Unisabana

Tenemos $\dfrac{\partial^2 u}{\partial t^2} = \alpha^2\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right)$

Para $r = \sqrt{x^2+y^2}$ y $x = r\cos\theta$ tq $\dfrac{\partial r}{\partial x} = \dfrac{+x}{\sqrt{x^2+y^2}} = \dfrac{+r\cos\theta}{r}$

$\theta = \arctan\left(\dfrac{y}{x}\right)$ y $y = r\operatorname{Sen}\theta$

$\dfrac{\partial r}{\partial y} = \dfrac{+y}{\sqrt{x^2+y^2}} = \dfrac{+r\operatorname{Sen}\theta}{r}$ ; $\dfrac{\partial \theta}{\partial x} = \dfrac{-y}{x^2+y^2} = \dfrac{-r\operatorname{Sen}\theta}{r^2}$ ; $\dfrac{\partial \theta}{\partial y} = \dfrac{x}{x^2+y^2} = \dfrac{r\cos\theta}{r^2}$

Para $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial u}{\partial r}\cdot\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial \theta}\cdot\dfrac{\partial \theta}{\partial x}\right] = \dfrac{\partial}{\partial x}\left[\dfrac{\partial u}{\partial r}\cdot\cos\theta + \dfrac{\partial u}{\partial \theta}\left(\dfrac{-\operatorname{Sen}\theta}{r}\right)\right]$

$*$ $\dfrac{\partial}{\partial r}\left[\dfrac{\partial u}{\partial r}\cos\theta + \dfrac{\partial u}{\partial \theta}\left(\dfrac{-\operatorname{Sen}\theta}{r}\right)\right]\cdot\dfrac{\partial r}{\partial x} = \left[\cos\theta\dfrac{\partial^2 u}{\partial r^2} - \dfrac{\partial^2 u}{\partial r\partial\theta}\dfrac{\operatorname{Sen}\theta}{r} + \dfrac{\partial u}{\partial \theta}\cdot\dfrac{\operatorname{Sen}\theta}{r^2}\right]\cos\theta$

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = \cos^2\theta\dfrac{\partial^2 u}{\partial r^2} + \dfrac{\operatorname{Sen}\theta\cos\theta}{r^2}\cdot\dfrac{\partial u}{\partial \theta} - \dfrac{\operatorname{Sen}\theta\cos\theta}{r}\dfrac{\partial^2 u}{\partial r\partial\theta} + \dfrac{\partial}{\partial \theta}\left[\dfrac{\partial u}{\partial r}\cos\theta + \dfrac{\partial u}{\partial \theta}\left(-\dfrac{\operatorname{Sen}\theta}{r}\right)\right]\partial\theta/\partial x$

Para $\dfrac{\partial \theta}{\partial x}\cdot\dfrac{\partial}{\partial \theta}\left[\dfrac{\partial u}{\partial r}\cos\theta - \dfrac{\partial u}{\partial \theta}\dfrac{\operatorname{Sen}\theta}{r}\right] = -\dfrac{\operatorname{Sen}\theta}{r}\left[\dfrac{\partial^2 u}{\partial r\partial\theta}\cos\theta - \operatorname{Sen}\theta\dfrac{\partial u}{\partial r} - \dfrac{\partial^2 u}{\partial \theta^2}\dfrac{\operatorname{Sen}\theta}{r} - \dfrac{\cos\theta}{r}\dfrac{\partial u}{\partial \theta}\right]$

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = \left[\dfrac{\partial^2 u}{\partial r^2}\cos^2\theta + \dfrac{\partial u}{\partial \theta}\dfrac{\operatorname{Sen}\theta\cos\theta}{r^2} - \dfrac{\partial^2 u}{\partial r\partial\theta}\dfrac{\operatorname{Sen}\theta\cos\theta}{r} - \dfrac{\partial^2 u}{\partial r\partial\theta}\dfrac{\operatorname{Sen}\theta\cos\theta}{r} + \dfrac{\partial u}{\partial r}\dfrac{\operatorname{Sen}^2\theta}{r}\right.$
$\left. + \dfrac{\partial^2 u}{\partial \theta^2}\dfrac{\operatorname{Sen}^2\theta}{r^2} + \dfrac{\operatorname{Sen}\theta\cos\theta}{r^2}\dfrac{\partial u}{\partial \theta}\right]$

