# Prelims. Multivariable Calculus

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Using extensive material from lecture notes by Richard Earl
and also material from lecture notes by Ruth Baker and Eamonn Gaffney.

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# Recommended Texts

- Erwin Kreyszig, *Advanced Engineering Mathematics* (Wiley, 8th Edition, 1999).
- D. E. Bourne & P. C. Kendall, *Vector Analysis and Cartesian Tensors* (Stanley Thornes, 1992).
- H. M. Schey, *Div, Grad and Curl and all that* (W. W. Norton, Third Edition, 1996).

# Further Reading

- Mary L. Boas, *Mathematical Models in the Physical Sciences* (Wiley 2nd Edition, 1983).
- Gerald B. Folland, *Advanced Calculus* (Prentice Hall, 2002)
- Murray R Spiegel, *Vector Analysis* (Schaum's Outline Series, McGraw-Hill, 1974).

# Website

https://www.maths.ox.ac.uk/courses/material

# 1. MULTIPLE INTEGRALS IN 2D

## 1.1 Preliminaries

**Definition 1** *By a **scalar field** $\phi$ on $\mathbb{R}^3$ we shall mean a map $\phi : \mathbb{R}^3 \to \mathbb{R}$.*

**Definition 2** *By a **vector field** $\mathbf{F}$ on $\mathbb{R}^3$ we shall mean a map $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$.*

- We will typically assume that scalar and vector fields are smooth – their partial derivatives exist with respect to $x$, $y$ and $z$ to all orders – for brevity, this will not always be stated.
- Occasionally we may consider more general scalar fields $\phi : \mathbb{R}^n \to \mathbb{R}$ and vector fields $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$.

**Example 3** *The three co-ordinates $x, y, z$ are each scalar fields on $\mathbb{R}^3$. The position vector $\mathbf{r} = (x, y, z)$ is a vector field; its magnitude $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ is a scalar field, though note it is not smooth at $(0, 0, 0)$.*

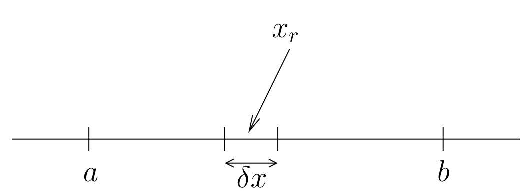
We shall also consider scalar and vector fields defined on proper subsets of $\mathbb{R}^3$ (or more generally $\mathbb{R}^n$). The domains of these fields will usually be *open,* so we can define their partial derivatives.

**Definition 4** *A set $U \subseteq \mathbb{R}^n$ is said to be **open** if for every $\mathbf{x} \in U$ there exists $\varepsilon > 0$ such that*

$$B(\mathbf{x}; \varepsilon) = \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{y} - \mathbf{x}| < \varepsilon\} \subset U, \qquad \text{where} \qquad |\mathbf{y} - \mathbf{x}| = \left[\sum_{i=1}^{n} (y_i - x_i)^2\right]^{1/2},$$

*where $x_i$, $y_i$ are the $i^{th}$ components of $\mathbf{x}$, $\mathbf{y}$. We refer to $B(\mathbf{x}; \varepsilon)$ as the **open ball of radius** $\varepsilon$ **about** $\mathbf{x}$.*

### 1.1.1 Integrals in one dimension




An informal definition of the integral

$$\int_a^b f(x)\mathrm{d}x, \tag{1.1}$$

would be as follows.

Suppose $f : [a, b] \to \mathbb{R}$ is a function. Subdivide $[a, b]$ into $m$ sub-intervals of equal length $\delta x$ and let $x_1, \ldots, x_m$ be points in the respective intervals. On partitioning with smaller and smaller intervals by taking the limit $\delta x \to 0$, we have

$$\int_a^b f(x)\mathrm{d}x = \lim_{\delta x \to 0} \sum_{r=1}^{m} f(x_r)\delta x, \tag{1.2}$$

provided the limit exists. This will always be the case if $f$ is continuous (also, alternative subdivisions will not change the limit when $f$ is continuous).

## 1.2 Multiple Integrals in Two Dimensions: A Brief Introduction

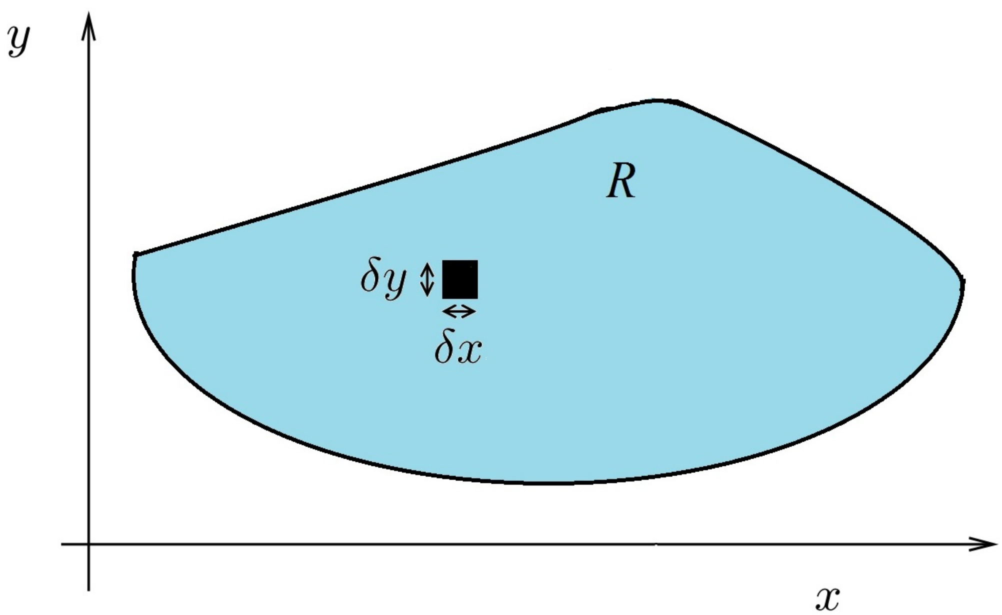



Figure 1-1 A region $R$ for a double integration, with a square element of area $\delta x \delta y$.

Consider a region of the plane, $R$, such as depicted in Figure 1-1, together with a scalar field $\psi(x, y)$.

- Partition the region into $N$ square elements of equal area $\delta A = \delta x \delta y$; difficulties with boundary elements extending outside the region $R$ will disappear below.
- Suppose the scalar field $\psi(\mathbf{r})$ takes the value $\psi_i$ at the centre of the $i^{th}$ element, $i \in \{1, \ldots N\}$.

- On partitioning with smaller and smaller squares, by taking the limit $\delta A \to 0$, we have

$$\lim \sum_{i=1}^{N} \psi_i \delta A = \lim \sum_{i=1}^{N} \psi_i \delta x \delta y \stackrel{def}{=} \iint\limits_R \psi \mathrm{d}A = \iint\limits_R \psi \mathrm{d}x \mathrm{d}y, \tag{1.3}$$

  noting the region with boundary elements extending outside the region $R$ yields no contribution to the integral in the limit.

- This is an informal definition of a **double integral**; a rigorous approach would, for example, verify the limit is independent of the details of the elements used to decompose $R$. If $\psi$ is piecewise continuous and $R$ is a suitable region (we only ever consider such cases), there are no difficulties. In general, which is not the objective here, complexities can emerge (e.g. in three dimensions, The Banach-Tarski paradox).

**Note:** $\psi(\mathbf{r}) = 1 \; \forall \; \mathbf{r} \;\; \Rightarrow \;\; \iint_R \psi \mathrm{d}A = \int_R \mathrm{d}x\mathrm{d}y =$ (area of $R$).

**Properties of double integrals.** The following properties are inherited from integration with respect to one variable (i.e., integrals of the form $\int f(x)\mathrm{d}x$.).

- *Linearity.* Let $a,\ b$ be constants.

$$\iint\limits_R \left(af(x,y) + bg(x,y)\right) \mathrm{d}A = a \iint\limits_R f(x,y)\mathrm{d}A + b \iint\limits_R g(x,y)\mathrm{d}A.$$

- *Order.* If $f(x,y) \geq g(x,y) \; \forall \; (x,y) \in R$ then $\iint_R f(x,y)\mathrm{d}A \geq \iint_R g(x,y)\mathrm{d}A$.

- *Domain Splitting.* If $R = R_1 \cup R_2$, and $R_1 \cap R_2$ the empty set, then

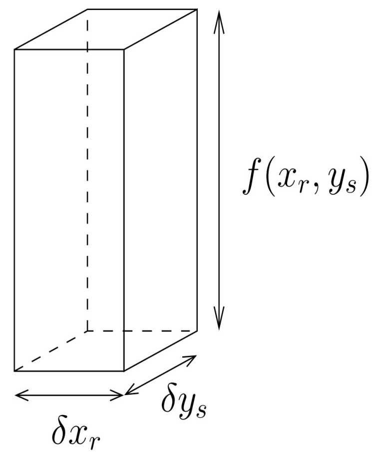
$$\iint\limits_R f(x,y)\mathrm{d}A = \iint\limits_{R_1} f(x,y)dA + \iint\limits_{R_2} f(x,y)\mathrm{d}A.$$

**Interpretation.** We can view the integral as the weight of a plate $R$ in Fig. 1-1, given an area density $\psi(x,y)$ at $(x,y)$. The mass inside the element with sides $\delta x$, and $\delta y$ is $\psi_i \delta x \delta y$ and the total mass is given by summing and taking the limit.

Alternatively, we can interpret the integral as the volume under the surface $\psi(x,y)$ (provided $\psi(x,y) \geq 0$). The height of the cuboid associated with element $i$ is $\psi_i$ and the area of its base is $\delta x \delta y$, so the volume of the cuboid is $\psi_i \delta x \delta y$. The total volume is given by summing and taking the limit.

**Example 5** *Take* $R = [0,2] \times [1,3]$ *and evaluate*

$$I = \iint_R (x + y^2)\mathrm{d}x\mathrm{d}y. \tag{1.4}$$




**Solution.** We compute (first integrating with respect to $x$ and keeping $y$ constant)

$$\begin{aligned} I &= \int_{y=1}^{y=3}\left(\int_{x=0}^{x=2}(x+y^2)\mathrm{d}x\right)\mathrm{d}y = \int_{y=1}^{y=3}\left[\frac{x^2}{2}+y^2x\right]_{x=0}^{x=2}\mathrm{d}y, \\ &= \int_{y=1}^{y=3}\left(\frac{4}{2}+2y^2-0\right)\mathrm{d}y = \left[2y+\frac{2y^3}{3}\right]_{y=1}^{y=3} = \frac{64}{3}, \end{aligned}$$

or (first integrating with respect to $y$ and keeping $x$ constant)

$$\begin{aligned} I &= \int_{x=0}^{x=2}\left(\int_{y=1}^{y=3}(x+y^2)\mathrm{d}y\right)\mathrm{d}x, \\ &= \int_{x=0}^{x=2}\left[xy+\frac{y^3}{3}\right]_{y=1}^{y=3}\mathrm{d}x = \int_{x=0}^{x=2}\left(3x+\frac{27}{3}-\left(x+\frac{1}{3}\right)\right)\mathrm{d}x, \\ &= \int_{x=0}^{x=2}\left(2x+\frac{26}{3}\right)\mathrm{d}x = \left[\frac{2x^2}{2}+\frac{26x}{3}\right]_{x=0}^{x=2} = \frac{64}{3}. \end{aligned}$$

■

**Note:** We get the same answer regardless of the order in which we do the integrals (as one would expect!).

**Example 6** *Let $R$ be the unit square. Determine* $\displaystyle\iint_R y\cos^2(\pi xy)dA.$

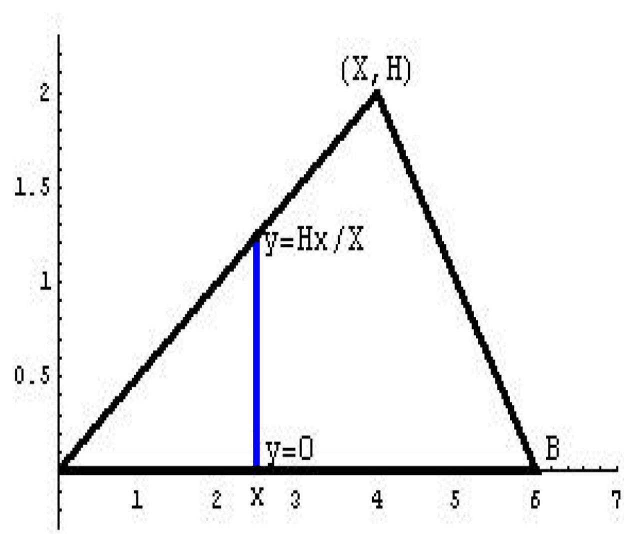
**Solution.** We can represent the domain by $x \in [0,1],\ y \in [0,1]$, and thus

$$\begin{aligned}
\iint\limits_R y\cos^2(\pi xy)\mathrm{d}A &= \iint\limits_R y\cos^2(\pi xy)\mathrm{d}x\mathrm{d}y = \int_0^1 \left\{ \int_0^1 y\cos^2(\pi xy)\mathrm{d}x \right\} \mathrm{d}y \\
&= \frac{1}{2\pi}\int_0^1 \{\pi y + \cos(\pi y)\sin(\pi y)\}\,\mathrm{d}y = \frac{1}{4} \\
&\equiv \int_0^1 \left\{ \int_0^1 \cos^2(\pi xy)\mathrm{d}y \right\} \mathrm{d}x. \qquad (1.5)
\end{aligned}$$

■

**Note:** It is much easier to do the $x$ integral first, even though the answer is independent of the order of integration.

**Important:** For general domains, we need to be careful with the limits of the integrals, as they depend on the order of integration.We illustrate with a trianglular domain.




**Example 7** *Calculate the area of the triangle with vertices* $(0,0)$ , $(B,0)$ *and* $(X,H)$.

**Solution.** We know that the answer is "$\frac{1}{2}BH$", but we shall demonstrate this via a double integral. The three bounding lines of the triangles are

$$y = 0, \qquad y = \frac{H}{X}x, \qquad y = \frac{H}{X-B}(x-B).$$

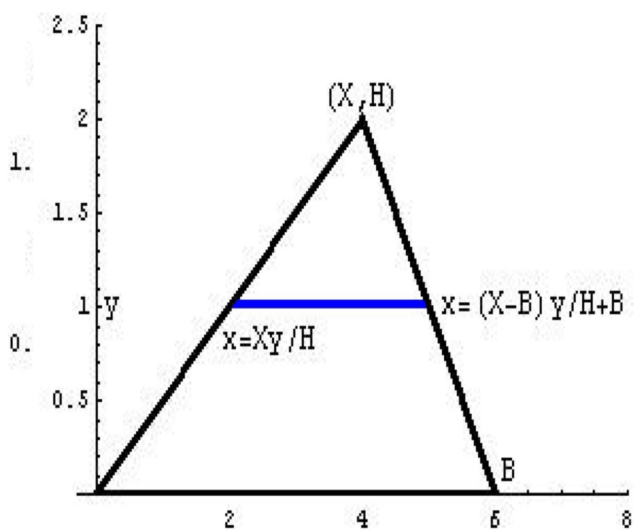
We'll assume first that $0 < X < B$. In order to "pick up" all of the triangle's area, $x$ must range from $x = 0$ to $x = B$ and $y$ from $y = 0$ to the bounding line above $(x,0)$. As $x$ and $y$

sweep out the triangle, we need to pick up an infinitesimal piece of area ($\mathrm{d}x\ \mathrm{d}y$) at each point. We calculate the triangle's area as

$$
\begin{aligned}
A &= \int_{x=0}^{x=X}\int_{y=0}^{y=Hx/X} \mathrm{d}y\,\mathrm{d}x + \int_{x=X}^{x=B}\int_{y=0}^{y=H(x-B)/(X-B)} \mathrm{d}y\,\mathrm{d}x \\
&= \int_{x=0}^{x=X} \frac{Hx}{X}\,\mathrm{d}x + \int_{x=X}^{x=B} \frac{H\,(x-B)}{X-B}\,\mathrm{d}x \\
&= \frac{H}{X}\left[\frac{x^2}{2}\right]_0^X + \frac{H}{X-B}\left[\frac{(x-B)^2}{2}\right]_X^B \\
&= \frac{H}{X}\frac{X^2}{2} - \frac{H}{X-B}\frac{(X-B)^2}{2} = \frac{HX}{2} - \frac{H\,(X-B)}{2} \\
&= \frac{HB}{2}.
\end{aligned}
$$

■

Integrating this way (i.e., first $y$, then $x$), we need to treat $X < 0$ and $X > B$ as two further cases. Alternatively, we could determine the area by letting $y$ range from $y = 0$ to $y = H$ and letting $x$ range over the interior points of the triangle at height $y$ (i.e., first $x$, then $y$)




**Solution.** This method is easier as we don't have to treat the cases $X < 0$, $0 < X < B$ and

$B < X$ separately.

$$\begin{aligned}
A &= \int_{y=0}^{y=H} \int_{x=Xy/H}^{x=y(X-B)/H+B} \mathrm{d}x\, \mathrm{d}y \\
&= \int_{y=0}^{y=H} \left( \frac{(X-B)\,y}{H} + B - \frac{yX}{H} \right) \mathrm{d}y \\
&= B \int_{y=0}^{y=H} \left( 1 - \frac{y}{H} \right) \mathrm{d}y = B \left[ y - \frac{y^2}{2H} \right]_{y=0}^{y=H} = B \left( H - \frac{H^2}{2H} \right) \\
&= \frac{BH}{2}
\end{aligned}$$

■

More generally, we choose the limits to suit the domain:

**Example 8** *Calculate the area of the disc* $x^2 + y^2 \leqslant a^2$.

**Solution.** Again, we know the answer: $\pi a^2$. If we wish to pick up all of the disc's area we can let $x$ vary over the range $-a$ to $a$ and, at each $x$, we need to let $y$ vary from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$. So we have

$$\begin{aligned}
A &= \int_{x=-a}^{x=a} \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \mathrm{d}y\, \mathrm{d}x \\
&= \int_{x=-a}^{x=a} 2\sqrt{a^2 - x^2}\, \mathrm{d}x \\
&= \int_{\theta=-\pi/2}^{\theta=\pi/2} 2\sqrt{a^2 - a^2 \sin^2\theta}\, a\cos\theta\, d\theta \qquad [x = a\sin\theta] \\
&= a^2 \int_{-\pi/2}^{\pi/2} 2\cos^2\theta\, \mathrm{d}\theta = a^2 \int_{-\pi/2}^{\pi/2} 1 + \cos 2\theta\ \mathrm{d}\theta = a^2 \left[ \theta + \frac{1}{2}\sin 2\theta \right]_{-\pi/2}^{\pi/2} \\
&= \pi a^2.
\end{aligned}$$

■

In the first example of the triangle, it is simpler to do $x$-integration before $y$-integration (to avoid consdering special cases). For the example of the disc, it would be more natural to use polar co-ordinates — *if* we knew how to calculate areas with them!

## 1.3 Change of Variables and Jacobians.

The Jacobian, or rather its modulus, is a measure of how a general mapping stretches space locally, near a particular point, even when this stretching effect varies from point to point.

The Jacobian takes its name from the German mathematician Carl Jacobi (1804-1851).

**Definition 9** *Given two co-ordinates $u(x,y)$ and $v(x,y)$ which depend on variables $x$ and $y$, we define the* ***Jacobian***

$$\frac{\partial(u,v)}{\partial(x,y)}$$

*to be the determinant*

$$\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

In 3D, we define the Jacobian

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}$$

to be the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

**Example 10** *Let $x = r\cos\theta$ and $y = r\sin\theta$ where $r$ and $\theta$ are polar co-ordinates. Then*

$$\begin{aligned} \frac{\partial(x,y)}{\partial(r,\theta)} &= \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \\ &= \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= r\left(\cos^2\theta + \sin^2\theta\right) = r. \end{aligned}$$

**Example 11** *In reverse, $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$ and*

$$\begin{aligned} \frac{\partial(r,\theta)}{\partial(x,y)} &= \det\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} \\ &= \det\begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} \\ &= \frac{x^2+y^2}{(x^2+y^2)^{3/2}} \\ &= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r} \end{aligned}$$

**Theorem 12** *Let $\mathbf{f} : R \to S$ be a bijection between two regions of $\mathbb{R}^2$, which is differentiable and has differentiable inverse with*

$$\frac{\partial(u,v)}{\partial(x,y)}, \quad \frac{\partial(x,y)}{\partial(u,v)},$$

*defined and non-zero everywhere. Further, write $(u,v) = \mathbf{f}(x,y)$ and let $\psi(x,y) = \Psi(u,v)$. Then*

$$\begin{aligned}
\iint\limits_{(u,v)\in S} \Psi(u,v)du\, dv &= \iint\limits_{(x,y)\in R} \psi(x,y)\left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\, dy,\\
\iint\limits_{(x,y)\in R} \psi(x,y)dx\, dy &= \iint\limits_{(u,v)\in S} \Psi(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.
\end{aligned}$$

**Proof.** (Sketch proof of preceding theorem) It is sufficient to prove the first integral identity. Divide the region $R$ into $N$ square elements of equal area $\delta x\delta y$ as previously with $\psi_i$ the scalar field at the centre of the $i^{th}$ element, $i \in \{1, \ldots N\}$.

Consider the mapping of the $i^{th}$ element, which is bounded by the co-ordinate lines $x = x_i$ and $x = x_i + \delta x$ and $y = y_i$ and $y = y_i + \delta y$. The value of the scalar field at the mapped element centre is

$$\Psi_i = \Psi(\mathbf{f}(x_i + \delta x/2,\ y_i + \delta y/2)) = \psi_i = \psi(x_i + \delta x/2,\ y_i + \delta y/2).$$

Also, given sufficiently small $\delta x,\ \delta y$ the $i^{th}$ element maps to an image region, denoted $IM_i$, which is a deformed parallelogram spanned by the vectors

$$\begin{aligned}
\mathbf{a} &= \mathbf{f}(x_i + \delta x, y_i) - \mathbf{f}(x_i, y_i) \approx \frac{\partial \mathbf{f}}{\partial x}(x_i, y_i)\ \delta x,\\
\mathbf{b} &= \mathbf{f}(x_i, y_i + \delta y) - \mathbf{f}(x_i, y_i) \approx \frac{\partial \mathbf{f}}{\partial y}(x_i, y_i)\ \delta y,
\end{aligned}$$

and, thus, of area

$$\left|\frac{\partial \mathbf{f}}{\partial x}\delta x \wedge \frac{\partial \mathbf{f}}{\partial y}\delta y\right| = \left|\frac{\partial \mathbf{f}}{\partial x} \wedge \frac{\partial \mathbf{f}}{\partial y}\right| \delta x\, \delta y.$$

Now $\mathbf{f} = (u,v)$, so $\mathbf{f}_x = (u_x, v_x)$, $\mathbf{f}_y = (u_y, v_y)$ and thus

$$\left|\frac{\partial \mathbf{f}}{\partial x} \wedge \frac{\partial \mathbf{f}}{\partial y}\right| \delta x\delta y = |(u_x, v_x) \wedge (u_y, v_y)|\, \delta x\delta y = |(u_x v_y - u_y v_x)\, \mathbf{k}|\, \delta x\delta y = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \delta x\delta y.$$

Thus, before taking limits, partitioning $S$ by the images $IM_i$, we have an approximation for

$$\iint\limits_{(u,v)\in S} \Psi(u,v)\mathrm{d}u\,\mathrm{d}v$$

which is

$$\sum_{i=1}^{N} \Psi_i \mathrm{Area}(IM_i) = \sum_{i=1}^{N} \Psi_i \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \delta x\delta y = \sum_{i=1}^{N} \psi_i \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \delta x\delta y.$$

Taking limits gives

$$\iint\limits_{(u,v)\in S} \Psi(u,v)\mathrm{d}u\,\mathrm{d}v = \iint\limits_{(x,y)\in R} \psi(x,y)\left|\frac{\partial(u,v)}{\partial(x,y)}\right| \mathrm{d}x\,\mathrm{d}y.$$

■

If the previous motivation for this formula seems somewhat non-rigorous, note that the chain rule for Jacobians ensures that it is impossible that we might determine two different answers for a double integral by using different sets of variables.

**Note:** to evaluate integrals over an infinite domain (e.g., $\int_{-\infty}^{\infty} \exp(-x^2)\mathrm{d}x$), we define

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{\substack{X\to\infty \\ Y\to-\infty}} \int_{Y}^{X} f(x)\mathrm{d}x. \tag{1.6}$$

**Exercise 13** *Evaluate*

$$\iint\limits_{\mathbb{R}^2} \exp[-(x^2+y^2)]\mathrm{d}A.$$

*Hence, determine* $\int_{-\infty}^{\infty} \exp[-p^2]\mathrm{d}p$.

**Solution.**

$$\begin{aligned} I &= \iint\limits_{\mathbb{R}^2} \exp[-(x^2+y^2)]\mathrm{d}A = \iint\limits_{\mathbb{R}^2} \exp[-(x^2+y^2)]\mathrm{d}x\mathrm{d}y \\ &= \iint\limits_{\mathbb{R}^2} \exp[-(r^2)] \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \mathrm{d}r\mathrm{d}\theta = \int_0^\infty \left\{\int_0^{2\pi} \mathrm{d}\theta\right\} \exp[-(r^2)]r\mathrm{d}r = \pi \int_0^\infty 2r\exp[-(r^2)]\mathrm{d}r \\ &= \pi\left[\exp[-(r^2)]\right]_0^\infty = \pi. \end{aligned}$$

Furthermore let $J = \int_{-\infty}^{\infty} \exp[-p^2]\mathrm{d}p$. Then

$$J^2 = \int_{-\infty}^{\infty} \exp[-p^2]\mathrm{d}p \int_{-\infty}^{\infty} \exp[-q^2]\mathrm{d}q = \iint\limits_{\mathbb{R}^2} \exp[-(p^2+q^2)]\mathrm{d}p\mathrm{d}q = I.$$

Noting $J > 0$ to give the sign of the root, we thus have $J = \sqrt{\pi}$. ■

**Example 14** *Calculate the area bounded by the curves*

$$2x = 1-y^2, \qquad 2x = y^2-1, \qquad 8x = 16-y^2, \qquad 8x = y^2-16.$$

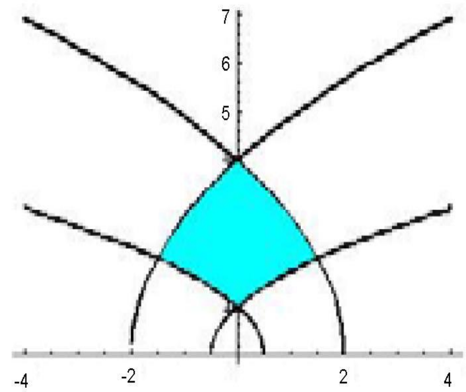
*as shown in the diagram.*

**Solution.** If we change to $(u,v)$, parabolic coordinates

$$x = \frac{1}{2}\left(u^2-v^2\right), \qquad y = uv,$$

then the region of interest is $1 \leqslant u \leqslant 2, \ \ 1 \leqslant v \leqslant 2$. The Jacobian is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} u & -v \\ v & u \end{pmatrix} = u^2+v^2.$$

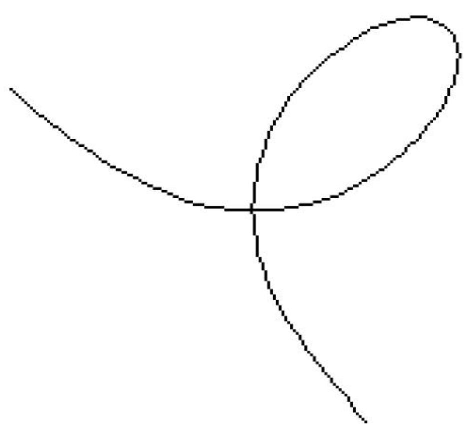



Hence, the area is given by

$$\begin{aligned} A &= \int_{u=1}^{u=2}\int_{v=1}^{v=2}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}v\,\mathrm{d}u = \int_{u=1}^{u=2}\int_{v=1}^{v=2}\left(u^2+v^2\right)\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_{u=1}^{u=2}\left[u^2v+\frac{v^3}{3}\right]_{v=1}^{v=2}\mathrm{d}u = \int_{u=1}^{u=2}\left(u^2+\frac{7}{3}\right)\,\mathrm{d}u \\ &= \left[\frac{u^3}{3}+\frac{7u}{3}\right]_1^2 = \frac{7}{3}+\frac{7}{3} = \frac{14}{3}. \end{aligned}$$

■

**Example 15** *Find the area bounded by the Folium of Descartes (see diagram) which has equation* $x^3+y^3=3xy$.



**Solution.** In polar co-ordinates we have

$$r^3\cos^3\theta + r^3\sin^3\theta = 3r^2\sin\theta\cos\theta$$

which we can rearrange to

$$r = \frac{3\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}.$$

The curve passes through the origin at $\theta = 0$ and $\theta = \pi/2$ and hence the area bounded is

$$A = \int\int r dr d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{9\sin^2\theta\cos^2\theta}{\left(\cos^3\theta + \sin^3\theta\right)^2}\,\mathrm{d}\theta = \frac{3}{2}.$$

■

**Exercise 16** *How would you calculate the above integral? The substitution* $t = \tan\frac{1}{2}\theta$ *is one option: it converts the integrand to a rational function.*

# 2. VOLUME INTEGRATION

## 2.1 An Informal Definition of The Volume Integral

Consider a scalar field $\psi(x, y, z)$, and a three-dimensional region $R$.

