Chapter 5:
Implementations of the Simplex Method

(cp. Bertsimas & Tsitsiklis, Chapter 3.3)

Revised Simplex Method

Observation 5.1.

To execute one iteration of the simplex method efficiently, it suffices to know $B(1), \dots, B(m)$, the inverse B^{-1} of the basis matrix, and the input data A, b, and c. It is then easy to compute:

$$x_B = B^{-1} \cdot b \qquad \bar{c}^T = c^T - {c_B}^T \cdot B^{-1} \cdot A$$

$$u = B^{-1} \cdot A_j \qquad \theta^* = \min_{i: u_i > 0} \frac{x_{B(i)}}{u_i} = \frac{x_{B(\ell)}}{u_\ell}$$

The new basis matrix is then

$$\bar{B} = \left(A_{B(1)}, \dots, A_{B(\ell-1)}, A_j, A_{B(\ell+1)}, \dots, A_{B(m)}\right)$$

Critical question: How to obtain $\bar{B}^{-1}$ efficiently?

Computing the Inverse of the Basis Matrix

- Notice that $B^{-1} \cdot \bar{B} = (e_1, \dots, e_{\ell-1}, u, e_{\ell+1}, \dots, e_m)$.
- Find a matrix Q such that $Q \cdot B^{-1} \cdot \bar{B} = I_m$. Then $\bar{B}^{-1} = Q \cdot B^{-1}$.
- The following elementary row operations are needed to turn $B^{-1} \cdot \bar{B}$ into the identity matrix:
  - multiply ℓth row with $1/u_\ell$;
  - for $i \neq \ell$, subtract u_i times resulting ℓth row from ith row.
- Elementary row operations are the same as multiplying the matrix with corresponding elementary matrices from the left-hand side.
- $\bar{B}^{-1}$ is obtained by applying these elementary row operations to B^{-1}.

Obtaining $\bar{B}^{-1}$ from B^{-1}

Apply elementary row operations to the matrix $(B^{-1} \mid u)$ to make the last column equal to the unit vector e_ℓ. The first m columns of the resulting matrix form the inverse $\bar{B}^{-1}$ of the new basis matrix $\bar{B}$.

An Iteration of the Revised Simplex Method

Given: $B = \big(A_{B(1)}, \dots, A_{B(m)}\big)$, corresp. basic feasible sol. x, and B^{-1}.

1. Let $p^T := {c_B}^T \cdot B^{-1}$ and $\bar{c}_j := c_j - p^T \cdot A_j$, $j \neq B(1), \dots, B(m)$;
   if $\bar{c} \geq 0$, then STOP; else choose j with $\bar{c}_j < 0$.
2. Let $u := B^{-1} \cdot A_j$. If $u \leq 0$, then STOP (optimal cost is $-\infty$).
3. Let $\theta^* := \min_{i : u_i > 0} \frac{x_{B(i)}}{u_i} = \frac{x_{B(\ell)}}{u_\ell}$ for some $\ell \in \{1, \dots, m\}$.
4. Form new basis by replacing $A_{B(\ell)}$ with A_j; corresponding basic feasible solution y is given by
   $$y_j := \theta^* \qquad \text{and} \qquad y_{B(i)} = x_{B(i)} - \theta^* \, u_i \quad \text{for } i \neq \ell.$$
5. Apply elementary row operations to the matrix $(B^{-1} \mid u)$ to make the last column equal to the unit vector e_ℓ.
   The first m columns of the resulting matrix yield $\bar{B}^{-1}$.

Full Tableau Implementation

Main idea

Instead of maintaining and updating the matrix B^{-1}, we maintain and update the $m \times (n+1)$-matrix

$$B^{-1} \cdot (b \mid A) = (B^{-1} \cdot b \mid B^{-1} \cdot A)$$

which is called simplex tableau.

- The zeroth column $B^{-1} \cdot b$ contains x_B.
- For $i = 1, \ldots, n$, the ith column of the tableau is $B^{-1} \cdot A_i$.
- The column $u = B^{-1} \cdot A_j$ corresponding to the variable x_j that is about to enter the basis is the pivot column.
- If the ℓth basic variable $x_{B(\ell)}$ exits the basis, the ℓth row of the tableau is the pivot row.
- The element $u_\ell > 0$ is the pivot element.

Full Tableau Implementation (Cont.)

