Optimality of Extreme Points

Theorem 3.31.

Let $P \subseteq \mathbb{R}^n$ a pointed polyhedron and $c \in \mathbb{R}^n$. If $\min\{c^T \cdot x \mid x \in P\}$ is bounded, there is a vertex that is optimal.

Proof:. . . □

Corollary 3.32.

Every linear programming problem is either infeasible or unbounded or there exists an optimal solution.

Proof: Every linear program is equivalent to an LP in standard form. The claim thus follows from Corollary 3.24 and Theorem 3.31. □

Proof of Theorem 3.31:

wrong proof: Let $\gamma \overset{?}{:=} \min\{c^T x \mid x \in P\}$ and consider face $F := \{x \in P \mid c^T x = \gamma\}$. Since P is pointed, F is pointed and contains vertex which is optimal LP solution.

is this an inf?

correct proof: We show that for every $x_0 \in P$ there is a vertex $y \in P$ with $c^T y \le c^T x_0$.

Assume that $P = \{x \in \mathbb{R}^n \mid a_i^T x \ge b_i, i \in M\}$

Let $x_0 \in P$ and $I := \{i \in M \mid a_i^T x_0 = b_i\}$. If $\operatorname{rank}\{a_i \mid i \in I\} = n$, then x_0 is a vertex, set $y := x_0$.

Otherwise, there is $d \in \mathbb{R}^n \setminus \{0\}$ with $a_i^T d = 0 \ \forall i \in I$.

Assume that $c^T \cdot d \le 0$ (otherwise replace with $-d$).

1. Case: $c^T d < 0 \Rightarrow c^T(x_0 + \lambda \cdot d) \xrightarrow{\lambda \to \infty} -\infty$

Thus, there is $j \notin I$: $a_j^T d < 0$

Let $\lambda_0 := \min\limits_{j : a_j^T d < 0} \frac{b_j - a_j^T x_0}{a_j^T \cdot d}$

$a_j^T\left(x_0 + \frac{b_j - a_j^T x_0}{a_j^T d} \cdot d\right) = \cancel{a_j^T x_0} + b_j - \cancel{a_j^T x_0} = b_j$

Then $x_0 + \lambda_0 \cdot d \in P$ and $\{i \mid a_i^T(x_0 + \lambda_0 d) = b_i\} \supseteq I \cup \{j_0\}$ where $\lambda_0 = \frac{b_{j_0} - a_{j_0}^T x_0}{a_{j_0}^T \cdot d}$. Thus, we have found a point $x_0 + \lambda_0 \cdot d$ with more linearly indep. active constraints and $c^T(x_0 + \lambda_0 d) \le c^T x_0$

Repeat this process until you find vertex.

2. Case: $c^T \cdot d = 0 \Rightarrow c^T(x_0 + \lambda \cdot d) = c^T x_0 \ \forall \lambda \in \mathbb{R}$

Since P does not contain a line, we must find an additional constraint in direction d or $-d$. Proceed as above in Case 1. ... $\square$

Number of Vertices

Corollary 3.33.

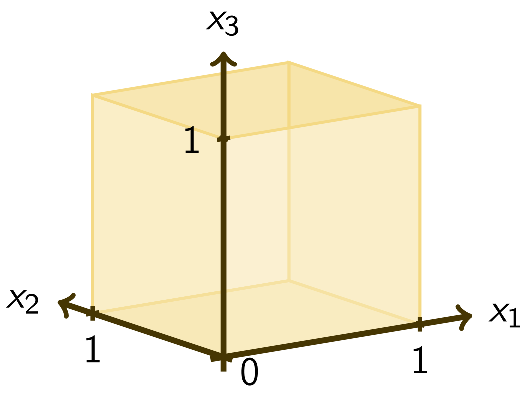
a) A polyhedron has a finite number of vertices and basic solutions.

b) For a polyhedron in $\mathbb{R}^n$ given by m linear inequalities, this number is at most $\binom{m}{n}$.

Example:

$P := \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \ i = 1, \dots, n\}$
(n-dimensional unit cube)

- number of constraints: $m = 2n$
- number of vertices: 2^n



Adjacent Basic Solutions and Edges

Definition 3.34.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

a) Two distinct basic solutions are adjacent if there are $n-1$ linearly independent constraints that are active at both of them.

b) If both solutions are feasible, the line segment that joins them is an edge of P.

Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$ a polyhedron in standard form representation.