- Partition $R$ into $N$ cubic elements, of volume $\delta V = \delta x \delta y \delta z$ and let $\psi_i$ denote the value of $\psi$ at the centre of the $i^{th}$ cubic element, $i \in \{1, \dots N\}$.

- Then, on partitioning with smaller and smaller cubes, and taking the limit $\delta V \to 0$,

$$\lim \sum_{i=1}^{N} \psi_i \delta V \stackrel{def}{=} \iiint\limits_R \psi \mathrm{d}V = \iiint\limits_R \psi \mathrm{d}x \mathrm{d}y \mathrm{d}z \tag{2.1}$$

- This is an informal definition. For the regions $R$ and functions $\psi$ we shall meet there will never be any issue about whether the integral exists and we will usually determine such integrals by calculating three definite integrals separately over co-ordinates $x, y, z$ (or other more appropriate co-ordinates). Again, the Banach-Tarski paradox illustrates the complexities that can emerge but these are not the focus.

**Note:** If the scalar field $\psi(x, y, z) = 1$ then the integral $\int_R \psi \mathrm{d}V = \int_R \mathrm{d}V$ gives the volume of the region $R$.

## 2.2 Examples

**Example 17** *A cone of height $h$ occupies the region*

$$x^2 + y^2 \leqslant z^2, \qquad 0 \leqslant z \leqslant h.$$

*and has density $\rho(x, y, z) = (x^2 + y^2) z$ at each point. Find the mass of the cone using Cartesian co-ordinates.*

**Solution.** First, we must divide up the cone and find the corresponding limits. There are many ways to proceed. For example, we might take cross-sections of the cones, decomposing the cone into two- and then one-dimensional sections. Poorly chosen co-ordinates, whilst they could be used in principle to determine the mass, may require calculation of a complicated set of integrals.

If we use $z$ as our first variable to divide up the cone, then the cross-section of the cone with a plane $z = z_0$ gives a disc $x^2 + y^2 \leqslant (z_0)^2$, $z = z_0$ (at least if $0 \leqslant z_0 \leqslant h$). We can then, say, take cross-sections of such discs with the line $y = y_0$ to produce horizontal slithers

$$x^2 \leqslant (z_0)^2 - (y_0)^2, \; y = y_0, \; z = z_0 \quad \text{(at least if } |y_0| \leqslant z_0)$$

and calculating their contribution to the mass is a simple one-dimensional integral.

So, the mass $M$ is given by the triple integral

$$\begin{aligned} M &= \int_{z=0}^{h} \int_{y=-z}^{z} \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} \rho(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \int_{z=0}^{h} \int_{y=-z}^{z} \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} \left(x^2 + y^2\right) z \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \end{aligned}$$

We calculate the internal integrals in turn. We define

$$\begin{aligned} I_1(z, y) = \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} \left(x^2 + y^2\right) z \, \mathrm{d}x &= \left[ z \left( \frac{x^3}{3} + y^2 x \right) \right]_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} \\ &= 2z \left( \frac{\left(z^2 - y^2\right)^{3/2}}{3} + y^2 \sqrt{z^2 - y^2} \right) \\ &= \frac{2z}{3} \left(z^2 + 2y^2\right) \sqrt{z^2 - y^2}. \end{aligned}$$

The next internal integral is then

$$I_2(z) = \int_{y=-z}^{z} \frac{2z}{3} \left(z^2 + 2y^2\right) \sqrt{z^2 - y^2} \, \mathrm{d}y.$$

If we make the substitution $y = z \sin t$, where $-\pi/2 \leqslant t \leqslant \pi/2$, then $I_2(z)$ becomes

$$\begin{aligned} I_2(z) &= \int_{t=-\pi/2}^{\pi/2} \frac{2z}{3} \left(z^2 + 2z^2 \sin^2 t\right) \sqrt{z^2 - z^2 \sin^2 t} \; (z \cos t \, \mathrm{d}t) \\ &= \int_{t=-\pi/2}^{\pi/2} \frac{2z^3}{3} \left(1 + 2\sin^2 t\right) |z \cos t| \; (z \cos t \, \mathrm{d}t) \\ &= \frac{2z^5}{3} \int_{t=-\pi/2}^{\pi/2} \left(1 + 2\sin^2 t\right) \cos^2 t \, \mathrm{d}t. \end{aligned}$$

Now

$$\int_{t=-\pi/2}^{\pi/2} \left(\cos^2 t + 2\sin^2 t \cos^2 t\right) \, \mathrm{d}t = \int_{t=-\pi/2}^{\pi/2} \left( \frac{1}{2}(1 + \cos 2t) + \frac{1}{2} \sin^2 2t \right) \mathrm{d}t$$

$$= \int_{t=-\pi/2}^{\pi/2} \left[ \frac{1}{2}(1 + \cos 2t) + \frac{1}{4}(1 - \cos 4t) \right] \mathrm{d}t = \left[ \frac{3t}{4} + \frac{1}{4} \sin 2t - \frac{1}{8} \sin 4t \right]_{-\pi/2}^{\pi/2} = \frac{3\pi}{4}.$$

Finally, we have

$$M = \int_{z=0}^{h} I_2(z)\,\mathrm{d}z = \int_{z=0}^{h} \frac{2z^5}{3} \times \frac{3\pi}{4}\,\mathrm{d}z = \frac{\pi}{2}\left[\frac{z^6}{6}\right]_0^h = \frac{\pi h^6}{12}.$$

■

**Solution.** (Alternative method with planar polars). The mass $M$ could be written as

$$M = \int_{z=0}^{h} \iint\limits_{x^2+y^2\leqslant z^2} \rho\,(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{z=0}^{h} \iint\limits_{x^2+y^2\leqslant z^2} \left(x^2+y^2\right) z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

We know how to change from Cartesian co-ordinates $(x,y)$ to planar polar co-ordinates $(r,\theta)$: the change of variable rule is

$$\iint \phi\,\mathrm{d}A = \iint \phi\,\mathrm{d}x\,\mathrm{d}y = \iint \phi\ (r\,\mathrm{d}r\,\mathrm{d}\theta)\,,$$

where the $r$ appears as it is the Jacobian $\partial\,(x,y)\,/\partial\,(r,\theta)$ . Thus

$$\iint\limits_{x^2+y^2\leqslant z^2} \left(x^2+y^2\right) z\,\mathrm{d}x\,\mathrm{d}y = \int_{r=0}^{z}\int_{\theta=0}^{2\pi} r^2 z\ (r\,\mathrm{d}\theta\,\mathrm{d}r) = 2\pi z \int_{r=0}^{z} r^3\,\mathrm{d}r = 2\pi z\left[\frac{r^4}{4}\right]_0^z = \frac{\pi z^5}{2}.$$

Hence,

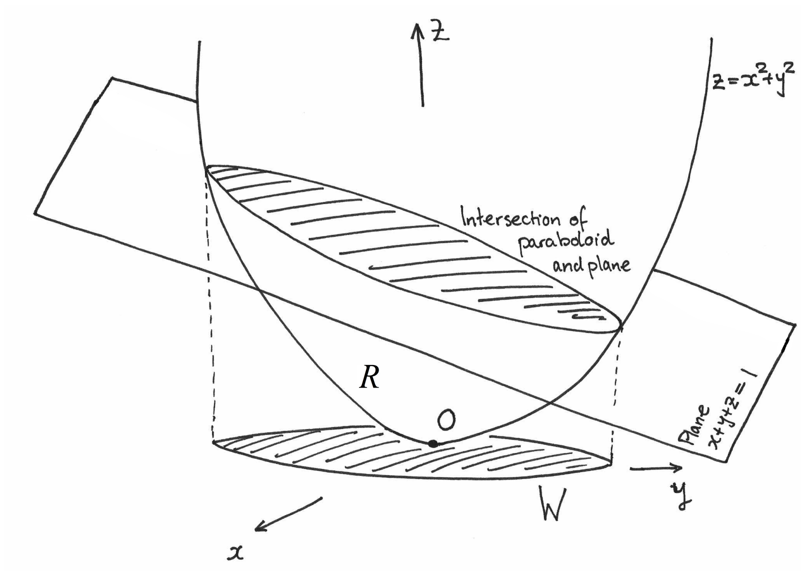
$$M = \int_{z=0}^{h} \frac{\pi z^5}{2}\,\mathrm{d}z = \left[\frac{\pi z^6}{12}\right]_0^h = \frac{\pi h^6}{12}.$$

**Note:** This change of variable makes the calculation much simpler!

■

**Remark 18** *In the above example, we essentially used cylindrical polar co-ordinates, appreciating that the volume element dV is given by*

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$




**Example 19** *Find the volume of the region $R$ that lies above the paraboloid $z = x^2 + y^2$ and beneath the plane $x + y + z = 1$.*

**Solution.** The $(x, y)$ co-ordinates of a point $(x, y, z)$ on the top planar surface of $R$ satisfy

$$1 - x - y \geqslant x^2 + y^2$$

and, so, the top planar surface projects vertically down to the set

$$\begin{aligned} W &= \left\{(x, y) \in \mathbb{R}^2 : x + y + x^2 + y^2 \leqslant 1\right\} \\ &= \left\{(x, y) \in \mathbb{R}^2 : \left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \leqslant \frac{3}{2}\right\} \end{aligned}$$

which is a disc, centre $(-1/2, -1/2)$ and radius $\sqrt{3/2}$. Hence we can determine the volume of the region $R$ as

$$\begin{aligned} V &= \iint_{(x,y)\in W} \left(\int_{z=x^2+y^2}^{1-x-y} \mathrm{d}z\right) \mathrm{d}A \\ &= \iint_{(x,y)\in W} \left(1 - x - y - x^2 - y^2\right) \mathrm{d}A \\ &= \iint_{(x,y)\in W} \left(\frac{3}{2} - \left(x + \frac{1}{2}\right)^2 - \left(y + \frac{1}{2}\right)^2\right) \mathrm{d}A. \end{aligned}$$

Natural co-ordinates for parameterising $W$ are $(r, \theta)$ where

$$x = -\frac{1}{2} + r\cos\theta, \qquad y = -\frac{1}{2} + r\sin\theta, \qquad 0 \leqslant \theta < 2\pi,\ 0 \leqslant r \leqslant \sqrt{3/2}.$$

Hence, we have

$$\begin{aligned} V &= \int_{r=0}^{\sqrt{3/2}} \int_{\theta=0}^{2\pi} \left(\frac{3}{2} - r^2\cos^2\theta - r^2\sin^2\theta\right)(r\,\mathrm{d}\theta\,\mathrm{d}r) \\ &= 2\pi \int_{r=0}^{\sqrt{3/2}} \left(\frac{3r}{2} - r^3\right)\mathrm{d}r = 2\pi\left[\frac{3r^2}{4} - \frac{r^4}{4}\right]_0^{\sqrt{3/2}} \\ &= 2\pi\left(\frac{9}{8} - \frac{9}{16}\right) \\ &= \frac{9\pi}{8}. \end{aligned}$$

■

Once again we have used polar coordinates. We should, therefore, consider what happens with a general change of variable.

## 2.3 Changing Co-ordinates in Volume Integrals

**Theorem 20** *Let $\mathbf{f} : R \to S$ be a bijection between subsets of $\mathbb{R}^3$ which is differentiable and has differentiable inverse. Take co-ordinates $x_i$ on $R$ and $u_i$ on $S$ related by the formula*

$$(u_1, u_2, u_3) = \mathbf{f}(x_1, x_2, x_3),$$

*and let $\psi(x_1, x_2, x_3) = \Psi(u_1, u_2, u_3)$ be scalar fields. Then*

$$\iiint\limits_S \Psi(u_1, u_2, u_3)\, du_1\, du_2 du_3 = \iiint\limits_R \psi(x_1, x_2, x_3)\left|\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)}\right| dx_1\ \ dx_2 dx_3.$$

**Proof.** (Sketch) Partition the region $R$ into $N$ cubic elements and let the $i^{th}$ cubic element be given by

$$[x_1, x_1 + \delta x_1] \times [x_2, x_2 + \delta x_2] \times [x_3, x_3 + \delta x_3]$$

which has volume $\delta x_1\, \delta x_2\, \delta x_3$. The value of the scalar field at the centre of the mapped element is

$$\Psi_i = \Psi(\mathbf{f}(x_1 + \delta x_1/2,\ x_2 + \delta x_2/2,\ x_3 + \delta x_3/2)) = \psi_i = \psi(x_1 + \delta x_1/2,\ x_2 + \delta x_2/2,\ x_3 + \delta x_3/2).$$

Also, under the map $\mathbf{f}$, the $i^{th}$ cubic element maps to a deformed parallelepiped, denoted $IM_i$ below, with sides

$$\begin{aligned} \mathbf{f}(x_1 + \delta x_1, x_2, x_3) - \mathbf{f}(x_1, x_2, x_3) &\approx \frac{\partial \mathbf{f}}{\partial x_1}\delta x_1; \\ \mathbf{f}(x_1, x_2 + \delta x_2, x_3) - \mathbf{f}(x_1, x_2, x_3) &\approx \frac{\partial \mathbf{f}}{\partial x_2}\delta x_2; \\ \mathbf{f}(x_1, x_2, x_3 + \delta x_3) - \mathbf{f}(x_1, x_2, x_3) &\approx \frac{\partial \mathbf{f}}{\partial x_3}\delta x_3. \end{aligned}$$

The volume of a parallelepiped with sides $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is $|\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})\,|$ and, so, as a first approximation, the volume of the image of the $i^{th}$ element is

$$\left|\left[\frac{\partial \mathbf{f}}{\partial x_1}, \frac{\partial \mathbf{f}}{\partial x_2}, \frac{\partial \mathbf{f}}{\partial x_3}\right]\right| \delta x_1\,\delta x_2\,\delta x_3 = \left|\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1,x_2,x_3\right)}\right| \delta x_1\,\delta x_2\,\delta x_3.$$

Thus, before taking limits, partitioning $S$ by the images $IM_i$, an approximation for

$$\iiint\limits_S \Psi\left(u_1,u_2,u_3\right) \mathrm{d}u_1\,\mathrm{d}u_2\mathrm{d}u_3$$

is

$$\sum_{i=1}^{N} \Psi_i \mathrm{Volume}(IM_i) = \sum_{i=1}^{N} \Psi_i \left|\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1,x_2,x_3\right)}\right| \delta x_1 \delta x_2 \delta x_3 = \sum_{i=1}^{N} \psi_i \left|\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1,x_2,x_3\right)}\right| \delta x_1 \delta x_2 \delta x_3.$$

Taking limits gives

$$\iiint\limits_{(u_1,u_2,u_3)\in S} \Psi\left(u_1,u_2,u_3\right) \mathrm{d}u_1\,\mathrm{d}u_2\mathrm{d}u_3 = \iiint\limits_{(x_1,x_2,x_3)\in R} \psi\left(x_1,x_2,x_3\right) \left|\frac{\partial\left(u_1,u_2,u_3\right)}{\partial\left(x_1,x_2,x_3\right)}\right| \mathrm{d}x_1\,\mathrm{d}x_2\mathrm{d}x_3.$$

■

If the previous motivation for this formula seems non-rigorous, note that the chain rule for Jacobians ensures that it is impossible to obtain different answers for a volume integral by using different sets of variables.

**Example 21** *(Cylindrical Polar Co-ordinates)*

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

*Then*

$$\frac{\partial\left(x,y,z\right)}{\partial\left(r,\theta,z\right)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial r} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

**Example 22** *(Spherical Polar Co-ordinates)*

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta, \qquad \theta\in[0,\pi], \quad \phi\in[0,2\pi).$$

*Then*

$$\begin{aligned}
\frac{\partial (x,y,z)}{\partial (r,\phi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\
&= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \\
&= \cos\theta \begin{vmatrix} r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} \\
&= r^2\cos\theta\,\sin\theta\cos\theta\left(\cos^2\phi+\sin^2\phi\right) + r^2\sin^3\theta\left(\cos^2\phi+\sin^2\phi\right) \\
&= r^2\sin\theta\left(\cos^2\theta+\sin^2\theta\right) \\
&= r^2\sin\theta \geq 0 \quad \textit{since } \theta\in[0,\pi].
\end{aligned}$$

Thus, for spherical polar co-ordinates, we have

$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = r^2\,\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

**Definition 23** *The **centre of mass** of a body occupying a region $R$ and with density $\rho(\mathbf{r})$ at the point with position vector $\mathbf{r}$ is*

$$\bar{\mathbf{r}} = (\bar{x},\bar{y},\bar{z}) = \frac{1}{M}\iiint_R \mathbf{r}\,\rho(\mathbf{r})\,\mathrm{d}V$$

*where $M$ is the total mass of the body.*

**Example 24** *Find the centre of mass of a uniform hemisphere*

$$\left\{(x,y,z) : x^2+y^2+z^2 < a^2,\ z>0\right\}.$$

**Solution.** By symmetry, the centre of mass lies at a point $(0,0,\bar{z})$ on the $z$-axis. Let $\rho$ denote the density of the hemisphere. Then its mass is $\frac{2}{3}\pi a^3\rho$ and

$$\begin{aligned}
\bar{z} &= \frac{3}{2\pi a^3\rho}\int_{r=0}^{a}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\theta=\pi/2}(r\cos\theta)\left(\rho r^2\sin\theta\right)\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \\
&= \frac{3}{2\pi a^3\rho}\times 2\pi\rho\int_{r=0}^{a} r^3\,\mathrm{d}r\int_{\theta=0}^{\theta=\pi/2}\cos\theta\sin\theta\,\mathrm{d}\theta \\
&= \frac{3}{a^3}\times\left[\frac{r^4}{4}\right]_0^a\times\left[\frac{-\cos 2\theta}{4}\right]_0^{\pi/2} = \frac{3}{a^3}\times\frac{a^4}{4}\times\frac{1}{2} \\
&= \frac{3a}{8}.
\end{aligned}$$

■

**Example 25** *Evaluate the integral*

$$\iiint\limits_{R} x \, \mathrm{d}V$$

*over the intersection of the unit ball* $x^2 + y^2 + z^2 \leqslant 1$ *with the half-space* $x + y \geqslant 3\sqrt{2}/5$.

**Solution.** If we take as an orthonormal basis

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1, -1, 0), \qquad \mathbf{e}_2 = (0, 0, 1), \qquad \mathbf{e}_3 = \frac{1}{\sqrt{2}}(1, 1, 0),$$

with corresponding co-ordinates $X, Y, Z$ so that

$$X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

then we have $x = (X + Z)/\sqrt{2}$ and $x + y = Z\sqrt{2}$.

So we are left considering the integral

$$\iiint\limits_{\substack{X^2+Y^2+Z^2 \leqslant 1 \\ Z \geqslant 3/5}} \left(\frac{X + Z}{\sqrt{2}}\right) \mathrm{d}V.$$

By symmetry

$$\iiint\limits_{\substack{X^2+Y^2+Z^2 \leqslant 1 \\ Z \geqslant 3/5}} X \, \mathrm{d}V = 0,$$

so we are left to calculate

$$\begin{aligned}
\frac{1}{\sqrt{2}} \iiint\limits_{\substack{X^2+Y^2+Z^2 \leqslant 1 \\ Z \geqslant 3/5}} Z \, \mathrm{d}V &= \frac{1}{\sqrt{2}} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\cos^{-1}(3/5)} \int_{r=(3/5)\sec\theta}^{1} (r\cos\theta)\, r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\
&= \frac{2\pi}{\sqrt{2}} \int_{\theta=0}^{\cos^{-1}(3/5)} \int_{r=(3/5)\sec\theta}^{1} r^3 \cos\theta \, \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \pi\sqrt{2} \int_{\theta=0}^{\cos^{-1}(3/5)} \left[\frac{r^4}{4}\right]_{(3/5)\sec\theta}^{1} \cos\theta \, \sin\theta \, \mathrm{d}\theta \\
&= \frac{\pi\sqrt{2}}{4} \int_{\theta=0}^{\cos^{-1}(3/5)} \left(1 - \frac{81}{625}\sec^4\theta\right) \cos\theta \, \sin\theta \, \mathrm{d}\theta \\
&= \frac{\pi\sqrt{2}}{4} \int_{\theta=0}^{\cos^{-1}(3/5)} \left(\cos\theta \, \sin\theta - \frac{81}{625}\frac{\sin\theta}{\cos^3\theta}\right) \mathrm{d}\theta \\
&= \frac{\pi\sqrt{2}}{4} \left[-\frac{1}{2}\cos^2\theta - \frac{81}{1250}\frac{1}{\cos^2\theta}\right]_0^{\cos^{-1}(3/5)} \\
&= \frac{\pi\sqrt{2}}{4} \left[-\frac{1}{2} \times \frac{9}{25} - \frac{81}{1250} \times \frac{25}{9} + \frac{1}{2} + \frac{81}{1250}\right] \\
&= \frac{\pi\sqrt{2}}{4} \left[\frac{128}{625}\right] = \frac{32\pi\sqrt{2}}{625}.
\end{aligned}$$

■

**Example 26** *By a suitable rotation of co-ordinates, or otherwise, evaluate*

$$\iiint (ax+by+cz)^4 \,\mathrm{d}V$$

*taken over the region* $x^2+y^2+z^2 \leqslant 1$, *where* $a, b, c$ *are positive constants.*

**Solution.** Consider instead this integral as

$$\iiint_{x^2+y^2+z^2\leqslant 1} ((a,b,c)\cdot \mathbf{r})^4 \,\mathrm{d}V.$$

We can rotate the axes to create new co-ordinates $X, Y, Z$ in such a way that the $Z$ axis points in the direction of vector $(a,b,c)$ and under this change of co-ordinates the integral becomes

$$\iiint_{X^2+Y^2+Z^2\leqslant 1} \left(\sqrt{a^2+b^2+c^2}\,\mathbf{e}_3\cdot\mathbf{r}\right)^4 \mathrm{d}V = \iiint_{X^2+Y^2+Z^2\leqslant 1} \left(a^2+b^2+c^2\right)^2 Z^4 \,\mathrm{d}V$$

where $\mathbf{e}_3$ is the unit vector pointing down the $Z$-axis; notice that we are still integrating over the unit sphere as we have simply rotated the axes about the origin.

Now, by a change to spherical polar co-ordinates,

$$\begin{aligned}
&\iiint_{X^2+Y^2+Z^2\leqslant 1} \left(a^2+b^2+c^2\right)^2 Z^4 \,\mathrm{d}V \\
=\;& \left(a^2+b^2+c^2\right)^2 \int_{r=0}^{1}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (r\cos\theta)^4 \left(r^2\sin\theta\right) \mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \\
=\;& 2\pi\left(a^2+b^2+c^2\right)^2 \left[\frac{r^7}{7}\right]_0^1 \left[-\frac{\cos^5\theta}{5}\right]_0^{\pi} \\
=\;& 2\pi\left(a^2+b^2+c^2\right)^2 \times \frac{1}{7}\times\frac{2}{5} \\
=\;& \frac{4\pi}{35}\left(a^2+b^2+c^2\right)^2.
\end{aligned}$$

■

**Example 27** *Let* $a,b,c\in\mathbb{R}$. *Determine*

$$I = \iiint_{x^2+y^2+z^2\leqslant 1} \cos(ax+by+cz)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

*Show that your answer is consistent with the fact the volume of the unit sphere is* $4\pi/3$.

**Solution.** We can rewrite the integral as

$$I = \iiint\limits_{x^2+y^2+z^2\leqslant 1} \cos(\mathbf{a}\cdot\mathbf{r})\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

where $\mathbf{a} = (a, b, c)$.

The vector

$$\mathbf{e}_3 = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$$

is of unit length and so we can extend it to an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, with associated co-ordinates $(X, Y, Z)$. In terms of these co-ordinates

$$\mathbf{a}\cdot\mathbf{r} = (0, 0, |\mathbf{a}|)\cdot(X, Y, Z) = |\mathbf{a}|\ Z.$$

The unit sphere $x^2+y^2+z^2 \leqslant 1$ is still given as $X^2+Y^2+Z^2 \leqslant 1$ as $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are an orthonormal basis and $\mathrm{d}V = \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z$. So

$$I = \iiint\limits_{X^2+Y^2+Z^2\leqslant 1} \cos(|\mathbf{a}|\ Z)\ \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z.$$

If we now change to spherical polar co-ordinates $(r, \theta, \phi)$ associated with $(X, Y, Z)$, then

$$\begin{aligned}
I &= \int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \cos(|\mathbf{a}|\ r\cos\theta)\ r^2\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\
&= 2\pi\int_{r=0}^{1}\int_{\theta=0}^{\pi} \cos(|\mathbf{a}|\ r\cos\theta)\ r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r \\
&= 2\pi\int_{r=0}^{1}\left[\frac{-\sin(|\mathbf{a}|\ r\cos\theta)}{|\mathbf{a}|\ r}\,r^2\right]_0^{\pi}\mathrm{d}r \\
&= 2\pi\int_{r=0}^{1}\frac{r}{|\mathbf{a}|}\left(2\sin(|\mathbf{a}|\ r)\right)\,\mathrm{d}r \\
&= \frac{4\pi}{|\mathbf{a}|}\left\{\left[\frac{-r\cos(|\mathbf{a}|\ r)}{|\mathbf{a}|}\right]_{r=0}^{1} + \int_{r=0}^{1}\frac{\cos(|\mathbf{a}|\ r)}{|\mathbf{a}|}\,\mathrm{d}r\right\} \\
&= \frac{4\pi}{|\mathbf{a}|}\left[\frac{-r\cos(|\mathbf{a}|\ r)}{|\mathbf{a}|} + \frac{\sin(|\mathbf{a}|\ r)}{|\mathbf{a}|^2}\right]_{r=0}^{1} \\
&= \frac{4\pi}{|\mathbf{a}|}\left[\frac{-\cos|\mathbf{a}|}{|\mathbf{a}|} + \frac{\sin(|\mathbf{a}|)}{|\mathbf{a}|^2}\right] \\
&= \frac{4\pi}{|\mathbf{a}|^3}\left(\sin|\mathbf{a}| - |\mathbf{a}|\cos|\mathbf{a}|\right).
\end{aligned}$$

We determine the volume of the unit sphere by setting $a = b = c = 0$. So letting $|\mathbf{a}| \to 0$ we see

$$\begin{aligned}
\frac{4\pi}{|\mathbf{a}|^3}\left(\sin|\mathbf{a}| - |\mathbf{a}|\cos|\mathbf{a}|\right) &= \frac{4\pi}{|\mathbf{a}|^3}\left(\left(|\mathbf{a}| - \frac{|\mathbf{a}|^3}{6} + O\left(|\mathbf{a}|^5\right)\right) - |\mathbf{a}|\left(1 - \frac{|\mathbf{a}|^2}{2} + O\left(|\mathbf{a}|^4\right)\right)\right) \\
&= \frac{4\pi}{|\mathbf{a}|^3}\left(\frac{|\mathbf{a}|^3}{3} + O\left(|\mathbf{a}|^4\right)\right) \\
&= \frac{4\pi}{3} + O\left(|\mathbf{a}|\right) \to \frac{4\pi}{3} \quad \text{as } |\mathbf{a}| \to 0.
\end{aligned}$$

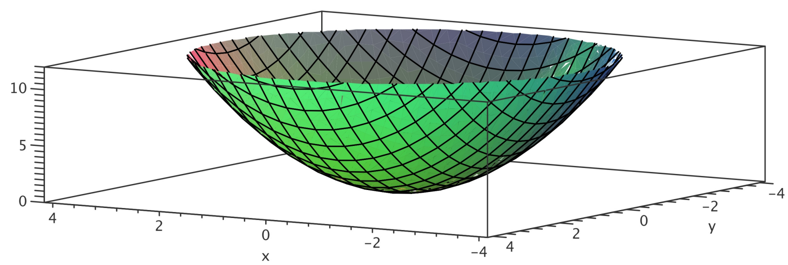
■

# 3. SURFACE INTEGRALS

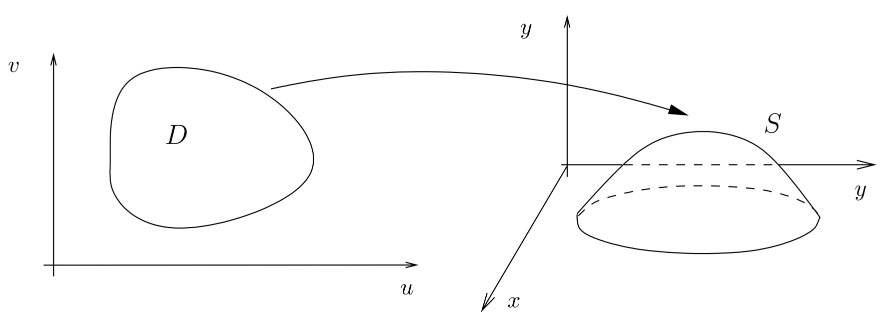
## 3.1 Parameterised Surfaces

We will now be interested in finding integrals over surfaces. We begin by considering different ways to represent surfaces.

