

# Numerical performance of discontinuous and stabilized continuous Galerkin methods for convection–diffusion problems

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We compare the performance of two classes of numerical methods for the approximation of linear steady–state convection–diffusion equations, namely, the discontinuous Galerkin (DG) method and the continuous streamline upwind Petrov–Galerkin (SUPG) method. We present a fair comparison of such schemes considering both diffusion–dominated and convection–dominated regimes, and present numerical results obtained on a series of test problems including smooth solutions, and test cases with sharp internal and boundary layers.

## 1 INTRODUCTION

Convection–diffusion equations occur in the mathematical modeling of a wide range of phenomena as semiconductor devices modeling, magnetostatics and electrostatic flows, heat and mass-transfer, and flows in porous media related to oil and groundwater applications. The linear steady–state convection–diffusion equation with constant coefficients posed on a domain  $\Omega \subseteq \mathbb{R}^d$ ,  $d = 2, 3$ , is a boundary value problem of the form

$$\begin{aligned} -\varepsilon \Delta u + \operatorname{div}(\beta u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where  $f \in L^2(\Omega)$  is a given real-valued function,  $\varepsilon > 0$  is the diffusion parameter and  $\beta \in \mathbb{R}^d$ ,  $d = 2, 3$ , represents a constant (for simplicity) velocity field. In many physical phenomena the convection has much greater magnitude than the diffusion, i.e.,  $\|\beta\|/\varepsilon \gg 1$ . In such cases, problem (1) is an example of a singularly perturbed problem (with respect to  $\varepsilon$ ): that is, the solution in the case  $\varepsilon = 0$  (with boundary conditions prescribed not on the whole boundary  $\partial\Omega$  but on  $\partial\Omega_{\text{in}} = \{x \in \partial\Omega : \beta \cdot \mathbf{n}(x) < 0\}$  with  $\mathbf{n}(x)$  denoting the unit outward normal vector to  $\partial\Omega$  at the point  $x \in \partial\Omega$ , see (3) below) is not equal, at all points, to the limit of the solutions as  $\varepsilon \searrow 0^+$ . From the numerical viewpoint, the design of robust numerical schemes for the solution of (1) represents

a challenging problem, and indeed there has been an extensive development of discretizations methods for convection–diffusion equations that are robust in all the regimes, see (Roos et al. 1996), for example (cf. also (Ferziger and Perić 1999), (LeVeque 2007), (LeVeque 2002)). Among the numerical methods developed so far, we are interested here in discontinuous and stabilized continuous Galerkin finite element methods which restore two opposite paradigms within the finite element approach, namely non-conforming (discontinuous) versus conforming (continuous) approximation spaces.

Stabilized conforming finite element methods have been extensively developed for hyperbolic and parabolic conservation laws. The original conforming stabilized method for advection–diffusion equations, the Streamline-Upwind Petrov–Galerkin (SUPG), was developed by (Brooks and Hughes 1982) and then analyzed by (Johnson et al. 1984) (see also (Burman and Smith 2010) for the analysis of the SUPG method for transient convection–diffusion equations). It provides an upwinding effect to standard finite element methods using the Petrov–Galerkin framework. It is well known that the SUPG method has the capability of improving numerical stability for convection–dominated flows, while satisfying a strong consistency property. Like many other conforming stabil-

ized methods, the SUPG scheme contains an element-wise stabilization parameter  $\tau$  that has to be tuned in practice, and, except for simplified situations, the “optimal” value is not known. More precisely, for linear finite element discretizations on a uniform decomposition of a one-dimensional problem with constant coefficients  $\varepsilon \in \mathbb{R}^+$ ,  $\beta \in \mathbb{R}$ , and constant data, it can be shown that the choice

$$\tau = \frac{h}{2|\beta|} \left( \coth(\text{Pe}_T) - \frac{1}{\text{Pe}_T} \right), \quad (2)$$

where  $\text{Pe}_T$  is the mesh Péclet number (cf. also (6), below), leads to a nodally exact approximate solution (Brooks and Hughes 1982), see also (Christie et al. 1976). In the multi-dimensional case, many different choices of the stabilization parameter  $\tau$  have been proposed see (Franca et al. 2001), and the references therein. It is also known that the choice of the stabilization parameter  $\tau$  also influences the behavior of the iterative solvers employed to solve the resulting linear system of equations. For example, GMRES converges more slowly for worse stabilization parameters, as pointed out in (Liesen and Strakos 2005).

On the other hand, discontinuous Galerkin (DG) methods are parameters free since they do not rely on the addition of *ad-hoc tuned* streamline stabilization terms. Since their introduction in 1970s (Reed and Hill 1973), (Douglas and Dupont 1976), (Arnold 1982), the discontinuous Galerkin method has become a topic of extensive research for the numerical solution of differential equations. DG methods employ piecewise polynomial spaces which may be discontinuous across elements boundaries. Therefore, DG can accommodate non-matching meshes as well as variable polynomial approximation orders. These features make them ideally suited for both geometrical and functional adaptive discretizations. Moreover, DG methods are locally mass conservative and can capture possible discontinuities in the solution thanks to the discontinuous approximation spaces. For such capabilities, DG methods have been extensively developed for convection–diffusion problems, cf. for example (Cockburn 1999), (Baumann and Oden 1999), (Houston et al. 2002), (Buffa et al. 2006), (Antonietti and Houston 2008), (Ayuso de Dios and Marini 2009), and the references therein.

In this paper we aim at comparing DG and SUPG solutions, working, for the sake of simplicity, on a linear steady–state convection–diffusion equation with constant coefficients. A detailed assessment of the performance of such schemes (employing the lowest order elements) with respect to accuracy is presented in a number of test problems featuring smooth solutions, and other problems featuring sharp layers

(either internal or boundary’s).

The rest of the paper is organized as follows. In Section 2 we introduce some notation and recall the variational formulation of problem (1). In Section 3 and Section 4 we recall the DG and SUPG formulations and the corresponding error estimates, respectively. Section 4 contains an extensive set of numerical experiments to compare the DG and SUPG solutions. Finally, in Section 6 we draw some conclusions.

## 2 MODEL PROBLEM

We consider the linear steady–state convection–diffusion equation with constant coefficients (1) posed on  $\Omega$ , a bounded convex open polygonal (resp. polyhedral) domain in  $\mathbb{R}^2$  (resp.  $\mathbb{R}^3$ ). The corresponding weak formulation reads as: Find  $u \in H_0^1(\Omega)$  such that

$$\int_{\Omega} (\varepsilon \nabla u - \beta u) \cdot \nabla v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H_0^1(\Omega).$$

The existence of a unique solution of the above problem follows from the Lax-Milgram Theorem. For the analysis of more general convection–diffusion equations see, e.g., (Quarteroni and Valli 1994). For the design and analysis of DG methods in the case of a diffusion tensor whose entries are bounded, piecewise continuous real-valued functions defined on  $\bar{\Omega}$  and a piecewise continuous real-valued velocity field, we refer to (Ayuso de Dios and Marini 2009). By  $\mathbf{n} \equiv \mathbf{n}(x)$  we denote the unit outward normal vector to  $\partial\Omega$  at the point  $x \in \partial\Omega$ , and define the sets  $\partial\Omega_{\text{in}}$  and  $\partial\Omega_{\text{out}}$  of inflow and outflow boundary, respectively, as

$$\begin{aligned} \partial\Omega_{\text{in}} &= \{x \in \Gamma : \beta \cdot \mathbf{n} < 0\}, \\ \partial\Omega_{\text{out}} &= \{x \in \Gamma : \beta \cdot \mathbf{n} \geq 0\}. \end{aligned} \quad (3)$$

## 3 DG AND SUPG DISCRETISATIONS

The aim of this section is to introduce the DG and SUPG discretisations.

