Chapter 3:
The Geometry of Linear Programming

(cp. Bertsimas & Tsitsiklis, Chapter 2)

Linear, Conic, Affine, and Convex Combinations and Hulls

Definition 3.1.

Let $x^1, \dots, x^k \in \mathbb{R}^n$, $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$, and $x := \sum_{i=1}^k \lambda_i x^i$.

a x is a linear combination of $x^1, \dots, x^k$.

b If $\lambda \geq 0$, then x is a conic combination of $x^1, \dots, x^k$.

c If $\sum_{i=1}^k \lambda_i = 1$, then x is an affine combination of $x^1, \dots, x^k$.

d If $\lambda \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$, x is a convex combination of $x^1, \dots, x^k$.

If neither $\lambda = 0$ nor $\lambda = e_i$ for some $i \in \{1, \dots, n\}$, then x is a proper linear / conic / affine /convex combination.

Definition 3.2.

For a non-empty subset $S \subseteq \mathbb{R}^n$, the linear / conic / affine / convex hull of S, denoted by $\text{lin}(S)$ / $\text{cone}(S)$ / $\text{aff}(S)$ / $\text{conv}(S)$, is the set of all vectors that can be written as a linear / conic / affine / convex combination of finitely many vectors from S.
Moreover, let $\text{lin}(\emptyset) := \text{cone}(\emptyset) := \{0\}$ and $\text{aff}(\emptyset) := \text{conv}(\emptyset) := \emptyset$.

Linear Subspace, Cone, Affine Subspace, Convex Subset

Definition 3.3.

- a $S \subseteq \mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$, if $S = \text{lin}(S)$.
- b $S \subseteq \mathbb{R}^n$ is a (convex) cone, if $S = \text{cone}(S)$.
- c $S \subseteq \mathbb{R}^n$ is an affine subspace of $\mathbb{R}^n$, if $S = \text{aff}(S)$.
- d $S \subseteq \mathbb{R}^n$ is called convex, if $S = \text{conv}(S)$ (cp. Definition 1.1).

Lemma 3.4.

- a Let $S \subseteq \mathbb{R}^n$. Then $\text{lin}(S)$ / $\text{cone}(S)$ / $\text{aff}(S)$ / $\text{conv}(S)$ is the inclusion-wise smallest linear subspace / cone / affine subspace / convex subset containing S.
- b The sets of linear subspaces / cones / affine subspaces / convex sets in $\mathbb{R}^n$ are closed under taking intersections.
- c Every linear subspace and every cone in $\mathbb{R}^n$ contains the zero-vector 0.

Proof of Lemma 3.4

a) $\text{conv}(S)$ is a convex set since $\text{conv}\big(\text{conv}(S)\big) = \text{conv}(S)$. (Why?)

For any convex set $X \supseteq S$, it holds that $X = \text{conv}(X) \supseteq \text{conv}(S)$.

Similar for $\text{lin}(S)$, $\text{cone}(S)$, and $\text{aff}(S)$.

b) For $j \in J$, let $C_j \subseteq \mathbb{R}^n$ with $C_j = \text{cone}(C_j)$; (family of cones)

then, $\text{cone}\Big(\bigcap_{j\in J} C_j\Big) = \bigcap_{j\in J} C_j$ because:

'$\supseteq$': clear; '$\subseteq$': $\text{cone}\Big(\bigcap_{j\in J} C_j\Big) \subseteq \bigcap_{j\in J} \text{cone}(C_j) = \bigcap_{j\in J} C_j$

Similar for linear subspaces, affine subspaces, and convex sets.

c) A linear subspace or cone X is non-empty, as $\text{lin}(\emptyset) = \text{cone}(\emptyset) = \{0\}$.

Thus there is an element $x \in X$ and therefore also $0 = 0 \cdot x \in X$. $\square$

Linear and Affine Independence, Dimension

Definition 3.5.

A finite non-empty subset $S \subseteq \mathbb{R}^n$ is linearly (affinely) independent, if no element of S can be written as a proper linear (affine) combination of elements from S.

Definition 3.6.

The dimension $\dim(S)$ of a subset $S \subseteq \mathbb{R}^n$ is the largest cardinality of an affinely independent subset of S minus 1.

Remark.

- If $S \subseteq \mathbb{R}^n$ is a linear subspace, $\dim(S)$ according to Definition 3.6 is equal to the maximal number of linearly independent vectors in S.
- Adding the zero-vector to a linearly independent set of vectors yields an affinely independent set.

Hyperplanes and Halfspaces

Definition 3.7.

Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$:

a. set $\{x \in \mathbb{R}^n \mid a^T \cdot x = b\}$ is called hyperplane;
b. set $\{x \in \mathbb{R}^n \mid a^T \cdot x \geq b\}$ is called halfspace.

Remarks.

