Algorithms

Lecture 21: Maximum Flow (Part 1)

Anxiao (Andrew) Jiang

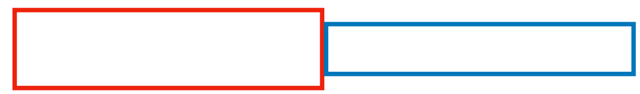
CH 26. Maximum Flow

Capacity of a flow “pipe"

Capacity: 3 units of flow / second

Capacity: 1 unit of flow / second

What is its overall capacity?



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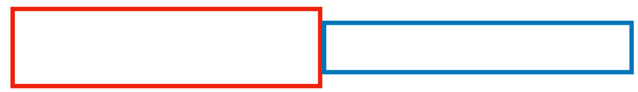
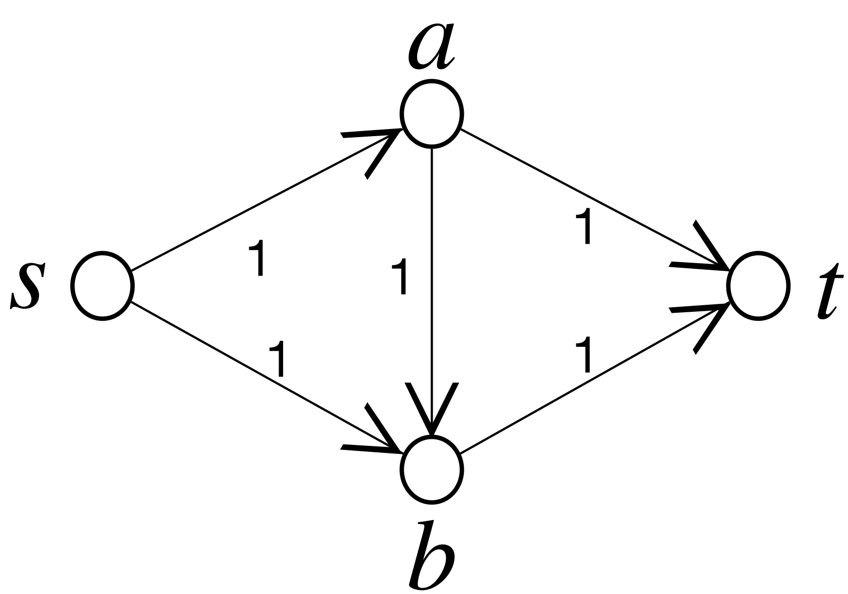


$$min\{3,1\} = 1$$

CH 26. Maximum Flow

How much flow can be transmitted from s to t,
assuming every edge has capacity 1?



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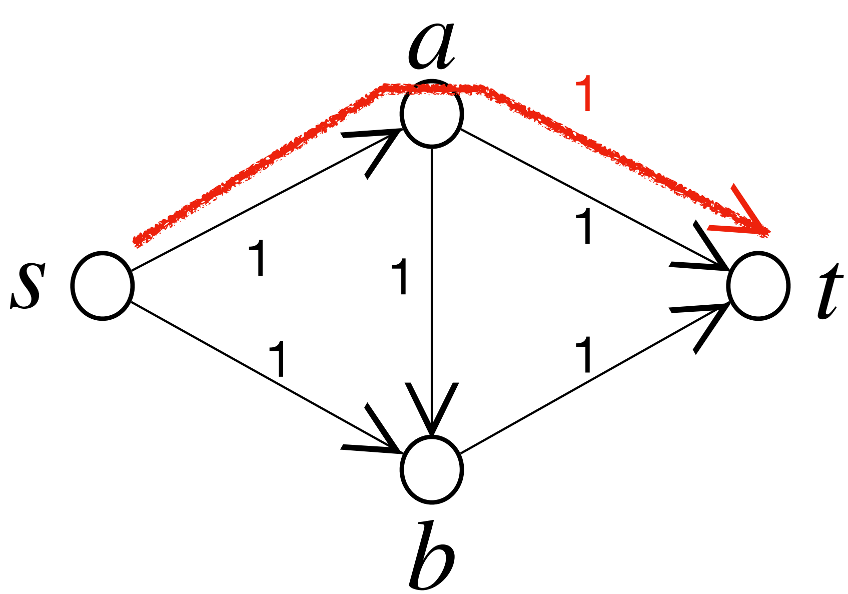

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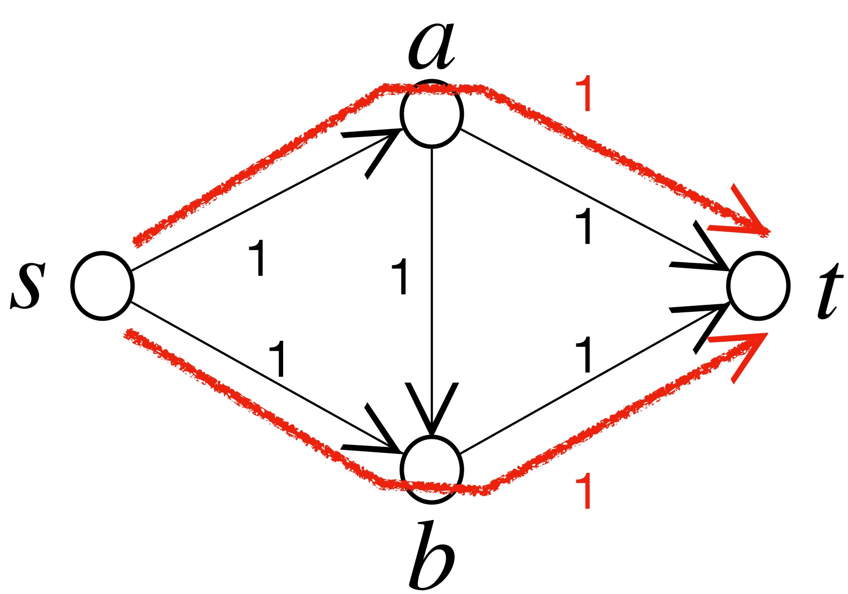
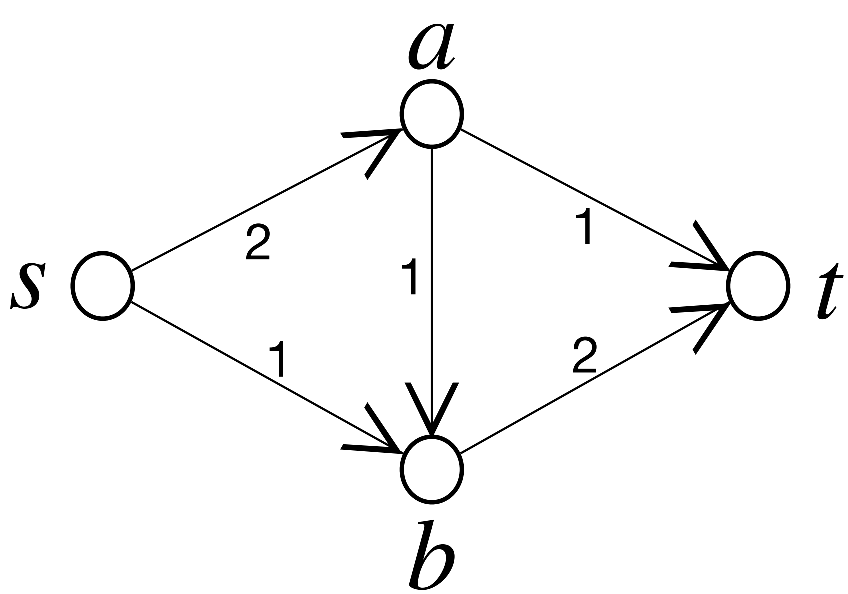


Maximum flow from s to t :

2

CH 26. Maximum Flow

How much flow can be transmitted from s to t, assuming the capacities of edges are as shown?



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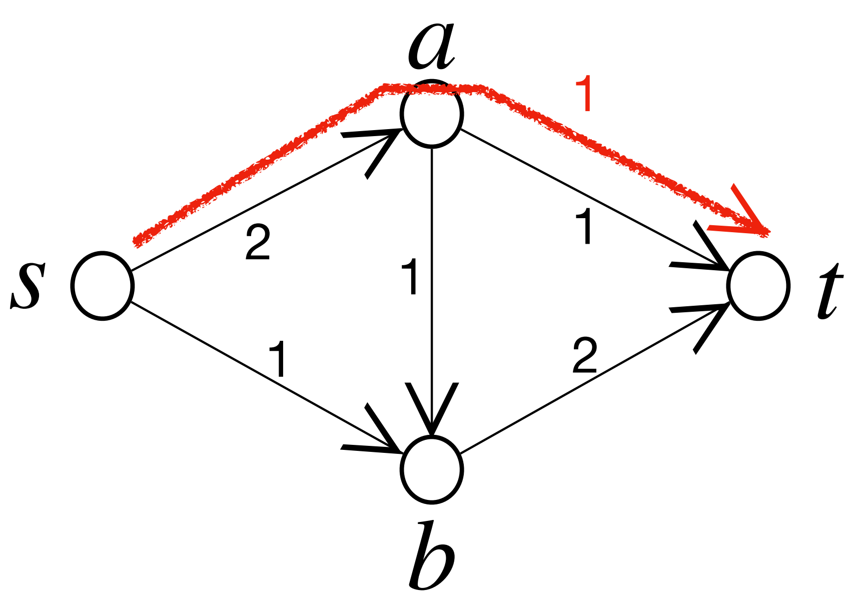

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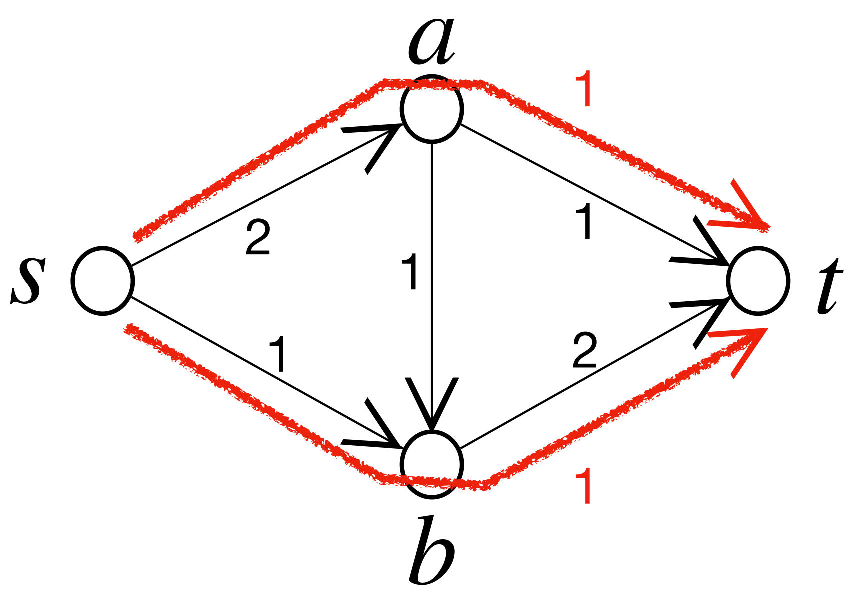

