Mathematical Physics I (Week 12)

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Theorem 1. *If a flow of a vector field X on $\mathbb{R}^{2n}(x,p)$ preserves the canonical two form $\omega = \sum_{k=1}^{n} dx_k \wedge dp_k$, then (locally) there exists $H : \mathbb{R}^{2n} \to \mathbb{R}$ such that $X = X_H = \begin{pmatrix} \mathrm{grad}_p H \\ -\mathrm{grad}_x H \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathrm{grad} H$.*

Proof. Let

$$X = \begin{pmatrix} f_1(x,p) \\ \vdots \\ f_n(x,p) \\ g_1(x,p) \\ g_n(x,p) \end{pmatrix}$$

so that $\begin{cases} \dot{x}_k = f_k(x,p) \\ \dot{p}_k) g_k(x,p) \end{cases}$. Assume that

$$\begin{aligned} L_x\omega = 0 &\iff \sum_{k=1}^{n} (df_k(x,p) \wedge dp_k + dx_k \wedge dg_k(x,p)) = 0 \\ &\iff \sum_{k=1}^{n} (\sum_{j=1}^{n} (\frac{\partial f_k}{\partial x_j} dx_j + \frac{\partial f_k}{\partial p_j} dp_j) \wedge dp_k + dx_k \wedge (\sum_{j=1}^{n} \frac{\partial g_k}{\partial x_j} dx_j + \frac{\partial g_k}{\partial p_j} \wedge dp_j)) = 0. \end{aligned}$$

Coefficients by $dx_k \wedge dx_j$; $\frac{\partial g_k}{\partial x_j} - \frac{\partial g_j}{\partial x_k} = 0$. By $dp_j \wedge dp_k$: $\frac{\partial f_k}{\partial p_j} - \frac{\partial f_j}{\partial p_k} = 0$. By $dx_j \wedge dp_k$: $\frac{\partial f_k}{\partial x_j} - \frac{\partial g_j}{\partial p_k} = 0$. The first condition is that there exists a function H_1 (defined up to a function of p) such that locally it holds $g_k = \frac{\partial H_1}{\partial x_k}$. The second condition is that there exists a function H_2 (defined up to a function of x) such that locally it holds $f_k = \frac{\partial H_2}{\partial p_k}$.

Additionally

$$\frac{\partial}{\partial x_j}(\frac{\partial H_2}{\partial p_k}) + \frac{\partial}{\partial p_k}(\frac{\partial H_1}{\partial x_j}) = 0 \iff \frac{\partial^2}{\partial x_j \partial p_k}(H_1 + H_{"}) = 0$$
$$\iff H_1(x,p) + H_2(x,p) = a(x) + b(p)$$
$$\iff H_1(x,p) - b(p) = -H_{"}(x,p) + a(x) \coloneqq -H(x,p).$$

Then $f_k = \frac{\partial H}{\partial p_k}, g_k = -\frac{\partial H}{\partial x_k}$. $\square$

An example of a locally Hamiltonian vector field which is not globally Hamiltonian. Consider a vector field on a non-simply-connected manifold - a torus[1]. A torus is analytically described by $(\mathbb{R}\backslash\mathbb{Z})^2 = S^1 \times S^1 = \{(x(\mod 1), p(\mod 1))\}$. The canonical two form of the torus is

$$\omega = dx \wedge dp \quad \text{it measures the area of the torus.}$$

The simples vector field is $\begin{cases} \dot{x} = 1 \\ \dot{p} = 0 \end{cases}$. It just describes a translation. We claim that this vector field is locally Hamiltonian $\begin{pmatrix} \frac{\partial H}{\partial p} = 1 \\ \frac{\partial H}{\partial x} = 0 \end{pmatrix}$, but $H(x,p) = p$ is not a smooth function on the torus since it gets an increment $\neq 0$ along closed non-null-homotopic curves $x = \text{const}$.

Definition 2. *A map $\varphi : \mathbb{R}^{2n}_{(x,p)} \to \mathbb{R}^{2n}_{(x,p)}$ preserving $\omega = \sum_{k=1}^n dx_k \wedge dp_k$ (that is $\varphi^*\omega = \omega$) is called a **symplectic map** (so that the flow of X_H consists of symplectic maps).*

Definition 3. *A matrix $D \in \mathrm{mat}_{2n\times 2n}(\mathbb{R})$ is symplectic, i.e. $D \in \mathrm{Sp}_{2n}(\mathbb{R})$, if it satisfies*

$$D^T J D = J \qquad \text{with } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Theorem 4. *A map φ is symplectic if and only if $d\varphi \in \mathrm{mat}_{2n\times 2n}(\mathbb{R})$ is symplectic.*

Proof. Let $(\tilde{x}, \tilde{p}) = \varphi(x,p)$.