Let  $\mathcal{T}_h$  be a conforming shape-regular quasi-uniform partition of  $\Omega$  into disjoint open triangles  $T$  where each  $T \in \mathcal{T}_h$  is the affine image of the reference open unit  $d$ -simplex in  $\mathbb{R}^d$ ,  $d = 2, 3$ . Denoting by  $h_T$  the diameter of an element  $T \in \mathcal{T}_h$ , we define the mesh size  $h = \max_{T \in \mathcal{T}_h} h_T$ .

### 3.1 THE DG METHOD

We start by introducing suitable trace operators, defined in the usual way (Arnold, Brezzi, Cockburn, and Marini 2002). Let  $\mathcal{F}_h^I$  and  $\mathcal{F}_h^B$  be the sets of all the interior and boundary  $(d-1)$ -dimensional (open) faces (if  $d = 2$ , “face” means “edge”), respectively, we set  $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^B$ . We also decompose  $\mathcal{F}_h^B = \mathcal{F}_h^{B_{\text{out}}} \cup \mathcal{F}_h^{B_{\text{in}}}$ , where  $\mathcal{F}_h^{B_{\text{out}}} = \{F \in \mathcal{F}_h^B : F \subset \partial\Omega_{\text{out}}\}$

and  $\mathcal{F}_h^{B_{\text{in}}} = \{F \in \mathcal{F}_h^B : F \subset \partial\Omega_{\text{in}}\}$ . Implicit in these definitions is the assumption that  $\mathcal{T}_h$  respects the decomposition of  $\partial\Omega$  in the sense that each  $F \in \mathcal{F}_h$  that lies on  $\partial\Omega$  belongs to the interior of exactly one of  $\partial\Omega_{\text{out}}$ ,  $\partial\Omega_{\text{in}}$ . Let  $F \in \mathcal{F}_h^I$  be an interior face shared by the elements  $T^+$  and  $T^-$  with outward unit normal vectors  $\mathbf{n}^+$  and  $\mathbf{n}^-$ , respectively. For piecewise smooth vector-valued and scalar functions  $\boldsymbol{\tau}$  and  $z$ , respectively, let  $\boldsymbol{\tau}^\pm$  and  $z^\pm$  be the traces of  $\boldsymbol{\tau}$  and  $z$  on  $\partial T^\pm$  taken within the interior of  $T^\pm$ , respectively. We define the *jump* and the *average* across  $F$  by

$$[\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+ \cdot \mathbf{n}^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^-, \quad [z] = z^+ \mathbf{n}^+ + z^- \mathbf{n}^-,$$

$$\{\{\boldsymbol{\tau}\}\} = (\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-)/2, \quad \{\{z\}\} = (z^+ + z^-)/2.$$

On a boundary face  $F \in \mathcal{F}_h^B$ , we set, analogously,

$$[z] = z \mathbf{n}, \quad \{\{\boldsymbol{\tau}\}\} = \boldsymbol{\tau}.$$

We do not need either  $[\![\boldsymbol{\tau}]\!]$  or  $\{\{v\}\}$  on boundary faces, and leave them undefined. On each interior face  $F \in \mathcal{F}_h^I$ , we introduce an *upwind* average  $\{\{\cdot\}\}_\beta$  defined as

$$\{\{\beta u\}\}_\beta = \begin{cases} \beta u^+ & \text{if } \beta \cdot \mathbf{n}^+ > 0 \\ \beta u^- & \text{if } \beta \cdot \mathbf{n}^+ < 0 \\ \beta \{\{u\}\} & \text{if } \beta \cdot \mathbf{n}^+ = 0 \end{cases}$$

On  $F \in \mathcal{F}_h^B$ , we set  $\{\{\beta u\}\}_\beta = \beta u$ , if  $\beta \cdot \mathbf{n} > 0$ ,  $\{\{\beta u\}\}_\beta = 0$  otherwise. For a given approximation order  $\ell \geq 1$ , we introduce the DG finite element space  $V_h^{\text{DG}}$  as

$$V_h^{\text{DG}} = \{v \in L^2(\Omega) : v|_T \in \mathbb{P}^\ell(T) \quad \forall T \in \mathcal{T}_h\},$$

where  $\mathbb{P}^\ell(T)$  is the set of polynomials of total degree  $\ell$  on  $T$ .

We define the bilinear form  $\mathcal{A}_h^{\text{DG}}(\cdot, \cdot) : V_h^{\text{DG}} \times V_h^{\text{DG}} \rightarrow \mathbb{R}$  as

$$\begin{aligned} \mathcal{A}_h^{\text{DG}}(u, v) &= \sum_{T \in \mathcal{T}_h} \int_T (\varepsilon \nabla u - \beta u) \cdot \nabla v \, dx \\ &\quad - \sum_{F \in \mathcal{F}_h} \int_F (\{\{\varepsilon \nabla u\}\} \cdot [v] + [u] \cdot \{\{\varepsilon \nabla v\}\}) \, ds \\ &\quad + \sum_{F \in \mathcal{F}_h} \int_F (\sigma_F [u] \cdot [v] + \{\{\beta u\}\}_\beta \cdot [v]) \, ds \end{aligned}$$

with the convention that the application of the operator  $\nabla$  has to be intended elementwise whenever the

function to which is applied is elementwise discontinuous. Here  $\sigma_F$  is defined for all  $F \in \mathcal{F}_h$  as

$$\sigma_F = \alpha \frac{\varepsilon \ell^2}{h_F} \quad \forall F \in \mathcal{F}_h, \quad (4)$$

with  $\alpha$  a positive real number at our disposal, and  $h_F$  the diameter of the face  $F \in \mathcal{F}_h$ . The DG approximation of problem (1) is defined as follows : find  $u_h^{\text{DG}} \in V_h^{\text{DG}}$  such that

$$\mathcal{A}_h^{\text{DG}}(u_h^{\text{DG}}, v_h) = \int_\Omega f v_h \, dx \quad \forall v_h \in V_h^{\text{DG}}.$$

**Remark 3.1.** For the approximation of the convection and diffusion terms we have employed a DG method with upwinded fluxes (Brezzi et al. 2004) and the symmetric interior penalty (SIPG) method (Arnold 1982), respectively. With minor changes, any other (possibly unsymmetric) DG approximation of the second order term could be considered.

We close this section observing that, denoting by  $\mathbf{n}_F$  the normal to the face  $F \in \mathcal{F}_h$ , it holds

$$\begin{aligned} \sum_{F \in \mathcal{F}_h} \int_F \{\{\beta u\}\}_\beta \cdot [v] \, ds &= \\ \sum_{F \in \mathcal{F}_h^I \cup \mathcal{F}_h^{B_{\text{out}}}} \int_F \{\{\beta u\}\} \cdot [v] \, ds &+ \sum_{F \in \mathcal{F}_h^I} \int_F \frac{|\beta \cdot \mathbf{n}_F|}{2} [u] \cdot [v] \, ds, \end{aligned}$$

for all  $u, v \in V_h^{\text{DG}}$ . Therefore, the *upwind* value of  $\beta u$ , can be written as the sum of the usual (symmetric) average plus a jump penalty term. Such equivalent representation has several advantages. For example, notice that both terms on the right hand side are of the same kind as the ones already present in the treatment of the diffusive part of the operator. This can be favorably exploited in the implementation process.

