Exercise 2.1

(i) Let x_1, x_2 denote the position of the masses relative to the quilibrium position. Let F_1, F_2 be the spring force on m_1, m_2 respectively. The forces are described by the Hooke law:

$$F_1 = \underbrace{-kx_1}_{\text{left spring}} + \underbrace{\frac{k}{3}(x_2 - x_1)}_{\text{center srping}}$$

$$F_2 = \underbrace{-kx_2}_{\text{right spring}} + \underbrace{\frac{k}{3}(x_1 - x_2)}_{\text{center spring}}$$

Newton's second law of motion is $F = ma = m\ddot{x}$. Applying this law gives the following system of differential equations:

$$\begin{cases} m\ddot{x}_1 = -\frac{4}{3}kx_1 + \frac{k}{3}x_2 \\ m\ddot{x}_2 = \frac{k}{3}x_1 - \frac{4}{3}kx_2 \end{cases}$$

We write that system in matrix form:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\frac{k}{3m}\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1}$$

We call this the coupled system.

(ii) We introduce a new vector

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} := \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}}_{:=T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Calculate T^{-1} by Cramer's rule:

$$T^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \tilde{x}_1 + \tilde{x}_2 \\ \tilde{x}_1 - \tilde{x}_2 \end{pmatrix}. \tag{2}$$

Substitute (1) into (2) yields

$$\frac{1}{2}\begin{pmatrix} \ddot{\tilde{x}}_1 + \ddot{\tilde{x}}_2 \\ \ddot{\tilde{x}}_1 - \ddot{\tilde{x}}_2 \end{pmatrix} = -\frac{k}{3m}\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \cdot (-\frac{1}{2})\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \frac{k}{6m}\begin{pmatrix} -3 & -5 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}.$$

We obtain the equations:

$$\frac{1}{2}(\ddot{\tilde{x}}_1 + \ddot{\tilde{x}}_2) = \frac{k}{6m}(-3\tilde{x}_1 - 5\tilde{x}_2), \tag{3}$$

$$\frac{1}{2}(\ddot{\tilde{x}}_1 - \ddot{\tilde{x}}_2) = \frac{k}{6m}(-3\tilde{x}_1 + 5\tilde{x}_2) \tag{4}$$

Now (3) + (4) and (3) − (4) results in

$$\ddot{\tilde{x}}_1 = -\frac{k}{m}\tilde{x}_1, \quad \ddot{\tilde{x}}_2 = -\frac{5k}{3m}\tilde{x}_2$$

We have got the uncoupled differential equation:

$$\begin{pmatrix} \ddot{\tilde{x}}_1 \\ \ddot{\tilde{x}}_2 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m}\tilde{x}_1 \\ -\frac{5k}{3m}\tilde{x}_2 \end{pmatrix}.$$

This is a linear differential equation with constant coefficients. Calculate $\tilde{x}_1$: So we get a solution u by $u(t) = e^{\lambda t}$.

$$\ddot{\tilde{x}}_1 + \frac{k}{m}\tilde{x}_1 = 0 \implies \lambda^2 + \frac{k}{m} = 0 \implies \lambda_{1,2}^2 = \pm i\sqrt{\frac{k}{m}} \implies u(t) = e^{\pm it\sqrt{\frac{k}{m}}}.$$

Let's obtain the real solution. Consider $u(t) = e^{it\sqrt{\frac{k}{m}}}$ (we ignore the negative solution):

$$u(t) = e^{it\sqrt{\frac{k}{m}}} = \cos\left(\sqrt{\frac{k}{m}}t\right) + i\sin\left(\sqrt{\frac{k}{m}}t\right).$$

We get the real solutions u_1, u_2 by taking the real and imaginary part of u:

$$u_1(t) = \cos\left(\sqrt{\frac{k}{m}}t\right), \quad u_2(t) = \sin\left(\sqrt{\frac{k}{m}}t\right)$$

General solution of the differential equation is

$$u_{gen}(t) = c_1\cos\left(\sqrt{\frac{k}{m}}t\right) + c_2\sin\left(\sqrt{\frac{k}{m}}t\right), \quad c_1, c_2 \in \mathbb{R}.$$

