SHARING PIZZA IN N DIMENSIONS

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ABSTRACT. We introduce and prove the n-dimensional Pizza Theorem: Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{R}^n$. If K is a measurable set of finite volume, the *pizza quantity* of K is the alternating sum of the volumes of the regions obtained by intersecting K with the arrangement $\mathcal{H}$. We prove that if $\mathcal{H}$ is a Coxeter arrangement different from A_1^n such that the group of isometries W generated by the reflections in the hyperplanes of $\mathcal{H}$ contains the map $-\,\mathrm{id}$, and if K is a translate of a convex body that is stable under W and contains the origin, then the pizza quantity of K is equal to zero. Our main tool is an induction formula for the pizza quantity involving a subarrangement of the restricted arrangement on hyperplanes of $\mathcal{H}$ that we call the *even restricted arrangement*. More generally, we prove that for a class of arrangements that we call *even* (this includes the Coxeter arrangements above) and for a *sufficiently symmetric* set K, the pizza quantity of $K+a$ is polynomial in a for a small enough, for example if K is convex and $0 \in K + a$. We get stronger results in the case of balls, more generally, convex bodies bounded by quadratic hypersurfaces. For example, we prove that the pizza quantity of the ball centered at a having radius $R \geq \|a\|$ vanishes for a Coxeter arrangement $\mathcal{H}$ with $|\mathcal{H}| - n$ an even positive integer. We also prove the Pizza Theorem for the surface volume: When $\mathcal{H}$ is a Coxeter arrangement and $|\mathcal{H}| - n$ is a nonnegative even integer, for an n-dimensional ball the alternating sum of the $(n-1)$-dimensional surface volumes of the regions is equal to zero.

1. INTRODUCTION

Given a disc in the plane select any point in the disc. Cut the disc by four lines through this point that are equally spaced. We obtain eight slices of the disc, each having angle $\pi/4$ at the point. The alternating sum of the areas of these eight slices is equal to zero. This is known as the Pizza Theorem and was first stated as a problem in Mathematics Magazine by Upton [17] and solved by Goldberg [8]. There are many two-dimensional extensions of this result; see [7, 13] and the references therein. An especially interesting solution to the original problem is by Carter and Wagon [4], who prove the result by a dissection.

In hyperplane arrangement terminology four equally spaced lines through one point is equivalent to the B_2 arrangement. Goldberg [8] pointed out that the Pizza Theorem also holds for $2k$ equally spaced lines through a point inside a disc, that is, the dihedral arrangement $I_2(2k)$. We extend these results to hyperplane arrangements in higher dimensions. Given a base chamber, every chamber T of an arrangement has a natural sign $(-1)^T$. We define the *pizza quantity* of a measurable set K with finite volume to be the alternating sum $P(\mathcal{H}, K) = \sum (-1)^T \cdot \mathrm{Vol}(K \cap T)$, where the sum ranges over all chambers T of the arrangement $\mathcal{H}$; see equation (2.1).

We first establish an expression that relates the variation of the pizza quantity $P(\mathcal{H}, K)$ when we translate a set K to pizza quantities of arrangements on hyperplanes in $\mathcal{H}$ known as the even restricted arrangements; see Theorem 3.4. Using this formula and induction on the dimension of V, we show that if $\mathbb{B}(a, R)$ is the closed ball with center a and radius R and if $|\mathcal{H}|$ and $\dim(V)$ have the same parity, then the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ is polynomial in the pair (R, a), homogeneous of degree $\dim(V)$ and only having terms of even degree in R as long as $\mathbb{B}(a, R)$ contains the origin; see Theorem 1.1. When $|\mathcal{H}|$ and $\dim(V)$ do not have have the same parity then we can only say that

$P(\mathcal{H}, \mathbb{B}(a, R))$ is a real analytic function of (a, R). If the arrangement $\mathcal{H}$ has enough symmetries, we can use them to show that the lower degree terms of $P(\mathcal{H}, \mathbb{B}(a, R))$ vanish, and in some cases, that the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ itself vanishes. More precisely, we obtain the following result (see Theorem 7.6 and Corollary 7.10 for more general statements):

Theorem 1.1. *Let $\mathcal{H}$ be a Coxeter arrangement on a finite-dimensional inner product space V such that $|\mathcal{H}| \geq \dim(V)$. Assume that the ball $\mathbb{B}(a, R)$ contains the origin, that is, $R \geq \|a\|$.*

  (i) *If the number of hyperplanes is strictly greater than the dimension of V and has the same parity as that dimension then the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ vanishes.*
 (ii) *If the number of hyperplanes is strictly greater than the dimension of V and does not have the same parity as that dimension then the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ tends to 0 as the radius R tends to infinity (when the center a is fixed).*
(iii) *If the number of hyperplanes is equal to the dimension of V then $P(\mathcal{H}, \mathbb{B}(a, R))$ is independent of the radius R.*

In the case when V is 3-dimensional and $\mathcal{H}$ is the arrangement of type $A_1 \times I_2(2k)$, this corollary specializes to the Calzone Theorem; see [1] and [7, page 32].

Note that if V is 1-dimensional, if $\mathcal{H}$ consists of the hyperplane $\{0\}$ and if K is a segment centered at 0, then the pizza quantity $P(\mathcal{H}, K + a)$ is equal to $2a$ as long as $0 \in K + a$, so it is polynomial in a and independent of K. Building on this observation, we prove a similar result in higher dimensions for an *even* arrangement $\mathcal{H}$ (see Definition 4.2) and a measurable set that is *sufficiently symmetric* with respect to $\mathcal{H}$ (see Definition 5.1). In particular, if $\mathcal{H}$ is a Coxeter arrangement then it is even if and only if $-\operatorname{id}_V$ is in its Coxeter group (Corollary 4.7). Furthermore, we show any measurable set stable by its Coxeter group is sufficiently symmetric (Corollary 5.4). We then conclude the following result, which is a particular case of Theorem 6.9 (recall that a convex body is a compact convex set):

Theorem 1.2. *Let $\mathcal{H}$ be a Coxeter arrangement on a finite-dimensional inner product space V such that the map $-\operatorname{id}_V$ belongs to the Coxeter group. Equivalently, let $\mathcal{H}$ be a product arrangement where the factors are from the types A_1, B_n for $n \geq 3$, D_{2m} for $m \geq 2$, E_7, E_8, F_4, H_3, H_4 and $I_2(2k)$ for $k \geq 2$. Let K be a convex body stable under reflections in the hyperplanes of the arrangement $\mathcal{H}$ such that the translate $K + a$ contains the origin. Then the pizza quantity $P(\mathcal{H}, K + a)$ is given by*

$$P(\mathcal{H}, K + a) = \begin{cases} 2^n \cdot a_1 \cdots a_n & \text{if } \mathcal{H} \text{ is of type } A_1^n, \\ 0 & \text{otherwise.} \end{cases}$$

When the arrangement $\mathcal{H}$ has type A_1^n, we assume that it is given by the coordinate hyperplanes $\{x_i = 0 : 1 \leq i \leq n\}$, the base chamber is $T_0 = (\mathbb{R}_{>0})^n$ and the point a is given by $a = (a_1, \ldots, a_n)$.

The paper is organized as follows. In Section 2 we introduce the pizza quantity, review Coxeter and product arrangements, and point out some basic properties of these notions. In Section 3 we define the even restricted arrangement. This is a subarrangement of the restricted arrangement. Using this notion we develop a recursion to compute the pizza quantity; see Theorem 3.4. In Section 4 we introduce the notion of an even restriction sequence, that is, a sequence iterating the notion of the even restricted arrangement. This yields the notion of an *even arrangement*, that is, an arrangement such that every even restriction sequence extends to a sequence that has length equal to the dimension of the space. In Corollary 4.7 we classify all the even Coxeter arrangements. One equivalent condition is that the negative of the identity map belongs to the Coxeter group. In Section 5 we introduce the notion of a sufficiently symmetric measurable set. For sufficiently symmetric measurable sets K of finite volume and for $a \in V$ satisfying some conditions, we show for example that the pizza quantity of $K + a$ only depends on the shift a and is given by the associated

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polynomial of the even arrangement; see Theorem 6.3. In Section 7 we restrict our attention to balls and to convex bodies bounded by quadratic surfaces. For a Coxeter arrangement $\mathcal{H}$ such that $|\mathcal{H}| > \dim(V)$, we show that the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ is equal to 0 if $|\mathcal{H}|$ and $\dim(V)$ have the same parity, and that otherwise $P(\mathcal{H}, \mathbb{B}(a, R)) \longrightarrow 0$ as $R \longrightarrow +\infty$ (with the center a fixed). Moreover if $\mathcal{H} = \dim(V)$ we show that $P(\mathcal{H}, \mathbb{B}(a, R))$ is independent of R. In all these cases, we always assume that the ball $\mathbb{B}(a, R)$ contains the origin. In Section 8 we look at the case of the n-dimensional ball and consider the alternating sum of the surface volume of the regions. This is the Pizza Theorem for the $(n - 1)$st intrinsic volume; see Theorem 8.2. Finally, in Section 9 we briefly consider the problem of sharing pizza among more than two people, and state some concluding remarks and open questions.

2. Initial remarks and definitions

Let V be an n-dimensional real vector space endowed with an inner product: for $v, w \in V$, we denote their inner product by (v, w). Let E be a collection of *unit* vectors in V such that no two vectors of E are linearly dependent. In other words, the intersection $E \cap (-E)$ is empty. By restricting to unit vectors, we will avoid having normalization factors in our expressions. Let $\mathcal{H}$ be the central hyperplane arrangement corresponding to E, that is, $\mathcal{H} = \{H_e : e \in E\}$ where $H_e = \{x \in V : (e, x) = 0\}$. Note that each hyperplane contains the origin. Let $|\mathcal{H}|$ denote the number of hyperplanes in the arrangement $\mathcal{H}$, that is, the cardinality of the index set E. A *chamber* of a hyperplane arrangement is a connected component of the complement of the arrangement in V. Let $\mathscr{T} = \mathscr{T}(\mathcal{H})$ be the collection of chambers of $\mathcal{H}$. For two chambers T_1 and T_2 in $\mathscr{T}$, define the *separation set* $S(T_1, T_2)$ to be the set of all indices $e \in E$ such that the two chambers lie on different sides of the hyperplane H_e.

The arrangement $\mathcal{H}$ is oriented by the data of the set E: if $e \in E$ then the hyperplane H_e cuts V into a positive and a negative half-space, where the positive half-space is the one containing the vector e. If T is a chamber of $\mathcal{H}$ then it gives rise to a sign vector in $\{\pm\}^E$ whose e-component, for $e \in E$, is $+$ if and only if T is included in the positive half-space bounded by H_e. We assume that there is a chamber T_0 whose corresponding sign vector only has $+$ components, and we call it the *base chamber*. The base chamber also determines the direction of each vector in the set E: for each $H \in \mathcal{H}$, the corresponding vector is the unique normal unit vector e of H such that T_0 and e are on the same side of H. Each chamber T has a *sign* $(-1)^T$, which is -1 to the number of hyperplanes of the arrangement one must pass through when walking from T_0 to T, that is, $(-1)^T = (-1)^{|S(T_0, T)|}$. For a reference on hyperplane arrangements, see [2, Chapter 1] or [15, Section 3.11].

Let a be a point in the vector space V and let R be a nonnegative real number. Let $\mathbb{B}(a, R)$ be the ball of radius R centered at a, that is,

$$\mathbb{B}(a, R) = \{x \in V : \|x - a\| \leq R\}.$$

The space V with its inner product is isomorphic to $\mathbb{R}^n$, and we denote by Vol_V or just Vol the pullback by such an isomorphism of Lebesgue measure on $\mathbb{R}^n$. This does not depend on the choice of the isomorphism, because Lebesgue measure is invariant under isometries. For every Lebesgue measurable subset K of V that has finite volume, define the *pizza quantity* for K to be the alternating sum

$$(2.1) \qquad P(\mathcal{H}, K) = \sum_{T \in \mathscr{T}} (-1)^T \cdot \mathrm{Vol}(K \cap T).$$

There is a slight abuse of notation here: the quantity in (2.1) not only depends on the arrangement $\mathcal{H}$, but also on the base chamber T_0. More generally, if $f : V \longrightarrow \mathbb{C}$ is any L^1 function, we

define the *pizza quantity* for f to be

$$P(\mathcal{H}, f) = \sum_{T \in \mathscr{T}} (-1)^T \cdot \int_T f(x) \, dV.$$

We then have $P(\mathcal{H}, K) = P(\mathcal{H}, \mathbb{1}_K)$, where $\mathbb{1}_K$ is the characteristic function of K.

The 2-dimensional Pizza Theorem is as follows:

Theorem 2.1 (Goldberg [8]). *Let $\mathcal{H}$ be the dihedral arrangement $I_2(2k)$ in $\mathbb{R}^2$ for $k \geq 2$. For every $a \in \mathbb{R}^2$ and every $R \geq \|a\|$, the pizza quantity for the ball $\mathbb{B}(a, R)$ vanishes:*

$$P(\mathcal{H}, \mathbb{B}(a, R)) = 0.$$

A hyperplane arrangement $\mathcal{H}$ is a *Coxeter arrangement* if the group W generated by the orthogonal reflections in the hyperplanes of $\mathcal{H}$ is finite and the arrangement is closed under all such reflections. There is a vast literature concerning Coxeter arrangements and their associated root systems; see [3, 11]. The group W is known as the *Coxeter group* of the arrangement.

We say a subset L of V is *stable* with respect to a Coxeter group W acting on V if W leaves it invariant, that is, $w(L) = L$ for all $w \in W$. We define the translate of L by $a \in V$ to be the set $L + a = \{x + a : x \in L\}$. If $f : V \longrightarrow \mathbb{C}$ is a function, we say that it is *stable* with respect to a Coxeter group W if $f(w(x)) = f(x)$ for every $w \in W$ and $x \in V$. We then have that a subset L of V is stable by W if and only if $\mathbb{1}_L$ is stable. For $a \in V$, we denote the shift of function $x \longmapsto f(x - a)$ by f_a. Then we have $(\mathbb{1}_L)_a = \mathbb{1}_{L+a}$ for every $a \in V$.

