Research Group
Applied Functional Analysis
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Functional Analysis I

Tutorial Assignment 3

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Exercise 1: Let V, W be vector spaces, $F : V \to W$ be a linear map, and $U \subseteq \ker F$. Then there exists a unique mapping

$$\widetilde{F} : V/U \to W, \qquad [v] \mapsto F(v).$$

Exercise 2: A closed subspace F of a normed space E is called *complemented* if there exists another closed subspace $G \subset E$ such that E is the direct sum of F and G, i.e. $E = F \oplus G$. Is each subspace of E complemented if E is finite-dimensional? Let F be complemented such that $E = F \oplus G$ with a closed subspace G. Show that there exists a bijective linear mapping $\Phi : G \to E/F$. Prove that Φ is bounded, i.e. $\|\Phi x\| \leq c\|x\|$ for some $c > 0$ and all $x \in G$. What is Φ^{-1}? Show that Φ^{-1} is bounded if G is finite-dimensional.

Exercise 3: Let $T : E \to F$ be a bijective linear mapping between two vector spaces E and F. Prove that T^{-1} is also linear.

Exercise 4: Let E be a normed space and let $S_1, S_2 \in L(E) := L(E, E)$. Define an operator $T : L(E) \to L(E)$ by

$$TX := S_1 X S_2, \quad X \in L(E).$$

Show that T is linear and bounded and that $\|T\| \leq \|S_1\|\|S_2\|$.

Exercise 5: We say that $T \in L(E)$ is *boundedly invertible* if it maps E bijectively onto itself and if $T^{-1} \in L(E)$[1]. Let $S_1, S_2 \in L(E)$ be boundedly invertible. Show that $S_1 S_2$ and $S_2 S_1$ are also boundedly invertible, and that

$$(S_1 S_2)^{-1} = S_2^{-1} S_1^{-1}, \qquad (S_2 S_1)^{-1} = S_1^{-1} S_2^{-1}.$$

[1] Later in the lecture we will see that this condition is not necessary if E is a Banach space (open mapping theorem), i.e. bijectivity and boundedness imply the boundedness of the inverse in this case.