Exercise 1

(i) Idea: Show that $\gamma(x) := |x|^{\frac{p}{q}}$ is Lipschitz continuous for $p > q$.

Proof. Let $p > q$ and $c := \frac{p}{q} > 1$. For a given $x_0 \in \mathbb{R}$ we must provide a neighborhood $U_\epsilon(x_0)$ and a constant $L > 0$ such that the Lipschitz property holds. Let $\epsilon := 1$. Since it holds $\gamma(x) = \gamma(-x)$ for all $x \in \mathbb{R}$, it follows that for $x \neq 0$ and $c > 1$:

$$\gamma'(x) := \frac{d}{dx}|x|^c = |cx^{c-1}| \cdot \frac{x}{|x|}. \tag{1}$$

We see that γ' is bounded on every $A \subsetneq \mathbb{R}\backslash\{0\}$ because of $c - 1 > 0$. Therefore, choose L as

$$L := \max_{u \in U_1(x_0)\backslash\{0\}} |\gamma'(u)|.$$

Let $u_1, u_2 \in U_1(x_0)$ with $u_1 \cdot u_2 > 0$. Using the Mean Value Theorem we show that

$$\frac{\||u_1|^c - |u_2|^c\|}{|u_1 - u_2|} = |\gamma'(\xi)| \leqslant L, \quad \xi \in (u_1, u_2)$$

and therefore

$$\||u_1|^c - |u_2|^c\| \leqslant L|u_1 - u_2|.$$

Let $u_1, u_2 \in U_1(x_0)$ with $u_1 \cdot u_2 < 0$. Without loss of generality, let $|u_2| > |u_1|$. Using triangle inequality, it holds:

$$|u_1 - u_2| = |u_1| + |u_2| \geqslant |u_1 + u_2|.$$

Again, we know that $\gamma(x) = \gamma(-x)$ and thus

$$\frac{\||u_1|^c - |u_2|^c\|}{|u_1 - u_2|} \leqslant \frac{\||u_1|^c - |-u_2|^c\|}{|u_1 - (-u_2)|} = \frac{\||u_1|^c - |u_2|^c\|}{|u_1 + u_2|} = |\gamma'(\xi)| \leqslant L, \quad \xi \in (u_1, -u_2).$$

Let $u_1, u_2 \in U_1(x_0)$ with $u_1 \cdot u_2 = 0$ and w.l.o.g. $u_2 \neq 0$ (if $u_1 = u_2 = 0$, the Lipschitz inequality is obiously true). Here we need $c > 1$.

$$\frac{\||u_1|^c - |u_2|^c\|}{|u_1 - u_2|} = \frac{|u_1|^c}{|u_1|} = |u_1|^{c-1} \leqslant |cu_1^{c-1}| \leqslant L.$$

We showed that γ is Lipschitz continuous and after Picard Lindelöf, a unique solution exists. □

(ii) Let $p < q$. Using seperation of variables results in

$$\int \frac{1}{x^{\frac{p}{q}}} dx = t \implies (1 - \frac{p}{q})^{-1} x^{1-\frac{p}{q}} + c = t \implies x^{1-\frac{p}{q}} = (t - c)(\frac{q-p}{q})$$

$$\implies x(t) = ((t-c)\frac{q-p}{q})^{\frac{q}{q-p}}, \quad c \in \mathbb{R}.$$

A solution of the IVP is found for $c = 0$. Now let for all $k > 0$:

$$x_k(t) := \begin{cases} 0, & \text{if } t < k \\ \left((t-k)\frac{q-p}{q}\right)^{\frac{q}{q-p}} & \text{if } t \geqslant k \end{cases}$$

It holds that $x_k(0) = 0$ and x_k is a solution for each $k \in \mathbb{R}$.

(iii) If $p = q$, we get the absolute value function. This function ist Lipschitz continuous (by using the reverse triangle inequality):

$$\||x| - |y|\| \leqslant L|x-y|, \quad \forall x, y \in \mathbb{R}.$$

After Lindelöf, a unique solution exists. The solution ist $\varphi = 0$.

Exercise 2

Proof. If $\epsilon = 0$, we have the differential equation

$$x' = |x|^{\frac{1}{2}}.$$

As shown in exercise 1, there are infinite number of solutions for the IVP.

For $\epsilon > 0$, we will calculate the derivative of $\gamma(x) := \frac{x^2}{x^2+\epsilon}\sqrt{|x|}$ for $x \in \mathbb{R}$. If the derivative is bounded for any $A \subsetneq \mathbb{R}$, γ is Lipschitz continuous and hence a unique solution exists.

