Algorithms

Lecture 10: Single-Source Shortest Paths

Anxiao (Andrew) Jiang

CH. 24 Single-Source Shortest Paths

Input: A directed graph G=(V,E) where every edge $e \in E$ has a weight $w(e)$.

Let $s \in V$ be a node called the “source”.

Output: Find a shortest path from s to every other node.

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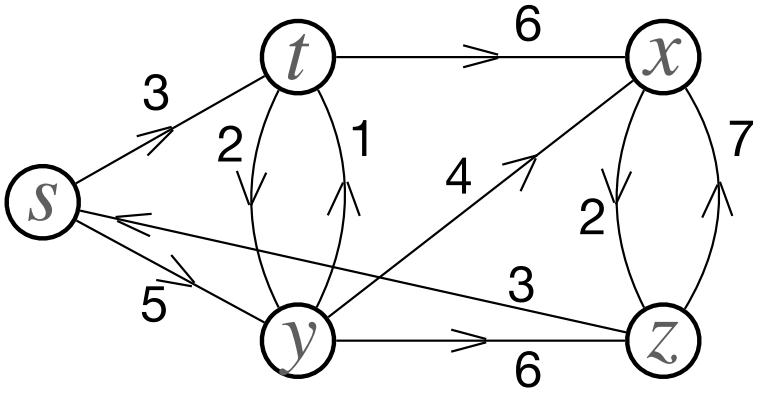
Output: Find a shortest path from s to every other node.

Byproduct: Those shortest paths will form a shortest-path tree rooted at s.


t
x
s
y
z
6
3
2
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4
7
2
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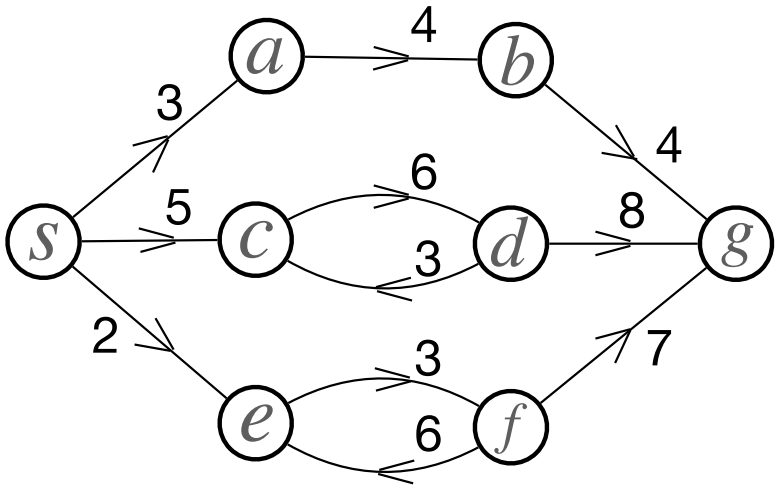
5
6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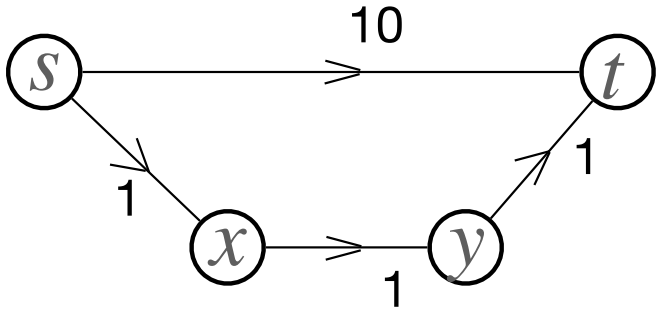
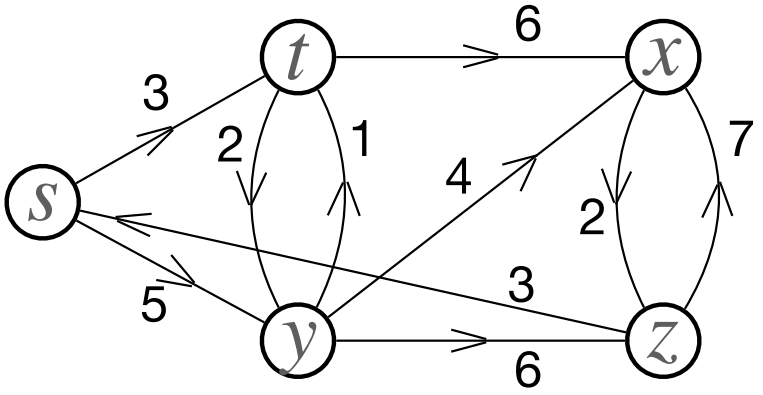
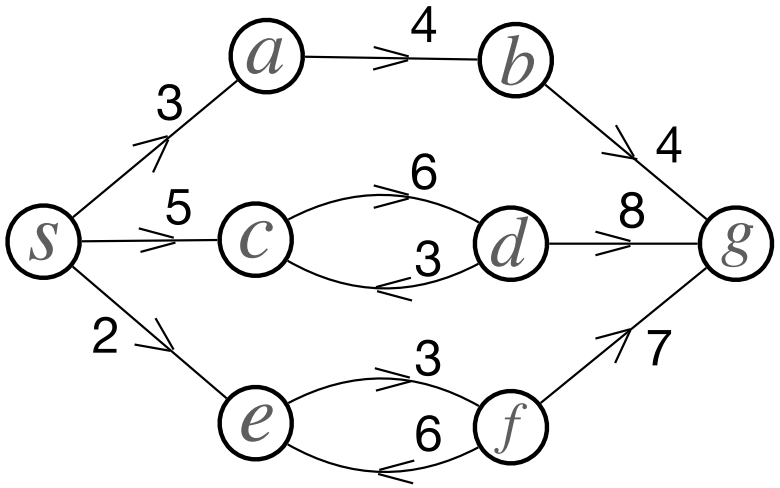
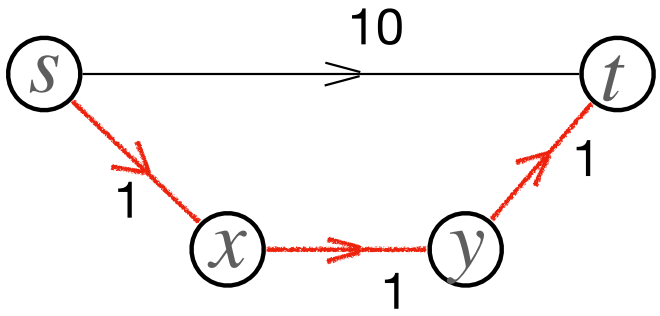

s
t
x
y
10
1
1
1
1

CH. 24 Single-Source Shortest Paths





A shortest path may have many edges.



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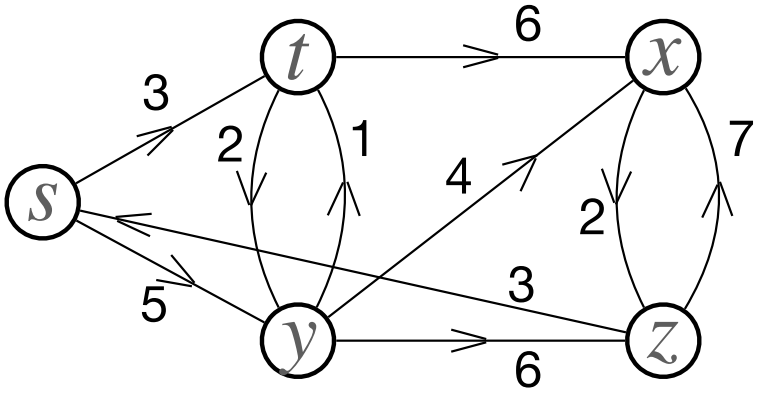

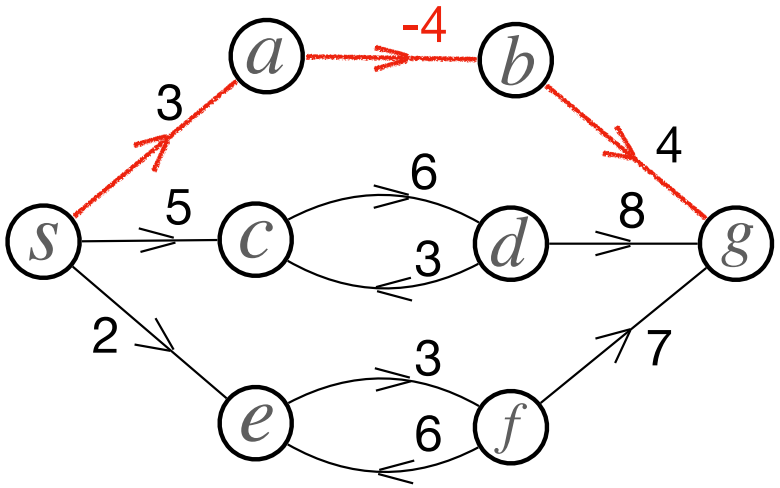

Edge weights may be negative.

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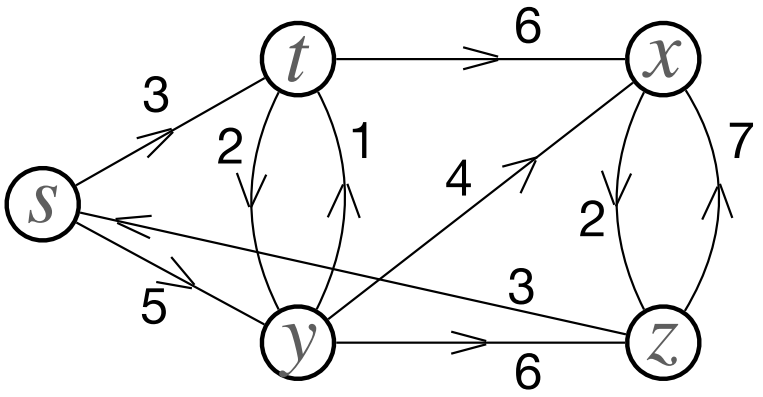

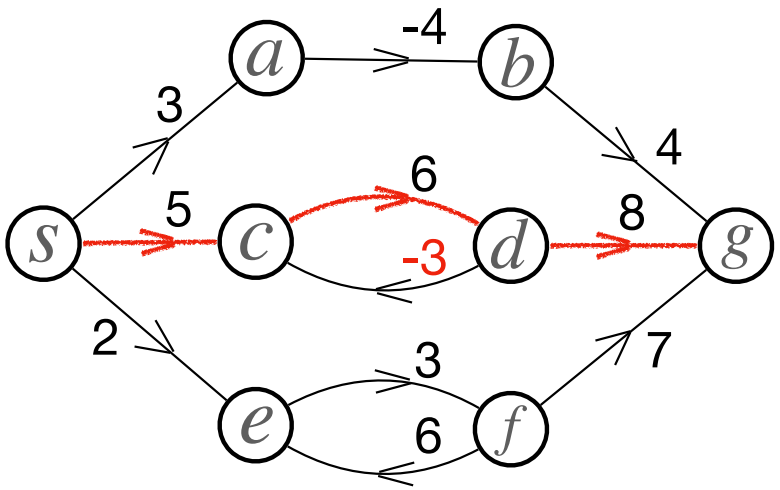

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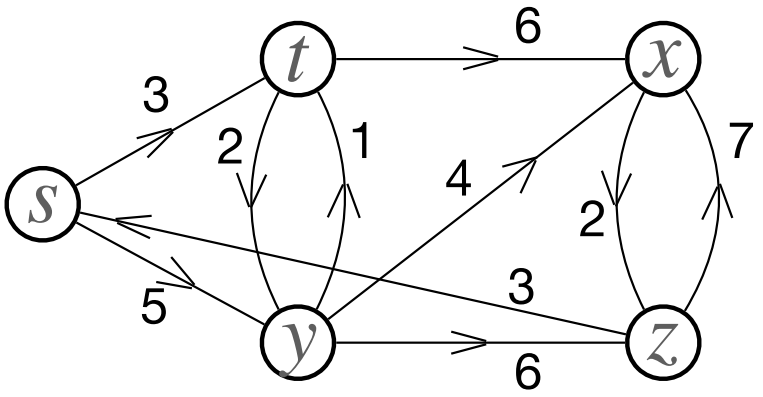

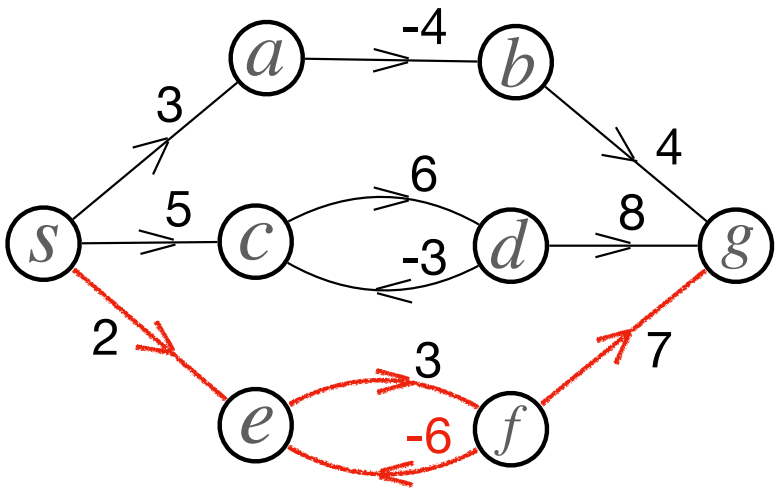

There can be negative cycles.

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Output: Find a shortest path from s to every other node if G has no negative cycle that s can reach.

(Otherwise, we return "There is a negative cycle that s can reach".)

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Theorem: A subpath of a shortest path is also a shortest path.

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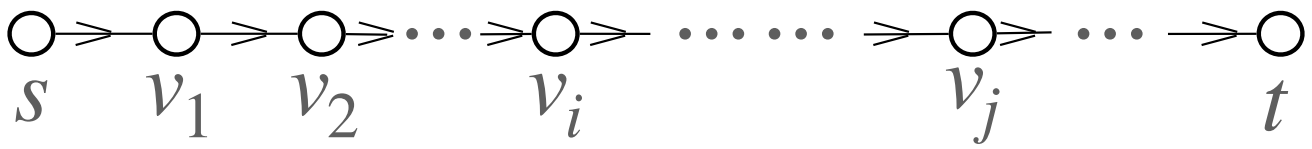
Proof: Consider a shortest path from s to t.



