Research Group
Applied Functional Analysis
www.math.tu-berlin.de/afg

Functional Analysis I

Homework Assignment 1

Martin Genzel, Mones Raslan Summer Term 2019

Exercise 1: **6 Points**

Two metrics d_1 and d_2 on a set X are called *equivalent* if for each $x \in X$ and each $\varepsilon > 0$ there exists $r > 0$ such that

$$U_1(x, r) \subset U_2(x, \varepsilon) \quad \text{and} \quad U_2(x, r) \subset U_1(x, \varepsilon),$$

where $U_j(x, r) := \{y \in X : d_j(y, x) < r\}$, $j = 1, 2$. Show that (X, d_1) and (X, d_2) are equivalent if and only if the convergent sequences in (X, d_1) and those in (X, d_2) coincide.

Now let (X, d) be a metric space. Show that

$$\delta(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in X,$$

provides another metric on X and that d and δ are equivalent metrics on X. *Hint:* Show and use that the function $t \mapsto t/(1 + t)$ is monotonically increasing on $(0, \infty)$.

Exercise 2: **5 Points**

Let $X := (0, 1]$ and define metrics d_1 and d_2 on X by

$$d_1(x, y) := \left| \frac{1}{x} - \frac{1}{y} \right| \quad \text{and} \quad d_2(x, y) := |x - y|, \quad x, y \in X.$$

Show that d_1 and d_2 are *equivalent* metrics, i.e., that d_1-open sets are d_2-open and vice versa (see Exercise 1). Furthermore, prove that (X, d_1) is complete, but (X, d_2) is not. Note that this shows that completeness cannot be characterized in terms of equivalence only.

Exercise 3: **5 Points**

Let $\mathcal{S}$ be the space of *all* sequences $(x_i)_{i=1}^{\infty}$ of complex numbers. Verify that

$$d(x, y) := \sum_{i=1}^{\infty} \frac{2^{-i}|x_i - y_i|}{1 + |x_i - y_i|}, \quad x, y \in \mathcal{S},$$

provides a metric on the space $\mathcal{S}$. Show that a sequence $(x^{(n)})_{n \in \mathbb{N}} \subset \mathcal{S}$ converges to $x \in \mathcal{S}$ in $(\mathcal{S}, d)$ if and only if $x_i^{(n)} \to x_i$ for all $i \in \mathbb{N}$. Prove that $\mathcal{S}$ is complete.

Exercise 4: **4 Points**

Prove that $\ell_p \subset \ell_q$ for $1 \leq p < q \leq \infty$. Moreover, show that this is a proper inclusion.

Since April, 22 is a public holiday, please submit your homework in a tutorial of your choice between April, 23 and April, 26.