Notice: The simplex tableau $B^{-1} \cdot (b \mid A)$ represents the linear equation

$$B^{-1} \cdot b = B^{-1} \cdot A \cdot x$$

which is equivalent to $A \cdot x = b$.

Updating the simplex tableau

- At the end of an iteration, the simplex tableau $B^{-1} \cdot (b \mid A)$ has to be updated to $\bar{B}^{-1} \cdot (b \mid A)$.
- $\bar{B}^{-1}$ can be obtained from B^{-1} by elementary row operations, i.e., $\bar{B}^{-1} = Q \cdot B^{-1}$ where Q is a product of elementary matrices.
- Thus, $\bar{B}^{-1} \cdot (b \mid A) = Q \cdot B^{-1} \cdot (b \mid A)$, and new tableau $\bar{B}^{-1} \cdot (b \mid A)$ can be obtained by applying the same elementary row operations.

Zeroth Row of the Simplex Tableau

In order to keep track of the objective function value and the reduced costs, we consider the following augmented simplex tableau:

$-{c_B}^T B^{-1} b$	$c^T - {c_B}^T B^{-1} A$
$B^{-1} b$	$B^{-1} A$

or in more detail

$$
\begin{array}{c|ccc}
-{c_B}^T x_B & \bar{c}_1 & \cdots & \bar{c}_n \\
\hline
x_{B(1)} & | & & | \\
\vdots & B^{-1}A_1 & \cdots & B^{-1}A_n \\
x_{B(m)} & | & & |
\end{array}
$$

Update after one iteration

The zeroth row is updated by adding a multiple of the pivot row to the zeroth row to set the reduced cost of the entering variable to zero.

An Iteration of the Full Tableau Implementation

Given: Simplex tableau corresp. to feasible basis $B = \big(A_{B(1)}, \ldots, A_{B(m)}\big)$.

1. If $\bar{c} \geq 0$ (zeroth row), then STOP; else choose pivot column j with $\bar{c}_j < 0$.
2. If $u = B^{-1} A_j \leq 0$ (jth column), STOP (optimal cost is $-\infty$).
3. Choose pivot row ℓ with $\displaystyle\min_{i: u_i > 0} \frac{x_{B(i)}}{u_i} = \frac{x_{B(\ell)}}{u_\ell}$ (compare columns 0 and j).
4. Apply elementary row operations to the simplex tableau so that u_ℓ (pivot element) becomes one and all other entries of the pivot column become zero. The resulting tableau corresponds to new basis $\bar{B}$ in which $A_{B(\ell)}$ is replaced with A_j.

Full Tableau Implementation: An Example

A simple linear programming problem:

$$\begin{array}{lrcrcrcl}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & \leq & 20 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