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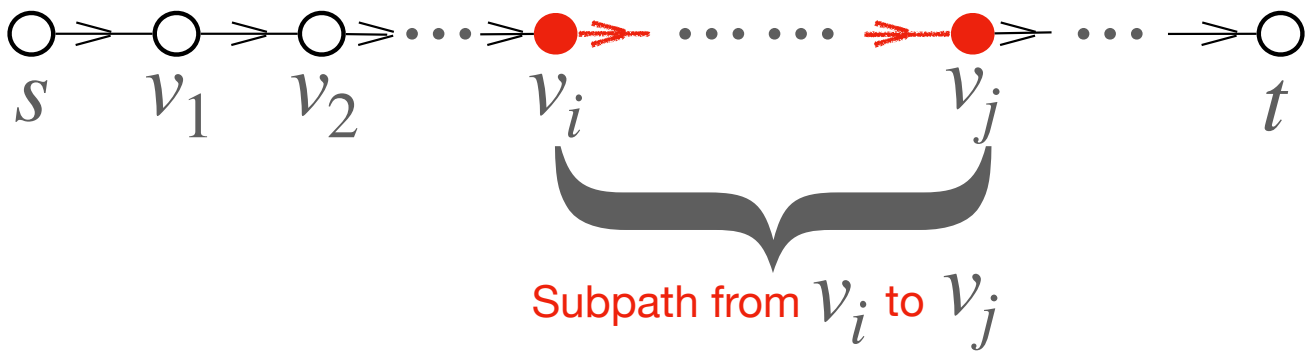

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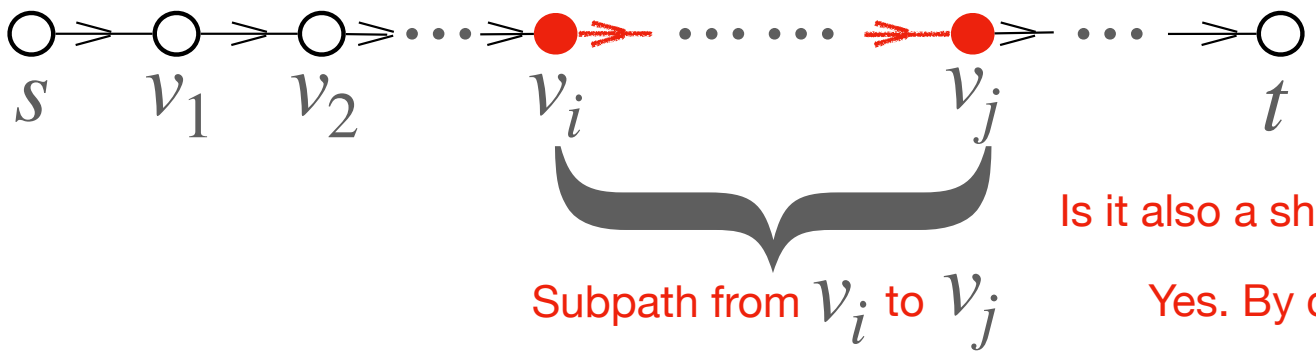

Is it also a shortest path?

Yes. By contradiction.

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For every node $v \in V$,

1) Let $d(v)$ denote the length of the shortest path we have found so far from s to v.

2) Let $\pi(v)$ be the node in front of v in the shortest path we have found so far from s to v.

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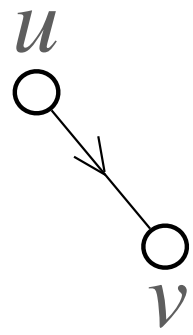
Idea of our algorithm:

1) Find shorter and shorter paths from s to each node v.



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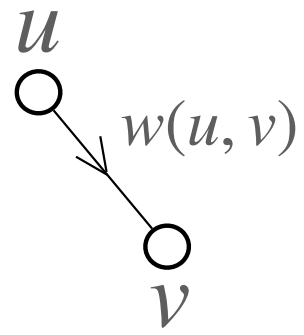

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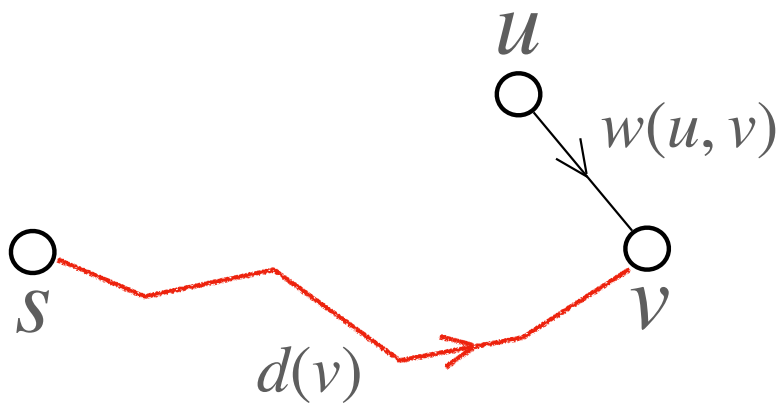

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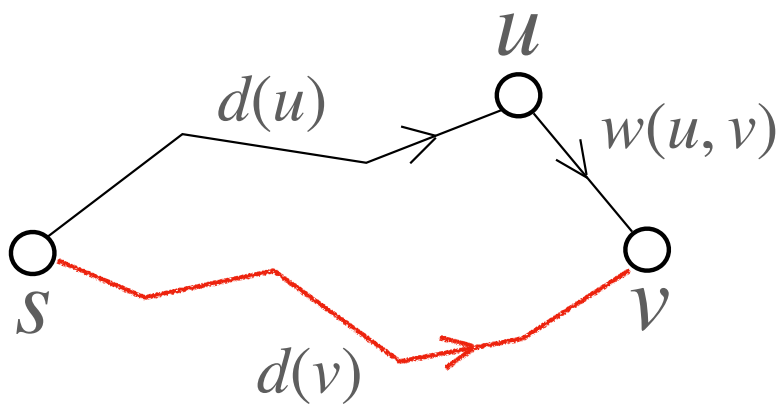

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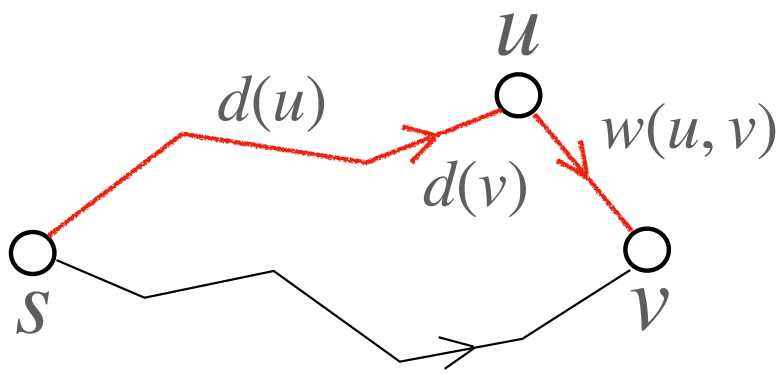

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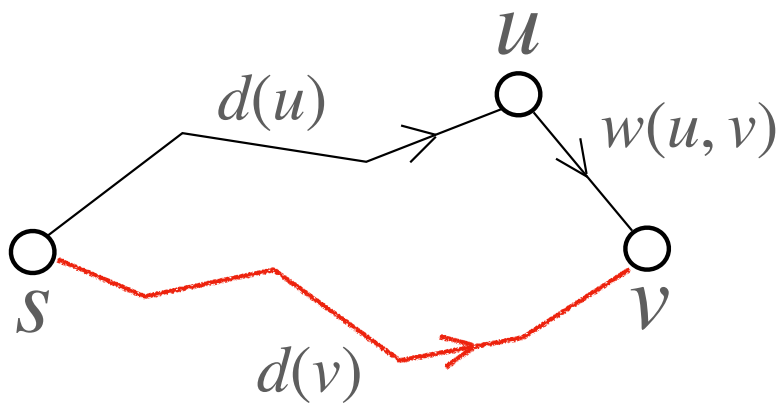
$d(v) = d(u) + w(u, v)$

$\pi(v) = u$



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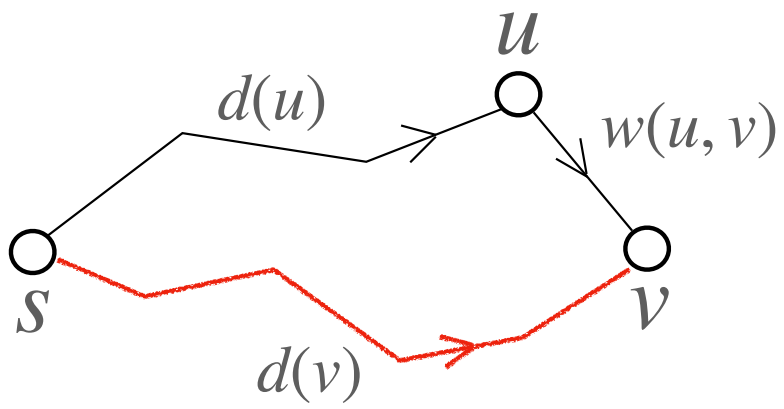
Relaxation



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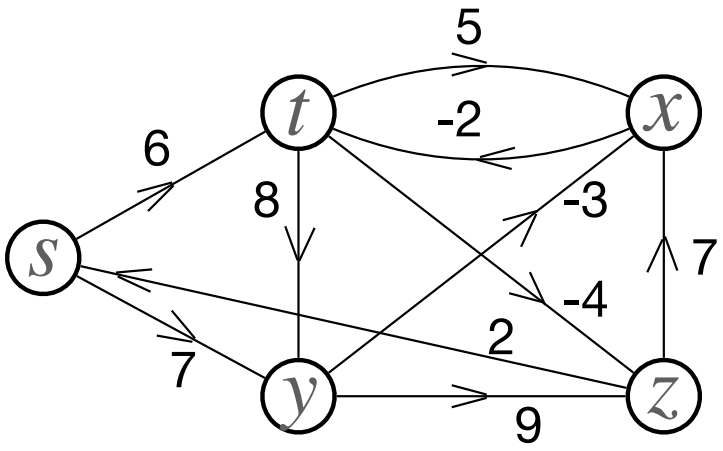

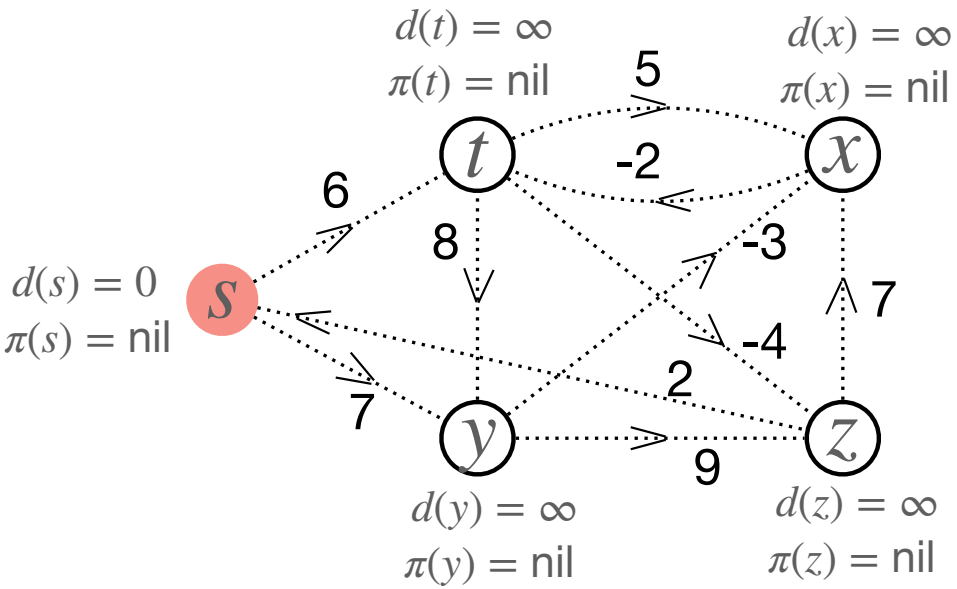
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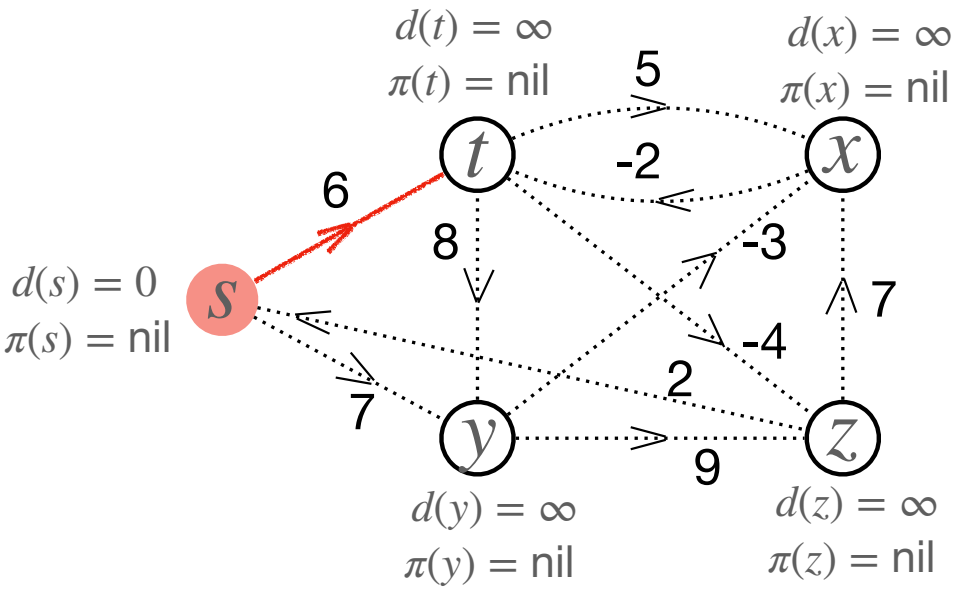
$\pi(t) = \text{nil}$
$d(x) = \infty$
$\pi(x) = \text{nil}$
5
t
x
-2
6
8
-3
$d(s) = 0$
$\pi(s) = \text{nil}$
s
7
-4
2
7
y
z
9
$d(y) = \infty$
$\pi(y) = \text{nil}$
$d(z) = \infty$
$\pi(z) = \text{nil}$


$d(t) = \infty$
$\pi(t) = \text{nil}$
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5
t
x
-2
6
8
-3
$d(s) = 0$
$\pi(s) = \text{nil}$
s
7
-4
2
7
y
z
9
$d(y) = \infty$
$\pi(y) = \text{nil}$
$d(z) = \infty$
$\pi(z) = \text{nil}$