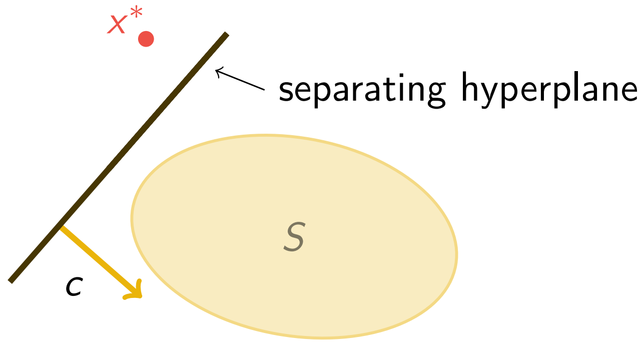
- A hyperplane is an affine subspace of dimension $n - 1$.
- A hyperplane with right-hand side $b = 0$ is a linear subspace.
- Hyperplanes and halfspaces are cones if and only if $b = 0$.
- Hyperplanes and halfspaces are convex sets.

Separating Hyperplane Theorem for Convex Sets

Theorem 3.8.

Let $S \subseteq \mathbb{R}^n$ closed and convex, and let $x^* \in \mathbb{R}^n \setminus S$. There exists a vector $c \in \mathbb{R}^n$ such that $c^T \cdot x^* < c^T \cdot x$ for all $x \in S$.

Illustration:



Corollary 3.9.

Every closed and convex set is the intersection of a family of halfspaces.

Proof of Theorem 3.8

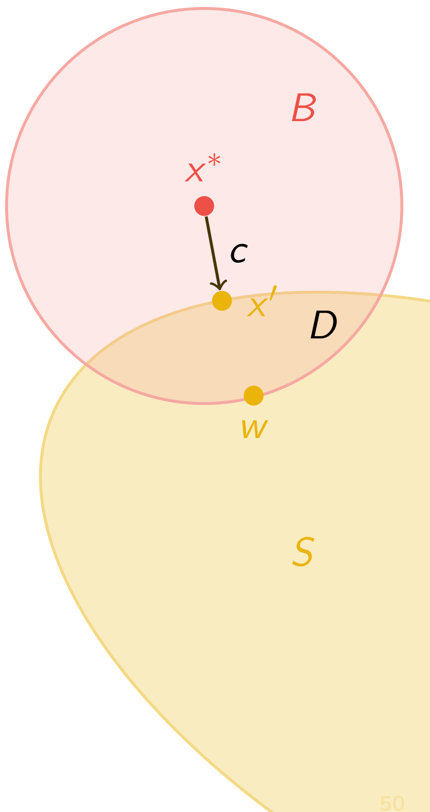
For arbitrary $w \in S$ let $B := \left\{x \,\middle|\, \|x - x^*\| \leq \|w - x^*\|\right\}$ (Euclidean norm)

$\Longrightarrow \quad D := S \cap B$ closed, bounded, non-empty

$\Longrightarrow \quad \exists\, x' \in D: \ \|x' - x^*\| \leq \|x - x^*\| \quad \forall x \in D$

$\Longrightarrow \quad \|x' - x^*\| \leq \|x - x^*\| \quad \forall x \in S \qquad (*)$

Let $c := x' - x^* \neq 0$.



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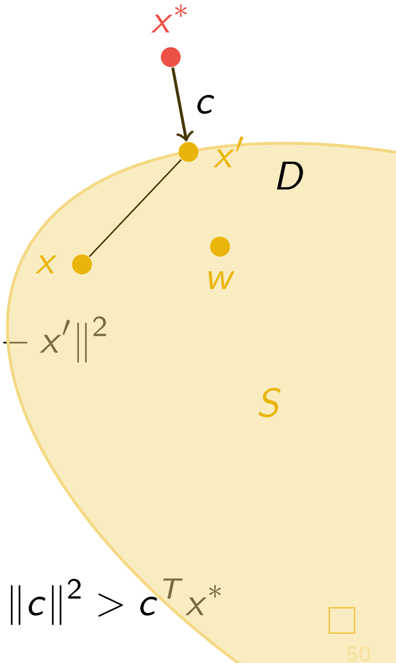
$\Longrightarrow \quad \|x' - x^*\| \leq \|x - x^*\| \quad \forall x \in S \qquad (*)$

Let $c := x' - x^* \neq 0$. Claim: $c^T x > c^T x^* \ \forall x \in S$.



Proof:

For all $x \in S$ and $\lambda \in (0,1]: \quad x' + \lambda(x - x') \in S$

$$\Longrightarrow \quad \|x' - x^*\|^2 \overset{(*)}{\leq} \|x' + \lambda(x - x') - x^*\|^2$$

$$= \|x' - x^*\|^2 + 2\lambda(x' - x^*)^T(x - x') + \lambda^2\|x - x'\|^2$$

$$\Longrightarrow \quad 0 \leq (x' - x^*)^T(x - x') + \tfrac{1}{2}\lambda\|x - x'\|^2$$

$$\overset{\lambda \to 0}{\Longrightarrow} \quad 0 \leq (x' - x^*)^T(x - x') = c^T(x - x')$$

$$\Longrightarrow \quad c^T x \geq c^T x' = c^T x^* + c^T(x' - x^*) = c^T x^* + \|c\|^2 > c^T x^*$$

$\square$

Polyhedra and Polytopes

Definition 3.10.

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$:

- a) set $\{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ is called polyhedron;
- b) $\{x \mid A \cdot x = b, \ x \geq 0\}$ is polyhedron in standard form representation.

That is, a polyhedron is an intersection of finitely many halfspaces.