### 3.2 THE SUPG METHOD

Let the SUPG discrete space be defined as the subspace of  $V_h^{\text{DG}}$  of continuous polynomials that also satisfy the boundary condition, i.e.,

$$V_h^{\text{SUPG}} = \{v \in C^0(\overline{\Omega}) : v|_T \in \mathbb{P}^\ell(T) \quad \forall T \in \mathcal{T}_h,$$

$$\text{and } v = 0 \text{ on } \partial\Omega\},$$

For piecewise positive parameters  $\tau_T$  that we will specify later, we define the bilinear form  $\mathcal{A}_h^{\text{SUPG}}(\cdot, \cdot) : V_h^{\text{SUPG}} \times V_h^{\text{SUPG}} \rightarrow \mathbb{R}$  as

$$\begin{aligned} \mathcal{A}_h^{\text{SUPG}}(u, v) &= \int_{\Omega} (\varepsilon \nabla u - \beta u) \cdot \nabla v \, dx \\ &+ \sum_{T \in \mathcal{T}_h} \int_T \tau_T (-\varepsilon \Delta u + \beta \cdot \nabla u) \beta \cdot \nabla v \, dx, \end{aligned}$$

and the functional  $\mathcal{F}_h^{\text{SUPG}}(\cdot) : V_h^{\text{SUPG}} \rightarrow \mathbb{R}$

$$\mathcal{F}_h^{\text{SUPG}}(v) = \sum_{T \in \mathcal{T}_h} \int_T f(v + \tau_T \beta \cdot \nabla v) \, dx.$$

Then, the SUPG formulation reads: find  $u_h^{\text{SUPG}} \in V_h^{\text{SUPG}}$  such that

$$\mathcal{A}_h^{\text{SUPG}}(u_h^{\text{SUPG}}, v_h) = \mathcal{F}_h^{\text{SUPG}}(v_h) \quad \forall v_h \in V_h^{\text{SUPG}}.$$

It is well known that the choice of the stabilization parameters  $\tau_T$ ,  $T \in \mathcal{T}_h$ , is a delicate matter since it may influence the performance of the method. For such a reason, the problem of finding the optimal stabilization parameter has been a subject of an extensive research in the last years, and many choices are available in the literature (cf. (John and Knobloch 2007), (John and Knobloch 2008), for example). Here, we consider a generalization of (2) to the multi-dimensional case and set

$$\tau|_T \equiv \tau_T = \frac{h_T}{2\|\beta\|\ell} \xi(\text{Pe}_T), \quad (5)$$

with  $\text{Pe}_T$  being the local Péclet number, i.e.,

$$\text{Pe}_T = \frac{\|\beta\| h_T}{2\varepsilon \ell}, \quad (6)$$

and where  $\|\beta\|$  is the Euclidean norm of  $\beta$ . The upwind function  $\xi(\cdot)$  can be chosen, for instance, as

$$\xi(\theta) = \coth(\theta) - 1/\theta, \quad \theta > 0, \quad (7)$$

cf. Figure 1. Notice that  $\xi(\theta) \rightarrow 1$  for  $\theta \rightarrow \infty$ , and  $\xi(\theta)/\theta \rightarrow 1/3$  for  $\theta \rightarrow 0^+$  (and the SUPG stabilization is not necessary for  $\theta \rightarrow 0^+$ ). Moreover, if convection strongly dominates diffusion, i.e.,  $\text{Pe}_T \gg 1$ , we have  $\tau_T \approx h_T/(2\|\beta\|\ell)$ . As already pointed out, many other definitions of  $\tau_T$  are possible. For example, in (5) and (6)  $h_T$  can be replaced by the element diameter in the direction of the convection  $\beta$ , or the upwind function  $\xi(\cdot)$  can be approximated either by  $\xi(\theta) \approx \max\{0, 1 - 1/\theta\}$  or by  $\xi(\theta) \approx \min\{1, \theta/3\}$ . We refer to (John and Knobloch 2007), (John and Knobloch 2008) (Brooks and Hughes 1982), for further details.

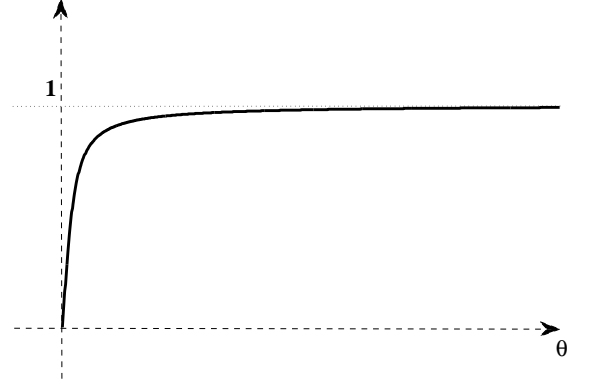


Figure 1: Stabilization function  $\xi(\theta) = \coth(\theta) - 1/\theta$ .

#### 4 ERROR ESTIMATES

In this section we briefly recall the error estimates for the DG and SUPG formulations.

To deal with the convection-dominated regime, we first introduce a norm which controls the streamline derivative:

$$\|v\|_{\beta} = \left( \sum_{T \in \mathcal{T}_h} \|\tau_T^{1/2} \beta \cdot \nabla v\|_{0,T}^2 \right)^{1/2} \quad \forall v \in H^1(\mathcal{T}_h), \quad (8)$$

where  $\tau$  is defined in (5), and where the piecewise broken Sobolev space  $H^1(\mathcal{T}_h)$  consists of functions  $v$  such that  $v|_T \in H^1(T)$  for any  $T \in \mathcal{T}_h$ . We endow the DG space with the norm

$$\|v\|_{\text{DG}} = (\|v\|^2 + \|v\|_{\beta}^2)^{1/2} \quad \forall v \in V_h^{\text{DG}}, \quad (9)$$

where  $\|\cdot\|_{\beta}$  is defined in (8), and where

$$\|v\|^2 = \sum_{T \in \mathcal{T}_h} \|\varepsilon^{1/2} \nabla v\|_{0,T}^2 + \sum_{F \in \mathcal{F}_h} \|\alpha_F^{1/2} \llbracket v \rrbracket\|_{0,F}^2$$

for all  $v \in V_h^{\text{DG}}$ . Here  $\alpha_F$  is defined as

$$\alpha_F = \sigma_F + |\beta \cdot \mathbf{n}| \quad \forall F \in \mathcal{F}_h,$$

with  $\sigma_F$  given in (4).

Provided the exact solution  $u$  is regular enough and the stability parameter  $\alpha$  in (4) is chosen sufficiently large, the following error estimates hold for the DG solution

$$\begin{aligned} \|u - u_h^{\text{DG}}\| &\lesssim h^{\ell} (\varepsilon^{1/2} + h^{1/2} \|\beta\|_{\infty}^{1/2}), \\ \|u - u_h^{\text{DG}}\|_{\text{DG}} &\lesssim h^{\ell} (\varepsilon^{1/2} + h^{1/2} \|\beta\|_{\infty}^{1/2}). \end{aligned} \quad (10)$$

We refer to (Houston et al. 2002) and (Ayuso de Dios and Marini 2009) for the proof (in a more general framework) and further details.

For the SUPG method, we define the norm

$$\|v\|_{\text{SUPG}} = \left( \|\varepsilon^{1/2} \nabla v\|_{L^2(\Omega)}^2 + \|v\|_{\beta}^2 \right)^{1/2} \quad (11)$$

for all  $v \in V_h^{\text{SUPG}}$ . Then, the following estimate holds:

$$\|u - u_h^{\text{SUPG}}\|_{\text{SUPG}} \lesssim h^{\ell} (\varepsilon^{1/2} + h^{1/2} \|\beta\|_{\infty}^{1/2}), \quad (12)$$

see (Quareroni and Valli 1994), for example.

