

Now, we use the general Lipschitz continuity property of f: since $K := \overline{B(u_0, r)}$ is a compact set, there exists a Lebesgue integrable function $l_K \in L^1[0,T]$ such that

$$\|f(t,v) - f(t,w)\| \leq l_K(t)\|v - w\|, \quad \forall t \in I_\alpha, \forall v, w \in K. \tag{5}$$

Then, we adjust $\bar{\alpha}$ for a second time such that

$$\exists \xi \in [0,1), \forall t \in I_{\bar{\alpha}} : \int_{t_0}^{t} l_K(s)ds \leq \xi.$$

Using (4) and (5) yields

$$\forall t \in I_{\bar{\alpha}} : \|(T\Phi)(t) - (T\Psi)(t)\| \leq \underbrace{\int_{t_0}^{t} l_K(s)ds}_{\leq \xi} \|\Phi - \Psi\|_\infty \leq \xi \|\Phi - \Psi\|_\infty.$$

Finally, T is contraction operating on Ω and from the Banach fixed-point theorem follows the existence of an unique solution u with $Tu = u$. The fixed point u has the following properties:

$$u : I_{\bar{\alpha}} \to \mathbb{R}^d \text{ with } u(t_0) = u_0 \quad \text{and} \quad u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds, \quad u \in AC(I_{\bar{\alpha}}, \mathbb{R}^d)^{1}.$$

Thus, u is the unique solution after Carathéodory. □

Summary

We extended our notion of solutions by functions which are "just" absolutely continuous. This allows us to state the existence of solutions even if f is not continuous; hereof f only needs to suffice the Carathéodory and majorant condition. In addition, if f is even general lipschitz continuous, the solution after Carathéodory is unique.

[1] $AC(A, B)$ is the vector space of all absolutely continuous functions $u : A \to B$