Definition 3.11.

- a) Set $S \subseteq \mathbb{R}^n$ is bounded if there is $K \in \mathbb{R}$ such that

$$\|x\|_\infty \leq K \qquad \text{for all } x \in S.$$

- b) A bounded polyhedron is called polytope.

Remark: The convex hull of finitely many points in $\mathbb{R}^n$ is a polytope (later).

Polyhedral Cones and Recession Cones

Definition 3.12.

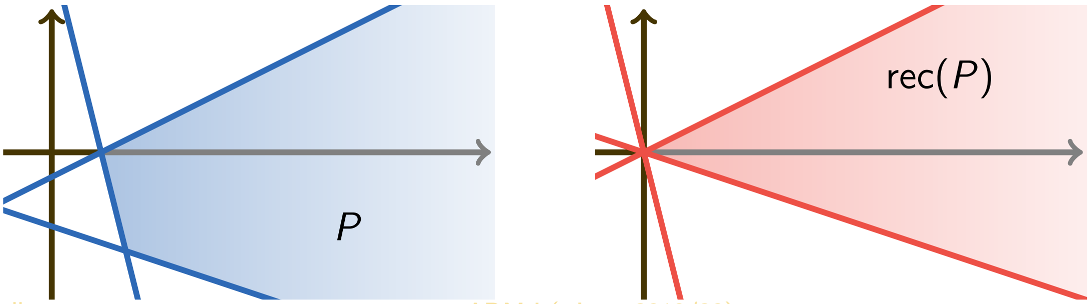
For $A \in \mathbb{R}^{m \times n}$ the polyhedron $\{x \in \mathbb{R}^n \mid A \cdot x \geq 0\}$ is a polyhedral cone.

Remark. Cone $\{0\}$ is the only polyhedral cone in $\mathbb{R}^n$ that is a polytope.

Definition 3.13.

For a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$, the recession cone $\text{rec}(P)$ is the polyhedral cone $\{x \in \mathbb{R}^n \mid A \cdot x \geq 0\}$. The non-zero elements of the recession cone are the rays of P.

Example in $\mathbb{R}^2$:



Characterization of Recession Cones

Lemma 3.14.

For a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ and $y \in P$,

$$\text{rec}(P) = \{x \in \mathbb{R}^n \mid y + \lambda \cdot x \in P \text{ for all } \lambda \geq 0\} \ .$$

In particular, if P is a polytope, $\text{rec}(P) = \{0\}$.

Proof:

For a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$ and $y \in P$,

$$\begin{aligned}\{x \in \mathbb{R}^n \mid y + \lambda \cdot x \in P \ \forall \lambda \geq 0\} &= \{x \in \mathbb{R}^n \mid A \cdot (y + \lambda \cdot x) \geq b \ \forall \lambda \geq 0\} \\ &= \{x \in \mathbb{R}^n \mid A \cdot x \geq 0\}. \qquad \square\end{aligned}$$

Observation. The recession cone of $P = \{x \in \mathbb{R}^n \mid A \cdot x = b, \ x \geq 0\}$ is

$$\{x \in \mathbb{R}^n \mid A \cdot x = 0, \ x \geq 0\}.$$

Extreme Points and Vertices of Polyhedra

Definition 3.15.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a $x \in P$ is an extreme point of P if

$$x \neq \lambda \cdot y + (1-\lambda) \cdot z \qquad \text{for all } y, z \in P \setminus \{x\},\ 0 \leq \lambda \leq 1,$$

i.e., x is not a convex combination of two other points in P.

b $x \in P$ is a vertex of P if there is some $c \in \mathbb{R}^n$ such that

$$c^T \cdot x < c^T \cdot y \qquad \text{for all } y \in P \setminus \{x\},$$

i.e., x is the unique optimal solution to the LP $\min\{c^T \cdot z \mid z \in P\}$.

Remark.

The only possible extreme point or vertex of a polyhedral cone C is $0 \in C$.

Active and Binding Constraints

In the following, let $P \subseteq \mathbb{R}^n$ be a polyhedron defined by

$$\begin{aligned} {a_i}^T \cdot x &\geq b_i && \text{for } i \in M, \\ {a_i}^T \cdot x &= b_i && \text{for } i \in N, \end{aligned}$$

with $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, for all i.

Definition 3.16.

If $x^* \in \mathbb{R}^n$ satisfies ${a_i}^T \cdot x^* = b_i$ for some i, then the corresponding constraint is active (or binding) at x^*.

Lemma 3.17.

If $P \neq \emptyset$ and none of the constraints ${a_i}^T \cdot x \geq b_i$ for $i \in M$ is active for all $x \in P$, then $\dim(P) = k := n - \operatorname{rank}\big\{a_i \mid i \in N\big\}$.

Proof of Lemma 3.17

'$\leq$': $\dim(P) \leq k$ since $\emptyset \neq P \subseteq \underbrace{\{x \in \mathbb{R}^n \mid {a_i}^T \cdot x = b_i \;\forall\, i \in N\}}_{\dim(\cdots)=k}$.

'$\geq$': Let $d_j \in \mathbb{R}^n$, $j = 1, \dots, k$, linearly indep. with ${a_i}^T \cdot d_j = 0 \;\forall\, i \in N$.

