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The Pre-WDVV Ring of Physics and its Topology

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Dedicated to Louis Billera in honor of his 60th Birthday

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Abstract. We show how a simplicial complex arising from the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations of string theory is the Whitehouse complex. Using discrete Morse theory, we give an elementary proof that the Whitehouse complex Δ_n is homotopy equivalent to a wedge of $(n-2)!$ spheres of dimension $n-4$. We also verify the Cohen-Macaulay property. Additionally, recurrences are given for the face enumeration of the complex and the Hilbert series of the associated pre-WDVV ring.

Key words: WDVV equations, Keel's presentation, moduli space, pre-WDVV complex, Morse matching, Cohen-Macaulay, Whitehouse complex.

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1. Introduction

The moduli space of smooth n-pointed stable curves of genus g, denoted $\bar{M}_{g,n}$, was introduced by Deligne and Mumford [6] Knudsen [15] and Mumford [20] to give a natural compactification of Mumford's [19] moduli space of nonsingular curves of genus g. A new construction of the genus 0 zero case is due to Keel [14] using blowups. Keel also gives the presentation for the generators and relations of the cohomology ring of $\bar{M}_{0,n}$.

The Associativity Equations in physics, also known as the WDVV equations [17, Section 0.2], are a system of partial differential equations due to Witten et al. [7], Dubrovin [9] and Witten [32]. Their solutions are generating functions Φ having coefficients which encode "potential" or "free energy" and are related to understanding quantum gravity via the topological quantum theory approach [31].

In [16] Kontsevich and Manin determine the potential function Φ in the case of Fano manifolds. Underlying their work is the need to construct Gromov-Witten classes. These are linear maps between the cohomology of a projective algebraic manifold and the cohomology of the moduli space $\bar{M}_{g,n}$. Kontsevich and Manin develop a cohomological field theory in terms of the Gromov-Witten class language and the language of operads. From this theory and Keel's presentation, they derive all the linear relations between homology classes of boundary strata of any codimension [16, Sections 6 and 7]. Since Keel's presentation and the splitting axiom for the Gromov-Witten classes [17, Eq. (0.3) and (0.4)]) imply the WDVV equations (see [16] for details), it is natural to refer to the cohomology ring given by Keel's presentation as the WDVV ring.

In a previous paper, the author studied an analogue of the WDVV ring [21]. This ring, known as the Losev-Manin ring, arises instead from the Commutativity Equations of physics [18]. Like the WDVV ring, the defining ideal for the Losev-Manin ring is composed of linear and quadratic relations. Since the quadratic relations determine the interesting behavior of the Losev-Manin ring, it is again natural to take a closer look at the quadratic relations defining the WDVV ring. We call this new ring the pre-WDVV ring.

In this paper we show the pre-WDVV ring can be realized as the Stanley-Reisner ring of the Whitehouse complex Δ_n. This is a well-studied complex which goes back to work of Boardman [4]. See Section 2 for further references. As a result, many of our results are rediscoveries, but with simpler proofs. Since our overall goal is to find a natural combinatorial object corresponding to the minimal generators of the WDVV ring, we expect our current approach to better lead to its understanding.

In Section 3 we study the links of faces in the Whitehouse complex via a forest representation of the faces. We then describe five injective maps which map Δ_n onto Δ_{n+1}. As new results, this allows us to deduce recurrences for the face vectors and Hilbert series of Δ_n. In Section 4 we construct a Morse matching on the complex to give an elementary proof that the Whitehouse complex Δ_n is homotopy equivalent to a wedge of $(n-2)!$ spheres of dimension $n-4$; see Theorem 4.7. In Section 5 we again return to the geometric description of the links of faces in Δ_n to give an elementary proof that the Whitehouse complex is Cohen-Macaulay. In the last section we indicate some work in progress about the related WDVV ring.

2. Definitions and notation

Throughout we will assume familiarity with basic poset terminology and combinatorial concepts. An excellent reference for the uninitiated is Stanley's text [24].