Note that the two error estimates (10) and (12) feature the same right hand side (apart from the hidden multiplicative constants).

**Remark 4.1.** In the error estimates (10) and (12) the hidden constants depend on the domain  $\Omega$ , the shape regularity constant of the partition  $\mathcal{T}_h$ , and the polynomial approximation degree  $\ell$ , but are independent of the mesh size  $h$  and the physical parameters  $\varepsilon$  and  $\beta$  of the problem.

## 5 NUMERICAL RESULTS

We compare the numerical performance of the DG and SUPG methods on four test cases. In the first example, we choose a smooth exact solution and compare the approximation properties of the two schemes. In the second and third examples, the exact solutions exhibit an internal and an exponential boundary layer, respectively. Finally, in the last example the (unknown) analytical solution features sharp internal and boundary layers. Throughout this section

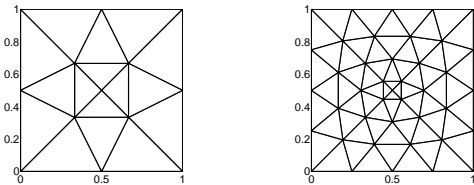


Figure 2: Left: Initial triangulations with mesh size  $h_0$ . Right: triangulation obtained after  $R = 2$  levels of refinement (mesh size  $h_0/4$ ).

we set  $\Omega = (0, 1) \times (0, 1)$  and consider a sequence of uniform unstructured triangulations obtained by uniformly refining the initial mesh shown in Figure 2 (left). Denoting by  $h_0$  the mesh size of this initial grid, after  $R = 1, 2, \dots$  refinements the corresponding mesh size is given by  $h = h_0/2^R$  (cf. Figure 2 (right), for  $R = 2$ ). Since the grids are quasi uniform, it holds  $h \approx 1/\sqrt{N}$ ,  $N$  being the number of elements of the partition. Finally, throughout the section, we restrict ourselves to piecewise linear elements, i.e.,  $\ell = 1$ .

### 5.1 EXAMPLE 1

In this section, we aim at comparing the approximation properties of the DG and SUPG schemes on a smooth analytical solution. To this aim, we chose  $\beta = (1, 1)^T$ , and the source term  $f$  such that the solution of problem (1) is  $u(x, y) = \sin(2\pi x)(y - y^2)$ , for any positive diffusion coefficient  $\varepsilon$ . Figure 3 (log-log scale) shows the computed errors in the energy norms (9) and (11) versus  $1/h$ , for different choices of the diffusion coefficient  $\varepsilon$ ,  $\varepsilon = 10^{-k}$ ,  $k = 0, 1, 5, 9$ . The expected convergence rates are clearly observed:  $O(h)$  and  $O(h^{3/2})$  convergence rates are attained in the diffusion- and convection-dominated regimes, respectively. We have also compared the errors com-

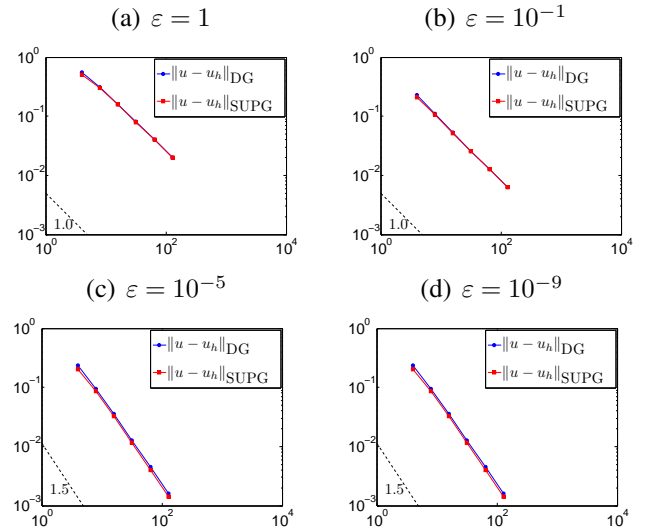


Figure 3: Example 1. Errors in the energy norms (9) and (11) for  $\varepsilon = 10^{-k}$ ,  $k = 0, 1, 5, 9$ .

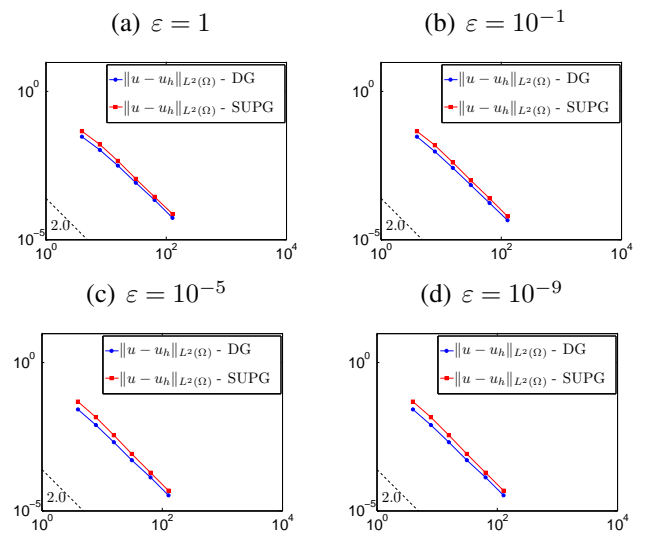


Figure 4: Example 1. Errors in the  $L^2(\Omega)$  norm for  $\varepsilon = 10^{-k}$ ,  $k = 0, 1, 5, 9$ .

puted in the  $L^2(\Omega)$  and  $L^\infty(\Omega)$  norms. The results are

reported in Figure 4 (log-log scale) and in Figure 5 (log-log scale), respectively. For all the cases, a quadratic convergence rate is clearly observed, for any value of the diffusion coefficient  $\varepsilon$ . We have run the same set of experiments on a sequence of structured triangular grids, the results (not reported here, for the sake of brevity) are completely analogous. From the

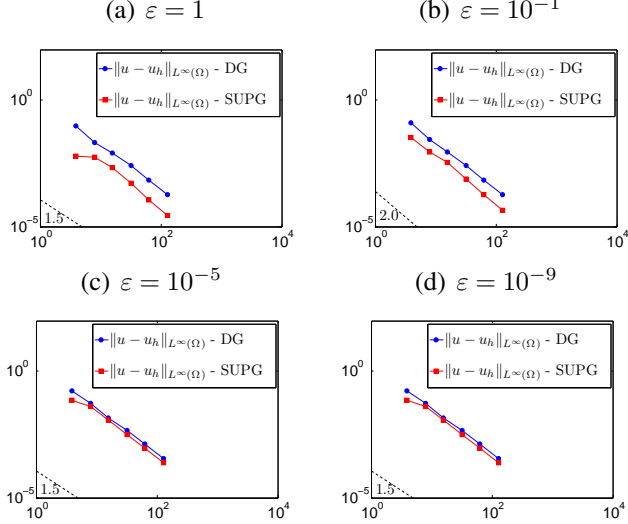


Figure 5: Example 1. Errors in the  $L^\infty(\Omega)$  norm for  $\varepsilon = 10^{-k}$ ,  $k = 0, 1, 5, 9$ .