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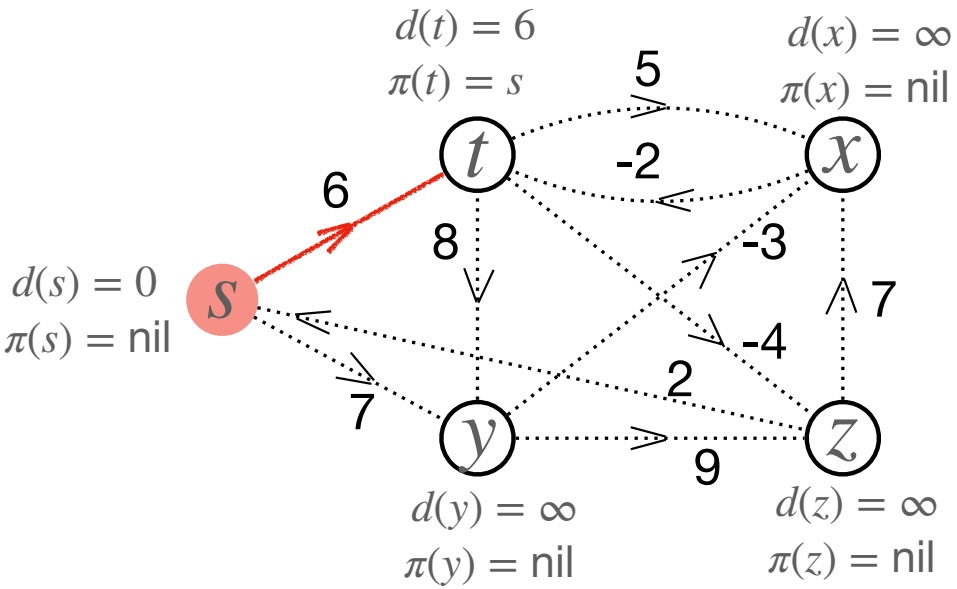

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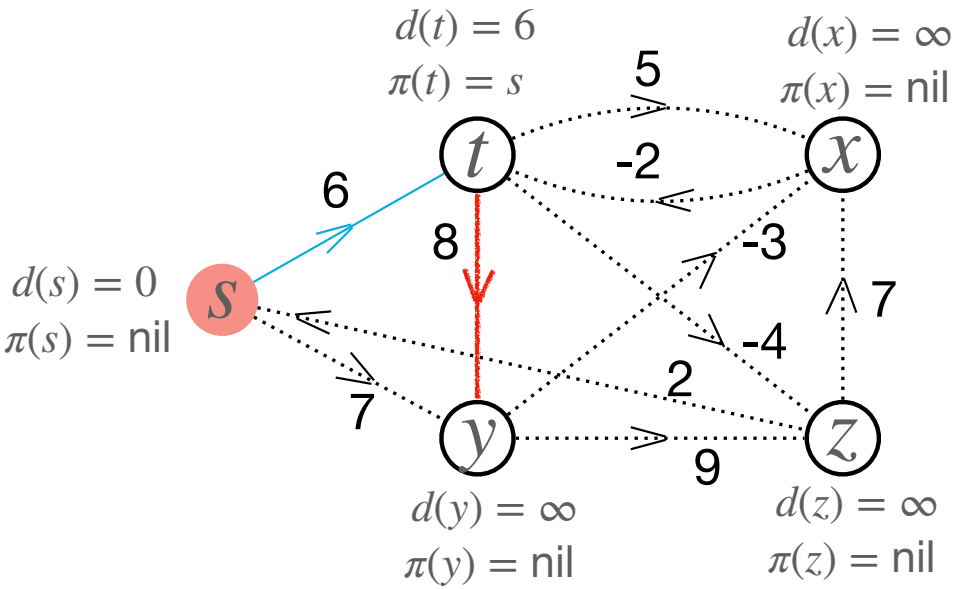

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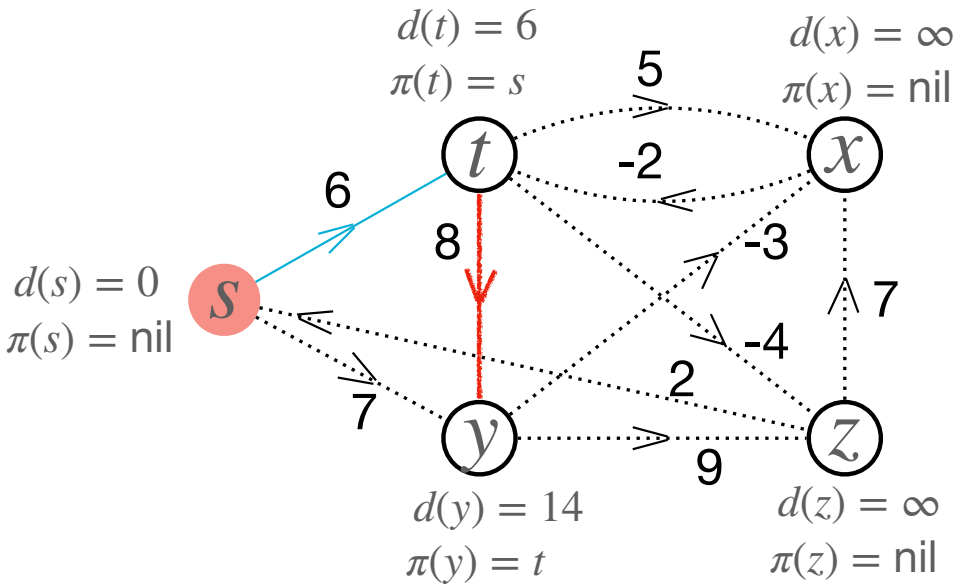
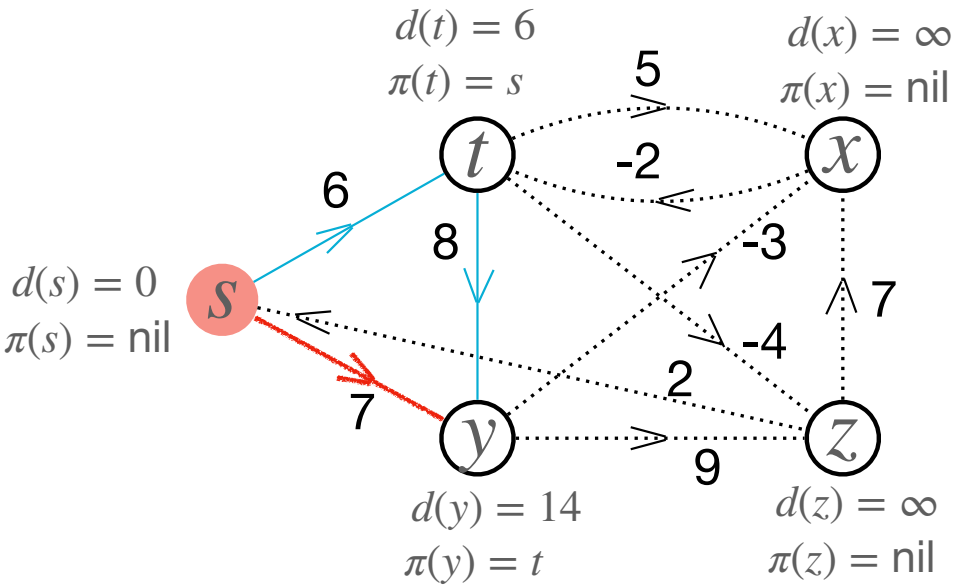
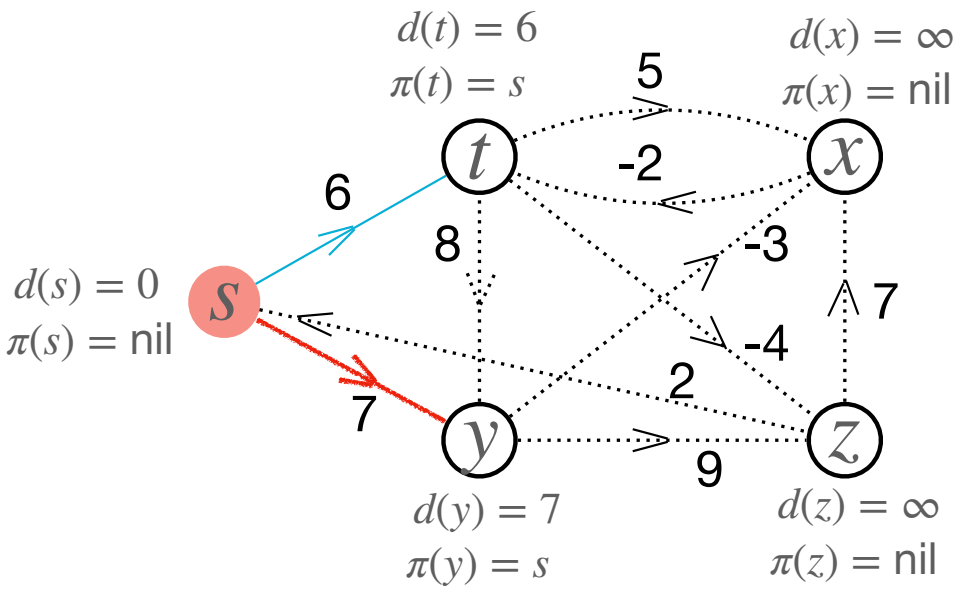



$d(t) = 6$
$\pi(t) = s$
$d(x) = \infty$
$\pi(x) = \text{nil}$
5
-2
6
8
-3
7
$d(s) = 0$
$\pi(s) = \text{nil}$
-4
2
7
9
$d(y) = 14$
$\pi(y) = t$
$d(z) = \infty$
$\pi(z) = \text{nil}$
t
x
s
y
z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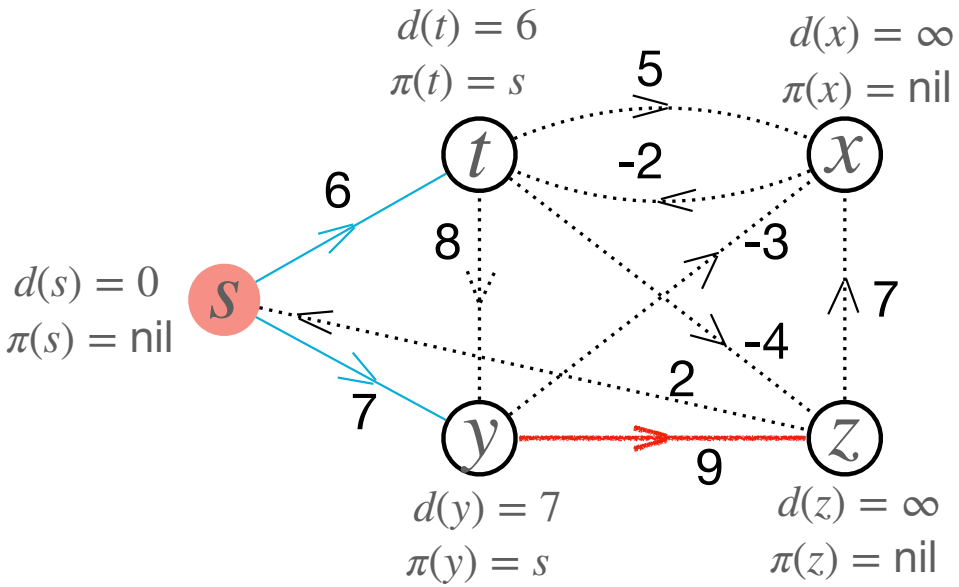
5
t
x
-2
6
8
-3
$d(s) = 0$
$\pi(s) = \text{nil}$
s
7
-4
2
7
y
z
9
$d(y) = 7$
$\pi(y) = s$
$d(z) = \infty$
$\pi(z) = \text{nil}$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7
-4
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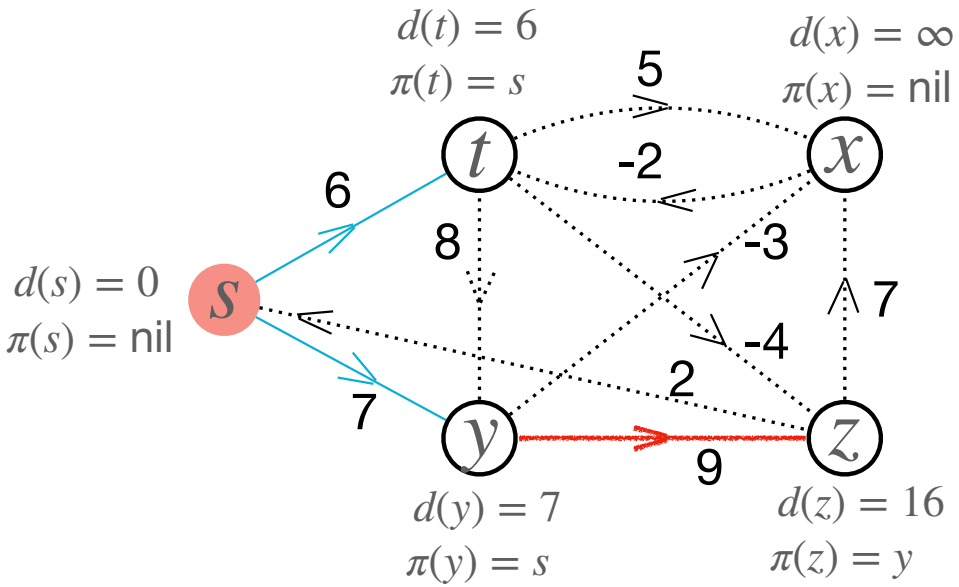
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5
t
x
-2
6
8
-3
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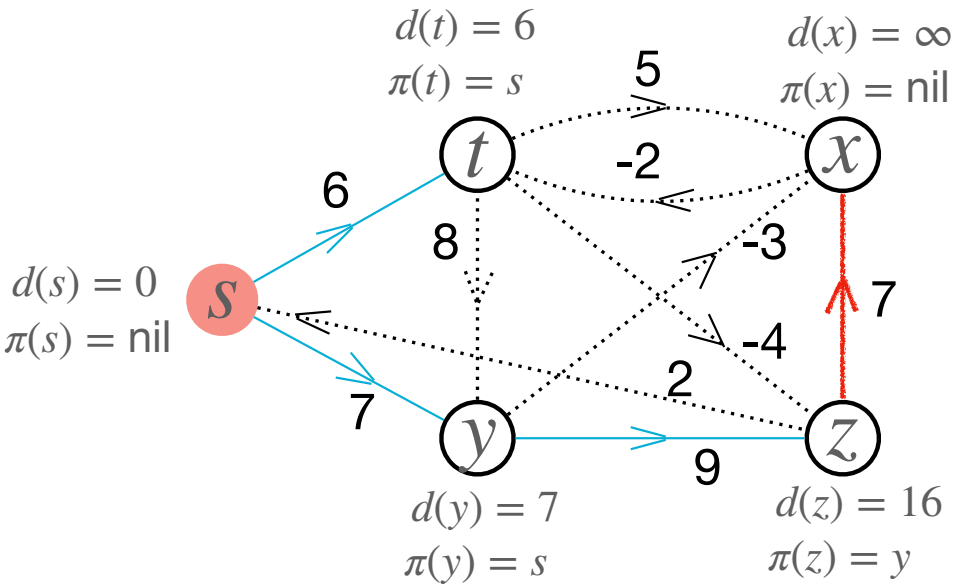
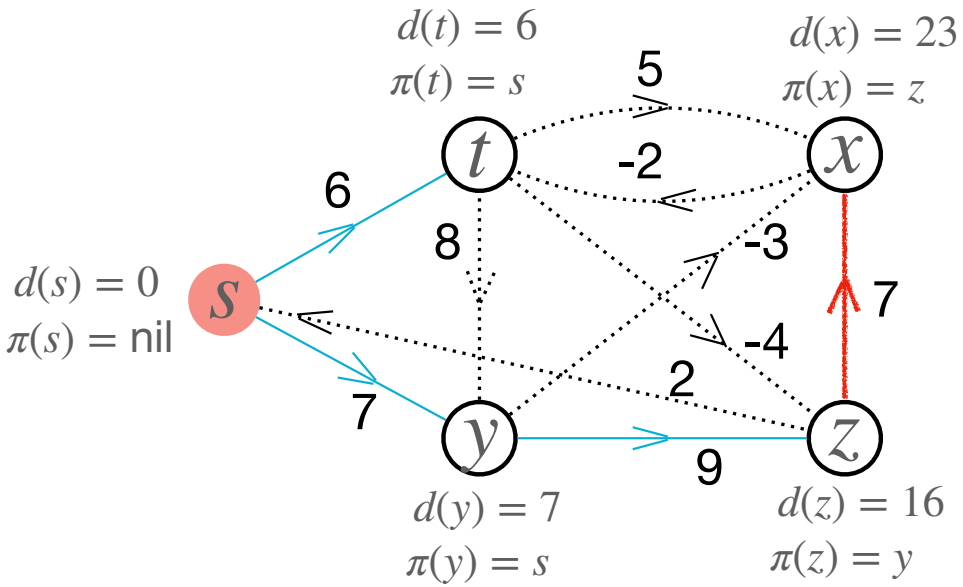
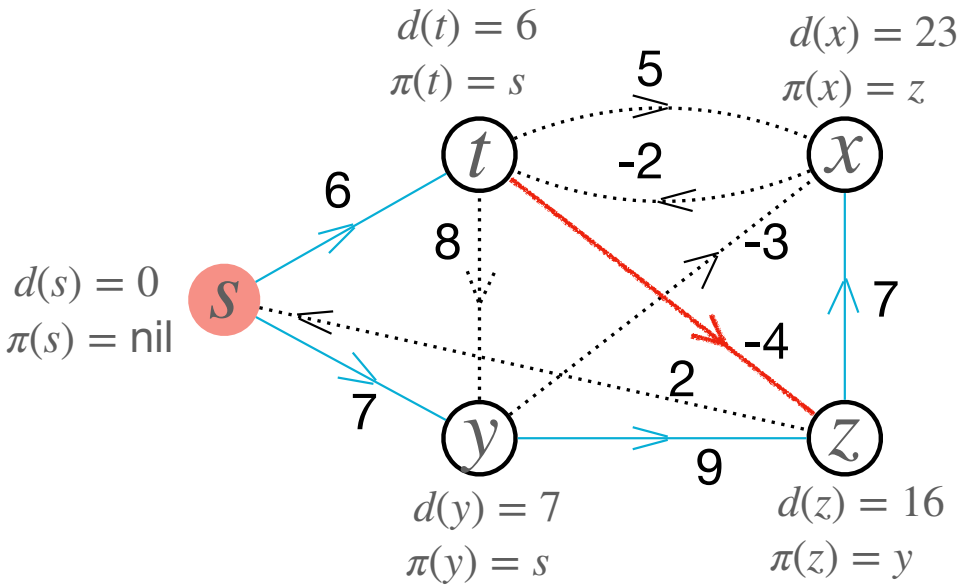
s
7
-4
2
7
y
z
9
$d(y) = 7$
$\pi(y) = s$
$d(z) = 16$
$\pi(z) = y$