Let V be a finite vertex set. A simplicial complex Δ is a collection of subsets of V called faces satisfying $\emptyset \in V$ and $\{v\} \in \Delta$ for all $v \in V$, and if $F \subseteq G \in \Delta$ then $F \in \Delta$. The *dimension* of a face $F \in \Delta$ is given by $|F| - 1$. A face F in Δ is a *facet* if there is no face in Δ strictly containing F, while a simplicial complex is *pure* if all the facets have the same dimension. The *link* of a face F in the complex Δ is defined to be $link_{\Delta}(F) = \{G \subseteq V : F \cap G = \emptyset \text{ and } F \cup G \in \Delta\}$. Finally, for Δ and Γ two simplicial complexes having disjoint vertex sets V and W, the *join* of Δ and Γ, denoted $\Delta * \Gamma$, is the simplicial complex on the vertex set $V \cup W$ consisting of the faces $\Delta * \Gamma = \{F \cup G : F \in \Delta$ and $G \in \Gamma\}$.

For completeness, we give the physicist's definition of the WDVV ring [17, Section 0.10] based on Keel's presentation [14, Theorem 1]. Let $n \geq 3$ and let k be a field of characteristic zero. We say σ is a *stable* 2*-partition of* $\{1, \ldots, n\}$ if σ is an unordered partition of the elements $\{1, \ldots, n\}$ into two blocks $\sigma = S_1/S_2$ with $|S_i| \geq 2$. Denote by P_n the set of stable 2-partitions of $\{1, \ldots, n\}$. For each $\sigma \in P_n$, the element x_σ corresponds to a cohomology class of $H^*(\bar{M}_{0,n})$, that is, the cohomology ring of the moduli space of n-pointed stable curves of genus zero. Throughout it will be convenient for fixed σ to think of x_σ as simply an indeterminate. For $\sigma = S_1/S_2$ and $\tau = T_1/T_2$ from the index set P_n, let $a(\sigma, \tau)$ be the number of nonempty pairwise distinct sets among $S_i \cap T_j$, where $1 \leq i, j \leq 2$. Define the ideal I_n in the polynomial ring $k[x_\sigma \ : \ \sigma \in P_n]$ by:

1. (linear relations) For i, j, k, l distinct:

$$R_{ijkl} = \sum_{ij\sigma kl} x_\sigma - \sum_{kj\tau il} x_\tau, \tag{2.1}$$

   where the summand $ij\sigma kl$ means to sum over all stable 2-compositions $\sigma = S_1/S_2$ with the elements $i, j \in S_1$ and the elements $k, l \in S_2$.
2. (quadratic relations) For each pair σ and τ with $a(\sigma, \tau) = 4$,

$$x_\sigma \cdot x_\tau \tag{2.2}$$

The *WDVV ring* (or *Associativity ring*) W_n is defined to be the quotient ring $W_n = k[x_\sigma \ : \ \sigma \in P_n]/I_n$.

Rather than working with the complex associated with the the entire WDVV ring, we now consider the pre-Associativity or pre-WDVV ring defined by taking the WDVV ring modulo the quadratic relations only. More formally, the *pre-WDVV ring* R_n is defined to be $R_n = k[x_\sigma \ : \ \sigma \in P_n]/J_n$, where J_n is the ideal generated by the quadratic relations (2.2) of I_n.

Keel's presentation of the cohomology ring of the moduli space $\bar{M}_{0,n}$ has twice as many variables as he instead indexes the cohomology classes by subsets S of $\{1, \ldots, n\}$. However, Keel makes the further requirement that $x_S = x_{\bar{S}}$, where $\bar{S}$ is the complement of S taken with respect to $\{1, \ldots, n\}$, making his variables correspond to stable 2-partitions. Hence, it is natural to think of the variables x_σ for $\sigma \in P_n$ to be indexed instead by subsets $S \subseteq \{2, \ldots, n\}$ having cardinality satisfying $2 \leq |S| \leq n - 2$. See [14, Theorem 1].