For $i \in M$, let $x^i \in P$ with ${a_i}^T \cdot x^i > b_i$; let $y := \frac{1}{|M|} \sum_{i \in M} x^i \in P$.

Notice that ${a_i}^T \cdot y > b_i$ for all $i \in M$. Thus, for $\varepsilon > 0$ small enough,

$$ {a_i}^T \cdot (y + \varepsilon\, d_j) \geq b_i \quad \text{for all } i \in M,\ j = 1, \dots, k. $$

Thus, $y + \varepsilon\, d_j \in P$ for $j = 1, \dots, k$, and

$$ \{y\} \cup \{y + \varepsilon\, d_j \mid j \in \{1, \dots, k\}\} \qquad \text{affinely independent.} $$

Therefore $\dim(P) \geq k$. □

Reminder: Basic Facts from Linear Algebra

Theorem 3.18.

Let $x^* \in \mathbb{R}^n$ and $I = \{i \mid {a_i}^T \cdot x^* = b_i\}$. The following are equivalent:

i) there are n vectors in $\{a_i \mid i \in I\}$ which are linearly independent;

ii) the vectors in $\{a_i \mid i \in I\}$ span $\mathbb{R}^n$;

iii) x^* is the unique solution to the system of equations ${a_i}^T \cdot x = b_i$, $i \in I$.

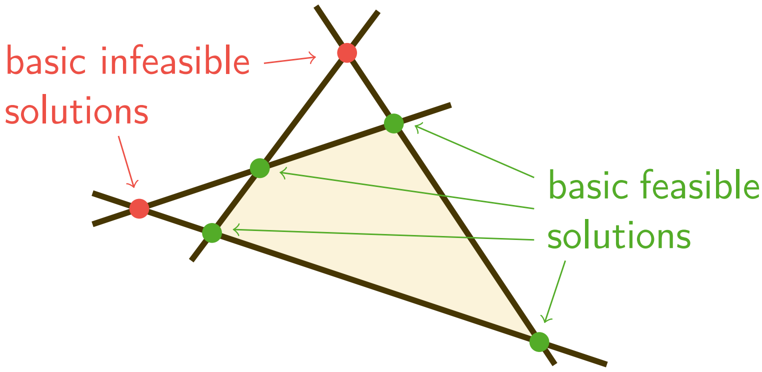
Vertices, Extreme Points, and Basic Feasible Solutions

Definition 3.19.

a $x^* \in \mathbb{R}^n$ is a basic solution of P if
- all equality constraints are active and
- there are n linearly independent constraints that are active.

b A basic solution satisfying all constraints is a basic feasible solution.

Example:



Vertices, Extreme Points, and Basic Feasible Solutions

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a $x^* \in \mathbb{R}^n$ is a basic solution of P if
- all equality constraints are active and
- there are n linearly independent constraints that are active.

b A basic solution satisfying all constraints is a basic feasible solution.

Theorem 3.20.

For $x^* \in P$, the following are equivalent:

i x^* is a vertex of P;

ii x^* is an extreme point of P;

iii x^* is a basic feasible solution of P.

Proof of Theorem 3.20

'(iii)$\Longrightarrow$(i)': x^* basic feasible solution, $I := \left\{i \in M \cup N \mid {a_i}^T \cdot x^* = b_i\right\}$

$$c := \sum_{i \in I} a_i, \qquad c^T \cdot x^* = \sum_{i \in I} {a_i}^T \cdot x^* = \sum_{i \in I} b_i.$$

For $x \in P$: $\quad c^T \cdot x = \sum_{i \in I} \underbrace{{a_i}^T \cdot x}_{\geq b_i} \geq \sum_{i \in I} b_i \quad$ and

$$c^T \cdot x = \sum_{i \in I} b_i \iff {a_i}^T \cdot x = b_i \ \forall\, i \in I \iff x = x^*$$

since there are n linearly independent vectors in $\{a_i \mid i \in I\}$.

'(i)$\Longrightarrow$(ii)': x^* vertex $\Longrightarrow \exists\, c \in \mathbb{R}^n:\ c^T \cdot x^* < c^T \cdot y \quad \forall\, y \in P \setminus \{x^*\}$

By contradiction assume that x^* is *not* an extreme point, i.e.,

$$x^* = \lambda y + (1-\lambda) z \qquad \text{with } y, z \in P \setminus \{x^*\},\ \lambda \in [0,1].$$

Then, $\quad c^T \cdot x^* = \lambda \underbrace{c^T \cdot y}_{> c^T \cdot x^*} + (1-\lambda) \underbrace{c^T \cdot z}_{> c^T \cdot x^*} > c^T \cdot x^*.$ ↯

Proof of Theorem 3.20 (Cont.)

'(ii)$\Longrightarrow$(iii)': Let x^* extreme point, $I := \left\{i \in M \cup N \mid {a_i}^T \cdot x^* = b_i\right\}$.

Assume by contradiction that $\operatorname{rank}\{a_i \mid i \in I\} < n$.

Then, there exists $d \in \mathbb{R}^n \setminus \{0\}$ with ${a_i}^T \cdot d = 0$ for all $i \in I$.