**Representation of surfaces**



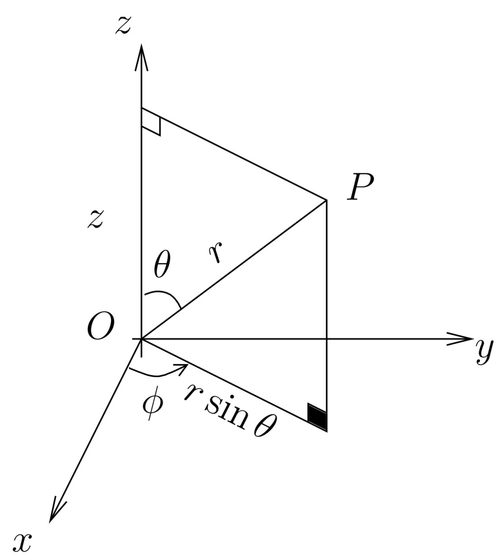

**Cartesian representation.** In Cartesian coordinates we represent a surface by $z = f(x, y)$, *e.g.* the paraboloid $z = x^2 + y^2$.




**Parametric representation.** In parametric coordinates we represent a surface by $\mathbf{r} = \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ where $(u, v) \in D \subset \mathbb{R}^2$. So $\mathbf{r} = \mathbf{r}(u, v)$ maps $D$ in the $(u, v)$ plane to a surface $S$ in $\mathbb{R}^3$.

**Note** If $z = f(x, y)$ we can use $x$ and $y$ as parameters: $\mathbf{r}(x, y) = (x, y, f(x, y))$.

**Example 28 (Parameterisation of a spherical surface)**




To represent the point $P = (x, y, z)$ in spherical polar coordinates $(r, \theta, \phi)$ we have

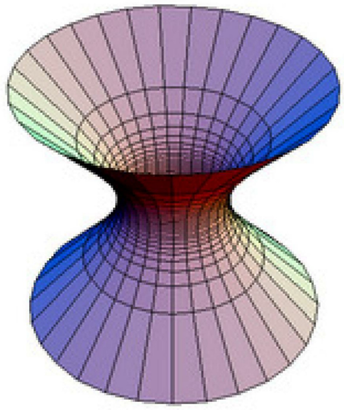
$$x = r \sin\theta \cos\phi, \tag{3.1}$$
$$y = r \sin\theta \sin\phi, \tag{3.2}$$
$$z = r \cos\theta, \tag{3.3}$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$. Thus we can parameterise the sphere $x^2 + y^2 + z^2 = a^2$ in $u = \theta$, $v = \phi$ by

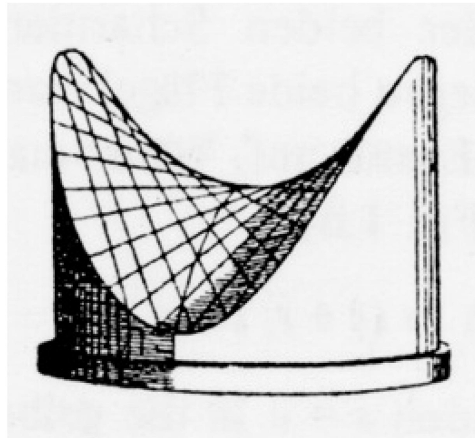
$$\mathbf{r} = \mathbf{r}(\theta, \phi) = (a \sin\theta \cos\phi, a \sin\theta \sin\phi, a \cos\theta). \tag{3.4}$$

**Example 29** *The quadrics are standard parameterised surfaces:*

- ***Sphere:*** $x^2 + y^2 + z^2 = a^2$ *;*
- ***Ellipsoid:*** $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$*;*
- ***Hyperboloid of One Sheet:*** $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$*;*
- ***Hyperboloid of Two Sheets:*** $x^2/a^2 - y^2/b^2 - z^2/c^2 = 1$*;*
- ***Paraboloid:*** $z = x^2 + y^2$*;*
- ***Hyperbolic Paraboloid:*** $z = x^2 - y^2$*;*
- ***Cone:*** $x^2 + y^2 = z^2$.



*one sheet hyperboloid*



*hyperbolic paraboloid*

**Definition 30** *A* ***smooth parameterised surface*** *is a map* $\mathbf{r}$, *known as the* ***parametrisation***

$$\mathbf{r} : U \to \mathbb{R}^3 : (u, v) \mapsto (x(u, v), y(u, v), z(u, v))$$

*from a* ***subset*** $U \subseteq \mathbb{R}^2$ *to* $\mathbb{R}^3$ *such that*

- $x, y, z$ *have continuous partial derivatives with respect to* $u$ *and* $v$ *at all orders*
- $\mathbf{r}$ *is a bijection between its domain and co-domain with* $\mathbf{r}^{-1}$ *continuous and also possessing continuous partial derivatives at all orders*
- *at each point the vectors*
$$\frac{\partial \mathbf{r}}{\partial u} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}$$
*are linearly independent (i.e. are not scalar multiples of one another). Equivalently,*
$$\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \neq \mathbf{0}.$$

We will not treat this definition with any generality. We shall simply parameterise some of the "standard" surfaces previously described.

**Definition 31** *Let* $\mathbf{r} : U \to \mathbb{R}^3$ *be a smooth parameterised surface and let* $\mathbf{p}$ *be a point on the surface. The plane containing* $\mathbf{p}$ *and which is parallel to the vectors*

$$\frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

*is called the* ***tangent plane*** *to* $\mathbf{r}(U)$ *at* $\mathbf{p}$. *Because these vectors are independent the tangent plane is well-defined.*

**Definition 32** *Any vector in the direction*

$$\frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \wedge \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

*is said to be* ***normal*** *to the surface at* $\mathbf{p}$. *There are two* ***unit normals*** *of length one which we denote as* $\mathbf{n}$ *and* $-\mathbf{n}$.

## 3.2 Surface Integrals

Let $\mathbf{r} : U \to \mathbb{R}^3$ be a smooth parameterised surface with

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

and consider the small element of the plane that is bounded by the co-ordinate lines $u = u_0$ and $u = u_0 + \delta u$ and $v = v_0$ and $v = v_0 + \delta v$. Then $\mathbf{r}$ maps this to a small region of the surface $\mathbf{r}(U)$ and we are interested in calculating the surface area of this small region. Note

$$\begin{aligned} \mathbf{r}(u + \delta u, v) - \mathbf{r}(u, v) &\approx \frac{\partial \mathbf{r}}{\partial u}(u, v)\, \delta u, \\ \mathbf{r}(u, v + \delta v) - \mathbf{r}(u, v) &\approx \frac{\partial \mathbf{r}}{\partial v}(u, v)\, \delta v. \end{aligned}$$

Recall that the area of a parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \wedge \mathbf{b}|$. So the element of surface area we are considering is approximately

$$\left| \frac{\partial \mathbf{r}}{\partial u} \delta u \wedge \frac{\partial \mathbf{r}}{\partial v} \delta v \right| = \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \delta u\, \delta v.$$

Thus, at this point we proceed as with double integrals. Partitioning $U$ by smaller and smaller elements, of area $\delta A = \delta u \delta v$, we have in the limit $\delta A \to 0$

$$\lim \sum_{\text{elements}} \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \delta u\, \delta v \stackrel{def}{=} \iint_U \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v.$$

This gives the surface area of the parameterised surface; as with double integrals the pre-limit summation can be weighted with a scalar function $\psi(\mathbf{r}(u, v))$ evaluated at the centre of the elements to yield

$$\iint_U \psi(\mathbf{r}(u, v)) \left| \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v.$$

**Definition 33** *We will often write*

$$\mathrm{d}S = \left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v$$

*to denote an infinitesimal part of surface area. We will also write*

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\,\mathrm{d}u\,\mathrm{d}v.$$

*This is also commonly written as* $\mathbf{n}\ \mathrm{d}S$. *Note there is a sign ambiguity in general until one defines whether the normal is in the direction of*

$$\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} \qquad \textit{or} \qquad \frac{\partial \mathbf{r}}{\partial v} \wedge \frac{\partial \mathbf{r}}{\partial u}.$$

**Proposition 34** *The surface area of* $\mathbf{r}(U)$ *is independent of the choice of parametrisation.*

**Proof.** Let $\Sigma = \mathbf{r}(U) = \mathbf{s}(W)$ be two different parametrisations of a surface $X$; take $u, v$ as the co-ordinates on $U$ and $p, q$ as the co-ordinates on $W$. Let $f = (f_1, f_2) : U \to W$ be the co-ordinate change map – i.e. for any $(u, v) \in U$ we have

$$\mathbf{r}(u,v) = \mathbf{s}(f(u,v)) = \mathbf{s}(f_1(u,v), f_2(u,v)) = \mathbf{s}(p,q).$$

Then

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial \mathbf{s}}{\partial p}\frac{\partial f_1}{\partial u} + \frac{\partial \mathbf{s}}{\partial q}\frac{\partial f_2}{\partial u}, \qquad \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial \mathbf{s}}{\partial p}\frac{\partial f_1}{\partial v} + \frac{\partial \mathbf{s}}{\partial q}\frac{\partial f_2}{\partial v}.$$

Hence

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v} &= \frac{\partial \mathbf{s}}{\partial p}\frac{\partial f_1}{\partial u} \wedge \frac{\partial \mathbf{s}}{\partial q}\frac{\partial f_2}{\partial v} + \frac{\partial \mathbf{s}}{\partial q}\frac{\partial f_2}{\partial u} \wedge \frac{\partial \mathbf{s}}{\partial p}\frac{\partial f_1}{\partial v} \\
&= \left(\frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}\right)\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q} \\
&= \frac{\partial(p,q)}{\partial(u,v)}\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q}.
\end{aligned}$$

Finally

$$\begin{aligned}
\iint_U \left|\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}\right| \mathrm{d}u\,\mathrm{d}v &= \iint_U \left|\frac{\partial(p,q)}{\partial(u,v)}\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q}\right| \mathrm{d}u\,\mathrm{d}v \\
&= \iint_U \left|\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q}\right| \left|\frac{\partial(p,q)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v \\
&= \iint_W \left|\frac{\partial \mathbf{s}}{\partial p} \wedge \frac{\partial \mathbf{s}}{\partial q}\right| \mathrm{d}p\,\mathrm{d}q
\end{aligned}$$

by the two-dimensional rule for the change of variables in integrals. ■

**General method of evaluation of surface integrals**

1. find a suitable parametrisation, e.g. in terms of some $u, v$;
2. find the range of $u, v$, *i.e.* a set $D \subset \mathbb{R}^2$;
3. evaluate $\partial \mathbf{r}/\partial u \wedge \partial \mathbf{r}/\partial v$ for $u, v \in D$;
4. substitute into the relevant integral and integrate.

**Example 35** *Let $0 < a < b$. Find the area of the torus obtained by revolving the circle $(x-b)^2 + z^2 = a^2$ in the $xz$-plane about the $z$-axis.*

**Solution.** We can parameterise the torus as

$$\mathbf{r}(\theta, \phi) = ((b + a\sin\theta)\cos\phi,\ (b + a\sin\theta)\sin\phi, a\cos\theta) \qquad 0 \leqslant \theta, \phi \leqslant 2\pi.$$

We have

$$\mathbf{r}_\theta = (a\cos\theta\cos\phi, a\cos\theta\sin\phi, -a\sin\theta), \qquad \mathbf{r}_\phi = (-(b + a\sin\theta)\sin\phi, (b + a\sin\theta)\cos\phi, 0)$$

and

$$\begin{aligned}
\mathbf{r}_\theta \wedge \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -(b + a\sin\theta)\sin\phi & (b + a\sin\theta)\cos\phi & 0 \end{vmatrix} \\
&= a(b + a\sin\theta) \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{vmatrix} \\
&= a(b + a\sin\theta)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).
\end{aligned}$$

The surface area of the torus is

$$\begin{aligned}
&\int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} a(b + a\sin\theta)\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}\, \mathrm{d}\phi\, \mathrm{d}\theta \\
= \;& 2\pi a \int_{\theta=0}^{2\pi} (b + a\sin\theta)\ \mathrm{d}\theta \\
= \;& 4\pi^2 ab.
\end{aligned}$$

■

**Example 36** *Let $z = f(x, y)$ denote the graph of a function $f$ defined on a subset $S$ of the $xy$-plane. Show that the graph has surface area*

$$\iint_S \sqrt{1 + (f_x)^2 + (f_y)^2}\, \mathrm{d}x\, \mathrm{d}y.$$

*Deduce that a sphere of radius $a$ has surface area $4\pi a^2$.*

**Solution.** We can parameterise the surface as

$$\mathbf{r}\left(x,y\right)=\left(x,y,f\left(x,y\right)\right)\qquad\left(x,y\right)\in S.$$

Then

$$\mathbf{r}_x\wedge\mathbf{r}_y=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&f_x\\0&1&f_y\end{vmatrix}=\left(-f_x,-f_y,1\right).$$

Hence the graph has surface area

$$\iint_S\left|\mathbf{r}_x\wedge\mathbf{r}_y\right|\,\mathrm{d}x\,\mathrm{d}y=\iint_S\sqrt{1+\left(f_x\right)^2+\left(f_y\right)^2}\,\mathrm{d}x\,\mathrm{d}y.$$

We can calculate the area of a hemisphere of radius $a$ by setting

$$f\left(x,y\right)=\sqrt{a^2-x^2-y^2}\qquad x^2+y^2<a^2.$$

We then have

$$f_x=\frac{-x}{\sqrt{a^2-x^2-y^2}},\qquad f_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$$

and so the hemisphere's area is

$$\begin{aligned}&\iint_{x^2+y^2<a^2}\sqrt{1+\frac{x^2+y^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y\\=&\iint_{x^2+y^2<a^2}\frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y\\=&\int_{r=0}^{a}\int_{\theta=0}^{2\pi}\frac{a}{\sqrt{a^2-r^2}}r\,\mathrm{d}\theta\,\mathrm{d}r\\=&\,2\pi\left[-a\sqrt{a^2-r^2}\right]_0^a\\=&\,2\pi a^2.\end{aligned}$$

Hence the area of the whole sphere is $4\pi a^2$. ■

**Example 37** *Calculate the area of a sphere of radius $a$ using spherical polar co-ordinates.*

**Solution.** We can parameterise the sphere by

$$\mathbf{r}\left(\theta,\phi\right)=\left(a\sin\theta\cos\phi,a\sin\theta\sin\phi,a\cos\theta\right)\qquad0\leqslant\theta\leqslant\pi,\ 0\leqslant\phi<2\pi.$$

Then

$$
\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \theta} \wedge \frac{\partial \mathbf{r}}{\partial \phi} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix} \\
&= a^2\left(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta\right) \\
&= a^2\sin\theta\left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right).
\end{aligned}
$$

Hence

$$
\mathrm{d}S = \left|a^2\sin\theta\left(-\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)\right| \,\mathrm{d}\theta\,\mathrm{d}\phi = a^2\left|\sin\theta\right| \,\mathrm{d}\theta\,\mathrm{d}\phi.
$$

Finally

$$
\begin{aligned}
A &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} a^2\left|\sin\theta\right| \,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= 2\pi a^2\int_{\theta=0}^{\pi}\left|\sin\theta\right| \,\mathrm{d}\theta \\
&= 4\pi a^2.
\end{aligned}
$$

■

**Definition 38** *If* $\mathbf{F}$ *and* $\phi$ *are a vector field and scalar field defined on a surface* $\Sigma$*, parameterised as* $\mathbf{r}(U)$*, then we may define the following* ***surface integrals****:*

$$
\begin{aligned}
\iint\limits_{\Sigma} \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \iint\limits_{U} \mathbf{F}\left(\mathbf{r}(u,v)\right)\cdot\left(\frac{\partial \mathbf{r}}{\partial u}\wedge\frac{\partial \mathbf{r}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v; \\
\iint\limits_{\Sigma} \mathbf{F}\,\mathrm{d}S &= \iint\limits_{U} \mathbf{F}\left(\mathbf{r}(u,v)\right)\left|\frac{\partial \mathbf{r}}{\partial u}\wedge\frac{\partial \mathbf{r}}{\partial v}\right|\mathrm{d}u\,\mathrm{d}v; \\
\iint\limits_{\Sigma} \phi\,\mathrm{d}\mathbf{S} &= \iint\limits_{U} \phi\left(\mathbf{r}(u,v)\right)\left(\frac{\partial \mathbf{r}}{\partial u}\wedge\frac{\partial \mathbf{r}}{\partial v}\right)\mathrm{d}u\,\mathrm{d}v; \\
\iint\limits_{\Sigma} \phi\,\mathrm{d}S &= \iint\limits_{U} \phi\left(\mathbf{r}(u,v)\right)\left|\frac{\partial \mathbf{r}}{\partial u}\wedge\frac{\partial \mathbf{r}}{\partial v}\right|\mathrm{d}u\,\mathrm{d}v.
\end{aligned}
$$

**Note:** When the surface $\Sigma$ encloses a 3D region such as the surface of the unit sphere, the **standard sign convention** is that

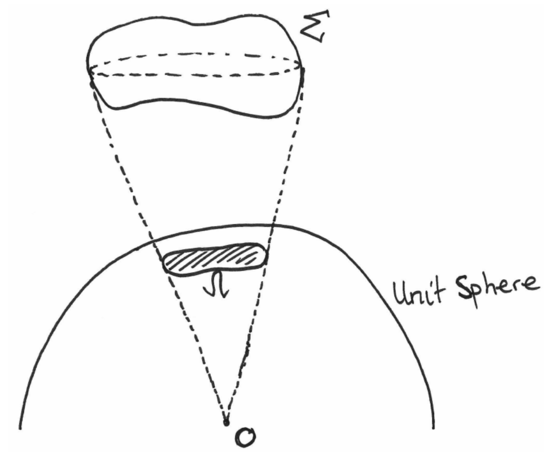
$$
\mathrm{d}\mathbf{S} = \mathbf{n}\mathrm{d}S
$$

has the normal, $\mathbf{n}$ pointing *outwards* from the body.

We will most commonly meet integrals of the first type

$$
\iint\limits_{\Sigma} \mathbf{F}\cdot\mathrm{d}\mathbf{S}
$$

which are known as **flux integrals**.




**Definition 39** *The **solid angle** is the angle an object subtends at a point in three-dimensional space. More precisely, half-lines from a fixed point (or observer) will either intersect with the object in question or not; those lines of sight that are blocked by the object represent a subset of the unit sphere centred on the observer. The solid angle is the area of this subset (strictly it is the area of this subset divided by the unit of length squared to ensure the solid angle is dimensionless). The unit of solid angle is the **steradian**. Given that the surface area of a sphere is $4\pi\,(radius)^2$ then a whole solid angle is $4\pi$ steradian.*

*If $\Sigma_*$ is a surface and $\Sigma$ is the subset of $\Sigma_*$ facing the unit sphere, then the solid angle $\Omega$ subtended at $O$ by $\Sigma_*$ equals, by definition,*

$$\Omega = \iint_{\Sigma} \frac{\mathbf{e}_r \cdot \mathrm{d}\mathbf{S}}{r^2} = \iint_{\Sigma} \frac{\mathbf{r} \cdot \mathrm{d}\mathbf{S}}{r^3},$$

*where $r = |\mathbf{r}|$.*

**Example 40** *Find the solid angle at the apex of a right pyramid with square base of side $2d$ and height $h$.*

**Solution.** Place the apex of the pyramid at the origin and orientate the axis of the pyramid along the positive $z$-axis so that the square base of the pyramid has vertices $(\pm d, \pm d, h)$. By symmetry the solid angle at the apex is 4 times the solid angle subtended by the smaller square with vertices

$$(0,0,h)\,, \qquad (0,d,h)\,, \qquad (d,d,h) \qquad (0,d,h)\,.$$

Hence the solid angle is

$$\Omega = 4\int_{y=0}^{d}\int_{x=0}^{d} \frac{(x,y,h)\cdot\mathbf{k}\,\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2+h^2)^{3/2}} = 4h\int_{y=0}^{d}\int_{x=0}^{d} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2+y^2+h^2)^{3/2}}.$$

If we set $x = \sqrt{y^2+h^2}\tan t$ and $\tan\tau = d/\left(\sqrt{y^2+h^2}\right)$ then

$$\begin{aligned}
\Omega &= 4h\int_{y=0}^{d}\int_{t=0}^{\tau}\frac{\sqrt{y^2+h^2}\sec^2 t\,\mathrm{d}t\,\mathrm{d}y}{\left((y^2+h^2)\tan^2 t+y^2+h^2\right)^{3/2}} \\
&= 4h\int_{y=0}^{d}\int_{t=0}^{\tau}\frac{\sqrt{y^2+h^2}\sec^2 t\,\mathrm{d}t\,\mathrm{d}y}{(y^2+h^2)^{3/2}\left(\tan^2 t+1\right)^{3/2}} \\
&= 4h\int_{y=0}^{d}\int_{t=0}^{\tau}\frac{\cos t\,\mathrm{d}t\,\mathrm{d}y}{(y^2+h^2)} = 4h\int_{y=0}^{d}\frac{\sin\tau\,\mathrm{d}y}{(y^2+h^2)} \\
&= 4hd\int_{y=0}^{d}\frac{\mathrm{d}y}{(y^2+h^2)\sqrt{y^2+h^2+d^2}}
\end{aligned}$$

as $\sin\tau = d\,(y^2+h^2+d^2)^{-1/2}$.

If we make a similar substitution again, namely $y = \sqrt{h^2+d^2}\tan\phi$ and $\tan\alpha = d/\sqrt{h^2+d^2}$ then

$$\begin{aligned}
\Omega &= 4hd\int_{\phi=0}^{\alpha}\frac{\sqrt{h^2+d^2}\sec^2\phi\,\mathrm{d}\phi}{\left((h^2+d^2)\tan^2\phi+h^2\right)\sqrt{h^2+d^2}\sec\phi} \\
&= 4hd\int_{\phi=0}^{\alpha}\frac{\cos\phi\,\mathrm{d}\phi}{\left((h^2+d^2)\sin^2\phi+h^2\cos^2\phi\right)} = 4hd\int_{\phi=0}^{\alpha}\frac{\cos\phi\,\mathrm{d}\phi}{\left((h^2+d^2)\sin^2\phi+h^2\cos^2\phi\right)} \\
&= 4hd\int_{\phi=0}^{\alpha}\frac{\cos\phi\,\mathrm{d}\phi}{d^2\sin^2\phi+h^2}.
\end{aligned}$$

Our final substitution is $u = \sin\phi$ so that $\sin\alpha = d/\sqrt{h^2+2d^2}$ and

$$\Omega = 4hd\int_{u=0}^{\sin\alpha}\frac{\mathrm{d}u}{d^2u^2+h^2} = \frac{4h}{d}\left[\frac{d}{h}\tan^{-1}\left(\frac{du}{h}\right)\right]_0^{\sin\alpha} = 4\tan^{-1}\left(\frac{d^2}{h\sqrt{h^2+2d^2}}\right).$$

■

**Example 41** *Evaluate the integral*

$$\iint_{\Sigma}\mathbf{F}\wedge\mathrm{d}\mathbf{S}$$

*where in each case $\Sigma$ is the closed hemispherical surface made up of points $(x,y,z)$ such that either $x^2+y^2+z^2=1$ and $z>0$,* ***or*** *$x^2+y^2\leqslant 1$ and $z=0$;. orient $\Sigma$ so that $\mathrm{d}\mathbf{S}$ is in the direction of the outward-pointing normal and $\mathbf{F}=(zx,zy,z^2)$.*

**Solution.** We could proceed to parameterise the two parts of $\Sigma$, the upper part of the hemispherical surface $\Sigma_1$ and the planar disc $\Sigma_2$. These can be respectively parameterised as

$$\begin{aligned}\mathbf{r}_1(\theta,\phi) &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \qquad 0 \leqslant \phi \leqslant 2\pi,\ 0 \leqslant \theta \leqslant \pi/2; \\ \mathbf{r}_2(r,\theta) &= (r\cos\theta, r\sin\theta, 0) \qquad 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1.\end{aligned}$$

With some further calculation we would find

$$\frac{\partial \mathbf{r}_1}{\partial \theta} \wedge \frac{\partial \mathbf{r}_1}{\partial \phi} = \sin\theta\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \qquad \frac{\partial \mathbf{r}_1}{\partial r} \wedge \frac{\partial \mathbf{r}_1}{\partial \theta} = (0,0,r)\,.$$

In order to get the correct outward-pointing direction we need to set

$$\mathrm{d}\mathbf{S} = \sin\theta\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\ \mathrm{d}\theta\,\mathrm{d}\phi, \qquad \mathrm{d}\mathbf{S} = -r\mathbf{k},$$

respectively.

**However**, if we stop to think a little, we can straight away see that ... On $\Sigma_1$ we have $\mathbf{F} = z\mathbf{r}$ and $\mathbf{n} = \mathbf{r}$ so that $\mathbf{F} \wedge \mathrm{d}\mathbf{S} = z\mathbf{r} \wedge \mathbf{n}\,\mathrm{d}S = z\mathbf{r} \wedge \mathbf{r}\,\mathrm{d}S = \mathbf{0}$ on all of $\Sigma_1$, and further as $z = 0$ on all of $\Sigma_2$ then it's also true that $\mathbf{F} = \mathbf{0}$ on all of $\Sigma_2$.