Based upon Keel's presentation, we next construct a simplicial complex intimately related to the pre-WDVV ring. Fix an integer $n \geq 3$. Let Δ_n be the simplicial complex with vertex set $V_n = \{S \subseteq \{2, \ldots, n\} \ : \ 2 \leq |S| \leq n - 2\}$. A subset $F \subseteq V_n$ is a face if for all $S, T \in F$ either $S \subseteq T$, $S \supseteq T$ or $S \cap T = \emptyset$.

Observe that the complex Δ_n is described by its minimal non-faces, and that each minimal non-face has cardinality 2. Also, a set S in the vertex set of Δ_n corresponds to a stable 2-partition $S/\bar{S}$, where the complement is taken with respect to the set $\{1, \ldots, n\}$. Hence the condition of being a face F corresponds to $a(\sigma, \tau) < 4$ for all $S, T \in F$ where $\sigma = S/\bar{S}$ and $\tau = T/\bar{T}$.

The complex Δ_n coincides with the Boardman's space of fully grown n-trees [4]. This space was rediscovered by Whitehouse in her thesis [30] and independently by Culler and Vogtmann [5]. In the literature it is commonly referred to as the Whitehouse complex. The Whitehouse complex also has connections with phylogenetic trees. See the work of Billera et al. [2].

Recall for a finite simplicial complex Δ with vertex set V the *Stanley-Reisner ring* $k[\Delta]$ is defined to be $k[\Delta] = k[x_v : v \in V]/I(\Delta)$, where $I(\Delta)$ is the ideal generated by the non-faces of the complex Δ; see [25]. Using this terminology, we immediately have the following result.

Proposition 2.1. *The pre-WDVV ring R_n is the Stanley-Reisner ring of the Whitehouse complex Δ_n, that is, $R_n = k[\Delta_n]$.*

3. Facial structure

We now proceed with a more formal study of the Whitehouse complex Δ_n which will lead to understanding its topology. In particular, as new results we derive recurrences for its face f-vector and h-vector.

Observe the complex Δ_3 consists of the empty set. It will be convenient to view this complex as a (-1)-dimensional sphere. The complex Δ_4 consists of 3 isolated vertices, while the complex Δ_5 is the Peterson graph. Note the two last examples are homotopy equivalent to the wedge of two 0-dimensional spheres, respectively, the wedge of six 1-dimensional spheres.

Each face F of Δ_n can be described by a forest. The leaves of the forest are $2, \ldots, n$. The internal nodes are the elements of the face F. The cover relation of the forests is defined as follows: For $S, T \in F$, we say S covers T if $S \supset T$ and there is no $U \in F$ such that $S \supset U \supset T$. Moreover, $S \in F$ covers a leaf $i \in \{2, \ldots, n\}$ if there is no $U \in F$ such that $i \in U \subset S$. (Here we use $\subset$ and $\supset$ to mean strict containment and reverse containment, respectively.) See figure 1 for the example $F = \{\{2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}\}$. From the forest representation it is straightforward to see that the complex Δ_n is pure of dimension $n - 4$.

We have the following results about the links of faces in the complex Δ_n.

Lemma 3.1. *Let T be a vertex of the Whitehouse complex Δ_n. Then*

$$\mathrm{link}_{\Delta_n}(T) \cong \Delta_{|T|+1} * \Delta_{n-|T|+1},$$

where $*$ denotes the join operation.

Proof: First observe that for T a vertex of Δ_n, the link of T consists of all faces that contain T. Each face in the $\mathrm{link}_{\Delta_n}(T)$ can be described as a forest on the elements $\{2, \ldots, n\}$.

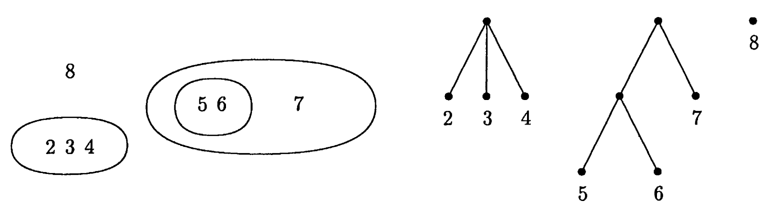

Figure 1. The forest representation of the face $F = \{\{2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}\}$ in Δ_8.