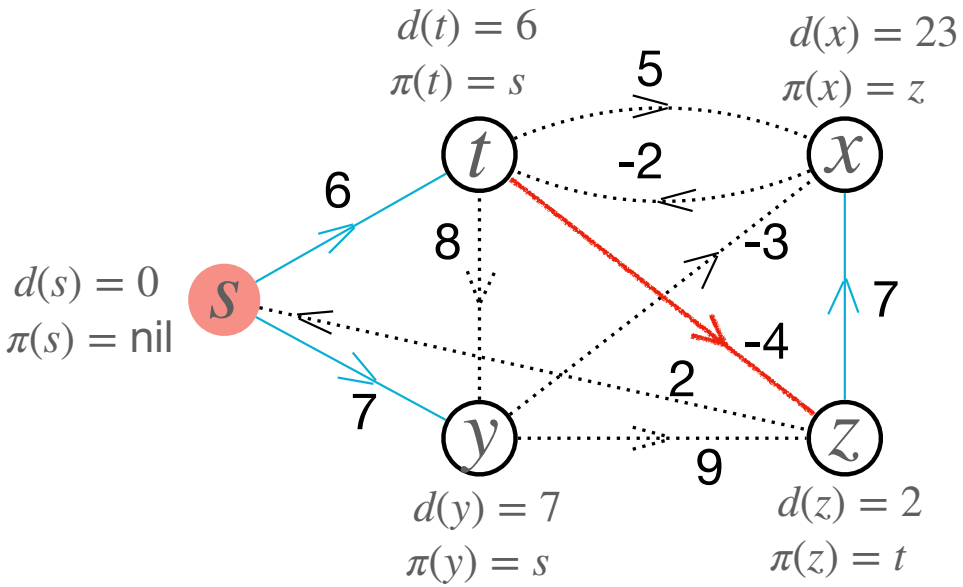

$d(t) = 6$
$\pi(t) = s$
$d(x) = \infty$
$\pi(x) = \text{nil}$
5
t
x
-2
6
8
-3
$d(s) = 0$
$\pi(s) = \text{nil}$
s
7
-4
2
7
y
z
9
$d(y) = 7$
$\pi(y) = s$
$d(z) = 16$
$\pi(z) = y$


$d(t) = 6$
$\pi(t) = s$
$d(x) = 23$
$\pi(x) = z$
5
t
x
-2
6
8
-3
$d(s) = 0$
$\pi(s) = \text{nil}$
s
7
-4
2
7
y
z
9
$d(y) = 7$
$\pi(y) = s$
$d(z) = 16$
$\pi(z) = y$