**Moral of the story: take some time to consider (i) the nature of your function and the region, (ii) what integrals need to be calculated, and (iii) what co-ordinates are best for the problem under consideration.**

■

**Remark 42** *(A point of notation in relation to sheet 4). In question 4, $\partial\Sigma$ refers to the boundary of $\Sigma$, which is formed from three curves which can be parameterised*

$$\begin{aligned}C_1 &= \{(t,0,t) : 0 \leqslant t \leqslant 2\}\,, \\ C_2 &= \{(2\cos t, 2\sin t, 2) : 0 \leqslant t \leqslant \pi\}\,, \\ C_3 &= \{(2-t)(-1,0,1) : 0 \leqslant t \leqslant 2\}\,.\end{aligned}$$

*Be sure to orient the curves consistently in a loop to find $\int_{\partial\Sigma} f\,\mathrm{d}\mathbf{r}$ (which is defined only up to sign).*

# 4. LINE INTEGRALS & CONSERVATIVE FIELDS

## 4.1 Curves

**Definition 43** *By a* ***curve*** *we shall mean a piecewise smooth function* $\gamma\mathbb{R}^3$ *defined on an interval* $I$ *of* $\mathbb{R}$*. Notice that order on* $I$ *also gives the curve* $\gamma$ *an* ***orientation****.*

*We shall also use the term curve to describe the images of such maps* $\gamma$*. Given such an image then it will be the image of more than one such map* $\gamma$ *and we will talk about* ***parametrisations*** $\gamma_1$ *and* $\gamma_2$ *of the curve. These parametrisations of the image come in two different possible orientations.*

**Definition 44** *We say a curve* $\gamma : [a, b] \to \mathbb{R}^3$ *is* ***simple*** *if* $\gamma$ *is 1-1, with the one possible exception that* $\gamma(a) = \gamma(b)$ *may be true; this means that the curve does not cross itself except possibly by its endpoints meeting.*

**Definition 45** *We say a curve* $\gamma : [a, b] \to \mathbb{R}^3$ *is* ***closed*** *if* $\gamma(a) = \gamma(b)$*.*

**Example 46** *The* ***line*** *through points* $\mathbf{p}$ *and* $\mathbf{q}$ *can be parameterised as*

$$\gamma(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}).$$

*When* $0 < t < 1$ *then* $\gamma(t)$ *lies between* $\mathbf{p}$ *and* $\mathbf{q}$*, for* $t > 1$ *beyond* $\mathbf{q}$ *and for* $t < 0$ *before* $\mathbf{p}$*.*

**Example 47** *A curve of the form*

$$\gamma(t) = (a\cos t, a\sin t, ct) \tag{4.1}$$

*is known as a* ***circular helix****.*

**Example 48** *Parameterise the parabola formed by intersecting the plane* $3x + 4y + 5z = 1$ *with the cone* $x^2 + y^2 = z^2$*,* $z > 0$*.*

**Solution.** A general point on the cone can be written as $\mathbf{r}(\theta, z) = (z\cos\theta, z\sin\theta, z)$ where $z > 0$ and $0 \leqslant \theta < 2\pi$. If this point also lies in the plane $3x + 4y + 5z = 1$ then

$$z(3\cos\theta + 4\sin\theta + 5) = 1$$

and so we see

$$\gamma(\theta) = \left(\frac{\cos\theta}{3\cos\theta + 4\sin\theta + 5}, \frac{\sin\theta}{3\cos\theta + 4\sin\theta + 5}, \frac{1}{3\cos\theta + 4\sin\theta + 5}\right)$$

lies on the parabola. What values should $\theta$ range through? We need

$$0 \neq 3\cos\theta + 4\sin\theta + 5 = 5\cos(\theta - \alpha) + 5$$

where $\alpha = \tan^{-1}\frac{4}{3}$. Hence

$$\gamma(\theta) = \frac{1}{5}\left(\frac{\cos\theta}{\cos(\theta-\alpha)+1}, \frac{\sin\theta}{\cos(\theta-\alpha)+1}, \frac{1}{\cos(\theta-\alpha)+1}\right) \qquad \alpha - \pi < \theta < \alpha + \pi$$

is a parameterisation.

■

**Aside.** In the above example, let

$$\mathbf{e}_1 = \frac{1}{\sqrt{50}}(-3,-4,5), \quad \mathbf{e}_2 = \frac{1}{5}(-4,3,0), \quad \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 = \frac{1}{\sqrt{50}}(3,4,5)$$

be a set of basis vectors. With

$$X = \mathbf{e}_1 \cdot \gamma(\theta), \quad Y = \mathbf{e}_2 \cdot \gamma(\theta))$$

we find that

$$Y^2/X$$

is constant for all $\theta$, explicitly confirming we have a parabola in the plane spanned by $\mathbf{e}_1$, $\mathbf{e}_2$.

**Example 49** *Show that the plane with equation $Ax + By + Cz = D$ intersects the unit sphere $x^2 + y^2 + z^2 = 1$ in a circle if and only if $A^2 + B^2 + C^2 > D^2$. Parameterise the intersection of $x + y + z = 1$ with the unit sphere.*

**Solution.** The plane $Ax + By + Cz = D$ has normal $(A, B, C)$ and so the point closest to the origin is the point with position vector $\lambda(A, B, C)$ which lies on the plane; by substitution, we see $\lambda = D/(A^2 + B^2 + C^2)$. This point is within unit distance of the origin if and only if

$$1 > \left|\frac{D}{A^2+B^2+C^2}(A,B,C)\right| = \frac{|D|\sqrt{A^2+B^2+C^2}}{A^2+B^2+C^2}$$

i.e. if and only if $A^2 + B^2 + C^2 > D^2$.

By this criterion the plane $x + y + z = 1$ intersects with the unit sphere. The centre of the circle which makes the intersection is at

$$\frac{1}{1^2+1^2+1^2}(1,1,1) = \frac{(1,1,1)}{3}.$$

By Pythagoras' Theorem the radius $r$ of the circle satisfies

$$r^2 + \left|\frac{(1,1,1)}{3}\right|^2 = 1 \implies r = \sqrt{\frac{2}{3}}.$$

As $\mathbf{e}_1 = (1,-1,0)/\sqrt{2}$ and $\mathbf{e}_2 = (1,1,-2)/\sqrt{6}$ are two orthonormal vectors parallel to the plane then every point of the circle can be written in the form $\gamma(t)$ for $0 \leqslant t < 2\pi$ where

$$\begin{aligned}\gamma(t) &= \frac{(1,1,1)}{3} + \mathbf{e}_1\sqrt{\frac{2}{3}}\cos t + \mathbf{e}_2\sqrt{\frac{2}{3}}\sin t \\ &= \left(\frac{1}{3} + \frac{1}{\sqrt{3}}\cos t + \frac{1}{3}\sin t, \frac{1}{3} - \frac{1}{\sqrt{3}}\cos t + \frac{1}{3}\sin t, \frac{1}{3} - \frac{2}{3}\sin t\right).\end{aligned}$$

■

## 4.2 Line Integrals

**Definition 50** *Let $C$ be a curve in $\mathbb{R}^3$ parameterised by $\gamma : [a,b] \to \mathbb{R}^3$, and let $\mathbf{F}$ be a vector field, whose domain includes $C$. We define the* ***line integral of*** $\mathbf{F}$ ***along*** *$C$ as*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \;\; \mathrm{d}t,$$

*where $\cdot$ denotes the scalar product.*

**Proposition 51** *If oriented the same, the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of the choice of parametrisation.*

**Proof.** Suppose that $\gamma_1 : [a_1, b_1] \to \mathbb{R}^3$ and $\gamma_2 : [a_2, b_2] \to \mathbb{R}^3$ are two parametrisations of $C$ with $\gamma_1(a_1) = \gamma_2(a_2)$ and $\gamma_1(b_1) = \gamma_2(b_2)$, so that $\gamma_1$ and $\gamma_2$ give $C$ the same orientation. Then $\gamma_2 = \gamma_1 \circ \psi$ where $\psi : [a_2, b_2] \to [a_1, b_1]$ associates the $\gamma_2$ co-ordinates of points on $C$ with their $\gamma_1$ co-ordinate.

We now define $I : [a_1, b_1] \to \mathbb{R}$ and $J : [a_2, b_2] \to \mathbb{R}$ by

$$I(t) = \int_{a_1}^t \mathbf{F}(\gamma_1(s)) \cdot \gamma_1'(s) \; \mathrm{d}s, \qquad J(t) = \int_{a_2}^t \mathbf{F}(\gamma_2(s)) \cdot \gamma_2'(s) \; \mathrm{d}s.$$

By the Fundamental Theorem of Calculus

$$I'(t) = \mathbf{F}(\gamma_1(t)) \cdot \gamma_1'(t), \qquad J'(t) = \mathbf{F}(\gamma_2(t)) \cdot \gamma_2'(t).$$

Further, for $\psi(t)$ a function of $t$ we have, by the chain rule,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} I(\psi(t)) &= \psi'(t) \; I'(\psi(t)) \\ &= \psi'(t) \; \mathbf{F}(\gamma_1(\psi(t))) \cdot \gamma_1'(\psi(t)).\end{aligned}$$

Recall

$$\gamma_2(t) = \gamma_1(\psi(t)).$$

Thus $\gamma_2'(t) = \psi'(t)\gamma_1'(\psi(t))$ and hence

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} I(\psi(t)) &= \mathbf{F}(\gamma_2(t)) \cdot (\gamma_1'(\psi(t))\psi'(t)) \\
&= \mathbf{F}(\gamma_2(t)) \cdot \gamma_2'(t) = J'(t).
\end{aligned}$$

It follows that $I(\psi(t))$ and $J(t)$ differ by a constant and, as they agree at $t = a_2$ (when they are both zero), then $I(\psi(t)) = J(t)$ and in particular when $t = b_2$ we have

$$\int_{a_1}^{b_1} \mathbf{F}(\gamma_1(s)) \cdot \gamma_1'(s)\ \mathrm{d}s = \int_{a_2}^{b_2} \mathbf{F}(\gamma_2(s)) \cdot \gamma_2'(s)\ \mathrm{d}s.$$

■

**Remark 52** *If we parameterised $C$ in the reverse orientation then the integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ would give negative what had been previously calculated.*

**Example 53** *Let $\mathbf{F} = \mathbf{c} \wedge \mathbf{r}$ where $\mathbf{c}$ is a constant vector and let $C$ be the circular helix parameterised by*

$$\mathbf{r}(t) = (\cos t, \sin t, t), \qquad 0 \leqslant t \leqslant 2\pi.$$

*Then*

$$\begin{aligned}
\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ \cos t & \sin t & t \end{vmatrix} \cdot (-\sin t, \cos t, 1)\ \mathrm{d}t \\
&= \int_0^{2\pi} \begin{vmatrix} -\sin t & \cos t & 1 \\ c_1 & c_2 & c_3 \\ \cos t & \sin t & t \end{vmatrix}\ \mathrm{d}t \\
&= \int_0^{2\pi} \left(-c_2 t \sin t + c_3 \cos^2 t + c_1 \sin t - c_2 \cos t + c_3 \sin^2 t - c_1 t \cos t\right)\ \mathrm{d}t \\
&= \int_0^{2\pi} (-c_2 t \sin t + c_3 - c_1 t \cos t)\ \mathrm{d}t \\
&= 2\pi c_3 - c_1 [t \sin t + \cos t]_0^{2\pi} - c_2 [-t \cos t + \sin t]_0^{2\pi} \\
&= 2\pi (c_2 + c_3).
\end{aligned}$$

**Definition 54** *If $\phi$ is a scalar field defined on a curve $C$, with parametrisation $\gamma : [a, b] \to \mathbb{R}^3$, then we also define the line integral*

$$\int_C \phi \,\mathrm{d}s = \int_a^b \phi(\gamma(t)) |\gamma'(t)|\ \mathrm{d}t.$$

*If $\mathbf{F} = (f_1, f_2, f_3)$ is a vector field defined on the curve $C$ then we define*

$$\int_C \mathbf{F}\,\mathrm{d}s = \left(\int_C f_1\,\mathrm{d}s, \int_C f_2\,\mathrm{d}s, \int_C f_3\,\mathrm{d}s\right).$$

**Remark 55** *Note that if* $\mathbf{t}$ *is the unit tangent vector field along* $C$*, in the same direction as the parametrisation, then*

$$\int_C \phi \, \mathrm{d}s = \int_C (\phi \mathbf{t}) \cdot \mathrm{d}\mathbf{r}$$

*and so, by inheritance, these line integrals do not depend on the choice of parametrisation. Note further that the two types of integral defined above are also independent of the choice of orientation of* $C$*, as they are independent of the sign of* $\gamma'(t)$*.*

**Definition 56** *If* $C$ *is a curve with parametrisation* $\gamma : [a, b] \to \mathbb{R}^3$ *then the* ***arc length*** *of the curve is*

$$\int_C \mathrm{d}s = \int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

**Example 57** *Find the arc length of the circular helix* $\mathbf{r}(t) = (\cos t, \sin t, t)$ *where* $0 \leqslant t \leqslant 2\pi$*.*

**Solution.** We have

$$s = \int_0^{2\pi} |(-\sin t, \cos t, 1)| \, \mathrm{d}t = \int_0^{2\pi} \sqrt{1 + \sin^2 t + \cos^2 t} \, \mathrm{d}t = \int_0^{2\pi} \sqrt{2} \, \mathrm{d}t = 2\sqrt{2}\,\pi.$$

We could also parameterise the helix "in reverse" by setting $\mathbf{s}(t) = \mathbf{r}(2\pi - t) = (\cos t, -\sin t, 2\pi - t)$ where $0 \leqslant t \leqslant 2\pi$. We would still find

$$s = \int_0^{2\pi} |(-\sin t, -\cos t, -1)| \, \mathrm{d}t = \int_0^{2\pi} \sqrt{1 + \sin^2 t + \cos^2 t} \, \mathrm{d}t = \int_0^{2\pi} \sqrt{2} \, \mathrm{d}t = 2\sqrt{2}\,\pi.$$

■

**Notation 58** *We will standardly use the notation*

$$\mathrm{d}\mathbf{r} = (\mathrm{d}x, \mathrm{d}y, \mathrm{d}z) \qquad \text{and} \qquad \mathrm{d}s = |\mathrm{d}\mathbf{r}|,$$

*even though this may seem a little non-rigorous and any analysis course would insist on such differentials only appearing as part of a limit or within an integral. Rest assured that these differentials can be rigorously defined, though doing so is not a primary concern of this course.*

**Definition 59** *With notation as in the Definition 50, if the vector field* $\mathbf{F}$ *represents a force on a particle then*

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \tag{4.2}$$

*is the* ***work done by the force in moving the particle along*** $C$*.*

This is a generalization of the formula

$$\text{Work} = \text{Force} \times \text{Distance}$$

which applies to constant forces acting parallel to the direction of the motion. More generally, we have

$$\text{Work} = \text{Component of force in direction of travel} \times \text{Distance}$$

if the force and movement are not parallel. The work integral (4.2 ) is just an integral of such infinitesimal contributions of work.

**Example 60** *Show that the work done by gravity,* $\mathbf{F} = -mg\mathbf{k}$, *in moving a particle along a straight line from* $(x_1, y_1, z_1)$ *to* $(x_2, y_2, z_2)$ *equals* $mg(z_1 - z_2)$.

**Solution.** We can parameterise the line segment as

$$\mathbf{r}(t) = (x_1, y_1, z_1) + t(x_2 - x_1, y_2 - y_1, z_2 - z_1), \qquad 0 \leqslant t \leqslant 1,$$

so that

$$\mathrm{d}\mathbf{r} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)\ \mathrm{d}t.$$

Then

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^1 -mg\mathbf{k} \cdot (x_2 - x_1, y_2 - y_1, z_2 - z_1)\ \mathrm{d}t \\ &= \int_0^1 -mg(z_2 - z_1)\ \mathrm{d}t \\ &= mg(z_1 - z_2).\end{aligned}$$

■

In fact, the work done by gravity – which is positive if $z_1 > z_2$, i.e. the particle has dropped – is the loss in gravitational potential energy which, commonly, will have been converted into kinetic energy. The work done is dependent only on the endpoints $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ and is independent of the path taken between these endpoints. This is common to certain types of field which are known as *conservative.*

## 4.3 Conservative Fields

**Proposition 61** *If* $\mathbf{F} = \nabla\phi$ *and* $\gamma : [a, b] \to S$ *is any curve such that* $\gamma(a) = \mathbf{p}$, $\gamma(b) = \mathbf{q}$ *then*

$$\int_\gamma \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \phi(\mathbf{q}) - \phi(\mathbf{p}).$$

*In particular, the integral* $\int_\gamma \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$ *depends only on the endpoints of the curve* $\gamma$.

**Proof.** If $\gamma(t) = (x(t), y(t), z(t))$ then

$$\begin{aligned}
\int_\gamma \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} &= \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\ \mathrm{d}t \\
&= \int_a^b \nabla\phi(\gamma(t)) \cdot \gamma'(t)\ \mathrm{d}t \\
&= \int_a^b \left( \frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \right) \mathrm{d}t \\
&= \int_a^b \left[ \frac{\mathrm{d}}{\mathrm{d}t}(\phi(\gamma(t))) \right] \mathrm{d}t \qquad \text{[by the chain rule]} \\
&= \phi(\gamma(b)) - \phi(\gamma(a)) \\
&= \phi(\mathbf{q}) - \phi(\mathbf{p}).
\end{aligned}$$

■

Motivated by the above proposition, we now introduce the following definitions:

**Definition 62** *Let $S$ be an open subset of $\mathbb{R}^3$. A vector field $\mathbf{F} : S \to \mathbb{R}^3$ is said to be* ***conservative*** *if there exists a scalar field $\phi : S \to$ such that $\mathbf{F} = \nabla\phi$.*

**Definition 63** *If $\mathbf{F} = \nabla\phi$ then $\phi$ is said to be a* ***potential*** *(or* ***scalar potential****) for $\mathbf{F}$.*

**Definition 64** *A subset $S$ of $\mathbb{R}^3$ is said to be* ***path-connected*** *if for any points $\mathbf{p}, \mathbf{q} \in S$ there exists a curve $\gamma : [a, b] \to S$ such that $\gamma(a) = \mathbf{p}$ and $\gamma(b) = \mathbf{q}$.*

**Corollary 65** *If $\nabla\phi = 0$ on a path-connected set $S$ then $\phi$ is constant. In particular, if $\phi$ and $\psi$ are potentials of the conservative field $\mathbf{F}$, defined on $S$, then $\phi$ and $\psi$ differ by a constant.*

**Proof.** If $\nabla\phi = 0$ then for any curve, with endpoints $\mathbf{p}$, $\mathbf{q}$, we have

$$\phi(\mathbf{q}) - \phi(\mathbf{p}) = \int_\gamma \nabla\phi \cdot \mathrm{d}\mathbf{r} = 0.$$

Given a fixed point $\mathbf{p}$ in $S$, any $\mathbf{q}$ in $S$ is connected to $\mathbf{p}$ by a curve, and so $\phi$ is constant. If $\mathbf{F} = \nabla\phi = \nabla\psi$ then $\nabla(\phi - \psi) = 0$ and the result follows. ■

**Theorem 66** *Let $S$ be an open path connected subset of $\mathbb{R}^3$ and let $\mathbf{F} : S \to \mathbb{R}^3$ be a vector field. Then the following three statements are equivalent.*

*(i) $\mathbf{F}$ is conservative – i.e. $\mathbf{F} = \nabla\phi$ for some scalar field $\phi : S \to \mathbb{R}$.*

*(ii) Given any two points* $\mathbf{p}, \mathbf{q} \in S$ *and curve* $\gamma$ *in* $S$, *starting at* $\mathbf{p}$ *and ending at* $\mathbf{q}$, *then the integral*

$$\int_\gamma \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

*is independent of the choice of curve* $\gamma$.

*(iii) For any simple closed curve* $\gamma$ *then*

$$\int_\gamma \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0.$$

**Proof.**

$(i) \implies (ii)$ : This was just proved in Proposition 61 as $\phi(\mathbf{q}) - \phi(\mathbf{p})$ is dependent only on the endpoints $\mathbf{p}$ and $\mathbf{q}$ and not on the path taken.

$(ii) \implies (i)$ : Let $\mathbf{p} \in S$ be a fixed point and define for any $\mathbf{q} \in S$

$$\phi(\mathbf{q}) = \int_{\mathbf{p}}^{\mathbf{q}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

which, by assumption, is independent of the curve taken and is a function only of $\mathbf{q}$.

As $S$ is open there is an $r > 0$ small enough that $\mathbf{q} + t\mathbf{i}$ is in $S$ for $0 \leqslant t \leqslant r$. Then

$$\phi(\mathbf{q} + t\mathbf{i}) - \phi(\mathbf{q}) = \int_{\mathbf{q}}^{\mathbf{q}+t\mathbf{i}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

where the curve from $\mathbf{q}$ to $\mathbf{q} + t\mathbf{i}$ can be taken to be a straight line. So

$$\phi(\mathbf{q} + t\mathbf{i}) - \phi(\mathbf{q}) = \int_{s=0}^{s=t} \mathbf{F}(\mathbf{q} + s\mathbf{i}) \cdot (\mathbf{i}\,\mathrm{d}s) = \int_{s=0}^{s=t} F_1(\mathbf{q} + s\mathbf{i})\ \mathrm{d}s$$

where $\mathbf{F} = (F_1, F_2, F_3)$. Hence

$$\frac{\partial \phi}{\partial x}(\mathbf{q}) = \lim_{t\to 0} \frac{\phi(\mathbf{q} + t\mathbf{i}) - \phi(\mathbf{q})}{t} = \lim_{t\to 0} \frac{1}{t} \int_{s=0}^{s=t} F_1(\mathbf{q} + s\mathbf{i})\ \mathrm{d}s = F_1(\mathbf{q})$$

by the continuity of $F_1$ at $\mathbf{q}$. Similarly

$$\frac{\partial \phi}{\partial y}(\mathbf{q}) = F_2(\mathbf{q}), \qquad \frac{\partial \phi}{\partial z}(\mathbf{q}) = F_3(\mathbf{q})$$

and so $\nabla\phi = \mathbf{F}$ as required.

$(ii) \implies (iii)$ : Suppose now that for any two points $\mathbf{p}$ and $\mathbf{q}$ the line integral $\int_{\mathbf{p}}^{\mathbf{q}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$ is independent of the curve taken.

Let $\gamma$ be a simple closed curve in $S$ and let $\mathbf{p}$ and $\mathbf{q}$ be two distinct points on $\gamma$. Then $\mathbf{p}$ and $\mathbf{q}$ split $\gamma$ into two curves $\gamma_1$ and $\gamma_2$, with $\gamma_1$ running from $\mathbf{p}$ to $\mathbf{q}$ and $\gamma_2$ running from $\mathbf{q}$ to $\mathbf{p}$.

So, by $(ii)$ and Remark 52,

$$\int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\gamma_1} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} + \int_{\gamma_2} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\mathbf{p}}^{\mathbf{q}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} - \int_{\mathbf{p}}^{\mathbf{q}} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0$$

$(iii) \implies (ii)$ : Let $\mathbf{p}, \mathbf{q} \in S$ and let $\gamma_1, \gamma_2$ be two curves in $S$, starting at $\mathbf{p}$ and ending at $\mathbf{q}$. If $\gamma$ is the curve which follows $\gamma_1$ and comes back around $\gamma_2$, so that we have returned to $\mathbf{p}$ then, by assumption and Remark 52,

$$0 = \int_{\gamma} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\gamma_1} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} - \int_{\gamma_2} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} \implies \int_{\gamma_1} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \int_{\gamma_2} \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

and the line integral of $\mathbf{F}$ from $\mathbf{p}$ to $\mathbf{q}$ is independent of the choice of path taken.

■

# 5. DIV, GRAD AND CURL

Recall:

**Definition 67** *By a* ***scalar field*** $\phi$ *on* $\mathbb{R}^3$ *we shall mean a map* $\phi : \mathbb{R}^3 \to \mathbb{R}$.

**Definition 68** *By a* ***vector field*** $\mathbf{F}$ *on* $\mathbb{R}^3$ *we shall mean a map* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$. *Occasionally we may consider more general scalar fields* $\phi : \mathbb{R}^n \to \mathbb{R}$ *and vector fields* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$.

We shall also consider scalar and vector fields defined on proper subsets of $\mathbb{R}^3$ (or more generally $\mathbb{R}^n$). The domains of these fields will usually be *open,* so that we can define their partial derivatives.

## 5.1 Definitions and Properties

**Definition 69** *(Gradient) Let* $\phi : \mathbb{R}^3 \to \mathbb{R}$ *be a scalar field. Then the* ***gradient*** *of* $\phi$*, written* $\operatorname{grad}\phi$*, is*

$$\operatorname{grad}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$

*Note:* grad *takes scalar fields to vector fields.*

**Definition 70** *(Divergence) Let* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *be a vector field where*

$$\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)),$$

*with respect to a* ***Cartesian*** *coordinate system. Then the* ***divergence*** *of* $\mathbf{F}$*, written* $\operatorname{div}\mathbf{F}$*, is*

$$\operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

*Note:* div *takes vector fields to scalar fields.*

**Definition 71** *(Curl) Let* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *be a vector field where*

$$\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)).$$

*with respect to a **Cartesian** coordinate system. Then the **curl** of* $\mathbf{F}$*, written* $\operatorname{curl}\mathbf{F}$*, is*

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right).$$

*Note:* curl *takes vector fields to vector fields.*

**Example 72** *Let* $\mathbf{r} = (x, y, z)$ *and* $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$*. Then*

$$\begin{aligned} \operatorname{div}\mathbf{r} &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3; \\ \operatorname{curl}\mathbf{r} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}. \end{aligned}$$

**Example 73** *With* $r$ *and* $\mathbf{r}$ *as above, and* $f$ *a differentiable function on* $(0, \infty)$ *note*

$$\begin{aligned} \operatorname{curl}(f(r)\,\mathbf{r}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)\,x & f(r)\,y & f(r)\,z \end{vmatrix} \\ &= \left( zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z}, xf'(r)\frac{\partial r}{\partial z} - zf'(r)\frac{\partial r}{\partial x}, yf'(r)\frac{\partial r}{\partial x} - xf'(r)\frac{\partial r}{\partial y} \right) \\ &= f'(r)\left( \frac{zy}{r} - \frac{yz}{r}, \frac{xz}{r} - \frac{zx}{r}, \frac{yx}{r} - \frac{xy}{r} \right) = \mathbf{0}. \end{aligned}$$

**Example 74** *Let* $\mathbf{c} = (c_1, c_2, c_3)$ *be a constant vector. Then*

$$\begin{aligned} \operatorname{div}(\mathbf{c} \wedge \mathbf{r}) &= \frac{\partial}{\partial x}(c_2 z - c_3 y) + \frac{\partial}{\partial y}(c_3 x - c_1 z) + \frac{\partial}{\partial z}(c_1 y - c_2 x) = 0; \\ \operatorname{curl}(\mathbf{c} \wedge \mathbf{r}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ c_2 z - c_3 y & c_3 x - c_1 z & c_1 y - c_2 x \end{vmatrix} = (2c_1, 2c_2, 2c_3) = 2\mathbf{c}. \end{aligned}$$

**Definition 75** *The differential operator*

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

*is called **del** or **nabla**.*

**Notation 76** *By a slight abuse of notation we can write*

$$
\begin{aligned}
\operatorname{grad}\phi &= \left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right)=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\phi;\\
\operatorname{div}\mathbf{F} &= \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(F_1,F_2,F_3)\,;\\
\operatorname{curl}\mathbf{F} &= \left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z},\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x},\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\\
&= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\wedge(F_1,F_2,F_3)\,.
\end{aligned}
$$

*So we often write*

$$\operatorname{grad}\phi=\nabla\phi,\qquad \operatorname{div}\mathbf{F}=\nabla\cdot\mathbf{F},\qquad \operatorname{curl}\mathbf{F}=\nabla\wedge\mathbf{F}. \tag{5.1}$$

**Remark 77** *The notation* grad, div, curl *and that of (5.1) are equally common in mathematical texts, and so these lecture notes and the problem sheets will deliberately make use of both notations.*

**Remark 78** *For any scalar field $\phi$ we have*

$$\operatorname{div}(\operatorname{grad}\phi)=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2},$$

*which is the **Laplacian**. As*

$$\operatorname{div}\operatorname{grad}=\nabla\cdot\nabla$$

*then we often write*

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}.$$

*Also, for vector fields $\mathbf{F}=(F_1,F_2,F_3)$ with respect to a **Cartesian** coordinate system, it is common to write*

$$\nabla^2\mathbf{F}=\left(\nabla^2F_1,\nabla^2F_2,\nabla^2F_3\right).$$

*Thus the Laplacian can both take scalar fields to scalar fields and vector fields to vector fields.*