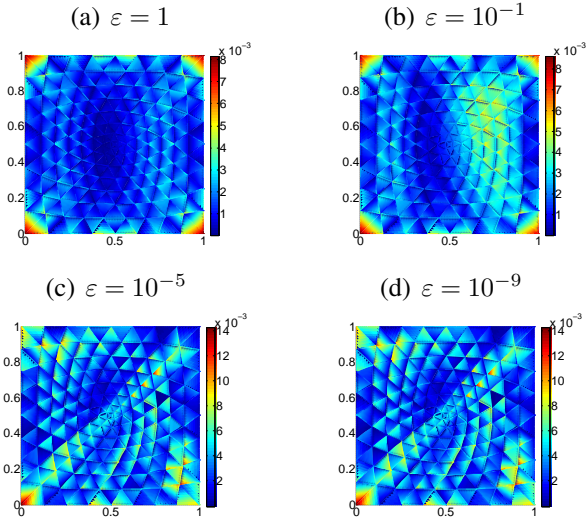


Figure 6: Example 1. Absolute value of the pointwise difference between the DG and SUPG approximate solutions for  $\varepsilon = 10^{-k}$ ,  $k = 0, 1, 5, 9$ . Grids with mesh sizes  $h = h_0/4$ .

results reported in Figures 3, 4, 5, we can infer that in the case of a smooth analytical solution, the DG and SUPG enjoy almost the same accuracy, for any value of the diffusion coefficient  $\varepsilon$ . Indeed, the error curves reported in Figures 3 show that the errors computed in the energy norms (9) and (11) are almost identical.

Concerning the error in average (cf. Figure 4) the DG method seems to be slightly more accurate than the SUPG; whereas if we measure the error in the  $L^\infty(\Omega)$  norm the SUPG method seems to be slightly better than the DG method (cf. Figure 5). Nevertheless, the difference between the DG and SUPG solutions are negligible, as highlighted by the results reported in Figure 6 where we have plotted, for the same choices of  $\varepsilon$ , the pointwise absolute value of the difference between the DG and SUPG numerical solutions, on a grid with mesh size  $h = h_0/4$ .

## 5.2 EXAMPLE 2

In the second example, we set again  $\beta = (1, 1)$  and let  $\varepsilon$  vary, however now the exact solution reads

$$u(x, y) = -\text{atan} \left( \frac{(x - 1/2)^2 + (y - 1/2)^2 - 1/16}{\sqrt{\varepsilon}} \right),$$

(the source term  $f$  and the non-homogeneous boundary conditions being set accordingly). We notice that, as  $\varepsilon \searrow 0^+$ , the solution exhibits a sharp internal layer, cf. Figure 7 (left) for  $\varepsilon = 10^{-3}$  and Figure 7 (right) for  $\varepsilon = 10^{-5}$ .

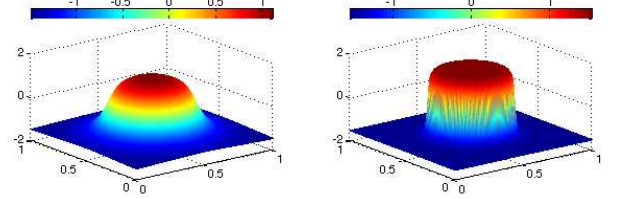


Figure 7: Example 2. Exact solutions for  $\varepsilon = 10^{-3}$  (left) and  $\varepsilon = 10^{-5}$  (right).

Figure 8 compare the DG and SUPG discrete solutions, respectively, obtained on an unstructured triangular grid of mesh size  $h = h_0/4$ , for  $\varepsilon = 10^{-k}$ ,  $k = 1, 2, 3, 4, 5$ . We can observe that the discrete solutions approximate quite well the corresponding analytical solution. Nevertheless, in the convection-dominated regime both methods exhibit spurious oscillations near the internal layer.

We next compare the numerical solutions provided by the two methods. In Figure 9 the pointwise absolute error of the DG and SUPG approximate solutions is reported on a grid with mesh size  $h = h_0/4$ , for  $\varepsilon = 10^{-k}$ ,  $k = 1, 2, 3, 4, 5$ . Notice that, in this case, the approximate solutions provided by the two methods are almost identical in the diffusion-dominated regime (cf. Figure 9, top), whereas they differ substantially near the internal layer in the convection-dominated regime (cf. Figure 9, bottom). Indeed, as  $\varepsilon$  becomes small, they both exhibit oscillations near the internal layer, but of different kind.



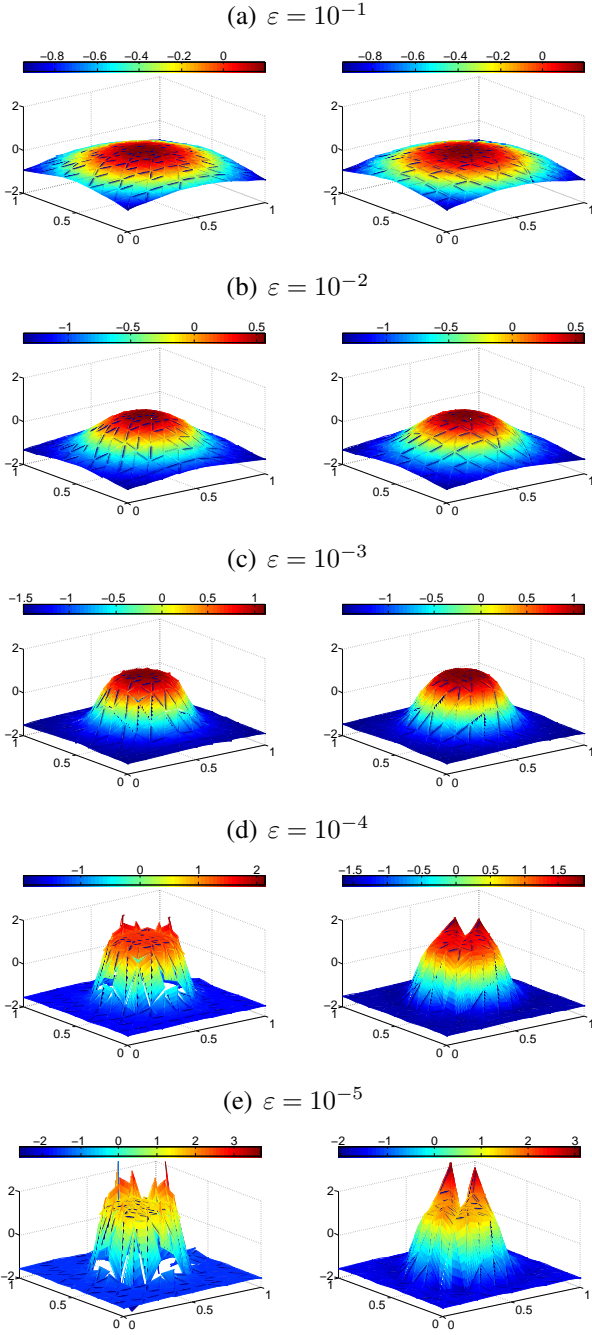


Figure 8: Example 2. DG (left) and SUPG (right) approximate solutions on an grid of mesh size  $h = h_0/4$  for  $\varepsilon = 10^{-k}$ ,  $k = 1, 2, 3, 4, 5$ .