Set of Feasible Solutions

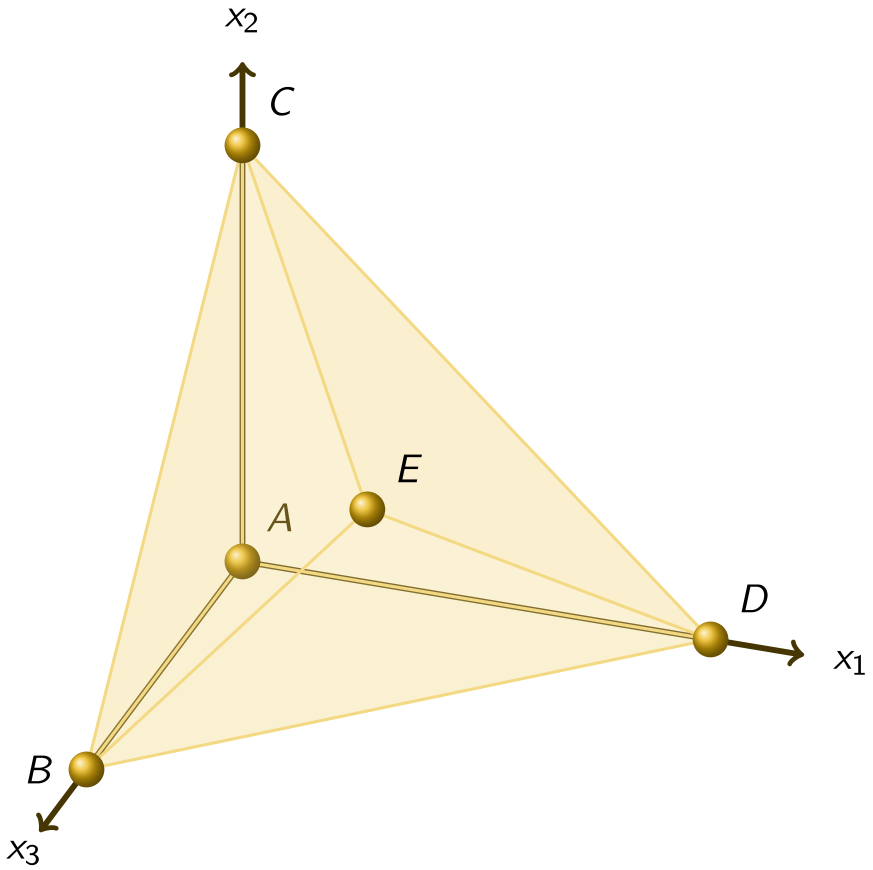
$A = (0,0,0)^T$

$B = (0,0,10)^T$

$C = (0,10,0)^T$

$D = (10,0,0)^T$

$E = (4,4,4)^T$



Introducing Slack Variables

$$
\begin{array}{llllllll}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & \leq & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & \leq & 20 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

LP in standard form

$$
\begin{array}{llllllllllll}
\min & -10\,x_1 & - & 12\,x_2 & - & 12\,x_3 & & & & & & \\
\text{s.t.} & x_1 & + & 2\,x_2 & + & 2\,x_3 & + x_4 & & & & = & 20 \\
 & 2\,x_1 & + & x_2 & + & 2\,x_3 & & + x_5 & & & = & 20 \\
 & 2\,x_1 & + & 2\,x_2 & + & x_3 & & & + x_6 & & = & 20 \\
 & & & & & & & & x_1, \ldots, x_6 & & \geq & 0
\end{array}
$$

Observation

The right-hand side of the system is non-negative. Therefore the point $(0, 0, 0, 20, 20, 20)^T$ is a basic feasible solution and we can start the simplex method with basis $B(1) = 4$, $B(2) = 5$, $B(3) = 6$.

Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	0	-10	-12	-12	0	0	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column (e. g., take smallest subscript rule).

Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	$\frac{x_{B(i)}}{u_i}$
	0	− 10	− 12	− 12	0	0	0	
$x_4 =$	20	1	2	2	1	0	0	20
$x_5 =$	20	2	1	2	0	1	0	10
$x_6 =$	20	2	2	1	0	0	1	10

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.

Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6	$\frac{x_{B(i)}}{u_i}$
	0	− 10	− 12	− 12	0	0	0	
$x_4 =$	20	1	2	2	1	0	0	20
$x_5 =$	20	2	1	2	0	1	0	10
$x_6 =$	20	2	2	1	0	0	1	10

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\Longrightarrow$ x_5 leaves the basis.

Setting Up the Simplex Tableau

	x_1	x_2	x_3	x_4	x_5	x_6
0	-10	-12	-12	0	0	0
$x_4 =$ 20	1	2	2	1	0	0
$x_5 =$ 20	2	1	2	0	1	0
$x_6 =$ 20	2	2	1	0	0	1

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Setting Up the Simplex Tableau

	x_1	x_2	x_3	x_4	x_5	x_6
100	0	-7	-2	0	5	0
$x_4 =$ 20	1	2	2	1	0	0
$x_5 =$ 20	2	1	2	0	1	0
$x_6 =$ 20	2	2	1	0	0	1

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	20	1	2	2	1	0	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
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Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	20	2	2	1	0	0	1

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	0	0	1	-1	0	-1	1

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_5 =$	20	2	1	2	0	1	0
$x_6 =$	0	0	1	-1	0	-1	1

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.

Setting Up the Simplex Tableau

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- Determine pivot column (e. g., take smallest subscript rule).
- $\bar{c}_1 < 0$ and x_1 enters the basis.
- Find pivot row with $u_i > 0$ minimizing $\frac{x_{B(i)}}{u_i}$.
- Rows 2 and 3 both attain the minimum.
- Choose $i = 2$ with $B(i) = 5$. $\implies$ x_5 leaves the basis.
- Perform basis change: Eliminate other entries in the pivot column.
- Obtain new basic feasible solution $(10, 0, 0, 10, 0, 0)^T$ with cost -100.