Let $x := x^* + \varepsilon d$ and $y := x^* - \varepsilon d$ for some $\varepsilon > 0$.

Claim: $x, y \in P$ for $\varepsilon > 0$ small enough. ($\Longrightarrow$ contradiction!)

Proof: For $i \in I$, ${a_i}^T \cdot x = \underbrace{{a_i}^T \cdot x^*}_{=b_i} + \varepsilon \underbrace{{a_i}^T \cdot d}_{=0} = b_i$;

for $i \notin I$, ${a_i}^T \cdot x = \underbrace{{a_i}^T \cdot x^*}_{>b_i} + \varepsilon\, {a_i}^T \cdot d \geq b_i$ for $\varepsilon > 0$ small enough.

The same holds for y instead of x. □

Existence of Extreme Points

Definition 3.21.

A polyhedron P is called pointed if it contains at least one vertex.

Definition 3.22.

A polyhedron $P \subseteq \mathbb{R}^n$ contains a line if there is $x \in P$ and a direction $d \in \mathbb{R}^n \setminus \{0\}$ such that

$$x + \lambda \cdot d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$

Theorem 3.23.

Consider non-empty $P \subseteq \mathbb{R}^n$ as on Slide 55. The following are equivalent:

- i P is pointed.
- ii P does not contain a line.
- iii $\operatorname{rank}\{a_i \mid i \in M \cup N\} = n$.

Proof of Theorem 3.23

'(i)$\Longrightarrow$(iii)': $x^* \in P$ vertex $\Longrightarrow$ x^* basic feasible solution

$\Longrightarrow$ There are n linearly independent constraints that are active at x^*.

$\Longrightarrow$ There are n linearly independent vectors in $\{a_i \mid i \in M \cup N\}$.

'(iii)$\Longrightarrow$(ii)': By contradiction assume that P contains a line.

That is, there are $x \in P$, $d \in \mathbb{R}^n \setminus \{0\}$ with

$$x + \lambda\, d \in P \qquad \text{for all } \lambda \in \mathbb{R}.$$

$$\Longrightarrow \quad {a_i}^T \cdot (x + \lambda\, d) = {a_i}^T \cdot x + \lambda\, {a_i}^T \cdot d \geq b_i \quad \forall\, i \in M \cup N,\ \lambda \in \mathbb{R}$$

$$\Longrightarrow \quad {a_i}^T \cdot d = 0 \quad \forall\, i \in M \cup N.$$

$$\Longrightarrow \quad \operatorname{rank}\{a_i \mid i \in M \cup N\} < n. \ ↯$$

Proof of Theorem 3.23 (Cont.)

'(ii)$\Longrightarrow$(i)': Choose $x \in P$ maximizing $\big|\underbrace{\{i \in M \cup N \mid {a_i}^T \cdot x = b_i\}}_{N \subseteq I :=}\big|$.

If $\operatorname{rank}\{a_i \mid i \in I\} = n$, then x is a vertex and we are done.

Otherwise, there is $d \in \mathbb{R}^n \setminus \{0\}$ with ${a_i}^T \cdot d = 0$ for all $i \in I$.

$\Longrightarrow \quad {a_i}^T \cdot (x + \lambda\, d) = {a_i}^T \cdot x = b_i \quad$ for all $i \in I$, $\lambda \in \mathbb{R}$.

Since P does not contain a line, $x + \lambda'\, d \notin P$ for some $\lambda' \in \mathbb{R}$.

$\Longrightarrow \quad {a_i}^T \cdot (x + \lambda'\, d) < b_i$ for some $i \in M \setminus I$.

Assume w.l.o.g. $\lambda' > 0$ (otherwise replace d with $-d$) and let

$$\lambda_0 := \max\{\lambda \mid {a_i}^T \cdot (x + \lambda\, d) \geq b_i \ \forall\, i \in M \cup N\} < \lambda'.$$

Notice: $\lambda_0 > 0$ since ${a_i}^T \cdot d = 0 \ \forall\, i \in I$ and ${a_i}^T \cdot x > b_i \ \forall\, i \in M \setminus I$.

Moreover, there is an $i \in M \setminus I$ with ${a_i}^T \cdot (x + \lambda_0\, d) = b_i$.

$\Longrightarrow$ At least $|I| + 1$ constraints active at $x + \lambda_0\, d \in P$. ↯ $\square$

Existence of Extreme Points (Cont.)

Corollary 3.24.

a. A non-empty polytope contains an extreme point.
b. A non-empty polyhedron in standard form contains an extreme point.
c. Polyhedron $P \neq \emptyset$ is pointed if and only if $\operatorname{rec}(P)$ is pointed.

Proof of b:

$$\begin{aligned} A \cdot x &= b \\ x &\geq 0 \end{aligned} \quad \Longleftrightarrow \quad \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \cdot x \geq \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix}$$ $\square$

Example:

$$P = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{rcrl} x_1 & + & x_2 & \geq 1 \\ x_1 & + & 2\,x_2 & \geq 0 \end{array} \right\}$$

contains a line since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in P \quad$ for all $\lambda \in \mathbb{R}$.

Characterization of Polytopes

Theorem 3.25.

A polytope is equal to the convex hull of its vertices.