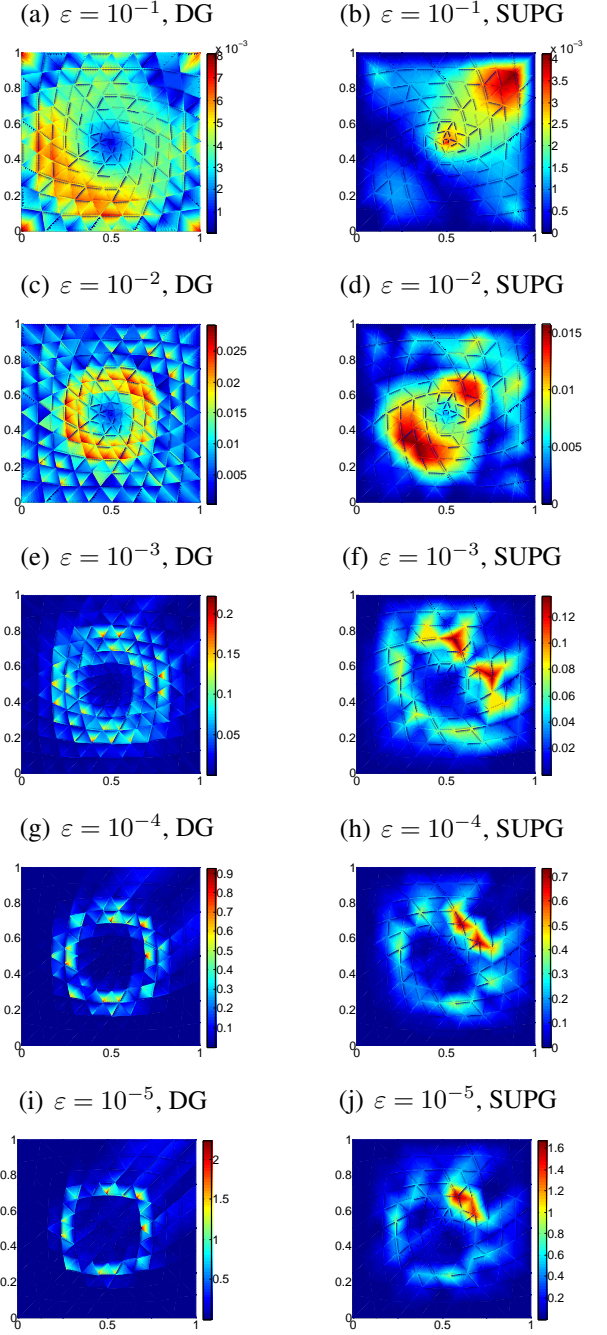


Figure 9: Example 2. Pointwise absolute error of the DG and SUPG approximate solutions for  $\varepsilon = 10^{-k}$ ,  $k = 1, 2, 3, 4, 5$ . Mesh size  $h = h_0/4$ .

Finally, for  $\varepsilon = 10^{-k}$ ,  $k = 1, 2, 3, 4$ , Figure 10 (log-log scale) and Figure 11 (log-log scale) show the computed errors in the energy norms (9) and (11) and in the  $L^2(\Omega)$  norm, respectively. The results reported in Figure 10 and Figure 11 have been obtained on unstructured triangular grids as the ones shown in Figure 2; analogous results (not reported here) have been obtained on structured triangular grids and the same convergence behavior has been observed. The error in the energy norms goes to zero at the predicted rate, namely linearly in the diffusion-dominated case and at a rate of  $3/2$  in the convection-dominated regime. Nevertheless, notice that such a rate seems to be achieved only asymptotically for  $\varepsilon = 10^{-4}$  by both the DG and SUPG methods. Concerning the mean error

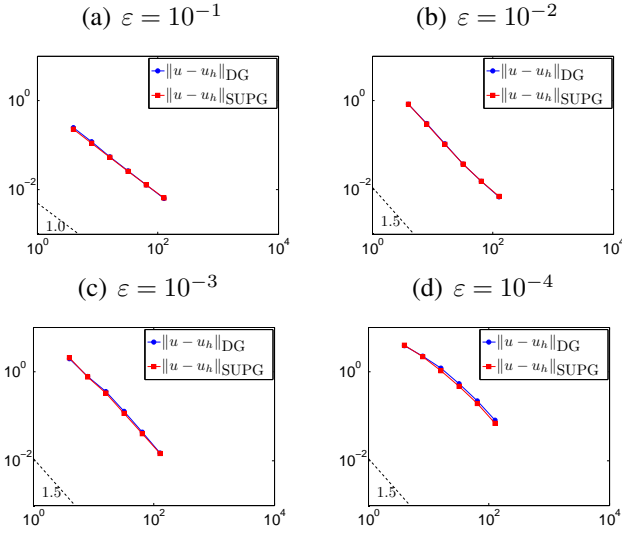


Figure 10: Example 2. Errors in the energy norms for  $\varepsilon = 10^{-k}$ ,  $k = 1, 2, 3, 4$ .

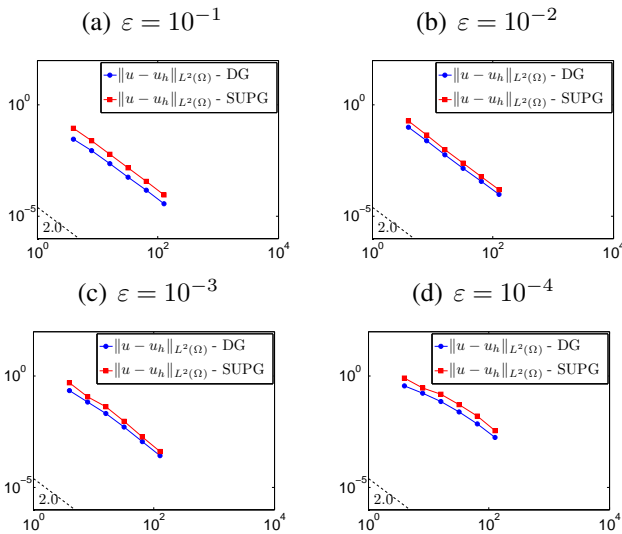


Figure 11: Example 2. Errors in the  $L^2(\Omega)$  norm for  $\varepsilon = 10^{-k}$ ,  $k = 1, 2, 3, 4$ .

(cf. Figure 11), we observe that both methods achieve

a quadratic convergence rate, and the DG scheme seems to be slightly more accurate than the SUPG method. If we measure the errors in the  $L^\infty(\Omega)$  norm (cf. Figure 12) we observe again a quadratic convergence rate but in this case the SUPG method seems to be slightly more accurate.

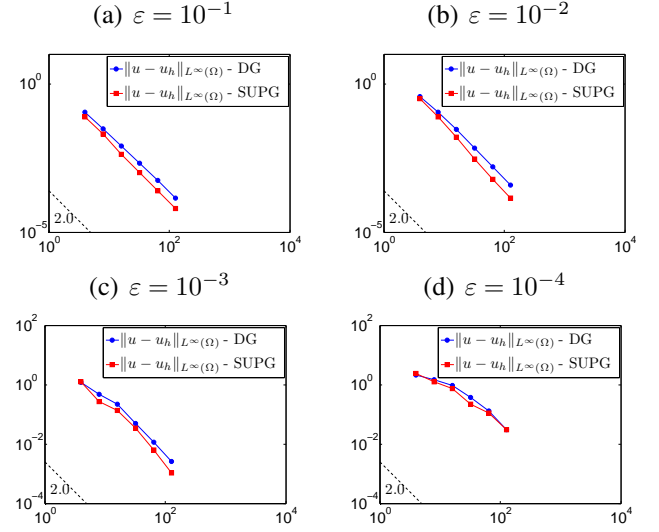


Figure 12: Example 2. Errors in the  $L^\infty(\Omega)$  norm for  $\varepsilon = 10^{-k}$ ,  $k = 1, 2, 3, 4$ .