In a given forest, the element T is an internal node. One can choose the structure beneath T in the forest in $\Delta_{|T|+1}$ ways. To build the rest of the forest, treat the tree built by T as a leaf. Then one sees there are $n - 1 - |T| + 1$ elements available. The resulting structure has the form $\Delta_{n-|T|+1}$. Hence, $\text{link}_{\Delta_n}(T) \cong \Delta_{|T|+1} * \Delta_{n-|T|+1}$. $\square$

Iterating Lemma 3.1 gives the following more general result.

Proposition 3.2. *Let F be a face of the Whitehouse complex Δ_n. Then*

$$link_{\Delta_n}(F) \cong \Delta_{c+1} * \prod_{T \in F} \Delta_{c(T)+1},$$

where $\prod$ denotes taking the join operation $*$ among the factors, c is the number of components in the forest representation of F, and $c(T)$ is the number of children the node T has in the forest representation of F.

We introduce five maps $A, B, C, D, E : \Delta_n \to \Delta_{n+1}$ which together map Δ_n onto Δ_{n+1}. For a face $F \in \Delta_n$ define

$$A(F) = F \quad \text{and} \quad B(F) = F \cup \{\{2, \ldots, n\}\}.$$

For $S \in F$ define

$$C(F, S) = \{T \cup \{n+1\} : S \subseteq T, T \in F\} \cup \{T : S \not\subseteq T, T \in F\}$$

and

$$D(F, S) = C(F, S) \cup \{S\}.$$

Finally, for $2 \leq i \leq n$ define

$$E(F, i) = \{\{i, n+1\}\} \cup \{T \cup \{n+1\} : i \in T, T \in F\} \cup \{T : i \notin T, T \in F\}.$$

Informally speaking, the A map adds the element $n+1$ to the set diagram of the face F whereas the B map creates a new set consisting of the entire set diagram of the face F and then adds the element $n+1$. The C map selects a set S from the face F and adds the element $n+1$ to it, whereas the D map selects a set S from the face F and creates a new set consisting of the set S and the element $n+1$. The E map replaces an element i with the set consisting of i and $n+1$. For F a face of dimension i the maps A and C leave the dimension of F unchanged while the maps B, D and E each increase the dimension by one.

It is easily seen that all of these maps are injective and that Δ_{n+1} is a disjoint union of their images. See figures 2–4 for an example of each of these maps.

Observe that for fixed i the image of $E(F, i)$ as F runs over all faces F in the complex Δ_n is the link of the vertex $\{i, n+1\}$ in Δ_{n+1} which is isomorphic to Δ_n. Hence we can decompose the face poset of the complex Δ_{n+1} into the images of the five maps A, B, C, D and E. In turn, the image of E further decomposes into $n - 1$ copies of the face poset of

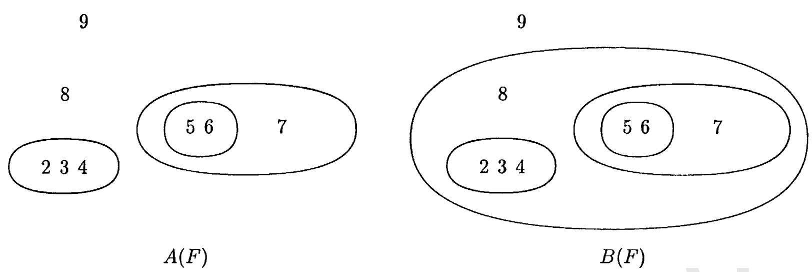

Figure 2. The maps A and B applied to the face $F = \{\{2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}\}$ in Δ_8.

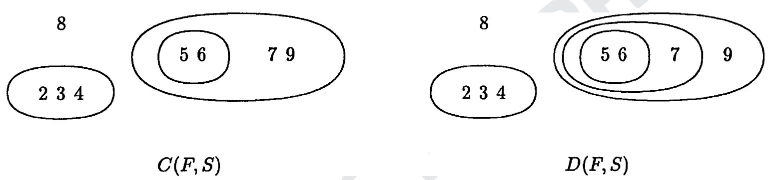

Figure 3. The maps C and D applied to the face $F = \{\{2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}\}$ and set $S = \{5, 6, 7\}$ in Δ_8.