**Remark 79** *Whilst $\nabla$ looks, and occasionally behaves, like a standard vector it is unwise to assume this behaviour of $\nabla$ without proof; most standard vector identities are not true when they involve $\nabla$. For example, the commutativity of the dot product does not extend:*

$$\nabla\cdot\mathbf{F}\neq\mathbf{F}\cdot\nabla.$$

*The LHS is* $\operatorname{div}\mathbf{F}$. *At first glance, the RHS doesn't mean anything; it certainly isn't a scalar. Rather*

$$\mathbf{F}\cdot\nabla=(F_1,F_2,F_3)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)=F_1\frac{\partial}{\partial x}+F_2\frac{\partial}{\partial y}+F_3\frac{\partial}{\partial z}$$

*which is a differential operator. In fact $\mathbf{F}\cdot\nabla$ is standard notation for it and it gives the directional derivative in the direction of $\mathbf{F}$.*

In summary, we have the following:

| operator | maps: |
|---|---|
| $\text{div} = \nabla\cdot$ | vector fields to scalar fields |
| $\text{grad} = \nabla$ | scalar fields to vector fields |
| $\text{curl} = \nabla\wedge$ | vector fields to vector fields |
| $\text{div}\,\text{grad} = \nabla^2$ | scalar fields to scalar fields |
| | vector fields to vector fields |
| $\mathbf{F}\cdot\nabla$ | scalar fields to scalar fields |
| | vector fields to vector fields |

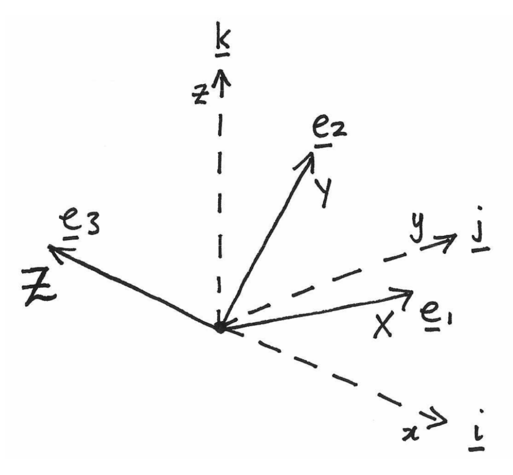
The next proposition shows that the formulae for calculating each of div, grad and curl do not depend on our choice $\mathbf{i},\mathbf{j},\mathbf{k}$ of a Cartesian co-ordinate system. Recall that:

**Definition 80** *The vectors $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ are said to be an **orthonormal basis** of $\mathbb{R}^3$ if*

$$\mathbf{e}_i\cdot\mathbf{e}_j=\delta_{ij}:=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$$

*So the vectors are all of unit length and mutually perpendicular.*

*Further it is said to be a **right-handed** orthonormal basis if $\mathbf{e}_1\wedge\mathbf{e}_2=\mathbf{e}_3$ and **left-handed** if $\mathbf{e}_1\wedge\mathbf{e}_2=-\mathbf{e}_3$. So $\mathbf{e}_1=\mathbf{i}$, $\mathbf{e}_2=\mathbf{j}$, $\mathbf{e}_3=\mathbf{k}$ is an example of a right-handed orthonormal basis.*




**Proposition 81** *Let $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ be a right-handed orthonormal basis of $\mathbb{R}^3$ with associated co-ordinates $X,Y,Z$ defined by the identity*

$$X\mathbf{e}_1+Y\mathbf{e}_2+Z\mathbf{e}_3=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}. \tag{5.2}$$

*Then*

$$\mathbf{e}_1\frac{\partial}{\partial X}+\mathbf{e}_2\frac{\partial}{\partial Y}+\mathbf{e}_3\frac{\partial}{\partial Z}=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}.$$

**Proof.** From (5.2) we have that

$$\begin{aligned} x &= (\mathbf{e}_1 \cdot \mathbf{i})\, X + (\mathbf{e}_2 \cdot \mathbf{i})\, Y + (\mathbf{e}_3 \cdot \mathbf{i})\, Z, \\ X &= (\mathbf{e}_1 \cdot \mathbf{i})\, x + (\mathbf{e}_1 \cdot \mathbf{j})\, y + (\mathbf{e}_1 \cdot \mathbf{k})\, z, \end{aligned}$$

and four other similar equations for $y, z, Y, Z$. It follows from the chain rule that

$$\frac{\partial}{\partial X} = \frac{\partial x}{\partial X}\frac{\partial}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial}{\partial y} + \frac{\partial z}{\partial X}\frac{\partial}{\partial z} = (\mathbf{e}_1 \cdot \mathbf{i})\,\frac{\partial}{\partial x} + (\mathbf{e}_1 \cdot \mathbf{j})\,\frac{\partial}{\partial y} + (\mathbf{e}_1 \cdot \mathbf{k})\,\frac{\partial}{\partial z}$$

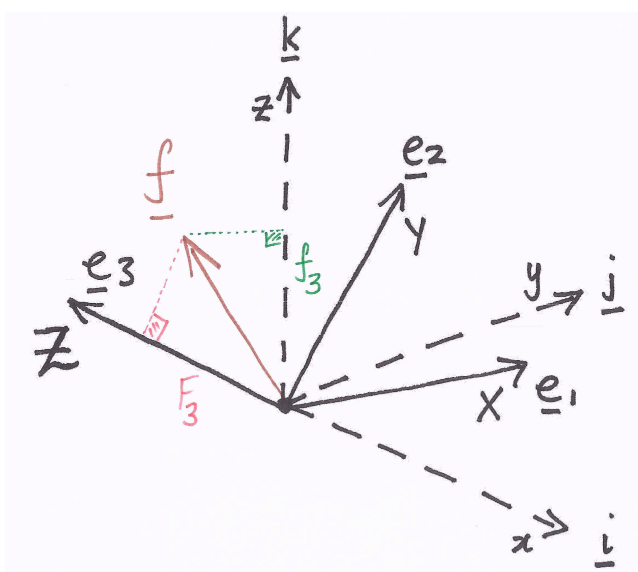
and hence

$$\begin{aligned} & \mathbf{e}_1\frac{\partial}{\partial X} + \mathbf{e}_2\frac{\partial}{\partial Y} + \mathbf{e}_3\frac{\partial}{\partial Z} \\ = \; & \mathbf{e}_1\left[(\mathbf{e}_1 \cdot \mathbf{i})\,\frac{\partial}{\partial x} + (\mathbf{e}_1 \cdot \mathbf{j})\,\frac{\partial}{\partial y} + (\mathbf{e}_1 \cdot \mathbf{k})\,\frac{\partial}{\partial z}\right] + \mathbf{e}_2\,[\ldots] + \mathbf{e}_3\,[\ldots] \\ = \; & [\mathbf{e}_1\,(\mathbf{e}_1 \cdot \mathbf{i}) + \mathbf{e}_2\,(\mathbf{e}_2 \cdot \mathbf{i}) + \mathbf{e}_3\,(\mathbf{e}_3 \cdot \mathbf{i})]\,\frac{\partial}{\partial x} + [\ldots]\,\frac{\partial}{\partial y} + [\ldots]\,\frac{\partial}{\partial z} \\ = \; & \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}; \end{aligned}$$

this last line follows from the orthonormality of the bases as

$$\mathbf{i} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3$$

leads to $\alpha = \mathbf{e}_1 \cdot \mathbf{i}$, $\beta = \mathbf{e}_2 \cdot \mathbf{i}$, $\gamma = \mathbf{e}_3 \cdot \mathbf{i}$, by dotting the equation with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively. ■




**Corollary 82** *Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field given by $\mathbf{f} = (f_1, f_2, f_3)$, let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a right-handed orthonormal basis of $\mathbb{R}^3$. Suppose that*

$$\begin{aligned} F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 &= f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}; \\ \Phi\,(X, Y, Z) &= \phi\,(x, y, z)\,. \end{aligned}$$

*That is,* $\mathbf{F}$ *and* $\Phi$ *represent* $\mathbf{f}$ *and* $\phi$ *in the new co-ordinates* $X, Y, Z$*. Then*

$$\begin{aligned}
\operatorname{grad}\phi &= \frac{\partial\Phi}{\partial X}\mathbf{e}_1 + \frac{\partial\Phi}{\partial Y}\mathbf{e}_2 + \frac{\partial\Phi}{\partial Z}\mathbf{e}_3; \\
\operatorname{div}\mathbf{f} &= \frac{\partial F_1}{\partial X} + \frac{\partial F_2}{\partial Y} + \frac{\partial F_3}{\partial Z}; \\
\operatorname{curl}\mathbf{f} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial X} & \frac{\partial}{\partial Y} & \frac{\partial}{\partial Z} \\ F_1 & F_2 & F_3 \end{vmatrix}.
\end{aligned}$$

*That is, we calculate* grad, div *and* curl *using the same formulae, irrespective of what right-handed orthonormal co-ordinate system we are using.*

**Proof.** *Sketch Proof* The previous proposition showed that the formula for grad does not change under an orthonormal change of co-ordinates. From Geometry, we have that the formula for the dot product is similarly invariant as is the formula for the cross product provided the change is to another right-handed system. Hence the formulae for div $= \nabla\cdot$ and curl $= \nabla\wedge$ do not change when we change to another right-handed orthonormal system. ■

Adjustments to the above formulae need to be made for co-ordinate systems that aren't orthonormal – e.g. cylindrical polar co-ordinates or spherical polar co-ordinates. The relevant formulae below are for reference only; certainly you are not expected to memorize them. *The main point to take away from this is that the* grad, div, curl *formulae are different for co-ordinate systems that are not Cartesian.*

**Definition 83** ***Cylindrical polar co-ordinates*** $r, \theta, z$ *are defined by the identity*

$$\mathbf{r} = (x, y, z) = (r\cos\theta, r\sin\theta, z)\,.$$

*If we note*

$$\mathrm{d}\mathbf{r} = (\cos\theta, \sin\theta, 0)\,\mathrm{d}r + (-r\sin\theta, r\cos\theta, 0)\,\mathrm{d}\theta + (0, 0, 1)\,\mathrm{d}z$$

*then we may write*

$$\mathrm{d}\mathbf{r} = \mathbf{e}_r\mathrm{d}r + r\mathbf{e}_\theta\mathrm{d}\theta + \mathbf{e}_z\mathrm{d}z$$

*where*

$$\mathbf{e}_r = (\cos\theta, \sin\theta, 0)\,, \qquad \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0)\,, \qquad \mathbf{e}_z = (0, 0, 1)\,.$$

*Note, for any values of* $r, \theta, z$*, the vectors* $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ *make a right-handed orthonormal basis of* $\mathbb{R}^3$*.*

**Proposition 84** *The formulae for* grad, div, curl *in terms of cylindrical polar co-ordinates of* *fields* $\psi(r,\theta,z)$ *and* $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z$ *are*

$$\begin{aligned}
\nabla\psi &= \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\psi}{\partial z}\mathbf{e}_z; \\
\nabla\cdot\mathbf{F} &= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}; \\
\nabla\wedge\mathbf{F} &= \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}.
\end{aligned}$$

**Proof.** By way of example we shall prove only the first identity here. Recall that Cartesian co-ordinates are given in terms of cylindrical polar ones by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

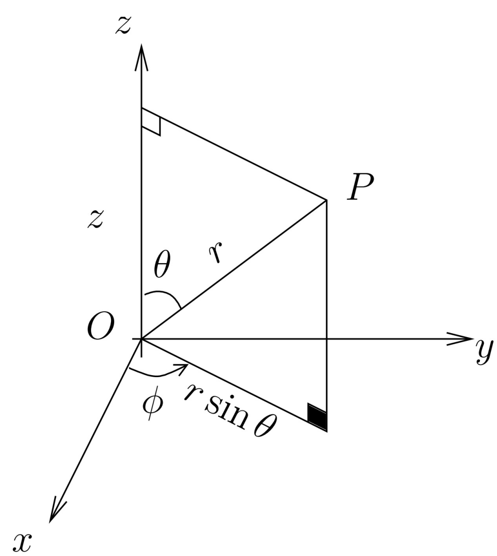
Say $\phi(x,y,z) = \psi(r,\theta,z)$. Then

$$\begin{aligned}
\frac{\partial\psi}{\partial r} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial r} = \frac{\partial\phi}{\partial x}\cos\theta + \frac{\partial\phi}{\partial y}\sin\theta, \\
\frac{\partial\psi}{\partial\theta} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial\theta} = -\frac{\partial\phi}{\partial x}r\sin\theta + \frac{\partial\phi}{\partial y}r\cos\theta, \\
\frac{\partial\psi}{\partial z} &= \frac{\partial\phi}{\partial z}.
\end{aligned}$$

Hence

$$\begin{aligned}
& \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{\partial\psi}{\partial z}\mathbf{e}_z \\
= & \left(\frac{\partial\phi}{\partial x}\cos\theta + \frac{\partial\phi}{\partial y}\sin\theta\right)(\cos\theta,\sin\theta,0) + \frac{1}{r}\left(-\frac{\partial\phi}{\partial x}r\sin\theta + \frac{\partial\phi}{\partial y}r\cos\theta\right)(-\sin\theta,\cos\theta,0) + \frac{\partial\phi}{\partial z}\mathbf{k} \\
= & \left(\frac{\partial\phi}{\partial x}\cos^2\theta + \frac{\partial\phi}{\partial x}\sin^2\theta\right)\mathbf{i} + \left(\frac{\partial\phi}{\partial y}\sin^2\theta + \frac{\partial\phi}{\partial y}\cos^2\theta\right)\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \\
= & \nabla\phi.
\end{aligned}$$

■




**Definition 85** ***Spherical polar co-ordinates*** *$r, \theta, \phi$ are defined by the identity*

$$\mathbf{r} = (x, y, z) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta).$$

*If we note* $\mathrm{d}\mathbf{r}$ *equals*

$$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,\mathrm{d}r + r\,(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)\,\mathrm{d}\theta + r\sin\theta\,(-\sin\phi, \cos\phi, 0)\,\mathrm{d}\phi$$

*then we may write*

$$\mathrm{d}\mathbf{r} = \mathbf{e}_r \mathrm{d}r + r\mathbf{e}_\theta \mathrm{d}\theta + r\sin\theta\mathbf{e}_\phi \mathrm{d}\phi$$

*where*

$$\mathbf{e}_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \quad \mathbf{e}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \quad \mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0).$$

*Note, for any values of $r, \theta, \phi$ the vectors $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$, make a right-handed orthonormal basis.*

**Proposition 86** *The formulae for* grad, div, curl *in terms of spherical polar co-ordinates of fields $\psi(r, \phi, \theta)$ and $\mathbf{F} = F_r\mathbf{e}_r + F_\phi\mathbf{e}_\phi + F_\theta\mathbf{e}_\theta$ are*

$$\begin{aligned}
\nabla\psi &= \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\mathbf{e}_\phi; \\
\nabla\cdot\mathbf{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta F_r\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta F_\theta\right) + \frac{\partial}{\partial\phi}\left(rF_\phi\right)\right]; \\
\nabla\wedge\mathbf{F} &= \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}.
\end{aligned}$$

Those interested in reading up on the more general theory of curvilinear co-ordinates (which explains the similarities in the previous two propositions) should see Boas (pp.427-435) or Kreyszig (pp.498-504).

## 5.2 Identities

**Proposition 87** *Let* $\mathbf{F}$ *be a vector field on* $\mathbb{R}^3$ *and* $\phi$ *be a scalar field on* $\mathbb{R}^3$*. Then*

$$\operatorname{curl}\operatorname{grad}\phi = \mathbf{0}; \qquad \operatorname{div}\operatorname{curl}\mathbf{F} = 0.$$

**Proof.** With the notation $\nabla\phi = \left(\phi_x, \phi_y, \phi_z\right)$ *where subscripts now denote partial differentiation*, we have

$$\operatorname{curl}(\nabla\phi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix} = \left(\phi_{zy} - \phi_{yz}, \phi_{xz} - \phi_{zx}, \phi_{yx} - \phi_{xy}\right) = \mathbf{0}.$$

Again with subscripts denoting partial differentiation, we also have

$$\begin{aligned} \operatorname{div}(\operatorname{curl}\mathbf{F}) &= \operatorname{div}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \left[(F_3)_{yx} - (F_2)_{zx}\right] + \left[(F_1)_{zy} - (F_3)_{xy}\right] + \left[(F_2)_{xz} - (F_1)_{yz}\right] = 0. \end{aligned}$$

■

**Proposition 88** *(Product Rules for div and curl) Let* $\mathbf{F}$ *be a vector field on* $\mathbb{R}^3$ *and* $\phi, \psi$ *be scalar fields on* $\mathbb{R}^3$*. Then*

$$\begin{aligned} \operatorname{div}(\phi\mathbf{F}) &= \nabla\phi \cdot \mathbf{F} + \phi\operatorname{div}\mathbf{F}; \\ \operatorname{curl}(\phi\mathbf{F}) &= \phi\operatorname{curl}\mathbf{F} + \nabla\phi \wedge \mathbf{F}. \end{aligned}$$

**Proof.** (i) With subscripts denoting partial differentiation we have

$$\begin{aligned} \operatorname{div}(\phi\mathbf{F}) &= (\phi F_1)_x + (\phi F_2)_y + (\phi F_3)_z \\ &= \phi\,(F_1)_x + \phi_x F_1 + \phi\,(F_2)_y + \phi_y F_2 + \phi\,(F_3)_z + \phi_z F_3 \\ &= \phi\left[(F_1)_x + (F_2)_y + (F_3)_z\right] + \left(\phi_x, \phi_y, \phi_z\right) \cdot (F_1, F_2, F_3) \\ &= \phi\operatorname{div}\mathbf{F} + (\nabla\phi) \cdot \mathbf{F}. \end{aligned}$$

(ii) $\operatorname{curl}(\phi\mathbf{F})$ equals

$$\begin{aligned} & \left(\frac{\partial(\phi F_3)}{\partial y} - \frac{\partial(\phi F_2)}{\partial z}, \frac{\partial(\phi F_1)}{\partial z} - \frac{\partial(\phi F_3)}{\partial x}, \frac{\partial(\phi F_2)}{\partial x} - \frac{\partial(\phi F_1)}{\partial y}\right) \\ = \;& \phi\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) + \left(\phi_y F_3 - \phi_z F_2, \phi_z F_1 - \phi_x F_3, \phi_x F_2 - \phi_y F_1\right) \\ = \;& \phi\operatorname{curl}\mathbf{F} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \phi_x & \phi_y & \phi_z \\ F_1 & F_2 & F_3 \end{vmatrix} \\ = \;& \phi\operatorname{curl}\mathbf{F} + (\nabla\phi) \wedge \mathbf{F}. \end{aligned}$$

■

**Proposition 89** *(Further Identities) For vector fields* $\mathbf{F}, \mathbf{G}$*:*

$$\begin{aligned}
\nabla(\mathbf{F}\cdot\mathbf{G}) &= (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\wedge(\nabla\wedge\mathbf{G}) + \mathbf{G}\wedge(\nabla\wedge\mathbf{F}) \\
\nabla\cdot(\mathbf{F}\wedge\mathbf{G}) &= \mathbf{G}\cdot(\nabla\wedge\mathbf{F}) - \mathbf{F}\cdot(\nabla\wedge\mathbf{G}) \\
\nabla\wedge(\mathbf{F}\wedge\mathbf{G}) &= \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} \\
\nabla\wedge(\nabla\wedge\mathbf{F}) &= \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}
\end{aligned}$$

**Remark 90** *All vector calculus can be proven by simply grinding out the calculations. But there are neater ways to write out the formula for* div, grad *and* curl *as given below:*

$$\nabla\phi = \sum_i \frac{\partial\phi}{\partial x_i}\mathbf{e}_i, \qquad \nabla\cdot\mathbf{F} = \sum_i \mathbf{e}_i\cdot\frac{\partial\mathbf{F}}{\partial x_i}, \qquad \nabla\wedge\mathbf{F} = \sum_i \mathbf{e}_i\wedge\frac{\partial\mathbf{F}}{\partial x_i}, \qquad \mathbf{F}\cdot\nabla = \sum_i (\mathbf{F}\cdot\mathbf{e}_i)\frac{\partial}{\partial x_i},$$

*where the dummy variable* $i$ *ranges over* $1,2,3$ *in each case and* $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ *is any right-handed orthonormal basis. For example*

$$\begin{aligned}
\sum_i \mathbf{e}_i\wedge\frac{\partial\mathbf{F}}{\partial x_i} &= \mathbf{e}_1\wedge\frac{\partial\mathbf{F}}{\partial x_1} + \mathbf{e}_2\wedge\frac{\partial\mathbf{F}}{\partial x_2} + \mathbf{e}_3\wedge\frac{\partial\mathbf{F}}{\partial x_3} \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 1 & 0 & 0 \\ \frac{\partial F_1}{\partial x_1} & \frac{\partial F_2}{\partial x_1} & \frac{\partial F_3}{\partial x_1} \end{vmatrix} + \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 1 & 0 \\ \frac{\partial F_1}{\partial x_2} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_3}{\partial x_2} \end{vmatrix} + \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 0 & 0 & 1 \\ \frac{\partial F_1}{\partial x_3} & \frac{\partial F_2}{\partial x_3} & \frac{\partial F_3}{\partial x_3} \end{vmatrix} \\
&= -\mathbf{e}_2\frac{\partial F_3}{\partial x_1} + \mathbf{e}_3\frac{\partial F_2}{\partial x_1} + \mathbf{e}_1\frac{\partial F_3}{\partial x_2} - \mathbf{e}_3\frac{\partial F_1}{\partial x_2} - \mathbf{e}_1\frac{\partial F_2}{\partial x_3} + \mathbf{e}_2\frac{\partial F_1}{\partial x_3} \\
&= \nabla\wedge\mathbf{F}.
\end{aligned}$$

*With these formulae the proofs of identities involving* div, grad *and* curl *are much shorter; on the other hand the formulae need to be applied with much more care and attention to detail than was previously the case.*

**Proof.** The second and third identities appear in the Exercise Sheets. To prove the first identity recall that $\mathbf{u}\wedge(\mathbf{v}\wedge\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ and hence

$$\begin{aligned}
&(\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F} + \mathbf{F}\wedge(\nabla\wedge\mathbf{G}) + \mathbf{G}\wedge(\nabla\wedge\mathbf{F}) \\
= \;& \sum_i (\mathbf{F}\cdot\mathbf{e}_i)\frac{\partial\mathbf{G}}{\partial x_i} + \sum_i (\mathbf{G}\cdot\mathbf{e}_i)\frac{\partial\mathbf{F}}{\partial x_i} + \mathbf{F}\wedge\left(\sum_i \mathbf{e}_i\wedge\frac{\partial\mathbf{G}}{\partial x_i}\right) + \mathbf{G}\wedge\left(\sum_i \mathbf{e}_i\wedge\frac{\partial\mathbf{F}}{\partial x_i}\right) \\
= \;& \sum_i (\mathbf{F}\cdot\mathbf{e}_i)\frac{\partial\mathbf{G}}{\partial x_i} + \sum_i (\mathbf{G}\cdot\mathbf{e}_i)\frac{\partial\mathbf{F}}{\partial x_i} + \sum_i \mathbf{F}\wedge\left(\mathbf{e}_i\wedge\frac{\partial\mathbf{G}}{\partial x_i}\right) + \sum_i \mathbf{G}\wedge\left(\mathbf{e}_i\wedge\frac{\partial\mathbf{F}}{\partial x_i}\right) \\
= \;& \sum_i \left\{(\mathbf{F}\cdot\mathbf{e}_i)\frac{\partial\mathbf{G}}{\partial x_i} + (\mathbf{G}\cdot\mathbf{e}_i)\frac{\partial\mathbf{F}}{\partial x_i} + \left[\left(\mathbf{F}\cdot\frac{\partial\mathbf{G}}{\partial x_i}\right)\mathbf{e}_i - (\mathbf{F}\cdot\mathbf{e}_i)\frac{\partial\mathbf{G}}{\partial x_i}\right] + \left[\left(\mathbf{G}\cdot\frac{\partial\mathbf{F}}{\partial x_i}\right)\mathbf{e}_i - (\mathbf{G}\cdot\mathbf{e}_i)\frac{\partial\mathbf{F}}{\partial x_i}\right]\right\} \\
= \;& \sum_i \mathbf{e}_i\left(\mathbf{F}\cdot\frac{\partial\mathbf{G}}{\partial x_i} + \mathbf{G}\cdot\frac{\partial\mathbf{F}}{\partial x_i}\right) \\
= \;& \sum_i \mathbf{e}_i\frac{\partial}{\partial x_i}(\mathbf{F}\cdot\mathbf{G}) = \nabla(\mathbf{F}\cdot\mathbf{G}).
\end{aligned}$$

To prove the fourth identity recall again that $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}$ and then

$$
\begin{aligned}
\nabla \wedge (\nabla \wedge \mathbf{F}) &= \sum_i \mathbf{e}_i \wedge \frac{\partial}{\partial x_i} \left( \sum_j \mathbf{e}_j \wedge \frac{\partial \mathbf{F}}{\partial x_j} \right) \\
&= \sum_i \sum_j \mathbf{e}_i \wedge \left( \mathbf{e}_j \wedge \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} \right) \\
&= \sum_i \sum_j \left[ \left( \mathbf{e}_i \cdot \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} \right) \mathbf{e}_j \right] - \sum_i \sum_j \left[ (\mathbf{e}_i \cdot \mathbf{e}_j)\, \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} \right] \\
&= \sum_j \mathbf{e}_j \frac{\partial}{\partial x_j} \left[ \sum_i \mathbf{e}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} \right] - \sum_i \sum_j \left[ \boldsymbol{\delta}_{ij} \frac{\partial^2 \mathbf{F}}{\partial x_i \partial x_j} \right] \\
&= \sum_j \mathbf{e}_j \frac{\partial}{\partial x_j} \left[ \sum_i \mathbf{e}_i \cdot \frac{\partial \mathbf{F}}{\partial x_i} \right] - \sum_i \frac{\partial^2 \mathbf{F}}{\partial x_i^2} = \nabla\,(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}
\end{aligned}
$$

■

## 5.3 Physical Interpretation

**Example 91** ***(Fourier's Law)*** *In an isotropic medium with constant thermal conductivity* $k$*, with temperature* $T(\mathbf{x},t)$ *at position* $\mathbf{x}$ *and at time* $t$*, Fourier's Law states that*

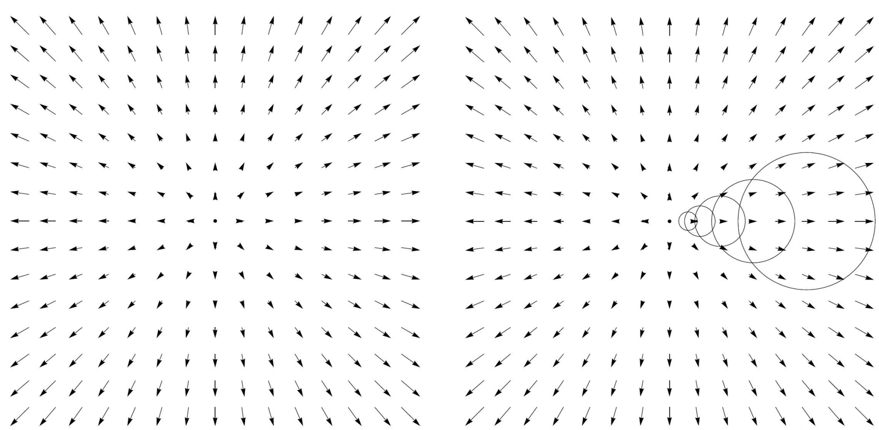
$$\mathbf{q} = -k\nabla T$$

*where* $\mathbf{q}$ *is the heat flux – that is the amount of energy that flows through a particular surface per unit area per unit time.*

This law contains much of the essence of grad. Recall from the Michaelmas Calculus course that, for a given scalar function $\phi$ and a unit vector $\mathbf{v}$, the directional derivative of $\phi$ in the direction $\mathbf{v}$ is $\nabla\phi\cdot\mathbf{v}$. In particular, $\phi$ increases fastest in the direction of $\nabla\phi$ and decreases fastest in the direction $-\nabla\phi$. So Fourier's law, loosely put, states that at each point the heat energy moves towards the coolest nearby points.

What about the physical motivation behind divergence?