Proposition 2.2. *Let $u : V \longrightarrow V'$ be an isometry, where V' is another real inner product space, and let $\mathcal{H}$ be a hyperplane arrangement in V. Let $u(\mathcal{H})$ be the hyperplane arrangement $\{u(H) : H \in \mathcal{H}\}$ with base chamber $u(T_0)$. For any L^1 function $f : V' \longrightarrow \mathbb{C}$, the following equality holds:*

$$P(u(\mathcal{H}), f) = P(\mathcal{H}, f \circ u).$$

Proof. The map u induces a bijection between the chambers of $\mathcal{H}$ and those of $u(\mathcal{H})$ that respects the cardinality of the separation set, that is, $|S(T_1, T_2)| = |S(u(T_1), u(T_2))|$. The conclusion follows from the fact that u, being an isometry, is volume-preserving. $\square$

Corollary 2.3. *Let $\mathcal{H}$ be a Coxeter arrangement with Coxeter group W and $f : V \longrightarrow \mathbb{C}$ be an L^1 function. Then for every $w \in W$ we have*

$$P(\mathcal{H}, f \circ w) = \det(w) \cdot P(\mathcal{H}, f).$$

In particular, if f is stable by a reflection in W then we have $P(\mathcal{H}, f) = 0$.

Proof. The first statement is just Proposition 2.2, because $w(\mathcal{H}) = \mathcal{H}$ and $(-1)^{|S(T_0, w(T_0))|} = \det(w)$ for every $w \in W$. The second statement follows from the first and from the fact that reflections have determinant -1. $\square$

Remark 2.4. Note that the situation where f is stable by a reflection s in W occurs if $f = g_a$, with $g : V \longrightarrow \mathbb{C}$ an L^1 function that is stable by W and a a point belonging to the hyperplane H of the arrangement $\mathcal{H}$, where s is the orthogonal reflection in H.

Given two hyperplane arrangements $\mathcal{H}_1 = \{H_e\}_{e \in E_1}$ and $\mathcal{H}_2 = \{H_e\}_{e \in E_2}$ in the vector space V_1, respectively V_2, define the product arrangement $\mathcal{H}_1 \times \mathcal{H}_2$ in $V_1 \times V_2$, where the index set of vectors is $E = (E_1 \times \{0\}) \cup (\{0\} \times E_2)$. Note that in this construction the hyperplanes inherited from $\mathcal{H}_1$ are orthogonal to the hyperplanes inherited from $\mathcal{H}_2$. Furthermore, if $T_{i,0}$ is the base chamber of $\mathcal{H}_i$ then $T_{1,0} \times T_{2,0}$ is the base chamber of $\mathcal{H}_1 \times \mathcal{H}_2$.

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3. The even restricted arrangement

In this section we obtain a recursion for evaluating the pizza quantity in terms of lower dimensional pizza quantities on certain subarrangements of the given arrangement. We begin by introducing two definitions.

Definition 3.1. (a) Let $V' \subseteq V$ be a subspace of codimension 2. The *intersection multiplicity* of the arrangement $\mathcal{H} = \{H_e : e \in E\}$ at the subspace V' is the cardinality

$$\mathrm{imult}(V') = |\{e \in E : H_e \supseteq V'\}|.$$

(b) For $e \in E$ the *even restricted arrangement* $\mathcal{H}_e$ is the arrangement inside the vector space H_e consisting of the hyperplanes $H_e \cap H_f$ (these are codimension 2 subspaces of V) with even intersection multiplicity, that is,

$$\mathcal{H}_e = \{H_e \cap H_f : f \in E, e \neq f, \mathrm{imult}(H_e \cap H_f) \equiv 0 \bmod 2\}.$$

The even restricted arrangement $\mathcal{H}_e$ is a subarrangement of the *arrangement $\mathcal{H}$ restricted to the hyperplane H_e*, for brevity known as the restricted arrangement, that is, $\mathcal{H}_e'' = \{H_e \cap H_f : f \in E, e \neq f\}$; see [15, Section 3.11.2]. In general these two arrangements are different. Even though the letter e appears in the notation for $\mathcal{H}_e$, the arrangement $\mathcal{H}_e$ only depends on H_e, and not on the choice of its normal vector e.

Proposition 3.2. *Suppose that $\mathcal{H}$ is a Coxeter arrangement, and let $e \in E$.*
   (i) *Let $e' \in E - \{e\}$. Then the intersection multiplicity of $\mathcal{H}$ at $H_e \cap H_{e'}$ is even if and only if there exists $f \in E - \{e\}$ such that $H_e \cap H_f = H_e \cap H_{e'}$ and that $(e, f) = 0$.*
   (ii) *Let $F = \{f \in E : (e, f) = 0\} = E \cap H_e$. Then $\mathcal{H}_e$ is the hyperplane arrangement $(H_e \cap H_f)_{f \in F}$ on H_e, and it is a Coxeter arrangement.*

Proof.   (i) Let $E' = \{f \in E - \{e\} : H_e \cap H_f = H_e \cap H_{e'}\}$. Then $|E'| + 1$ is the intersection multiplicity of $\mathcal{H}$ at the intersection $H_e \cap H_{e'}$, and we have $s_e(E') = E'$, where s_e is the orthogonal reflection in the hyperplane H_e. As $s_e^2 = 1$, the set E' has odd cardinality if and only if s_e has a fixed point on E'. But $f \in E'$ is fixed by s_e if and only if $s_e(H_f) = H_f$, which is equivalent to the condition that $(e, f) = 0$.
   (ii) The first statement follows from (i). It remains to prove that $\mathcal{H}_e$ is a Coxeter arrangement. For every $f \in F$, if s_f is the orthogonal reflection in the hyperplane H_f, then $s_f(H_e) = H_e$ and $s_f(F) = F$. In particular, s_f preserves the arrangement $\mathcal{H}_e$. We conclude that $\mathcal{H}_e$ is a Coxeter arrangement on H_e. $\qquad\qquad\square$

For a hyperplane H_e define the open half spaces H_e^+ and H_e^- to be

$$H_e^+ = \{x \in V : (e, x) > 0\} \quad \text{and} \quad H_e^- = \{x \in V : (e, x) < 0\}.$$

An *(open) face* of the arrangement $\mathcal{H} = \{H_e : e \in E\}$ is a non-empty intersection of the form

$$\bigcap_{e \in E_0} H_e \cap \bigcap_{e \in E_+} H_e^+ \cap \bigcap_{e \in E_-} H_e^-,$$

where $E = E_0 \sqcup E_+ \sqcup E_-$ and $\sqcup$ denotes disjoint union. For a face F of an arrangement $\mathcal{H}$ and a vector v, let the *composition $F \circ v$* denote the face G of the arrangement $\mathcal{H}$ such that $x + \varepsilon \cdot v \in G$ for all $x \in F$ and $\varepsilon > 0$ small enough. See [2, pages 8 and 102] where the composition is defined for signed vectors.

Lemma 3.3. *Let U_1 and U_2 be two chambers in the restricted arrangement $\mathcal{H}''_e$. Let Z_i be the unique chamber in the even restricted arrangement $\mathcal{H}_e$ containing U_i. Then the parities of the two separation sets $S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e)$ and $S_{\mathcal{H}_e}(Z_1, Z_2)$ agree, that is,*

$$|S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e)| \equiv |S_{\mathcal{H}_e}(Z_1, Z_2)| \bmod 2.$$

Proof. We denote by X the set of hyperplanes of $\mathcal{H}_e$ and by S the set of vectors $f \in E - \{e\}$ such that $H_e \cap H_f$ has even intersection multiplicity. Let $\iota : S \longrightarrow X$ the map sending $f \in S$ to $H_e \cap H_f$. We claim that:

    (a) The set $S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e) - S$ has even cardinality.
    (b) The image of $S \cap S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e)$ under ι is $S_{\mathcal{H}_e}(Z_1, Z_2)$.
    (c) The fibers of ι all have odd cardinality.

These three statements immediately imply the result, so it suffices to prove them.

Consider the equivalence relation $\sim$ on $E - \{e\}$ defined by $f \sim f'$ if and only if $H_e \cap H_f = H_e \cap H_{f'}$. Then S is a union of equivalence classes for $\sim$. We claim that $S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e)$ is also a union of equivalence classes for $\sim$. This follows from the fact that $e \notin S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e)$ and that if $f \in E - \{e\}$ then $f \in S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e)$ if and only if $H_f \cap H_e$ separates U_1 and U_2. In particular, the set $S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e) - S$ is also a union of equivalence classes of $\sim$. But if $f \in E - \{e\}$, then its equivalence class for $\sim$ is the set of $f' \in E - \{e\}$ such that $H_f \cap H_e \subset H_{f'}$, so it has even cardinality if and only if $f \notin S$. This shows that $S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e) - S$ is a disjoint union of sets of even cardinality and proves (a). Also, as the fibers of ι are exactly the equivalence classes of $\sim$ that are contained in S, we obtain (c). We finally prove (b). If $f \in S \cap S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e)$ then U_1 and U_2 are on opposite sides of $H_e \cap H_f$ and $H_e \cap H_f \in \mathcal{H}_e$, so Z_1 and Z_2 must be on opposite sides of $H_e \cap H_f$. Conversely, consider a hyperplane H of $\mathcal{H}_e$ such that Z_1 and Z_2 are on opposite sides of H. We can write $H = H_e \cap H_f$ with $f \in S$. The chambers U_1 and U_2 are on opposite sides of the hyperplane H_f, so $f \in S_{\mathcal{H}}(U_1 \circ e, U_2 \circ e)$. $\qquad\square$

For $e \in E$, let Z be a chamber of the even restricted arrangement $\mathcal{H}_e$. Note that the closure of Z is a union of closures of chambers in the restricted arrangement $\mathcal{H}''_e$, that is, $\overline{Z} = \cup_{U \in Q} \overline{U}$ for some subset Q of chambers of $\mathcal{H}''_e$. By Lemma 3.3 the sign $(-1)^{U \circ e}$ is independent of $U \in Q$ and hence we define $(-1)^{Z \circ e} = (-1)^{U \circ e}$ for any $U \in Q$.

Theorem 3.4. *Let $\mathcal{H} = \{H_e : e \in E\}$ be a central hyperplane arrangement with base chamber T_0. Let $f : V \longrightarrow \mathbb{C}$ be an L^1 function.*

    (i) *Let T be a chamber of $\mathcal{H}$. Then for every $a \in V$, we have*

$$(3.1) \qquad \int_T f_a(x)\, dV - \int_T f(x)\, dV = \sum_U (-1)^T (-1)^{U \circ e_U} (a, e_U) \int_0^1 \left(\int_U f_{ta}(y)\, dV_{H_{e_U}} \right) dt,$$

        where the sum runs over all facets U of T and, for each such U, the vector e_U is the unique element of E such that $U \subset H_{e_U}$.

    (ii) *For $e \in E$, let $Z_0(e)$ be an arbitrarily chosen base chamber of the even restricted arrangement $\mathcal{H}_e$. Let a be a vector in V. Then we have*

$$(3.2) \qquad P(\mathcal{H}, f_a) - P(\mathcal{H}, f) = 2 \cdot \sum_{e \in E} (-1)^{Z_0(e) \circ e} \cdot (a, e) \cdot \int_0^1 P(\mathcal{H}_e, f_{ta}|_{H_e})\, dt.$$

In particular, if K is a measurable subset of V that has finite volume, we obtain

$$(3.3) \qquad P(\mathcal{H}, K + a) - P(\mathcal{H}, K) = 2 \cdot \sum_{e \in E} (-1)^{Z_0(e) \circ e} \cdot (a, e) \cdot \int_0^1 P(\mathcal{H}_e, (K + ta) \cap H_e)\, dt.$$

Proof of Theorem 3.4. For this proof we assume that the vector space V is $\mathbb{R}^n$ with the usual inner product $(\cdot, \cdot)$. Hence we write $x = (x_1, \ldots, x_n)$ and $a = (a_1, \ldots, a_n)$. Suppose first that the function f is C^∞ with compact support. Next, let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the vector field $F(x) = f(x - ta) \cdot a$. Note that

$$\mathrm{div}(F) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} f(x - ta) = -\frac{\partial}{\partial t} f(x - ta).$$

Let T be a chamber of $\mathcal{H}$. The function $t \longmapsto \int_T f_{ta}(x)\, dV$ is also C^∞. By the Leibniz integral rule and Gauss's divergence theorem we have

(3.4)
$$\frac{d}{dt} \int_T f_{ta}\, dV = \int_T \frac{\partial}{\partial t} f(x - ta)\, dV$$
$$= \int_T \mathrm{div}(F)\, dV$$
$$= -\int_{\partial T} F \cdot m\, dS,$$

where m is the unit normal vector pointing outward from the chamber T.

Let $\mathscr{T}'$ be the collection of $(n-1)$-dimensional faces in the arrangement $\mathcal{H}$. We call the elements in $\mathscr{T}'$ *subchambers* since they are one dimension less than that of the chambers in $\mathscr{T}$. Note that each subchamber $U \in \mathscr{T}'$ is a subset of exactly one hyperplane in $\mathcal{H}$. Let e_U denote the unique vector f in E such that $U \subseteq H_f$.

The normal vector m at a point in $U \subseteq \partial T$ is $\pm e_U$. Since m points out from T we have $-(-1)^T (a, m) = (-1)^{U \circ m}(a, m) = (-1)^{U \circ e_U}(a, e_U)$, where the last equality is true since either both factors change sign or neither does. We conclude that the last integral appearing in equation (3.4) is the sum over all subchambers U of $\mathcal{H}$ included in ∂T, that is,

$$\frac{d}{dt} \int_T f_{ta}\, dV = (-1)^T (-1)^{U \circ e_U}(a, e_U) \int_U f(x - ta)\, dS.$$

Also note that the surface differential dS is the natural volume on U. We deduce equation (3.1) by integrating both sides, and this proves point (i) in the case when f is C^∞ with compact support.

To deduce the general case of (i) from the case of a C^∞ function with compact support, it suffices to prove that both sides of equation (3.1) are continuous for the L^1 norm, because the space of C^∞ functions with compact support is dense in the set of L^1 functions for that norm. This is clear for the left-hand side of equation (3.1). To show the continuity of the right-hand side, we need to see that for every subchamber U of $\mathcal{H}$ such that $(a, e_U) \neq 0$ the function $f \longmapsto \int_0^1 \int_U f(x - ta)\, dV_{H_{e_U}}(x) dt$ is continuous for the L^1 norm. As $(a, e_U) \neq 0$, we know that $a \notin H_{e_U}$. By Fubini's theorem, the double integral $\int_0^1 \int_U f(x - ta)\, dV_{H_{e_U}}(x) dt$ equals, up to a constant, the integral of f on the set $\{z + ta : z \in U,\ 0 \leq t \leq 1\}$ for the measure V on $\mathbb{R}^n$, so the result follows.