### 5.3 EXAMPLE 3

In this example, we consider a test case with an exponential boundary layer taken from (Houston et al. 2002) and (Ayuso de Dios and Marini 2009), which is the multi-dimensional version of the one-dimensional test proposed by (Melenk and Schwab 1999). We set  $\beta = (1, 1)$ , and vary the diffusion coefficient  $\varepsilon$ . The right hand side  $f$  and the non-homogeneous boundary condition are chosen in such a way that the exact solution is given by:

$$u(x, y) = x + y - xy + \frac{e^{-1/\varepsilon} - e^{-(1-x)(1-y)/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

Figure 13 shows the discrete solutions computed with the DG and SUPG methods for  $\varepsilon = 10^{-k}$ ,  $k = 0, 1, 2, 9$  on a grid with mesh size  $h = h_0/4$ . We can observe that, for  $\varepsilon = 1, 10^{-1}$  the two solutions are almost identical, whereas in the intermediate regime  $\varepsilon = 10^{-2}$  the DG method exhibits some spurious oscillations near the boundary layer, while the SUPG solution seems to be definitely more accurate. Finally, we observe that in the convection-dominated regime, the DG solution is totally free of spurious oscillations but the boundary layer is not captured. Such a behavior, already observed in (Ayuso de Dios and Marini 2009), is due to the fact that the boundary conditions are imposed weakly. On the other hand, for  $\varepsilon = 10^{-9}$  the SUPG method results in overshoot near the point



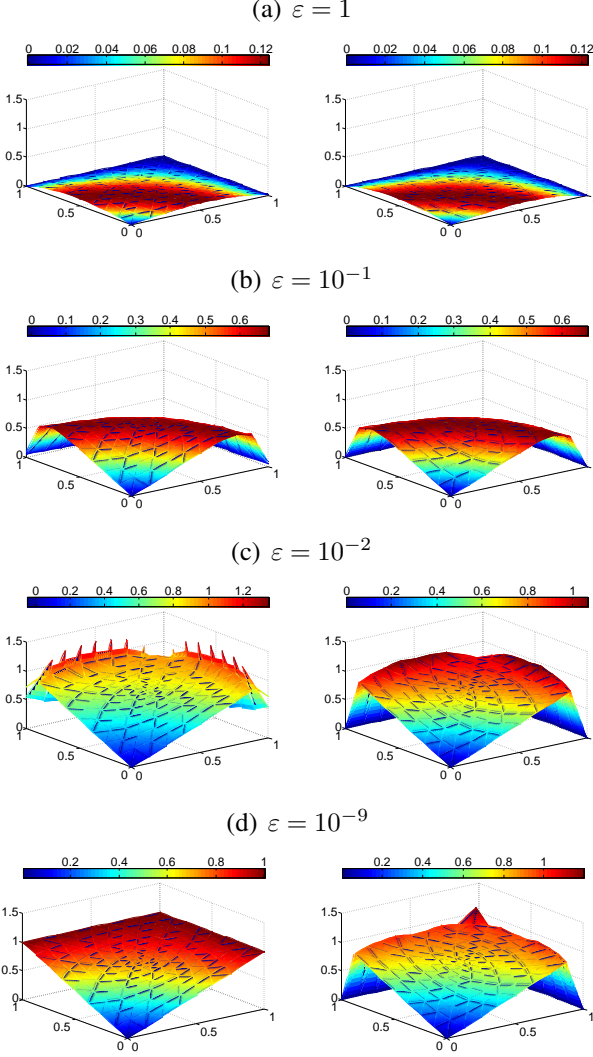


Figure 13: Example 3. DG (left) and SUPG (right) discrete solutions for  $\varepsilon = 10^{-k}$   $k = 0, 1, 2, 9$ . Mesh size  $h = h_0/4$ .

(1, 1). The above considerations are also confirmed by the results reported in Figure 14 where the absolute value of the pointwise error is shown on grid with mesh size  $h = h_0/4$ .

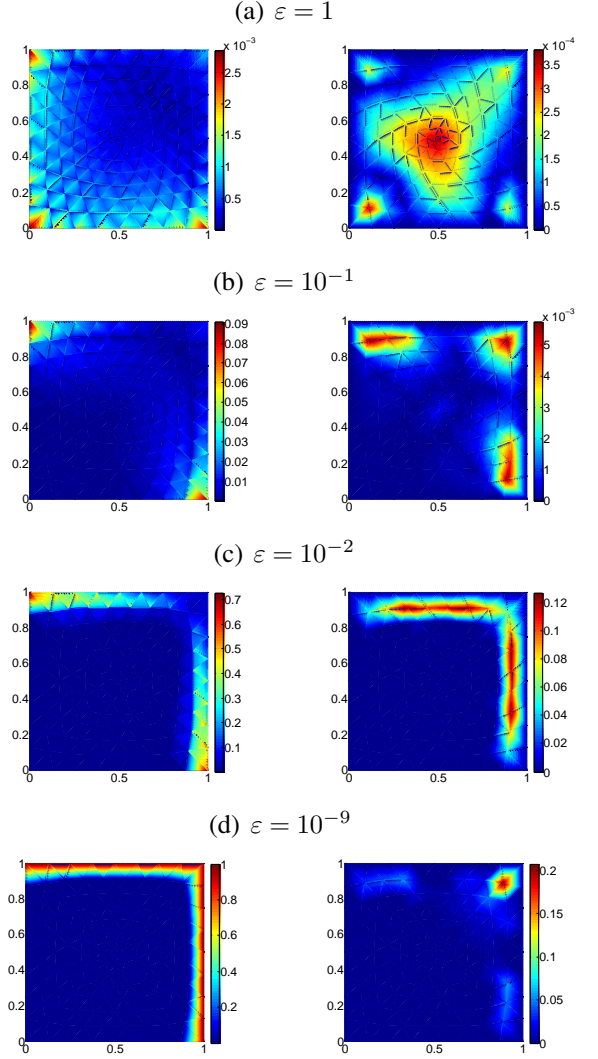


Figure 14: Example 3. Pointwise absolute error of the DG and SUPG approximate solutions for  $\varepsilon = 10^{-k}$ ,  $k = 0, 1, 2, 9$ . Mesh size  $h = h_0/4$ .

Finally, Figure 15 shows the computed errors in the  $L^2(\Omega)$  and  $L^1(\Omega)$  norms for  $\varepsilon = 10^{-k}$ ,  $k = 0, 1, 9$ . In the diffusion-dominated regime and for both the methods the errors go to zero quadratically as the mesh is refined, whereas for  $\varepsilon = 10^{-9}$  the DG method is quadratically convergent in both the  $L^2(\Omega)$  and  $L^1(\Omega)$  norm while the SUPG method convergences at a rate of order  $O(h^{1/2})$  and  $O(h)$  in the  $L^2(\Omega)$  and  $L^1(\Omega)$  norms, respectively.