Now, solve the IVP with $u_{gen}(0) = a$ and $\dot{u}_{gen}(0) = c$. So

$$a = c_1\cos(0) + c_2\sin(0) \implies c_1 = a.$$

and $\dot{u}_{gen}(t) = -c_1\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right) + c_2\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}t\right)$. Thus

$$c = -c_1\sin 0\sqrt{\frac{k}{m}} + c_2\sqrt{\frac{k}{m}}\cos 0 = c_2\sqrt{\frac{k}{m}} \implies c_2 = c\sqrt{\frac{m}{k}}.$$

The particular solution is

$$u_{1,part}(t) = a\cos\left(\sqrt{\frac{k}{m}}t\right) + c\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right)$$

Calculate $\tilde{x}_2$: Same approach as above. We obtain the characteristic polynomial:

$$\lambda^2 + \frac{5k}{3m} = 0 \implies \lambda_{1,2} = \pm i\sqrt{\frac{5k}{3m}}.$$

We get the general solution

$$u_{gen}(t) = d_1 \cos(\sqrt{\frac{5k}{3m}}t) + d_2 \sin(\sqrt{\frac{5k}{3m}}t), \quad d_1, d_2 \in \mathbb{R}.$$

Now, consider $u_{gen}(0) = b$ and $\dot{u}_{gen}(0) = d$. We get

$$\begin{aligned} b &= d_1 \cos(0) = d_1 \\ d &= -d_1 \sin(0)\sqrt{\frac{5k}{3m}} + d_2 \cos(0)\sqrt{\frac{5k}{3m}} \implies d_2 = d\sqrt{\frac{3m}{5k}}. \end{aligned}$$

Thus,

$$u_{2,part}(t) = b\cos(\sqrt{\frac{5k}{3m}}t) + d\sqrt{\frac{3m}{5k}}\sin(\sqrt{\frac{5k}{3m}}t).$$

Overall, the solution ist

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} a\cos(\sqrt{\frac{k}{m}}t) + c\sqrt{\frac{m}{k}}\sin(\sqrt{\frac{k}{m}}t) \\ b\cos(\sqrt{\frac{5k}{3m}}t) + d\sqrt{\frac{3m}{5k}}\sin(\sqrt{\frac{5k}{3m}}t) \end{pmatrix}. \tag{5}$$

(iii) By (2), we know that

$$\begin{aligned} x_1(t) &= \frac{1}{2}\Big(a\cos(\sqrt{\frac{k}{m}}t) + c\sqrt{\frac{m}{k}}\sin(\sqrt{\frac{k}{m}}t) + b\cos(\sqrt{\frac{5k}{3m}}t) + d\sqrt{\frac{3m}{5k}}\sin(\sqrt{\frac{5k}{3m}}t)\Big) \\ &= \frac{1}{2}\Big(a\cos(\sqrt{\frac{k}{m}}t) + b\cos(\sqrt{\frac{5k}{3m}}t) + \sqrt{\frac{m}{k}}[c\sin(\sqrt{\frac{k}{m}}t) + d\sqrt{\frac{3}{5}}\sin(\sqrt{\frac{5k}{3m}}t)]\Big) \\ x_2(t) &= \frac{1}{2}\Big(a\cos(\sqrt{\frac{k}{m}}t) + c\sqrt{\frac{m}{k}}\sin(\sqrt{\frac{k}{m}}t) - b\cos(\sqrt{\frac{5k}{3m}}t) - d\sqrt{\frac{3m}{5k}}\sin(\sqrt{\frac{5k}{3m}}t)\Big) \\ &= \frac{1}{2}\Big(a\cos(\sqrt{\frac{k}{m}}t) - b\cos(\sqrt{\frac{5k}{3m}}t) + \sqrt{\frac{m}{k}}[c\sin(\sqrt{\frac{k}{m}}t) - d\sqrt{\frac{3}{5}}\sin(\sqrt{\frac{5k}{3m}}t)]\Big) \\ \dot{x}_1(t) &= \frac{1}{2}\Big(-a\sqrt{\frac{k}{m}}\sin(\sqrt{\frac{k}{m}}t) - b\sqrt{\frac{5k}{3m}}\sin(\sqrt{\frac{5k}{3m}}t) + c\cos(\sqrt{\frac{k}{m}}t) + d\cos(\sqrt{\frac{5k}{3m}}t)]\Big) \\ \dot{x}_2(t) &= \frac{1}{2}\Big(-a\sqrt{\frac{k}{m}}\sin(\sqrt{\frac{k}{m}}t) + b\sqrt{\frac{5k}{3m}}\sin(\sqrt{\frac{5k}{3m}}t) + c\cos(\sqrt{\frac{k}{m}}t) - d\cos(\sqrt{\frac{5k}{3m}}t)]\Big) \end{aligned}$$