We now prove point (ii). Let $f : V \longrightarrow \mathbb{R}$ be an L^1 function. Summing equation (3.1) over all chambers T of $\mathcal{H}$ and noting that each subchamber U appears in the boundary of exactly two chambers, we obtain

(3.5)
$$P(\mathcal{H}, f_a) - P(\mathcal{H}, f) = 2 \cdot \sum_{U \in \mathscr{T}'} (-1)^{U \circ e_U} \cdot (a, e_U) \cdot \int_0^1 \int_U f(x - ta)\, dV_{H_{e_U}}\, dt.$$

To simplify the expression (3.5), consider a subchamber U. It lies in a unique hyperplane H_e, where $e = e_U$. Furthermore, the subchamber U is contained in a unique chamber Z of the even restricted arrangement $\mathcal{H}_e$. Pick a base chamber $U_0(e)$ of the restricted arrangement $\mathcal{H}_e''$ inside the

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$\mathcal{H}$	$\mathcal{H}_e$	$\mathcal{H}$	$\mathcal{H}_e$
A_n	A_{n-2}	E_7	D_6
B_2	A_1	E_8	E_7
B_3	B_2 or $A_1 \times A_1$	F_4	B_3
B_n	B_{n-1} or $A_1 \times B_{n-2}$, for $n \geq 4$	H_3	$A_1 \times A_1$
D_4	A_1^3	H_4	H_3
D_5	$A_1 \times A_3$	$I_2(2k)$	A_1
D_n	$A_1 \times D_{n-2}$, for $n \geq 6$	$I_2(2k+1)$	$\emptyset$

TABLE 1. The possible types of even restricted subarrangements for simple Coxeter arrangements of dimension ≥ 2. For the type A_n arrangement the even restricted subarrangements have type A_{n-2} inside an $(n-1)$-dimensional space, that is, the even restricted subarrangements are not essential. Similarly for the odd dihedral arrangement $I_2(2k+1)$, each of the even restricted subarrangements is the empty 1-dimensional arrangement.

chamber $Z_0(e)$. We then have by Lemma 3.3

$$(-1)^{U \circ e} = (-1)^{U_0(e) \circ e} \cdot (-1)^{S(U_0(e) \circ e, U \circ e)} = (-1)^{Z_0(e) \circ e} \cdot (-1)^{S(Z_0(e), Z)} = (-1)^{Z_0(e) \circ e} \cdot (-1)^{Z}.$$

Hence we can collect terms in the sum in equation (3.5) by first summing over vectors e in E and then chambers Z of the even restricted arrangement $\mathcal{H}_e$. This gives that

$$P(\mathcal{H}, f_a) - P(\mathcal{H}, f) = 2 \cdot \sum_{e \in E} \sum_{Z \in \mathcal{T}(\mathcal{H}_e)} (-1)^{Z_0(e) \circ e} \cdot (-1)^{Z} \cdot (a, e) \cdot \int_0^1 \int_Z f(x - ta) \, dV_{H_e} \, dt$$

$$= 2 \cdot \sum_{e \in E} (-1)^{Z_0(e) \circ e} \cdot (a, e) \cdot \int_0^1 P(\mathcal{H}_e, f_{ta}|_{H_e}) \, dt,$$

which is equation (3.2). $\qquad\square$

4. EVEN RESTRICTION SEQUENCES

For a sequence $(e_1, e_2, \ldots, e_r)$ in the Cartesian power E^r, define the subspace $H_{e_1, \ldots, e_r}$ to be the intersection $H_{e_1, \ldots, e_r} = \bigcap_{i=1}^r H_{e_i}$.

Definition 4.1. An *even restriction sequence* $(e_1, e_2, \ldots, e_r)$ and its associated hyperplane arrangement $\mathcal{H}_{e_1, \ldots, e_r}$ on $H_{e_1, \ldots, e_r}$ is defined recursively by:

(a) The empty sequence is an even restriction sequence. This is the case $r = 0$ and we set $\mathcal{H}_\emptyset = \mathcal{H}$.

(b) If $r \geq 1$, the sequence $(e_1, e_2, \ldots, e_{r-1})$ is an even restriction sequence and the subspace $H_{e_1, e_2, \ldots, e_r}$ is a hyperplane in the arrangement $\mathcal{H}_{e_1, e_2, \ldots, e_{r-1}}$ then $(e_1, e_2, \ldots, e_r)$ is an even restriction sequence. Furthermore, let $f \in H_{e_1, e_2, \ldots, e_{r-1}}$ be a unit normal vector of the hyperplane $H_{e_1, e_2, \ldots, e_r}$, that is, $H_{e_1, e_2, \ldots, e_r} = \{x \in H_{e_1, e_2, \ldots, e_{r-1}} : (f, x) = 0\}$. We then set $\mathcal{H}_{e_1, e_2, \ldots, e_r}$ to be the even restricted arrangement $\left(\mathcal{H}_{e_1, e_2, \ldots, e_{r-1}}\right)_f$ in the subspace $H_{e_1, e_2, \ldots, e_r}$.

Finally, let $P_r \subseteq E^r$ denote the set of all even restriction sequences of length r.

We introduce the following definitions.

Definition 4.2. Let $\mathcal{H} = \{H_e : e \in E\}$ be a hyperplane arrangement.

(a) We say that an even restriction sequence $(e_1, \ldots, e_r) \in P_r$ is *maximal* if it cannot be extended to a longer even restriction sequence. More formally, there is no $e_{r+1} \in E$ such that $(e_1, \ldots, e_r, e_{r+1})$ is an even restriction sequence. We denote by P the set of maximal even restriction sequences of the arrangement $\mathcal{H}$.

(b) We say that the arrangement $\mathcal{H}$ is *even* if for every $r \geq 0$ and every even restriction sequence $(e_1, \ldots, e_r) \in P_r$ the arrangement $\mathcal{H}_{e_1, \ldots, e_r}$ is essential.

Lemma 4.3. *Let $\mathcal{H}$ be an n-dimensional hyperplane arrangement.*

(i) *If $(e_1, \ldots, e_r) \in P_r$ then $\dim(\mathcal{H}_{e_1, \ldots, e_r}) = n - r$ (in particular, the vectors $e_1, \ldots, e_r$ are linearly independent) and every element of $\mathcal{H}_{e_1, \ldots, e_r}$ is an intersection of hyperplanes of $\mathcal{H}$.*

(ii) *We have $P_r = \varnothing$ for $r > n$.*

(iii) *The following statements are equivalent:*

    (a) *The arrangement $\mathcal{H}$ is even.*

    (b) *Every even restriction sequence can be extended to an even restriction sequence of length n, that is, for every $0 \leq r \leq n$ and every $(e_1, \ldots, e_r) \in P_r$, there exist $e_{r+1}, \ldots, e_n \in E$ such that $(e_1, e_2, \ldots, e_n) \in P_n$.*

    (c) *We have $P = P_n$.*

Proof. If $1 \leq i \leq r$ then by definition of $\mathcal{H}_{e_1, \ldots, e_i}$ we have $\dim(\mathcal{H}_{e_1, \ldots, e_i}) = \dim(\mathcal{H}_{e_1, \ldots, e_{i-1}}) - 1$. This implies the first statement of (i). We prove the second statement of (i) by induction on r. It is clear if $r = 0$, so assume that $r \geq 1$ and that the conclusion holds for $\mathcal{H}_{e_1, \ldots, e_{r-1}}$. As $\mathcal{H}_{e_1, \ldots, e_r}$ is a subarrangement of the restriction of $\mathcal{H}_{e_1, \ldots, e_{r-1}}$ to the hyperplane $H_{e_1, \ldots, e_r} = H_{e_1, \ldots, e_{r-1}} \cap H_{e_r}$ of $H_{e_1, \ldots, e_{r-1}}$, each element of $\mathcal{H}_{e_1, \ldots, e_r}$ is the intersection of H_{e_r} and an element of $\mathcal{H}_{e_1, \ldots, e_{r-1}}$. This implies (i).

Statement (ii) follows immediately from (i).

We now prove statement (iii). Suppose that (a) holds. We prove (b) by descending induction on r. If $r = n$, there is nothing to prove. Suppose that $r < n$ and that we know the statement for $r + 1$, and let $(e_1, \ldots, e_r) \in P_r$. As the arrangement $\mathcal{H}_{e_1, \ldots, e_r}$ is an essential arrangement in the $(n-r)$-dimensional space $H_{e_1, \ldots, e_r}$, and as $n - r > 0$, this arrangement has at least one hyperplane. By the second statement of (i), there exists $e_{r+1} \in E$ such that $(e_1, \ldots, e_{r+1}) \in P_{r+1}$. We finish the proof by applying the induction hypothesis to $(e_1, \ldots, e_{r+1})$.

Conversely, suppose that (b) holds, and let $(e_1, \ldots, e_r) \in P_r$, with $0 \leq r \leq n$. By (b), there exist vectors $e_{r+1}, \ldots, e_n \in E$ such that $(e_1, \ldots, e_n) \in P_n$. By (i), we know that $H_{e_1, \ldots, e_n} = \{0\}$ and that $H_{e_1, \ldots, e_n}$ is an intersection of hyperplanes of $\mathcal{H}_{e_1, \ldots, e_r}$. This implies that $\mathcal{H}_{e_1, \ldots, e_r}$ is essential, and hence (a) holds.

Finally, in light of statement (ii), it is clear that (b) and (c) are equivalent. $\qquad\square$

Definition 4.4. If $(e_1, \ldots, e_r) \in P$ then the arrangement $\mathcal{H}_{e_1, \ldots, e_r}$ is empty, so it only has one chamber $Z = H_{e_1, \ldots, e_r}$. By the discussion before Theorem 3.4, we have a well-defined sign $(-1)^{Z \circ e_r \circ \cdots \circ e_1}$, and we denote this sign by $(-1)^{e_r \circ \cdots \circ e_1}$.

Remark 4.5.   (1) Let $(e_1, \ldots, e_r) \in P$. Then $(-1)^{e_r \circ \cdots \circ e_1}$ is the sign of any chamber of $\mathcal{H}$ that contains the vector $v + \epsilon e_r + \epsilon^2 e_{r-1} + \cdots + \epsilon^r e_1$, for $v \in H_{e_1, \ldots, e_r}$ nonzero and $\epsilon > 0$ small enough.

(2) Suppose that $\mathcal{H}$ is an even arrangement. Then $P = P_n$ by Lemma 4.3. So for any $(e_1, \ldots, e_n) \in P_n$, we have a sign $(-1)^{e_n \circ \cdots \circ e_1}$, which is the sign of the chamber of $\mathcal{H}$ containing the vector $\epsilon e_n + \epsilon^2 e_{n-1} + \cdots + \epsilon^n e_1$ for $\epsilon > 0$ small enough.

For r a nonnegative integer, we introduce the following equivalence relation on the Cartesian power E^r: $(e_1, \ldots, e_r) \sim (f_1, \ldots, f_r)$ if and only if $H_{e_1, \ldots, e_i} = H_{f_1, \ldots, f_i}$ for every $0 \leq i \leq r$. The following facts are straightforward consequences of the definition of this equivalence relation.

(1) If $0 \leq i \leq r$, $(e_1, \ldots, e_r) \in E^r$ and $(f_1, \ldots, f_i) \in E^i$ then $(e_1, \ldots, e_i) \sim (f_1, \ldots, f_i)$ if and only if $(e_1, \ldots, e_r) \sim (f_1, \ldots, f_i, e_{i+1}, \ldots, e_r)$.

(2) If $(e_1, \ldots, e_r) \sim (f_1, \ldots, f_r)$ then $(e_1, \ldots, e_r) \in P_r$ if and only if $(f_1, \ldots, f_r) \in P_r$. If this condition holds then $\mathcal{H}_{e_1, \ldots, e_r} = \mathcal{H}_{f_1, \ldots, f_r}$.

(3) If $(e_1, \ldots, e_r) \sim (f_1, \ldots, f_r)$ then $(e_1, \ldots, e_r) \in P$ if and only if $(f_1, \ldots, f_r) \in P$. If this condition holds then $(-1)^{e_r \circ \cdots \circ e_1} = (-1)^{f_r \circ \cdots \circ f_1}$.

We now specialize these notions to the case of Coxeter arrangements. Suppose that $\mathcal{H}$ is a Coxeter arrangement and let r be a nonnegative integer. Then we denote by $E_r^{(0)} \subseteq E^r$ the set of sequences $(e_1, \ldots, e_r)$ of pairwise orthogonal elements of E.

Proposition 4.6. *Suppose that $\mathcal{H} = (H_e)_{e \in E}$ is a Coxeter arrangement and let r be a nonnegative integer.*

(i) *Let $(e_1, \ldots, e_r) \in P_r$. Then $\mathcal{H}_{e_1, \ldots, e_r}$ is a Coxeter arrangement on $H_{e_1, \ldots, e_r}$ given by the finite set of vectors $E \cap H_{e_1, \ldots, e_r}$.*

(ii) *If $(e_1, \ldots, e_r), (f_1, \ldots, f_r) \in E_r^{(0)}$ and $(e_1, \ldots, e_r) \sim (f_1, \ldots, f_r)$ then the equality $(e_1, \ldots, e_r) = (f_1, \ldots, f_r)$ holds.*

(iii) *Let $(e_1, \ldots, e_r) \in E^r$. Then $(e_1, \ldots, e_r)$ is in P_r if and only if there exists $(f_1, \ldots, f_r) \in E_r^{(0)}$ such that $(e_1, \ldots, e_r) \sim (f_1, \ldots, f_r)$. Furthermore, this sequence $(f_1, \ldots, f_r)$ is necessarily unique by (ii).*

   In particular, we have $E_r^{(0)} \subseteq P_r$, and this inclusion induces a bijection $E_r^{(0)} \simeq P_r / \sim$.

(iv) *Let $(e_1, \ldots, e_r) \in E_r^{(0)}$. Then $(e_1, \ldots, e_r) \in P$ if and only if $E \cap \{e_1, \ldots, e_r\}^{\perp} = \{0\}$.*

(v) *There exists an integer r with $0 \leq r \leq n$ such that $P = P_r$.*

Proof. We prove (i) by induction on r. The result is clear if $r = 0$. So assume that $r \geq 1$ and that we know the result for $r - 1$, and let $(e_1, \ldots, e_r) \in P_r$. By the induction hypothesis, the arrangement $\mathcal{H}_{e_1, \ldots, e_{r-1}}$ is a Coxeter arrangement, and its hyperplanes are the codimension one subspaces orthogonal to the vectors in $E \cap H_{e_1, \ldots, e_{r-1}}$. By Proposition 3.2, the arrangement $\mathcal{H}_{e_1, \ldots, e_r}$ is also Coxeter and the elements of $\mathcal{H}_{e_1, \ldots, e_r}$ are exactly the intersections $H_{e_1, \ldots, e_r} \cap H_f$, where $f \in E \cap H_{e_1, \ldots, e_{r-1}}$ is such that $(f, e_r) = 0$. In other words, they are the intersections $H_{e_1, \ldots, e_r} \cap H_f$, with $f \in H_{e_1, \ldots, e_r}$. This finishes the proof.

