Proofs from the Lebesgue theory

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Analysis III

December 23, 2018

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8 Properties of measurable functions

- Let $f, g : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Then $f+g$, f^2 and fg are measurable.

  Proof. We want to show that $f+g$ is measurable. It is measurable if

  $$\forall a \in \mathbb{R} : \{f + g < a\} \in \mathcal{F}.$$

  Idea: First, we will show that $\{h < i\} \in \mathcal{F}$ for measurable functions h and i. Setting $h := f$ and $i := a - g$ gives the result because f is measurable and $a - g$ is measurable.

  Note: We cannot directly set $h := f + g$ and $i := a$, for we do not know if $f + g$ is measurable!

  Now, let's show that $\{h < i\}$ is measurable if $h, i \in \mathcal{F}$. It holds

  $$\{h < i\} = \bigcup_{q \in \mathbb{Q}} \{h < q\} \cap \{q < i\} \in \mathcal{F},$$

  for $\{h < q\} \in \mathcal{F}$ and $\{q < i\} \in \mathcal{F}$ (after the assumption).

  Geometrically speaken, we have divided the half plane below the line $f + g = a$ in countably many boxes.

  Bonus: we show $\{h \leq i\}$ is measurable by using that $\{h < i\}$ is measurable. It holds

  $$\{h \leq i\} = \bigcap_{\substack{q \in \mathbb{Q} \\ q > 0}} \{h < i + q\} \in \mathcal{F}.$$

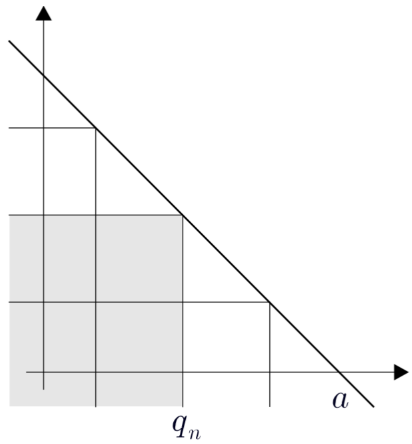

Figure 1: The subplane below the line is covered by boxes. Source: Lecture notes 2018, Prof. Charles Batty

We show that f^2 is measurable, and if that holds true, we can easily show that fg is measurable because $fg = \frac{1}{4}\Big((f+g)^2 - (f-g)^2\Big)$. Note that $f-g$ is measurable because $-g$ is measurable and thus $f+(-g)$.

Consider $\{f^2 > a\}$. If $a < 0$, then this set is Ω, which is indeed measurable. For $a > 0$:

$$\{f^2 > a\} = \{f > \sqrt{a}\} \cup \{f < -\sqrt{a}\} \in \mathcal{F},$$

for $\{f > \sqrt{a}\}$ and $\{f < -\sqrt{a}\}$ are measurable. □

- Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions in $\mathbb{R}$. Then, $\sup f_n$, $\inf f_n$, $\limsup f_n$ and $\liminf f_n$ are measurable.

  Proof. We show that $\sup f_n$ is measurable. For any $k \in \mathbb{N}_{\geq 1}$ and $a \in \mathbb{R}$, it holds:

$$\{x : (\sup_{n\geq k} f_n)(x) > a\} = \bigcup_{n\geq k} \{x : f_n(x) > a\} \in \mathcal{F}.$$

  Similarly,

$$\{x : (\inf_{n\geq k} f_n)(x) \geq a\} = \bigcap_{n\geq k} \{x : f_n(x) \geq a\} \in \mathcal{F}.$$

  Note that we must take $\geq$ since the intersection of open sets may not be open. Another idea to show that inf is measurable: $\inf(f_n) = -\sup(-f_n)$.

  Now,

$$\limsup f_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} f_n \in \mathcal{F} \quad \text{and} \quad \liminf f_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} f_n \in \mathcal{F},$$

  because as shown $\inf f_n$ and $\sup f_n$ is measurable for any sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions.

Alternatively, $\{x : (\sup_{n\geq k} f_n)(x) \leq a\} = \bigcup_{n\geq k}\{x : f_n(x) \leq a\} \in \mathcal{F}$ (note that we must use $\leq$).

Note that, we shall check that $\{x : (\sup f)(x) = \infty\} \in \mathcal{F}$, since $\sup f$ is a numerical function. The case is omitted, however it is easy to show. $\square$

- f^+, f^- and $|f|$ are measurable if f is measurable.

  Proof. The positive part is defined as $f^+(x) := \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) < 0 \end{cases} = \max\{f(x), 0\}$. Consider the characteristic function χ_A where $A = \{f \geq 0\}$. Since A is a measurable set, χ_A is a measurable function. Thus, the product $f \cdot \chi_A$ is measurable, since f is measurable, and so is $f^+(x) = f\chi_A \in \mathcal{F}$.

  Similarly, $f^- = (-f)^+$ is measurable, since $-f$ is measurable.

  The absolute value of f can be stated as $|f| = f^+ + f^-$, which is measurable as a sum of two measurable functions. $\square$

- $\max(f, g), \min(f, g)$ are measurable if f and g are measurable.

  Proof. $\{x : \max\{f(x), g(x)\} > a\} = \{x : f(x) > a\} \cup \{x : g(x) > a\} \in \mathcal{F}$. Now, it holds that $\min\{f, g\} = -\max\{-f, -g\} \in \mathcal{F}$. $\square$