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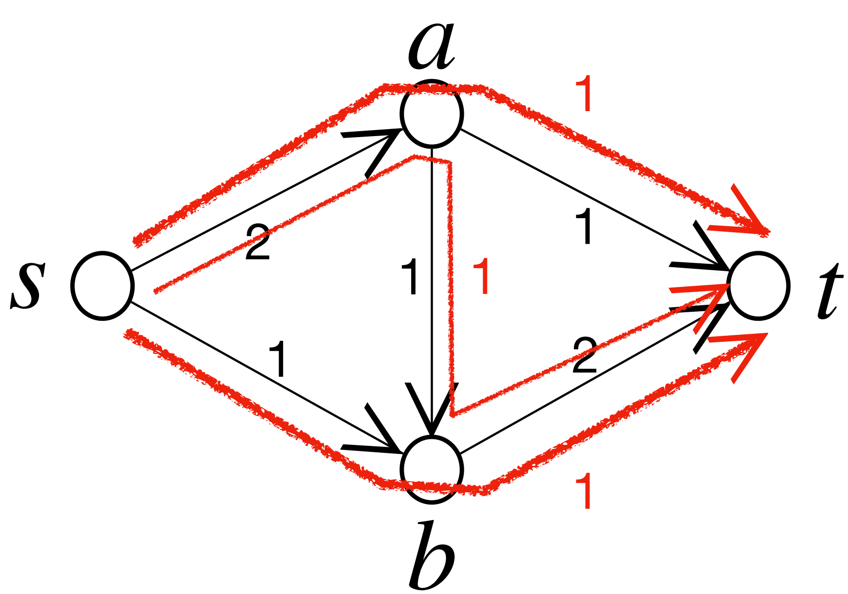


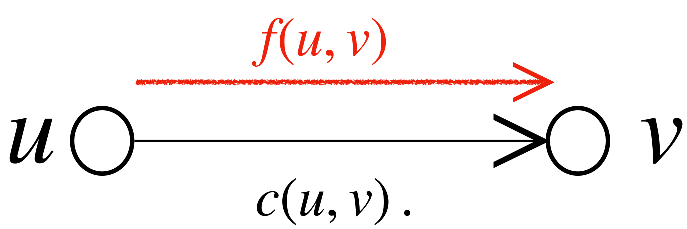
Maximum flow from s to t :

3

CH 26. Maximum Flow

Input: A directed graph G=(V,E), where every edge $(u, v) \in E$ has a non-negative capacity $c(u, v)$. Let $s \in V$ be a “source" node, and let $t \in V$ be a “sink" node.

Flow: $\forall\ (u, v) \in E$, let $f(u, v)$ be the flow from u to v in the edge (u, v).



Capacity constraint: $0 \leq f(u, v) \leq c(u, v)$

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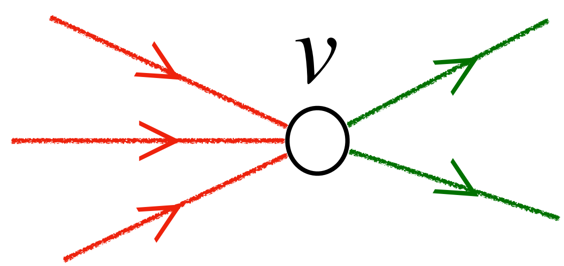
Constraints for flow f:

1) Capacity constraint: $\forall\ (u, v) \in E, \qquad 0 \leq f(u, v) \leq c(u, v)$

2) Flow-conservation constraint (also called “in-flow equals out-flow” constraint):

For every node that is not source s or sink t,
its total incoming flow equals its total outgoing flow.



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Size of flow f: The net amount of flow leaving the source s,
which is also the net amount of flow entering the sink t.

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Input: A directed graph G=(V,E), where every edge $(u, v) \in E$ has a non-negative capacity $c(u, v)$. Let $s \in V$ be a "source" node, and let $t \in V$ be a "sink" node.

Output: A maximum flow from s to t.

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Ford-Fulkerson Method:

Gradually add flow to the network G=(V,E) as follows:

Each time, look at the network with the residual capacities (i.e., “remaining capacities”), and find a path from s to t using only residual capacities, then add as much flow as possible along that path.

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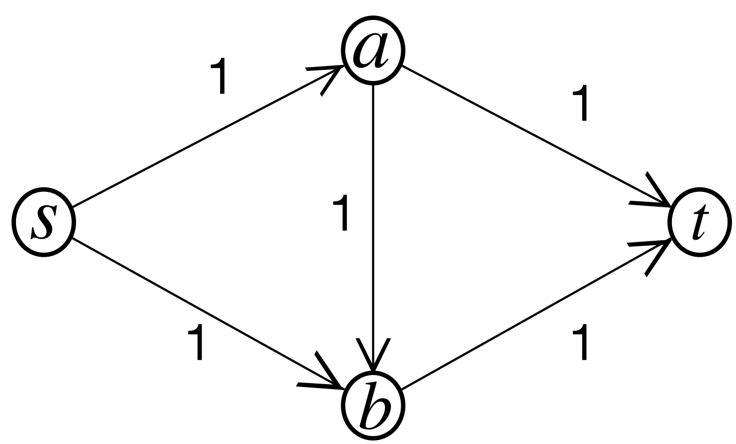

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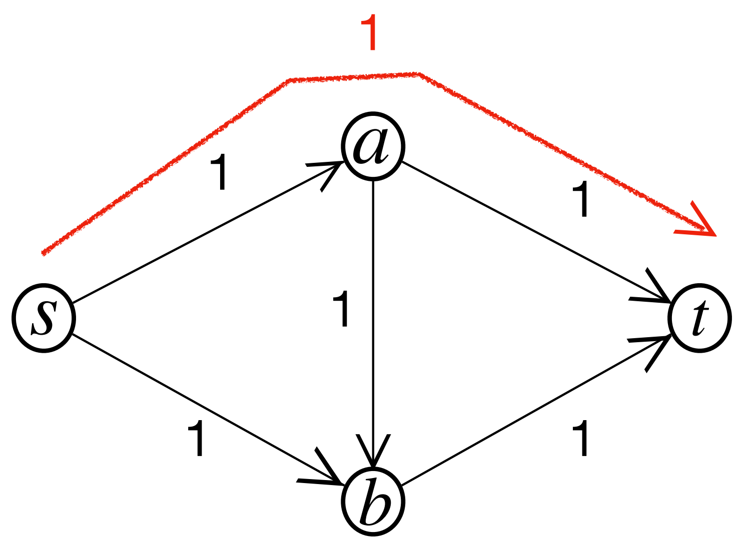

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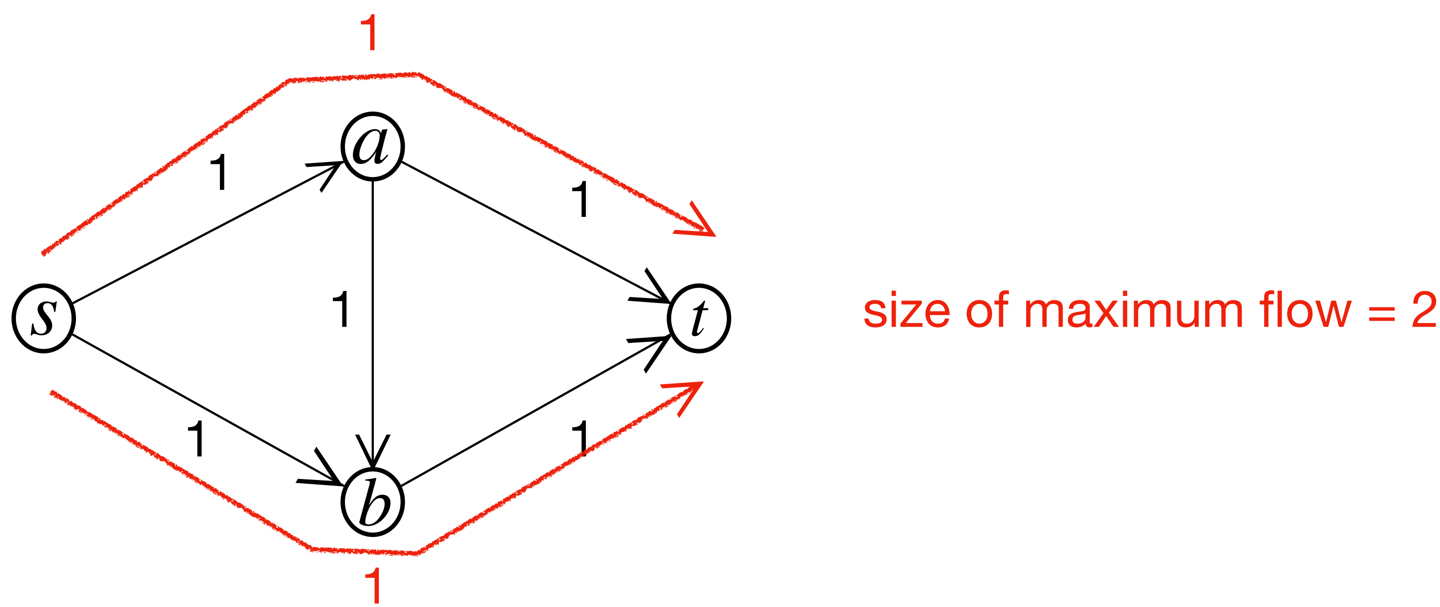

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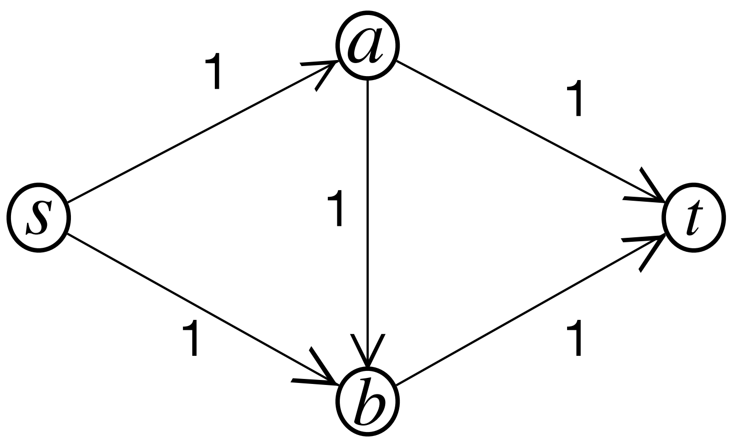

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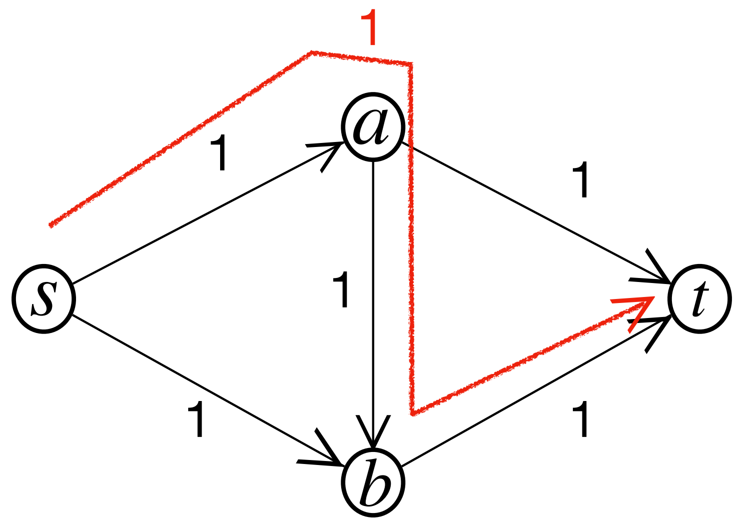

Does the Ford-Fulkerson Method still work?