We prove (ii) by induction on r. If $r \leq 1$ the result is clear, so assume that $r \geq 2$ and that we know the result for $r - 1$. Let $(e_1, \ldots, e_r), (f_1, \ldots, f_r) \in E_r^{(0)}$ such that $(e_1, \ldots, e_r) \sim (f_1, \ldots, f_r)$. Then $(e_1, \ldots, e_{r-1}), (f_1, \ldots, f_{r-1}) \in E_{r-1}^{(0)}$ and $(e_1, \ldots, e_{r-1}) \sim (f_1, \ldots, f_{r-1})$, so $e_i = f_i$ for $1 \leq i \leq r - 1$ by the induction hypothesis. The assumption that $H_{e_1, \ldots, e_r} = H_{e_1, \ldots, e_{r-1}, f_r}$ shows that the families $(e_1, \ldots, e_r)$ and $(e_1, \ldots, e_{r-1}, f_r)$ span the same subspace of V and, as $e_r, f_r \in \mathrm{Span}(e_1, \ldots, e_{r-1})^{\perp}$, we conclude that e_r and f_r are colinear, hence equal.

We prove (iii), again by induction on r. If $r \leq 1$ then $E^r = P_r = E_r^{(0)}$ and the result follows. Suppose that $r \geq 2$ and that we know the result for $r - 1$. Let $(e_1, \ldots, e_r) \in E^r$. Suppose that $(e_1, \ldots, e_r) \in P_r$. Then $(e_1, \ldots, e_{r-1}) \in P_{r-1}$, so by the induction hypothesis and observations (1) and (2) above we may assume that $(e_1, \ldots, e_{r-1}) \in E_{r-1}^{(0)}$. As $H_{e_1, \ldots, e_r}$ is an element of $\mathcal{H}_{e_1, \ldots, e_{r-1}}$, there exists by (i) an element f of $E \cap H_{e_1, \ldots, e_{r-1}}$ such that $H_{e_1, \ldots, e_r} = H_f \cap H_{e_1, \ldots, e_{r-1}}$. Then $(e_1, \ldots, e_{r-1}, f) \in E_r^{(0)}$ and $(e_1, \ldots, e_{r-1}, f) \sim (e_1, \ldots, e_r)$, so we are done. Now suppose that there exists $(f_1, \ldots, f_r) \in E_r^{(0)}$ such that $(e_1, \ldots, e_r) \sim (f_1, \ldots, f_r)$. By observation (2) above, it suffices to show that $(f_1, \ldots, f_r) \in P_r$, so we may assume that $(e_1, \ldots, e_r) \in E_r^{(0)}$. By the induction hypothesis, we have $(e_1, \ldots, e_{r-1}) \in P_{r-1}$ and by (i) the hyperplane $H_{e_1, \ldots, e_r}$ is an element of $\mathcal{H}_{e_1, \ldots, e_{r-1}}$, which shows that $(e_1, \ldots, e_r) \in P_r$.

Point (iv) immediately follows from point (iii) and from the definition of P.

To prove (v), we need to see that all maximal subsets of pairwise orthogonal elements of E have the same cardinality, which is a well-known fact. For crystallographic root systems this follows, for example, from Lemma 5.7 of [9] and from the fact that all 2-structures are conjugated under the Weyl group, as asserted before Theorem 5.5 of [9]; for the general case, see Appendix B of [5]. $\square$

Corollary 4.7. *Suppose $\mathcal{H}$ is a Coxeter arrangement with Coxeter group W in an n-dimensional vector space V. Then the following four statements are equivalent:*

(a) *The arrangement $\mathcal{H}$ is even.*

(b) *Every sequence of pairwise orthogonal elements of E can be extended to a sequence of pairwise orthogonal elements of length n, that is, for $0 \leq r \leq n$ and $(e_1, \ldots, e_r) \in E_r^{(0)}$, there exist vectors $e_{r+1}, \ldots, e_n \in E$ such that $(e_1, \ldots, e_n) \in E_n^{(0)}$.*

(c) *The map $-\operatorname{id}_V$ belongs to the Coxeter group W.*

(d) *The arrangement $\mathcal{H}$ is a Cartesian product of arrangements of type A_1, B_n for $n \geq 2$, D_n for $n \geq 4$ even, E_7, E_8, F_4, H_3, H_4 and $I_2(2k)$ for $k \geq 2$.*

The equivalence of (b), (c) and (d) is well-known. For completeness we include a proof.

Proof of Corollary 4.7. The equivalence of (a) and (b) follows from Lemma 4.3 (iii) by using statements (ii) and (iii) of Proposition 4.6.

We prove that (b) and (d) are equivalent. As both conditions hold for a Cartesian product of two arrangements if and only if they hold for each arrangement, we may assume that $\mathcal{H}$ is a simple Coxeter arrangement. If $\mathcal{H}$ is of type A_n for $n \geq 2$, D_n for $n \geq 5$ odd, $I_2(m)$ for $m \geq 3$ odd or E_6, then Φ does not contain a family of n pairwise orthogonal pseudo-roots, so a fortiori condition (b) does not hold. If $\mathcal{H}$ is of type A_1, B_2 or $I_2(2k)$ with $k \geq 2$, then condition (b) is clear. Suppose that $\mathcal{H}$ is of type B_n with $n \geq 3$, D_n with $n \geq 4$ even, E_7, E_8, F_4, H_3 or H_4. It suffices to show that if $e \in E$ then the arrangement on H_e given by the vectors of $E \cap H_e$, which is $\mathcal{H}_e$ by statement (i) of Proposition 4.6, satisfies condition (b). This follows from a straightforward induction using Table 1.

Suppose that $\mathcal{H}$ satisfies (b). Then there exists a family of pairwise orthogonal pseudo-roots $(\alpha_1, \ldots, \alpha_n) \in \Phi$; in particular, this family generates V. Denote by s_α the reflection corresponding to a root α. We obtain that $-\operatorname{id}_V = s_{\alpha_1} \cdots s_{\alpha_n} \in W$, which is (c).

Finally, we show by induction on n that (c) implies (a). If $n = 1$, then (c) implies that $\mathcal{H}$ is not empty, hence even. Suppose that $n \geq 2$ and that we know the result for Coxeter arrangements on vector spaces of dimension $n - 1$. Let $e \in E$ and set $W_e = \{w \in W : w(e) = e\}$. By a theorem of Steinberg [16, Theorem 1.5], the group W_e is a reflection subgroup of W, hence it is generated by reflections s_f, for $f \in E$, where s_f is as before the orthogonal reflection in the hyperplane H_f. As $s_f(e) = e$ if and only if $(e, f) = 0$, the group W_e is exactly the subgroup of W generated by the reflections s_f for $(e, f) = 0$, in other words, it is the Coxeter group of the even restricted arrangement $\mathcal{H}_e$. As $-\operatorname{id}_V \in W$, we have $-s_e \in W$ and, as $-s_e(e) = e$, we conclude that $-s_e \in W_e$. But $-s_e$ acts by $-\operatorname{id}_{H_e}$ on the hyperplane H_e, so we conclude that the Coxeter arrangement $\mathcal{H}_e$ satisfies condition (c), hence that it is even by the induction hypothesis. So we have shown that $\mathcal{H}_e$ is even for every $e \in E$, which implies that $\mathcal{H}$ is even. $\square$

5. Sufficiently symmetric sets

We now turn our attention to a collection of measurable sets that we call sufficiently symmetric for which the evaluation of the pizza quantity is easier. We conclude the section by showing that for Coxeter arrangements a measurable set of finite volume that is stable with respect to the reflections in the arrangement is sufficiently symmetric.

Definition 5.1. For a nonempty n-dimensional arrangement $\mathcal{H}$, we say that a measurable subset of V with finite volume K is *sufficiently symmetric* with respect to $\mathcal{H}$ if, for every $0 \leq r \leq n-1$, for

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every even restriction sequence $(e_1, \ldots, e_r) \in P_r$ that is not maximal, and for every $a \in H^{\perp}_{e_1,\ldots,e_r}$, the pizza quantity $P(\mathcal{H}_{e_1,\ldots,e_r}, (K + a) \cap H_{e_1,\ldots,e_r})$ vanishes.

We can check this condition inductively.

Lemma 5.2. *Let $\mathcal{H} = (H_e)_{e \in E}$ be a nonempty hyperplane arrangement and let K be a measurable subset of V with finite volume. Then the following conditions are equivalent:*

- (a) *The set K is sufficiently symmetric with respect to $\mathcal{H}$.*
- (b) *The pizza quantity $P(\mathcal{H}, K)$ is equal to zero and for every $e \in E$ and for every $b \in H^{\perp}_e$, the set $(K + b) \cap H_e$ is sufficiently symmetric with respect to $\mathcal{H}_e$.*

Proof. Suppose that (a) holds. As $\mathcal{H}$ is nonempty, the empty even restriction sequence is not maximal, and we conclude that $P(\mathcal{H}, K) = 0$ by taking $r = 0$ in Definition 5.1. Now let $e \in E$ and $b \in H^{\perp}_e$. Let $1 \leq r \leq n - 1$ and $e_2, \ldots, e_r \in E$ such that $(e, e_2, \ldots, e_r) \in P_r - P$, and let $a \in H_e \cap H^{\perp}_{e,e_2,\ldots,e_r}$. We wish to show that $P(\mathcal{H}_{e,e_2,\ldots,e_r}, L) = 0$ where $L = (((K + b) \cap H_e) + a) \cap H_{e,e_2,\ldots,e_r} = (K + a + b) \cap H_{e,e_2,\ldots,e_r}$. This follows from the hypothesis on K because $b \in H^{\perp}_e \subset H^{\perp}_{e,e_2,\ldots,e_r}$.

Suppose that (b) holds. Let $0 \leq r \leq n - 1$, let $(e_1, \ldots, e_r) \in P_r - P$ and let $a \in H^{\perp}_{e_1,\ldots,e_r}$. We wish to show that $P(\mathcal{H}_{e_1,\ldots,e_r}, (K+a) \cap H_{e_1,\ldots,e_r}) = 0$. If $r = 0$ then $\mathcal{H}_{e_1,\ldots,e_r} = \mathcal{H}$, $H_{e_1,\ldots,e_r} = V$ and $a \in V^{\perp} = \{0\}$, so the desired result follows from the hypothesis. Suppose that $r \geq 1$. We write $a = \lambda e_1 + b$, with $\lambda \in \mathbb{R}$ and $b \in e_1^{\perp} \cap H^{\perp}_{e_1,\ldots,e_r} = H_{e_1} \cap H^{\perp}_{e_1,\ldots,e_r}$. Let $L = (K + \lambda e_1) \cap H_{e_1}$. By condition (b) the set L is sufficiently symmetric with respect to $\mathcal{H}_{e_1}$, so $P(\mathcal{H}_{e_1,\ldots,e_r}, (L + b) \cap H_{e_1,\ldots,e_r}) = 0$. As $(K + a) \cap H_{e_1,\ldots,e_r} = (L + b) \cap H_{e_1,\ldots,e_r}$, we are done. $\square$

For Coxeter arrangements there is a more natural condition on measurable sets, namely, that of being stable by the action of the Coxeter group. We verify this also behaves well in the even restricted arrangement setting.

Proposition 5.3. *Let $\mathcal{H}$ be a Coxeter arrangement in the vector space V with Coxeter group W. Let L be a measurable subset of V stable under the action of W. Let H_e be a hyperplane in $\mathcal{H}$ and let b be a scalar multiple of the vector e. Then the measurable subset $(L + b) \cap H_e$ in the space H_e is stable by the action of the Coxeter group of the even restricted arrangement $\mathcal{H}_e$.*

Remember that the arrangement $\mathcal{H}_e$ is also a Coxeter arrangement by Proposition 3.2.

Proof of Proposition 5.3. It is suffices to show that $(L+b) \cap H_e$ is stable under any reflection in V', where V' is a hyperplane in $\mathcal{H}_e$, that is, V' is a codimension 2 subspace of $\mathcal{H}$ with even intersection multiplicity and is contained in H_e. Fix such a subspace V'. By Proposition 3.2 there exists a vector $f \in E$ such that $(e, f) = 0$ and $V' = H_e \cap H_f$. Since the vectors e and f are orthogonal, the vector b lies in the hyperplane H_f. Since L is stable under the reflection in H_f, so is the translate $L + b$. Note that reflecting $(L + b) \cap H_e$ in V' (this takes place in H_e) is equivalent to reflecting $(L + b) \cap H_e$ in H_f. Hence $(L + b) \cap H_e$ is stable under the action of the Coxeter group of $\mathcal{H}_e$. $\square$

Corollary 5.4. *Suppose that $\mathcal{H}$ is a nonempty Coxeter arrangement and that K is a measurable subset of V of finite volume that is stable under the action of the elements of the Coxeter group W of $\mathcal{H}$. Then K is sufficiently symmetric with respect to the arrangement $\mathcal{H}$.*

Proof. We prove the result by induction on $n = \dim V$. If $n = 1$ then we have to show that $P(\mathcal{H}, K) = 0$. This follows from the hypothesis and from Corollary 2.3. This corollary applies because we assume that the arrangement $\mathcal{H}$ is not empty. Suppose that $n \geq 2$ and that we know the result for $n - 1$. If $e \in E$ and $b \in H^{\perp}_e$ then the set $(K + b) \cap H_e$ is stable by the Coxeter group of the arrangement $\mathcal{H}_e$ by Proposition 5.3, hence sufficiently symmetric with respect to $\mathcal{H}_e$ by the induction hypothesis. The result now follows from Lemma 5.2. $\square$

6. Evaluating the pizza quantity

We now evaluate the pizza quantity for sufficiently symmetric measurable sets, and deduce that it is zero for even Coxeter arrangements and sufficiently symmetric measurable sets of finite volume.