For $x \neq 0$ we can easily calculate the derivative

$$\frac{d}{dx}\sqrt{|x|} = \frac{1}{2\sqrt{|x|}} \cdot \frac{x}{|x|} = \frac{x}{2\sqrt{|x|^3}}, \quad \frac{d}{dx}\frac{x^2}{x^2+\epsilon} = \frac{2x(x^2+\epsilon) - x^2 \cdot 2x}{(x^2+\epsilon)^2} = \frac{2x\epsilon}{(x^2+\epsilon)^2}$$

and with the aid of the product rule, it follows

$$\frac{d}{dx}\frac{x^2}{x^2+\epsilon}\sqrt{|x|} = \frac{x^2}{x^2+\epsilon}\frac{x}{2\sqrt{|x|^3}} + \frac{2x\epsilon}{(x^2+\epsilon)^2}\sqrt{|x|} = \frac{x^3(x^2+\epsilon) + 2x\epsilon\sqrt{|x|}2\sqrt{|x|^3}}{2(x^2+\epsilon)^2\sqrt{|x|^3}} = \underbrace{\frac{x^5 + x^3\epsilon + 4\epsilon x|x|^2}{2(x^2+\epsilon)^2\sqrt{|x|^3}}}_{=\frac{x^5+5x^3\epsilon}{2(x^2+\epsilon)^2\sqrt{|x|^3}}}.$$

Now, consider the differential quotient at $x = 0$:

$$\lim_{h \searrow 0}\frac{\gamma(0+h) - \gamma(0)}{h} = \lim_{h \searrow 0}\frac{h\sqrt{|h|}}{h^2+\epsilon} = \frac{0}{\epsilon} = 0 = \lim_{h \nearrow 0}\frac{h\sqrt{|h|}}{h^2+\epsilon} = \lim_{h \nearrow 0}\frac{\gamma(0+h)-\gamma(0)}{h}.$$

Therefore

$$\gamma'(x) = \begin{cases} \frac{x^5+5x^3\epsilon}{2(x^2+\epsilon)^2\sqrt{|x|^3}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

We see that γ' is indeed bounded on any $A \subsetneq \mathbb{R}$. Therefore, γ is Lipschitz continuous and a unique solution exists. □

Exercise 3

Let $y^{(0)} := 1$ and $P_y(t) := x_0 + \int_{t_0}^t s + y(s)ds$. Applying this on our IVP yields:

$$\begin{aligned}
y^{(1)}(t) &= P_{y^{(0)}}(t) = 1 + \int_0^t s + y^{(0)}(s)ds = 1 + \int_0^t s + 1ds = 1 + t + \frac{1}{2}t^2 \\
y^{(2)}(t) &= P_{y^{(1)}}(t) = 1 + \int_0^t s + y^{(1)}(s)ds = 1 + \int_0^t 1 + 2s + \frac{1}{2}s^2 ds = 1 + t + t^2 + \frac{1}{6}t^3 \\
y^{(3)}(t) &= P_{y^{(2)}}(t) = 1 + \int_0^t s1 + 2s + s^2 + \frac{1}{6}s^3 ds = 1 + t + t^2 + \frac{1}{3}t^3 + \frac{1}{24}t^4 \\
y^{(n)}(t) &= P_{y^{(n-1)}}(t) = -1 - t + 2\sum_{i=0}^{n-1} \frac{t^i}{i!} + \frac{t^n}{n!} \qquad (\star)
\end{aligned}$$

Verify that $(\star)$ is true for $n > 0$ by induction. For $n = 1$, we have $y^{(0)}(t) = 1 = -1 + 2 + 1$.
Consider: $n \rightsquigarrow n + 1$. Assume that $y^{(n)}(t) = -1 - t + 2\sum_{i=0}^{n-1} \frac{t^i}{i!} + \frac{t^n}{n!}$ is true for some $n > 0$.

$$\begin{aligned}
y^{(n+1)}(t) = P_{y^{(n)}}(t) = 1 + \int_0^t s + y^{(n)}(s)ds &= 1 + \int_0^t \left(s - 1 - s + 2\sum_{i=0}^{n-1} \frac{s^i}{i!} + \frac{s^n}{n!} \right) ds \\
&= 1 - t + 2\sum_{i=0}^{n-1} \left(\frac{1}{(i+1)!}t^{i+1}\right) + \frac{t^{n+1}}{(n+1)!} \\
&= 1 - t + 2\sum_{i=1}^{n} \frac{1}{i!}t^i + \frac{t^{n+1}}{(n+1)!} \\
&= -1 - t + 2\sum_{i=0}^{n} \frac{1}{i!}t^i + \frac{t^{n+1}}{(n+1)!}.
\end{aligned}$$

By mathematical induction, the assumption is true for all $n > 0$.
Now, let's find a solution φ for the IVP. If we let $n \to \infty$, we get the solution

$$\varphi(t) = \lim_{n\to\infty} y^{(n)}(t) = \lim_{n\to\infty} -1 - t + 2\sum_{i=0}^{n-1} \frac{t^i}{i!} + \underbrace{\frac{t^n}{n!}}_{\to 0} = -1 - t + 2\lim_{n\to\infty}\sum_{i=0}^{n} \frac{t^i}{i!} = -1 - t + 2\exp t.$$

Exercise 4

(i) $x^3 - x$ is Lipschitz continuous for its derivative $3x^2 - 1$ is bounded for any $A \subsetneq \mathbb{R}$. Therefore, after Picard Lindelöf a unique solution exists for the IVP.

(ii) We have the ODE $\dot{x} = x^3 - x$. Hence, seperation of the variables results in

$$t = \int_{x_0=0.5}^{x} \frac{ds}{s^3 - s} \leqslant \int_{0.5}^{x} \frac{ds}{s^2} = -\frac{1}{x} + 2.$$

If t goes to infinity, it must hold that $-\frac{1}{x} + 2$ goes to infinity. This is only the case if $x \to 0$. Therefore, if $\phi(t)$ is a solution, it follows

$$\lim_{t\to\infty} \phi(t) = 0.$$