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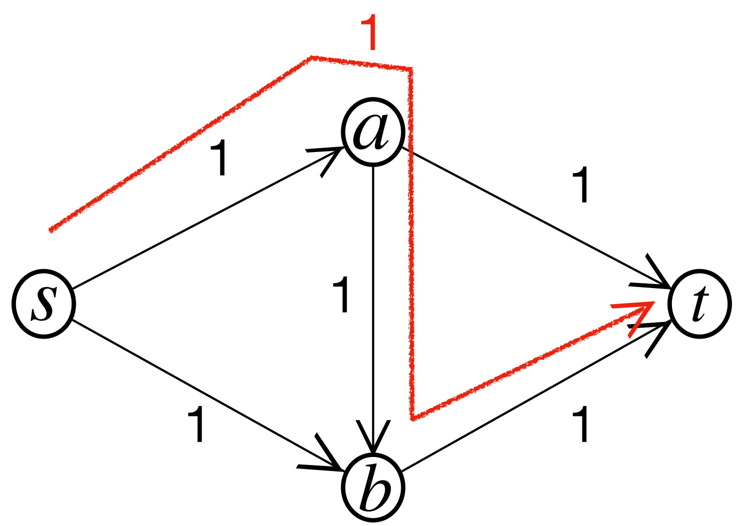

Does the
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Yes, but we need to understand
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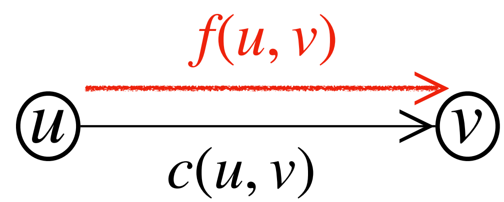
Residual capacity:

1) How much can the flow from u to v be increased by?



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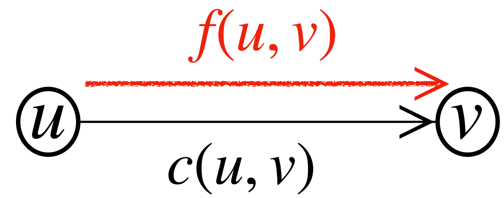

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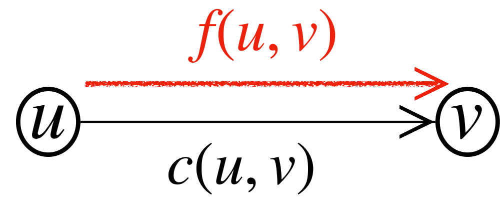

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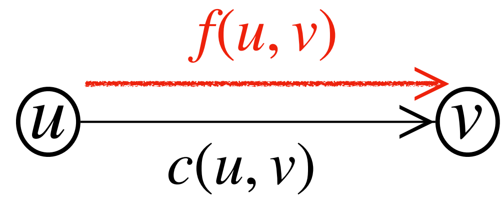

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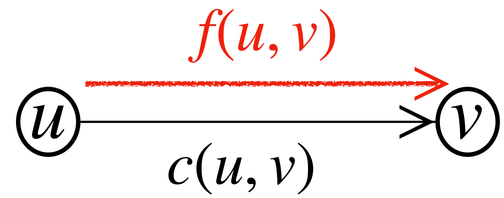

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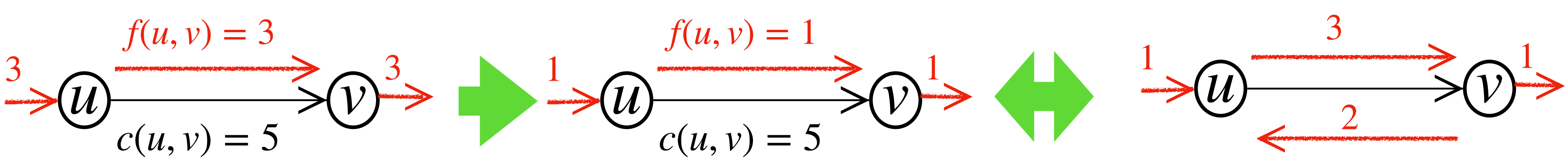
3) Decreasing the flow from u to v is equivalent to increasing the flow from v to u.



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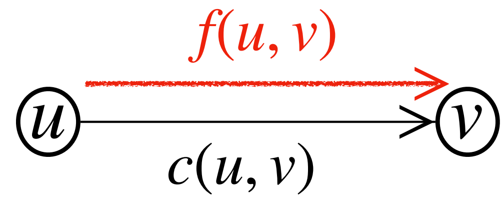

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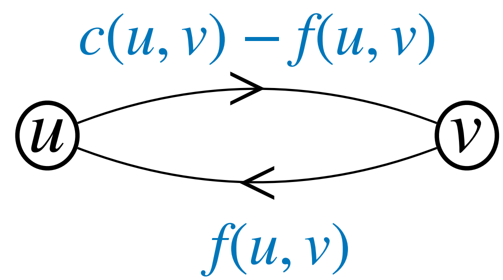

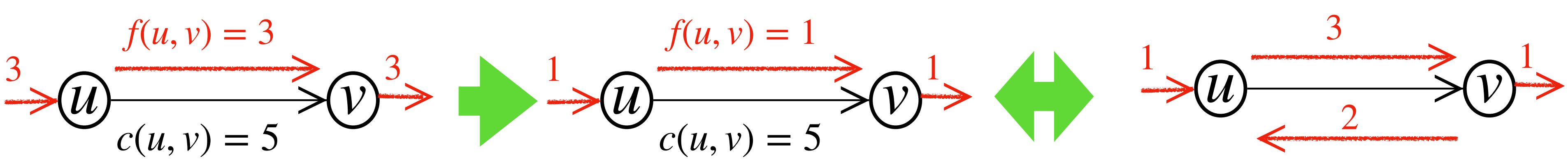

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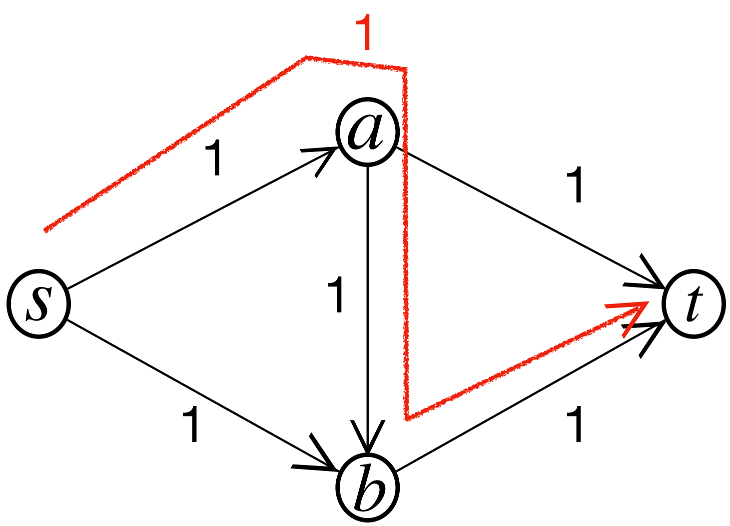

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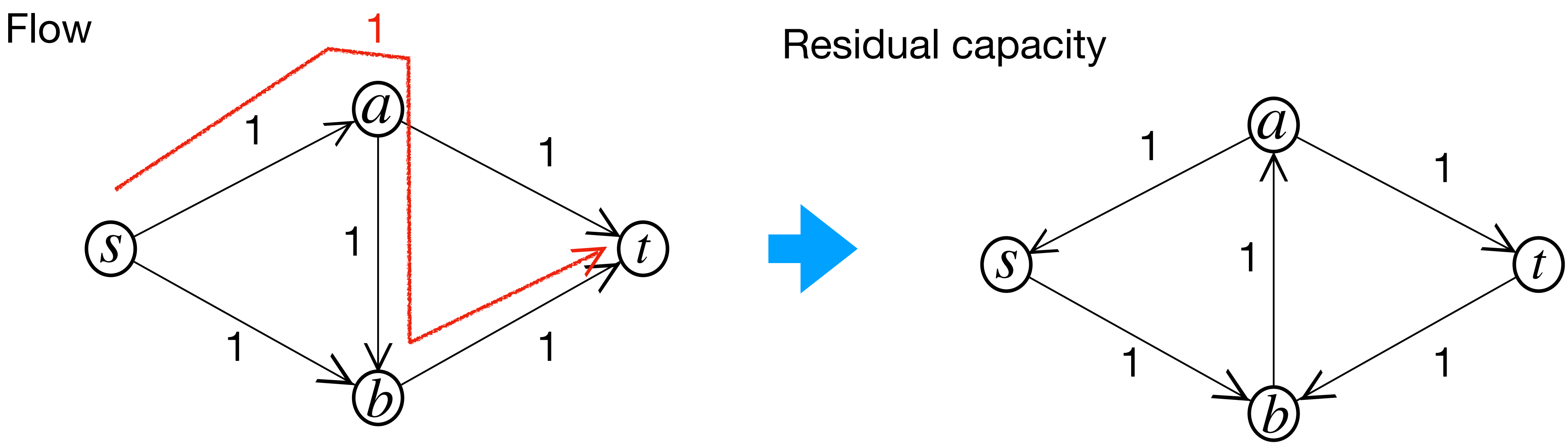

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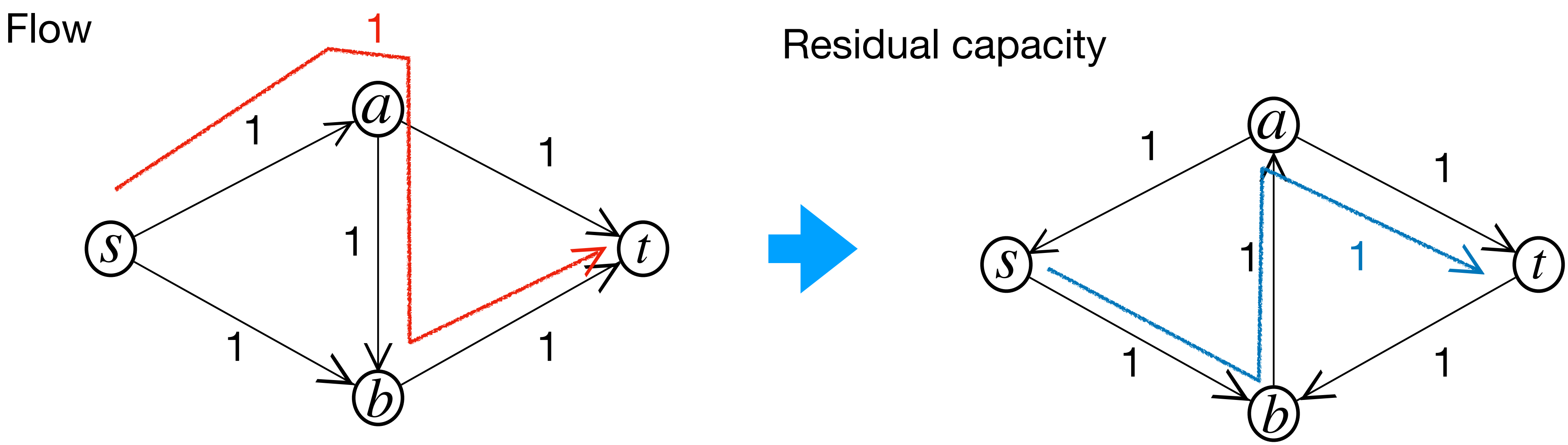