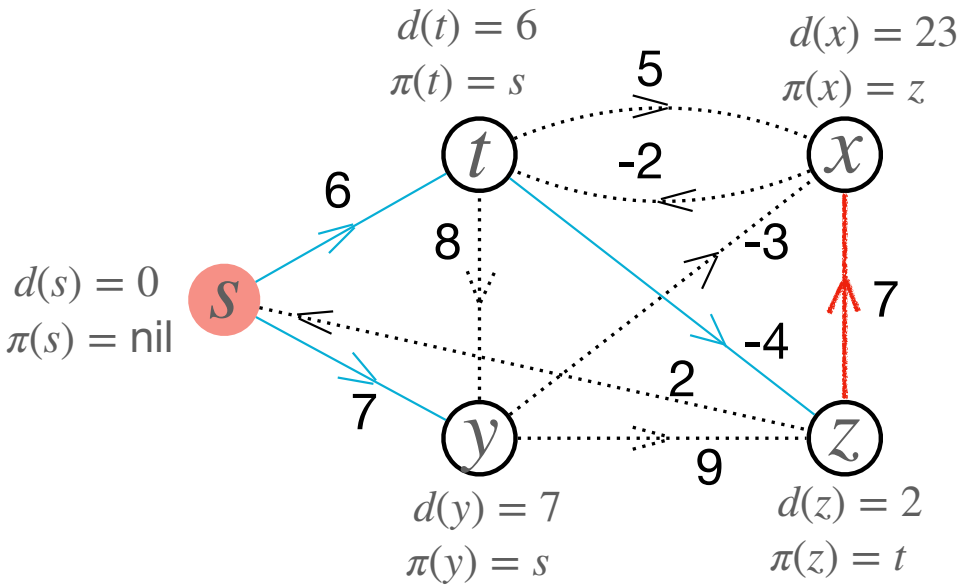
CH. 24 Single-Source Shortest Paths



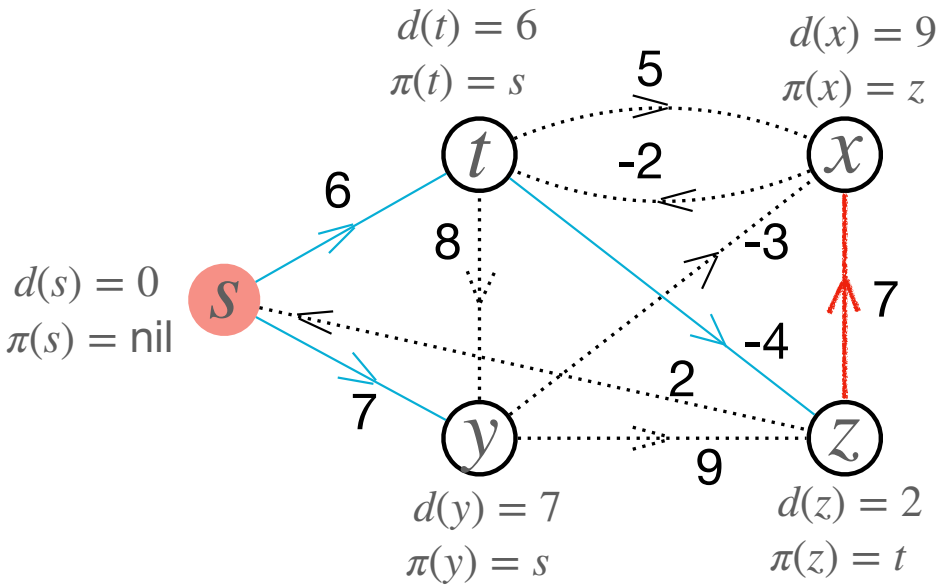
CH. 24 Single-Source Shortest Paths



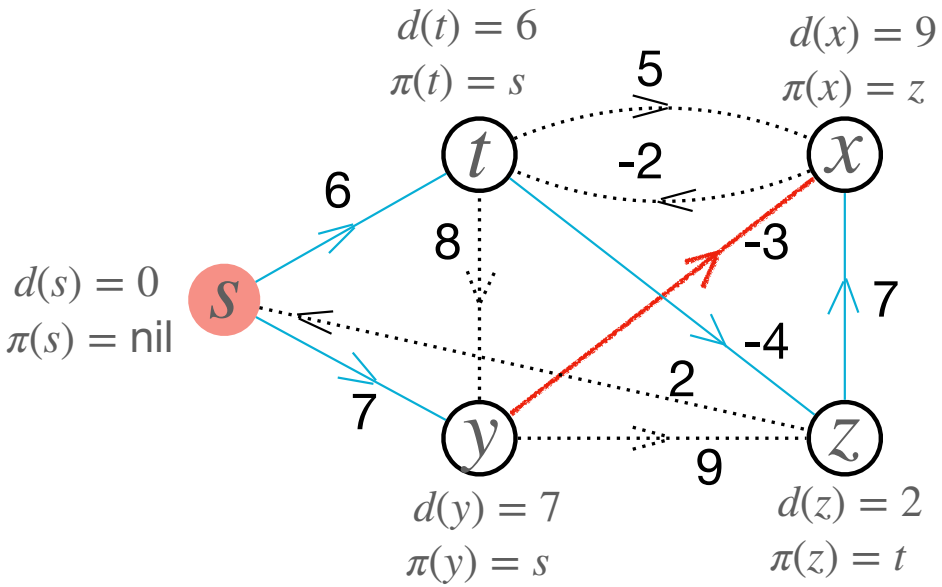
CH. 24 Single-Source Shortest Paths



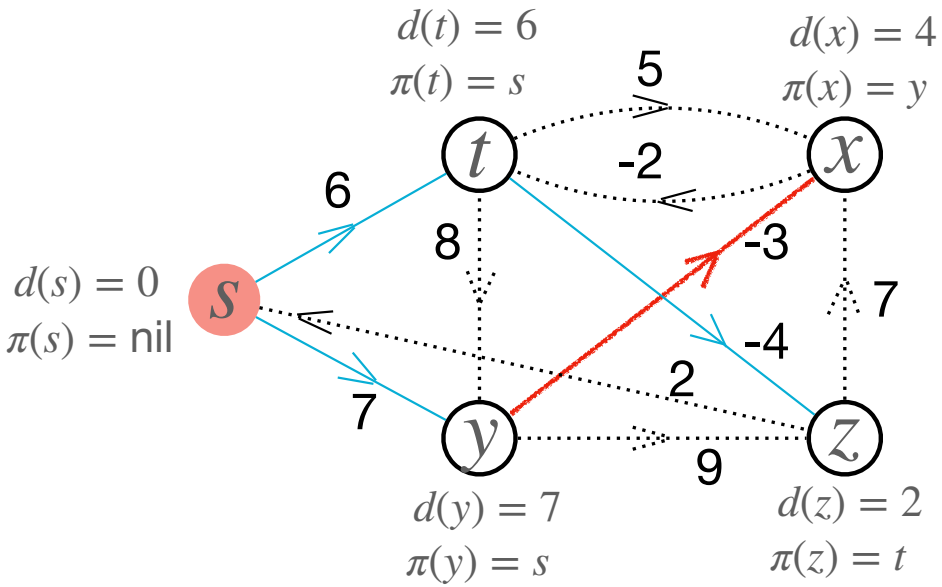
CH. 24 Single-Source Shortest Paths



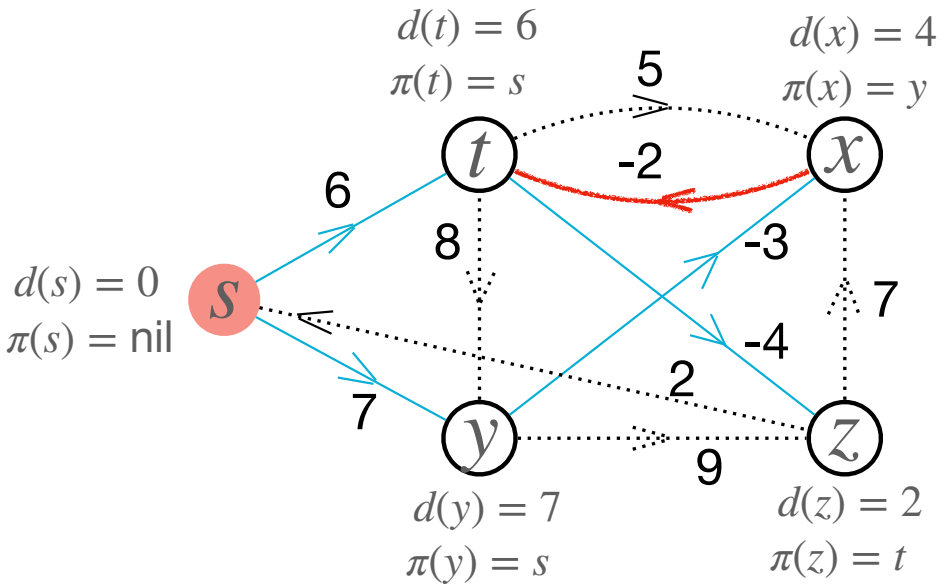
CH. 24 Single-Source Shortest Paths



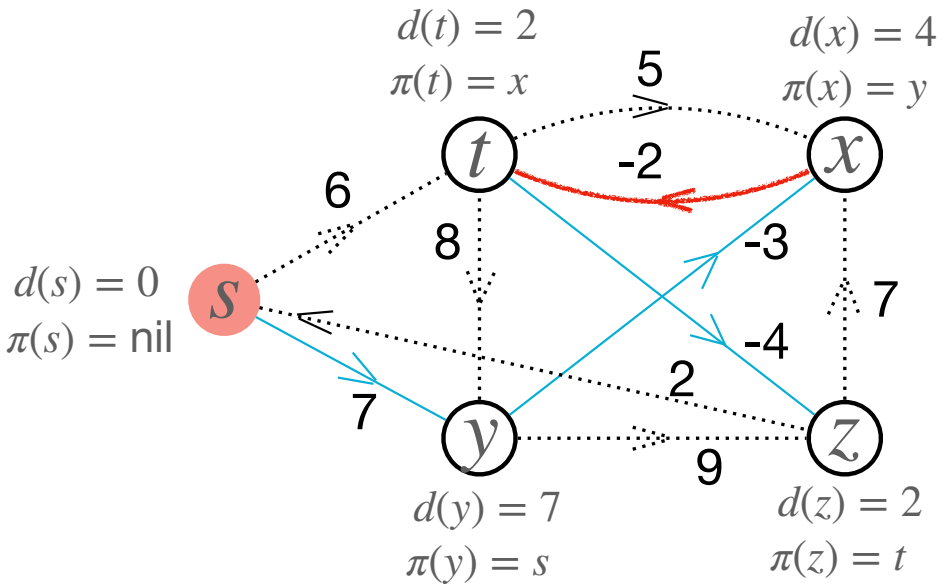
CH. 24 Single-Source Shortest Paths



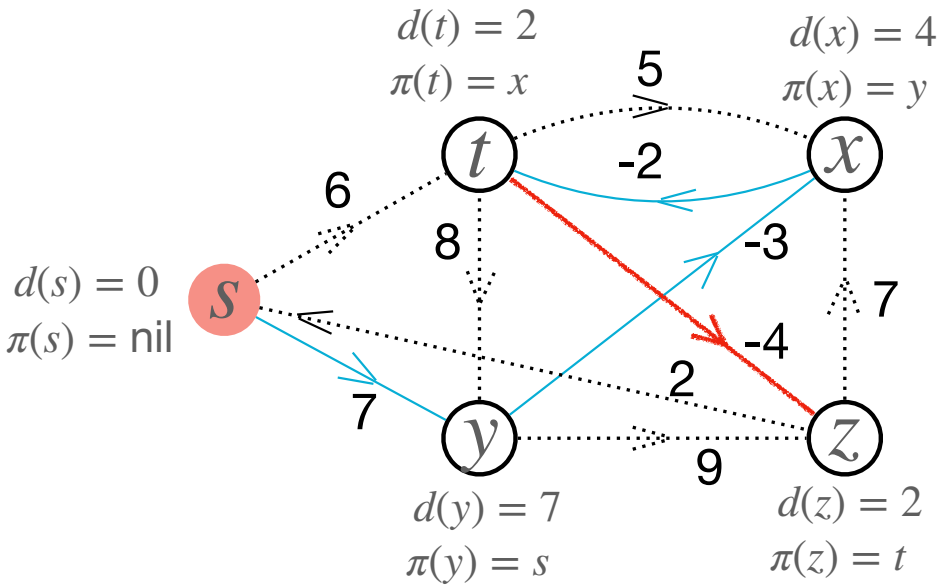
CH. 24 Single-Source Shortest Paths

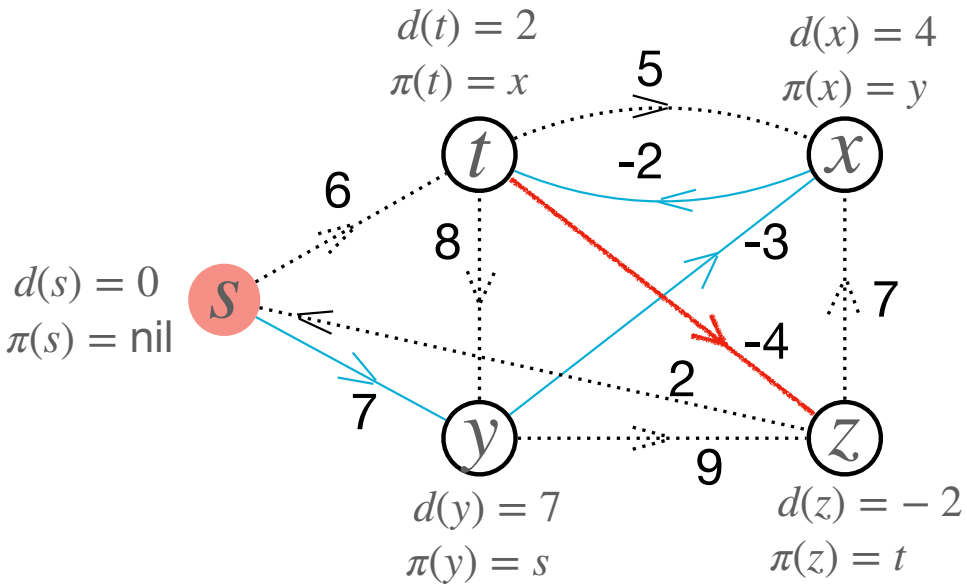


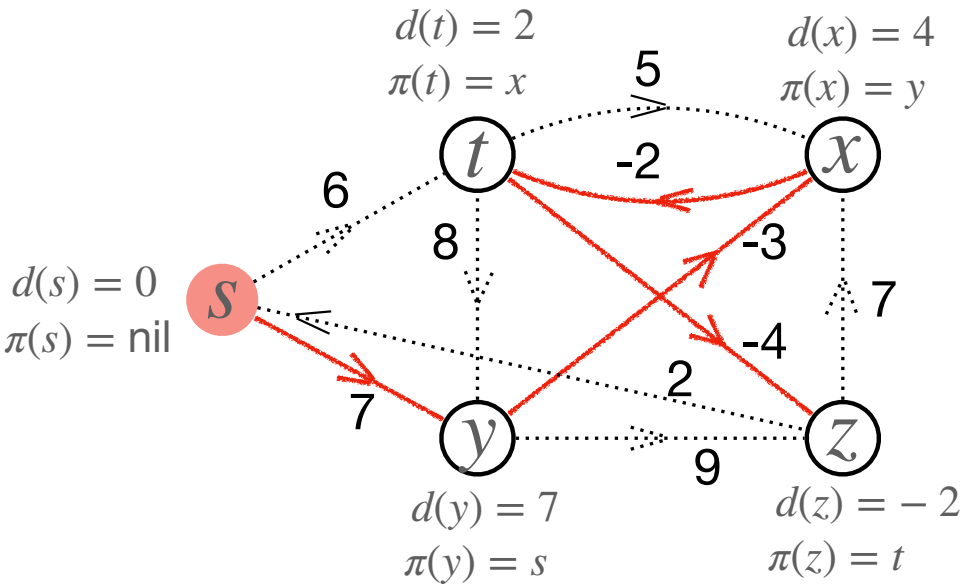
CH. 24 Single-Source Shortest Paths



CH. 24 Single-Source Shortest Paths




$d(t) = 2$
$\pi(t) = x$
$d(x) = 4$
$\pi(x) = y$
5
t
x
-2
6
8
-3
$d(s) = 0$
$\pi(s) = \text{nil}$
s
7
-4
2
7
y
z
9
$d(y) = 7$
$\pi(y) = s$
$d(z) = -2$
$\pi(z) = t$



Shortest-Path Tree

CH. 24 Single-Source Shortest Paths

Bellman-Ford Algorithm

1) Initialization: $\forall\ v \in V,\ \pi(v) = \text{nil}$

$d(s) = 0 \quad \forall\ v \neq s,\ d(v) = \infty$

2) for $i = 1$ to $|V| - 1$:

for every edge $(u, v) \in E$:

if $d(u) + w(u, v) < d(v)$:

$d(v) = d(u) + w(u, v)$

$\pi(v) = u$

Relaxation

3) for every edge $(u, v) \in E$:

if $d(u) + w(u, v) < d(v)$:

Return “there is a negative cycle that s can reach”

4) Return $d(v)$ and $\pi(v)$ for every node v

(which form a shortest-path tree).

CH. 24 Single-Source Shortest Paths

Bellman-Ford Algorithm

1) Initialization: $\forall\ v \in V,\ \pi(v) = \text{nil}$

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Return “there is a negative cycle that s can reach”

4) Return $d(v)$ and $\pi(v)$ for every node v

(which form a shortest-path tree).

Do Relaxation for |V|-1 rounds.

In each round, do Relaxation for all edges.

CH. 24 Single-Source Shortest Paths

Bellman-Ford Algorithm

1) Initialization: $\forall\ v \in V,\ \pi(v) = \text{nil}$

$d(s) = 0 \quad \forall\ v \neq s,\ d(v) = \infty$

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3) for every edge $(u, v) \in E$:

if $d(u) + w(u, v) < d(v)$:

Return “there is a negative cycle that s can reach”

4) Return $d(v)$ and $\pi(v)$ for every node v

(which form a shortest-path tree).

Why are |V|-1 rounds enough?

Do Relaxation for |V|-1 rounds.

In each round, do Relaxation for all edges.

Does their order matter?

CH. 24 Single-Source Shortest Paths

Bellman-Ford Algorithm

1) Initialization: $\forall\ v \in V,\ \pi(v) = \text{nil}$

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2) for $i = 1$ to $|V| - 1$:

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Relaxation

3) for every edge $(u, v) \in E$:

if $d(u) + w(u, v) < d(v)$:

Return “there is a negative cycle that s can reach”

Are the above |V|-1 rounds of Relaxation still not enough?
If still not enough, then something is wrong (i.e, there is a negative cycle).

4) Return $d(v)$ and $\pi(v)$ for every node v

(which form a shortest-path tree).

CH. 24 Single-Source Shortest Paths

Bellman-Ford Algorithm

1) Initialization: $\forall\ v \in V,\ \pi(v) = \text{nil}$

$d(s) = 0 \quad \forall\ v \neq s,\ d(v) = \infty$

2) for $i = 1$ to $|V| - 1$:

for every edge $(u, v) \in E$:

if $d(u) + w(u, v) < d(v)$:

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$\pi(v) = u$

Relaxation

3) for every edge $(u, v) \in E$:

if $d(u) + w(u, v) < d(v)$:

Return “there is a negative cycle that s can reach”

4) Return $d(v)$ and $\pi(v)$ for every node v (which form a shortest-path tree).

|V|-1 rounds of Relaxation turn out to be enough.
Let’s output the shortest-path tree.

CH. 24 Single-Source Shortest Paths

Lemma: Assume graph G has no negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges), for every node v, d(v) has become the shortest-path distance from s to v.

Step 2 of the
Bellman-Ford algorithm

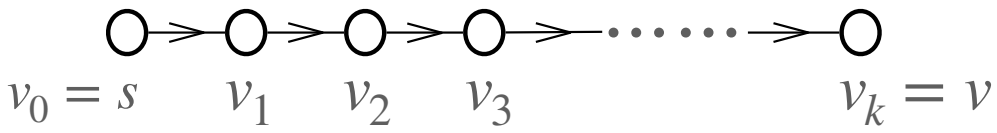
Lemma: Assume graph G has no negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
for every node v, d(v) has become the shortest-path distance from s to v.

Proof:

Step 2 of the
Bellman-Ford algorithm

Lemma: Assume graph G has no negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
for every node v, d(v) has become the shortest-path distance from s to v.

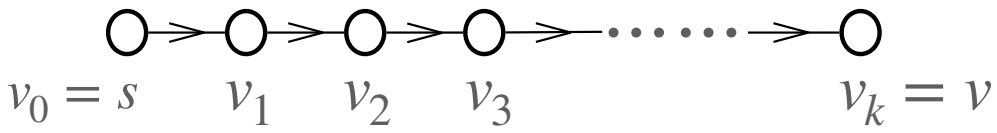
Proof: Consider a shortest path from s to v in graph G=(V,E).



Step 2 of the
Bellman-Ford algorithm

Lemma: Assume graph G has no negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
for every node v, d(v) has become the shortest-path distance from s to v.

Proof: Consider a shortest path from s to v in graph G=(V,E).
Since there is no negative cycle, we can safely assume this path contains no cycle.



Step 2 of the
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Lemma: Assume graph G has no negative cycle that s can reach.
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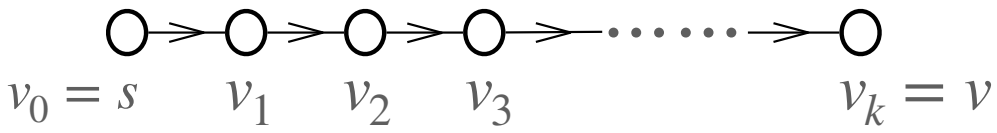
Proof: Consider a shortest path from s to v in graph G=(V,E).
Since there is no negative cycle, we can safely assume this path contains no cycle.
This path contains at most |V|-1 edges.

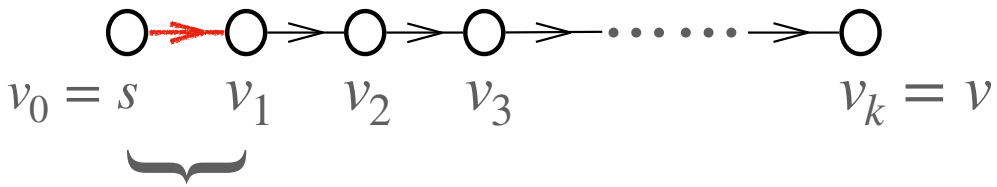


Step 2 of the
Bellman-Ford algorithm

Lemma: Assume graph G has no negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
for every node v, d(v) has become the shortest-path distance from s to v.

Proof: Consider a shortest path from s to v in graph G=(V,E).
Since there is no negative cycle, we can safely assume this path contains no cycle.
This path contains at most |V|-1 edges.

1) After round 1 of Relaxation, $d(v_1)$ will be the shortest-path distance from s to v_1 .



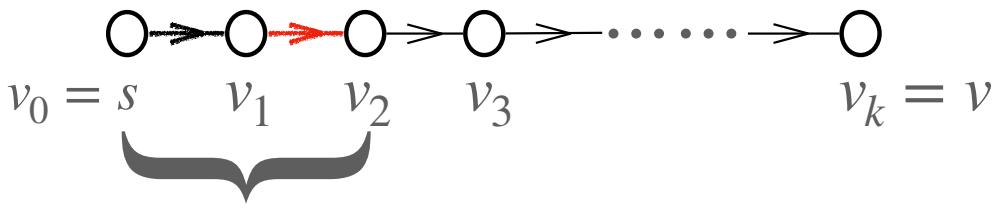
This sub-path is a shortest path.

