Characterization of Faces

Theorem 3.28.

Consider a polyhedron $P \subseteq \mathbb{R}^n$ be defined by

$$\begin{aligned} {a_i}^T \cdot x &\geq b_i && \text{for } i \in M, \\ {a_i}^T \cdot x &= b_i && \text{for } i \in N, \end{aligned}$$

and let $F \neq \emptyset$ be a face of P.

- a There exists $K \subseteq M$ with $F = \{x \in P \mid {a_i}^T \cdot x = b_i \text{ for } i \in K\}$.
- b For $K \subseteq M$, the subset $\{x \in P \mid {a_i}^T \cdot x = b_i \text{ for } i \in K\}$ is a face of P.
- c $G \subseteq F$ is a face of F if and only if it is a face of P.
- d There is a chain of faces $F = F_0 \subset F_1 \subset \cdots \subset F_q = P$ such that $\dim(F_{i+1}) = \dim(F_i) + 1$, for $i = 0, \ldots, q-1$.

Proof of Theorem 3.28: $F = \{x \in P \mid c^T x = \gamma\}$ with $c^T x \geq \gamma$ valid for P

a) Let $K := \{i \in M \mid a_i^T \cdot x = b_i \text{ for all } x \in F\}$

Claim: $F = \{x \in P \mid a_i^T \cdot x = b_i \text{ for all } i \in K\}$

"$\subseteq$": Clear by definition of K.

"$\supseteq$": Assume by contradiction that $y \in P \setminus F$ with $a_i^T \cdot y = b_i$ for all $i \in K$.

$\Rightarrow c^T \cdot y > \gamma$ since $y \notin F$

For each $i \in M \setminus K$ there is $x^i \in F$ with $a_i^T \cdot x^i > b_i$.

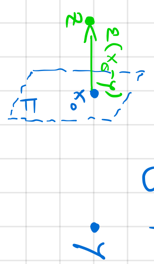
Let $x_0 := \frac{1}{|M \setminus K|} \sum_{i \in M\setminus K} x^i \in F$ (since F convex)

$\Rightarrow c^T x_0 = \gamma$

Notice that $a_i^T x_0 = b_i$ for all $i \in K$ and $a_i^T \cdot x_0 > b_i$ for all $i \in M \setminus K$

(because: $a_i^T x_0 = \frac{1}{|M\setminus K|} \cdot \sum_{j \in M \setminus K} \underbrace{a_i^T x^j}_{\substack{\geq b_i \\ > b_i \text{ for } i=j}} > b_i$)

For $\varepsilon > 0$ small enough, $z := x_0 + \varepsilon(x_0 - y)$



Then $c^T z = \underbrace{c^T x_0}_{=\gamma} + \underbrace{\varepsilon}_{>0} \cdot \underbrace{(\underbrace{c^T x_0}_{=\gamma} - \underbrace{c^T y}_{>\gamma})}_{<0} < \gamma$

$\Rightarrow z \notin P$

On the other hand: $z \in P$ because:

„$\supseteq$": Assume by contradiction that $y \in P \setminus F$ with $a_i^T \cdot y = b_i$ for all $i \in K$.

$\Rightarrow c^T \cdot y > \gamma$ since $y \notin F$

For each $i \in M \setminus K$ there is $x^i \in F$ with $a_i^T \cdot x^i > b_i$.

Let $x_0 := \frac{1}{|M \setminus K|} \sum_{i \in M\setminus K} x^i \in F$ (since F convex)

$\Rightarrow c^T x_0 = \gamma$

Notice that $a_i^T x_0 = b_i$ for all $i \in K$ and

$a_i^T \cdot x_0 > b_i$ for all $i \in M \setminus K$

(because: $a_i^T x_0 = \frac{1}{|M\setminus K|} \cdot \sum_{j \in M\setminus K} \underbrace{a_i^T x^j}_{\substack{\geq b_i \\ > b_i \text{ for } i=j}} > b_i$)

For $\varepsilon > 0$ small enough, $z := x_0 + \varepsilon(x_0 - y)$

Then $c^T z = \underbrace{c^T x_0}_{=\gamma} + \underbrace{\varepsilon}_{>0} \cdot \underbrace{(\underbrace{c^T x_0}_{=\gamma} - \underbrace{c^T y}_{>\gamma})}_{<0} < \gamma$

$\Rightarrow z \notin P$

On the other hand $z \in P$ because: ↯

$a_i^T \cdot z = (1+\varepsilon) \underbrace{a_i^T x_0}_{=b_i} - \varepsilon \underbrace{a_i^T \cdot y}_{=b_i} = b_i$ for $i \in N \cup K$

$a_i^T \cdot z = \underbrace{a_i^T x_0}_{>b_i} + \varepsilon \cdot (a_i^T x_0 - a_i^T \cdot y) \geq b_i$ for $i \in M \setminus K$, ε small $\square$

b) Let $c := \sum_{i \in K} a_i$ and $\gamma := \sum_{i \in K} b_i$

Then $c^T x \geq \gamma$ for all $x \in P$, because

$c^T x = \sum_{i \in K} \underbrace{a_i^T x}_{\geq b_i} \geq \sum_{i \in K} b_i = \gamma$

and: $c^T x = \gamma \Leftrightarrow a_i^T x = b_i \ \forall i \in K$

$\Rightarrow \{x \in P \mid c^T x = \gamma\} = \{x \in P \mid a_i^T x = b_i \ \forall i \in K\}$

c) „$\Leftarrow$": $G = \{x \in P \mid c^T x = \gamma\} \subseteq F$ with $c^T x \geq \gamma$ valid for P

then $G = \{x \in F \mid c^T x = \gamma\}$ and $c^T x \geq \gamma$ valid for F

$\Rightarrow G$ face of F.