Proof: Follows from Theorem 3.20 (see exercise). □

Theorem 3.26.

A set $P \subseteq \mathbb{R}^n$ is a polytope if and only if there exists a finite set $V \subseteq \mathbb{R}^n$ such that P is the convex hull of V.

Proof: '$\Longrightarrow$': Follows from Theorem 3.25 since a polytope has only finitely many basic feasible solutions (vertices).

'$\Longleftarrow$': See exercise session. □

Valid Inequalities, Supporting Hyperplanes, Faces, Facets

Definition 3.27.

Let $c \in \mathbb{R}^n \setminus \{0\}$ and $\gamma \in \mathbb{R}$.

a The linear inequality $c^T \cdot x \geq \gamma$ is valid for P if $P \subseteq \{x \mid c^T \cdot x \geq \gamma\}$.

b The hyperplane $H = \{x \in \mathbb{R}^n \mid c^T \cdot x = \gamma\}$ is a supporting hyperplane of P if $c^T \cdot x \geq \gamma$ is valid for P and $P \cap H \neq \emptyset$.

c The intersection of P with a supporting hyperplane is a face of P. Also P and $\emptyset$ are faces of P; the others are called proper faces.

d The inclusion-wise maximal proper faces are called facets.

Remarks.

- Every face of P is itself a polyhedron.
- Every vertex of P is a 0-dimensional face of P.
- The optimal LP solutions form a face of the underlying polyhedron.

Characterization of Faces

Theorem 3.28.

Consider a polyhedron $P \subseteq \mathbb{R}^n$ be defined by

$$\begin{aligned} {a_i}^T \cdot x &\geq b_i && \text{for } i \in M, \\ {a_i}^T \cdot x &= b_i && \text{for } i \in N, \end{aligned}$$

and let $F \neq \emptyset$ be a face of P.

- a There exists $K \subseteq M$ with $F = \{x \in P \mid {a_i}^T \cdot x = b_i \text{ for } i \in K\}$.
- b For $K \subseteq M$, the subset $\{x \in P \mid {a_i}^T \cdot x = b_i \text{ for } i \in K\}$ is a face of P.
- c $G \subseteq F$ is a face of F if and only if it is a face of P.
- d There is a chain of faces $F = F_0 \subset F_1 \subset \cdots \subset F_q = P$ such that $\dim(F_{i+1}) = \dim(F_i) + 1$, for $i = 0, \ldots, q-1$.

Proof of Theorem 3.28 a

Let $K := \{i \in M \mid {a_i}^T \cdot x = b_i \text{ for all } x \in F\}$.

Claim: $F = \{x \in P \mid {a_i}^T \cdot x = b_i \text{ for } i \in K\}$

'$\subseteq$': Clear by definition of K.

'$\supseteq$': Assume by contradiction that $y \in P \setminus F$ with ${a_i}^T \cdot y = b_i \ \forall\, i \in K$.

Let $c^T \cdot x \geq \gamma$ valid for P such that $F = \{x \in P \mid c^T \cdot x = \gamma\}$.

In particular, $c^T \cdot y > \gamma$ as $y \in P \setminus F$.

For each $i \in M \setminus K$ there is an $x^i \in F$ with ${a_i}^T \cdot x^i > b_i$.

Let $x_0 := \frac{1}{|M \setminus K|} \sum_{i \in M \setminus K} x^i \in F$ (convex), thus $c^T \cdot x_0 = \gamma$.

Notice that ${a_i}^T \cdot x_0 = b_i \ \forall\, i \in K$ and ${a_i}^T \cdot x_0 > b_i \ \forall\, i \in M \setminus K$.

For $\varepsilon > 0$ small enough, $z := x_0 + \varepsilon(x_0 - y) \in P$ because:

$${a_i}^T \cdot z = (1+\varepsilon)\, {a_i}^T \cdot x_0 - \varepsilon\, {a_i}^T \cdot y \begin{cases} = b_i & \text{for } i \in N \cup K, \\ \geq b_i & \text{for } i \in M \setminus K \end{cases}$$

But $c^T \cdot z = (1+\varepsilon)\, c^T \cdot x_0 - \varepsilon\, c^T \cdot y < \gamma$. $\lightning$

Proof of Theorem 3.28 b–d

b) Let $c := \sum_{i \in K} a_i$ and $\gamma := \sum_{i \in K} b_i$.

Then, $c^T \cdot x \geq \gamma$ is a valid inequality for P and for $x \in P$

$$c^T \cdot x = \gamma \qquad \Longleftrightarrow \qquad {a_i}^T \cdot x = b_i \text{ for all } i \in K.$$

c) '$\Longleftarrow$': If $G = \{x \in P \mid c^T \cdot x = \gamma\} \subseteq F$ with $c^T \cdot x \geq \gamma$ valid for P, then $G = \{x \in F \mid c^T \cdot x = \gamma\}$ and $c^T \cdot x \geq \gamma$ valid for F.