The global flow Φ is defined as follows:

$$\Phi_t : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}^4, (t, \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}) \mapsto \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}$$

Exercise 2.2

(i) First, we show that $\Phi_0(x,y,z) = (x,y,z)$ for all $(x,y,z) \in \mathbb{R}^3$.

$$\Phi_0(x,y,z) = (x+0, y, e^0 z) = (x,y,z).$$

Next, we show that $\Phi_{s+t}(x,y,z) = \Phi_s(x,y,z) \circ \Phi_t(x,y,z)$.

$$\begin{aligned}
\Phi_{s+t}(x,y,z) &= (x + 2(s+t)y, y, e^{-(s+t)x-(s+t)^2y}z), \\
\Phi_t(x,y,z) &= (x+2ty, y, e^{-tx-t^2y}z) \\
\Phi_s(x+2ty, y, e^{-tx-t^2y}z) &= (x+2ty+2sy, y, e^{-s(x+2ty)-s^2y}e^{-tx-t^2y}z) \\
&= (x+2(s+t)y, y, e^{-s(x+2ty)-s^2y-tx-t^2y}z) \\
&= (x+2(s+t)y, y, e^{-sx-tx-2ty-s^2y-t^2y}z) \\
&= (x+2(s+t)y, y, e^{-(s+t)x-(s+t)^2y}z)
\end{aligned}$$

Indeed, $\Phi_{s+t}(x,y,z) = \Phi_s(\Phi_t(x,y,z))$.

(ii) We know that $\dot{x} = f(x) \iff \frac{d}{dt}\Big|_{t=0}\Phi_t(x) = f(x)$.

$$\frac{d}{dt}|_{t=0}\Phi_t(x,y,z) = (2y, 0, e^{-tx-t^2y}z(-x-2ty))|_{t=0} = (2y, 0, -xz) = f(x,y,z).$$

Therefore, the vector field is $f(x,y,z) = (2y, 0, -xz)$.

(iii) The corresponding IVP for $\Phi_t(x_0,y_0,z_0)$ is

$$\begin{cases} \dot{x} = 2y, \\ \dot{y} = 0, \\ \dot{z} = -xz \\ x(0) = x_0, y(0) = y_0, z(0) = z_0 \end{cases}$$

Exercise 2.3

(i) Let $\Gamma := \{(x,y) \in \mathbb{R}^2 : x \neq \alpha^{-1}\}$. We want to show that $\Phi(\Gamma) \subset \Gamma$. Let $p = (x,y) \in \Gamma$. First, we show that for $\Phi(p) = (\tilde{x}, \tilde{y})$ it yields $\tilde{x} \neq \alpha^{-1}$. We know that

$$\tilde{x} = \frac{x}{\alpha x - 1}.$$

It yields: $\tilde{x} = \frac{x}{\alpha x - 1} = \frac{1}{\alpha}$ iff $x = 1$ but then it follows that $\tilde{x} = \frac{1}{\alpha - 1}$ which is not equal to $\frac{1}{\alpha}$.

Now we show that $\tilde{y} \in \mathbb{R}$ which is indeed true for $\tilde{y} = \frac{y + \alpha x(x-y)}{\alpha x - 1}, x \neq \frac{1}{\alpha}$. Therefore, $\Phi(p) \in \Gamma$.