Step 2 of the
Bellman-Ford algorithm

Lemma: Assume graph G has no negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
for every node v, d(v) has become the shortest-path distance from s to v.

Proof: Consider a shortest path from s to v in graph G=(V,E).
Since there is no negative cycle, we can safely assume this path contains no cycle.
This path contains at most |V|-1 edges.

1) After round 1 of Relaxation, $d(v_1)$ will be the shortest-path distance from s to v_1.
2) After round 2 of Relaxation, $d(v_2)$ will be the shortest-path distance from s to v_2.



This sub-path is a shortest path.

And so on.
Q.E.D.

Lemma: Assume graph G has no negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
for every node v, d(v) has become the shortest-path distance from s to v.

Then the "if" condition in Step 3
of Bellman-Ford Algorithm will not be true.
The algorithm will return the shortest-path tree.

Bellman-Ford Algorithm

3) for every edge $(u, v) \in E$:
if $d(u) + w(u, v) < d(v)$:
Return "there is a negative cycle
that s can reach"

4) Return $d(v)$ and $\pi(v)$ for every node v
(which form a shortest-path tree).

Lemma: Assume graph G has a negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
there will still be some edge $(u, v) \in E$ that has $d(u) + w(u, v) < d(v)$.

So the algorithm will return
"There is a negative cycle that s can reach".

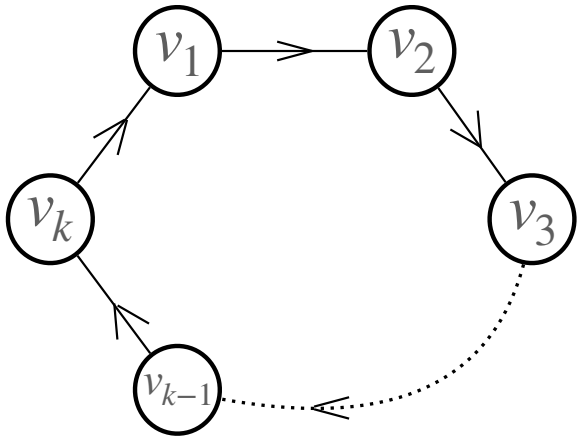
Bellman-Ford Algorithm

3) for every edge $(u, v) \in E$:
    if $d(u) + w(u, v) < d(v)$:
        Return "there is a negative cycle
        that s can reach"

CH. 24 Single-Source Shortest Paths

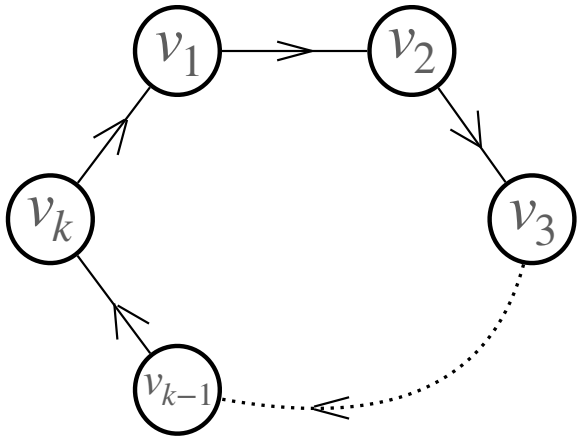
Lemma: Assume graph G has a negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
there will still be some edge $(u, v) \in E$ that has $d(u) + w(u, v) < d(v)$.

Proof: Consider a negative cycle that s can reach.



Lemma: Assume graph G has a negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
there will still be some edge $(u, v) \in E$ that has $d(u) + w(u, v) < d(v)$.

Proof: Consider a negative cycle that s can reach.

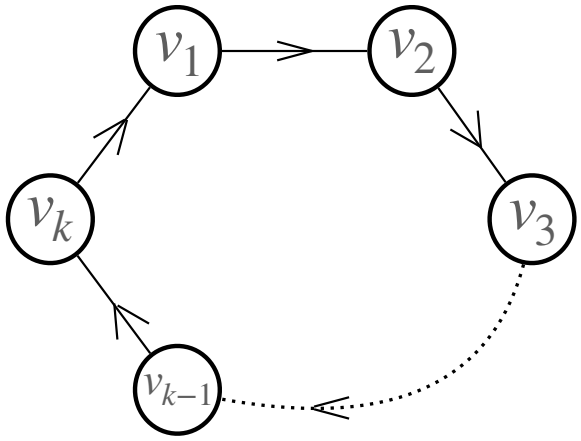


Assume:

$$
\begin{aligned}
d(v_1) + w(v_1, v_2) &\geq d(v_2) \\
d(v_2) + w(v_2, v_3) &\geq d(v_3) \\
d(v_3) + w(v_3, v_4) &\geq d(v_4) \\
&\vdots \\
d(v_{k-1}) + w(v_{k-1}, v_k) &\geq d(v_k) \\
d(v_k) + w(v_k, v_1) &\geq d(v_1)
\end{aligned}
$$

Lemma: Assume graph G has a negative cycle that s can reach.
Then after |V|-1 rounds of Relaxation (where each round does Relaxation for all edges),
there will still be some edge $(u, v) \in E$ that has $d(u) + w(u, v) < d(v)$.

Proof: Consider a negative cycle that s can reach.



Assume:

$$
\begin{aligned}
d(v_1) &+ w(v_1, v_2) \geq d(v_2) \\
d(v_2) &+ w(v_2, v_3) \geq d(v_3) \\
d(v_3) &+ w(v_3, v_4) \geq d(v_4) \\
&\vdots \\
d(v_{k-1}) &+ w(v_{k-1}, v_k) \geq d(v_k) \\
d(v_k) &+ w(v_k, v_1) \geq d(v_1)
\end{aligned}
$$

Sum them: $w(v_1, v_2) + w(v_2, v_3) + \cdots + w(v_k, v_1) \geq 0$

The total weight of the negative cycle is non-negative! Contradiction.

CH. 24 Single-Source Shortest Paths

Bellman-Ford Algorithm

1) Initialization: $\forall\ v \in V,\ \pi(v) = \text{nil}$

$d(s) = 0 \quad \forall\ v \neq s,\ d(v) = \infty$

2) for $i = 1$ to $|V| - 1$:

for every edge $(u, v) \in E$: Relaxation

if $d(u) + w(u, v) < d(v)$:

$d(v) = d(u) + w(u, v)$

$\pi(v) = u$

3) for every edge $(u, v) \in E$:

if $d(u) + w(u, v) < d(v)$:

Return "there is a negative cycle that s can reach"

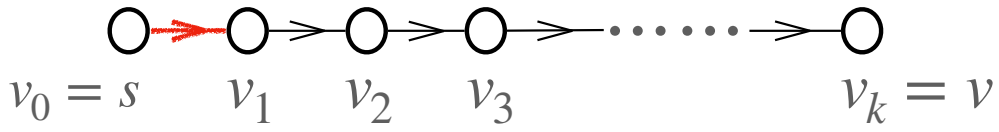
4) Return $d(v)$ and $\pi(v)$ for every node v (which form a shortest-path tree).

Time Complexity: $O(VE)$

This time complexity is actually relatively high for very large graphs.

Difficulty: we do not know the order of edges in the "shortest paths". If we do, We can do relaxation following the order.

Are there graphs where the "order" can be known?



CH. 24 Single-Source Shortest Paths

Shortest-Path Problem for Directed Acyclic Graph (DAG)

Algorithm:

1) Topologically sort the graph G=(V,E)

2) Initialization: $\forall\ v \in V,\ \pi(v) = \text{nil}$

$$d(s) = 0 \qquad \forall\ v \neq s,\ d(v) = \infty$$

3) For each node u, taken in the topologically sorted order:

for every edge (u, v) leaving u :

if $d(u) + w(u, v) < d(v)$:

$$d(v) = d(u) + w(u, v)$$

$$\pi(v) = u$$

Relaxation

4) Return $d(v)$ and $\pi(v)$ for every node v

(which form a shortest-path tree).

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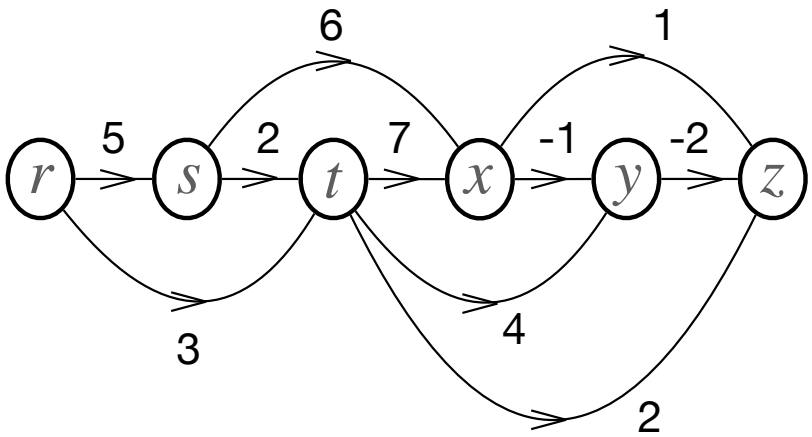
A simplified Bellman-Ford Algorithm where:

1) Every edge is relaxed only once.

2) No need to check negative cycle.

CH. 24 Single-Source Shortest Paths

Shortest-Path Problem for Directed Acyclic Graph (DAG)



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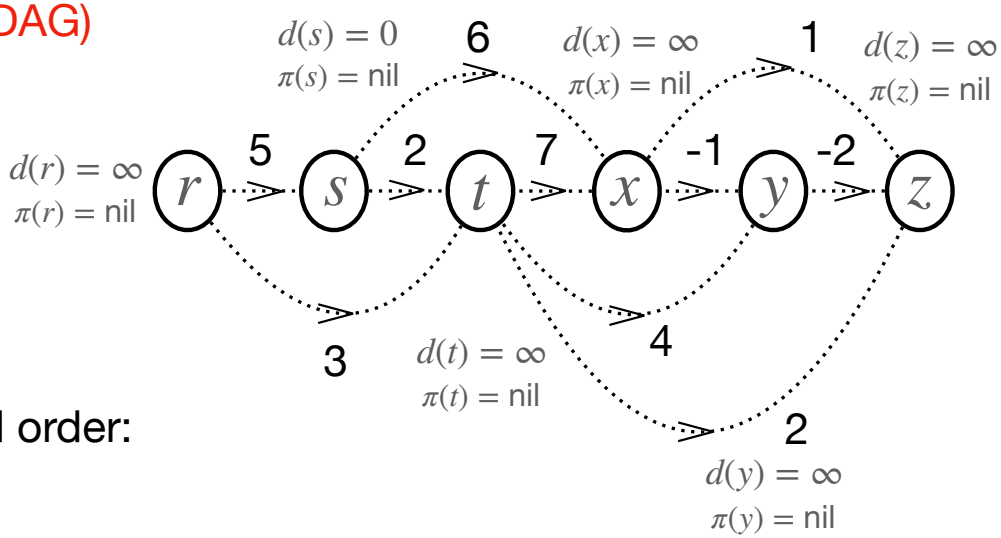

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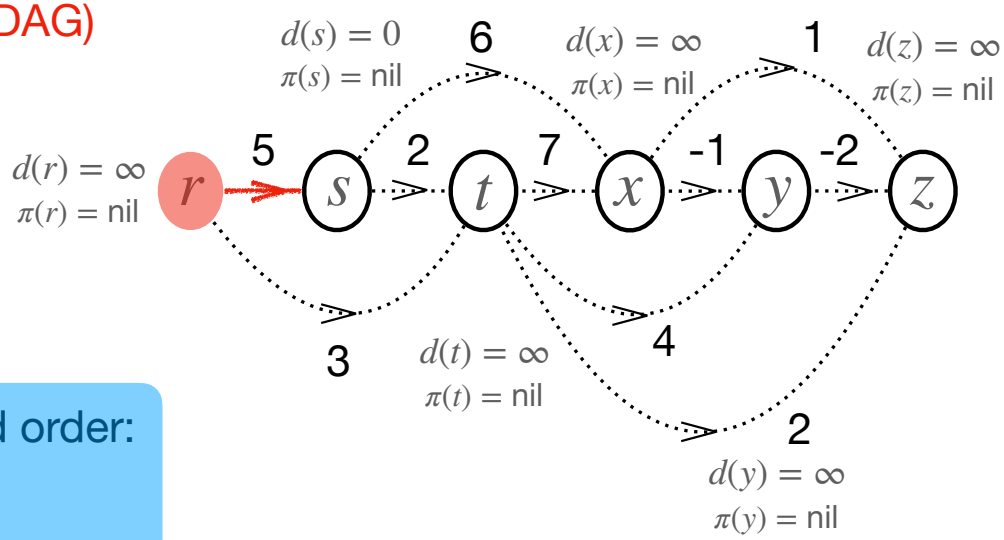

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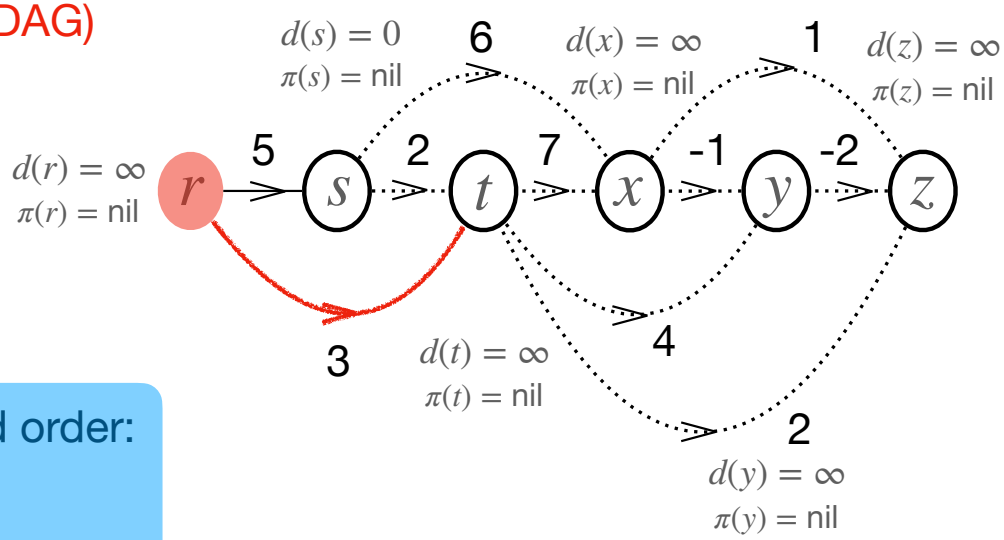