**Example 92** *Consider fluid particles in the xy-plane flowing according to the steady flow*



$\mathbf{u}(x,y) = (x,y)$.    $\varepsilon = 0.5,\ t = 0, 0.5, 1, 1.5, 2$

*We have a source at* $(0,0)$ *from which the fluid emanates. However, at every* $(x,y)$, *we have that*

$$\operatorname{div}\mathbf{u} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2.$$

*Suppose a particle starts off at* $(x_0,y_0)$*; then the particle's position* $(x(t),y(t))$ *at time* $t$ *is determined by the differential equations*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = y, \qquad x(0) = x_0, \qquad y(0) = y_0$$

*and, hence,*

$$(x(t),y(t)) = \left(x_0e^t, y_0e^t\right).$$

*Consider now a blob of fluid of area $\pi\varepsilon^2$ which at $t = 0$ occupies*

$$(x-1)^2 + y^2 < \varepsilon^2.$$

*Where will the particles which make up this blob have moved by time $t$? Initially (at $t = 0$) the particles were at $x_0 = 1 + r\cos\theta$, $y = r\sin\theta$ where $r < \varepsilon$; at time $t > 0$ they have moved to*

$$(x(t), y(t)) = \left(e^t + re^t\cos\theta, re^t\sin\theta\right) \qquad r < \varepsilon,$$

*which comprises a disc with centre $(e^t, 0)$ and area $A(t) = \pi\varepsilon^2 e^{2t}$. So, throughout the motion we have*

$$\frac{\mathrm{d}A}{\mathrm{d}t} = (\operatorname{div}\mathbf{u})\, A.$$

*More generally, for a two dimensional flow, it is the case that*

$$\nabla\cdot\mathbf{u}(\mathbf{x}) = \lim_{\varepsilon\to 0}\left[\frac{1}{\textit{area}\,(\textit{Flow image of } B(\mathbf{x},\varepsilon) \textit{ at time } t)}\frac{\mathrm{d}}{\mathrm{d}t}\textit{area}\,(\textit{Flow image of } B(\mathbf{x},\varepsilon) \textit{ at time } t)\right],$$

*with the analogous result for 3D flows in terms of volume rather than area. In physical terms, then, the divergence of a vector field is a measure of the flow's source density, the extent to which the vector field flow behaves like a source or a sink at a given point.*

If you choose to take the second year option in fluid dynamics you will meet:

**Example 93** ***Euler's Equations for an ideal fluid*** *(Second year course in Fluid Dynamics). Euler's equations for an incompressible, inviscid fluid state that*

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\frac{\nabla p}{\rho} + \mathbf{g}, \qquad \nabla\cdot\mathbf{u} = 0,$$

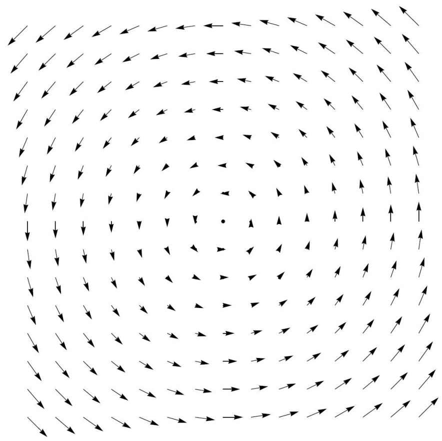
*where $\mathbf{u}(\mathbf{x}, t)$ is the flow velocity, $p$ is the pressure, $\rho$ is the density and $\mathbf{g}$ is acceleration due to gravity. It is the equation*

$$\nabla\cdot\mathbf{u} = 0$$

*which encodes the incompressibility of the fluid.*

Such vector calculus are found in many applications of mathematics, including Maxwell's equations of electromagnetism, fluid and solid mechanics, and cellular behaviour.

Roughly speaking the curl of a vector field relates to the rotation of the vector field, *but very much in a local sense.*



$\mathbf{u}(x, y) = (-y, x)$

**Example 94** ***(Rotation of a rigid body)*** *Recall from the Michaelmas Dynamics course that if* $\mathbf{r} = r\mathbf{e}_r$ *then*

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta.$$

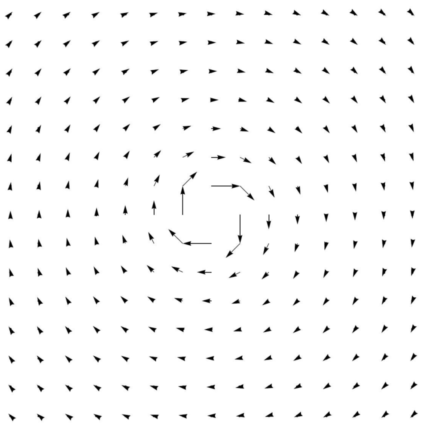
*In particular, if a rigid body is rotating with constant angular velocity* $\omega$ *about the* $z$*-axis then the distance of a point* $r$ *from the* $z$ *-axis will remain constant and* $\dot{\theta} = \omega$ *so that*

$$\dot{\mathbf{r}} = r\omega\mathbf{e}_\theta = \omega\mathbf{k}\wedge(r\mathbf{e}_r) = \omega\mathbf{k}\wedge\mathbf{r}.$$

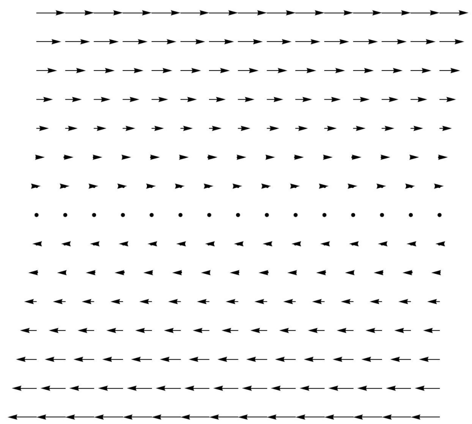
*We have already seen from Example 74 that*

$$\operatorname{curl}\dot{\mathbf{r}} = \operatorname{curl}(\omega\mathbf{k}\wedge\mathbf{r}) = 2\omega\mathbf{k}.$$

**Example 95** *It is important to stress though that* curl *is very subtly a measure of local rotation. Consider the following two examples:*



$\mathbf{u}_1(x,y) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$



$\mathbf{u}_2(x,y) = (cy, 0)$

*A simple calculation shows that* $\operatorname{curl}\mathbf{u}_2 = -c\mathbf{k}$ *yet there appears to be no sense of global rotation, whilst in Exercise 5 on Sheet 1 you are asked to show that* $\operatorname{curl}\mathbf{u}_1 = \mathbf{0}$*, yet there appears to be a clear sense of global rotation. Despite particles moving in circles in the flow* $\mathbf{u}_1$ *different circles are moving around the origin with different angular velocities and with a greater angular velocity towards the origin; in fact this differential is such that a circular paddle wheel inserted into the fluid would not spin as it moves around the origin. In contrast, with the shear flow* $\mathbf{u}_2$*, whilst the particles are moving in lines, a small paddle wheel inserted into the flow would spin as the flow for greater* $y$ *is moving faster. In general, the axis of this spinning is in the direction of* curl.

# 6. DIVERGENCE AND STOKES' THEOREMS

## 6.1 The Divergence Theorem

**Theorem 96** *(**Divergence Theorem** – Lagrange 1762, Gauss 1813, Green 1825) Let $R$ be a region of $\mathbb{R}^3$ with a piecewise smooth boundary $\partial R$, and let $\mathbf{F}$ be a differentiable vector field on $R$. Then*

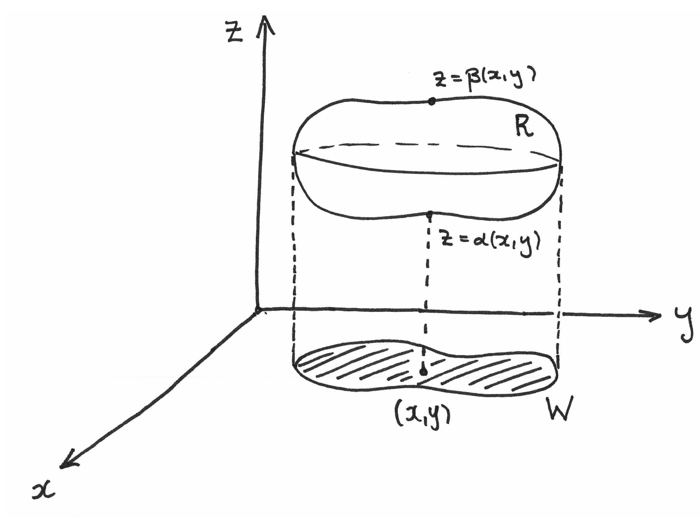
$$\iiint_R \operatorname{div} \mathbf{F} \, \mathrm{d}V = \iint_{\partial R} \mathbf{F} \cdot \mathrm{d}\mathbf{S},$$

*where $\mathrm{d}\mathbf{S}$ is oriented in the direction of the outward pointing normal from $R$.*

**Definition 97** *We say that a region $R$ is **convex** if for any $\mathbf{p}, \mathbf{q} \in R$ the line segment connecting $\mathbf{p}$ and $\mathbf{q}$ is contained in $R$.*

We will only prove the Divergence Theorem for convex regions, though the proof extends with very little extra work to any region that can be decomposed into convex regions.

**Proof.** (**Not examinable.**) Let $W$ be a subset of the $xy$-plane and $\alpha, \beta$ continuous functions on $W$.




Let

$$R = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in W, \, \alpha(x, y) \leqslant z \leqslant \beta(x, y)\right\}$$

and $\mathbf{F} = F\mathbf{k}$ is a differentiable vector field on $R$. (That is $R$ is $z$-convex in the sense that the intersection of $R$ with any line parallel with the $z$-axis is an interval.) The Divergence Theorem in this case reads as

$$\iiint\limits_R \frac{\partial F}{\partial z}\,\mathrm{d}V = \iint\limits_{\partial R} F\mathbf{k}\cdot\mathrm{d}\mathbf{S}.$$

Note that $\partial R$ is made up of three surfaces:

$$\begin{aligned}
\Sigma^+ &= \left\{(x,y,z)\in\mathbb{R}^3 : (x,y)\in W,\ z=\beta(x,y)\right\},\\
\Sigma &= \left\{(x,y,z)\in\mathbb{R}^3 : (x,y)\in \partial W,\ \alpha(x,y)\leqslant z\leqslant\beta(x,y)\right\},\\
\Sigma^- &= \left\{(x,y,z)\in\mathbb{R}^3 : (x,y)\in W,\ z=\alpha(x,y)\right\}.
\end{aligned}$$

Then

$$\iiint\limits_R \frac{\partial F}{\partial z}\,\mathrm{d}V = \iint\limits_W \int_{\alpha(x,y)}^{\beta(x,y)} \frac{\partial F}{\partial z}\,\mathrm{d}z\,\mathrm{d}A = \iint\limits_W \left(F(x,y,\beta(x,y)) - F(x,y,\alpha(x,y))\right)\mathrm{d}A.$$

Regarding the boundary integral:

$$\text{on } \Sigma^+ \quad : \qquad \mathrm{d}\mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \beta_x \\ 0 & 1 & \beta_y \end{vmatrix}\,\mathrm{d}x\,\mathrm{d}y = \left(-\beta_x, -\beta_y, 1\right)\,\mathrm{d}x\,\mathrm{d}y;$$

$$\text{on } \Sigma^- \quad : \qquad \mathrm{d}\mathbf{S} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \alpha_x \\ 0 & 1 & \alpha_y \end{vmatrix}\,\mathrm{d}x\,\mathrm{d}y = \left(\alpha_x, \alpha_y, -1\right)\,\mathrm{d}x\,\mathrm{d}y;$$

$$\text{on } \Sigma \quad : \qquad \mathbf{k}\cdot\mathrm{d}\mathbf{S} = 0.$$

Hence we have

$$\begin{aligned}
\iint\limits_{\Sigma^+} F\mathbf{k}\cdot\mathrm{d}\mathbf{S} &= \iint\limits_W F(x,y,\beta(x,y))\,\mathrm{d}x\,\mathrm{d}y;\\
\iint\limits_{\Sigma^-} F\mathbf{k}\cdot\mathrm{d}\mathbf{S} &= -\iint\limits_W F(x,y,\alpha(x,y))\,\mathrm{d}x\,\mathrm{d}y;\\
\iint\limits_{\Sigma} F\mathbf{k}\cdot\mathrm{d}\mathbf{S} &= 0;
\end{aligned}$$

and we have shown

$$\iiint\limits_R \frac{\partial F}{\partial z}\,\mathrm{d}V = \iint\limits_{\partial R} F\mathbf{k}\cdot\mathrm{d}\mathbf{S}$$

for the given region $R$. However, any convex body has the property of being in the form of $R$ when viewed from each of the $x$, $y$, $z$ directions. Hence we have shown

$$\iiint_R \operatorname{div}\mathbf{F}\,\mathrm{d}V = \iint_{\partial R} \mathbf{F}\cdot\mathrm{d}\mathbf{S}$$

for any convex body. ■

**Corollary 98** *Let $\phi$ be a smooth scalar field defined on a region $R\subseteq\mathbb{R}^3$ with a piecewise smooth boundary. Then*

$$\iiint_R \nabla\phi\,\mathrm{d}V = \iint_{\partial R} \phi\,\mathrm{d}\mathbf{S}.$$

**Proof.**

Let $\mathbf{c}$ be a constant vector and $\mathbf{F}=\mathbf{c}\phi$. Then

$$\operatorname{div}(\mathbf{c}\phi) = \mathbf{c}\cdot\nabla\phi + \phi\operatorname{div}\mathbf{c} = \mathbf{c}\cdot\nabla\phi$$

as $\mathbf{c}$ is constant. By the Divergence Theorem

$$\iiint_R \mathbf{c}\cdot\nabla\phi\,\mathrm{d}V = \iint_{\partial R}\mathbf{c}\phi\cdot\mathrm{d}\mathbf{S}, \implies \mathbf{c}\cdot\left(\iiint_R \nabla\phi\,\mathrm{d}V\right) = \mathbf{c}\cdot\left(\iint_{\partial R}\phi\mathrm{d}\mathbf{S}\right).$$

As $\mathbf{c}$ is an arbitrary vector then

$$\iiint_R \nabla\phi\,\mathrm{d}V = \iint_{\partial R} \phi\,\mathrm{d}\mathbf{S}.$$

■

**Example 99** *Verify the Divergence Theorem when $R$ is the finite region bounded by the paraboloid $z=x^2+y^2$ and the plane $x+y+z=1$, and $\mathbf{F}=(x,y,z)$.*

**Solution.** We already met this region in Example 19. We showed that the top surface of $R$ projects vertically down to the set

$$W=\left\{(x,y)\in\mathbb{R}^2:(x+1/2)^2+(y+1/2)^2\leqslant 3/2\right\}$$

which is a disc, centre $(-1/2,-1/2)$ and radius $\sqrt{3/2}$. Further we calculated the volume of the region to be $9\pi/8$. As $\operatorname{div}\mathbf{F}=3$ then

$$\iiint_R \operatorname{div}F\,\mathrm{d}V = 3\operatorname{Vol}(R) = \frac{27\pi}{8}.$$

The boundary $\partial R$ is made up of two surfaces

$$\begin{aligned}
\Sigma_1 &= \left\{(x,y,z) : z \geqslant x^2+y^2,\ x+y+z=1\right\} = \{(x,y,z) : (x,y) \in W,\ x+y+z=1\}\,; \\
\Sigma_2 &= \left\{(x,y,z) : z = x^2+y^2,\ x+y+z \leqslant 1\right\} = \left\{(x,y,z) : (x,y) \in W,\ z=x^2+y^2\right\}.
\end{aligned}$$

We can parameterise $\Sigma_1$ and $\Sigma_2$ as

$$\mathbf{r}_1(u,v) = (u, v, 1-u-v)\,; \qquad \mathbf{r}_2(u,v) = \left(u, v, u^2+v^2\right).$$

Note

$$(\mathbf{r}_1)_u \wedge (\mathbf{r}_1)_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1,1,1)$$

which is outward-pointing from $R$, and

$$(\mathbf{r}_2)_u \wedge (\mathbf{r}_2)_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u,-2v,1)$$

which is inward-pointing and so we will take its negative. So

$$\iint\limits_{\Sigma_1} \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \iint\limits_{W} (u,v,1-u-v)\cdot(1,1,1)\ \mathrm{d}u\,\mathrm{d}v = \iint\limits_{W} \mathrm{d}u\,\mathrm{d}v = \pi\left(\sqrt{3/2}\right)^2 = 3\pi/2.$$

For the second surface

$$\begin{aligned}
\iint\limits_{\Sigma_2} \mathbf{F}\cdot \mathrm{d}\mathbf{S} &= \iint\limits_{W} \left(u,v,u^2+v^2\right)\cdot(2u,2v,-1)\ \mathrm{d}A \\
&= \iint\limits_{W} \left(u^2+v^2\right)\ \mathrm{d}A \\
&= \int_{r=0}^{\sqrt{3/2}} \int_{\theta=0}^{2\pi} \left\{(-1/2+r\cos\theta)^2 + (-1/2+r\sin\theta)^2\right\}\ r\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_{r=0}^{\sqrt{3/2}} \int_{\theta=0}^{2\pi} \left\{r^2 - r\cos\theta - r\sin\theta + 1/2\right\}\ r\,\mathrm{d}\theta\,\mathrm{d}r \\
&= 2\pi \int_{r=0}^{\sqrt{3/2}} \left\{r^3 + \frac{r}{2}\right\}\ \mathrm{d}\theta\,\mathrm{d}r \\
&= 2\pi\left[\frac{r^4}{4} + \frac{r^2}{4}\right]_0^{\sqrt{3/2}} \\
&= 2\pi\,(9/16+3/8) = \frac{15\pi}{8}.
\end{aligned}$$

Hence, in total, we have

$$\iint_{\partial R} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \frac{3\pi}{2} + \frac{15\pi}{8} = \frac{12\pi}{8} + \frac{15\pi}{8} = \frac{27\pi}{8}.$$

■

**Example 100** *The twice continuously differentiable function $f(x, y, z)$ is homogeneous of degree $n$, so that*

$$f(tx, ty, tz) = t^n f(x, y, z) \quad \textit{for all } t \in \mathbb{R}. \tag{6.1}$$

*Prove* ***Euler's Theorem****, which states that*

$$\mathbf{r} \cdot \nabla f = nf.$$

*Let $B$ be the unit ball $x^2 + y^2 + z^2 < 1$ and let $\partial B$ be its boundary. Show that*

$$\iiint_B f \,\mathrm{d}V = \frac{1}{n(n+3)} \iiint_B \nabla^2 f \,\mathrm{d}V,$$

*and hence show that*

$$\iiint_B (x - y + z)^4 \,\mathrm{d}V = \frac{36\pi}{35}.$$

**Solution.** If we differentiate the identity (6.1) with respect to $t$ then we find

$$x\frac{\partial f}{\partial x}(tx, ty, tz) + y\frac{\partial f}{\partial y}(tx, ty, tz) + z\frac{\partial f}{\partial z}(tx, ty, tz) = nt^{n-1} f(x, y, z).$$

If we set $t = 1$ then we arrive at Euler's Theorem $\mathbf{r} \cdot \nabla f = nf$.

By the Divergence Theorem, with $B$ as the unit ball, and $\partial B$ its boundary, then

$$\iiint_B \nabla^2 f \,\mathrm{d}V = \iiint_B \nabla \cdot \nabla f \,\mathrm{d}V = \iint_{\partial B} \nabla f \cdot \mathbf{n} \,\mathrm{d}S = \iint_{\partial B} \nabla f \cdot \mathbf{r} \,\mathrm{d}S = n \iint_{\partial B} f \,\mathrm{d}S$$

as $\mathbf{n} = \mathbf{r}$ on $\partial B$. On the other hand

$$\nabla \cdot (f\mathbf{r}) = (\nabla f) \cdot \mathbf{r} + f(\nabla \cdot \mathbf{r}) = nf + 3f = (n+3) f.$$

and so, by the Divergence Theorem again,

$$(n+3) \iiint_B f \,\mathrm{d}V = \iiint_B \nabla \cdot (f\mathbf{r}) \,\mathrm{d}V = \iint_{\partial B} f\mathbf{r} \cdot \mathbf{n} \,\mathrm{d}S = \iint_{\partial B} f \,\mathrm{d}S$$

as $\partial B$ is the unit sphere. Hence

$$\iiint_B f \,\mathrm{d}V = \frac{1}{n(n+3)} \iiint_B \nabla^2 f \,\mathrm{d}V,$$

The function $(x-y+z)^4$ is homogeneous of degree 4 in $x, y, z$ and so that

$$\begin{aligned}
& \iiint_B (x-y+z)^4 \, \mathrm{d}V \\
= & \frac{1}{4\times 7} \iiint_B \nabla^2 (x-y+z)^4 \, \mathrm{d}V \\
= & \frac{1}{4\times 7} \iiint_B 36\,(x-y+z)^2 \, \mathrm{d}V \qquad \text{[which is again homogeneous]} \\
= & \frac{36}{4\times 7} \times \frac{1}{2\times 5} \iiint_B \nabla^2 (x-y+z)^2 \, \mathrm{d}V \\
= & \frac{36}{4\times 7} \times \frac{1}{2\times 5} \iiint_B 6 \, \mathrm{d}V \\
= & \frac{36}{4\times 7} \times \frac{6}{2\times 5} \times \frac{4\pi}{3} \\
= & \frac{36\pi}{35}.
\end{aligned}$$

■

## 6.2 Applications of The Divergence Theorem

### 6.2.1 Boundary value problems

The Divergence Theorem also has important implications for certain boundary-value problems.

**Corollary 101** ***(Uniqueness of the Dirichlet Problem)*** *Let $R \subseteq \mathbb{R}^3$ be a path-connected region with piecewise smooth boundary $\partial R$ and let $f$ be a continuous function defined on $\partial R$. Suppose that $\phi_1$ and $\phi_2$ are such that*

$$\begin{aligned}
\nabla^2 \phi_1 &= 0 = \nabla^2 \phi_2 \quad \textit{in } R; \\
\phi_1 &= f = \phi_2 \quad \textit{on } \partial R.
\end{aligned}$$

*Then $\phi_1 = \phi_2$ in $R$.*

**Proof.** Let $\psi = \phi_1 - \phi_2$, so that

$$\nabla^2 \psi = 0 \;\text{ in } R, \qquad \psi = 0 \;\text{ on } \partial R.$$

If we consider the function $\psi^2$ then, by the Divergence Theorem and as $\nabla^2 = \nabla \cdot \nabla$,

$$\iint_{\partial R} \nabla\left(\psi^2\right)\cdot \mathrm{d}\mathbf{S} = \iiint_R \nabla^2\left(\psi^2\right)\,\mathrm{d}V.$$

Looking at the LHS,

$$\iint_{\partial R} \nabla\left(\psi^2\right)\cdot \mathrm{d}\mathbf{S} = \iint_{\partial R} 2\psi\nabla\psi\cdot \mathrm{d}\mathbf{S} = 0 \qquad \text{as } \psi = 0 \text{ on } \partial R.$$

On the RHS we have

$$\nabla^2\left(\psi^2\right) = 2\psi\nabla^2\psi + 2\nabla\psi\cdot\nabla\psi = 2\left|\nabla\psi\right|^2 \qquad \text{as } \nabla^2\psi = 0 \text{ in } R.$$

Hence

$$\iiint_R 2\left|\nabla\psi\right|^2\,\mathrm{d}V = 0 \implies \nabla\psi = \mathbf{0} \text{ in } R.$$

As $\nabla\psi = \mathbf{0}$ in $R$ and $R$ is path-connected then $\psi$ is constant throughout $R$. But $\psi = 0$ on $\partial R$ and so by continuity $\psi = 0$ throughout $R$.

■

**Corollary 102** ***(Uniqueness, up to a constant, of the Neumann Problem)*** *Let $R \subseteq \mathbb{R}^3$ be a path-connected region with piecewise smooth boundary $\partial R$ and let $f$ be a continuous function defined on $\partial R$. Suppose that $\phi_1$ and $\phi_2$ are such that*

$$\begin{aligned} \nabla^2\phi_1 &= 0 = \nabla^2\phi_2 \quad \textit{in } R; \\ \frac{\partial\phi_1}{\partial n} &= f = \frac{\partial\phi_2}{\partial n} \quad \textit{on } \partial R. \end{aligned}$$

*Then $\phi_1 - \phi_2$ is constant in $R$.*

**Proof.** Let $\psi = \phi_1 - \phi_2$, so that

$$\nabla^2\psi = 0 \text{ in } R, \qquad \frac{\partial\psi}{\partial n} = \nabla\psi\cdot\mathbf{n} = 0 \text{ on } \partial R.$$

If we consider the function $\psi^2$, then by the Divergence Theorem and as $\nabla^2 = \nabla\cdot\nabla$,

$$\iint_{\partial R} \nabla\left(\psi^2\right)\cdot \mathrm{d}\mathbf{S} = \iiint_R \nabla^2\left(\psi^2\right)\,\mathrm{d}V.$$

Looking at the LHS,

$$\iint_{\partial R} \nabla\left(\psi^2\right)\cdot \mathrm{d}\mathbf{S} = \iint_{\partial R} 2\psi\nabla\psi\cdot\mathbf{n}\,\mathrm{d}S = 0 \qquad \text{as } \nabla\psi\cdot\mathbf{n} = 0 \text{ on } \partial R.$$

On the RHS we have

$$\nabla^2\left(\psi^2\right)=2\psi\nabla^2\psi+2\nabla\psi\cdot\nabla\psi=2\left|\nabla\psi\right|^2 \qquad \text{as } \nabla^2\psi=0 \text{ in } R.$$

Hence

$$\iiint_R 2\left|\nabla\psi\right|^2 \,\mathrm{d}V=0 \implies \nabla\psi=\mathbf{0} \text{ in } R.$$

As $\nabla\psi=\mathbf{0}$ in $R$ then $\psi$ is constant throughout $R$. ■

**Corollary 103** ***(Robin (or Mixed) Boundary Problem)*** *Let $R\subseteq\mathbb{R}^3$ be a path-connected region with piecewise smooth boundary $\partial R$ and let $f$ be a continuous function defined on $\partial R$. Further let $\beta$ be a real constant. Suppose that $\phi_1$ and $\phi_2$ are such that*

$$\begin{aligned}\nabla^2\phi_1 &= 0=\nabla^2\phi_2 \quad \textit{in } R;\\ \beta\phi_1+\frac{\partial\phi_1}{\partial n} &= f=\beta\phi_2+\frac{\partial\phi_2}{\partial n} \quad \textit{on } \partial R.\end{aligned}$$

*(i) If $\beta>0$ then $\phi_1=\phi_2$ in $R$.*
*(ii) If $\beta=0$ then $\phi_1-\phi_2$ is constant in $R$.*
*(iii) If $\beta<0$ then the solution need not be unique, even up to a constant.*

**Proof.** Let $\psi=\phi_1-\phi_2$, so that

$$\nabla^2\psi=0 \ \text{ in } R, \qquad \beta\psi+\frac{\partial\psi}{\partial n}=\beta\psi+\nabla\psi\cdot\mathbf{n}=0 \ \text{ on } \partial R.$$

If we consider the function $\psi^2$ then, by the Divergence Theorem and as $\nabla^2=\nabla\cdot\nabla$,

$$\iint_{\partial R}\nabla\left(\psi^2\right)\cdot\mathrm{d}\mathbf{S}=\iiint_R\nabla^2\left(\psi^2\right)\,\mathrm{d}V.$$

Looking at the LHS and noting $\beta\psi+\nabla\psi\cdot\mathbf{n}=0$ on $\partial R$ we have

$$\iint_{\partial R}\nabla\left(\psi^2\right)\cdot\mathrm{d}\mathbf{S}=\iint_{\partial R}2\psi\left(\nabla\psi\cdot\mathbf{n}\right)\,\mathrm{d}S=\iint_{\partial R}2\psi\left(-\beta\psi\right)\,\mathrm{d}S=-2\beta\iint_{\partial R}\psi^2\,\mathrm{d}S.$$

On the RHS we have as before

$$\nabla^2\left(\psi^2\right)=2\psi\nabla^2\psi+2\nabla\psi\cdot\nabla\psi=2\left|\nabla\psi\right|^2 \qquad \text{as } \nabla^2\psi=0 \text{ in } R.$$

Hence

$$-2\beta\iint_{\partial R}\psi^2\,\mathrm{d}S=\iiint_R 2\left|\nabla\psi\right|^2\,\mathrm{d}V.$$

If $\beta > 0$ then the LHS is non-positive and the RHS is non-negative – consequently both are zero and $\nabla\psi = \mathbf{0}$ in $R$ and $\psi = 0$ on $\partial R$. Hence $\psi = 0$ throughout $R$.

