Exploring Aliasing and the Sampling Theorem

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1 Introduction

In signal processing, ...

An analog signal refers to a signal that varies continuously over time. The complexity of analog signal processing, their susceptibility to noise and signal degradation over time, as well as their limited reproductibility and scalability makes them inconvenient to work with in practice. Therefore, digital signals are used – signals that vary discretely over time and can take only a finite number of distinct values.

Sampling refers to the process of converting an analog signal into a digital signal. If we let $x(t)$ be a continuous time signal, the sampled signal $x[n]$ is defined as

$$x[n] = x(nT_s)$$

where n represents discrete time sampling points and T_s represents the sampling period, such that the sampling frequency $f_s = \frac{1}{T_s}$.

Sampling can also be represented as

$$x_s(t) = x(t) \sum_{-\infty}^{\infty} \delta(t - nT_s)$$

where $x_s(t)$ is the sampled points, and δ is the dirac delta distribution, taking value 1 if $t = nT_s$ and 0 otherwise. Therefore, it is clear that when $t = nT_s$, $x_s(t) = x(t)$; otherwise $x_s(t) = 0$.

In practice, sampling allows us to discretize a continuous input, facilitating the handling of signals. After sampling is done, it is natural that the signal must be reconstructed in order to recover the original (time continuous) signal. However, recovery is not always perfect – the Shannon-Nyquist condition must be met.

The Shannon-Nyquist theorem states that a signal can be perfectly reconstructed if the sampling frequency is greater than two times the maximum frequency of the continuous signal:

$$f_s > 2B$$

where B is the maximum frequency in Hertz. For example, if we consider a sine wave with frequency 5 Hertz, $\sin(2\pi \cdot 5 \cdot t)$, once sampled it can be perfectly reconstructed if the sampling frequency is greater than 10.

In the time domain, a sampled signal looks like discrete spikes (Figure 1). To visualize what sampling looks like in the frequency domain, we make use of the Inverse Convolution Theorem.

The Inverse Convolution Theorem states that if we have two functions $x(t)$ and $h(t)$ whose Fourier Transforms $\mathcal{F}(x(t))$ and $\mathcal{F}(h(t))$ are absolutely integrable in the frequency domain, then

$$X(f) * H(f) = \mathcal{F}(x(t) \cdot h(t))$$

Therefore, because we can express sampling as $x_s(t) = x(t) \cdot \sum_{-\infty}^{\infty} \delta(t - nT_s)$ in the time domain, the Inverse Convolution Theorem tells us that

$$\mathcal{F}\left(x(t) \cdot \sum_{-\infty}^{\infty} \delta(t - nT_s)\right) = \mathcal{F}(x(t)) * \mathcal{F}\left(\sum_{-\infty}^{\infty} \delta(t - nT_s)\right)$$

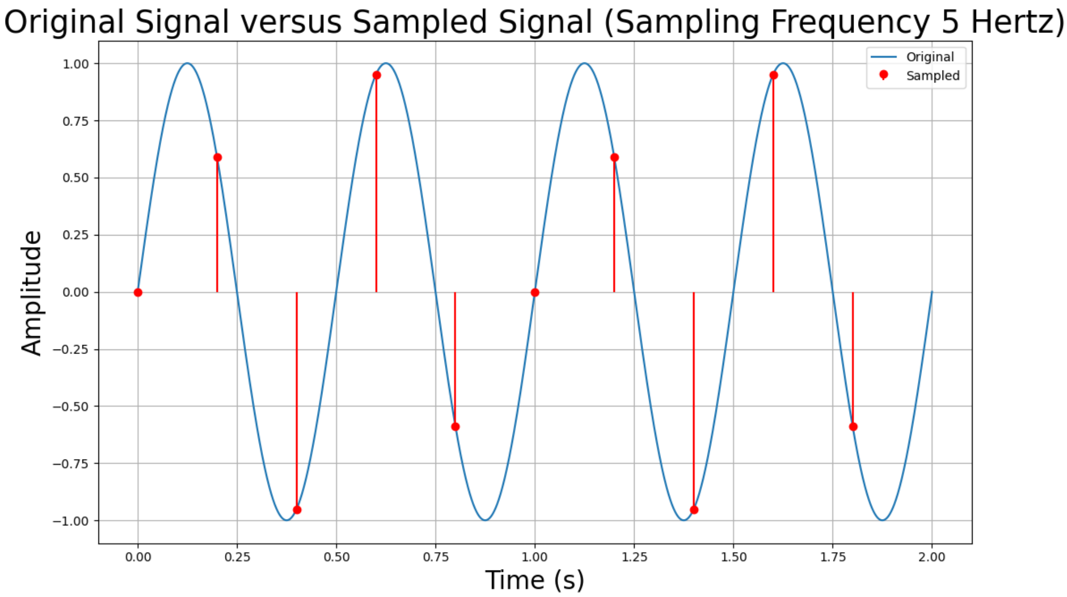

Figure 1: Representation of sampling of $\sin(4\pi t)$ in the time domain, sampled at a frequency of 5 Hertz

2 Implementation

3 Discussion

4 Conclusion

References

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