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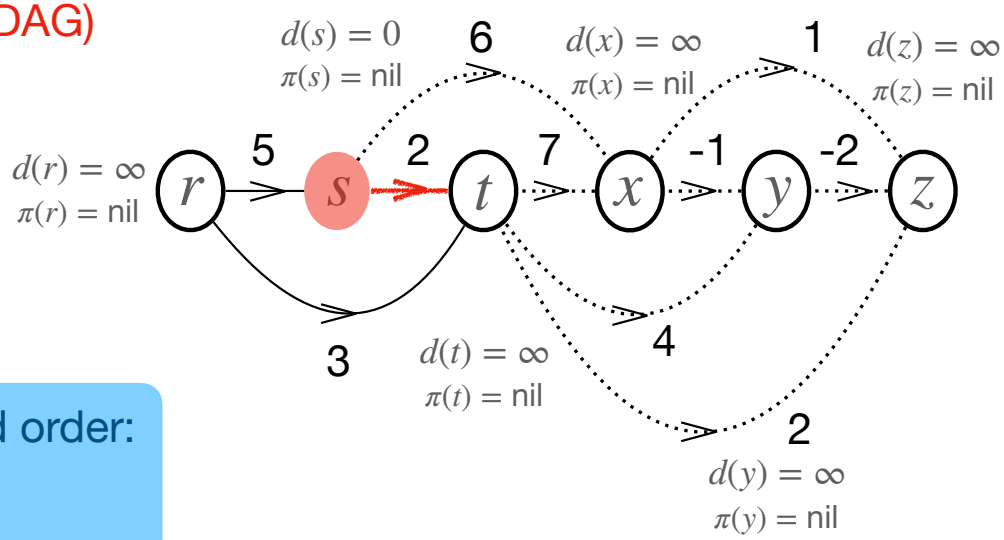

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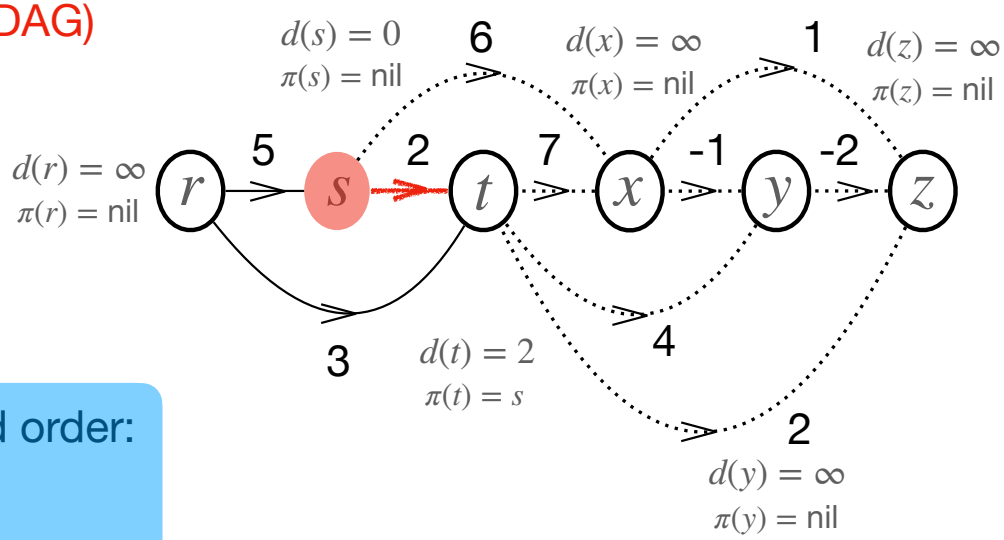

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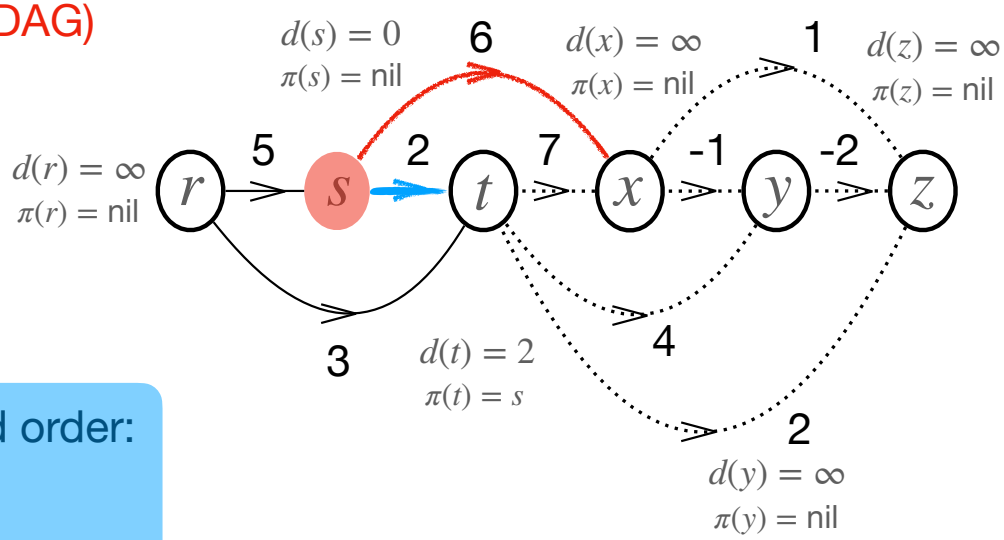

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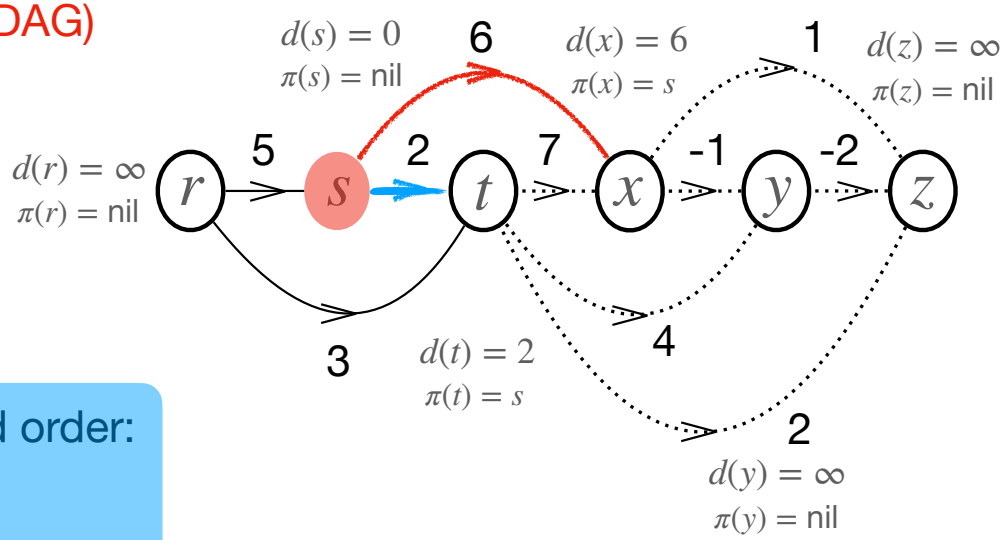

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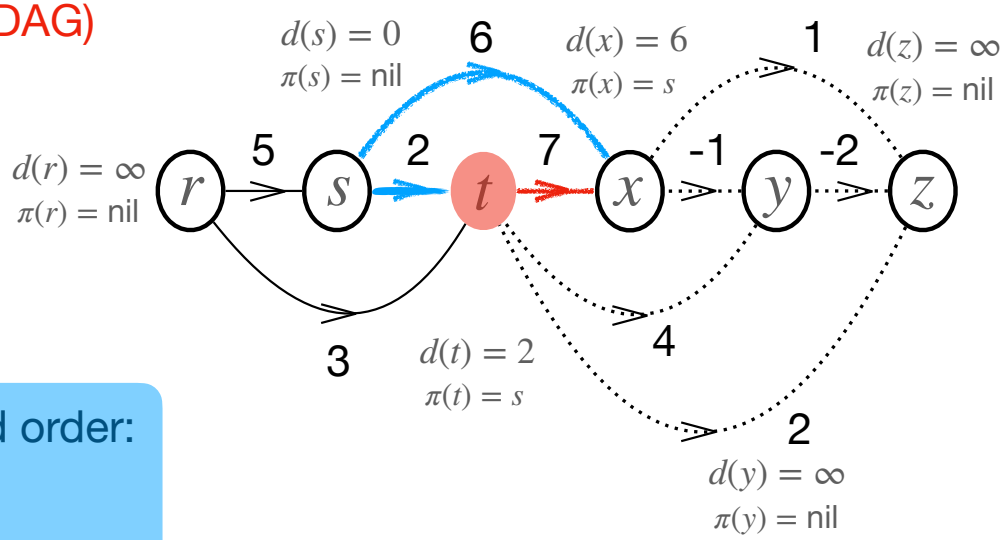

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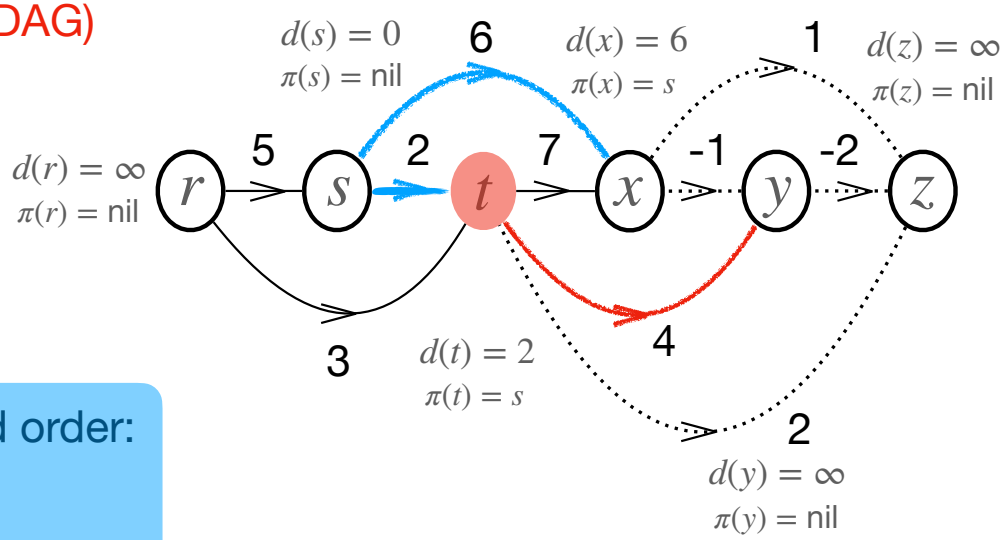

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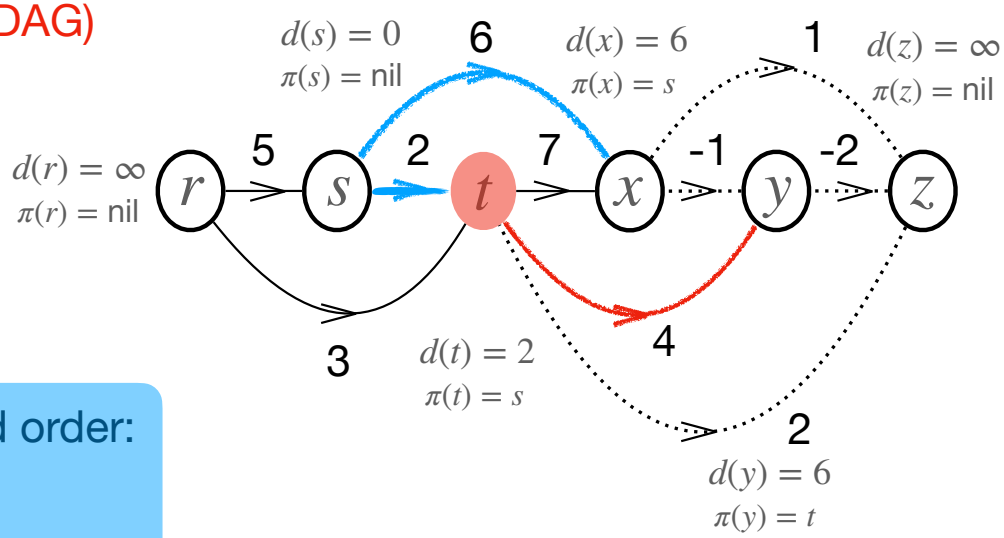

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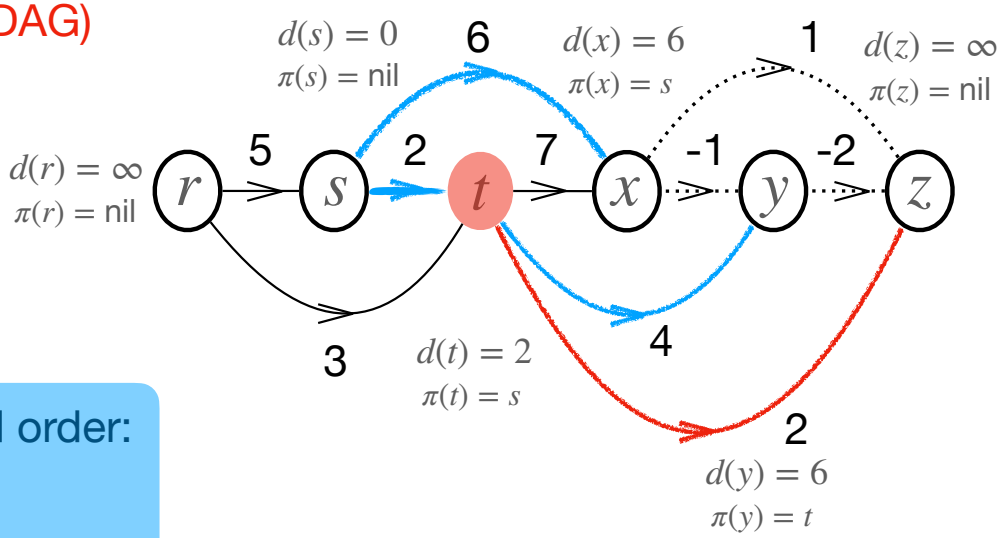

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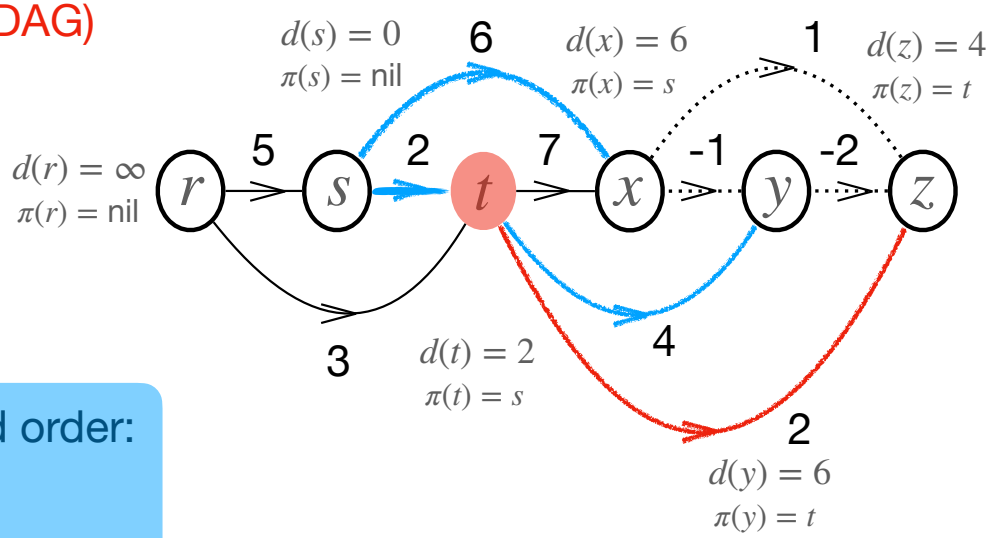