If $\beta = 0$ then the Robin Boundary Problem is just the Neumann Problem, with which we have already dealt.

The following example shows that solutions need not be unique if $\beta < 0$. ■

**Example 104** *Let $R$ be the interior of the cylinder $x^2 + y^2 = 1$. Find two solutions to the Robin Boundary Problem*

$$\nabla^2\psi = 0 \quad \textit{in } R, \qquad \psi = \frac{\partial\psi}{\partial n} \quad \textit{on } \partial R.$$

**Solution.** Laplace's equation in cylindrical polars reads

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} = 0.$$

We know for $\psi = r^n \sin n\theta$, where $n$ is a non-negative integer, that $\nabla^2\psi = 0$ and on the boundary $r = 1$ we have

$$\frac{\partial\psi}{\partial n} = \frac{\partial\psi}{\partial r} = nr^{n-1}\sin n\theta = n\sin n\theta; \qquad \psi = \sin n\theta.$$

Hence when $n = 1$, we see $r\sin\theta$ satisfies the given boundary problem. Other solutions are 0, $r\cos\theta$ and $r\sin(\theta - \alpha)$ for arbitrary $\alpha$ so we see this solution is far from unique. ■

### 6.2.2 Derivation of balance equations in 3D: the heat equation.

Finally, as a last application, we use the Divergence Theorem to rederive the heat equation, but now in 3D. We need to following lemma first.

**Lemma 105** *If $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a continuous scalar field such that*

$$\iiint_R \phi \,\mathrm{d}V = 0$$

*for all bounded subsets $R$, then $\phi \equiv 0$.*

**Proof.**

Suppose, for a contradiction that $\phi(\mathbf{p}) \neq 0$ for some point $p$. Without any loss of generality let's suppose $\phi(\mathbf{p}) > 0$. Let $\varepsilon = \frac{1}{2}\phi(\mathbf{p}) > 0$. Then, by the continuity of $\phi$ at $\mathbf{p}$, there exists $\delta > 0$ such that

$$|\phi(\mathbf{x}) - \phi(\mathbf{p})| < \varepsilon \quad \text{when} \quad |\mathbf{x} - \mathbf{p}| < \delta.$$

In particular, $\phi(\mathbf{x}) > \frac{1}{2}\phi(\mathbf{p})$ on all of $B(\mathbf{p}, \delta) = \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x} - \mathbf{p}| < \delta\}$ and so

$$\iiint\limits_{B(\mathbf{p},\delta)} \phi \,\mathrm{d}V > \frac{1}{2}\phi(\mathbf{p}) \times \mathrm{Vol}(B(\mathbf{p}, \delta)) > 0$$

which is a contradiction.

■

**Theorem 106** *Let $T(\mathbf{x}, t)$ denote the temperature at position $\mathbf{x}$ and at time $t$ in an isotropic medium $R$ with thermal conductivity $k$, density $\rho$, specific heat $c$ with heat flow determined by Fourier's Law which states that*

$$\mathbf{q} = -k\nabla T$$

*where $\mathbf{q}$ is the **heat flux**. Then $T$ satisfies the **heat equation***

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c}\nabla^2 T.$$

**Proof.** Let $S \subseteq R$ be an arbitrary subset of $R$. The total heat energy in $S$ equals

$$\iiint\limits_{S} \rho c T \,\mathrm{d}V.$$

The amount of heat flowing *out* of $S$ is given by the flux integral

$$\iint\limits_{\partial S} \mathbf{q} \cdot \mathrm{d}\mathbf{S}.$$

So if no heat is generated within $S$ then the only heat loss or gain is via the boundary $\partial S$. Hence we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\iiint_S \rho c T \,\mathrm{d}V &= -\iint_{\partial S} \mathbf{q}\cdot \mathrm{d}\mathbf{S} \\
&= \iint_{\partial S} k\nabla T\cdot \mathrm{d}\mathbf{S} \qquad \text{[by Fourier's Law]} \\
&= \iiint_S \nabla\cdot(k\nabla T)\ \mathrm{d}V \qquad \text{[by the Divergence Theorem]} \\
&= \iiint_S k\,\nabla^2 T\,\mathrm{d}V.
\end{aligned}$$

So

$$\begin{aligned}
\iiint_S \rho c\frac{\partial T}{\partial t}\,\mathrm{d}V &= \iiint_S k\,\nabla^2 T\,\mathrm{d}V, \\
\Longrightarrow \iiint_S \left(\rho c\frac{\partial T}{\partial t} - k\,\nabla^2 T\right)\mathrm{d}V &= 0.
\end{aligned}$$

As this is true for an arbitrary subset $S$ of $R$ then it follows, from Lemma 105, that

$$\rho c\frac{\partial T}{\partial t} - k\,\nabla^2 T = 0 \qquad \text{throughout } R.$$

■

### 6.2.3 Greens' Theorem in The Plane

Below, recall that a closed set contains its boundary and a simple closed curve is a curve that has its endpoints meet but otherwise does not cross itself.

**Theorem 107 (Green's Theorem in The Plane (Divergence Theorem Form))** *Let $D$ be a closed bounded region in the $(x, y)$ plane, whose boundary $C$ is a piecewise smooth simple closed curve, and that $p(x, y), q(x, y)$ have continuous first-order derivatives in $D$. Then*

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \int_C (p, q)\cdot\mathbf{n}\mathrm{d}s, \tag{6.2}$$

*where $\mathbf{n}$ is the outward pointing unit normal to $C$ in the $(x, y)$ plane.*

**Proof.** This is a simple corollary of the divergence theorem and is effectively the divergence theorem in a plane. Let the vector field $\mathbf{F}$ be given by

$$\mathbf{F} = (p(x, y), q(x, y), 0)$$

and define the three dimensional region $R$ to be

$$R = \left\{(x, y, z) \in \mathbb{R}^3 \mid \ (x, y) \in D, \quad z \in [0, 1]\right\},$$

with boundary $\partial R$. Noting $\nabla \cdot \mathbf{F}$ has no dependence on $z$, we have

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) \mathrm{d}x\mathrm{d}y \ = \ \iint_D \nabla \cdot \mathbf{F} \mathrm{d}x\mathrm{d}y \int_0^1 dz = \iint_R \nabla \cdot \mathbf{F} \mathrm{d}V = \int_{\partial R} \mathbf{F} \cdot \mathbf{n} \mathrm{d}S.$$

The contribution to the surface integral from the surfaces at $z = 0, 1$ are zero as the integrand is zero. For the surface

$$\left\{(x, y, z) \in \mathbb{R}^3 \mid \ (x, y) \in D, \quad z \in [0, 1]\right\}$$

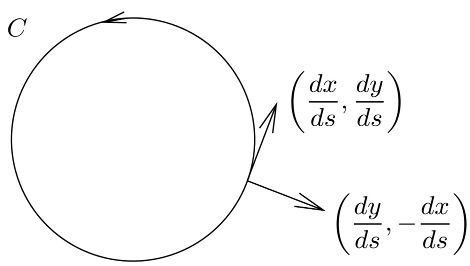
the outward pointing unit normal to $R$ coincides with the outward pointing unit normal to $C$; also we can write the surface element $\mathrm{d}S$ as

$$\mathrm{d}S = \mathrm{d}s\mathrm{d}z,$$

where $\mathrm{d}s$ is the arclength element of the curve $C$ since the $\mathbf{k}$ direction and the plane of the region $D$ are perpendicular. Hence

$$\int_{\partial R} \mathbf{F} \cdot \mathbf{n} \mathrm{d}S = \int_0^1 \left\{\int_C \mathbf{F} \cdot \mathbf{n} \mathrm{d}s\right\} \mathrm{d}z = \int_0^1 \left\{\int_C (p, q) \cdot \mathbf{n} \mathrm{d}s\right\} \mathrm{d}z = \int_C (p, q) \cdot \mathbf{n} \mathrm{d}s,$$

with the final equality from the fact the integrand $(p, q) \cdot \mathbf{n}$ has no dependence on $z$. ■




**Remark 108** *By parameterising the curve $C$ in the form $(x(s), y(s))$, where $s$ is the arclength of $C$, increasing on moving* **anticlockwise** *around $C$ when viewed from above, one has the unit tangent, $\mathbf{t}$, and the outward unit normal $\mathbf{n}$ are given by*

$$\mathbf{t} = \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right), \qquad \mathbf{n} = \left(\frac{\mathrm{d}y}{\mathrm{d}s}, -\frac{\mathrm{d}x}{\mathrm{d}s}\right). \tag{6.3}$$

**Important** *Below, it is important to note the direction around $C$ is* **anticlockwise** *when viewed from above; this is a positive orientation of $C$, which we will consider in generality below.*

*Using the expression for $\mathbf{n}$, we have*

$$\int_C (p, q) \cdot \mathbf{n} \mathrm{d}s = \int_C (p, q) \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}s}, -\frac{\mathrm{d}x}{\mathrm{d}s}\right) \mathrm{d}s \stackrel{def}{=} \int_C p\mathrm{d}y - q\mathrm{d}x \tag{6.4}$$

*and Greens' theorem is often written in the equivalent form*

$$\iint_D \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \int_C p\mathrm{d}y - q\mathrm{d}x,$$

*where $C$ is positively oriented, i.e. the integration around $C$ is anticlockwise.*

Another *equivalent* form of Green's theorem in the plane is a precursor for Stokes' Theorem, which is our next subject of study.

**Theorem 109 (Green's Theorem in The Plane (Stokes' Theorem Form))** *An equivalent form of Green's theorem is as follows. Let $D$ be a closed bounded region in the $(x, y)$ plane, whose boundary $C$ is a piecewise smooth simple closed curve, and that $p(x, y), q(x, y)$ have continuous first-order derivatives in $D$. Then*

$$\iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) \mathrm{d}x\mathrm{d}y = \int_C (p, q) \cdot \mathrm{d}\mathbf{r} = \int_C p\mathrm{d}x + q\mathrm{d}y, \tag{6.5}$$

*where $C$ is positively oriented, i.e. the integration around $C$ is anticlockwise.*

**Proof.** Let $(p, q) \to (q, -p)$ in Green's theorem in the plane (divergence form), equation 6.2, and use equations 6.4. We then have

$$\begin{aligned}
\iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) \mathrm{d}x\mathrm{d}y &= \int_C (q, -p) \cdot \left(\frac{\mathrm{d}y}{\mathrm{d}s}, -\frac{\mathrm{d}x}{\mathrm{d}s}\right) \mathrm{d}s = \int_C (p, q) \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right) \mathrm{d}s \\
&= \int_C (p, q) \cdot \mathrm{d}\mathbf{r} = \int_C p\mathrm{d}x + q\mathrm{d}y,
\end{aligned}$$

with the direction of integration around $C$ inherited from Green's theorem in the plane (divergence form), via the relationship (6.3). ■

## 6.3 Stokes' Theorem

Let $\mathbf{F} = (p(x, y), q(x, y), 0)$. Then

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ p(x, y) & q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right) \mathbf{k},$$

and so

$$\iint\limits_D \left(\frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}\right) \mathrm{d}A = \iint\limits_D \operatorname{curl} \mathbf{F} \cdot (\mathbf{k}\, \mathrm{d}A) = \iint\limits_D \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Thus Greens theorem in the plane can be rephrased as

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}, \tag{6.6}$$

Here

$$\mathrm{d}\mathbf{S} = \mathbf{k}\mathrm{d}S = \mathbf{n}\mathrm{d}S, \tag{6.7}$$

where now, $\mathbf{n}$ is the normal to the planar region $D$ in $\mathbb{R}^3$ [rather than the in-plane normal to the boundary $C$ as used in the previous section deducing Green's theorem in the plane].

The identity 6.6 is purely a rewriting of Green's Theorem, but it is part of a larger identity known as *Stokes' Theorem* which applies more generally to surfaces in $\mathbb{R}^3$ with boundary.

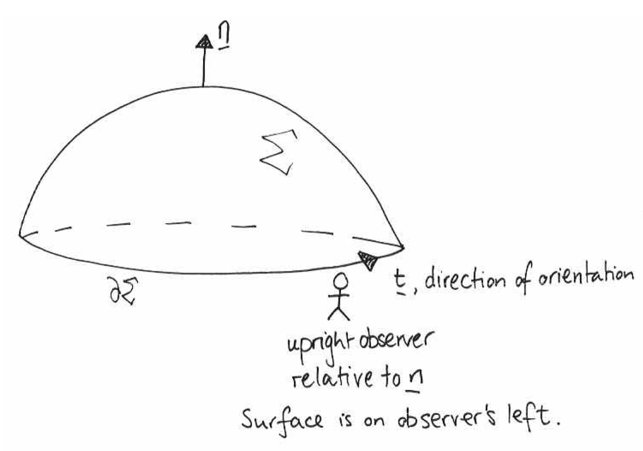
However, before we can properly understand the identity in its fullest generality, we need to appreciate how to consistently orient a curve $C$ which bounds a surface $\Sigma$.

**Definition 110** *At each point of a smooth surface there are two unit normal vectors* $\pm\mathbf{n}$. *Given a smooth surface* $\Sigma$, *then by an **orientation** of* $\Sigma$ *we shall mean a continuous choice of unit normal* $\mathbf{n}$ *for the whole surface.*

*For the bounding curve, which we shall denote as* $\partial\Sigma$, *there are two possible orientations. Given a certain orientation of* $\partial\Sigma$ *and an oriented tangent vector* $\mathbf{t}$ *along* $\partial\Sigma$ *then the vector* $\mathbf{t} \wedge \mathbf{n}$ *lies in the tangent plane of* $\Sigma$ *at the boundary point and is normal to the bounding curve* $\partial\Sigma$. *So* $\mathbf{t} \wedge \mathbf{n}$ *points either towards the surface* $\Sigma$ *or away from* $\Sigma$.

*We shall always choose to orient a surface* $\Sigma$ *and its boundary* $\partial\Sigma$ *in such a way that* $\mathbf{t} \wedge \mathbf{n}$ *points away from* $\Sigma$.

**Remark 111** *Note that this is consistent in the plane with taking positively oriented curves and normal* $\mathbf{k}$, *so that equations (6.6,6.7) are equivalent to Green's theorem in the plane, equation (6.5).*



In practice it is fairly clear how to consistently orient a surface and its boundary. If one imagines an observer moving along the boundary $\partial\Sigma$ (in the direction of the orientation) and upright (in the sense of the surface normal $\mathbf{n}$) then the surface $\Sigma$ will be to his/her left.

**Example 112** *Given the hemisphere* $\Sigma = \{(x,y,z) : x^2+y^2+z^2 = 1,\ z \geqslant 0\}$ *with boundary* $\partial\Sigma = \{(x,y,0) : x^2+y^2 = 1\}$ *then we can either orient* $\Sigma$ *as*

$$\mathbf{n}(x,y,z) = (x,y,z) \qquad \text{or} \qquad -\mathbf{n}(x,y,z) = (-x,-y,-z).$$

*We can parameterise* $\partial\Sigma$ *as* $(\cos\theta, \sin\theta, 0)$. *So one choice of unit tangent is*

$$\mathbf{t} = (-\sin\theta, \cos\theta, 0).$$

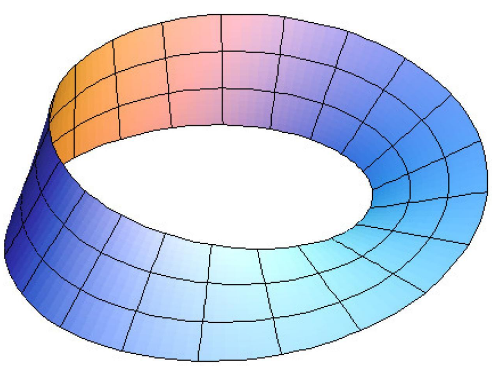
*At* $(\cos\theta, \sin\theta, 0)$ *we have*

$$\mathbf{t} \wedge \mathbf{n} = (-\sin\theta, \cos\theta, 0) \wedge (\cos\theta, \sin\theta, 0) = (0,0,-1)$$

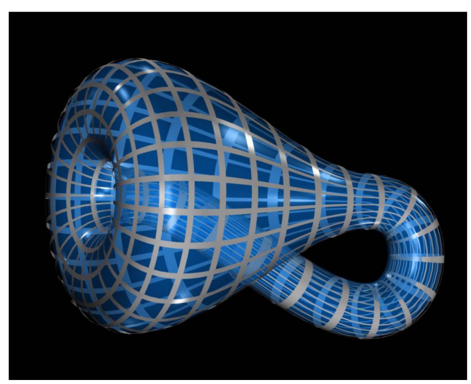
*which points down, away from the hemisphere. The two consistent ways to orient the surface and boundary are*

$$(\mathbf{n} \text{ on } \Sigma \text{ and } \mathbf{t} \text{ on } \partial\Sigma) \quad \text{or} \quad (-\mathbf{n} \text{ on } \Sigma \text{ and } -\mathbf{t} \text{ on } \partial\Sigma).$$

**Example 113** *Not all surfaces have an orientation, such surfaces being called* ***non-orientable***. *Examples of non-orientable surfaces are the* ***Möbius strip*** *and the* ***Klein bottle***. *We shall only be interested in* ***orientable*** *surfaces though.*



*Möbius Strip*



*Klein Bottle*

**Theorem 114** *(**Stokes' Theorem** c. 1850. The first known appearance of this result though is in fact in a letter from Kelvin to Stokes.) Let* $\Sigma$ *be a smooth oriented surface in* $\mathbb{R}^3$ *whose boundary is the curve* $\partial\Sigma$. *Let* $\mathbf{F}$ *be a smooth vector field defined on* $\Sigma \cup \partial\Sigma$. *Then*

$$\int_{\partial\Sigma} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\Sigma} \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}. \tag{6.8}$$

**Remark 115** *The following proof applies only to patches of surface – we shall see in later in Example (122) that Stokes' Theorem applies generally to more complicated surfaces and their boundaries. In any event* ***the proof of Stokes' Theorem is not examinable****.*

**Proof.** Suppose that $\Sigma$ is parameterised as $\mathbf{r}(u,v)$ where $(u,v)$ ranges over some $S \subseteq \mathbb{R}^2$, with $\partial\Sigma$ being the image of $\partial S$ under $\mathbf{r}$. We shall first prove Stokes Theorem for the vector field $\mathbf{F} = (F,0,0)$, noting that similar proofs apply for fields $(0,F,0)$ and $(0,0,F)$ and then the more general (6.8) follows by linearity.

So, if $\mathbf{F} = (F,0,0)$, then (6.8) reads as

$$\int_{\partial\Sigma} F\,\mathrm{d}x = \iint_{\Sigma} (0, F_z, -F_y)\cdot \mathrm{d}\mathbf{S}. \tag{6.9}$$

Now using the parametrisation $\mathbf{r}(u,v)$ we have

$$\begin{aligned}
\iint_{\Sigma} (0,F_z,-F_y)\cdot\mathrm{d}\mathbf{S} &= \iint_{S} (0,F_z,-F_y)\cdot(\mathbf{r}_u \wedge \mathbf{r}_v)\,\mathrm{d}u\,\mathrm{d}v \\
&= \iint_{S} (0,F_z,-F_y)\cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}\,\mathrm{d}u\,\mathrm{d}v \\
&= \iint_{S} \left[F_z\left(z_u x_v - x_u z_v\right) - F_y\left(x_u y_v - y_u x_v\right)\right]\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

On the other hand if we apply Green's Theorem to the LHS of (6.9) we have

$$\begin{aligned}
\int_{\partial\Sigma} F\,\mathrm{d}x &= \int_{\partial S} F\,\left(x_u\mathrm{d}u + x_v\mathrm{d}v\right) \\
&= \iint_{S} \left[(Fx_v)_u - (Fx_u)_v\right]\mathrm{d}u\,\mathrm{d}v \\
&= \iint_{S} \left[F_u x_v + F x_{vu} - F_v x_u - F x_{uv}\right]\mathrm{d}u\,\mathrm{d}v \\
&= \iint_{S} \left[F_u x_v - F_v x_u\right]\mathrm{d}u\,\mathrm{d}v \\
&= \iint_{S} \left[\left(F_x x_u + F_y y_u + F_z z_u\right)x_v - \left(F_x x_v + F_y y_v + F_z z_v\right)x_u\right]\mathrm{d}u\,\mathrm{d}v \\
&= \iint_{S} \left[F_z\left(z_u x_v - x_u z_v\right) - F_y\left(x_u y_v - y_u x_v\right)\right]\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

The result follows. ■

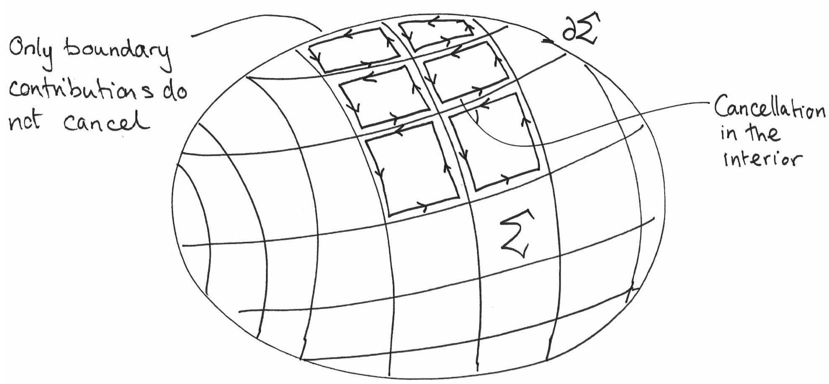
**Remark 116** *Stokes' Theorem and the Divergence Theorem are in fact statements of a more general theorem, also known as Stokes' Theorem, which applies in all dimensions. It more* *generally reads:*

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$

*where $M$ is a (compact) oriented $n$-dimensional manifold (i.e. the $n$-dimensional equivalent of a surface) with boundary $\partial M$ and $\omega$ is a differential form of degree $n-1$. A proper appreciation of this result is approximately at fourth year undergraduate level. Note, though, that when $n=1$ Stokes' Theorem reads*

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a),$$

*when $M = [a,b]$ and $\partial M = \{a,b\}$ oriented to sum positively the contribution at $b$ and negatively at $a$.*




**Remark 117** *For an intuitive appreciation of why Stokes' Theorem is true, recall that* curl $\mathbf{F}$, *when physically interpreted, is a measure of the local spin of the field $\mathbf{F}$. So one might envisage summing up all the* curl *over the surface $\Sigma$ with most contributions to this total spin cancelling out, in a manner akin to intermeshing cogs. The only contributions that wouldn't be cancelled out are those at the boundary, these remaining contributions making up the integral around the boundary.*

Note that Stokes' Theorem is a scalar identity, that is both sides of the identity are scalar quantities. There is also a vector identity related to Stokes' Theorem. But firstly we note:

**Lemma 118** *If*

$$\mathbf{c}\cdot\mathbf{v} = \mathbf{c}\cdot\mathbf{w} \quad \textit{for all } \mathbf{c} \in \mathbb{R}^3$$

*then $\mathbf{v} = \mathbf{w}$.*

**Proof.** We have $\mathbf{c}\cdot(\mathbf{v}-\mathbf{w}) = 0$ for all $\mathbf{c}$ and thus for all basis vectors. Thus all the components of $\mathbf{v}$, $\mathbf{w}$ are equal, hence so are the vectors. ■

**Corollary 119** *Let $\Sigma$ be a smooth oriented surface in $\mathbb{R}^3$ whose boundary is the curve $\partial\Sigma$. Let $\psi$ be a smooth scalar field defined on $\Sigma \cup \partial\Sigma$. Then*

$$\iint_\Sigma \nabla\psi \wedge \mathrm{d}\mathbf{S} = -\int_{\partial\Sigma} \psi\,\mathrm{d}\mathbf{r}.$$

**Proof.** Let $\mathbf{c}$ be an arbitrary constant vector and set $\mathbf{F} = \psi\mathbf{c}$. In this case, and using the curl product rule, Stokes Theorem reads

$$\iint_{\Sigma} \operatorname{curl}(\psi\mathbf{c}) \cdot \mathrm{d}\mathbf{S} = \iint_{\Sigma} (\psi \operatorname{curl}\mathbf{c} + \nabla\psi \wedge \mathbf{c}) \cdot \mathrm{d}\mathbf{S} = \int_{\partial\Sigma} \psi\mathbf{c} \cdot \mathrm{d}\mathbf{r}.$$

As $\mathbf{c}$ is constant then $\operatorname{curl}\mathbf{c} = \mathbf{0}$ and we have

$$\mathbf{c} \cdot \left( -\iint_{\Sigma} \nabla\psi \wedge \mathrm{d}\mathbf{S} \right) = \iint_{\Sigma} \nabla\psi \wedge \mathbf{c} \cdot \mathrm{d}\mathbf{S} = \mathbf{c} \cdot \left( \int_{\partial\Sigma} \psi \, \mathrm{d}\mathbf{r} \right).$$

As $\mathbf{c}$ is arbitrary then by Lemma 118 we have

$$-\iint_{\Sigma} \nabla\psi \wedge \mathrm{d}\mathbf{S} = \int_{\partial\Sigma} \psi \, \mathrm{d}\mathbf{r}$$

and the result follows.

■

**Example 120** *Verify Stokes' Theorem when* $\mathbf{F} = (y, z, x)$ *on the hemisphere*

$$\Sigma = \left\{ (x, y, z) : x^2 + y^2 + z^2 = a^2, \ z \geqslant 0 \right\}.$$

**Solution.** We can parameterise the sphere as

$$\mathbf{r}(\theta, \phi) = (a \sin\theta \cos\phi, a \sin\theta \sin\phi, a \cos\theta) \qquad 0 \leqslant \theta \leqslant \pi, \ 0 \leqslant \phi \leqslant 2\pi.$$

Then

$$\frac{\partial \mathbf{r}}{\partial \theta} \wedge \frac{\partial \mathbf{r}}{\partial \phi} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix} = a^2 \sin\theta (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

Now $\operatorname{curl}\mathbf{F} = (-1, -1, -1)$ and hence

$$\begin{aligned}
\iint_{\Sigma} \operatorname{curl}\mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (-1, -1, -1) \cdot a^2 \sin\theta (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= -a^2 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin\theta (\sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \\
&= -2\pi a^2 \int_{\theta=0}^{\pi/2} \sin\theta\cos\theta \, \mathrm{d}\theta = -2\pi a^2 \left[ \frac{\sin^2\theta}{2} \right]_0^{\pi/2} \\
&= -\pi a^2.
\end{aligned}$$

On the other hand, once we parameterise and orient $\partial\Sigma$ as $\mathbf{r}(\theta) = (a\cos\theta, a\sin\theta, 0)$, we have

$$\begin{aligned}
\int_{\partial\Sigma} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} (a\sin\theta, 0, a\cos\theta)\cdot(-a\sin\theta, a\cos\theta, 0)\ \mathrm{d}\theta \\
&= -a^2\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta = -\pi a^2.
\end{aligned}$$

■

**Example 121** *Verify Stokes' Theorem when* $\mathbf{F} = (2y, 3x, x^2+y^2+z^2)$ *and* $\Sigma$ *is the lower half of the ellipsoid* $x^2/4 + y^2/9 + z^2/27 = 1$.

**Solution.** Here $\partial\Sigma$ is the ellipse $x^2/4+y^2/9 = 1$ in the $xy$-plane. If $\partial\Sigma$ is oriented anti-clockwise then we need to take the upward pointing normal on $\Sigma$.