'$\Longrightarrow$': $F = \{x \mid {a_i}^T \cdot x \geq b_i \ \forall i \in M \setminus K, \ {a_i}^T \cdot x = b_i \ \forall i \in K \cup N\}$

for some $K \subseteq M$ due to a). Since G is a face of F, again due to a),

$$G = \{x \mid {a_i}^T \cdot x \geq b_i \ \forall i \in M \setminus L, \ {a_i}^T \cdot x = b_i \ \forall i \in L \cup N\}$$

for some $K \subseteq L \subseteq M$. Thus, due to b), G is a face of P.

d) Follows from a–c and Lemma 3.17. □

Characterization of Faces (Cont.)

Corollary 3.29.

Consider a polyhedron $P \subseteq \mathbb{R}^n$ be defined by

$$\begin{aligned} {a_i}^T \cdot x &\geq b_i \qquad && \text{for } i \in M, \\ {a_i}^T \cdot x &= b_i && \text{for } i \in N. \end{aligned}$$

- a P has finitely many distinct faces.
- b If F is a facet of P, then $\dim(F) = \dim(P) - 1$.
- c An inclusion-wise minimal proper face F of P can be written as
  $$F = \{x \in \mathbb{R}^n \mid {a_i}^T \cdot x = b_i \text{ for all } i \in K \cup N\} \qquad \text{for some } K \subseteq M$$
  with $\operatorname{rank}\{a_i \mid i \in K \cup N\} = \operatorname{rank}\{a_i \mid i \in M \cup N\}$.
- d If P is pointed, every minimal nonempty face of P is a vertex.

Proof: Exercise. □

Edges, Extreme Rays, Extreme Lines

Definition 3.30.

A one-dimensional face F of polyhedron P is

- a an edge if F has two vertices, i.e., $F = \mathrm{conv}(\{x, y\})$ with $x, y \in \mathbb{R}^n$, $x \neq y$;
- b an extreme ray if F has one vertex, i.e., $F = x + \mathrm{cone}(\{z\})$ with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n \setminus \{0\}$;
- c an extreme line if F has no vertex, i.e., $F = x + \mathrm{lin}(\{z\})$ with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n \setminus \{0\}$.

Optimality of Extreme Points

Theorem 3.31.

Let $P \subseteq \mathbb{R}^n$ a pointed polyhedron and $c \in \mathbb{R}^n$. If $\min\{c^T \cdot x \mid x \in P\}$ is bounded, there is a vertex that is optimal.

Proof:... □

Corollary 3.32.

Every linear programming problem is either infeasible or unbounded or there exists an optimal solution.

Proof: Every linear program is equivalent to an LP in standard form. The claim thus follows from Corollary 3.24 and Theorem 3.31. □

Number of Vertices

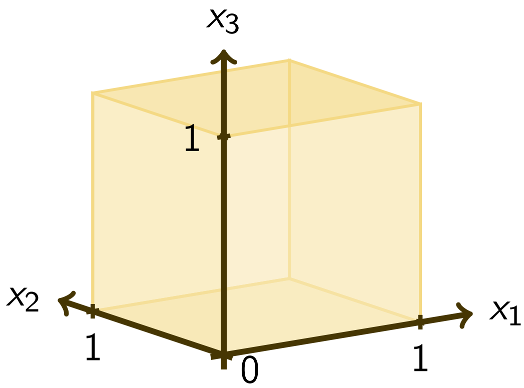
Corollary 3.33.

- a) A polyhedron has a finite number of vertices and basic solutions.
- b) For a polyhedron in $\mathbb{R}^n$ given by m linear inequalities, this number is at most $\binom{m}{n}$.

Example:

$P := \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1,\ \ i = 1, \dots, n\}$
(n-dimensional unit cube)

- number of constraints: $m = 2n$
- number of vertices: 2^n



Adjacent Basic Solutions and Edges

Definition 3.34.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

- a) Two distinct basic solutions are adjacent if there are $n - 1$ linearly independent constraints that are active at both of them.
- b) If both solutions are feasible, the line segment that joins them is an edge of P.

Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ a polyhedron in standard form representation.

Observation.
One can assume without loss of generality that $\text{rank}(A) = m$.

Proof: Let $a_1, \dots, a_m \in \mathbb{R}^n$ rows of A. Assume that $a_i = \sum_{j\neq i} \lambda_j \cdot a_j$.
Case 1: $b_i = \sum_{j\neq i} \lambda_j\, b_j$. Then,

$$a_j{}^T \cdot x = b_j \quad \forall j \neq i \quad \Longrightarrow \quad a_i{}^T \cdot x = \sum_{j\neq i} \lambda_j \cdot (a_j{}^T \cdot x) = \sum_{j\neq i} \lambda_j\, b_j = b_i .$$

Thus the ith constraint is redundant and can be deleted.
Case 2: $b_i \neq \sum_{j\neq i} \lambda_j\, b_j \quad \Longrightarrow \quad A \cdot x = b$ has no solution. □

Basic Solutions of Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = m$, $b \in \mathbb{R}^m$, and

$$P = \left\{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\right\}$$

Theorem 3.35.
A point $x \in \mathbb{R}^n$ is a basic solution of P if and only if $A \cdot x = b$ and there are indices $B(1), \dots, B(m) \in \{1, \dots, n\}$ such that

- columns $A_{B(1)}, \dots, A_{B(m)}$ of matrix A are linearly independent, and
- $x_i = 0$ for all $i \notin \{B(1), \dots, B(m)\}$.