Geometric Interpretation in the Original Polyhedron

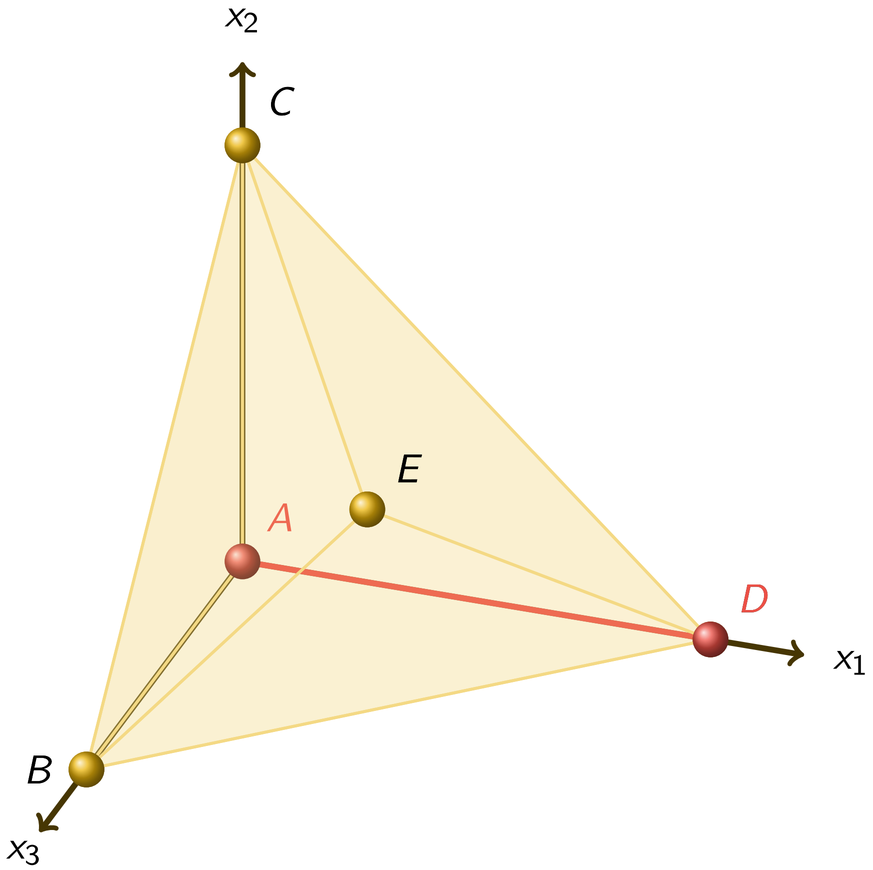
$A = (0, 0, 0)^T$

$B = (0, 0, 10)^T$

$C = (0, 10, 0)^T$

$D = (10, 0, 0)^T$

$E = (4, 4, 4)^T$



Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- $\bar{c}_2, \bar{c}_3 < 0 \implies$ two possible choices for pivot column.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6	$\frac{x_{B(i)}}{u_i}$
	100	0	-7	-2	0	5	0	
$x_4 =$	10	0	1.5	1	1	-0.5	0	10
$x_1 =$	10	1	0.5	1	0	0.5	0	10
$x_6 =$	0	0	1	-1	0	-1	1	—

- $\bar{c}_2, \bar{c}_3 < 0 \implies$ two possible choices for pivot column.
- Choose x_3 to enter the new basis.
- $u_3 < 0 \implies$ third row cannot be chosen as pivot row.
- Choose x_4 to leave basis.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6
	100	0	-7	-2	0	5	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- $\bar{c}_2, \bar{c}_3 < 0 \implies$ two possible choices for pivot column.
- Choose x_3 to enter the new basis.
- $u_3 < 0 \implies$ third row cannot be chosen as pivot row.
- Choose x_4 to leave basis.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- $\bar{c}_2, \bar{c}_3 < 0 \implies$ two possible choices for pivot column.
- Choose x_3 to enter the new basis.
- $u_3 < 0 \implies$ third row cannot be chosen as pivot row.
- Choose x_4 to leave basis.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	10	1	0.5	1	0	0.5	0
$x_6 =$	0	0	1	-1	0	-1	1

- $\bar{c}_2, \bar{c}_3 < 0 \implies$ two possible choices for pivot column.
- Choose x_3 to enter the new basis.
- $u_3 < 0 \implies$ third row cannot be chosen as pivot row.
- Choose x_4 to leave basis.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	0	0	1	-1	0	-1	1

