TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Prof. Dr. Yuri B. Suris
Jan Techter

Mathematical Physics I - WS 2018/2019

http://www3.math.tu-berlin.de/geometrie/Lehre/WS18/MP1/

Exercise Sheet 7

Exercise 1: (5 pts)

Consider the differential equation

$$x^{(n)} + a_{n-1}(t)x^{(n-1)} + \ldots + a_1(t)\dot{x} + a_0(t)x = 0,$$

where $a_0, \ldots, a_{n-1} : J \to \mathbb{R}$ are continuous functions on an intervall $J \subset \mathbb{R}$. For n arbitrary solutions $x_1, \ldots, x_n$ we set

$$W(t) := W(x_1(t), \ldots, x_n(t)) := \begin{vmatrix} x_1(t) & \cdots & x_n(t) \\ \dot{x}_1(t) & \cdots & \dot{x}_n(t) \\ \vdots & & \vdots \\ x_1^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{vmatrix} \qquad (t \in J).$$

Prove that

$$W(t) = W(t_0) \exp\left(-\int_{t_0}^{t} a_{n-1}(\tau)d\tau\right) \qquad (t_0 \in J \text{ arbitrary}).$$

Exercise 2: (8 pts)

Consider the differential equation

$$\ddot{x} + a(t)\dot{x} + b(t)x = 0, \tag{1}$$

where $a, b : \mathbb{R} \to \mathbb{R}$ are continuous and T-periodic functions. Let x_1, x_2 be a fundamental system of solutions of (1).

(i) Prove that $W(t+T) = W(t) \det C$, where

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

is the monodromy matrix of (1) and W is defined as in Exercise 1 with $n = 2$.

(ii) From now on set $a(t) \equiv 0$. Prove that for the multipliers μ_1 and μ_2 there hold:

$$\mu_1\mu_2 = 1 \qquad \text{and} \qquad \mu_1 + \mu_2 = c_{11} + c_{22}.$$

(iii) Consider a fundamental system x_1, x_2 of solutions of (1) ($a(t) \equiv 0$) with

$$x_1(0) = 1, \dot{x}(0) = 0 \qquad \text{and} \qquad x_2(0) = 0, \dot{x}_2(0) = 1.$$

Prove that there exists an $r \in \mathbb{C}$ such that

$$\mu_1 = e^r, \qquad \mu_2 = e^{-r}, \qquad \text{and} \qquad \cosh(r) = \frac{x_1(T) + \dot{x}_2(T)}{2}.$$

Exercise 3: (7 pts)

Consider the following system of ODEs in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1 = -x_1, \\ \dot{x}_2 = x_2 + x_1^2. \end{cases}$$

(i) Determine $\alpha \in \mathbb{R}$ such that

$$\Phi^t(x_1(0), x_2(0)) = \left(x_1(0)\,\mathrm{e}^{-t}, x_2(0)\,\mathrm{e}^{t} + \alpha x_1^2(0)\left(\mathrm{e}^{t} - \mathrm{e}^{-2\,t}\right)\right).$$

is the flow of the ODE.

(ii) Linearize the system around the fixed point $(0, 0)$ and find the stable and unstable linear subspaces.

(iii) Construct the stable and unstable manifolds of the nonlinear system.

Due Monday, December 10.