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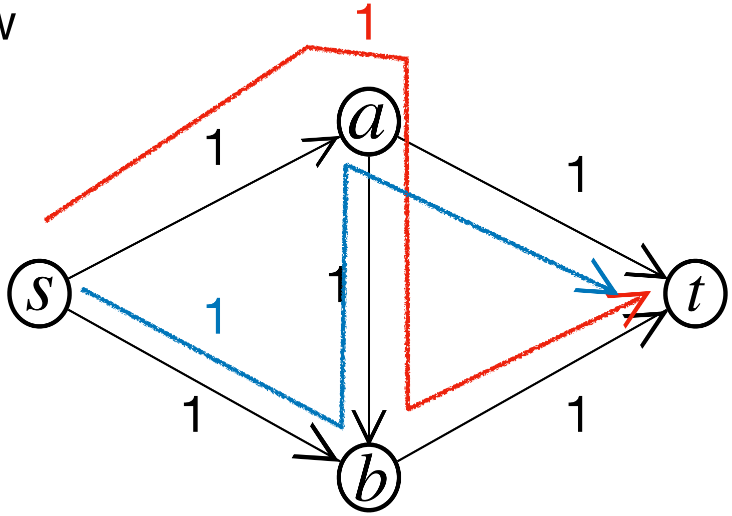
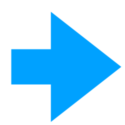
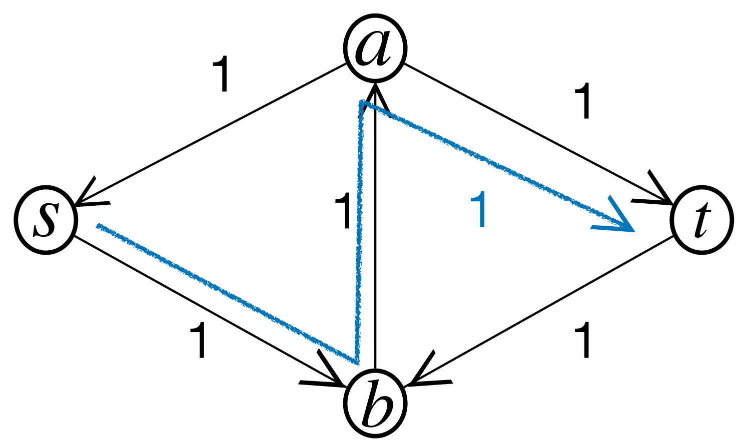
Flow





Residual capacity



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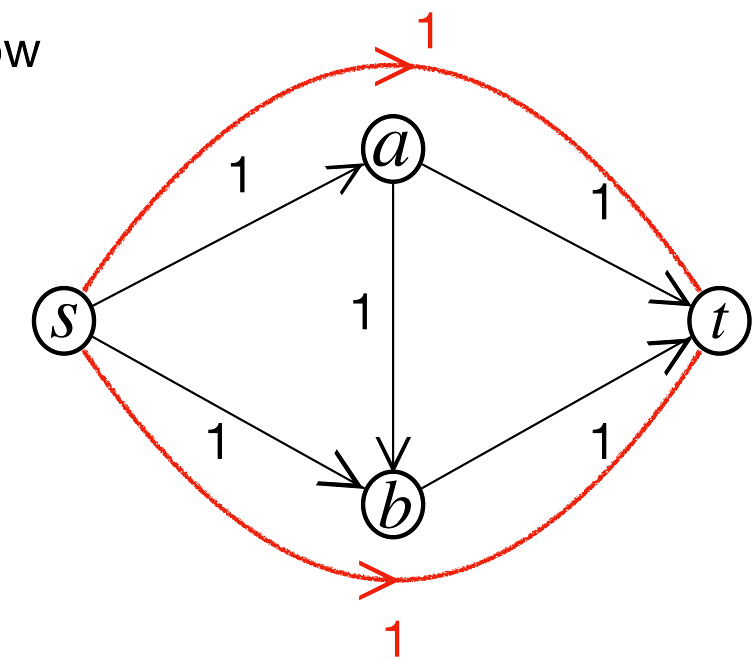

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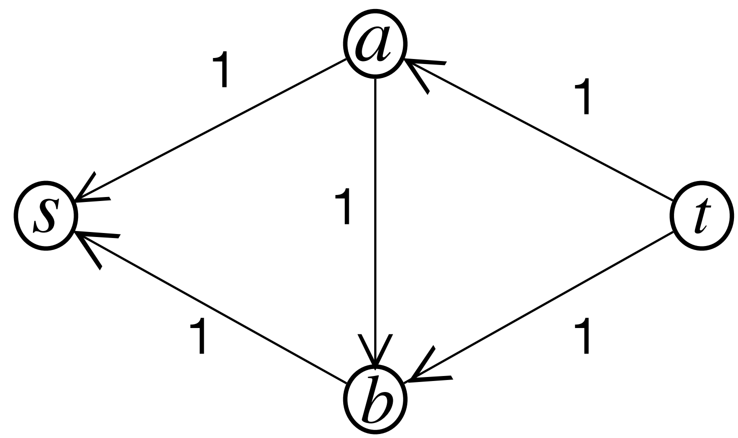
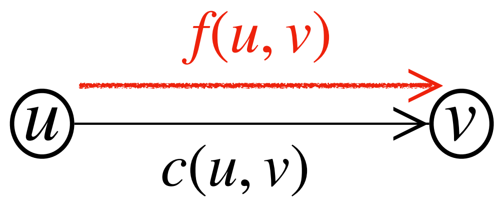
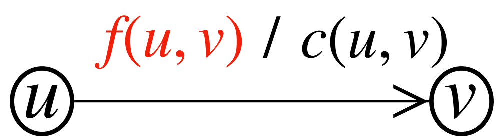


No more path from s to t

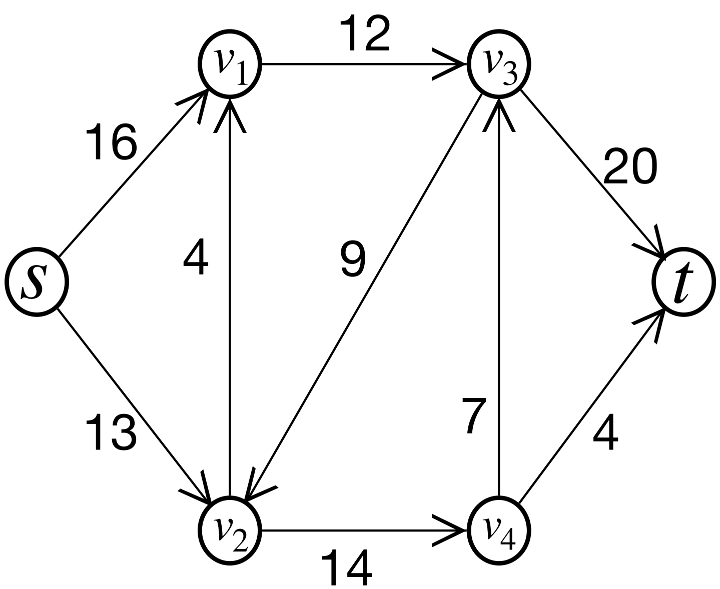


Simpler notation:



Ford-Fulkerson Method

Original network



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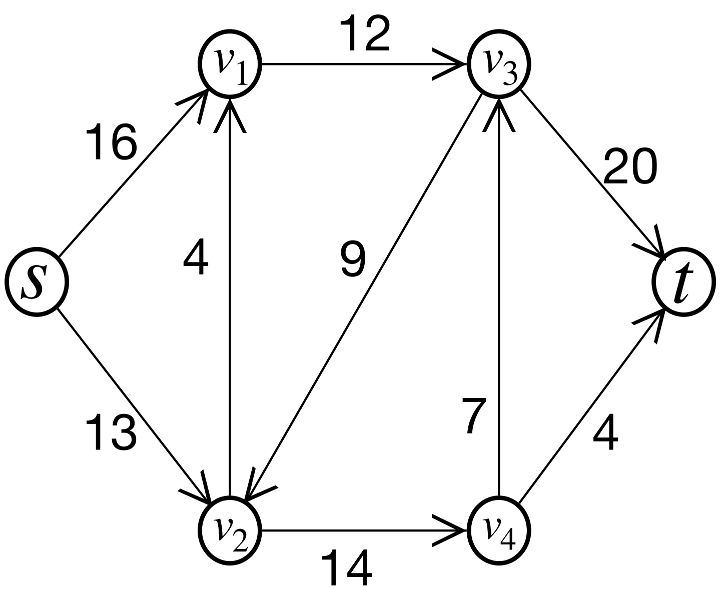
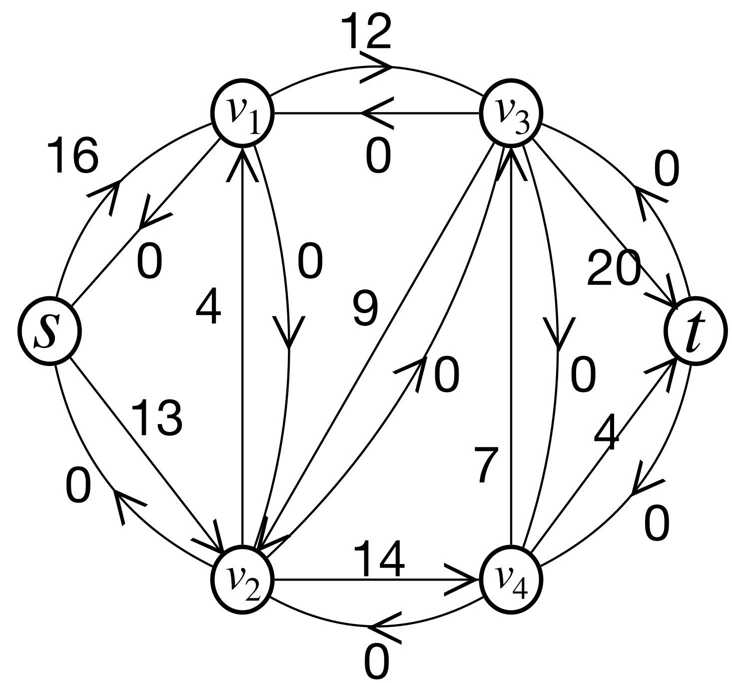


Residual network



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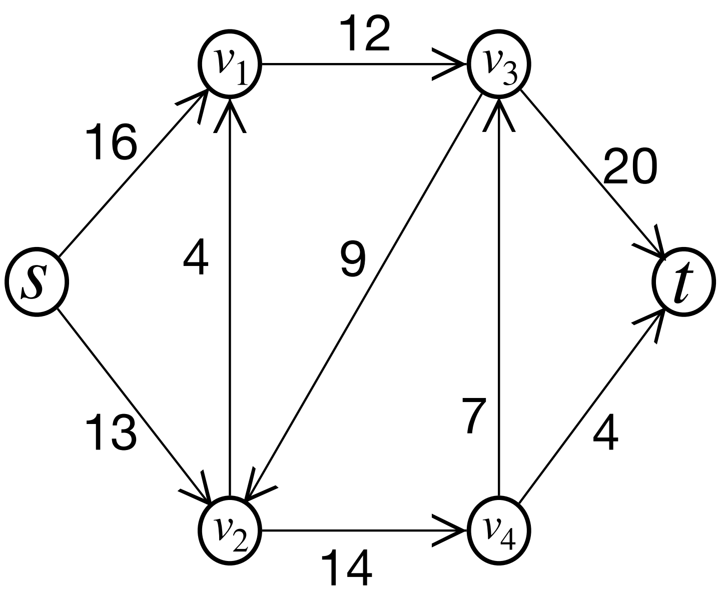