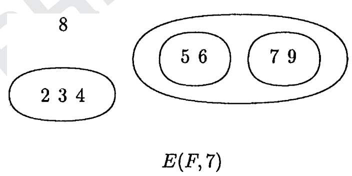

Figure 4. The map $E(F, 7)$ where $F = \{\{2, 3, 4\}, \{5, 6\}, \{5, 6, 7\}\}$ in Δ_8.

Δ_n. It is straightforward to see each of the $n - 1$ copies of the Δ_n in the face poset of Δ_{n+1} is an upper order ideal.

From the maps we have just defined, we can give a recurrence for the number of faces in the Whitehouse complex.

Theorem 3.3. *Let $f_{n,i}$ denote the number of faces in the Whitehouse complex Δ_n having dimension i. Then $f_{n,i}$ satisfies the recursion*

$$f_{n,i} = (i + 2) \cdot f_{n-1,i} + (n + i - 1) \cdot f_{n-1,i-1},$$

where $-1 \leq i \leq n - 4$ and $f_{n,-1} = 1$.

Proof: Easily $f_{n,-1} = 1$ for all $n \geq 3$, as there is one (-1)-dimensional face in Δ_n, namely the empty set. To show the recurrence, note that a face having dimension i has $i + 1$ sets occurring in its set representation. The maps A, B, C, D and E each either leave the number of sets unchanged or increase the number by one. Since they map Δ_{n-1} bijectively onto Δ_n, we can build all the i-dimensional faces in Δ_n. First, we take a face from Δ_{n-1}

that has $i + 1$ sets. The A map simply adds the element n in 1 way, while the C map adds it in the number of sets ways, that is, $i + 1$ ways . Overall we have constructed $(i + 2) \cdot f_{n-1,i}$ new faces in Δ_n. The other way to build an i-dimensional face is to take a face from Δ_{n-1} having i sets and add the element n in such a way to increase the number of sets. The B map does this in 1 way, the D map in number of sets ways, that is, i ways, and the E map does this in $n - 2$ ways. Hence we have constructed $(n + i - 1) \cdot f_{n-1,i-1}$ new faces in Δ_n, and the recurrence holds. □

Corollary 3.4. *The Whitehouse complex Δ_n is pure of dimension $n - 4$ with $(2n - 5)!! = (2n - 5) \cdot (2n - 7) \cdots 3 \cdot 1$ facets.*

This corollary can be found in [23] and [29].

Recall the *h-vector* of a $(d-1)$-dimensional simplicial complex Δ is the vector $(h_0, \ldots, h_d)$ where $h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} \cdot f_{d-1,i-1}$. The Hilbert series of the Stanley-Reisner ring of Δ is thus given by $\mathcal{H}(k[\Delta]) = (h_0 + h_1 \cdot t + \cdots + h_d \cdot t^d)/(1 - t)^d$. See [25]. Using the expression for the h-vector in terms of the f-vector and Theorem 3.3, we obtain the following identity.

Corollary 3.5. *The h-vector of the Whitehouse complex Δ_n satisfies the recurrence*

$$h_{n,k} = (k + 1) \cdot h_{n-1,k} + (2n - k - 5) \cdot h_{n-1,k-1},$$

where $0 \le k \le n - 3$.

The Hilbert series of the pre-WDVV ring R_n for $3 \le n \le 8$ (and thus the h-vector values for the Whitehouse complex Δ_n) are displayed in Table 1.

4. Morse matching and the topology of Δ_n

R. Forman devised a discrete version of Morse theory to study the topology of simplicial complexes, and more generally, CW-complexes. We give a brief overview of this. More details can be found in [10]. We then describe a Morse matching of the Whitehouse complex and verify its homotopy type.