- $\bar{c}_2, \bar{c}_3 < 0 \implies$ two possible choices for pivot column.
- Choose x_3 to enter the new basis.
- $u_3 < 0 \implies$ third row cannot be chosen as pivot row.
- Choose x_4 to leave basis.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	0	0	1	-1	0	-1	1

- $\bar{c}_2, \bar{c}_3 < 0 \implies$ two possible choices for pivot column.
- Choose x_3 to enter the new basis.
- $u_3 < 0 \implies$ third row cannot be chosen as pivot row.
- Choose x_4 to leave basis.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_4 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5	0	1	-1.5	1

- $\bar{c}_2, \bar{c}_3 < 0 \implies$ two possible choices for pivot column.
- Choose x_3 to enter the new basis.
- $u_3 < 0 \implies$ third row cannot be chosen as pivot row.
- Choose x_4 to leave basis.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5	0	1	-1.5	1

- $\bar{c}_2, \bar{c}_3 < 0 \implies$ two possible choices for pivot column.
- Choose x_3 to enter the new basis.
- $u_3 < 0 \implies$ third row cannot be chosen as pivot row.
- Choose x_4 to leave basis.
- New basic feasible solution $(0, 0, 10, 0, 0, 10)^T$ with cost -120, corresponding to point B in the original polyhedron.

