CSCE 411 Design and Analysis of Algorithms
Sketch of Solution to In-class Test 10

Answer to Problem 1

Main idea of the algorithm: we select nodes for the independent set greedily one by one. Each time, we select an arbitrary node from the (remaining) graph for the independent set, and then remove that node as well as all its neighboring nodes from the graph. We keep selecting nodes as above until the graph becomes empty.

Pseudo code:

1) $S \leftarrow \emptyset$, $V' \leftarrow V$.
2) while $V' \neq \emptyset$:
3) let v be a node in V'
4) $S \leftarrow S \cup \{v\}$
5) $V' \leftarrow V' - \{v$ and all neighboring nodes of v in $V'\}$
6) return S

Time complexity: $O(V + E)$

Proof of approximation ratio: let's say that the algorithm returns k nodes

$$v_1,\ v_2,\ \cdots, v_k$$

as the independent set. For $i = 1, 2, \cdots, k$, let U_i denote the set of nodes – including v_i and its neighbors – removed from V' in Step 5 of the algorithm. Then U_1, U_2, $\cdots$, U_k form a partition of all the nodes V; so we have

$$|V| = \sum_{i=1}^{k} |U_i|.$$

Since d is the maximum degree of nodes, we have $|U_i| \leq d + 1$ for all i. So

$$|V| \leq k(d + 1).$$

Let C^* be the maximum size of an independent set (and clearly we have $C^* \leq |V|$); and let $C = k$ be the size of the independent set returned by the algorithm. Then we have

$$\frac{C^*}{C} \leq \frac{|V|}{k} \leq d + 1.$$