Theorem 6.1. *Suppose that K is a measurable subset of V with finite volume, and that it is sufficiently symmetric with respect to $\mathcal{H}$. Then for every $a \in V$ we have*

$$P(\mathcal{H}, K + a) = \sum_{(e_1,\ldots,e_r) \in P/\sim} 2^r (-1)^{e_r \circ \cdots \circ e_1} (a, e_1)(\pi_{e_1}(a), e_2) \cdots (\pi_{e_1,\ldots,e_{r-1}}(a), e_r)$$

$$\cdot \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} \mathrm{Vol}_{H_{e_1,\ldots,e_r}}(K_{e_1,\ldots,e_r}(t_1,\ldots,t_r))dt_r \cdots dt_2 dt_1,$$

where, for every $(e_1,\ldots,e_r) \in E^r$, we denote by $\pi_{e_1,\ldots,e_r}$ the orthogonal projection on $H_{e_1,\ldots,e_r}$ and by $K_{e_1,\ldots,e_r}(t_1,\ldots,t_r)$ the measurable subset

$$H_{e_1,\ldots,e_r} \cap (K + t_1(a - \pi_{e_1}(a)) + t_2(\pi_{e_1}(a) - \pi_{e_1,e_2}(a)) + \cdots + t_r(\pi_{e_1,\ldots,e_{r-1}}(a) - \pi_{e_1,\ldots,e_r}(a)))$$

of $H_{e_1,\ldots,e_r}$.

Proof. Iterate Theorem 3.4 in the following form: $P(\mathcal{H}, K + a) = P(\mathcal{H}, K) + \cdots$. At each step use the sufficiently symmetric condition to note that the $P(\mathcal{H}, K)$ terms vanish. □

Remark 6.2. We could use other projections instead of the orthogonal projections $\pi_{e_1,\ldots,e_r}$. For Coxeter arrangements, it is natural to use orthogonal projections, but for other arrangements a different choice might be more appropriate. We keep the discussion to the orthogonal projections because there is no canonical general choice and the statements are more straightforward.

Theorem 6.3. *Suppose that $\mathcal{H}$ is an even hyperplane arrangement, and define the homogenous degree n polynomial function $f_{\mathcal{H}} : V \longrightarrow \mathbb{R}$ by*

$$f_{\mathcal{H}}(a) = \frac{2^n}{n!} \sum_{(e_1,\ldots,e_n) \in P_n/\sim} (-1)^{e_r \circ \cdots \circ e_1} (a, e_1)(\pi_{e_1}(a), e_2) \cdots (\pi_{e_1,\ldots,e_{n-1}}(a), e_n).$$

Then for any measurable subset K of V of finite volume that is sufficiently symmetric with respect to $\mathcal{H}$, and for every $a \in V$ such that

$$-(t_1(a - \pi_{e_1}(a)) + t_2(\pi_{e_1}(a) - \pi_{e_1,e_2}(a)) + \cdots + t_r(\pi_{e_1,\ldots,e_{r-1}}(a) - \pi_{e_1,\ldots,e_r}(a))) \in K$$

for $0 \le t_r \le t_{r-1} \le \cdots \le t_1 \le 1$, we have

$$P(\mathcal{H}, K + a) = f_{\mathcal{H}}(a).$$

Remark 6.4. The conditions of Theorem 6.3 on K and a hold in the following cases:

(1) The set K is convex (of finite volume), sufficiently symmetric with respect to $\mathcal{H}$ and $0 \in K + a$.
(2) The arrangement $\mathcal{H}$ is Coxeter with Coxeter group W, and the set K is stable by W and contains the convex hull of the finite set $\{w(-a) : w \in W\}$ (as W contains $-\mathrm{id}_V$ because $\mathcal{H}$ is even, it is equivalent to say that K contains the convex hull of the set $\{w(a) : w \in W\}$).
(3) The arrangement $\mathcal{H}$ is Coxeter, the set K is convex (of finite volume) and stable by its Coxeter group, and $0 \in K + a$.

Proof of Theorem 6.3. As $\mathcal{H}$ is even, we have $P/\sim = P_n/\sim$. Moreover, by the conditions on K and a, for every $(e_1,\ldots,e_r) \in P_r$ and every $(t_1,\ldots,t_r) \in \mathbb{R}^r$ such that $0 \le t_r \le t_{r-1} \le \cdots \le$

$t_1 \leq 1$, the subset $K_{e_1,\ldots,e_r}(t_1,\ldots,t_r)$ of $H_{e_1,\ldots,e_n}$ contains 0. Thus, if $r = n$, we obtain that $K_{e_1,\ldots,e_n}(t_1,\ldots,t_n) = H_{e_1,\ldots,e_n} = \{0\}$, and hence

$$\mathrm{Vol}_{H_{e_1,\ldots,e_n}}(K_{e_1,\ldots,e_n}(t_1,\ldots,t_n)) = 1.$$

The corollary then follows from Theorem 6.1 and from the fact that

$$\int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \cdots dt_2 dt_1 = \mathrm{Vol}(\{(t_1,t_2,\ldots,t_n) : 0 \leq t_n \leq \cdots \leq t_2 \leq t_1 \leq 1\}) = \frac{1}{n!}. \qquad \square$$

Corollary 6.5. *Suppose that $\mathcal{H}$ is a Coxeter arrangement, and let $0 \leq r \leq n$ be the integer such that $P = P_r$. Let K be a measurable subset of V of finite volume that is sufficiently symmetric with respect to $\mathcal{H}$ (for example, K could be stable by the Coxeter group of $\mathcal{H}$). Then for every $a \in V$ we have*

$$P(\mathcal{H}, K + a) = 2^r \sum_{(e_1,\ldots,e_r) \in E_r^{(0)}} (-1)^{e_r \circ \cdots \circ e_1}(a, e_1)(a, e_2) \cdots (a, e_r)$$

$$\int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} \mathrm{Vol}_{H_{e_1,\ldots,e_r}}(H_{e_1,\ldots,e_r} \cap (K \cap t_1(a,e_1) + t_2(a,e_2) + \cdots + t_r(a,e_r)))dt_r \cdots dt_2 dt_1.$$

Remark 6.6. *If K is the ball $\mathbb{B}(0,R)$ and $\|a\| \leq R$ then for every $(e_1,\ldots,e_r) \in P$ and $(t_1,\ldots,t_r) \in \mathbb{R}^r$ such that $0 \leq t_r \leq \cdots \leq t_1 \leq 1$, the set $K_{e_1,\ldots,e_r}(t_1,\ldots,t_r)$ of Theorem 6.1 is a ball of radius $\sqrt{R(t_1,\ldots,t_r)}$, where*

$$R(t_1,\ldots,t_r) = R^2 - t_1^2\|a - \pi_{e_1}(a)\|^2 - t_2^2\|\pi_{e_1}(a) - \pi_{e_1,e_2}(a)\|^2 - \cdots - t_r^2\|\pi_{e_1,\ldots,e_{r-1}}(a) - \pi_{e_1,\ldots,e_r}(a)\|^2.$$

So we can conclude the following corollary.

Corollary 6.7. *Suppose that the ball $\mathbb{B}(0,R)$ is sufficiently symmetric with respect to $\mathcal{H}$. Then for every $R \geq 0$ and every $a \in V$ such that $\|a\| \leq R$, we have*

$$P(\mathcal{H}, \mathbb{B}(a,R)) = \frac{2^r \pi^{r/2}}{\Gamma(r/2 + 1)}$$

$$\cdot \sum_{(e_1,\ldots,e_r) \in P/\sim} (-1)^{e_r \circ \cdots \circ e_1}(a,e_1)(\pi_{e_1}(a),e_2) \cdots (\pi_{e_1,\ldots,e_{r-1}}(a),e_r)$$

$$\cdot \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} R(t_1,\ldots,t_r)^{(n-r)/2} dt_r \cdots dt_2 dt_1,$$

where $R(t_1,\ldots,t_r)$ is given in Remark 6.6. In particular, if $\mathcal{H}$ is a Coxeter arrangement, let $0 \leq r \leq n$ be the integer such that $P = P_r$. Then the ball $\mathbb{B}(0,R)$ is sufficiently symmetric with respect to $\mathcal{H}$, and for every $R \geq 0$ and every $a \in V$ such that $\|a\| \leq R$, we have

$$P(\mathcal{H}, \mathbb{B}(a,R)) = \frac{2^r \pi^{r/2}}{\Gamma(r/2 + 1)} \sum_{(e_1,\ldots,e_r) \in E_r^{(0)}} (-1)^{e_r \circ \cdots \circ e_1}(a,e_1)(a,e_2) \cdots (a,e_r)$$

$$\cdot \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{r-1}} (R^2 - t_1^2(a,e_1)^2 - t_2^2(a,e_2)^2 - \cdots - t_r^2(a,e_r)^2)^{(n-r)/2} dt_r \cdots dt_2 dt_1.$$

Proposition 6.8. *Suppose that $\mathcal{H} = (H_e)_{e \in E}$ is an even Coxeter arrangement. Let $n = \dim(V)$. Then the polynomial $f_{\mathcal{H}}$ of Theorem 6.3 is given by*

$$f_{\mathcal{H}}(a) = \begin{cases} 2^n \cdot \prod_{e \in E}(a,e) & \text{if } \mathcal{H} \text{ is of type } A_1^n, \\ 0 & \text{otherwise.} \end{cases}$$

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More generally, if $\mathcal{H}$ is an even arrangement that has a Coxeter subarrangement $\mathcal{H}'$ whose Coxeter group preserves $\mathcal{H}$, then

(i) if $\mathcal{H}'$ has at least $n+1$ hyperplanes then $f_{\mathcal{H}} = 0$;
(ii) if $\mathcal{H}'$ has n hyperplanes then $f_{\mathcal{H}}$ is a scalar multiple of the function $a \longmapsto \prod_{e \in E'}(a, e)$, where E' is the subset of E corresponding to $\mathcal{H}'$.

Proof. All even Coxeter arrangements in V have at least $n+1$ hyperplanes, except for the arrangement of type A_1^n, which has n hyperplanes. So it suffices to prove statements (i) and (ii), and to calculate the leading coefficient of $f_{\mathcal{H}}$ for $\mathcal{H}$ of type A_1^n.

Suppose that $f_{\mathcal{H}} \neq 0$. Let X be the hypersurface $\{f_{\mathcal{H}} = 0\}$. Then X is a hypersurface of degree n, and it contains all the hyperplanes of $\mathcal{H}'$, so the arrangement $\mathcal{H}'$ has at most n hyperplanes. Suppose that $\mathcal{H}'$ has exactly n hyperplanes. Then $X = \bigcup_{e \in E'} H_{e'}$, thus the polynomial $f_{\mathcal{H}}$ is of the form $a \longmapsto c \cdot \prod_{e \in E}(a, e)$ where c is a constant. Now suppose that $\mathcal{H} = \mathcal{H}'$ is of type A_1^n. To calculate c, we use Theorem 6.3. Let $K = [-1,1]^n$ and $a = (a_1, \dots, a_n) = (1, \dots, 1)$. Then the cube $K + a$ is entirely in the first orthant and hence $c = P(\mathcal{H}, K + a) = \text{Vol}(K) = 2^n$. $\square$

Theorem 6.9. *Suppose that $\mathcal{H}$ is an even Coxeter arrangement in $\mathbb{R}^n$ and that K is a measurable subset of V of finite volume and stable by the Coxeter group of $\mathcal{H}$. Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ such that K contains the convex hull of the set $\{w(a) : w \in W\}$. Then the pizza quantity $P(\mathcal{H}, K + a)$ is given by*

$$P(\mathcal{H}, K + a) = \begin{cases} 2^n \cdot a_1 \cdots a_n & \text{if } \mathcal{H} \text{ is of type } A_1^n, \\ 0 & \text{otherwise.} \end{cases}$$

Here if $\mathcal{H}$ has type A_1^n we assume that it is given by the hyperplanes $\{x_i = 0 : 1 \le i \le n\}$ and that the base chamber is $T_0 = (\mathbb{R}_{>0})^n$.

Proof. The set K satisfies the conditions of Theorem 6.3, so the result follows from Proposition 6.8. $\square$

Example 6.10. Suppose that $|E| = n$ and that $\mathcal{H}$ is essential. This implies that $\mathcal{H}$ is even and that for every $0 \le r \le n$ the set P_r is the set of $(e_1, \dots, e_r) \in E^r$ such that $e_i \neq e_j$ for $i \neq j$. [1] Write $E = \{e_1, \dots, e_n\}$. Then for any $a \in V$ we have

$$f_{\mathcal{H}}(a) = \frac{2^n}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^{e_{\sigma(n)} \circ \cdots \circ e_{\sigma(1)}}$$
$$\cdot (a, e_{\sigma(1)})(\pi_{e_{\sigma(1)}}(a), e_{\sigma(2)})(\pi_{e_{\sigma(1)}, e_{\sigma(2)}}(a), e_{\sigma(3)}) \cdots (\pi_{e_{\sigma(1)}, \dots, e_{\sigma(n-1)}}(a), e_{\sigma(n)}).$$

For arrangements that are not Coxeter, we do not know a general way to find sufficiently symmetric sets, or even to decide whether the ball $\mathbb{B}(0,1)$ is sufficiently symmetric. We end this section with two low-dimensional examples.

Example 6.11. Suppose that $V = \mathbb{R}^2$ and $\mathcal{H}$ is a line arrangement. Then $\mathcal{H}$ is even if and only if it is nonempty and has an even number of lines, say $2m$ where m is a positive integer. Suppose that this is the case, and that one of the lines in $\mathcal{H}$ is the horizontal axis $\{x_2 = 0\}$. Let $\theta_1, \dots, \theta_{2m-1}$ be the angles between the other lines and the horizontal axis, ordered so that $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_{2m-1} < \theta_{2m} = \pi$. If $\ell \in \mathcal{H}$ and $a \in \ell^\perp$ then $\ell \cap (\mathbb{B}(0,1) + a)$ is always a segment centered at the origin (or empty). Thus the disc $\mathbb{B}(0,1)$ is sufficienty symmetric with respect to $\mathcal{H}$ if and only if $P(\mathcal{H}, \mathbb{B}(0,1)) = 0$. This is equivalent to the condition that $\sum_{i=1}^m (\theta_{2i} - \theta_{2i-1}) = \sum_{i=1}^m (\theta_{2i-1} - \theta_{2i-2})$, that is, $\sum_{j=1}^{2m} (-1)^j \cdot \theta_j = \pi/2$. If this condition is satisfied then the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ is given by $P(\mathcal{H}, \mathbb{B}(a, R)) = f_{\mathcal{H}, T_0}(a)$ if $0 \in \mathbb{B}(a, R)$, that is, when $\|a\| \le R$, and in particular it is independent of the radius R of the ball.

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[1] More generally, these two statements hold for any simple hyperplane arrangement.

