Cheat sheet: Multiple Riemann integral

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Contents

0 Preface

A concise overview of the multiple integral theory for the Analysis III class at the Technical University in the winter term of 2018. Bullet points with a ($\star$) mean that a proof exists on the auxiliary proof sheet. This sheet is primarily written for me as a learning guide but may be useful for others. Feel free to use it.

1 One dimensional integral

- Let $f : [a, b] \times U$ be continuous and continuously partially differentiable in $U \subset \mathbb{R}^n$. Then,

$$\frac{d}{d\mathbf{u}} \int_a^b f(x, \mathbf{u}) dx = \int_a^b \frac{d}{d\mathbf{u}} f(x, \mathbf{u}) dx$$

  This derivative is continuous.

  In words, one may differentiate under the integral sign ($\star$).

2 Multiple integral on compact cuboids

- Let $f : [a, b] \times U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$ be continuous. Then, the function

$$(u_1, ..., u_n) \mapsto \int_b^a f(x, u_1, ..., u_n) dx$$

  is continuous ($\star$).

- Let Q be a compact cuboid $Q = [a_1, b_1] \times ... \times [a_n, b_n] \subset \mathbb{R}^d$. The **integral** of a continuous function $f : Q \to \mathbb{R}$ on a compact cuboid Q is defined as

$$\int_Q f(\mathbf{x}) d\mathbf{x} = \int_{a_n}^{b_n} ... \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} f(x_1, ..., x_n) dx_1 \right) dx_2 \right) ... dx_n.$$

  Such integral is also called *iterated integral*. The integral is well defined due to the previous theorem; integration of a continuous function with respect to a single variable yields a continuous function which can be further integrated.

- **Theorem of Fubini:** Let $f : Q \to \mathbb{R}$ be continuous. Then, one may change the order of integration ($\star$).

3 Multiple integral on $\mathbb{R}^d$

We generalise the integral of a continuous functions f on the whole $\mathbb{R}^d$ space.

- The **support** is defined as

$$\text{supp} f = \overline{\{\mathbf{x} \in \mathbb{R}^d : f(\mathbf{x}) \neq 0\}}.$$

  So, all x for which $f(x) \neq 0$ is contained in the support.

- The class of functions for which the integral is defined is given by the **space of continuous functions with compact support**:

$$\mathcal{C}_{\mathrm{c}} = \{f : f \text{ is continuous and } \mathrm{supp} f \text{ is compact}\}.$$

In $\mathbb{R}^d$, one can also write

$$\mathcal{C}_{\mathrm{c}} = \{f : f \text{ is continuous and } \mathrm{supp} f \text{ is bounded}\}.$$

In words, we do only integrate those functions which are continuous on $\mathbb{R}^d$ and whose values are zero outside a compact cuboid.

- The **integral** for $f \in \mathcal{C}_{\mathrm{c}}$ on the whole $\mathbb{R}^d$ space is defined as:

$$\int_{\mathbb{R}^d} f(\mathbf{x})d\mathbf{x} = \int_Q f(\mathbf{x})d\mathbf{x}, \tag{1}$$

where Q is a compact cuboid that contains the support of f.

4 Linear, monotonic and translation-invariant functionals

A *functional* maps a function to a real number. One example is the integral. The integral takes a function and assigns a real value to the function.

- The integral $\int_{\mathbb{R}^d} : \mathcal{C}_{\mathrm{c}} \to \mathbb{R}$ defined in (1) is a *linear*, *monotonic* and *translation-invariant* functional. Translational invariance means

$$\forall \mathbf{x} \in \mathbb{R}^d : \int \theta_{\mathbf{x}} f(\mathbf{u})d\mathbf{u} = \int f(\mathbf{u})d\mathbf{u}$$