$$\sum_{k=1}^n d\tilde{x}_k \wedge d\tilde{p}_k = \sum_{k=1}^n \sum_{j=1}^n (\frac{\partial \tilde{x}_k}{\partial x_j} dx_j + \frac{\partial \tilde{x}_k}{\partial p_j} dp_j) \wedge \sum_{i=1}^n (\frac{\partial \tilde{p}_k}{\partial x_i} dx_i + \frac{\partial \tilde{p}_k}{\partial p_i} dp_i)$$

$$\sum_{j=1}^n \sum_{i=1}^n (\sum_{k=1}^n \frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{p}_k}{\partial x_i} dx_j \wedge dx_i) + \sum_{j=1}^n \sum_{i=1}^n (\sum_{k=1}^n \frac{\partial \tilde{x}_k}{\partial p_j}\frac{\partial \tilde{p}_k}{\partial p_i} dp_j \wedge dp_i) + \sum_{j=1}^n \sum_{i=1}^n (\frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{p}_k}{\partial p_i} - \frac{\partial \tilde{x}_k}{\partial p_i}\frac{\partial \tilde{p}_k}{\partial x_j}) dx_j \wedge dx_i$$

This should be equal to $\sum_{j=1}^n dx_j \wedge dp_j$.

Coefficients by $dx_j \wedge dx_i$: $\sum_{k=1}^n \frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{p}_k}{\partial x_i} - \frac{\partial \tilde{x}_k}{\partial x_i}\frac{\partial \tilde{p}_j}{\partial x_k} = 0 \iff (\frac{\partial \tilde{x}}{\partial x})^T \frac{\partial \tilde{p}}{\partial x} - (\frac{\partial \tilde{p}}{\partial x})^T (\frac{\partial \tilde{x}}{\partial x}) = 0_{n\times n}$.

Coefficients by $dp_j \wedge dp_i$: $\sum_{k=1}^n \frac{\partial \tilde{x}_k}{\partial p_j}\frac{\partial \tilde{p}_k}{\partial p_i} - \frac{\partial \tilde{x}_k}{\partial p_i}\frac{\partial \tilde{p}_k}{\partial p_j} = 0 \iff (\frac{\partial \tilde{x}}{\partial p})^T \frac{\partial \tilde{p}}{\partial p} - (\frac{\partial \tilde{p}}{\partial p})^T (\frac{\partial \tilde{x}}{\partial p}) = 0_{n\times n}$.

Coefficients by $dx_j \wedge dp_i$: $\sum_{k=1}^n \frac{\partial \tilde{x}_k}{\partial x_j}\frac{\partial \tilde{p}_k}{\partial p_i} - \frac{\partial \tilde{x}_k}{\partial p_i}\frac{\partial \tilde{p}_k}{\partial x_j} = \delta_{ij} \iff (\frac{\partial \tilde{x}}{\partial x})^T \frac{\partial \tilde{p}}{\partial p} - (\frac{\partial \tilde{p}}{\partial x})^T \frac{\partial \tilde{x}}{\partial p} = I_{n\times n}$

This is is equivalent to

$$\begin{pmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial p} \\ \frac{\partial \tilde{p}}{\partial x} & \frac{\partial \tilde{p}}{\partial p} \end{pmatrix}^T \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial p} \\ \frac{\partial \tilde{p}}{\partial x} & \frac{\partial \tilde{p}}{\partial p} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

□

Definition 5. *$Sp_{2n} \coloneqq \{D \in mat_{2n\times 2n}(\mathbb{R}) : D^T J D = J\}$ is called the symplectic group.*

Based on

- $D_1^T J D_1 = J, D_2^T J D_2 = J \implies (D_1 D_2)^T J (D_1 D_2) = J$
- $\det D \neq 0$: $\det(D^T J D) = \det(D)\det(J)\det(D) = \det(J) \implies (\det(D))^2 = 1 \implies \det(D) = \pm 1$.
- $D^T J D = J \iff (D^{-1})^T J (D^{-1}) = J$.

Thus, $\mathrm{Sp}_{2n}(\mathbb{R})$ is a group, indeed.

Some side remarks: Classical matrix Lie groups

$$\begin{aligned} \mathrm{GL}_n(\mathbb{R}) &= \{A \in \mathrm{mat}_{n\times n}(\mathbb{R}) : \det(A) \neq 0\} \\ O_n(\mathbb{R}) &= \{A \in \mathrm{mat}_{n\times n}(\mathbb{R}) : A^T A = I\} \\ \mathrm{Sp}_{2n}(\mathbb{R}) &= \{A \in \mathrm{mat}_{2n\times 2n}(\mathbb{R}) : A^T J A = J\} \end{aligned}$$

The group $\mathrm{Sp}_{2n}(\mathbb{R})$ preserves skew symmetric two forms.