Example 6.12. In $V = \mathbb{R}^3$ consider the arrangement $\mathcal{H}$ given by the following seven hyperplanes:

$$H_1 = \{x_1 = 0\}, \quad H_2 = \{x_1 = \alpha \cdot x_2\}, \quad H_4 = \{x_2 = 0\}, \quad H_5 = \{x_3 = \beta \cdot x_2\}, \quad H_7 = \{x_3 = 0\},$$
$$H_3 = \{x_1 = -\alpha \cdot x_2\}, \qquad\qquad\qquad H_6 = \{x_3 = -\beta \cdot x_2\},$$

where α and β are positive real numbers. We also fix a base chamber T_0 of $\mathcal{H}$. We claim that this arrangement $\mathcal{H}$ is even, and that the unit ball $\mathbb{B}(0,1)$ is sufficiently symmetric with respect to $\mathcal{H}$.

First, for every chamber T of $\mathcal{H}$, the set $-T$ is also a chamber and $(-1)^{-T} = -(-1)^T$ because $\mathcal{H}$ has an odd number of hyperplanes. For any centrally symmetric measurable set of finite volume K, and in particular for the ball, we have $P(\mathcal{H}, K) = 0$. We can then apply Lemma 5.2 to check the second statement once we know that $\mathcal{H}$ is even. Note also that for every $H \in \mathcal{H}$ and for every $a \in H^\perp$, the intersection $H \cap (\mathbb{B}(0,1) + a)$ is a disk centered at the origin in H.

For every $1 \le i \le 7$, let $\mathcal{H}_i$ be the even restricted arrangement induced by $\mathcal{H}$ on H_i. We need to check that for every i the arrangement $\mathcal{H}_i$ is even and a disk centered at the origin is sufficiently symmetric with respect to $\mathcal{H}_i$.

- The three arrangements $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are all isometric to a four line arrangement, as in Example 6.11. However, the adjacent angles for $\mathcal{H}_1$ are given by $\eta_1 = \eta_4 = \arctan(\beta)$ and $\eta_2 = \eta_3 = \pi/2 - \arctan(\beta)$, where $\theta_i = \eta_1 + \cdots + \eta_i$, whereas, the angles for $\mathcal{H}_2$ and $\mathcal{H}_3$ are $\eta_1 = \eta_4 = \arctan(\beta/\sqrt{\alpha^2 + 1})$ and $\eta_2 = \eta_3 = \pi/2 - \arctan(\beta/\sqrt{\alpha^2 + 1})$.
- The cases of the arrangements $\mathcal{H}_5$, $\mathcal{H}_6$ and $\mathcal{H}_7$ are symmetric to $\mathcal{H}_3$, $\mathcal{H}_2$ and $\mathcal{H}_1$.
- The arrangement $\mathcal{H}_4$ has two lines and has type A_1^2.

We conclude that the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ is given by $P(\mathcal{H}, \mathbb{B}(a, R)) = f_{\mathcal{H},T_0}(a)$, and hence it is independent of the radius R of the ball. Furthermore, Proposition 6.8 (ii) implies that $f_{\mathcal{H},T_0}(a) = c \cdot a_1 a_2 a_3$ where c is a constant. Note that we could also get this result directly from Theorem 7.6.

7. THE CASE OF THE BALL AND CONVEX BODIES BOUNDED BY QUADRATIC SURFACES

We now revisit the case when the measurable set K is an n-dimensional ball. We show in particular that the pizza quantity vanishes for the arrangements E_6 and A_n where $n \equiv 0, 1 \bmod 4$. Recall that V is an n-dimensional vector space endowed with an inner product $(\cdot, \cdot)$ and that $\mathcal{H} = (H_e)_{e \in E}$ is an arrangement with base chamber T_0.

Definition 7.1. We say that the hyperplane arrangement $\mathcal{H}$ satisfies *the parity condition* if $|\mathcal{H}|$ and $\dim(V)$ have the same parity.

Lemma 7.2.   (i) *Suppose that $\mathcal{H}$ is not the empty arrangement. Then the following three statements are equivalent:*
- (a) *The arrangement $\mathcal{H}$ satisfies the parity condition.*
- (b) *There exists $e \in E$ such that the even restricted arrangement $\mathcal{H}_e$ satisfies the parity condition.*
- (c) *For every $e \in E$, the even restricted arrangement $\mathcal{H}_e$ satisfies the parity condition.*

(ii) *The following statements are equivalent:*
- (a) *The arrangement $\mathcal{H}$ satisfies the parity condition.*
- (b) *There exists $(e_1, \ldots, e_r) \in P$ such that $\dim(H_{e_1,\ldots,e_r})$ is even.*
- (c) *For every $(e_1, \ldots, e_r) \in P$, we have that $\dim(H_{e_1,\ldots,e_r})$ is even.*

(iii) *If $\mathcal{H}$ is an even arrangement then it satisfies the parity condition.*

(iv) *Suppose that $\mathcal{H}$ is a Coxeter arrangement. We write $V = V_0 \oplus V_1$, where $V_0 = \bigcap_{H \in \mathcal{H}} H$ and $V_1 = V_0^\perp$, and we denote by $\mathcal{H}_1$ the restriction of $\mathcal{H}$ to V_1, that is, the arrangement $\{H \cap V_1 : H \in \mathcal{H}\}$. Note that $\mathcal{H}_1$ is an essential Coxeter arrangement and that $\mathcal{H}$ is the Cartesian product of $\mathcal{H}_1$ and of the empty arrangement on V_0. We can also decompose $\mathcal{H}_1$*

as a product of simple Coxeter arrangements. Let r be the number of arrangements in this decomposition that are of type A_n with $n \equiv 2, 3 \bmod 4$, D_n for $n \geq 5$ odd or $I_2(2k + 1)$ for $k \geq 2$. Then $\mathcal{H}$ satisfies the parity condition if and only if $r + \dim(V_0)$ is even.

In particular, if $\mathcal{H}$ is an essential Coxeter arrangement then it satisfies the parity condition if and only it has an even number of simple factors of types A_n with $n \equiv 2, 3 \bmod 4$, D_n for $n \geq 5$ odd or $I_2(2k + 1)$ for $k \geq 2$.

Proof. We begin by proving (i). As E is nonempty, condition (c) clearly implies condition (b). Let $e \in E$. We denote by $\sim$ the equivalence relation on $E - \{e\}$ defined by $e' \sim e''$ if and only if $H_e \cap H_{e'} = H_e \cap H_{e''}$. Let E_1, respectively E_2, be the set of $e' \in E - \{e\}$ such that the intersection multiplicity of $H_e \cap H_{e'}$ is odd, respectively even. Then E_1 and E_2 are unions of equivalence classes of the relation $\sim$. Since the element e is not in the set $E - \{e\}$, the intersection multiplicity of $H_e \cap H_{e'}$ has the opposite parity of the cardinality of the equivalence class e' belongs to. As all the equivalence classes contained in E_1 have even cardinality, the set E_1 also has even cardinality. Moreover, we have a surjective map from E_2 to $\mathcal{H}_e$ sending $e' \in E_2$ to $H_e \cap H_{e'}$, and the fibers of this map are the equivalence classes contained in E_2, all of which have odd cardinality. Thus we obtain $|E_2| \equiv |\mathcal{H}_e| \bmod 2$. We deduce that $|\mathcal{H}| = |E| = 1 + |E_1| + |E_2| \equiv 1 + |\mathcal{H}_e| \bmod 2$. This shows that $\mathcal{H}$ satisfies the parity condition if and only if $\mathcal{H}_e$ does. As e was arbitrary, this argument proves that (a) implies (c) and that (b) implies (a).

Next we prove (ii) by induction on $|\mathcal{H}|$. Note that (c) always implies (b), so we just need to prove that (a) implies (c) and that (b) implies (a). If $\mathcal{H}$ is empty then $P = \varnothing$, and the three conditions (a), (b) and (c) are equivalent to the fact that $\dim(V)$ is even. Suppose that $|\mathcal{H}| \geq 1$. If $\mathcal{H}$ satisfies the parity condition, let $(e_1, \ldots, e_r) \in P$. We have $r \geq 1$ because $\mathcal{H}$ is not empty, so $\mathcal{H}_{e_1}$ satisfies the parity condition by (ii). As $(e_2, \ldots, e_r)$ is a maximal even restriction sequence for $\mathcal{H}_{e_1}$, the induction hypothesis implies that $\dim(H_{e_1, \ldots, e_r})$ is even. Conversely, suppose that there exists $(e_1, \ldots, e_r) \in P$ such that $\dim(H_{e_1, \ldots, e_r})$ is even. By the induction hypothesis $\mathcal{H}_{e_1}$ satisfies the parity condition so $\mathcal{H}$ satisfies the parity condition by (ii).

If $\mathcal{H}$ is even then $P = P_n$, so $H_{e_1, \ldots, e_n} = \{0\}$ for every $(e_1, \ldots, e_n) \in P$. Hence (iii) follows from (ii).

Finally, we prove (iv). Simple Coxeter arrangements satisfy the parity condition if and only if they are of types A_n with $n \equiv 0, 1 \bmod 4$, B_n, D_n with $n \geq 4$ even, E_6, E_7, E_8, F_4, or $I_2(2k)$ with $k \geq 2$. Thus $|\mathcal{H}| + \dim(V) \equiv r + \dim(V_0) \bmod 2$, which implies the result. $\qquad\square$

Remark 7.3. By Lemma 7.2 the evenness condition implies the parity condition, but it is much stronger condition. (In dimensions at most 2, the two conditions are equivalent). For example, the Coxeter arrangements of type $A_2 \times A_3$, E_6 and A_n for $n \equiv 0, 1 \bmod 4$ satisfy the parity condition but not the evenness condition.

If $\mathcal{H}$ satisfies the parity condition then $(n - r)/2$ is an integer for every $(e_1, \ldots, e_r) \in P$, so if the ball $\mathbb{B}(0, R)$ is sufficiently symmetric with respect to $\mathcal{H}$, Corollary 6.7 implies that the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ is polynomial in a and R as long as $\|a\| \leq R$.

In this section we will see that we do not need the condition that the ball is sufficiently symmetric for this result. We will actually consider slightly more general "balls" that are convex bodies bounded by quadratic hypersurfaces. Let q be a positive definite quadratic form on V, so that $(q(x + y) - q(x) - q(y))/2$ is an inner product on V. For $a \in V$ and $R \geq 0$, we write

$$\mathbb{B}_q(a, R) = \{x \in V : q(x - a) \leq R^2\}.$$

If q is the quadratic form defined by $q(x) = (x, x)$ then $\mathbb{B}_q(a, R)$ is the usual ball $\mathbb{B}(a, R)$ with center a and radius R. Note also that $\mathbb{B}_q(a, R) = a + \mathbb{B}_q(0, R)$ and $\mathbb{B}_q(0, R) = R \cdot \mathbb{B}_q(0, 1)$.

Theorem 7.4. *Let C be a closed convex polyhedral cone, that is, an intersection of closed half-spaces in V. Then there exists a polynomial function $g_{C,q}$ on $\mathbb{R} \times V$ such that:*

(a) *The polynomial $g_{C,q}$ is homogeneous of degree $n = \dim(V)$.*
(b) *For every $(R, a) \in \mathbb{R} \times V$, we have $g_{C,q}(R, a) = g_{C,q}(-R, a)$, that is, $g_{C,q}$ contains only terms of even degree in the first variable R.*
(c) *For every $(R, a) \in \mathbb{R} \times V$ such that $q(a) \leq R^2$, we have*

$$\mathrm{Vol}(C \cap \mathbb{B}_q(a, R)) + (-1)^{\dim(V)} \mathrm{Vol}((-C) \cap \mathbb{B}_q(a, R)) = g_{C,q}(R, a).$$

Proof. We prove the result by induction on the dimension of V. If $\dim(V) = 0$, then $V = C = \{0\}$ and $\mathrm{Vol}(C \cap \mathbb{B}_q(a, R)) + (-1)^{\dim(V)} \mathrm{Vol}((-C) \cap \mathbb{B}_q(a, R)) = 2$ if $q(a) \leq R^2$, so we can take $g_{C,q} = 2$.

Suppose that $\dim(V) \geq 1$ and that we know the result for lower-dimensional inner product spaces. Let $(R, a) \in \mathbb{R} \times V$ such that $q(a) \leq R^2$. Let $\mathcal{H}$ be the set of hyperplanes containing a facet of C. For every $H \in \mathcal{H}$, we denote by e_H the unit normal vector of H that points to the half-space containing C, and we set $E = \{e_H : H \in \mathcal{H}\}$. By Theorem 3.4(i) we have

$$\mathrm{Vol}(C \cap \mathbb{B}_q(a, R)) + (-1)^{\dim(V)} \mathrm{Vol}((-C) \cap \mathbb{B}_q(a, R))$$
$$- \left(\mathrm{Vol}(C \cap \mathbb{B}_q(0, R)) + (-1)^{\dim(V)} \mathrm{Vol}((-C) \cap \mathbb{B}_q(0, R)) \right)$$
$$= 2 \cdot \sum_{e \in H} (a, e) \cdot \int_0^1 \left(\mathrm{Vol}(C \cap H_e \cap \mathbb{B}_q(sa, R)) + (-1)^{\dim(H_e)} \mathrm{Vol}((-C) \cap H_e \cap \mathbb{B}_q(sa, R)) \right) ds.$$

Indeed, the facets of C and $-C$ are exactly the relative interiors of the intersections $C \cap H_e$, respectively $(-C) \cap H_e$, for $e \in E$. Moreover, for every $e \in E$, as the vector e points towards C, hence away from $-C$, we have $(-1)^{\mathring{C}}(-1)^{U \circ e} = 1$ if U is the relative interior of $C \cap H_e$ and $(-1)^{\mathring{C}}(-1)^{U \circ e} = -1$ if U is the relative interior of $(-C) \cap H_e$.