- Let J be a linear and monotonic functional. Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{C}_c(\mathbb{R}^d)$ such that there is a compact cuboid $Q \supset \mathrm{supp}(f_n)$ for all n and $f_n \to f \in \mathcal{C}_c(\mathbb{R}^d)$ uniformly. Then $\lim J(f_n) = J(f)$. $(\star)$
- Let J be a linear and translation-invariant function in $\mathbb{R}^d$. It holds

$$J(\Psi_{\frac{\epsilon}{2}}) = \frac{1}{2^d} J(\Phi_\epsilon).$$

- Every function $f \in \mathcal{C}_c(\mathbb{R}^d)$ can be uniformly approximated by

$$f - \sum_{k\in\mathbb{Z}^d} f(k\epsilon)\theta_{k\epsilon}\Psi_\epsilon.$$

- Every linear, monotonic and translation-invariant functional is unique up to a constant c. For every $f \in \mathcal{C}_c(\mathbb{R}^d)$ there exists a constant c such that $J(f) = cI(f)$. It even holds that there is one c for all f such that $J(f) = cI(f)$. This c can be chosen as $c = J(\Psi)$. $(\star)$

5 Integration by substitution

- $J(Af)$ is a linear, monotonic and translation-invariant functional if $A \in \mathrm{GL}(\mathbb{R}^d)$.
- Some linear substituion (all A must be invertible):
  - if A is orthogonal, $\int_{\mathbb{R}^d} f(Ax)dx = \int_{\mathbb{R}^d} f(x)dx$. $(\star)$
  - If $A = \mathrm{diag}(a_1, ..., a_d)$ then $\int f(Ax)dx = \frac{1}{a_1 \cdot ... \cdot a_d} \int f(x)dx$.

6 Integral of semicontinuous functions

- Let A be non-degenerate. Then $\int f(Ax+b)dx = \frac{1}{|\det A|} \int f(x)dx$ for every $f \in \mathcal{H}^{\uparrow}(\mathbb{R}^d) \cup \mathcal{H}^{\downarrow}(\mathbb{R}^d)$.
- Theorem of Fubini for semicontinuous functions. Let $f \in \mathcal{H}^{\uparrow}(\mathbb{R}^d)$. The map $(x_{k+1}, ..., x_d) \mapsto f(x)$ is in $\mathcal{H}^{\uparrow}(\mathbb{R}^k)$. Then

$$F(x_{k+1}, ..., x_d) \coloneqq \int_{\mathbb{R}^k} f(x)dx_1...dx_d.$$

  F is well defined and can be integrated:

$$\int_{\mathbb{R}^{d-k}} F(x_{k+1}, ..., x_d)dx_{k+1}...dx_d = \int_{\mathbb{R}^d} f(x)dx.$$

7 Volumes

- The volume of a compact set K is defined as $\mathrm{vol}(K) \coloneqq \int_K 1d\mathbf{x} = \int_{\mathbb{R}^d} \chi_K(\mathbf{x})d\mathbf{x}$.
- $\mathrm{vol}(K_1 \times K_2) = \mathrm{vol}K_1 \cdot \mathrm{vol}K_2$
- $\mathrm{vol}(AK+b) = |\det A|\mathrm{vol}(K)$. Be careful, it is not $|\det A|^{-1}$. $(\star)$
- $\mathrm{vol}(rK) = r^d \mathrm{vol}K$. $(\star)$
- Cavalieri's principle: Let K be compact and

$$\mathrm{Vol}_d(K) = \int_{\mathbb{R}} \mathrm{Vol}_{d-1} K_t dt$$

  and $K_t \coloneqq \{(x_1, ..., x_{d-1}) : (x_1, ..., x_{d-1}, t) \in K\}$. $(\star)$
- Let $A \subset \mathbb{R}^{d-1}$ be compact and $f : A \to [0, \infty)$. The volume of $K = \{(x,y) : x \in A, y \leq f(x)\}$ is given by $\mathrm{vol}_d(K) = \int_A f(x)dx$.
- The volume of a general cylinder $B \times [0,h]$ is given by $\mathrm{vol}(Z) = \mathrm{vol}(B) \cdot h$.
- The volume of a parallelepiped $K = \{\sum \lambda_i a_i : \lambda_i \leq 1\}$ is given by $\mathrm{vol}(K) = |\det A|$.