Table 1. The Hilbert series of the pre-WDVV ring R_n for $3 \le n \le 8$

n	$(1 - t)^{n-3} \cdot \mathcal{H}(W_n)$
3	1
4	$1 + 2t$
5	$1 + 8t + 6t^2$
6	$1 + 22t + 58t^2 + 24t^3$
7	$1 + 52t + 328t^2 + 444t^3 + 120t^4$
8	$1 + 114t + 1452t^2 + 4400t^3 + 3708t^4 + 720t^5$

Let P be an arbitrary poset. Begin by orienting all of the edges in the Hasse diagram of P, that is, the cover relations of P, downwards. Next, form a matching M on the elements of P. Reverse the orientation of the edges in the matching to be upwards. Such a matching M is a *Morse matching* if the resulting directed graph is acyclic. An unmatched element of P is called *critical*.

For a simplicial complex Δ the *face poset* P is the poset formed by taking the faces of the complex as elements and ordering them by inclusion. The face poset is ranked with the rank of an element $x \in P$ given by i if $|x| = i$. In the case we do not wish to include the empty face in the face poset, we denote the resulting face poset by $\bar{P}$.

The following is Forman's result [10].

Theorem 4.1. *For a simplicial complex Δ with face poset P, let M be a Morse matching of $\bar{P}$. For $i \geq 0$ let u_i denote the number of critical i-dimensional simplices. Then Δ is homotopy equivalent to a CW-complex consisting of u_i i-cells, where $i \geq 0$.*

As it will be convenient for us to include the empty face in our matching, we will use the following corollary of Forman's theorem.

Corollary 4.2. *Let Δ be a simplicial complex having Morse matching on the face poset P with exactly m critical k-dimensional simplices. Then Δ is homotopy equivalent to a wedge of m k-dimensional spheres.*

There is a straightforward criterion to determine when a matching is a Morse matching.

Lemma 4.3. *For a ranked poset P to determine a given matching M is a Morse matching, it is enough to verify the Morse condition on the elements that are in the matching M from adjacent ranks in P.*

Proof: We first show it is enough to check the Morse acyclicity condition on elements from ranks i and $i + 1$ in the poset P for i fixed. As it is impossible to construct a cycle composed entirely of edges oriented downward, any cycle in P must include at least two elements from the matching M So suppose x and y are two elements from the matching of ranks i and $i + 1$ with the edge oriented from x to y. If the edge $x \to y$ is in a cycle, then the edge $y \to z$ following this one cannot have $\rho(z) = i + 1$ (and hence, pointing upwards) since the element y is already matched. Similarly, if the edge $x \to y$ is in a cycle, then the edge $w \to x$ proceeding this one cannot have $\rho(w) = i - 1$ since x is already matched. Hence we have shown if there is a cycle in P, all the elements are from adjacent ranks. Finally, if we have a cycle in P on adjacent ranks, the edges in the cycle alternate pointing upwards and downwards, implying every other edge is from the matching M and hence all the elements in the cycle are elements of M. □

What follows is a result which will be helpful when we construct a Morse matching of the Whitehouse complex.

Lemma 4.4. *Let P be a ranked poset which is the disjoint union of a lower order ideal L and an upper order ideal U. If M_1 is a Morse matching of L and M_2 is a Morse matching of U, then $M_1 \cup M_2$ is a Morse matching of P.*

Proof: By Lemma 4.3 without loss of generality we may assume all the elements we consider here are those elements from $M_1 \cup M_2$ of ranks i and $i+1$, where $i \geq 1$. As there is no element of U matched with an element of L, all the edges from rank $i+1$ elements in U to rank i elements of L are oriented downwards. Furthermore, as U is an upper order ideal, there is no edge between any rank $i+1$ element in L to a rank i element of U. Hence, it is impossible to construct a cycle on the elements of $M_1 \cup M_2$, implying the matching $M_1 \cup M_2$ is indeed a Morse matching of P. □

We now describe a Morse matching on the face poset of the Whitehouse complex.

