Analysis III Problem Sheet 02

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Exercise 2

Consider the following IVP

$$\begin{cases} x'' + 2ax' + bx = 0 \\ x(0) = c, \; x'(0) = 0 \end{cases}$$

Formulate it as a system of linear differential equations:

$$\begin{pmatrix} 0 & 1 \\ -b & -2a \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x' \\ x'' \end{pmatrix}.$$

The eigenvalues read

$$\det \begin{pmatrix} -\lambda & 1 \\ -b & -2a-\lambda \end{pmatrix} = \lambda(2a+\lambda) + b = 0 \implies \lambda_{1,2} = -a \pm \sqrt{a^2 - b}$$

Case: $a^2 > b$. The eigenvectors are given by

$$\begin{pmatrix} a + \sqrt{a^2-b} & 1 \\ -b & -a + \sqrt{a^2-b} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \text{ and } \begin{pmatrix} a - \sqrt{a^2-b} & 1 \\ -b & -a - \sqrt{a^2-b} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

Let $b \neq 0$. For $\lambda = -a + \sqrt{a^2-b}$ we get

$$\mathbf{v} = \begin{pmatrix} -\frac{a+\sqrt{a^2-b}}{b} \\ 1 \end{pmatrix}$$

and for $\lambda = -a - \sqrt{a^2-b}$ we obtain

$$\mathbf{v} = \begin{pmatrix} -\frac{a-\sqrt{a^2-b}}{b} \\ 1 \end{pmatrix}.$$

The system of solutions is

$$\mathbf{x}(t) = \underbrace{\exp((-a+\sqrt{a^2-b})t) \begin{pmatrix} -\frac{a+\sqrt{a^2-b}}{b} \\ 1 \end{pmatrix}}_{=\varphi_1(t)} + \underbrace{\exp((-a-\sqrt{a^2-b})t) \begin{pmatrix} -\frac{a-\sqrt{a^2-b}}{b} \\ 1 \end{pmatrix}}_{=\varphi_2(t)}$$

Find coefficients α and β such that $\alpha\varphi_1 + \beta\varphi_2$ solves the IVP. From $x'(0) = 0$ it follows that $\alpha = -\beta$. With $x(0) = c$ we get

$$-\alpha\frac{a + \sqrt{a^2 - b}}{b} - \beta\frac{a - \sqrt{a^2 - b}}{b} = c.$$

Solving the system yields

$$\alpha = -\frac{bc}{2\sqrt{a^2 - b}}, \quad \beta = \frac{bc}{2\sqrt{a^2 - b}}$$

So, the final solution of the linearised IVP is

$$\mathbf{x}(t) = -\frac{bc}{2\sqrt{a^2 - b}}\varphi_1(t) + \frac{bc}{2\sqrt{a^2 - b}}\varphi_2(t).$$

and we get the solution for the original IVP by inspecting the first component of $\mathbf{x}(t)$.

Let $b = 0$. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -2a$, and the corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2a} \\ 1 \end{pmatrix}$. So for the fundamental system

$$\mathbf{x}(t) = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \exp(-2at) \begin{pmatrix} -\frac{1}{2a} \\ 1 \end{pmatrix}$$

From $x'(0) = 0$ we see that $\beta = 0$. Thus, $\alpha = c$ so that $x(0) = c$. The solution of the original IVP is given by

$$x(t) = c.$$

Case: $a^2 = b$. Because the characteristic polynomial has a double root in λ we get the fundamental system for the original IVP:

$$x(t) = \alpha \exp(-at) + t\beta \exp(-at)$$

and the derivative

$$x'(t) = -a\alpha \exp(-at) - at\beta \exp(-at) + \beta \exp(-at)$$

$x'(0) = 0 \implies -\alpha a + \beta = 0$ and $x(0) = c \implies \alpha = c$. So, $\beta = ac$. We get the solution

$$x(t) = c \exp(-at) + act \exp(-at) = c \exp(-at)(1 + at).$$

Case: $a^2 < b$.

Exercise 3

Let $f : \mathbb{R} \times \mathbb{R}^2, (t, x, y) \mapsto (y, -x)$. We see that f is locally Lipschitz-continuous since it is partial differentiable w.r.t x and y. After Picard-Lindelof there exists a unique solution of the IVP

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(t, x, y), \quad x(0) = a,\ y(0) = b,$$

which can be obtained by the iteration

$$\mathbf{x}_0 \equiv \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \mathbf{x}_i(t) = \mathbf{x}_0 + \int_0^t f(s, \mathbf{x}_{i-1}(s))ds.$$

We will calculate the first iterations so that we get a feeling how the solution might look:

$$\mathbf{x}_1(t) = \begin{pmatrix} a \\ b \end{pmatrix} + \int_0^t \begin{pmatrix} b \\ -a \end{pmatrix} dx = \begin{pmatrix} a \\ b \end{pmatrix} + [\begin{pmatrix} bs \\ -as \end{pmatrix}]_0^t = \begin{pmatrix} a + bt \\ b - at \end{pmatrix}$$

and

$$\begin{aligned} \mathbf{x}_2(t) = \begin{pmatrix} a \\ b \end{pmatrix} + \int_0^t \begin{pmatrix} b - as \\ -a - bs \end{pmatrix} ds &= \begin{pmatrix} a \\ b \end{pmatrix} + [\begin{pmatrix} bs - \frac{1}{2}as^2 \\ -as - \frac{1}{2}bs^2 \end{pmatrix}]_0^t \\ &= \begin{pmatrix} a + bt - \frac{1}{2}at^2 \\ b - at - \frac{1}{2}bt^2 \end{pmatrix} \end{aligned}$$