„$\Rightarrow$": $F = \{x \mid a_i^T x \geq b_i \ \forall i \in M \setminus K, a_i^T x = b_i \ \forall i \in K \cup N\}$

for some $K \subseteq M$ (by part a)). Since G is face of F, we can write

$G = \{x \mid a_i^T x \geq b_i \ \forall i \in M \setminus L, a_i^T x = b_i \ \forall i \in N \cup L\}$

for some $K \subseteq L \subseteq M$ by part a). Thus

$G = \{x \in P \mid a_i^T x = b_i \ \ \forall i \in L\}$

and G is face of P by b).

d) Follows from a)–c) and Lemma 3.17.

Proof of Theorem 3.28 a

Let $K := \{i \in M \mid {a_i}^T \cdot x = b_i \text{ for all } x \in F\}$.

Claim: $F = \{x \in P \mid {a_i}^T \cdot x = b_i \text{ for } i \in K\}$

'$\subseteq$': Clear by definition of K.

'$\supseteq$': Assume by contradiction that $y \in P \setminus F$ with ${a_i}^T \cdot y = b_i \ \forall\, i \in K$.

Let $c^T \cdot x \geq \gamma$ valid for P such that $F = \{x \in P \mid c^T \cdot x = \gamma\}$.

In particular, $c^T \cdot y > \gamma$ as $y \in P \setminus F$.

For each $i \in M \setminus K$ there is an $x^i \in F$ with ${a_i}^T \cdot x^i > b_i$.

Let $x_0 := \frac{1}{|M \setminus K|} \sum_{i \in M \setminus K} x^i \in F$ (convex), thus $c^T \cdot x_0 = \gamma$.

Notice that ${a_i}^T \cdot x_0 = b_i \ \forall\, i \in K$ and ${a_i}^T \cdot x_0 > b_i \ \forall\, i \in M \setminus K$.

For $\varepsilon > 0$ small enough, $z := x_0 + \varepsilon(x_0 - y) \in P$ because:

$$
{a_i}^T \cdot z = (1+\varepsilon)\, {a_i}^T \cdot x_0 - \varepsilon\, {a_i}^T \cdot y \begin{cases} = b_i & \text{for } i \in N \cup K, \\ \geq b_i & \text{for } i \in M \setminus K \end{cases}
$$

But $c^T \cdot z = (1+\varepsilon)\, c^T \cdot x_0 - \varepsilon\, c^T \cdot y < \gamma$. $\lightning$

Proof of Theorem 3.28 b–d

b) Let $c := \sum_{i \in K} a_i$ and $\gamma := \sum_{i \in K} b_i$.

Then, $c^T \cdot x \geq \gamma$ is a valid inequality for P and for $x \in P$

$$
c^T \cdot x = \gamma \qquad \Longleftrightarrow \qquad {a_i}^T \cdot x = b_i \text{ for all } i \in K.
$$

c) '$\Longleftarrow$': If $G = \{x \in P \mid c^T \cdot x = \gamma\} \subseteq F$ with $c^T \cdot x \geq \gamma$ valid for P,

then $G = \{x \in F \mid c^T \cdot x = \gamma\}$ and $c^T \cdot x \geq \gamma$ valid for F.

'$\Longrightarrow$': $F = \{x \mid {a_i}^T \cdot x \geq b_i \ \forall\, i \in M \setminus K,\ {a_i}^T \cdot x = b_i \ \forall\, i \in K \cup N\}$

for some $K \subseteq M$ due to a). Since G is a face of F, again due to a),

$$
G = \{x \mid {a_i}^T \cdot x \geq b_i \ \forall\, i \in M \setminus L,\ {a_i}^T \cdot x = b_i \ \forall\, i \in L \cup N\}
$$

for some $K \subseteq L \subseteq M$. Thus, due to b), G is a face of P.

d) Follows from a–c and Lemma 3.17. $\square$

Characterization of Faces (Cont.)

Corollary 3.29.

Consider a polyhedron $P \subseteq \mathbb{R}^n$ be defined by

$$\begin{aligned} {a_i}^T \cdot x &\geq b_i && \text{for } i \in M, \\ {a_i}^T \cdot x &= b_i && \text{for } i \in N. \end{aligned}$$

- a P has finitely many distinct faces.
- b If F is a facet of P, then $\dim(F) = \dim(P) - 1$.
- c An inclusion-wise minimal proper face F of P can be written as

  $$F = \{x \in \mathbb{R}^n \mid {a_i}^T \cdot x = b_i \text{ for all } i \in K \cup N\} \qquad \text{for some } K \subseteq M$$

  with $\operatorname{rank}\{a_i \mid i \in K \cup N\} = \operatorname{rank}\{a_i \mid i \in M \cup N\}$.
- d If P is pointed, every minimal nonempty face of P is a vertex.

Proof: Exercise. □

Edges, Extreme Rays, Extreme Lines

Definition 3.30.

A one-dimensional face F of polyhedron P is

- a an edge if F has two vertices, i.e., $F = \operatorname{conv}(\{x, y\})$ with $x, y \in \mathbb{R}^n$, $x \neq y$;
- b an extreme ray if F has one vertex, i.e., $F = x + \operatorname{cone}(\{z\})$ with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n \setminus \{0\}$;
- c an extreme line if F has no vertex, i.e., $F = x + \operatorname{lin}(\{z\})$ with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n \setminus \{0\}$.