Proposition 4.5. *Let Q be the face poset given by the images of the maps A, B, C and D applied to the Whitehouse complex Δ_n. Let M be the matching described by orienting the edges from $A(F)$ to $B(F)$ and $C(F, S)$ to $D(F, S)$, where $S \subseteq F$ and F ranges over all faces of the Whitehouse complex Δ_n. Then M is a Morse matching on Q.*

Proof: Assume on the contrary that we can find a cycle between rank i and $i+1$ elements of Q. The edges oriented from a rank i to rank $i+1$ elements in this cycle are Morse matched edges. In terms of the forest representation of a face, such an edge corresponds to the element $n+1$ being moved up one level higher in the tree. (Here we are thinking of the leaves in the forest as being the lowest level.) In terms of the maps, such an edge corresponds to $C(F, S)$ and $D(F, S)$ for some face F and subset $S \subseteq F$ in Δ_n. The final move to raise the element $n+1$ corresponds to the maps $A(F)$ and $B(F)$. However, the path we have created cannot be continued, and more importantly, cannot be completed to form a cycle, since the element $n+1$ cannot be moved any higher. Hence we cannot construct a cycle on the elements of Q, so the matching M described is a Morse matching. □

Corollary 4.6. *The complex Q described in Proposition* 4.5 *is contractible.*

We are now ready to prove our main result. For other proofs, see [23, 26, 29].

Theorem 4.7. *The Whitehouse complex Δ_n is homotopy equivalent to a wedge of $(n-2)!$ spheres of dimension $n-4$.*

Proof: We proceed by induction on the dimension n. For $n = 3$, the complex Δ_3 consists solely of the empty set, so there is the trivial empty Morse matching. This complex is homotopy equivalent to one (-1)-dimensional sphere.

Begin to construct a Morse matching on the face poset P of Δ_{n+1} by first orienting the edges in the face poset $Q = \mathrm{Im}(A(\Delta_n)) \,\dot\cup\, \mathrm{Im}(B(\Delta_n)) \,\dot\cup\, \mathrm{Im}(C(\Delta_n)) \,\dot\cup\, \mathrm{Im}(D(\Delta_n))$ as described in Proposition 4.5. The remainder of the face poset of Δ_{n+1} is $\mathrm{Im}(E(\Delta_n))$. Recall the image of E applied to Δ_n is isomorphic to $(n-1)$ copies of Δ_n. By induction, we have a Morse matching in each of these $(n-1)$ copies of Δ_n. Each copy of Δ_n is an upper order ideal in the face poset P. Hence Lemma 4.4 applies, so we have a constructed a Morse matching on the face poset of Δ_{n+1}. In each of the $(n-1)$ copies of Δ_n there are $(n-2)!$ critical elements. Moreover, all the critical elements are facets of dimension $(n-3)$. Thus, by Corollary 4.2 the complex Δ_{n+1} is homotopy equivalent to a wedge of $(n-1)!$ spheres each having dimension $(n-3)$. □

5. The Cohen-Macaulay property

Recall that a simplicial complex Δ is *Cohen-Macaulay* if the associated Stanley-Reisner ring $k[\Delta]$ is Cohen-Macaulay [25]. In combinatorial commutative algebra Cohen-Macaulay complexes have some very nice enumerative properties. Reisner's Criterion [22, 25] gives a characterization of Cohen-Macaulay simplicial complexes in terms of reduced homology of their links.

Theorem 5.1. *A pure simplicial complex Δ is Cohen-Macaulay if and only if for all faces $F \in \Delta$ and for all $i < \dim(link(F))$ we have $\widetilde{H}_i(link(F); k) = 0$.*

We have the following result about the topology of the links of the faces in the Whitehouse complex.