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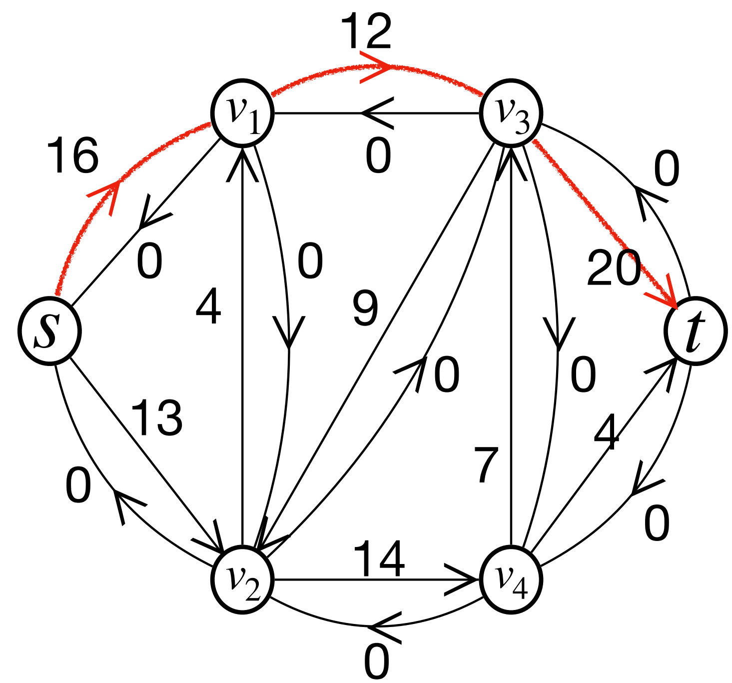

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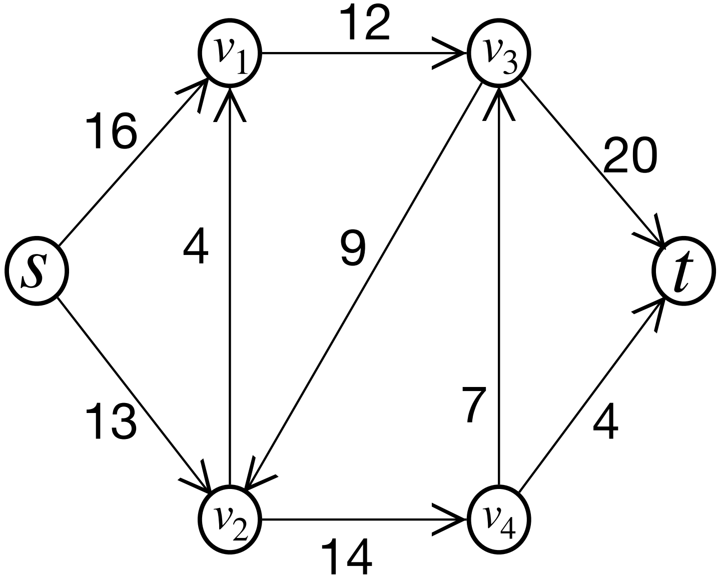
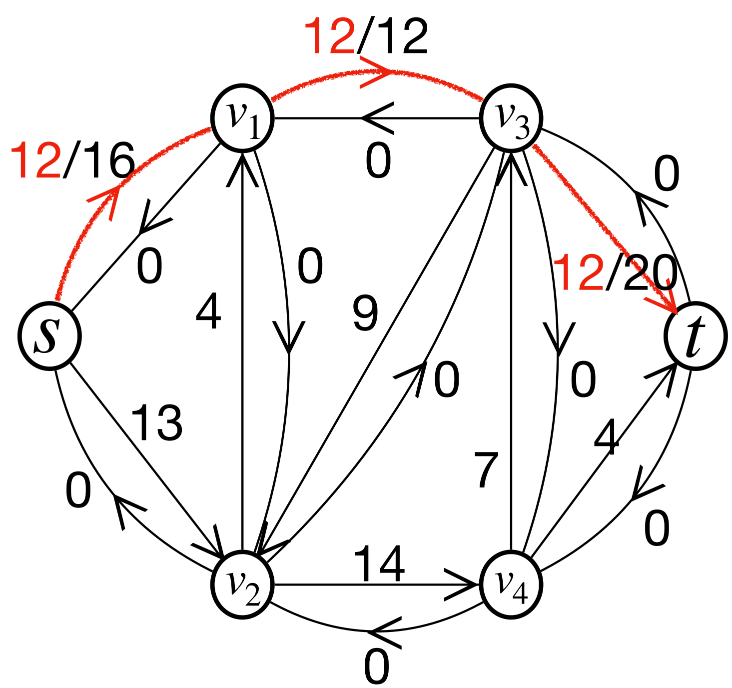


Residual network



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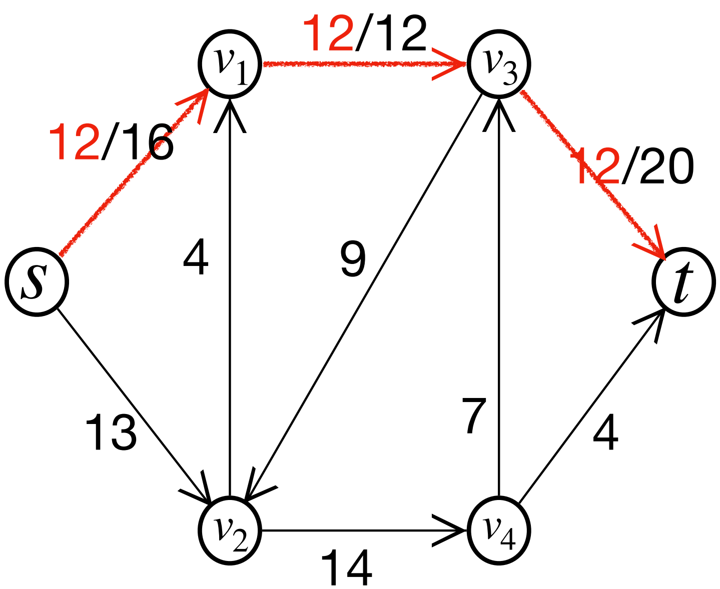

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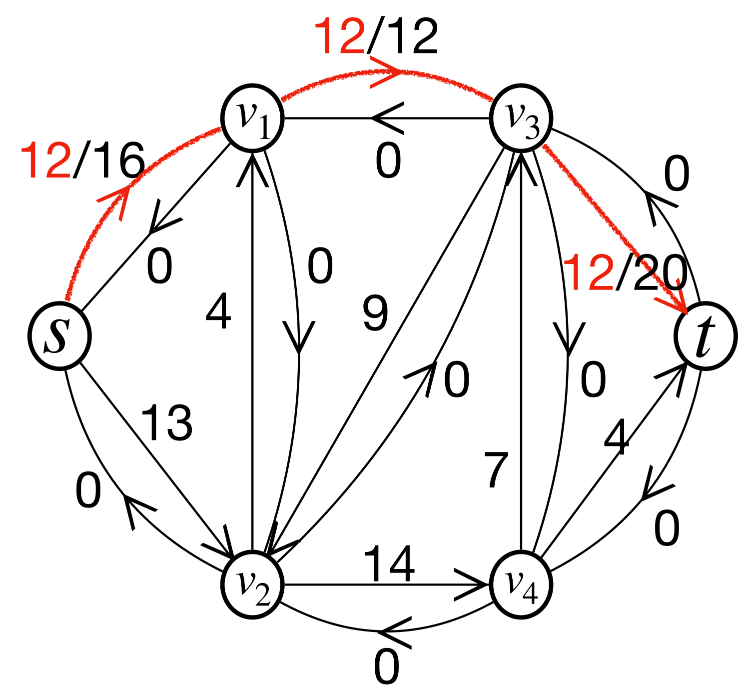
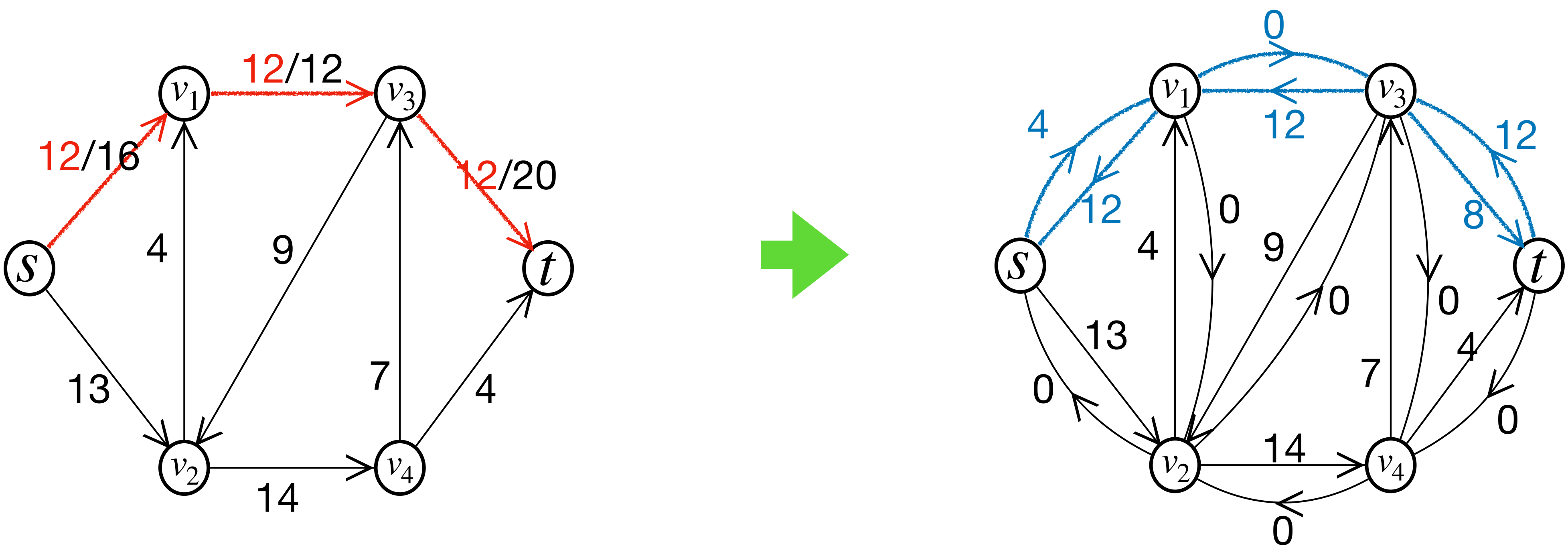


