Exercise 9.1

(a) The Lagrange function is given by $\mathcal{L}(y,y') = \frac{1}{\sqrt{2g}}\sqrt{\frac{1+y'^2}{y}}$. The Euler-Lagrange equation is defined as $E\mathcal{L} = 0$, and $E\mathcal{L}$ is $\frac{\partial\mathcal{L}}{\partial y} - \frac{d}{dx}\left(\frac{\partial\mathcal{L}}{\partial y'}\right)$. First, we calculate $\frac{\partial\mathcal{L}}{\partial y}$ and we get

$$\frac{\partial\mathcal{L}}{\partial y} = \frac{\partial}{\partial y}\frac{1}{\sqrt{2g}}\sqrt{\frac{1+y'^2}{y}} = \frac{1}{2\sqrt{2g}}\sqrt{\frac{y}{1+y'^2}}\cdot\left(-\frac{1+y'^2}{y^2}\right) = -\frac{1}{2\sqrt{2g}}\sqrt{\frac{1+y'^2}{y^3}} \overset{g=0.5}{=} -\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}}$$

For $\frac{\partial\mathcal{L}}{\partial y'}$ we calculate

$$\frac{\partial\mathcal{L}}{\partial y'} = \frac{\partial}{\partial \dot{y}}\frac{1}{\sqrt{2g}}\sqrt{\frac{1+y'^2}{y}} = \frac{1}{2\sqrt{2g}}\sqrt{\frac{y}{1+y'^2}}\frac{2y'}{y} = \frac{1}{\sqrt{2g}\sqrt{y}}\frac{y'}{\sqrt{1+y'^2}} \overset{g=0.5}{=} \frac{1}{\sqrt{y}}\frac{y'}{\sqrt{1+y'^2}}.$$

Let's derive the last term with respect to x. This will be fun!

$$\begin{aligned}\frac{d}{dx}\frac{y'}{\sqrt{y(1+y'^2)}} = \frac{y''\sqrt{y(1+y'^2)} - \frac{y'(y'+y'^3+2yy'y'')}{2\sqrt{y(1+y'^2)}}}{y(1+y'^2)} &= \frac{\frac{2y''y(1+y'^2)-y'^2-y'^4-2yy'^2y''}{2\sqrt{y(1+y'^2)}}}{y(1+y'^2)} \\ &= \frac{2yy''(1+y'^2)-y'^2-y'^4-2yy'^2y''}{2\sqrt{y^3(1+y'^2)^3}}\end{aligned}$$

Substitute everything in $E\mathcal{L} = 0$ yields the equation

$$\begin{aligned} -\frac{1}{2}\sqrt{\frac{1+y'^2}{y^3}} &= \frac{2yy''(1+y'^2)-y'^2-y'^4-2yy'^2y''}{2\sqrt{y^3(1+y'^2)^3}} \\ &\Downarrow \text{Multiply by } 2\sqrt{y^3} \\ -(1+y'^2) &= \frac{2yy''(1+y'^2)-y'^2-y'^4-2yy'^2y''}{\sqrt{(1+y'^2)^3}} \\ &\Downarrow \text{Multiply by } \sqrt{(1+y'^2)^3} \\ -(1+y'^2)^2 &= 2yy''(1+y'^2)-y'^2-y'^4-2yy'^2y'' \\ &\Downarrow \\ -1-2y'^2-y'^4 &= 2yy''+2yy'^2y''-y'^2-y'^4-2yy'^2y'' \\ &\Downarrow \\ -1-2y'^2 &= 2yy''-y'^2 \\ &\Downarrow \\ 0 &= y'^2+2yy''+1 \end{aligned}$$

The **Euler Lagrange equation** is therefore $0 = y'^2 + 2yy'' + 1$.