$\Rightarrow \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial r^2}\cos^2\theta + \dfrac{\partial^2 u}{\partial \theta^2}\dfrac{\operatorname{Sen}^2\theta}{r^2} + \dfrac{\partial u}{\partial \theta}\dfrac{\operatorname{Sen}(2\theta)}{r^2} + \dfrac{\partial u}{\partial r}\dfrac{\operatorname{Sen}^2\theta}{r^2} - \dfrac{\partial^2 u}{\partial r\partial\theta}\dfrac{\operatorname{Sen}(2\theta)}{r}$

Para $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}\right] = \frac{\partial}{\partial y}\underbrace{\left[\frac{\partial u}{\partial r}\operatorname{Sen}\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right]}_{\star} = \frac{\partial}{\partial r}\star\cdot\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\star\cdot\frac{\partial \theta}{\partial y}$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \left[\frac{\partial^2 u}{\partial r^2}\operatorname{Sen}\theta + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\cos\theta}{r} - \frac{\partial u}{\partial\theta}\cdot\frac{\cos\theta}{r^2}\right]\operatorname{Sen}\theta + \left[\frac{\partial^2 u}{\partial r\partial\theta}\operatorname{Sen}\theta + \frac{\partial u}{\partial r}\cos\theta + \frac{\partial^2 u}{\partial\theta^2}\frac{\cos\theta}{r} - \frac{\partial u}{\partial\theta}\frac{\operatorname{Sen}\theta}{r}\right]\frac{\cos\theta}{r}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\operatorname{Sen}^2\theta + \frac{\partial^2 u}{\partial\theta^2}\cdot\frac{\cos^2\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r} - \frac{\partial u}{\partial\theta}\cdot\frac{\operatorname{Sen}(2\theta)}{r^2} + \frac{\partial^2 u}{\partial r\partial\theta}\cdot\frac{\operatorname{Sen}(2\theta)}{r}$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = \alpha^2\Big[\frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{\partial^2 u}{\partial\theta^2}\frac{\operatorname{Sen}^2\theta}{r^2} + \cancel{\frac{\partial u}{\partial\theta}\frac{\operatorname{Sen}(2\theta)}{r^2}} + \frac{\partial u}{\partial r}\frac{\operatorname{Sen}^2\theta}{r} - \cancel{\frac{\partial^2 u}{\partial r\partial\theta}\frac{\operatorname{Sen}(2\theta)}{r}} + \frac{\partial^2 u}{\partial r^2}\operatorname{Sen}^2\theta + \frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r} - \cancel{\frac{\partial u}{\partial\theta}\frac{\operatorname{Sen}(2\theta)}{r^2}} + \cancel{\frac{\partial^2 u}{\partial r\partial\theta}\cdot\frac{\operatorname{Sen}(2\theta)}{r}}\Big]$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = \alpha^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial\theta^2}\cdot\frac{1}{r^2} + \frac{1}{r}\cdot\frac{\partial u}{\partial r}\right)$$

La discretizamos:

$$\frac{u^{l+1}_{i,j} - 2u^{l}_{i,j} + u^{l-1}_{i,j}}{(\Delta t)^2} = \alpha^2\left(\frac{u^l_{i+1,j} - 2u^l_{i,j} + u^l_{i-1,j}}{(\Delta r)^2} + \frac{1}{r_i^2}\cdot\frac{u^l_{i,j+1} - 2u^l_{i,j} + u^l_{i,j-1}}{(\Delta\theta)^2} + \frac{1}{r_i}\cdot\frac{u^l_{i,j} - u^l_{i-1,j}}{\Delta r}\right)$$

Reorganizando:

$$u^{l+1}_{i,j} = \left(\frac{\alpha\Delta t}{\Delta r}\right)^2\left[\left(u^l_{i+1,j} - 2u^l_{i,j} + u^l_{i-1,j}\right) + \left(\frac{\Delta r}{\Delta\theta}\right)^2\frac{\left(u^l_{i,j+1} - 2u^l_{i,j} + u^l_{i-1,j}\right)}{r_i^2} + \frac{\Delta r}{r_i}\left(u^l_{i,j} - u^l_{i-1,j}\right)\right] + 2u^l_{i,j} - u^{l-1}_{i,j}$$

$$\boxed{u^{l+1}_{i,j} = \vartheta^2\left[u^l_{i+1,j} - 2u^l_{i,j} + u^l_{i-1,j} + \frac{\Delta r}{r_i}\left(u^l_{i,j} - u^l_{i,j-1}\right) + \left(\frac{\lambda}{r_i}\right)^2\left(u^l_{i,j+1} - 2u^l_{i,j} + u^l_{i,j-1}\right)\right] + 2u^l_{i,j} - u^{l-1}_{i,j}}$$