Ford-Fulkerson Method

Original network   Residual network



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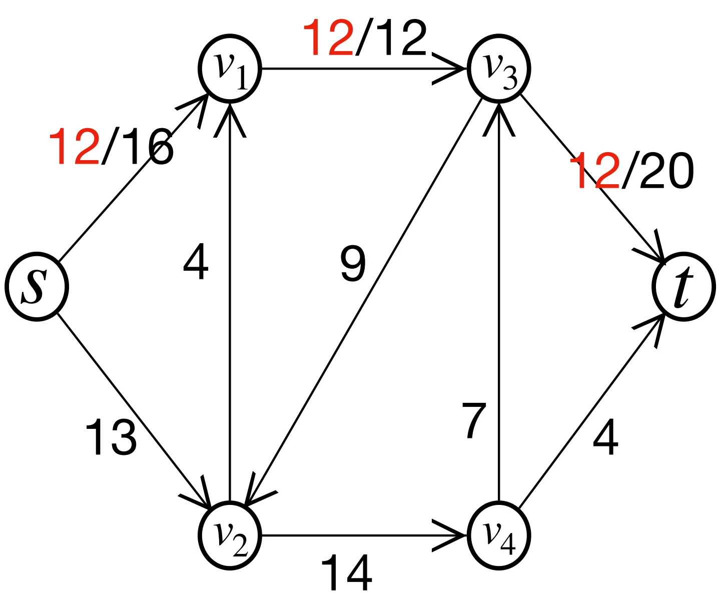
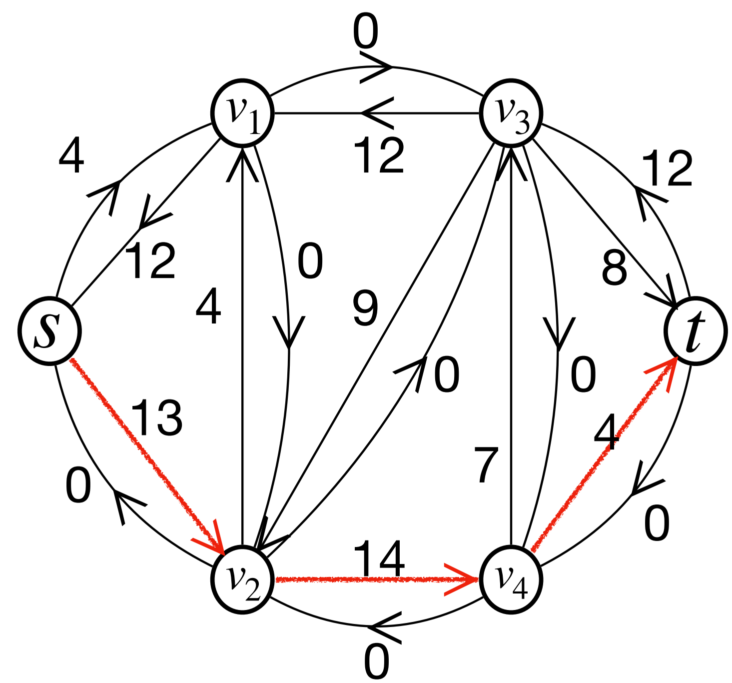
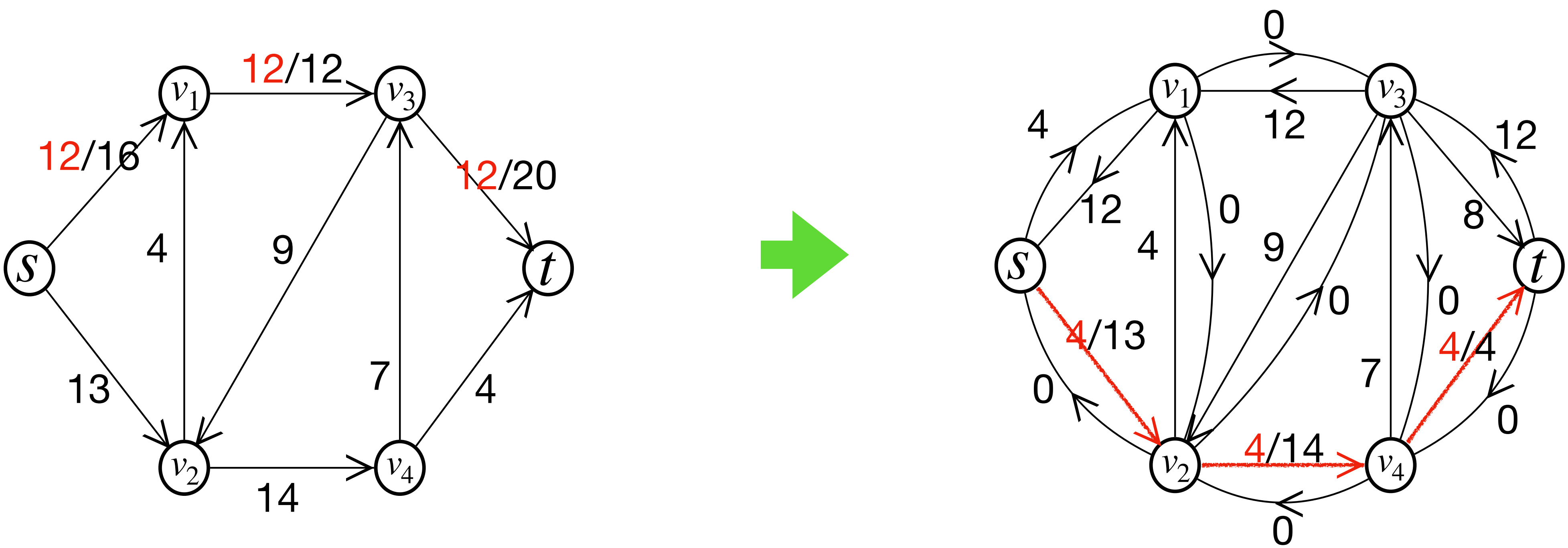


Residual network



Ford-Fulkerson Method

Original network | Residual network



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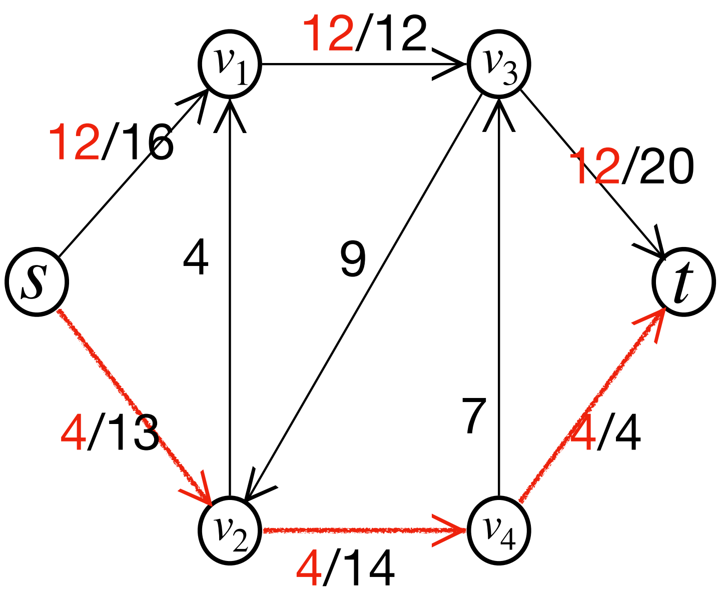
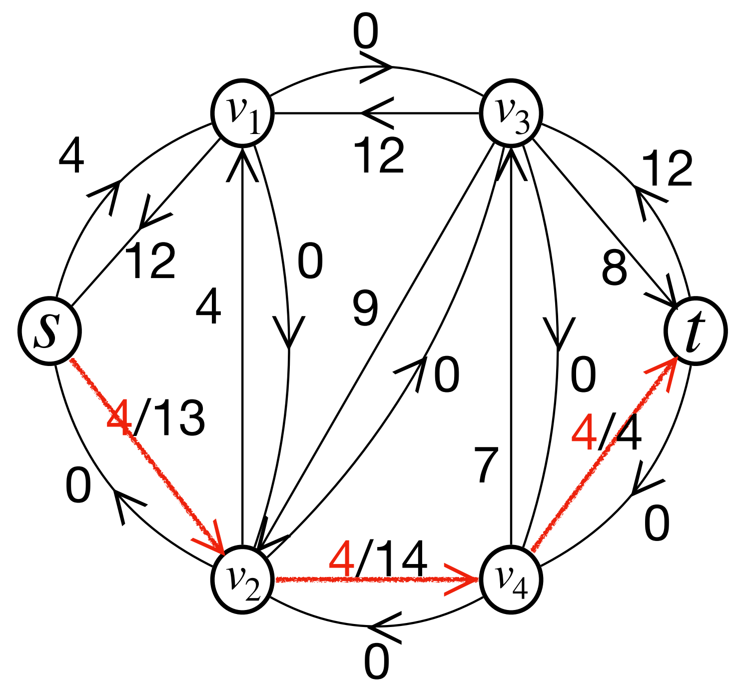
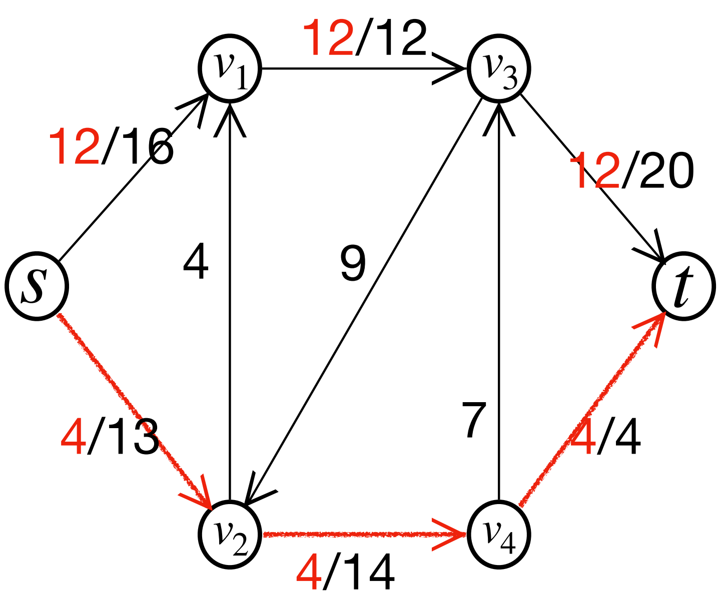