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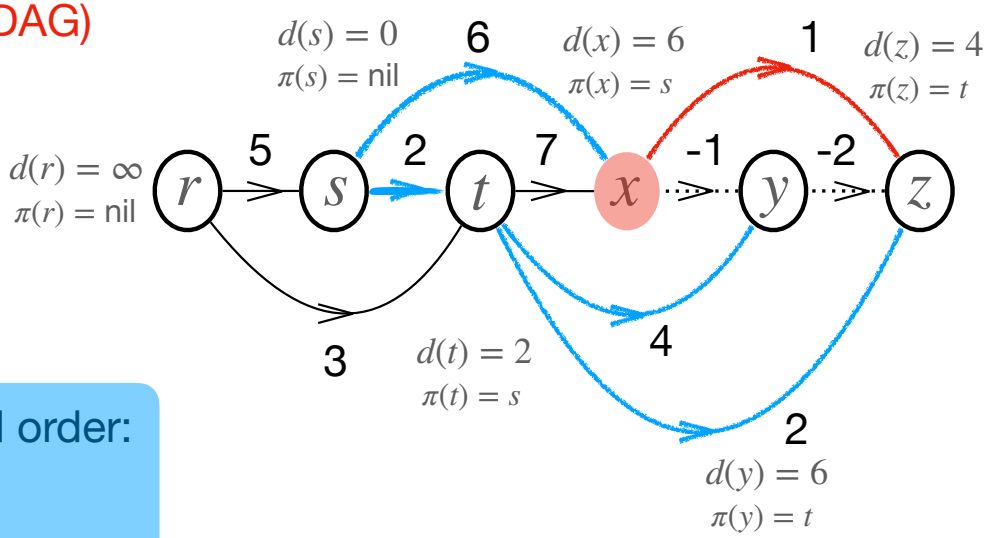

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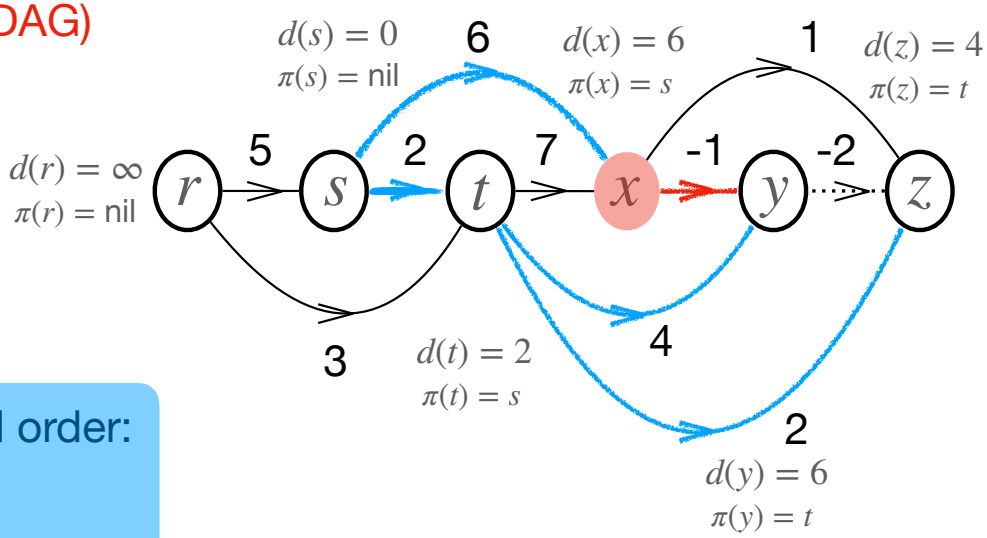

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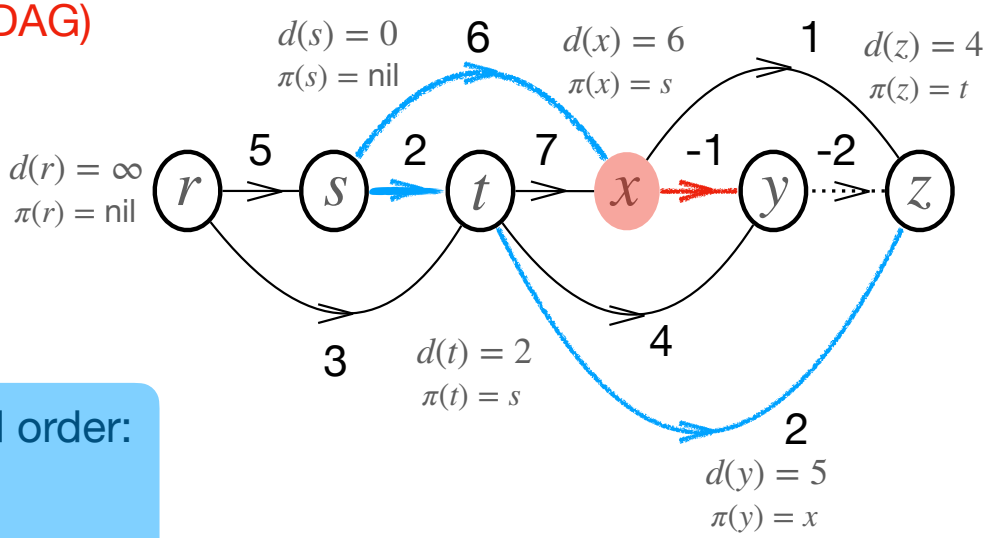

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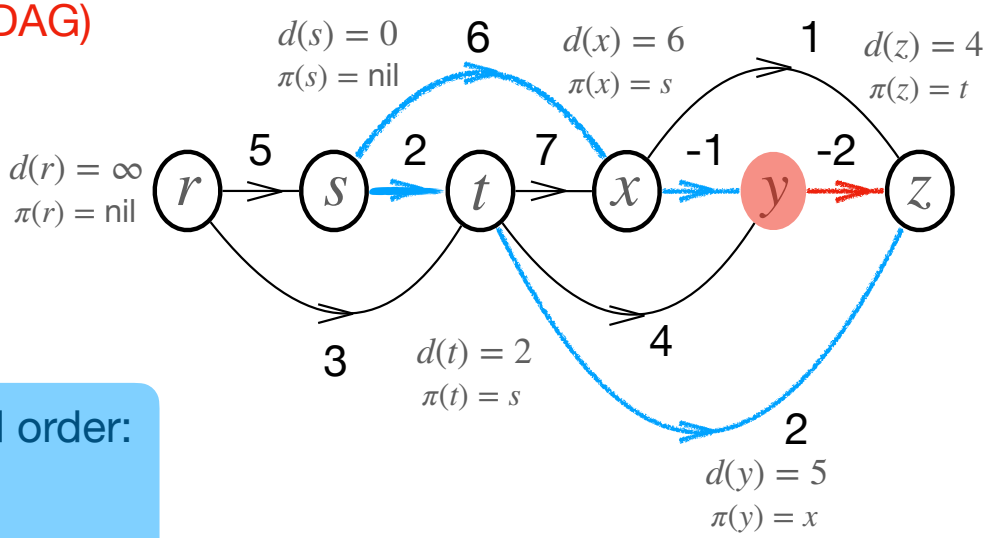

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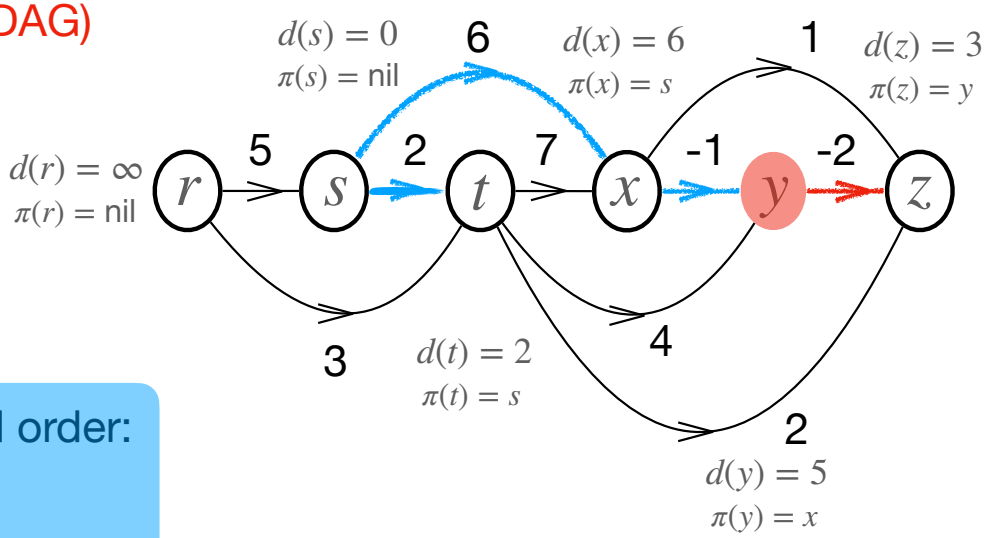

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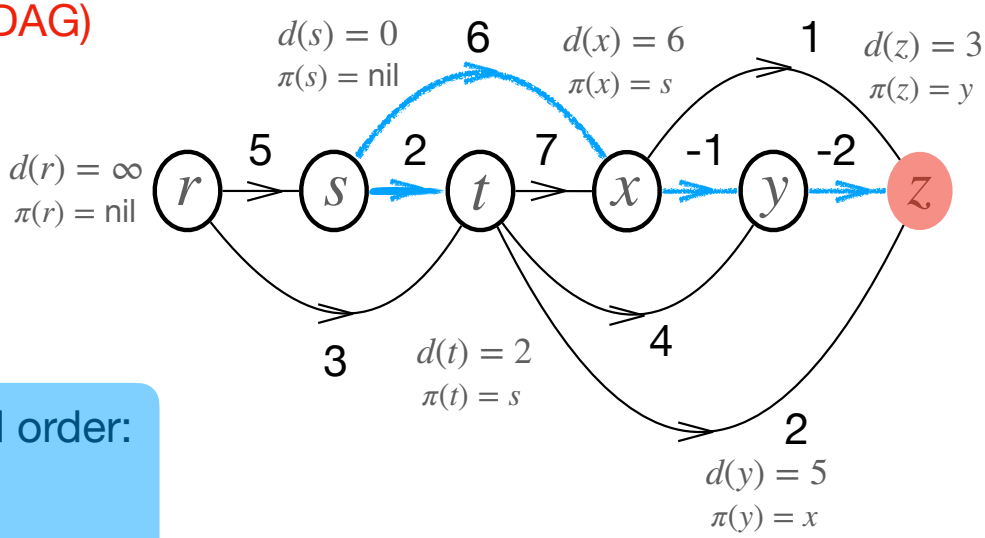

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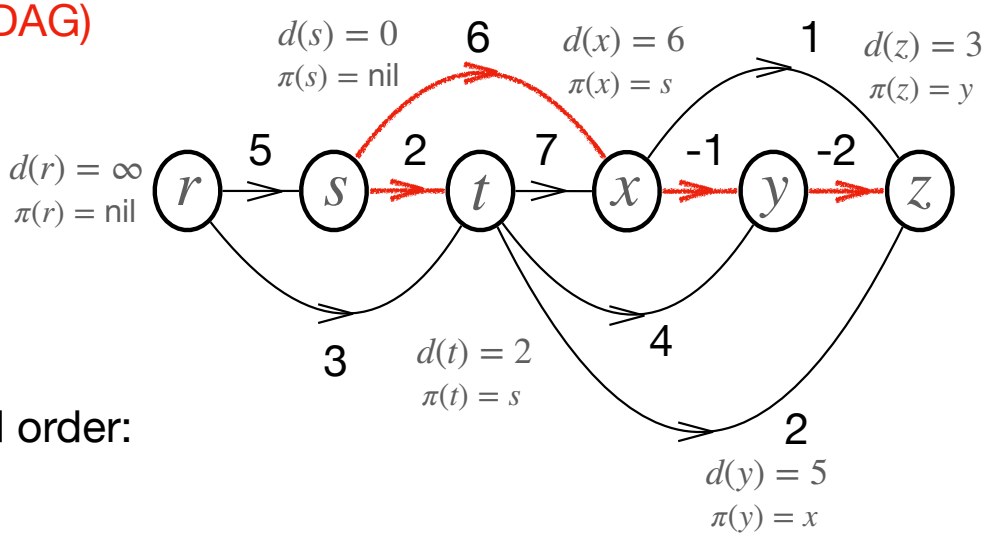

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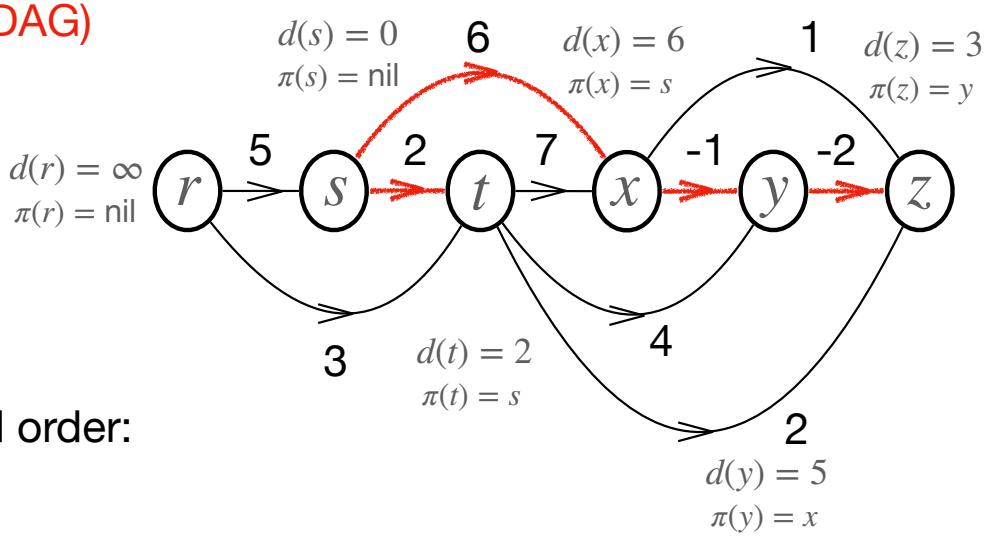

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Time Complexity:

$$O(V + E)$$

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