(b) We multiply the Euler Lagrange equation by y' and obtain $0 = y'^3 + 2yy'y'' + y'$. If we derive $y(1+y'^2)$, we see that $\frac{d}{dx}y(1+y'^2) = y'(1+y'^2) + 2yy'y'' = y' + y'^3 + 2yy'y''$. The derivation of $y(1+y'^2)$ is the right hand side of the Euler Lagrange equation. Thus we obtain

$$\frac{d}{dx}y(1+y'^2) = 0 \iff y(1+y'^2) = c, \quad c \in \mathbb{R}.$$

Transforming the equation yields

$$y' = \sqrt{\frac{c-y}{y}}$$

With separation of variables we get

$$\frac{d}{dx}y = \sqrt{\frac{c-y}{y}} \implies \sqrt{\frac{y}{c-y}}dy = dx \implies \int \sqrt{\frac{y}{c-y}}dy = x$$

(c) Substitute $y \coloneqq c\sin^2\varphi$. Then $dy = 2c\sin\varphi\cos\varphi d\varphi$. We obtain

$$x = \int \sqrt{\frac{c\sin^2\varphi}{c(1-\sin^2\varphi)}}2c\sin\varphi\cos\varphi d\varphi = \int \frac{\sin\varphi}{\cos\varphi}2c\sin\varphi\cos\varphi d\varphi = 2c\int \sin^2\varphi d\varphi.$$

Since $\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi)$ we get

$$x(\varphi) = c\int 1-\cos 2\varphi d\varphi = c(\varphi - \frac{1}{2}\sin 2\varphi) + d, \quad d \in \mathbb{R}$$

and

$$y(\varphi) = c\sin^2(\varphi) = c(\frac{1}{2} - \frac{1}{2}\cos 2\varphi)$$

Since $y(\varphi) = 0$ and $x(\varphi) = 0$, it follows that $d = 0$. The parametrised curve γ, which minimises T, is given by

$$\varphi \mapsto \begin{pmatrix} c(\varphi - \frac{1}{2}\sin 2\varphi) \\ c(\frac{1}{2} - \frac{1}{2}\cos 2\varphi) \end{pmatrix}.$$

The constant c is chosen such that (a, b) lies on the graph of γ.

Exercise 9.2

(a) $A(x)_{i,j}$ denotes the i,j-entry of the matrix $A(x)$.

$$\mathcal{L}(x,\dot{x}) = \frac{1}{2}\langle \dot{x}, A(x)\dot{x}\rangle - U(x) = \frac{1}{2}\sum_{i,j=1}^{n} A(x)_{i,j}\dot{x}_i\dot{x}_j - U(x).$$

The Euler Lagrange equation is $E\mathcal{L}_i = \frac{\partial\mathcal{L}}{\partial x_i} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}_i}) = 0$ for all $i = 1, ..., n$. First, we compute $\frac{\partial\mathcal{L}}{\partial x_i}$.

$$\frac{\partial\mathcal{L}}{\partial x_i} = \frac{1}{2}\sum_{h,j=1}^{n}\left(\frac{\partial}{\partial x_i}A(x)\right)_{h,j}\dot{x}_h\dot{x}_j - \frac{\partial U}{\partial x_i}.$$

Next,

$$\frac{\partial\mathcal{L}}{\partial \dot{x}_i} = \frac{1}{2}\sum_{j=1}^{n} A(x)_{i,j}\dot{x}_j.$$

Thus,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial \dot{x}_i} = \frac{d}{dt}\frac{1}{2}\sum_{j=1}^{n} A(x)_{i,j}\dot{x}_j = \frac{1}{2}\sum_{j=1}^{n}[\left(\frac{d}{dt}A(x)\right)_{i,j}\dot{x}_j + A(x)_{i,j}\ddot{x}_j].$$

Finally, the Euler Lagrange equation is

$$\sum_{h,j=1}^{n}\left(\frac{\partial}{\partial x_i}A(x)\right)_{h,j}\dot{x}_h\dot{x}_j - \frac{\partial U}{\partial x_i} = \sum_{j=1}^{n}[\left(\frac{d}{dt}A(x)\right)_{i,j}\dot{x}_j + A(x)_{i,j}\ddot{x}_j].$$

(b) From the lecture we know that $\tilde{E}(x,\dot{x}) = \sum_{i=1}^{n}\dot{x}_i\frac{\partial \mathcal{L}}{\partial \dot{x}_i} - \mathcal{L}(x,\dot{x})$ is a conserved quantitiy. We show that $\tilde{E}(x,\dot{x}) = \sum_{i=1}^{n}\dot{x}_i\frac{\partial L}{\partial \dot{x}_i} - \mathcal{L}(x,\dot{x}) = 0.5\langle\dot{x}, A(x)\dot{x}\rangle + U(x) = E(x,\dot{x})$. Thus, $E(x,\dot{x})$ is a conserved quantity.

