Let E be a Banach space, and let $T \in L(E)$.
To show: $\ker T^* = (\text{ran}T)^\perp$

Proof. Let $x \in E$ such that $T^*x = 0$. Let $y \in H$. One sees that

$$\langle y, T^*x \rangle = \langle y, 0 \rangle = 0$$

Using the adjoint operator one obtains

$$\langle Ty, x \rangle = 0.$$

This holds for all $y \in H$ and $x \in \ker T^*$, and hence $x \in (\text{ran}T)^\perp$.
For the reverse direction, let $x \in E$. Let $z \in \ker T^*$. Observe that

$$\langle z, Tx \rangle = \langle T^*z, x \rangle = 0,$$

implying $Tx \in (\ker T^*)^\perp$.
In the end, one obtains $\ker T^* = (\text{ran}T)^\perp$. □

Let H be a Hilbert space, and let $F \in L(E)$. **To show:** $\ker F = (\overline{\text{ran}F^*})^\perp$.

Proof. It is already known that $\ker T^* = (\text{ran}T)^\perp$ in a Banach space for any linear and bounded operator T. Now, set $T = F^*$, and one obtains $\ker F = (\text{ran}F^*)^\perp$, since it holds $F^{**} = F$ for a reflexive space. Knowing that $S^\perp = \overline{S}^\perp$ for any subset $S \subset H$, one immediately sees $\ker F = (\text{ran}F^*)^\perp = (\overline{\text{ran}F^*})^\perp$. □

To show: Let H be an inner product space. $S^\perp = \overline{S}^\perp$ for any subset $S \subset H$.

Proof. Let $x \in S^\perp$. Per definition, one has $\langle x, y \rangle = 0$ for all $y \in S$. We want to show that even $\langle x, y \rangle$ holds for all $y \in \bar{S}$. Let $y \in \bar{S}$ with $\lim y_n = y$ with $y_n \in S$ for all $n \in \mathbb{N}$. Observe that

$$0 = \lim \langle x, y_n \rangle = \langle x, y \rangle.$$

Thus, $S^\perp \subset \bar{S}^\perp$. For the other direction, note that $S \subset \bar{S}$, and as a result, one gets $\bar{S}^\perp \subset \bar{S}^\perp$.
In the end, $S^\perp = \overline{S}^\perp$. □