SVM

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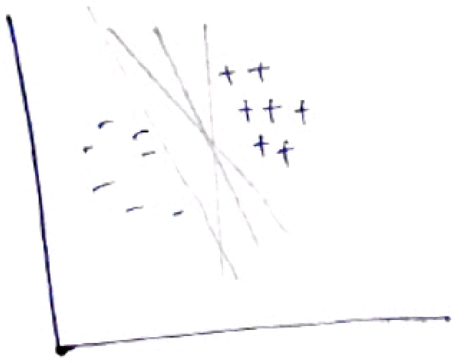
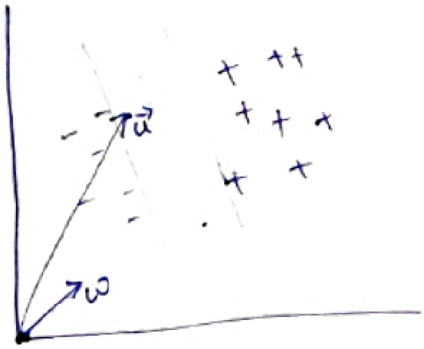
Let's say we have positive and negative data samples.



there are infinitely many lines we can draw to separate them. But how do we decide the best line?

→ widest street (margin) possible.



Let $\vec{w}$ be a vector perpendicular to the street.

Let $\vec{u}$ be an unknown, we would like to figure out whether $\vec{u}$ is in the left side of the street? or right side of the street?

Let us project $\vec{u}$ down onto the perpendicular of the street ($\vec{w}$) which would tell us whether it is in left or right side.

$$\vec{w} \cdot \vec{u} \geq c$$ (bigger the projection is more it crosses the street)

$\boxed{\text{decision rule.} \quad \vec{w} \cdot \vec{u} + b \geq 0 \quad \text{then } +}$ where $c = -b$

$\vec{w} \cdot \vec{x}_+ + b \geq 1$ -① $\because$ $\vec{x}_+$ is a positive sample.

$\vec{w} \cdot \vec{x}_- + b \leq -1$ -② $\because$ $\vec{x}_-$ is a negative sample.

for mathematical convenience,

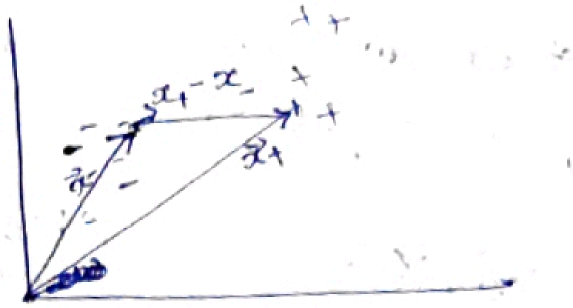
$\exists$ y_i such that $y_i = \begin{cases} +1, & \text{for + sample.} \\ -1, & \text{for − sample.} \end{cases}$

① × y_i.

$$\left.\begin{aligned} &y_i (\vec{w} \cdot \vec{x}_i + b) \geq 1 \\ \text{② × } y_i \quad &y_i (\vec{w} \cdot \vec{x}_i + b) \geq 1 \end{aligned}\right\} \text{ same. ☺}$$

$$y_i (\vec{w} \cdot \vec{x}_i + b) - 1 \geq 0$$

$$\boxed{y_i (\vec{w} \cdot \vec{x}_i + b) - 1 = 0}$$ for x in gutter (middle) —— ③



let vector to a positive sample be x_+

" " " negative " " x_-

difference of those vector $x_+ - x_-$

$$\text{width of street} = (x_+ - x_-) \cdot \frac{\vec{\omega}}{\|\omega\|}$$

$$= x_+ \cdot \frac{\vec{\omega}}{\|\omega\|} - x_- \cdot \frac{\vec{\omega}}{\|\omega\|}$$

ω is a normal to the street.

$$\text{~~margin~~} = \frac{1-b}{\|\omega\|} - \frac{-(1+b)}{\|\omega\|}$$

$$= \frac{1-b+1+b}{\|\omega\|}$$

$$= \frac{2}{\|\omega\|}$$ (using equation 3)

we would like to maximize the width of the street.

$$\max \frac{2}{\|\omega\|}$$

$$\Rightarrow \min \|\omega\|$$

$$\Rightarrow \min \frac{1}{2}\|\omega\|^2$$ (mathematical convenience)

to find extremum of functions with equality constraints we need to use Lagrange multiplier.

$$L(x,\lambda) = f(x) - \lambda g(x)$$

L = Lagrangian
λ = Lagrange multiplier
$g(x)$ = equality constraint
$f(x)$ = function
x = Integer.

that would give us a new expression, which we can maximize or minimize without thinking about the constraints any more.

$$L = \frac{1}{2}\|\vec{w}\|^2 - \sum_i \alpha_i\left[y_i(\vec{w}\cdot\vec{x}_i + b) - 1\right]$$

$\alpha_i = \lambda$

to find min or max, set derivative to zero.

$$\frac{\partial L}{\partial \vec{w}} = \vec{w} - \sum \alpha_i y_i x_i = 0$$

$$\Rightarrow \boxed{\vec{w} = \sum_i \alpha_i y_i \vec{x}_i}$$

IMP.

decision vector is a linear sum of all the samples.

$$\frac{\partial L}{\partial b} = \sum \alpha_i y_i = 0$$

$$\Rightarrow \boxed{\sum \alpha_i y_i = 0}$$

$$L = \frac{1}{2}\left(\sum \alpha_i y \bar{x}_i\right)\left(\sum \alpha_j y_j \bar{x}_j\right) - \left(\sum \alpha_i y_i x_i\right)\cdot\left(\sum \alpha_j y_j x_j\right) - \overbrace{\sum \alpha_i y_i b}^{=0} + \sum \alpha_i$$

$$= \sum \alpha_i - \frac{1}{2}\sum_i \sum_j \alpha_i \alpha_j y_i y_j \, x_i \cdot x_j$$

(depends on this)

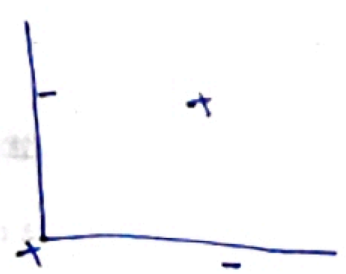
And now the decision rule is:

$$\sum \alpha_i y_i \vec{x}_i \cdot \vec{u} + b \geq 0 \text{ then } +.$$

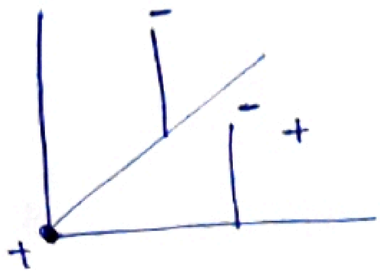
Summary

the whole thing depends on the dot product of all samples ($\vec{x}$) with the unknown ($\vec{u}$)

Till now we were talking about linearly separable data, But lets look at the following



this is not linearly separable so we take to a space where it is ~~separa~~ linearly separable.



transformation ϕ

$\phi(\bar{x})$

$\phi(x_i) \cdot \phi(\cancel{x}_j)$ to max.

$\phi(x_i) \cdot \phi(u)$

$K(x_i, x_j) = \phi(x_i) \cdot \phi(x_j)$

K is kernal function.

we need K and no need of ϕ.

different Kernels.

① choice 1.

$(\bar{u} \cdot \bar{u} + 1)^n$

② choice 2

$e^{-\frac{\| x_i - x_j \|}{\sigma}}$