$$\begin{aligned}\tilde{E}(x,\dot{x}) = \sum_{i=1}^{n}\dot{x}_i\frac{\partial L}{\partial \dot{x}_i} - \mathcal{L}(x,\dot{x}) &= \underbrace{\sum_{i=1}^{n}\dot{x}_i(\sum_{j=1}^{n}A(x)_{ij}\dot{x}_j)}_{=\langle\dot{x},A(x)\dot{x}\rangle} - 0.5\langle\dot{x}, A(x)\dot{x}\rangle + U(x)\\ &= \langle\dot{x}, A(x)\dot{x}\rangle - 0.5\langle\dot{x}, A(x)\dot{x}\rangle + U(x)\\ &= \frac{1}{2}\langle\dot{x}, A(x)\dot{x}\rangle + U(x)\\ &= E(x,\dot{x}).\end{aligned}$$

(c) We must show that $\langle\dot{x}, \Delta_{\dot{x}}\mathcal{L}(x,\dot{x})\rangle - \mathcal{L}(x,\dot{x}) = E(x,\dot{x})$. For the gradient it holds

$$\Delta_{\dot{x}}\mathcal{L} = \frac{1}{2}\Delta_{\dot{x}}\left(\sum_{i,j=1}^{n}A(x)_{i,j}\dot{x}_i\dot{x}_j - U(x)\right) = \frac{1}{2}\Delta_{\dot{x}}\sum_{i,j=1}^{n}A(x)_{i,j}\dot{x}_i\dot{x}_j = \frac{1}{2}\begin{pmatrix}\sum_{j=1}^{n}A(x)_{1,j}\dot{x}_j\\ \vdots\\ \sum_{j=1}^{n}A(x)_{n,j}\dot{x}_j\end{pmatrix}.$$

So it follows that

$$\begin{aligned}\langle\dot{x}, \Delta_{\dot{x}}\mathcal{L}(x,\dot{x})\rangle = \frac{1}{2}(\sum_{j=1}^{n}A(x)_{1,j}\dot{x}_j\dot{x}_1 + \sum_{j=1}^{n}A(x)_{2,j}\dot{x}_j\dot{x}_2 + ... + \sum_{j=1}^{n}A(x)_{n,j}\dot{x}_j\dot{x}_n) &= \frac{1}{2}\sum_{i,j=1}^{n}A(x)_{i,j}\dot{x}_i\dot{x}_j\\ &= \frac{1}{2}\langle\dot{x}, A(x)\dot{x}\rangle.\end{aligned}$$

We obtain $\frac{1}{2}\langle\dot{x}, A(x)\dot{x}\rangle - \mathcal{L}(x,\dot{x})$. Now the steps are the same as in 9.2(b) to show the equivalence.

Exercise 9.3

(a) Find the extrema of $\varphi(\gamma) = \int_0^T \dot{q}^2(t)dt$ for $\gamma = \{(t, q(t)) : 0 \leq t \leq T\}$. Use the Euler Lagrange equation. $\mathcal{L}(q,\dot{q}) = \dot{q}^2$.

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}) = 0 \iff -\frac{d}{dt}(2\dot{q}) = 0 \iff 2\ddot{q} = 0.$$

Thus, $q(t) = mt + n$ for $m, n \in \mathbb{R}$. We have the constraints $q(0) = 0 \implies n = 0$ and $q(T) = a = mT \implies m = \frac{a}{T}$. The solution curve γ_0 is given by

$$q_0(t) := \frac{a}{T}t.$$

It is a minimum, for we will show that $\varphi(\gamma_0 + h) - \varphi(\gamma_0) \geq 0$ for all curves h with $h(0) = h(T) = 0$. It holds

$$\varphi(\gamma_0) = \int_0^T \frac{a^2}{T^2} dt = \frac{a}{T}$$

and

$$\varphi(\gamma_0 + h) = \int_0^T (\frac{a}{T} + h'(t))^2 dt = \int_0^T \frac{a^2}{T^2} + 2\frac{a}{T} h'(t) + (h'(t))^2 dt = \frac{a}{T} + 2\frac{a}{T}(h(T) - h(0)) + \int_0^T (h'(t))^2 dt.$$

If we calculate the difference, we get

$$\varphi(\gamma_0 + h) - \varphi(\gamma_0) = 2\frac{a}{T}(\underbrace{h(T) - h(0)}_{=0}) + \underbrace{\int_0^T (h'(t))^2 dt}_{\geq 0} \geq 0.$$

Thus, γ_0 is a minimum.