Let $e \in E$. We denote by $\pi_{q,e}$ the orthogonal projection on H_e with respect to the inner product $(q(x + y) - q(x) - q(y))/2$ corresponding to the quadratic form q. Then for all $a \in V$ and $x \in H_e$, we have

$$q(x - a) = q(x - \pi_{q,e}(a)) + q(a - \pi_{q,e}(a)).$$

In particular, the convex body $\mathbb{B}_q(sa, R) \cap H_e$ is given by

$$\mathbb{B}_q(sa, R) \cap H_e = \mathbb{B}_{q_e} \left(s\pi_{q,e}(a), \sqrt{R^2 - s^2 q(a - \pi_{q,e}(a))} \right),$$

where q_e is the restriction of q to H_e. Hence the induction hypothesis applied to H_e yields that

$$\mathrm{Vol}(C \cap H_e \cap \mathbb{B}_q(sa, R)) + (-1)^{\dim(H_e)} \mathrm{Vol}((-C) \cap H_e \cap \mathbb{B}_q(sa, R))$$
$$= g_{C \cap H_e, q_e} \left(\sqrt{R^2 - s^2 \cdot q(a - \pi_{q,e}(a))}, s\pi_{q,e}(a) \right).$$

By conditions (a) and (b), there exist homogeneous polynomial functions $g_{C \cap H_e, i}$ of degree $n - 1 - 2i$ on H_e such that, for every $(t, x) \in \mathbb{R} \times H_e$, we have $g_{C \cap H_e}(t, x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} t^{2i} \cdot g_{C \cap H_e, i}(x)$. We obtain

$$\int_0^1 \left(\mathrm{Vol}(C \cap H_e \cap \mathbb{B}_q(sa, R)) + (-1)^{\dim(H_e)} \mathrm{Vol}((-C) \cap H_e \cap \mathbb{B}_q(sa, R)) \right) ds$$
$$= \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} g_{C \cap H_e, i}(\pi_{q,e}(a)) \cdot \int_0^1 \left(R^2 - s^2 \cdot q(a - \pi_{q,e}(a)) \right)^i \cdot s^{n-1-2i} \, ds.$$

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We define a function $g_C^{(0)} : \mathbb{R} \times V \longrightarrow \mathbb{R}$ by

$$g_C^{(0)}(t,x) = 2 \cdot \sum_{e \in E}(x,e) \cdot \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} g_{\mathcal{H}_e,i}(\pi_{q,e}(x)) \cdot \int_0^1 \left(t^2 - s^2 \cdot q(x - \pi_{q,e}(x))\right)^i \cdot s^{n-1-2i}\, ds,$$

for $(t,x) \in \mathbb{R} \times V$. As the functions $x \longmapsto (x,e)$ and $x \longmapsto \pi_{q,e}(x)$ are linear in x and the function $x \longmapsto q(x)$ is quadratic in x, the function $g_C^{(0)}$ is a homogeneous polynomial of degree $\dim(V)$ that satisfies condition (b), and we have

$$\mathrm{Vol}(C \cap \ \mathbb{B}_q(a,R)) + (-1)^{\dim(V)}\mathrm{Vol}((-C) \cap \ \mathbb{B}_q(a,R))$$
$$= \mathrm{Vol}(C \cap \ \mathbb{B}_q(0,R)) + (-1)^{\dim(V)}\mathrm{Vol}((-C) \cap \ \mathbb{B}_q(0,R)) + g_C^{(0)}(R,a).$$

Suppose that $n = \dim(V)$ is even. As $\mathrm{Vol}(Z \cap \ \mathbb{B}_q(0,R)) = R^n \cdot \mathrm{Vol}(Z \cap \ \mathbb{B}_q(0,1))$ for $Z = C$ or $Z = -C$, the function $g_{C,q}$ defined by

$$g_{C,q}(t,x) = t^n \cdot (\mathrm{Vol}(C \cap \ \mathbb{B}_q(0,1)) + \mathrm{Vol}((-C) \cap \mathbb{B}_q(0,1))) + g_C^{(0)}(t,x)$$

satisfies the desired properties. Suppose that $\dim(V)$ is odd. Then, as $\mathbb{B}_q(0,R)$ is centrally symmetric, we have $\mathrm{Vol}(C \cap \ \mathbb{B}_q(0,R)) - \mathrm{Vol}((-C) \cap \mathbb{B}_q(0,R)) = 0$ for every $R \geq 0$, so the function $g_{C,q} = g_C^{(0)}$ satisfies the desired properties. $\qquad\square$

Corollary 7.5. *Suppose that $\mathcal{H}$ satisfies the parity condition. Then there exists a polynomial function $g_{\mathcal{H},q}$ on $\mathbb{R} \times V$ such that:*

(a) *The polynomial $g_{\mathcal{H},q}$ is homogeneous of degree $n = \dim(V)$.*
(b) *For every $(R,a) \in \mathbb{R} \times V$, we have $g_{\mathcal{H},q}(R,a) = g_{\mathcal{H},q}(-R,a)$, that is, $g_{\mathcal{H},q}$ contains only terms of even degree in the first variable R.*
(c) *For every $(R,a) \in \mathbb{R} \times V$ such that $q(a) \leq R^2$, we have $P(\mathcal{H}, \mathbb{B}_q(a,R)) = g_{\mathcal{H},q}(R,a)$.*

Proof. As $\mathcal{H}$ satisfies the parity condition, we have $(-1)^{-T} = (-1)^{\dim(V)}(-1)^T$ for every chamber T of $\mathcal{H}$, hence $P(\mathcal{H}, \mathbb{B}_q(a,R))$ is an alternating sum of quantities as in Theorem 7.4. More precisely, choose a subset $\mathscr{T}'$ of $\mathscr{T}(\mathcal{H})$ such that, for every chamber T of $\mathcal{H}$, exactly one of T or $-T$ is in $\mathscr{T}'$. Then, for all $a \in V$ and $R \in \mathbb{R}_{\geq 0}$, we have

$$P(\mathcal{H}, \mathbb{B}_q(a,R)) = \sum_{T \in \mathscr{T}'}(-1)^T\left(\mathrm{Vol}(T \cap \ \mathbb{B}_q(a,R)) + (-1)^{\dim(V)}\mathrm{Vol}((-T) \cap \ \mathbb{B}_q(a,R))\right).$$

So it suffices to take

$$g_{\mathcal{H},q} = \sum_{T \in \mathscr{T}'}(-1)^T g_{\overline{T},q}. \qquad\qquad\square$$

Theorem 7.6. *Let q be a positive definite quadratic form on V. Suppose that $\mathcal{H}$ satisfies the parity condition and that it contains a Coxeter arrangement $\mathcal{H}'$ whose Coxeter group preserves $\mathcal{H}$ and the convex body $\mathbb{B}_q(0,1)$. (For example, these conditions are satisfied if $\mathcal{H} = \mathcal{H}'$ is a Coxeter arrangement and q is given by $q(x) = (x,x)$.) Let $(R,a) \in \mathbb{R} \times V$ such that $q(a) \leq R^2$, equivalently $0 \in \mathbb{B}_q(a,R)$.*

(i) *If $|\mathcal{H}'| > \dim(V)$, so in particular $\mathcal{H}'$ is not of type $A_1^{\dim(V)}$, then the pizza quantity $P(\mathcal{H}, \mathbb{B}_q(a,R))$ vanishes.*
(ii) *If $|\mathcal{H}'| = \dim(V)$ (for example if $\mathcal{H}'$ is of type $A_1^{\dim(V)}$, or the product of a type A_2 arrangement on $\mathbb{R}^2$ and the empty arrangement on $\mathbb{R}$), then there exists a constant c independent of the center a and the radius R such that $P(\mathcal{H}, \mathbb{B}_q(a,R)) = c \cdot \prod_{e \in E'}(a,e)$, where E' is the subset of E corresponding to the hyperplanes of $\mathcal{H}'$. In particular, the pizza quantity $P(\mathcal{H}, \mathbb{B}_q(a,R))$ is independent of the radius R.*

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Proof. The proof is very similar to that of Proposition 6.8. Let W be the Coxeter group of $\mathcal{H}'$. Fix $R > 0$, and define a function $h : V \longrightarrow \mathbb{R}$ by $h(x) = g_{\mathcal{H},q}(R, x)$. Then h is polynomial of degree at most $\dim(V)$, and we have $P(\mathcal{H}, \mathbb{B}_q(a, R)) = h(a)$ for every $a \in V$ such that $q(a) \leq R^2$. By Corollary 2.3 this implies that $h(w(x)) = (-1)^w \cdot h(x)$ for every $w \in W$ and $x \in V$, and in particular that $h(x) = 0$ if x is on one of the hyperplanes of $\mathcal{H}'$. If h is nonzero then the hypersurface $\{h = 0\}$ is of degree at most $\dim(V)$ and contains $\bigcup_{H \in \mathcal{H}'} H$. But this is impossible in the situation of (i), so we conclude that $h = 0$ in that situation. In the situation of (ii), this is possible and implies that h is of the form $x \longmapsto c(R) \cdot \prod_{e \in E'}(x, e)$, for some $c(R) \in \mathbb{R}$ not depending on x. By definition of h we obtain that $g_{\mathcal{H},q}(R, a) = c(R) \cdot \prod_{e \in E'}(a, e)$ for every $(R, a) \in \mathbb{R} \times V$ such that $q(a) \leq R^2$. As $g_{\mathcal{H},q}$ and the function $x \longmapsto \prod_{e \in E'}(x, e)$ are both polynomials homogeneous of degree $\dim(V) = |\mathcal{H}'|$, this implies that the function $R \longmapsto c(R)$ is constant, completing the proof. $\qquad\square$

Remark 7.7. If $\mathcal{H}$ is the product of the empty arrangement in $\mathbb{R}$ and a type $I_2(2k+1)$ arrangement in $\mathbb{R}^2$ with $k \geq 2$, then the corollary recovers the case of Entrée 2 on page 433 of [13] where $N \geq 5$ is odd.

If the arrangement $\mathcal{H}$ does not satisfy the parity condition, then it is not true in general that the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ is polynomial in a and R. See for example the calculations on page 429 of the paper [13]. We can, however, control its behavior as R tends to infinity in the case where $\mathcal{H}$ is a Coxeter arrangement.

Proposition 7.8. *Suppose that $\mathcal{H}$ is a Coxeter arrangement with Coxeter group W in an n-dimensional vector space V. Let K be a measurable subset of V with finite volume that is stable by W and contains a neighborhood of 0. Suppose that there exists an integer $k \geq 0$ and a function $h : V \longrightarrow \mathbb{R}$ of type C^k at 0 such that $P(\mathcal{H}, K + a) = h(a)$ for a in a neighborhood of 0.*

Fix the center $a \in V$. Then $0 \in a + rK$ for $r > 0$ large enough and the function $P(\mathcal{H}, a + rK)$ satisfies

$$P(\mathcal{H}, a + rK) = o\left(r^{n - \min(k, |\mathcal{H}| - 1)}\right)$$

as $r \longrightarrow +\infty$. In particular, if $k \geq n$ and $|\mathcal{H}| \geq n + 1$ then we have $\displaystyle\lim_{r \to +\infty} P(\mathcal{H}, a + rK) = 0$.

Proof. By the multivariate version of Taylor's theorem, there exist polynomial functions $h^{(i)} : V \longrightarrow \mathbb{R}$ for $0 \leq i \leq k$ such that $h^{(i)}$ is homogeneous of degree i and

$$h(a) = \sum_{i=0}^{k} \frac{1}{i!} \cdot h^{(i)}(a) + o\left(\|a\|^k\right).$$

We also have explicit formulas for the $h^{(i)}$ involving partial differentials of h. In particular the fact that $h(w(b)) = (-1)^w \cdot h(b)$ for $w \in W$ and b in a neighborhood of 0 implies that $h^{(i)}(w(a)) = (-1)^w \cdot h^{(i)}(a)$ for every i, every $w \in W$ and every $a \in V$. Let $0 \leq i \leq k$, and suppose that $i \leq |\mathcal{H}| - 1$. As in the proof of Proposition 6.8, if $h^{(i)} \neq 0$, then the hypersurface $\{h^{(i)} = 0\}$ is of degree i and contains $\bigcup_{H \in \mathcal{H}} H$. This is not possible hence $h^{(i)} = 0$. If $k \geq |\mathcal{H}|$, we deduce that $h(a) = h^{(|\mathcal{H}|)}(a) + o(\|a\|^{|\mathcal{H}|}) = o(\|a\|^{|\mathcal{H}|-1})$. If $k \leq |\mathcal{H}| - 1$, we deduce that $h(a) = o(\|a\|^k)$. So we conclude that $h(a) = o(\|a\|^{\min(k, |\mathcal{H}|-1)})$.

We now fix $a \in V$. If $r > 0$ then $a + rK = r \cdot (a/r + K)$, so $0 \in a + rK$ as soon as $-a/r \in K$. This holds for large enough r by the assumption on K. Also, we have $P(\mathcal{H}, a + rK) = r^n \cdot P(\mathcal{H}, a/r + K) = r^n \cdot h(a/r)$ for large enough r, so $P(\mathcal{H}, a + rK) = o(r^{n - \min(k, |\mathcal{H}|-1)})$. If $|\mathcal{H}| \geq n + 1$ and $k \geq n$, this implies in particular that $P(\mathcal{H}, a + rK) = o(1)$. $\qquad\square$

Remark 7.9. As in Theorem 7.6, we just need in Proposition 7.8 the fact that $\mathcal{H}$ contains a Coxeter subarrangement $\mathcal{H}'$ whose Coxeter groups stabilizes both the arrangement $\mathcal{H}$ and the set K.

Corollary 7.10. *Suppose that $\mathcal{H}$ is a Coxeter arrangement in an n-dimensional inner product space V that has at least $n+1$ hyperplanes. Let q be a positive definite quadratic form on V, and fix a point $a \in V$. Then the function $R \longmapsto P(\mathcal{H}, \mathbb{B}_q(a, R))$ is $o(R^{n-|\mathcal{H}|+1})$ as $R \longrightarrow +\infty$, and in particular, $\lim_{R \to +\infty} P(\mathcal{H}, \mathbb{B}_q(a, R)) = 0$.*

Proof. We apply Proposition 7.8 with $K = \mathbb{B}_q(0, 1)$. In that case, the function $a \longmapsto P(\mathcal{H}, a + K)$ is C^∞ (and even real analytic) in a. $\qquad\square$

8. Surface volume

We now change our discussion from measuring the volume of the regions $K \cap T$ to measuring the surface volume in the case when the convex body is a ball. For an n-dimensional convex set X, let $\mathrm{Vol}_{n-1}(\partial X)$ denote the $(n-1)$-dimensional surface volume of the set X.

Theorem 8.1. *Assume that $\mathcal{H}$ is an n-dimensional hyperplane arrangement such that the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ does not depend on the radius $R \geq \|a\|$. Then the alternating sum of the surface volumes of the regions $\mathbb{B}(a, R) \cap \overline{T}$ where T ranges over all chambers of the arrangement $\mathcal{H}$ is zero, that is,*

$$\tag{8.1} \sum_{T \in \mathscr{T}} (-1)^T \cdot \mathrm{Vol}_{n-1}(\partial(\mathbb{B}(a, R) \cap \overline{T})) = 0.$$

Proof. Since the ball $\mathbb{B}(a, R)$ grows uniformly in each direction as R increases, taking the derivative of the pizza quantity with respect to R yields

$$\sum_{T \in \mathscr{T}} (-1)^T \cdot \mathrm{Vol}_{n-1}(\mathbb{S}(a, R) \cap T) = 0,$$

where $\mathbb{S}(a, R)$ denotes the $(n-1)$-dimensional sphere having center a and radius R, that is, $\mathbb{S}(a, R) = \{x \in V : \|x - a\| = R\}$. The result follows by observing that each subchamber contributes its $(n-1)$-dimensional volume to two terms in (8.1) with opposite signs. $\qquad\square$

Combining Theorem 7.6 with Theorem 8.1 yields the following result.