Theorem 5.2. *For F a face of the Whitehouse complex Δ_n, $link_{\Delta_n}(F)$ is a wedge of $(c-1)! \cdot \prod_{T \in F}(c(T)-1)!$ spheres of dimension $n-4-|F|$, where c is the number of components in the forest representation of F and $c(T)$ is the number of children the node T has in the forest representation of F.*

Proof: Let $(S^n)^{\wedge k}$ denote the wedge of k n-dimensional spheres. The free join of an n-dimensional sphere with an m-dimensional sphere satisfies $S^n * S^m \cong S^{n+m+1}$. Additionally, the wedge and free join operations are distributive over simplicial complexes, that is, $(X \wedge Y) * Z \cong (X * Z) \wedge (Y * Z)$. For a proof of this fact, see [8, Lemma 3.14]. Hence it follows that $(S^n)^{\wedge k} * (S^m)^{\wedge l} \cong (S^{n+m+1})^{\wedge k \cdot l}$. See also [3, Lemma 2.5 (ii)]. By Proposition 3.2 and Theorem 4.7 we have

$$\begin{aligned} \operatorname{link}_{\Delta_n}(F) &\cong (S^{c-3})^{\wedge(c-1)!} * \prod_{T \in F} \left(S^{c(T)-3}\right)^{\wedge(c(T)-1)!} \\ &\cong \left(S^{|F|+c-3+\sum_{T\in F}(c(T)-3)}\right)^{\bigwedge(c-1)!\cdot\prod_{T\in F}(c(T)-1)!} \\ &\cong \left(S^{-2|F|-3+c+\sum_{T\in F} c(T)}\right)^{\bigwedge(c-1)!\cdot\prod_{T\in F}(c(T)-1)!} . \end{aligned}$$

But $c + \sum_{T\in F} c(T) = n - 1 + |F|$. Hence

$$\operatorname{link}_{\Delta_n}(F) \cong \left(S^{n-4-|F|}\right)^{\bigwedge(c-1)!\cdot\prod_{T\in F}(c(T)-1)!}$$

□

From Reisner's Criterion and Theorem 5.2 we have the following immediate result. This can also be found in work of Robinson-Whitehouse, Sundaram and Vogtmann [23, 26, 29].

Theorem 5.3. *The Whitehouse complex Δ_n is Cohen-Macaulay and hence the pre-WDVV ring R_n is Cohen-Macaulay.*

Trappmann and Ziegler [27] prove shellability of the k-tree complex. This is Hanlon's generalization of the Whitehouse tree complex corresponding to the case $k = 2$; see [12, 13]. In unpublished work, Wachs independently determined shellability of the Whitehouse complex using a different shelling order. As shellability implies Cohen-Macaulayness, this gives another proof of Theorem 5.3.

6. Concluding remarks

The author is currently studying the Hilbert series of the WDVV ring. The first few values are given in Table 2. The symmetry follows immediately from Poincaré duality and the fact the moduli space $\bar{M}_{g,n}$ is smooth and compact. It would be interesting to find a natural combinatorial object corresponding to these values.

Vic Reiner has asked if the WDVV ring and the pre-WDVV rings are Koszul. Evidence for this is that both rings are defined by quadratic relations and the reciprocal of the Hilbert series for each ring has alternating coefficients. From a result of Fröberg [11], it follows that the Stanley-Reisner ring of a simplicial complex with minimal non-faces having cardinality 2 is Koszul. This latter result applies to the pre-WDVV ring.

Theorem 6.1. *The pre-WDVV ring is Koszul.*

It remains to determine if either the pre-WDVV ring or the WDVV ring are Gorenstein.

The complex of not 1-connected graphs on $(n + 1)$ vertices is also homotopy equivalent to a wedge of $(n - 2)!$ spheres of dimension $n - 4$; see [1, 28]. Although the Whitehouse complex Δ_n and this complex are homotopy equivalent, they are not the same. By Corollary 3 the pre-WDVV complex is pure, while the the cardinality of a facet in the complex of not 1-connected graphs on $(n + 1)$ vertices ranges from $\lfloor \frac{n^2}{4} \rfloor$ to $\binom{n}{2}$.

Table 2. The Hilbert series of the WDVV ring W_n for $3 \leq n \leq 8$.

n	$\mathcal{H}(W_n)$
3	1
4	$1 + t$
5	$1 + 5t + t^2$
6	$1 + 16t + 16t^2 + t^3$
7	$1 + 42t + 127t^2 + 42t^3 + t^4$
8	$1 + 99t + 715t^2 + 715t^3 + 99t^4 + t^5$

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