Geometric Interpretation in the Original Polyhedron

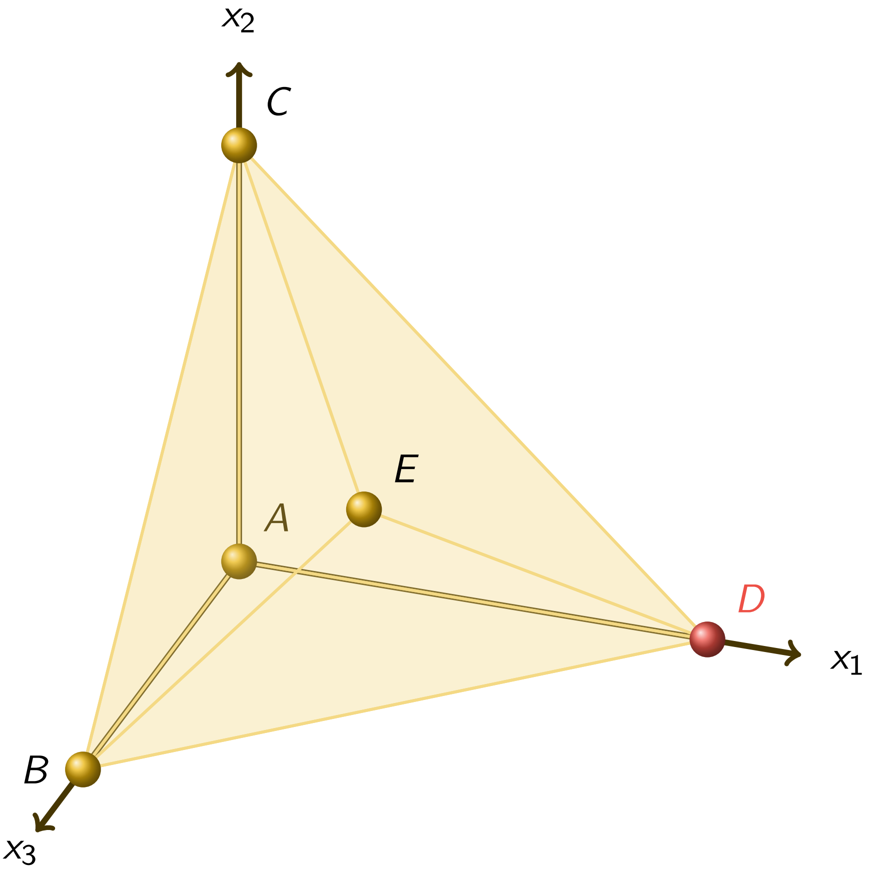
$A = (0, 0, 0)^T$

$B = (0, 0, 10)^T$

$C = (0, 10, 0)^T$

$D = (10, 0, 0)^T$

$E = (4, 4, 4)^T$



Geometric Interpretation in the Original Polyhedron

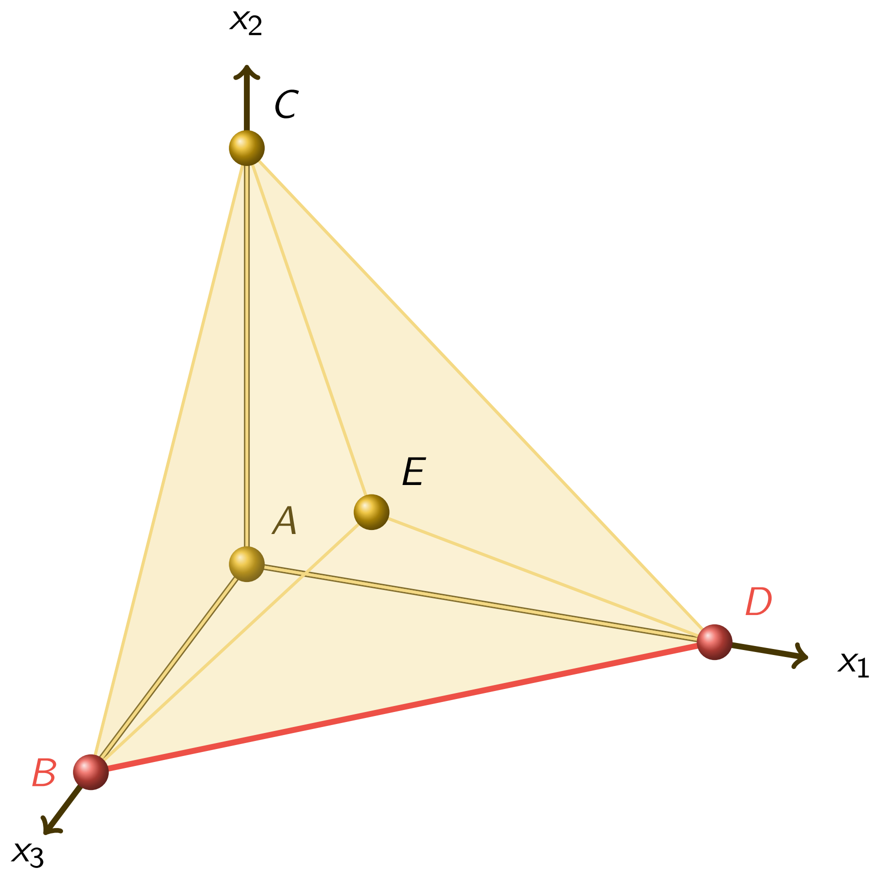
$A = (0, 0, 0)^T$

$B = (0, 0, 10)^T$

$C = (0, 10, 0)^T$

$D = (10, 0, 0)^T$

$E = (4, 4, 4)^T$



Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6	$\frac{x_{B(i)}}{u_i}$
	120	0	-4	0	2	4	0	
$x_3 =$	10	0	1.5	1	1	-0.5	0	$\frac{20}{3}$
$x_1 =$	0	1	-1	0	-1	1	0	—
$x_6 =$	10	0	2.5	0	1	-1.5	1	4

Thus $(4, 4, 4, 0, 0, 0)$ is an optimal solution with cost -136, corresponding to point $E = (4, 4, 4)$ in the original polyhedron.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6	$\frac{x_{B(i)}}{u_i}$	
	120	0	-4	0	2	4	0		
$x_3 =$	10	0	1.5	1	1	-0.5	0	$\frac{20}{3}$	
$x_1 =$	0	1	-1	0	-1	1	0	—	
$x_6 =$	10	0	2.5	0	1	-1.5	1	4	$< \frac{20}{3}$

Thus $(4, 4, 4, 0, 0, 0)$ is an optimal solution with cost -136, corresponding to point $E = (4, 4, 4)$ in the original polyhedron.

Next Iterations

		x_1	x_2	x_3	x_4	x_5	x_6
	120	0	-4	0	2	4	0
$x_3 =$	10	0	1.5	1	1	-0.5	0
$x_1 =$	0	1	-1	0	-1	1	0
$x_6 =$	10	0	2.5	0	1	-1.5	1

x_2 enters the basis, x_6 leaves it. We get

		x_1	x_2	x_3	x_4	x_5	x_6
	136	0	0	0	3.6	1.6	1.6
$x_3 =$	4	0	0	1	0.4	0.4	-0.6
$x_1 =$	4	1	0	0	-0.6	0.4	0.4
$x_2 =$	4	0	1	0	0.4	-0.6	0.4

and the reduced costs are all non-negative.