We know for othrogonal matrices if the row form an orthonormal basis so do the columns, i.e. $A^T A = I$ and $A A^T = I$. Something similar also holds for the symplectic matrices.

Theorem 6. *It holds: $D \in \mathrm{Sp}_{2n}(\mathbb{R}) \iff D J D^T = J$.*

Proof.

$$\begin{aligned} D^T J D = J \iff J D = (D^T)^{-1} J \overset{J^{-1}=-J}{\iff} D = -J(D^T)^{-1} J &\iff D J = J (D^T)^{-1} \\ &\iff D J D^T = J. \end{aligned}$$

□

Collorary 7. *Symplectic maps preserve Poisson brackets.*

Proof. For $D = d\varphi = \begin{pmatrix} \frac{\partial \tilde{x}}{\partial x} & \frac{\partial \tilde{x}}{\partial p} \\ \frac{\partial \tilde{p}}{\partial x} & \frac{\partial \tilde{p}}{\partial p} \end{pmatrix}$ the identitiy $DJD^T = J$ reads

$$\frac{\partial \tilde{x}}{\partial x}(\frac{\partial \tilde{x}}{\partial p})^T - \frac{\partial \tilde{x}}{\partial p}(\frac{\partial \tilde{x}}{\partial x})^T = 0 \iff \sum_{k=1}^{n}(\frac{\partial \tilde{x}_i}{\partial x_k}\frac{\partial \tilde{x}_j}{\partial p_k} - \frac{\partial \tilde{x}_i}{\partial p_k}\frac{\partial \tilde{x}_j}{\partial x_k}) = 0$$
$$\frac{\partial \tilde{p}}{\partial p}(\frac{\partial \tilde{x}}{\partial p})^T - \frac{\partial \tilde{p}}{\partial p}(\frac{\partial \tilde{x}}{\partial x})^T = 0 \iff \sum_{k=1}^{n}(\frac{\partial \tilde{p}_i}{\partial x_k}\frac{\partial \tilde{p}_j}{\partial p_k} - \frac{\partial \tilde{p}_i}{\partial p_k}\frac{\partial \tilde{p}_j}{\partial x_k}) = 0$$
$$\frac{\partial \tilde{x}}{\partial x}(\frac{\partial \tilde{p}}{\partial p})^T - \frac{\partial \tilde{x}}{\partial p}(\frac{\partial \tilde{p}}{\partial x})^T = I \iff \sum_{k=1}^{n}(\frac{\partial \tilde{x}_i}{\partial x_k}\frac{\partial \tilde{p}_j}{\partial p_k} - \frac{\partial \tilde{x}_i}{\partial p_k}\frac{\partial \tilde{p}_j}{\partial x_k}) = \delta_{i,j}.$$

This is equivalent to $\{\tilde{x}_i\tilde{x}_j\} = 0 = \{x_i, x_j\}, \{\tilde{p}_i\tilde{p}_j\} = 0 = \{p_i, p_j\}, \{\tilde{x}_i\tilde{p}_j\} = \delta_{i,j} = \{x_i, p_j\}$. □

This means that the map $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a **Poisson map**.

Definition 8. *A map $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called a **Poisson map** if it preserves the Poisson brackets: in coordinates,*

$$\{\tilde{x}_i, \tilde{x}_j\} = \{x_i, x_j\} = 0$$
$$\{\tilde{p}_i, \tilde{p}_j\} = \{p_i, p_j\} = 0$$
$$\{\tilde{x}_i, \tilde{p}_j\} = \{x_i, p_j\} = \delta_{i,j}$$

or intrinsically, $\{F \circ \varphi, G \circ \varphi\} = \{F, G\} \circ \varphi \forall F, G : \mathbb{R}^{2n} \to \mathbb{R}$.

Why is coordinate formulation equivalent to the intrinsic one? Suppose that there is a coordinate system $\{y_i\}$ such that the Poisson bracket of coordinate functions is given by $\{y_i, y_i\} = \omega_{ij} = -\omega_{ji}$. Suppose that $\varphi : y \mapsto \tilde{y}$ and $\{\tilde{y}_i, \tilde{y}_j\} = \omega_{ij}$. Then the poisson bracket is $\{F, G\} = \sum_{i=1}^{2n}\sum_{j=1}^{2n} \frac{\partial F}{\partial y_i}\frac{\partial G}{\partial y_j}\omega_{ij}$, in our case $(\omega_{ij}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. We have

$$\{F \circ \varphi, G \circ \varphi\} = \{F(\tilde{y}), G(\tilde{y})\} = \sum_{k,l=1}^{2n} \frac{\partial F(\tilde{y})}{\partial y_k}\frac{G(\tilde{y})}{\partial y_k}\omega_{kl}$$