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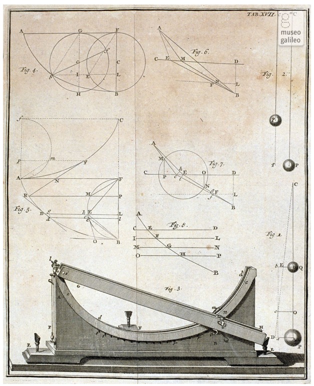
Mathematical Physics I - WS 2018/2019

http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/MP1/

Exercise Sheet 9

Exercise 1: (7 pts)

We look for the optimal shape of a wire that connects two fixed points A and B on a vertical plane. A bead of unit mass falls along this wire, without friction, under the influence of gravity. The shape of the wire is defined to be optimal if the bead falls from A to B in as short a time as possible.
Let $y = y(x)$ be the function which describes the shape of the wire on the (x, y)-plane, connecting $A := (0, 0)$ and $B := (a, b)$ with $a > 0$ and $b \geq 0$. We assume that the positive y-axis is pointing downward.



The associated falling time follows from elementary mechanics. It reads

$$T(y) = \frac{1}{\sqrt{2\,g}} \int_0^a \sqrt{\frac{1 + (y')^2}{y}}\, \mathrm{d}x, \qquad y' := \frac{\mathrm{d}y}{\mathrm{d}x}.$$

Here g is the constant gravitational acceleration (fix $g = 1/2$). To solve the problem one has to minimize the functional T over the set of all functions $y \in C^1([0, a], (0, \infty))$ with $(y(0), y(a)) = (0, b)$.

(i) Construct the Euler-Lagrange equation of the problem.

(ii) Reduce the second-order ODE obtained in 1. to a first-order ODE of the form

$$y\big(1 + (y')^2\big) = c,$$

where $c \in \mathbb{R}$ is a constant of integration.

(iii) Introduce the angular variable φ, which measures the angle that the tangent to the curve makes with the vertical. Find the family of parametric equations for the plane curve $y = y(x)$ which minimizes T (φ is the parameter).

Exercise 2: (6 pts)

In $\mathbb{R}^n$ consider the Lagrangian system with Lagrange function

$$\mathscr{L}(q,\dot{q}) = \frac{1}{2}\langle \dot{q}, A(q)\,\dot{q}\rangle - U(q).$$

where $k = 1, \dots, n$, $A := (A_{ij})_{1\le i,j\le n} \in C^2(\mathbb{R}^n, \mathbf{GL}(n,\mathbb{R}))$ is symmetric and positive definite and $U \in C^2(\mathbb{R}^n, \mathbb{R})$ is the potential energy.

(i) Derive the corresponding Euler-Lagrange equations.

(ii) Prove that the total energy

$$E(q,\dot{q}) = \frac{1}{2}\langle \dot{q}, A(q)\,\dot{q}\rangle + U(q)$$

is an integral of motion.

(iii) Prove that
$$E(q,\dot{q}) = \langle \dot{q}, \mathsf{grad}_{\dot{q}}\mathscr{L}(q,\dot{q})\rangle - \mathscr{L}(q,\dot{q}).$$

Exercise 3: (7 pts)

(i) Fix $T, \alpha > 0$. Consider the functional $\psi : K \to \mathbb{R}$ defined by

$$\psi(\gamma) := \int_0^T \dot{q}^2 \,\mathsf{d}t,$$

where K is the space of all C^1-curves

$$\gamma := \left\{(t,q) \,:\, q = q(t),\, q \in C^1([0,T],\mathbb{R}),\, q(0) = 0,\, q(T) = \alpha\right\}.$$

Find an extremal point of ψ. Is this extremal point a candidate to be a maximum or a minimum?

(ii) Consider the functional $\psi : K \to \mathbb{R}$ defined by

$$\psi(\gamma) := \int_0^1 \sqrt{q^2 + \dot{q}^2}\,\mathsf{d}t,$$

where K is the space of all C^1-curves

$$\gamma := \left\{(t,q) \,:\, q = q(t),\, q \in C^1([0,1],\mathbb{R}),\, q(0) = 0,\, q(1) = 1\right\}.$$

Prove that $\psi(\gamma) > 1$ for all $\gamma \in K$.

(iii) Consider a string of length ℓ with one end fixed at the origin of the (x,y)-plane, and the other end on the x-axis. Find the shape of the string maximizing the area enclosed between the string and the x-axis.

Due Monday, January 7.