Observation.
One can assume without loss of generality that $\text{rank}(A) = m$.

Proof: Let $a_1, \dots, a_m \in \mathbb{R}^n$ rows of A. Assume that $a_i = \sum_{j\neq i} \lambda_j \cdot a_j$.

Case 1: $b_i = \sum_{j\neq i} \lambda_j\, b_j$. Then,

$$a_j{}^T \cdot x = b_j \quad \forall j \neq i \quad \Longrightarrow \quad a_i{}^T \cdot x = \sum_{j\neq i} \lambda_j \cdot (a_j{}^T \cdot x) = \sum_{j\neq i} \lambda_j\, b_j = b_i.$$

Thus the ith constraint is redundant and can be deleted.

Case 2: $b_i \neq \sum_{j\neq i} \lambda_j\, b_j \quad \Longrightarrow \quad A \cdot x = b$ has no solution. $\square$

Basic Solutions of Polyhedra in Standard Form

Let $A \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(A) = m$, $b \in \mathbb{R}^m$, and

$$P = \{x \in \mathbb{R}^n \mid A \cdot x = b,\ x \geq 0\}$$

Theorem 3.35.

A point $x \in \mathbb{R}^n$ is a basic solution of P if and only if $A \cdot x = b$ and there are indices $B(1), \dots, B(m) \in \{1, \dots, n\}$ such that

- columns $A_{B(1)}, \dots, A_{B(m)}$ of matrix A are linearly independent, and
- $x_i = 0$ for all $i \notin \{B(1), \dots, B(m)\}$.

Proof: ... □

- $x_{B(1)}, \dots, x_{B(m)}$ are basic variables, the remaining variables non-basic.
- The vector of basic variables is denoted by $x_B := (x_{B(1)}, \dots, x_{B(m)})^T$.
- $A_{B(1)}, \dots, A_{B(m)}$ are basic columns of A and form a basis of $\mathbb{R}^m$.
- The matrix $B := (A_{B(1)}, \dots, A_{B(m)}) \in \mathbb{R}^{m \times m}$ is called basis matrix.

Proof of Theorem 3.35:

„$\Rightarrow$": $Ax = b$ clear from def. There are n linearly indep. constraints active at x, namely $Ax = b$ and $x_i = 0$ for $i \in \{1, \dots, n\} \setminus \{B(1), \dots, B(m)\}$

$$\begin{aligned} Ax &= b \\ I_n x &\geq 0 \end{aligned} \quad \text{where } I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

W.l.o.g. $B(i) = i \ \forall i \in \{1, \dots, m\}$ (reindex variables).

Active constraints (linearly indep.):

$$(*) \quad \begin{array}{c} m\{ \\ n-m\{ \end{array} \underbrace{\begin{pmatrix} A_1\ A_2 \cdots A_m & * \\ 0 & I_{n-m} \end{pmatrix}}_{\text{full rank matrix } \in \mathbb{R}^{n \times n}} \begin{pmatrix} x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

full rank matrix $\in \mathbb{R}^{n \times n}$ with determinant $\det((A_1, \dots, A_m)) \cdot \det(I_{n-m}) \neq 0$

$\Rightarrow A_1, \dots, A_m$ linearly indep.

„$\Leftarrow$": By assumption we have n active constr. at x. Since $A_{B(1)}, \dots, A_{B(m)}$ are linearly indep., the matrix $(*)$ has full rank. $\Rightarrow$ The n active constr. are linearly indep. □

Corollary: Carathéodory's Theorem

Theorem 3.36 (Carathéodory 1911).

For $S \subseteq \mathbb{R}^n$, every element of $\mathrm{conv}(S)$ can be written as a convex combination of at most $n+1$ points in S.

Proof: Consider $x \in \mathrm{conv}(S)$. Then x can be written as

$$x = \sum_{i=1}^{k} \lambda'_i \, y_i \qquad \text{with } y_1, \dots, y_k \in S \text{ and } \sum_{i=1}^{k} \lambda'_i = 1,\ \lambda' \geq 0.$$

Consider the following polyhedron in standard form:

$$P = \left\{ \lambda \in \mathbb{R}^k \;\middle|\; \underbrace{\sum_{i=1}^{k} \lambda_i \, y_i = x,\ \ \sum_{i=1}^{k} \lambda_i = 1,}_{n+1 \text{ equality constraints}}\ \lambda \geq 0 \right\}.$$

A basic feasible solution $\lambda^* \in P$ yields the desired representation of x. $\square$