Theorem 8.2. *Suppose that $\mathcal{H}$ satisfies the parity condition and that it contains a Coxeter arrangement $\mathcal{H}'$ whose Coxeter group preserves $\mathcal{H}$ and such that $|\mathcal{H}'| \geq \dim(V)$. Let $R \geq \|a\|$. Then the alternating sum of the surface volumes of the regions $\mathbb{B}(a, R) \cap \overline{T}$ where T ranges over all chambers of the arrangement $\mathcal{H}$ is zero, that is,*

$$\sum_{T \in \mathscr{T}} (-1)^T \cdot \mathrm{Vol}_{n-1}(\partial(\mathbb{B}(a, R) \cap \overline{T})) = 0.$$

Remark 8.3. If $\mathcal{H}$ is the product of the trivial arrangement on $\mathbb{R}$ and of a type A_2 or $I_2(2k + 1)$ with $k \geq 2$ arrangement on $\mathbb{R}^2$ then Theorem 8.2 is Confection 2 on page 433 of [13]. If $\mathcal{H}$ is a type $I_2(2k)$ for $k \geq 2$ arrangement on $\mathbb{R}^2$ then Theorem 8.2 is the "N even" part of Confection 3 on page 434 of [13].

Remark 8.4. We can also apply Theorem 8.1 to all the even hyperplanes arrangements for which a ball centered at the origin is sufficiently symmetric. See for example the arrangement of Example 6.11. Also, Theorem 8.2 applies to the arrangement of Example 6.12.

9. Concluding remarks

The n-dimensional volume and the $(n-1)$-dimensional surface volume are both examples of intrinsic volumes; see [12, 14]. In the paper [6], we have generalized Theorem 1.2 to all intrinsic volumes, and more generally, to all valuations on convex subsets of $\mathbb{R}^n$ that are invariant by affine isometries. The methods we develop are different from the ones used here. However, it is still an open question whether Theorems 1.1 and 8.1 can be generalized to all intrinsic volumes. Naturally, the result is true for the 0th intrinsic volume, that is, the Euler characteristic.

Another generalization of the Pizza Theorem is to consider the problem of sharing pizza among more than two people. We can use Theorem 7.4 to produce such pizza-sharing results for the ball. The general idea is that if we consider a sum of terms as in that theorem that has enough symmetries, we will be able to show that it vanishes by the method of Proposition 6.8 and Theorem 7.6. We state one such result in dimension 2, where we can eliminate the assumption that the pizza is a disc.

Proposition 9.1. *Let $\mathcal{H}$ be a Coxeter arrangement of type $I_2(k)$ in the plane V, and let $p < k$ be a positive integer dividing k. Let K be a measurable subset of V with finite volume that is stable by the Coxeter group W of $\mathcal{H}$. If k is odd, or if $k = 2p$ with p odd, we also suppose that K is stable by the Coxeter group of the Coxeter arrangement of type $I_2(2k)$ containing $\mathcal{H}$. Let $T_0, T_1, \ldots, T_{2k-1}$ be the chambers of $\mathcal{H}$. For $a \in V$ such that K contains the convex hull of the set $\{w(a) : w \in W\}^2$, the following sum is independent of $0 \leq r \leq p-1$:*

$$\sum_{i=0}^{2k/p-1} \mathrm{Vol}(T_{r+pi} \cap (K+a)) = \mathrm{Vol}(K)/p.$$

In short, p people can share a pizza and have $2k/p$ slices each.

Proof of Proposition 9.1. If T is a chamber of the arrangement $\mathcal{H}$ and K is a measurable subset with finite volume, we deduce from point (i) of Theorem 3.4 that

$$\mathrm{Vol}(T \cap (K+a)) + \mathrm{Vol}((-T) \cap (K+a)) - (\mathrm{Vol}(T \cap K) + \mathrm{Vol}(-T \cap K))$$

(9.1)
$$= (a, e) \int_0^1 P(\mathcal{H}_e, (K+ta) \cap H_e)\, dt + (a, e') \int_0^1 P(\mathcal{H}_{e'}, (K+ta) \cap H_{e'})\, dt,$$

where $e, e' \in E$ are such that the boundaries of T and $-T$ are contained in $H_e \cup H_{e'}$. If K satisfies the conditions of the proposition, then for every $e \in E$ the intersection $(K + a - \pi_e(a)) \cap H_e$ is centrally symmetric in H_e and contains the interval $[-\pi_e(a), \pi_e(a)]$. (To see that $(K+a-\pi_e(a))\cap H_e$ is centrally symmetric, we use the fact that K is stable under the orthogonal reflection in the line perpendicular to H_e. If k is even, this line is part of the arrangement $\mathcal{H}$, but if k is odd, it is only part of the larger arrangement of type $I_2(2k)$. This is why we assume that K is stable under the Coxeter group of that larger arrangement.) This implies that there exists a linear function $g_e : H_e \longrightarrow \mathbb{R}$ such that $P(\mathcal{H}_e, (K+a) \cap H_e) = g_e(\pi_e(a))$ for all K and a satisfying the conditions on the proposition. So we deduce that the right-hand side of (9.1) is a polynomial homogeneous of degree 2 in a that does not depend on K, as long as K contains the convex hull of the set $\{w(a) : w \in W\}$.

Recall that the chambers are labelled $T_0, T_1, \ldots, T_{2k-1}$, where the index is modulo $2k$. Note that $-T_i = T_{i+k}$. Let ℓ_i be the line that borders the chambers T_i and T_{i+1}; hence ℓ_i also borders T_{i+k} and T_{i+k+1}. We take the index i of the lines to be an integer modulo k. For every $i \in \mathbb{Z}/k\mathbb{Z}$ and every $j \in \mathbb{Z}/2k\mathbb{Z}$, the orthogonal reflection in the line ℓ_i sends T_j to T_{2i+1-j}.

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[2]This condition holds for example if K is convex and $0 \in K + a$.

For $0 \le r \le p-1$ define

$$S_r(K+a) = \sum_{i=0}^{2k/p-1} \mathrm{Vol}(T_{r+pi} \cap (K+a)).$$

By equation (9.1), for every r, there exists a homogeneous polynomial $f_r : V \longrightarrow \mathbb{R}$ of degree 2 such that $S_r(K+a) - S_r(K) = f_r(a)$ for all K and a satisfying the conditions in the proposition. Fix $0 \le r \le p-2$. For every $0 \le j \le k/p-1$ the orthogonal reflection in the line ℓ_{r+jp} sends $\bigcup_{i=0}^{2k/p-1} T_{r+pi}$ to $\bigcup_{i=0}^{2k/p-1} T_{r+1+pi}$, so $S_r(K+a) = S_{r+1}(K+a)$ if $a \in \ell_{r+jp}$. We deduce that the polynomial $f_r - f_{r+1}$ vanishes on $\bigcup_{j=0}^{k/p-1} \ell_{r+jp}$. If $k/p \ge 3$ then this union contains at least three lines. As the polynomial $f_r - f_{r+1}$ is homogeneous of degree 2, it has to be zero, so $S_r(K+a) - S_r(K) = S_{r+1}(K+a) - S_{r+1}(K)$ for all K and a satisfying the condition of the proposition. As $S_r(K) = S_{r+1}(K)$ because K is stable by W, we finally obtain that $S_r(K + a) = S_{r+1}(K + a)$.

We finally consider the case where $k/p = 2$, that is, $k = 2p$. If p is even then the orthogonal reflection in the line $\ell_{r+p/2}$ also send $\bigcup_{i=0}^{2k/p-1} T_{r+pi}$ to $\bigcup_{i=0}^{2k/p-1} T_{r+1+pi}$, so the polynomial $f_r - f_{r+1}$ vanishes on the line $\ell_{r+p/2}$, and we can again deduce that $f_r - f_{r+1} = 0$ and that $S_r(K + a) = S_{r+1}(K+a)$. If p is odd then the orthogonal reflection in the line ℓ bisecting $\ell_{r+(p-1)/2}$ and $\ell_{r+(p+1)/2}$ sends $\bigcup_{i=0}^{2k/p-1} T_{r+pi}$ to $\bigcup_{i=0}^{2k/p-1} T_{r+1+pi}$. The line ℓ is part of the arrangement $I_2(2k)$ containing $\mathcal{H}$, since we assumed that K is stable by the Coxeter group of this arrangement, we obtain that the polynomial $f_r - f_{r+1}$ vanishes on ℓ, and then we deduce as before that $S_r(K + a) = S_{r+1}(K + a)$. $\qquad\square$

If K is a disc centered at 0 and $k = 2p$ then Proposition 9.1 recovers the following result:

Corollary 9.2 (J., M. D., J. K., A. D. and P. M. Hirschhorn [10])**.** *Cut a disc with 2p lines through a point in the disc such the lines are equally spaced. Then p people can have 4 slices each so that they each have the same amount of pizza.*

It is natural to ask if there are systematic generalizations of this kind of result for higher-dimensional hyperplane arrangements that give ways to share a pizza equally between more than two people. We present two such results in the following two remarks.

Remark 9.3. Using an arrangement $\mathcal{H}$ of type F_4 in $\mathbb{R}^4$ we can divide a pizza evenly among 4 people. Let W denote the Coxeter group generated by $\mathcal{H}$. Let $a \in \mathbb{R}^4$ and let K be a measurable set of finite volume that contains the convex hull of $\{w(a) : w \in W\}$. Finally, let L be the translation $K + a$.

We first note that we can write the arrangement $\mathcal{H}$ as a disjoint union $\mathcal{H}_1 \sqcup \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are both of type D_4. This partition of $\mathcal{H}$ corresponds to the long and short roots in the crystallographic root system F_4. Now we can assign three signs $\vec{s}(T) = (s, s_1, s_2)$ to a chamber $T \in \mathscr{T}(\mathcal{H})$. First let s be $(-1)^T$ where the sign is computed with respect to the arrangement $\mathcal{H}$. For $i = 1, 2$ let $T^{(i)}$ be the unique chamber in $\mathscr{T}(\mathcal{H}_i)$ that contains T. Let s_i be the sign $(-1)^{T^{(i)}}$ with respect to $\mathcal{H}_i$. Observe that $s = s_1 \cdot s_2$ holds since the three separation sets satisfy $S_{\mathcal{H}}(T_0, T) = S_{\mathcal{H}_1}(T_0^{(1)}, T^{(1)}) \sqcup S_{\mathcal{H}_2}(T_0^{(2)}, T^{(2)})$. Hence there are only four possible sign patterns for $\vec{s}(T)$, which are the elements of $P = \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}$. We use the set P to label the four people sharing the pizza. For $p \in P$ let V_p be the amount of pizza that person p receives, that is,

$$V_p = \sum_{\substack{T \in \mathscr{T}(\mathcal{H}) \\ \vec{s}(T) = p}} \mathrm{Vol}(T \cap L).$$

Since the hyperplane $\mathcal{H}$ satisfies Theorem 6.9, we obtain

$$V_{(1,1,1)} + V_{(1,-1,-1)} = V_{(-1,1,-1)} + V_{(-1,-1,1)} = \mathrm{Vol}(L)/2.$$

Similarly, since $\mathcal{H}_1$ and $\mathcal{H}_2$ also satisfy Theorem 6.9, the following two equalities hold:

$$V_{(1,1,1)} + V_{(-1,1,-1)} = V_{(1,-1,-1)} + V_{(-1,-1,1)} = \mathrm{Vol}(L)/2,$$
$$V_{(1,1,1)} + V_{(-1,-1,1)} = V_{(1,-1,-1)} + V_{(-1,1,-1)} = \mathrm{Vol}(L)/2.$$

Solving this linear equation system yields

$$V_{(1,1,1)} = V_{(1,-1,-1)} = V_{(-1,1,-1)} = V_{(-1,-1,1)} = \mathrm{Vol}(L)/4.$$

Remark 9.4. Let $n \geq 2$ be even and let $\mathcal{H}$ be an arrangement of type B_n in $\mathbb{R}^n$. Using the same idea as the previous remark, we can divide the boundary (crust) of an n-dimensional ball $\mathbb{B}(a, R)$, where $\|a\| \leq R$, evenly among four people. Note that the arrangement $\mathcal{H}$ can be written as the disjoint union of two Coxeter arrangements of types D_n and A_1^n for $n \geq 4$ and a disjoint union of two arrangements of type A_1^2 when $n = 2$. Again, we can assign three signs to each chamber, and we obtain one of the four sign patterns in the set P. Let S_p denote the sum

$$S_p = \sum_{\substack{T \in \mathcal{T}(\mathcal{H}) \\ \vec{s}(T) = p}} \mathrm{Vol}_{n-1}(T \cap \mathbb{S}(a, R)).$$

By using Theorem 8.1 and reasoning similar to that of Remark 9.3, we obtain

$$S_{(1,1,1)} = S_{(1,-1,-1)} = S_{(-1,1,-1)} = S_{(-1,-1,1)} = \mathrm{Vol}_{n-1}(\mathbb{S}(a, R))/4.$$

Finally, let us end with a conjecture about the type A arrangement. The A_n arrangement lies inside the n-dimensional space $\{(x_1, x_2, \ldots, x_{n+1}) : x_1 + x_2 + \cdots + x_{n+1} = 0\}$ and consists of the $\binom{n+1}{2}$ hyperplanes $x_i = x_j$ where $1 \leq i < j \leq n + 1$.

Conjecture 9.5. *Let $\mathcal{H}$ be a hyperplane arrangement of type A_n with $n \equiv 2, 3 \bmod 4$, and let $a \in V$ such that $\|a\| \leq R$. Then the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ is zero if and only if the point a lies on one of the hyperplanes of $\mathcal{H}$. Furthermore, when the point a belongs to the interior of a chamber T of the arrangement, the sign of the pizza quantity $P(\mathcal{H}, \mathbb{B}(a, R))$ is $(-1)^T$, that is, the sign of the chamber T.*

This conjecture is true in dimension 2; see Theorem 1 in [13] in the case of 3 lines. What can be said about the other irreducible Coxeter arrangements, that is, type D_n where n is odd?

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