Residual network

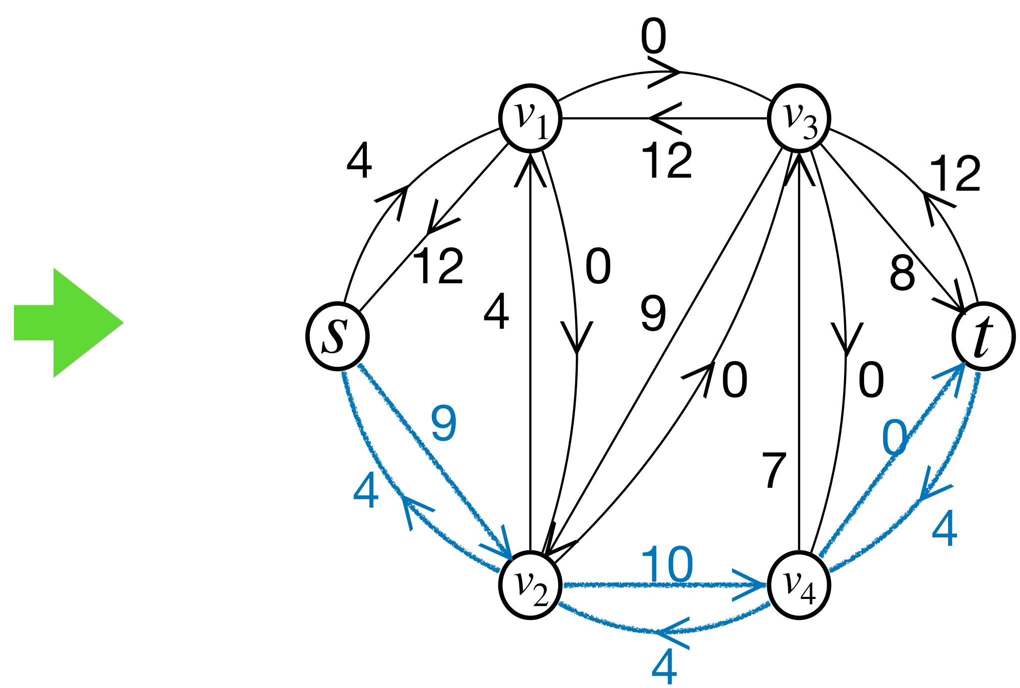


Ford-Fulkerson Method

Original network

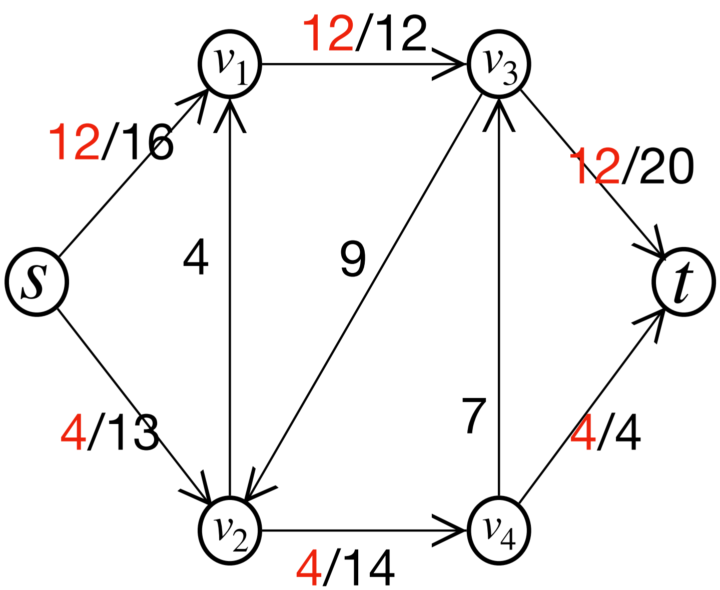


Residual network

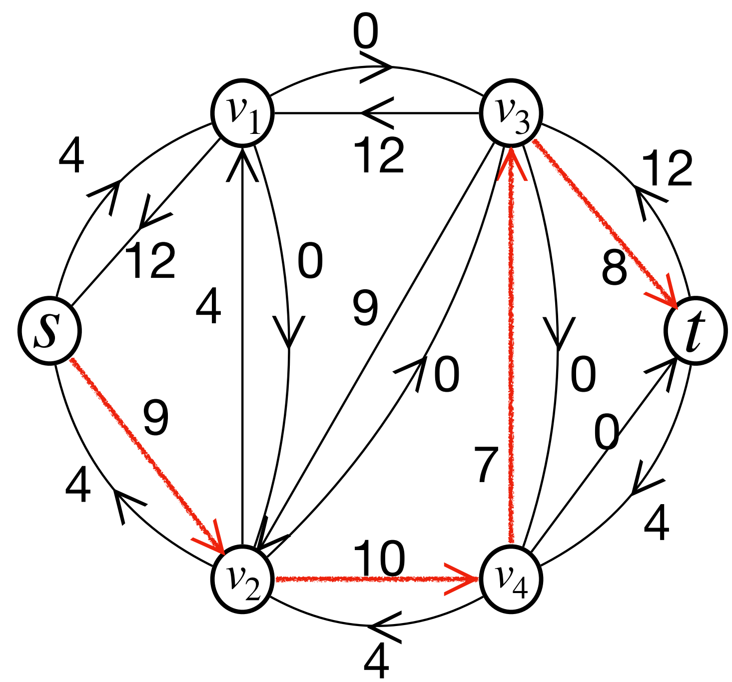


Ford-Fulkerson Method

Original network

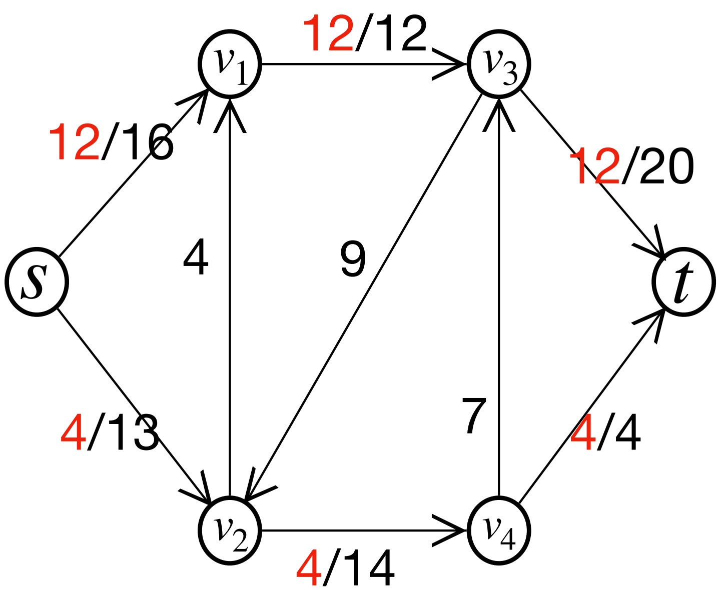


Residual network

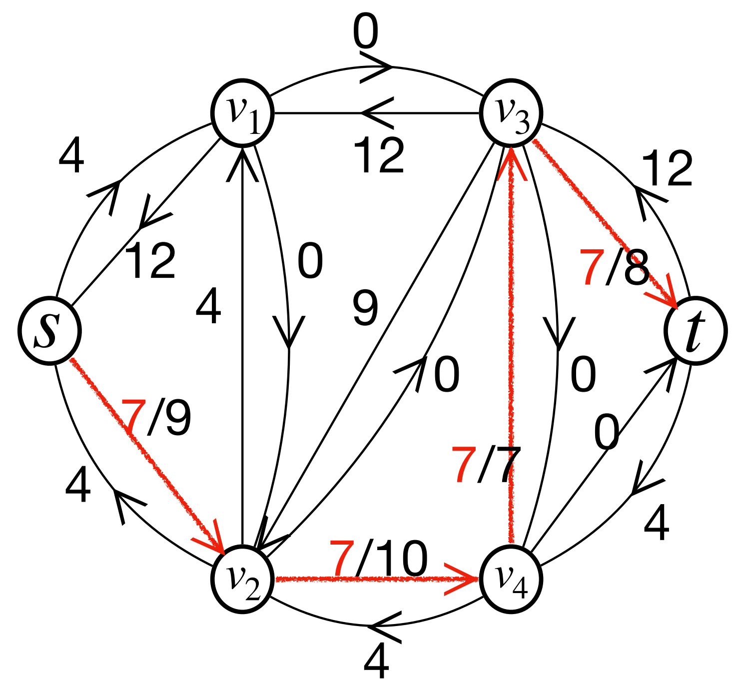


Ford-Fulkerson Method

Original network

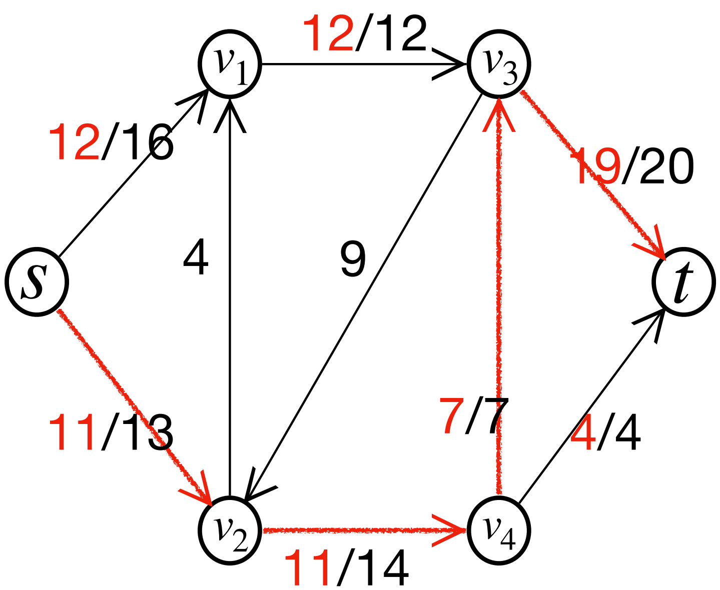


Residual network

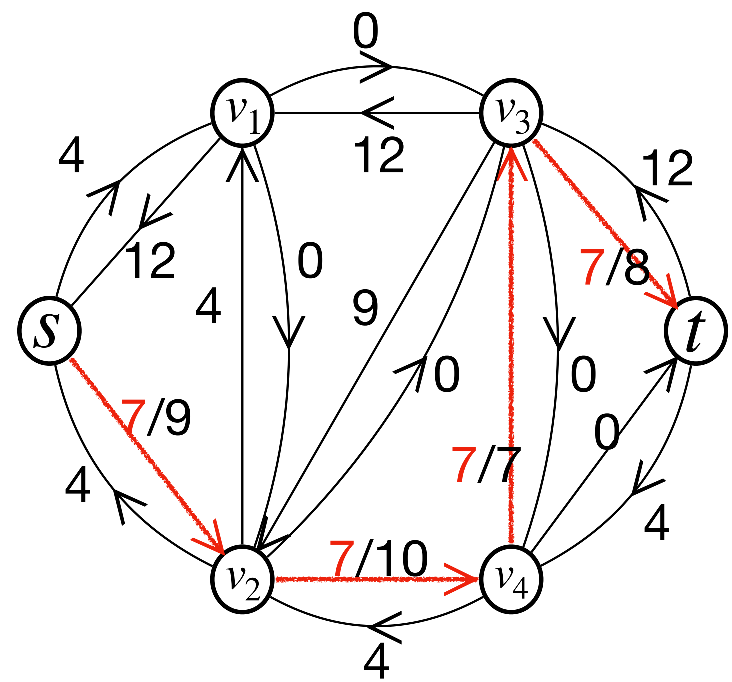


Ford-Fulkerson Method

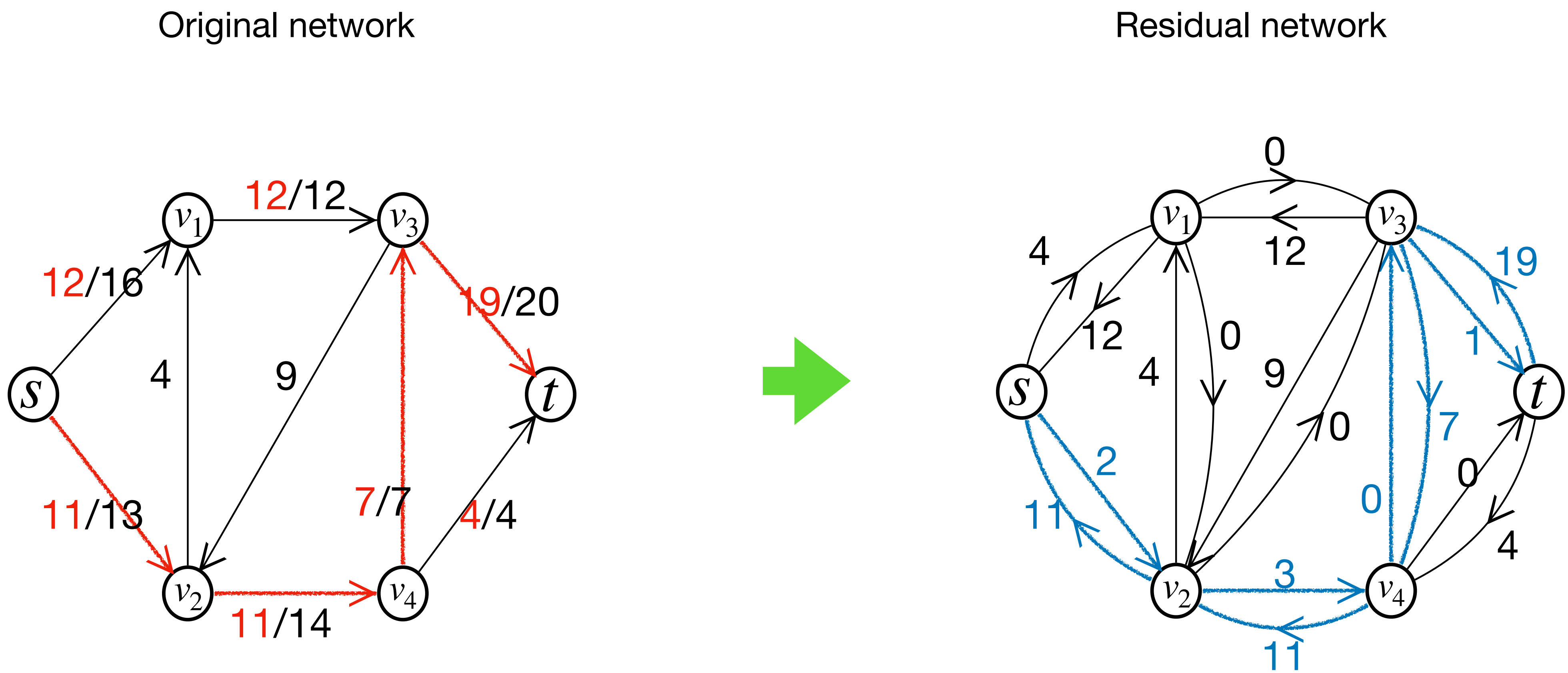
Original network



Residual network

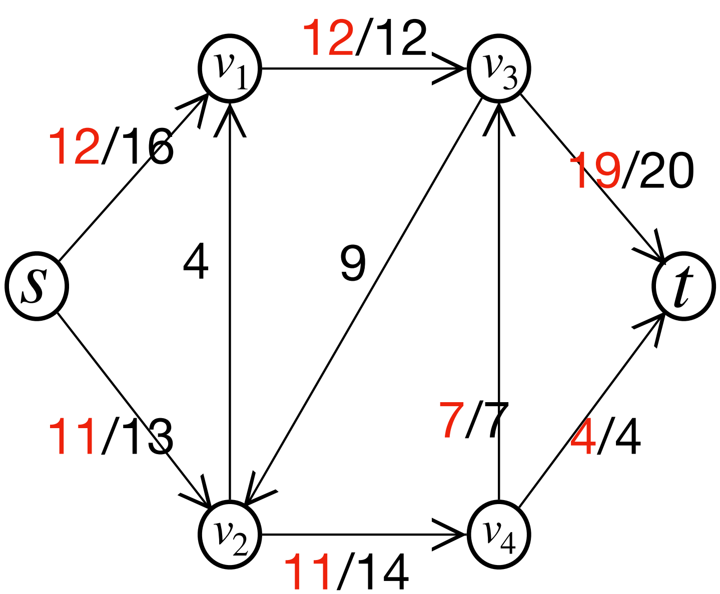