Thus $(4, 4, 4, 0, 0, 0)$ is an optimal solution with cost -136, corresponding to point $E = (4, 4, 4)$ in the original polyhedron.

All Iterations from Geometric Point of View

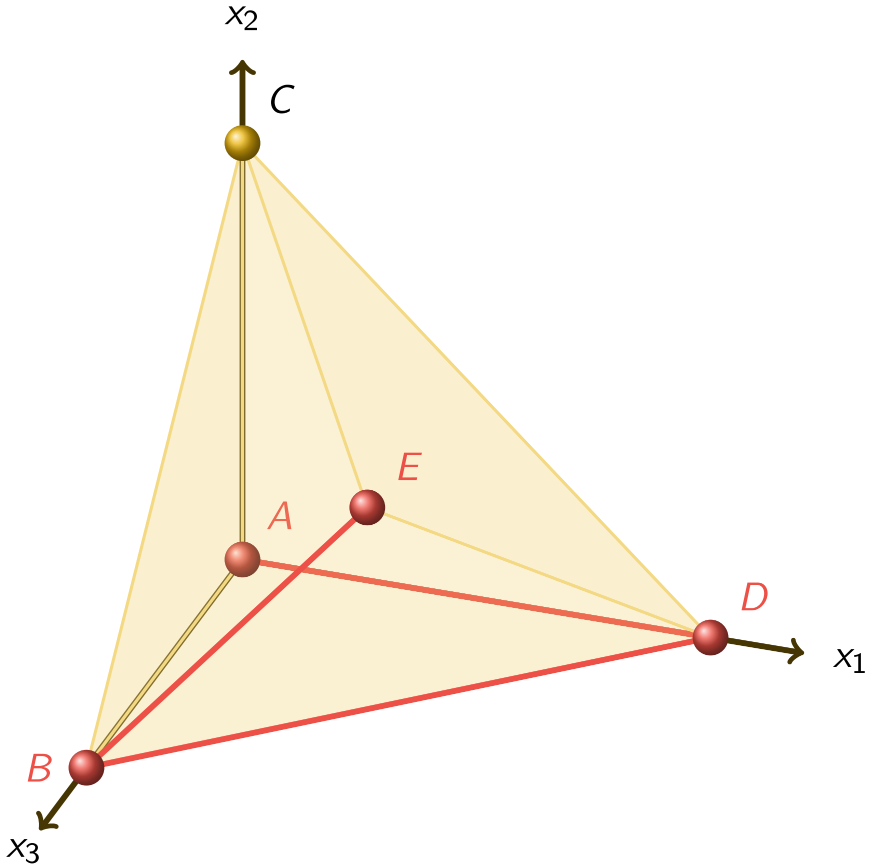
$A = (0, 0, 0)^T$

$B = (0, 0, 10)^T$

$C = (0, 10, 0)^T$

$D = (10, 0, 0)^T$

$E = (4, 4, 4)^T$



Comparison of Full Tableau and Revised Simplex Methods

The following table gives the computational cost of one iteration of the simplex method for the two variants introduced above.

	full tableau	revised simplex
memory	$O(mn)$	$O(m^2)$
worst-case time	$O(mn)$	$O(mn)$
best-case time	$O(mn)$	$O(m^2)$

Conclusion

- For implementation purposes, the revised simplex method is preferable due to its smaller memory requirement and average running time.
- The full tableau method is convenient for solving small LP instances by hand since all necessary information is readily available.

Practical Performance Enhancements

Numerical stability

The most critical issue when implementing the (revised) simplex method is numerical stability. In order to deal with this, a number of additional ideas from numerical linear algebra are needed.

- Every update of B^{-1} introduces roundoff or truncation errors which accumulate and might eventually lead to highly inaccurate results.
  Solution: Compute the matrix B^{-1} from scratch once in a while.

- Instead of computing B^{-1} explicitly, it can be stored as a product of matrices $Q_k \cdot Q_{k-1} \cdot \ldots \cdot Q_1$ where each matrix Q_i can be specified in terms of m coefficients. Then $\overline{B}^{-1} = Q_{k+1} \cdot B^{-1} = Q_{k+1} \cdot \ldots \cdot Q_1$. This might also save space.

- Instead of computing B^{-1} explicitly, compute and store an LR-decomposition.