Research Group
Applied Functional Analysis
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Functional Analysis I

Tutorial Assignment 9

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Exercise 1: Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{R}$. Prove that for all $x, y \in E$ it holds

$$(\|x\| + \|y\|) \cdot \langle x, y \rangle \leq \|x + y\| \cdot \|x\| \|y\|.$$

Exercise 2: Prove that a scalar product $\langle \cdot, \cdot \rangle$ on ℓ_2 is defined via

$$\langle (x_n)_n, (y_n)_n \rangle := \sum_{n \in \mathbb{N}} x_n \overline{y_n} \quad \text{for } (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \in \ell_2.$$

Exercise 3: Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be an inner product space. For $S \subset \mathcal{H}$ define

$$S^{\perp} := \Big\{ x \in \mathcal{H} \ : \ \langle x, y \rangle = 0 \text{ for all } y \in S \Big\}.$$

Show that $S^{\perp}$ is a closed subspace of $\mathcal{H}$. Moreover, prove the following relation:

$$\overline{S}^{\perp} = S^{\perp} = (\operatorname{span} S)^{\perp}.$$

Exercise 4: Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be an inner product space and let $A \subset \mathcal{H}$ be a set whose interior is non-empty. Prove that $A^{\perp} = \{0\}$.

Exercise 5: Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{K}$, and let $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ be Cauchy sequences in $\mathcal{H}$. Prove that then $(\langle f_n, g_n \rangle)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$.

Exercise 6: Let $\mathcal{H}$ be a Hilbert space and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{H}$. Show the following statements:

(a) If $(f(x_n))_{n\in\mathbb{N}}$ converges for each $f \in \mathcal{H}^*$, then there exists $x \in \mathcal{H}$ such that $(x_n)_{n\in\mathbb{N}}$ converges weakly to x.

(b) If $(x_n)_{n\in\mathbb{N}}$ converges weakly to x and $\limsup_{n\to\infty} \|x_n\| \leq \|x\|$, then $(x_n)_{n\in\mathbb{N}}$ converges to x in norm.

Exercise 7: Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator acting on a Hilbert space $\mathcal{H}$. Show if $\langle Tx, y\rangle = \langle x, Ty\rangle$ holds for all $x, y \in \mathcal{H}$, then $T \in L(\mathcal{H})$.

Exercise 8: Let $\mathcal{H}$ be an inner product space which is not complete. Then prove that there exists a closed subspace $U \subsetneq \mathcal{H}$ such that $U^\perp = \{0\}$. *Hint:* Assume the contrary and prove that the Riesz mapping is surjective.