We may parameterise $\partial\Sigma$ by $\mathbf{r}(t) = (2\cos t, 3\sin t, 0)$ and so

$$\begin{aligned}
\int_{\partial\Sigma} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \left(6\sin t, 6\cos t, 4\cos^2 t + 9\sin^2 t\right)\cdot(-2\sin t, 3\cos t, 0)\ \mathrm{d}t \\
&= \int_0^{2\pi} \left(-12\sin^2 t + 18\cos^2 t\right)\ \mathrm{d}t \\
&= \int_0^{2\pi} \left((-6-6\cos 2t) + (9+9\cos 2t)\right)\ \mathrm{d}t \\
&= 6\pi.
\end{aligned}$$

On the other hand we can parameterise the lower half of the ellipsoid as

$$\mathbf{r}(\theta,\phi) = \left(2\sin\theta\cos\phi, 3\sin\theta\sin\phi, 3\sqrt{3}\cos\theta\right).$$

So

$$\begin{aligned}
\mathbf{r}_\theta \wedge \mathbf{r}_\phi &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos\theta\cos\phi & 3\cos\theta\sin\phi & -3\sqrt{3}\sin\theta \\ -2\sin\theta\sin\phi & 3\sin\theta\cos\phi & 0 \end{vmatrix} \\
&= \left(9\sqrt{3}\sin^2\theta\cos\phi, 6\sqrt{3}\sin^2\theta\sin\phi, 6\sin\theta\cos\theta\right),
\end{aligned}$$

but this points downwards, so we will take its negative. Also

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & x^2+y^2+z^2 \end{vmatrix} = (2y, -2x, 1).$$

Finally, then

$$
\begin{aligned}
&\iint\limits_{\Sigma} \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} \\
= \; & \int_{\theta=\pi/2}^{\pi} \int_{\phi=0}^{2\pi} (6\sin\theta\sin\phi, -4\sin\theta\cos\phi, 1) \cdot \left(-9\sqrt{3}\sin^2\theta\cos\phi, -6\sqrt{3}\sin^2\theta\sin\phi, -6\sin\theta\cos\theta\right) \,\mathrm{d}\phi\,\mathrm{d}\theta \\
= \; & \int_{\theta=\pi/2}^{\pi} \int_{\phi=0}^{2\pi} \left(-54\sqrt{3}\sin^2\theta\sin\phi\cos\phi + 24\sqrt{3}\sin^3\theta\cos\phi\sin\phi - 6\sin\theta\cos\theta\right) \,\mathrm{d}\phi\,\mathrm{d}\theta \\
= \; & -2\pi \int_{\theta=\pi/2}^{\pi} 6\sin\theta\cos\theta\,\mathrm{d}\theta \\
= \; & 2\pi \left[\frac{3\cos 2\theta}{2}\right]_{\pi/2}^{\pi} \\
= \; & 3\pi\,(1+1) = 6\pi.
\end{aligned}
$$

■

**Example 122** *Verify Stokes' Theorem for* $\mathbf{F} = (x^2 + 3yz, -2xz, 7y)$ *on* $\Sigma$*, where* $\Sigma$ *is that part of the cylindrical surface* $x^2 + y^2 = a^2$ *between the planes* $x + y + z = 0$ *and* $x + y + z = h$*, where* $h \geqslant 0$*.*

**Solution.** We may parameterise the cylinder as

$$\mathbf{r}\,(\theta, z) = (a\cos\theta, a\sin\theta, z)$$

and orient it with outward pointing normal $\mathrm{d}\mathbf{S} = (\cos\theta, \sin\theta, 0)\; a\,\mathrm{d}\theta\,\mathrm{d}z$. So

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + 3yz & -2xz & 7y \end{vmatrix} = (7 + 2x, 3y, -5z)\,.$$

$$
\begin{aligned}
\iint\limits_{\Sigma} \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} &= \int_{\theta=0}^{2\pi} \int_{z=-a\cos\theta-a\sin\theta}^{h-a\cos\theta-a\sin\theta} (7 + 2a\cos\theta, 3a\sin\theta, -5z) \cdot (\cos\theta, \sin\theta, 0)\; a\,\mathrm{d}z\,\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi} \int_{z=-a\cos\theta-a\sin\theta}^{h-a\cos\theta-a\sin\theta} \left(7\cos\theta + 2a\cos^2\theta + 3a\sin^2\theta\right)\; a\,\mathrm{d}z\,\mathrm{d}\theta \\
&= \int_{\theta=0}^{2\pi} \left(7\cos\theta + 2a\cos^2\theta + 3a\sin^2\theta\right)\; ah\,\mathrm{d}\theta \\
&= (2a\pi + 3a\pi)\; ah \\
&= 5\pi a^2 h.
\end{aligned}
$$

On the other side, we have a disconnected boundary $\partial\Sigma$ made up of

$$\begin{aligned} C_1 &= \{(a\cos\theta, a\sin\theta, -a\cos\theta - a\sin\theta) : 0 \leqslant \theta \leqslant 2\pi\}\,; \\ C_2 &= \{(a\cos\theta, a\sin\theta, h - a\cos\theta - a\sin\theta) : 0 \leqslant \theta \leqslant 2\pi\}\,. \end{aligned}$$

As we used the outward pointing normal to orient $\Sigma$ then we need to orient $C_1$ in the direction of increasing $\theta$ but orient $C_2$ in the direction of decreasing $\theta$.

If we write $x(\theta) = a\cos\theta,\ y(\theta) = a\sin\theta,\ z(\theta) = -a\cos\theta - a\sin\theta$ then we have

$$\begin{aligned} \int_{C_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \left(x(\theta)^2 + 3y(\theta)\,z(\theta), -2x(\theta)\,z(\theta), 7y(\theta)\right)\cdot \mathrm{d}\mathbf{r}(\theta) \\ \int_{C_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= -\int_0^{2\pi} \left(x(\theta)^2 + 3y(\theta)\,[z(\theta)+h], -2x(\theta)\,[z(\theta)+h], 7y(\theta)\right)\cdot \mathrm{d}\mathbf{r}(\theta)\,. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\partial\Sigma} \mathbf{F}\cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} (-3hy(\theta), 2hx(\theta), 0)\cdot \mathrm{d}\mathbf{r}(\theta) \\ &= ah\int_0^{2\pi} (-3a\sin\theta, 2a\cos\theta, 0)\cdot(-\sin\theta, \cos\theta, \sin\theta - \cos\theta)\ \mathrm{d}\theta \\ &= a^2h\int_0^{2\pi} \left(3\sin^2\theta + 2\cos^2\theta\right)\ \mathrm{d}\theta \\ &= 5\pi a^2 h. \end{aligned}$$

■

**Example 123** *Let $C$ be a closed curve bounding a smooth surface $\Sigma$. Show that*

$$\int_C \mathbf{r}\wedge \mathrm{d}\mathbf{r} = 2\iint_\Sigma \mathrm{d}\mathbf{S}.$$

*Hence, or otherwise, evaluate* $\iint_\Sigma \mathrm{d}\mathbf{S}$ *where $\Sigma$ is the surface*

$$\Sigma = \left\{(x,y,z) : z = \frac{x^2}{a^2} + \frac{y^2}{b^2},\ x^2+y^2 < 1\right\}.$$

*[You may assume that $\nabla\wedge(\mathbf{u}\wedge\mathbf{v}) = (\mathbf{v}\cdot\nabla)\,\mathbf{u} - (\nabla\cdot\mathbf{u})\,\mathbf{v} + (\nabla\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{u}\cdot\nabla)\,\mathbf{v}$.]*

**Solution.** If we set $\mathbf{F} = \mathbf{c} \wedge \mathbf{r}$ where $\mathbf{c}$ is an arbitrary constant vector, then we have

$$\begin{aligned}
\nabla \wedge (\mathbf{c} \wedge \mathbf{r}) &= (\mathbf{r} \cdot \nabla)\,\mathbf{c} - (\nabla \cdot \mathbf{c})\,\mathbf{r} + (\nabla \cdot \mathbf{r})\,\mathbf{c} - (\mathbf{c} \cdot \nabla)\,\mathbf{r} \\
&= (\nabla \cdot \mathbf{r})\,\mathbf{c} - (\mathbf{c} \cdot \nabla)\,\mathbf{r} \qquad [\text{as } \mathbf{c} \text{ is constant}] \\
&= 3\mathbf{c} - \mathbf{c} \\
&= 2\mathbf{c}
\end{aligned}$$

as

$$(\mathbf{c} \cdot \nabla)\,\mathbf{r} = \left( c_1 \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y} + c_3 \frac{\partial}{\partial z} \right) (x, y, z) = (c_1, c_2, c_3) = \mathbf{c}.$$

So, by Stokes' Theorem,

$$\int_C (\mathbf{c} \wedge \mathbf{r}) \cdot \mathrm{d}\mathbf{r} = \iint_\Sigma 2\mathbf{c} \cdot \mathrm{d}\mathbf{S}$$

$$\implies \mathbf{c} \cdot \left( \int_C \mathbf{r} \wedge \mathrm{d}\mathbf{r} \right) = \mathbf{c} \cdot \left( \iint_\Sigma 2\mathrm{d}\mathbf{S} \right).$$

As $\mathbf{c}$ is arbitrary then by Lemma 118

$$\int_C \mathbf{r} \wedge \mathrm{d}\mathbf{r} = 2 \iint_\Sigma \mathrm{d}\mathbf{S}. \tag{6.10}$$

The bounding curve of the given surface $\Sigma$ is

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}, \quad x^2 + y^2 = 1$$

which can naturally be parameterised as

$$\mathbf{r}(\theta) = \left( \cos\theta, \sin\theta, \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \right), \quad 0 \leqslant \theta \leqslant 2\pi.$$

By (6.10) we have (writing $c = \cos\theta$, $s = \sin\theta$)

$$\begin{aligned}
\iint_\Sigma \mathrm{d}\mathbf{S} &= \frac{1}{2} \int_0^{2\pi} \left( c, s, \frac{c^2}{a^2} + \frac{s^2}{b^2} \right) \wedge \left( -s, c, \left( \frac{-1}{a^2} + \frac{1}{b^2} \right) 2cs \right) \mathrm{d}\theta \\
&= \frac{1}{2} \int_0^{2\pi} \left( \frac{cs^2}{b^2} + \frac{(-2cs^2 - c^3)}{a^2}, \frac{c^2 s}{a^2} + \frac{(-s^3 - 2c^2 s)}{b^2}, 1 \right) \mathrm{d}\theta \\
&= \frac{1}{2} \int_0^{2\pi} \left( \frac{cs^2}{b^2} + \frac{(-cs^2 - c)}{a^2}, \frac{c^2 s}{a^2} + \frac{(-s - c^2 s)}{b^2}, 1 \right) \mathrm{d}\theta \\
&= \frac{1}{2} \left[ \left( \frac{s^3}{3b^2} + \frac{(-s^3/3 - s)}{a^2}, \right), \frac{-c^3}{3a^2} + \frac{(c + c^3/3)}{b^2}, \theta \right]_0^{2\pi} \\
&= \frac{1}{2} \left[ (0, 0, 2\pi) \right] \\
&= \pi \mathbf{k}.
\end{aligned}$$

■

## 6.4 Stokes' Theorem and Conservative Fields

**Definition 124** *A region $R \subseteq \mathbb{R}^3$ is said to be* ***simply connected*** *if every simple closed curve $C$ can be continuously deformed to a single point. Specifically, if $\mathbf{r} : [0,1] \to R$ is a simple closed curve beginning and ending in $\mathbf{p}$ then there exists a map $H : [0,1] \times [0,1] \to R$ such that $H(t,1) = \mathbf{r}(t)$ and $H(t,0) = \mathbf{p}$.*

**Example 125** *$\mathbb{R}^3$ is simply connected, as is any plane or line. $\mathbb{R}^2 - \{\mathbf{0}\}$ is not simply connected, nor is the cylinder $x^2 + y^2 = a^2$, nor a torus. $\mathbb{R}^3 - \{\mathbf{0}\}$ is simply connected though.*

**Corollary 126** ***(Existence of a Potential)*** *Let $R$ be a simply connected region and let $\mathbf{F}$ be a smooth vector field on $R$ for which $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Then $\mathbf{F}$ is conservative, i.e. there exists a potential $\phi$ on $R$ such that $\mathbf{F} = \nabla\phi$.*

**Remark 127** *Firstly note that if $\mathbf{F}$ is conservative then $\mathbf{F} = \nabla\phi$ and hence $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Combining this observation with corollary 126 gives, by use of Theorem 66, that with a restriction to simply connected regions, $\operatorname{curl}\mathbf{F} = \mathbf{0}$ is equivalent to (i), (ii) and (iii):*

*(i) $\mathbf{F}$ is conservative – i.e. $\mathbf{F} = \nabla\phi$ for some scalar field $\phi : S \to \mathbb{R}$.*

*(ii) Given any two points $\mathbf{p}, \mathbf{q} \in S$ and curve $\gamma$ in $S$, starting at $\mathbf{p}$ and ending at $\mathbf{q}$, then the integral*

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r}$$

*is independent of the choice of curve $C$.*

*(iii) For any simple closed curve $C$ then*

$$\int_C \mathbf{F}(\mathbf{r}) \cdot \mathrm{d}\mathbf{r} = 0.$$

**Proof.** Let $\mathbf{p}$ be a fixed point in $R$ and define for any $\mathbf{q} \in R$,

$$\phi(\mathbf{q}) = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where $C_1$ is a curve in $S$ from $\mathbf{p}$ to $\mathbf{q}$. If $C_2$ is a second such curve then there is a simple closed curve $C$ formed by $C_1$ followed by $C_2$ in reverse orientation. As $R$ is simply connected then $C$ is the boundary of a surface $\Sigma$. By Stokes' Theorem then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_\Sigma \operatorname{curl}\mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0.$$

This means that

$$\phi(\mathbf{q}) = \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

and so $\phi$ is a well-defined function on $R$, dependent only on the variable $\mathbf{q}$ and not on the choice of curve from $\mathbf{p}$ to $\mathbf{q}$. Further, arguing as in Theorem 66, we can show that $\nabla\phi = \mathbf{F}$. ■

But this is **not** generally equivalent – for example,

$$\mathbf{F} = \left( \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right) \qquad (x,y) \neq (0,0)$$

satisfies $\operatorname{curl} \mathbf{F} = \mathbf{0}$ but there is no $\phi$ on $\mathbb{R}^2 - \{(0,0)\}$ such that $\mathbf{F} = \nabla\phi$. See the Exercise Sheets.

# 7. GAUSS' FLUX THEOREM, POISSON'S EQUATION AND GRAVITY

**Definition 128** *(**Newton's Law of Gravity**) The **gravitational field** associated with a point mass $M$ at the origin $O$ is*

$$\mathbf{f} = -\frac{GM}{r^3}\mathbf{r} = -\frac{GM}{r^2}\mathbf{e}_r = \nabla\left(\frac{GM}{r}\right).$$

*The **gravitational force** on a particle of mass $m$ at position vector $\mathbf{r}$ is*

$$\mathbf{F} = m\mathbf{f} = -\frac{GMm}{r^2}\mathbf{e}_r.$$

*Here $G$ is the **gravitational constant** $6.67300 \times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$.*

Note that the gravitational field $\mathbf{f}$ is conservative with **gravitational potential**

$$\phi = \frac{GM}{r}.$$

Recall that potentials are determined by the field only up to a constant. This choice of potential is such that $\phi \to 0$ as $r \to \infty$.

In addition to $\operatorname{curl}\mathbf{f} = \mathbf{0}$, we also have that that

$$\operatorname{div}\mathbf{f} = \nabla^2\phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\left(\frac{-GM}{r^2}\right)\right) = 0, \qquad r \neq 0,$$

and

$$\nabla^2\phi = 0 \qquad \text{for gravitational potential } \textit{in vacuo}.$$

This is physically intuitive as, aside from at the origin, there is no other mass contributing to the gravitational field.

**Proposition 129** *The gravitational potential $\phi = GM/r$ is the amount of work gravity does per unit mass to bring a point object from $\infty$ to $\mathbf{r}$.*

**Proof.** The work done by gravity to bring a unit point mass from $\infty$ to $\mathbf{r}$ is

$$\frac{1}{m}\int_\infty^{\mathbf{r}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_\infty^{\mathbf{r}} \mathbf{f}\cdot\mathrm{d}\mathbf{r} = [\phi(\mathbf{r}) - \phi(\infty)] = \phi(\mathbf{r}).$$

■

**Definition 130** *The **gravitational potential energy** of a point mass $m$ at $\mathbf{r}$ equals $-m\phi(\mathbf{r})$.*

**Remark 131** *In many texts the gravitational potential $\Phi$ is instead defined to be the gravitational energy per unit mass; in such texts then the potential used is $\Phi = -\phi$ and $\mathbf{f} = -\nabla\Phi$.*

Given $n$ particles of mass $M_i$ at points with position vectors $\mathbf{r}_i$, the gravitational potential $\phi$ and field $\mathbf{f}$ are then given by

$$\phi\left(\mathbf{p}\right) = \sum_{i=1}^{n} \frac{GM_i}{\left|\mathbf{p} - \mathbf{r}_i\right|}, \qquad \mathbf{f}\left(\mathbf{p}\right) = -\sum_{i=1}^{n} \frac{GM_i\left(\mathbf{p} - \mathbf{r}_i\right)}{\left|\mathbf{p} - \mathbf{r}_i\right|^3}.$$

Suppose instead, in the continuous case, we have matter of density $\rho\left(\mathbf{r}\right)$ occupying a region $R$; then the potential $\phi$ and field $\mathbf{f}$ are given by

$$\phi\left(\mathbf{p}\right) = \iiint_R \frac{G\rho\left(\mathbf{r}\right)\,\mathrm{d}V}{\left|\mathbf{p} - \mathbf{r}\right|}, \qquad \mathbf{f}\left(\mathbf{p}\right) = -\iiint_R \frac{G\rho\left(\mathbf{r}\right)\,\left(\mathbf{p} - \mathbf{r}\right)\,\mathrm{d}V}{\left|\mathbf{p} - \mathbf{r}\right|^3}.$$

In the above we have an integral relationship between the mass density $\rho$ and the potential $\phi$. Our objective is find an equivalent relationship in terms of a partial differential equation. To proceed we first need to informally consider the Dirac delta function.

**The Dirac Delta Function** Note that, for a ball $B$ radius $a$ centred at the origin, and with $\mathbf{f}\left(\mathbf{r}\right) = \mathbf{r}/r^3 = \mathbf{e}_r/r^2$,

$$\iint_{\partial B} \mathbf{f}\cdot\mathrm{d}\mathbf{S} = \iint_{\partial B} \frac{\mathbf{e}_r}{a^2}\cdot\left(\mathbf{e}_r\,\mathrm{d}S\right) = \frac{1}{a^2}\iint_{\partial B} \mathrm{d}S = \frac{4\pi a^2}{a^2} = 4\pi.$$

A direct calculation shows that

$$\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) = 0 \quad \text{for } r \neq 0,$$

but the Divergence Theorem (improperly applied to a discontinuous function) "should" give

$$\iiint_B \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)\,\mathrm{d}V = \iint_{\partial B} \frac{\mathbf{r}}{r^3}\cdot\mathrm{d}\mathbf{S} = 4\pi,$$

and this would be true for any size sphere (or indeed region) containing the origin. So, for a general region $R$, we have that

$$\iiint_R \nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right)\,\mathrm{d}V = \begin{cases} 4\pi & \text{if } 0 \in R, \\ 0 & \text{if } 0 \notin R. \end{cases}$$

The shorthand for this is to write

$$\nabla\cdot\left(\frac{\mathbf{r}}{r^3}\right) = 4\pi\delta\left(\mathbf{r}\right)$$

where $\delta$ is the **Dirac Delta Function**. This function has the important filtering property that

$$\iiint\limits_R \phi(\mathbf{r})\ \delta(\mathbf{r}-\mathbf{a})\ \mathrm{d}V = \phi(\mathbf{a}).$$

(All the above can, in fact, be made rigorous as part of the theory of **distributions** or **generalized functions** which were developed by Schwartz in the 1940s.)

**Theorem 132** ***(Poisson's Equation)*** *Let $\phi$ be the gravitational potential associated with a non-uniform material of density $\rho(\mathbf{r})$ occupying a region $R$. Then*

$$\nabla^2\phi = -4\pi G\rho(\mathbf{r}).$$

**Proof.** (**Non-examinable for the general case.**) The gravitational field associated with this matter is given by

$$\mathbf{f}(\mathbf{p}) = \nabla\phi(\mathbf{p}) = -\iiint\limits_R \frac{G\rho(\mathbf{r})\ (\mathbf{p}-\mathbf{r})}{|\mathbf{p}-\mathbf{r}|^3}\,\mathrm{d}V.$$

Then $\nabla^2\phi(\mathbf{p})$ equals

$$\nabla\cdot\mathbf{f}(\mathbf{p}) = -\iiint\limits_R G\rho(\mathbf{r})\ \nabla\cdot\left(\frac{(\mathbf{p}-\mathbf{r})}{|\mathbf{p}-\mathbf{r}|^3}\right)\mathrm{d}V = -\iiint\limits_R G\rho(\mathbf{r})\ \delta(\mathbf{r}-\mathbf{p})\ \mathrm{d}V = -4\pi G\rho(\mathbf{p}).$$

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**Example 133** *A spherical shell, centred at $O$ and with internal and external radii $a$ and $b$ respectively, has matter distribution with density $\rho$ given by*

$$\rho(r) = \begin{cases} 1/r & 0 < a \leqslant r \leqslant b, \\ 0 & \text{otherwise.} \end{cases}$$

*Find the gravitational potential at all points in space.*
*[You may assume that the spherically symmetric form of Laplace's equation is such that $\nabla^2\phi(r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right)$.]*

**Solution.** Method One – Poisson's Equation.

By Poisson's equation gravitational potential $\phi$ satisfies

$$\nabla^2\phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = \begin{cases} 0 & r < a, \\ -4\pi G/r & a < r < b, \\ 0 & b < r. \end{cases}$$

Solving $\nabla^2\phi = 0$ in the region $r < a$ we get

$$
\begin{aligned}
&\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = 0, \\
\implies\quad & r^2\frac{\mathrm{d}\phi}{\mathrm{d}r} = -A \implies \frac{\mathrm{d}\phi}{\mathrm{d}r} = \frac{-A}{r^2}, \\
\implies\quad & \phi = \frac{A}{r} + B.
\end{aligned}
$$

Similarly $\phi = E/r + B$ in the region $r > b$ for constants $E$ and $F$ .

In the middle region $a < r < b$ we have

$$
\begin{aligned}
&\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = \frac{-4\pi G}{r} \\
\implies\quad & \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = -4\pi G r \implies r^2\frac{\mathrm{d}\phi}{\mathrm{d}r} = -2\pi G r^2 - C \\
\implies\quad & \frac{\mathrm{d}\phi}{\mathrm{d}r} = -2\pi G - \frac{C}{r^2} \\
\implies\quad & \phi = -2\pi G r + \frac{C}{r} + D.
\end{aligned}
$$

So, for constants $A, B, C, D, E, F,$ we have

$$
\phi(r) = \begin{cases} \frac{A}{r} + B & r < a, \\ -2\pi G r + \frac{C}{r} + D & a < r < b, \\ \frac{E}{r} + F & b < r. \end{cases}
$$

We can use certain conditions now to determine these constants:

(i) because of physical considerations $\phi$ is finite at $r = 0$,

(ii) $\phi$ is both continuous and differentiable at both $r = a$ and at $r = b$,

(iii) $\phi$ is unique only up to addition by a constant – to specify $\phi$ uniquely we typically stipulate that $\phi(r) \to 0$ as $r \to \infty$.

From condition (i) we have that $A = 0$ and from (iii) that $F = 0$.

The two boundary conditions at $r = a$, namely that $\phi(a_-) = \phi(a_+)$ and $\phi'(a_-) = \phi'(a_+)$, tell us that

$$
B = -2\pi G a + \frac{C}{a} + D, \qquad 0 = -2\pi G - \frac{C}{a^2}.
$$

Similar conditions at $r = b$ give us that

$$
-2\pi G b + \frac{C}{b} + D = \frac{E}{b}, \qquad -2\pi G - \frac{C}{b^2} = -\frac{E}{b^2}.
$$

We have four linear equations in the four unknowns $B, C, D, E$ in terms of the radii $a, b$ and the gravitational constant $G$.

The second equation immediately gives $C$ and thus the fourth equation gives $E$. Then the third equation gives $D$ and the first finally gives $B$. Thus one can then deduce:

$$\phi(r) = \begin{cases} 4\pi G(b-a) & r < a, \\ -2\pi G r - \frac{2\pi G a^2}{r} + 4\pi G b & a < r < b, \\ \frac{2\pi G\left(b^2 - a^2\right)}{r} & b < r. \end{cases}$$

■

Because of the symmetry of the above problem, we can take another approach using Gauss' Flux Theorem.

**Theorem 134** ***(Gauss' Flux Theorem)*** *For a smooth and bounded region $R$, which contains matter of total mass $M$, then*

$$\iint_{\partial R} \mathbf{f} \cdot \mathrm{d}\mathbf{S} = -4\pi G M.$$

*This result is in fact equivalent to Poisson's equation.*

**Proof.** Poisson's equation gives us that

$$\nabla^2 \phi = -4\pi G \rho$$

where $\rho(\mathbf{r})$ is the density of the matter. Applying the Divergence Theorem we have

$$\iint_{\partial R} \mathbf{f} \cdot \mathrm{d}\mathbf{S} = \iiint_R \nabla \cdot \mathbf{f}\, \mathrm{d}V = \iiint_R \nabla^2 \phi\, \mathrm{d}V = -4\pi G \iiint_R \rho\, \mathrm{d}V = -4\pi G M.$$

Conversely suppose that we know

$$\iint_{\partial R} \mathbf{f} \cdot \mathrm{d}\mathbf{S} = -4\pi G M$$

for any bounded region $R$. Then

$$\iiint_R \nabla^2 \phi\, \mathrm{d}V = -4\pi G \iiint_R \rho\, \mathrm{d}V$$

and so

$$\iiint_R \left(\nabla^2 \phi + 4\pi G \rho\right)\, \mathrm{d}V = 0 \quad \text{for any bounded region } R.$$

Hence (at least if $\nabla^2 \phi$ and $\rho$ are piecewise continuous) we have

$$\nabla^2 \phi + 4\pi G \rho \equiv 0.$$

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**Solution.** (to Example 133) Method Two - Gauss' Flux Theorem.

Alternatively, we may use Gauss' Flux Theorem applied to concentric spheres centred on the shell's centre.

Now, $\phi$ is only dependent on $r$ and, so, is constant on the sphere $r = R$, which has surface area $4\pi R^2$.

So, if we apply the flux theorem to the region $r \leqslant R$ we have

$$\iint_{r=R} \nabla\phi \cdot \mathrm{d}\mathbf{S} = \iint_{r=R} \phi'(R)\,\mathbf{e}_r \cdot \mathrm{d}\mathbf{S} = \iint_{r=R} \phi'(R)\,\mathrm{d}S = 4\pi R^2 \phi'(R) = -4\pi G M(R)$$

where $M(R)$ is the total mass within the region $r \leqslant R$. For $a \leqslant R \leqslant b$ we have

$$\begin{aligned} M(R) &= \int_{r=a}^{R} \int_{\theta=0}^{\pi} \int_{\alpha=0}^{2\pi} \frac{1}{r} r^2 \sin\theta \,\mathrm{d}\alpha\,\mathrm{d}\theta\,\mathrm{d}r \\ &= 2\pi \times [-\cos\theta]_0^{\pi} \times \int_{r=a}^{R} r\,\mathrm{d}r \\ &= 2\pi \left(R^2 - a^2\right). \end{aligned}$$

Hence

$$\phi'(R) = \begin{cases} 0 & R < a, \\ 2\pi G \left(\dfrac{a^2}{R^2} - 1\right) & a < R < b, \\ \dfrac{2\pi G\left(a^2 - b^2\right)}{R^2} & b < R. \end{cases}$$

If we integrate we have

$$\phi(R) = \begin{cases} A & R < a, \\ B - 2\pi G \left(\dfrac{a^2}{R} + R\right) & a < R < b, \\ \dfrac{2\pi G\left(b^2 - a^2\right)}{R} + C & b < R. \end{cases}$$

As $\phi(\infty) = 0$ then $C = 0$. As $\phi(b_-) = \phi(b_+)$ then

$$B - 2\pi G \left(\frac{a^2}{b} + b\right) = \frac{2\pi G\left(b^2 - a^2\right)}{b} \implies B = 4\pi G b.$$

Finally as $\phi(a_-) = \phi(a_+)$ then

$$A = 4\pi G b - 2\pi G \left(\frac{a^2}{a} + a\right) = 4\pi G (b - a),$$

and we have the same expressions for $\phi$ as were achieved by the previous method.

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