Ford-Fulkerson Method

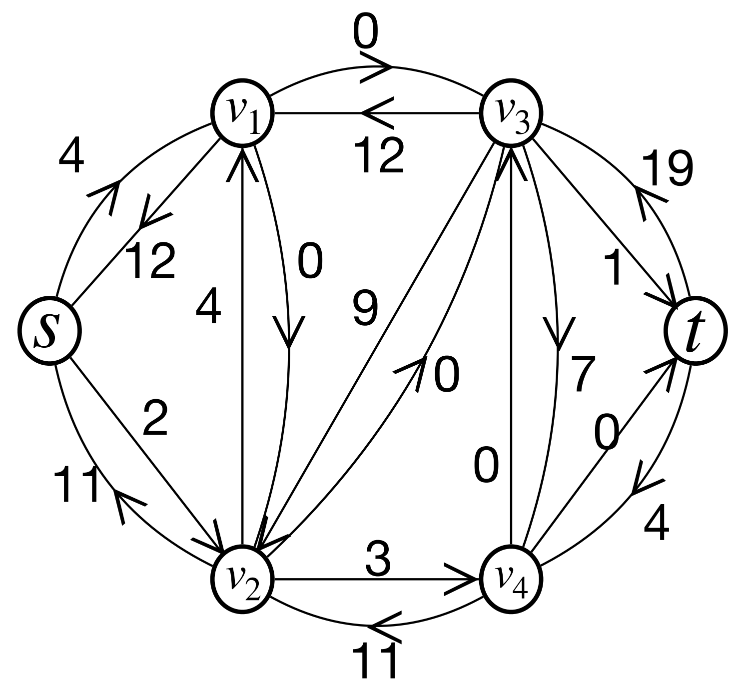


Ford-Fulkerson Method

Original network



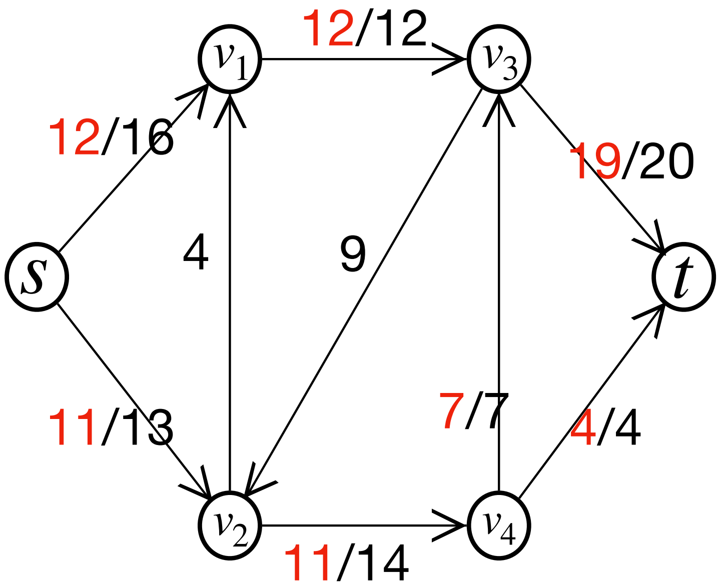
Residual network



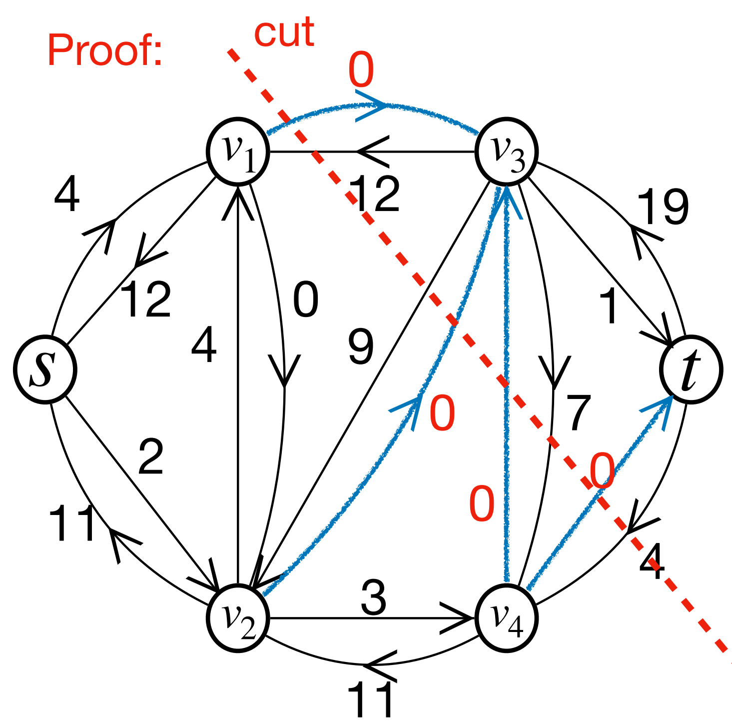
Is there still a path from s to t of residual capacity > 0?

Ford-Fulkerson Method

Original network



Residual network

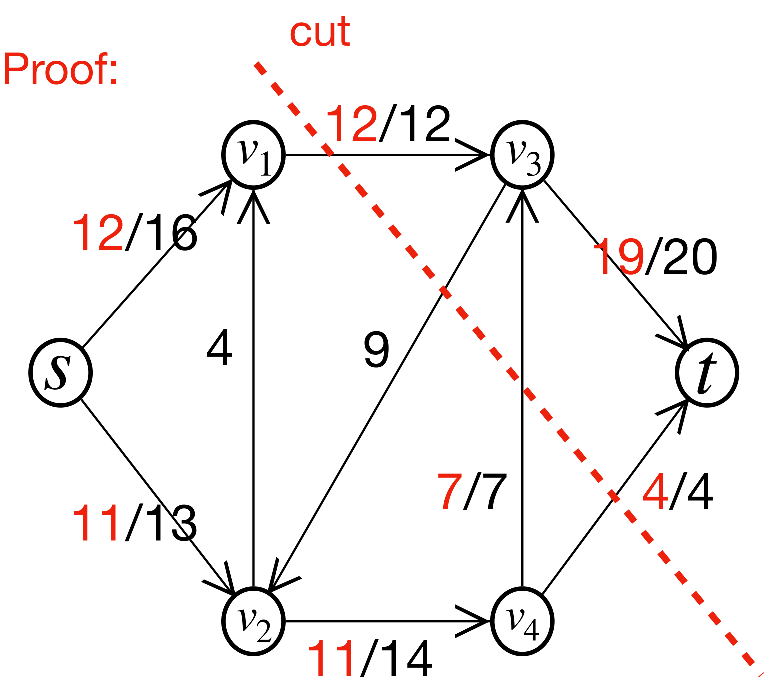


Is there still a path from s to t of residual capacity > 0?

NO

Ford-Fulkerson Method

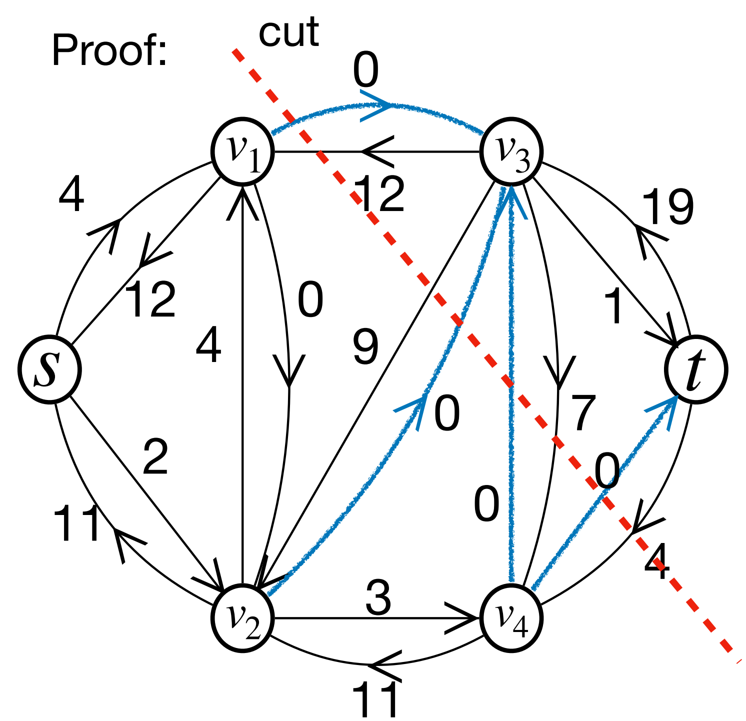
Original network



Size of flow: 12+11=19+4=23

Is it optimal? YES.

Residual network



Is there still a path from s to t of residual capacity > 0?

NO