(ii) Let $\mathcal{O}(x,y)$ be an orbit and let $(x_0,y_0) \in \mathcal{O}(x,y)$. Let $f(x) := \frac{x}{\alpha x-1}$ and $g(x,y) := \frac{y+\alpha x(x-y)}{\alpha x-1}$. Then $\Phi(x,y) = (f(x), g(x,y))$. We want to show that $f(f(x)) = x$ and therefore

$$f(x) = f(f(f(x))).$$

Proof. Show that $f(f(x)) = x, \forall x \in \mathbb{R}\backslash\{\alpha^{-1}\}$.

$$f(x) = \frac{x}{\alpha x-1} \implies f(f(x)) = f(\frac{x}{\alpha x-1}) = \frac{\frac{x}{\alpha x-1}}{\alpha\frac{x}{\alpha x-1}-1} = \frac{\frac{x}{\alpha x-1}}{\frac{\alpha x-\alpha x+1}{\alpha x-1}} = \frac{\frac{x}{\alpha x-1}}{\frac{1}{\alpha x-1}} = x.$$

□

Now, we want to show that $g(f(x), g(x,y)) = y$. Be prepared...

Proof.

$$\begin{aligned}
g(\frac{x}{\alpha x-1}) &= \frac{\frac{y+\alpha x(x-y)}{\alpha x-1} + \frac{\alpha x}{\alpha x-1}(\frac{x-y-\alpha x(x-y)}{\alpha x-1})}{\alpha\frac{x}{\alpha x-1}-1} \\
&= \frac{\frac{y+\alpha x(x-y)}{\alpha x-1} + \frac{\alpha x}{\alpha x-1}(\frac{x-y-\alpha x(x-y)}{\alpha x-1})}{\frac{1}{\alpha x-1}} \\
&= y + \alpha x(x-y) + \alpha x(\frac{x-y-\alpha x(x-y)}{\alpha x-1}) \\
&= \frac{(\alpha x-1)(y+\alpha x(x-y)) + \alpha x^2 - \alpha xy - \alpha^2 x^2(x-y)}{\alpha x-1} \\
&= \frac{\alpha xy + \alpha^2 x^2(x-y) - y - \alpha x(x-y) + \alpha x^2 - \alpha xy - \alpha^2 x^2(x-y)}{\alpha x-1} \\
&= \frac{\alpha x^2 - \alpha x(x-y) - y}{\alpha x-1} \\
&= \frac{\alpha xy - y}{\alpha x-1} \\
&= \frac{y(\alpha x-1)}{\alpha x-1} \\
&= y.
\end{aligned}$$

□

So we have

$$\Phi^3(x,y) = \Phi(\Phi^2(x,y)) = \Phi(\Phi(f(x), g(x,y))) = \Phi(f(f(x)), g(f(x), g(x,y)) = \Phi(x,y).$$

Each orbit is periodic and the period is three.

Exercise 2.4

Let $p \in \mathcal{O}(x_0), p \neq x_0$. We know that $\Phi^T(x_0) = x_0$. Due to $p \in \mathcal{O}(x_0)$, there exists $L < T$ such that $\Phi^L(x_0) = p$. So we get

$$\Phi^T(x_0) = \Phi^{T-L}(\underbrace{\Phi^L(x_0)}_{=p}) = x_0.$$

Applying Φ^L on both sides yields

$$\Phi^L(\Phi^{T-L}(p)) = \Phi^L(x_0) \iff \Phi^T(p) = p.$$

Assume $\Phi^{\tilde{T}}(p) = p$ for $\tilde{T} < T$. There is a $\tilde{L} < \tilde{T}$ such that $\Phi^{\tilde{L}}(p) = x_0$ since $p \in \mathcal{O}(x_0)$. Now

$$p = \Phi^{\tilde{T}-\tilde{L}}(\Phi^{\tilde{L}}(p)) = \Phi^{\tilde{T}-\tilde{L}}(x_0)$$

but applying $\Phi^{\tilde{L}}$ results in

$$x_0 = \Phi^{\tilde{L}}(p) = \Phi^{\tilde{L}}(\Phi^{\tilde{T}-\tilde{L}}(x_0)) = \Phi^{\tilde{T}}(x_0).$$

Contradiction: T is the smallest T such that $\Phi^T(x_0) = x_0$ but we have found a $\tilde{T} < T$ with such property. Therefore, $\tilde{T} < T$ with $\Phi^{\tilde{T}}(p) = p$ cannot exist and T is the period.