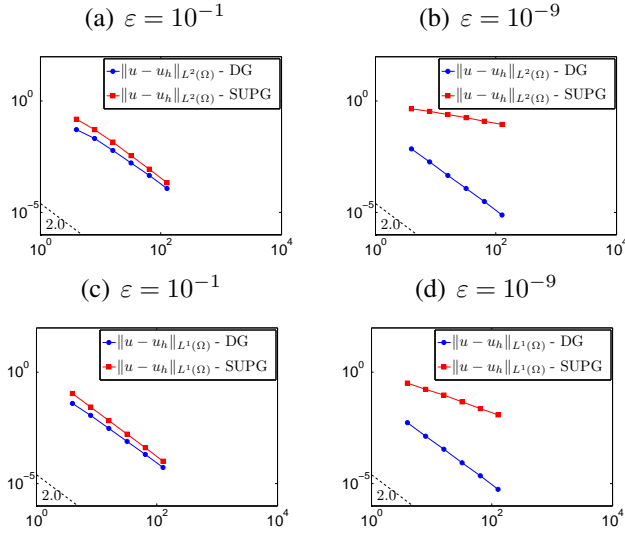


Figure 15: Example 3. Errors in the  $L^2(\Omega)$  (top) and  $L^1(\Omega)$  (bottom) norms for  $\varepsilon = 10^{-k}$ ,  $k = 1, 9$ .

#### 5.4 EXAMPLE 4

In the last example, always with  $\Omega = (0, 1)^2$ , we consider the following convection–diffusion equation:

$$-\varepsilon \Delta u + \beta \cdot \nabla u = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega.$$

The Dirichlet boundary conditions are set as follows:

$$g = \begin{cases} 1 & \text{if } y = 0, \\ 1 & \text{if } x = 0 \text{ and } y \leq 1/2, \\ 0 & \text{otherwise,} \end{cases}$$

the velocity  $\beta$  is chosen as  $\beta = (\cos(\pi/3), \sin(\pi/3))$ , and we let  $\varepsilon$  vary. Notice that in this case, a close expression for the exact solution is not available. We know however that, as  $\varepsilon$  tends to zero, the exact solution exhibits a strong internal and boundary layer.

We compare the discrete solutions computed by the DG and SUPG schemes for  $\varepsilon = 10^{-k}$ ,  $k = 0, 3, 9$ . The results on the unstructured triangular mesh with mesh size  $h = h_0/8$  are shown in Figure 16. The following conclusions can be drawn:

- i) in the diffusion–dominated regime, the DG and SUPG methods are substantially equivalent;
- ii) in the intermediate regime, the DG method exhibits spurious oscillations, whereas the SUPG method seems to be more stable and only an overshooting effect is observed near the point  $(1, 1)$ ;
- iii) in the convection–dominated regime, DG method exhibits some spurious oscillations on the internal layer whereas no oscillations along boundary layer are present, however unfortunately the

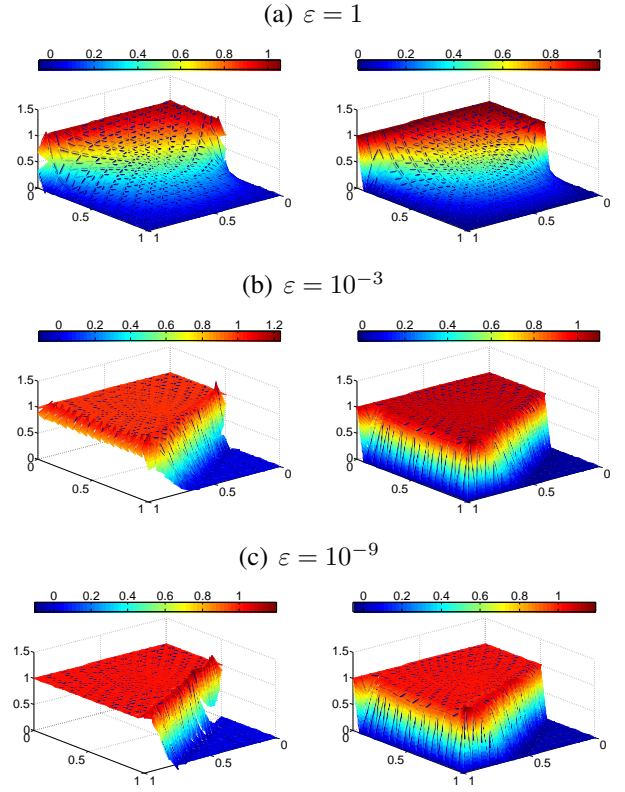


Figure 16: Example 4. DG (left) and SUPG (right) discrete solutions for  $\varepsilon = 10^{-k}$ ,  $k = 0, 3, 9$ . Mesh size  $h = h_0/8$ .

boundary layer is completely missed. On the other hand SUPG method exhibits again an overshooting phenomenon near the point  $(1, 1)$  and also some spurious oscillations along the internal layer.

The above considerations are also confirmed by the results reported in Figure 17 where the absolute value of the pointwise difference between the DG and SUPG approximate solutions for  $\varepsilon = 10^{-k}$ ,  $k = 0, 3, 9$ , are reported. Results shown in Figure 17 (left) have been obtained on a grid with mesh size  $h = h_0/8$ ; the analogous ones obtained on a finer triangulation with mesh size  $h = h_0/16$  are reported in Figure 17 (right).

## 6 CONCLUSIONS

In this paper we have recalled the formulation and principal properties of DG and SUPG approximations of linear steady–state convection–diffusion equations. The aim was to carry out a careful comparison of their performance (as of stability, accuracy and numerical robustness) in three different solution regimes: diffusion–dominated, intermediate, and convection–dominated. Our comparative analysis is made for piecewise linear elements on a number of significant test problems. As expected, DG and SUPG methods are substantially equivalent in the diffusion–dominated regime. In the intermediate regime, SUPG

looks more robust (even if overshooting may occur in the boundary layer). In the convection-dominated regime, SUPG still exhibits overshooting whereas DG is oscillation free, at least when the exact solution features sharp boundary layers. However, it is fair to notice that the DG method is not able to capture the boundary layers (indeed, the latter are completely missed because of the weak enforcement of boundary conditions). An in depth comparison between discontinuous and stabilized continuous Galerkin methods in the case of higher order polynomial approximations and transient convection-diffusion problems is currently under investigation (Antonietti and Quarteroni 2011).

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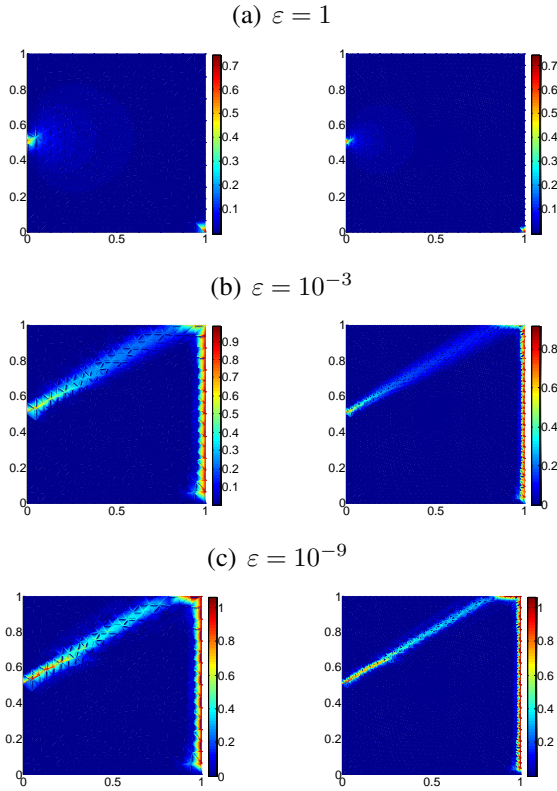


Figure 17: Example 4. Pointwise absolute difference between the DG and SUPG approximate solutions for  $\varepsilon = 10^{-k}$ ,  $k = 0, 3, 9$ . Mesh size  $h = h_0/8$  (left) and  $h = h_0/16$  (right).

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