Proof: ... □

- $x_{B(1)}, \dots, x_{B(m)}$ are basic variables, the remaining variables non-basic.
- The vector of basic variables is denoted by $x_B := \left(x_{B(1)}, \dots, x_{B(m)}\right)^T$.
- $A_{B(1)}, \dots, A_{B(m)}$ are basic columns of A and form a basis of $\mathbb{R}^m$.
- The matrix $B := \left(A_{B(1)}, \dots, A_{B(m)}\right) \in \mathbb{R}^{m\times m}$ is called basis matrix.

Corollary: Carathéodory's Theorem

Theorem 3.36 (Carathéodory 1911).

For $S \subseteq \mathbb{R}^n$, every element of $\mathrm{conv}(S)$ can be written as a convex combination of at most $n+1$ points in S.

Proof: Consider $x \in \mathrm{conv}(S)$. Then x can be written as

$$x = \sum_{i=1}^{k} \lambda'_i\, y_i \qquad \text{with } y_1, \dots, y_k \in S \text{ and } \sum_{i=1}^{k} \lambda'_i = 1,\ \lambda' \geq 0.$$

Consider the following polyhedron in standard form:

$$P = \Bigg\{ \lambda \in \mathbb{R}^k \;\Bigg|\; \underbrace{\sum_{i=1}^{k} \lambda_i\, y_i = x,\ \ \sum_{i=1}^{k} \lambda_i = 1,}_{n+1 \text{ equality constraints}}\ \ \lambda \geq 0 \Bigg\}.$$

A basic feasible solution $\lambda^* \in P$ yields the desired representation of x. □

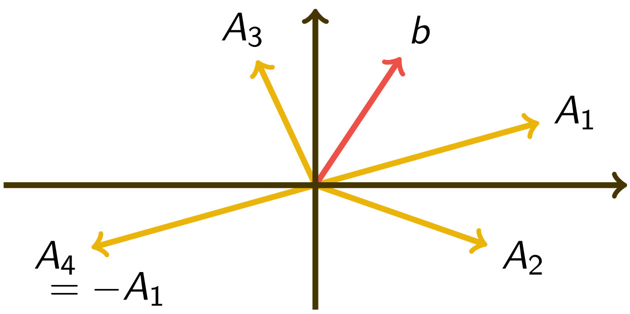
Basic Columns and Basic Solutions

Observation 3.37.

Let $x \in \mathbb{R}^n$ be a basic solution, then:

- $B \cdot x_B = b$ and thus $x_B = B^{-1} \cdot b$;
- x is a basic feasible solution if and only if $x_B = B^{-1} \cdot b \geq 0$.

Example: $m = 2$



- A_1, A_3 or A_2, A_3 form bases with corresp. basic feasible solutions.
- A_1, A_4 do not form a basis.
- A_1, A_2 and A_2, A_4 and A_3, A_4 form bases with infeasible basic solution.

Bases and Basic Solutions

Corollary 3.38.

- Every basis $A_{B(1)}, \ldots, A_{B(m)}$ determines a unique basic solution.
- Thus, different basic solutions correspond to different bases.
- But: two different bases might yield the same basic solution.

Example: If $b = 0$, then $x = 0$ is the only basic solution.

Adjacent Bases

Definition 3.39.

Two bases $A_{B(1)}, \ldots, A_{B(m)}$ and $A_{B'(1)}, \ldots, A_{B'(m)}$ are adjacent if they share all but one column.

Observation 3.40.

a. Two adjacent basic solutions can always be obtained from two adjacent bases.
b. If two adjacent bases lead to distinct basic solutions, then the latter are adjacent.

Degeneracy

Definition 3.41.

A basic solution x of a polyhedron $P \subseteq \mathbb{R}^n$ is degenerate if more than n constraints are active at x.

Observation 3.42.

Let $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ be a polyhedron in standard form with $A \in \mathbb{R}^{m \times n}$, $\operatorname{rank}(A) = m$, and $b \in \mathbb{R}^m$.

a. A basic solution $x \in P$ is degenerate if and only if more than $n - m$ components of x are zero.
b. For a non-degenerate basic solution $x \in P$, there is a unique basis.

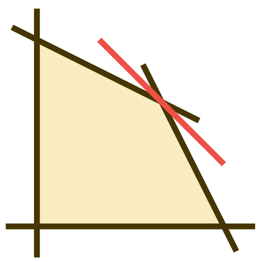
Three Different Reasons for Degeneracy

i. redundant variables

   Example:
$$\begin{array}{rcrcr} x_1 & + & x_2 & & = 1 \\ & & & x_3 & = 0 \\ & & x_1, & x_2,\ x_3 & \geq 0 \end{array} \quad \longleftrightarrow \quad A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

ii. redundant constraints

   Example:
$$\begin{array}{rcrr} x_1 & + & 2\,x_2 & \leq 3 \\ 2\,x_1 & + & x_2 & \leq 3 \\ x_1 & + & x_2 & \leq 2 \\ & & x_1,\ x_2 & \geq 0 \end{array}$$



iii. geometric reasons (non-simple polyhedra)

   Example: Octahedron

Observation 3.43.

Perturbing the right hand side vector b may remove degeneracy.