Claim:

$$\mathbf{x}_{2n}(t) = \begin{pmatrix} \sum_{j=0}^n (-1)^j \frac{1}{(2j)!} at^{2j} + \sum_{j=0}^n (-1)^j \frac{1}{(2j+1)!} bt^{2j+1} \\ \sum_{j=0}^n (-1)^j \frac{1}{(2j)!} bt^{2j} + \sum_{j=0}^n (-1)^{j+1} \frac{1}{(2j+1)!} at^{2j+1}) \end{pmatrix},$$

$$\mathbf{x}_{2n+1}(t) = \begin{pmatrix} \sum_{j=0}^{n+1} (-1)^j \frac{1}{(2j)!} at^{2j} + \sum_{j=0}^n (-1)^j \frac{1}{(2j+1)!} bt^{2j+1} \\ \sum_{j=0}^{n+1} (-1)^j \frac{1}{(2j)!} bt^{2j} + \sum_{j=0}^n (-1)^{j+1} \frac{1}{(2j+1)!} at^{2j+1}) \end{pmatrix}$$

for all $n \in \mathbb{N}$. This converges to $\begin{pmatrix} a\cos(t) + b\sin(t) \\ b\cos(t) - a\sin(t) \end{pmatrix}$.

Proof. By induction. We showed it for $n = 0$. Assume the claim is true for

$2n \in \mathbb{N}$. We want to show that the claim also holds for $2n+1$.

$$\begin{aligned}
\mathbf{x}_{2n+1}(t) &= \begin{pmatrix} a \\ b \end{pmatrix} + \int_0^t f(s, \mathbf{x}_{2n}(s)) ds \\
&= \begin{pmatrix} a \\ b \end{pmatrix} + \int_0^t \begin{pmatrix} \sum_{j=0}^{n} (-1)^j \frac{1}{(2j)!} bt^{2j} + \sum_{j=0}^{n} (-1)^{j+1} \frac{1}{(2j+1)!} at^{2j+1}) \\ - \sum_{j=0}^{n} (-1)^j \frac{1}{(2j)!} at^{2j} - \sum_{j=0}^{n} (-1)^j \frac{1}{(2j+1)!} bt^{2j+1} \end{pmatrix} dx \\
&= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \sum_{j=0}^{n} (-1)^j \frac{1}{(2j+1)!} bt^{2j+1} + \sum_{j=0}^{n} (-1)^{j+1} \frac{1}{(2j+2)!} at^{2j+2}) \\ - \sum_{j=0}^{n} (-1)^j \frac{1}{(2j+1)!} at^{2j+1} - \sum_{j=0}^{n} (-1)^j \frac{1}{(2j+2)!} bt^{2j+2} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=0}^{n+1} (-1)^j \frac{1}{(2j)!} at^{2j} + \sum_{j=0}^{n} (-1)^j \frac{1}{(2j+1)!} bt^{2j+1} \\ \sum_{j=0}^{n} (-1)^{j+1} \frac{1}{(2j+1)!} at^{2j+1} + \sum_{j=0}^{n+1} (-1)^j \frac{1}{(2j)!} bt^{2j} \end{pmatrix}
\end{aligned}$$

Assume the claim is true for $2n-1 \in \mathbb{N}$. We want to show that the claim also holds for $2n$.

$$\begin{aligned}
\mathbf{x}_{2n}(t) &= \begin{pmatrix} a \\ b \end{pmatrix} + \int_0^t f(s, \mathbf{x}_{2n-1}(s)) ds \\
&= \begin{pmatrix} a \\ b \end{pmatrix} + \int_0^t \begin{pmatrix} \sum_{j=0}^{n} (-1)^j \frac{1}{(2j)!} bt^{2j} + \sum_{j=0}^{n-1} (-1)^{j+1} \frac{1}{(2j+1)!} at^{2j+1}) \\ - \sum_{j=0}^{n} (-1)^j \frac{1}{(2j)!} at^{2j} - \sum_{j=0}^{n-1} (-1)^j \frac{1}{(2j+1)!} bt^{2j+1} \end{pmatrix} dx \\
&= \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \sum_{j=0}^{n} (-1)^j \frac{1}{(2j+1)!} bt^{2j+1} + \sum_{j=0}^{n-1} (-1)^{j+1} \frac{1}{(2j+2)!} at^{2j+2}) \\ - \sum_{j=0}^{n} (-1)^j \frac{1}{(2j+1)!} at^{2j+1} - \sum_{j=0}^{n-1} (-1)^j \frac{1}{(2j+2)!} bt^{2j+2} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=0}^{n} (-1)^j \frac{1}{(2j)!} at^{2j} + \sum_{j=0}^{n} (-1)^j \frac{1}{(2j+1)!} bt^{2j+1} \\ \sum_{j=0}^{n} (-1)^{j+1} \frac{1}{(2j+1)!} at^{2j+1} + \sum_{j=0}^{n} (-1)^j \frac{1}{(2j)!} bt^{2j} \end{pmatrix}
\end{aligned}$$

Due to the principle of induction the claim holds for all natural n. □

The solution of the IVP is given by

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a\cos(t) + b\sin(t) \\ b\cos(t) - a\sin(t) \end{pmatrix},$$

for it is the fixed point of the Picard iteration.