Algorithms

Lecture 5: Amortized Analysis

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CH. 17 Amortized Analysis

for i from 1 to x

$\{$

Operations

$O(y)$

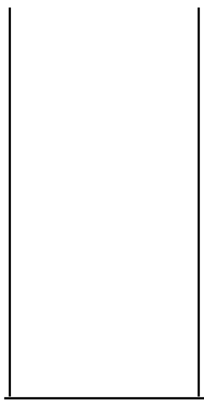
$\}$

$\Rightarrow$ $O(xy)$?

"Amortized Analysis" can sometimes help us get tighter bounds for time complexity.

Technique 1: Aggregate Analysis

Example: Stack Operations

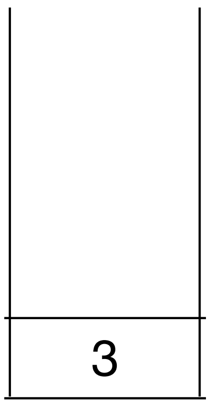


Size of Stack: $|S| = 0$

Stack: first-in-last-out (FILO)

Technique 1: Aggregate Analysis

Example: Stack Operations

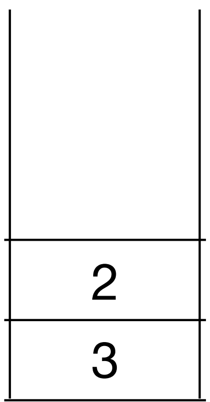
Push 

Size of Stack: $|S| = 1$

Technique 1: Aggregate Analysis

Example: Stack Operations

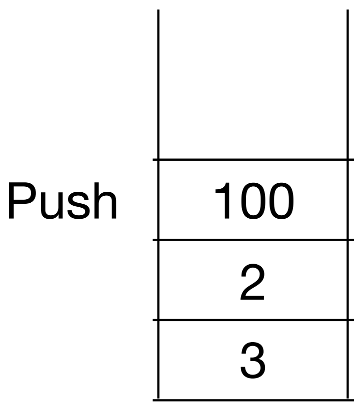
Push



Size of Stack: $|S| = 2$

Technique 1: Aggregate Analysis

Example: Stack Operations

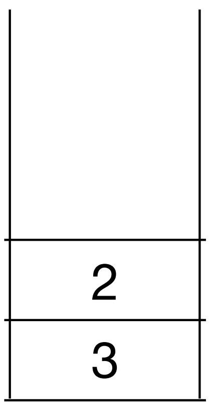


Size of Stack: $|S| = 3$

Technique 1: Aggregate Analysis

Example: Stack Operations

Pop

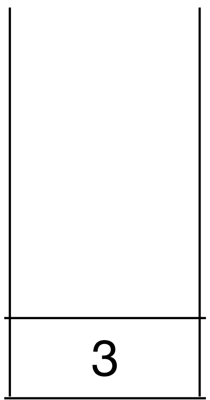


Size of Stack: $|S| = 2$

Technique 1: Aggregate Analysis

Example: Stack Operations

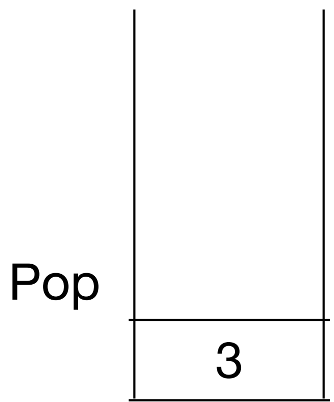
Pop



Size of Stack: $|S| = 1$

Technique 1: Aggregate Analysis

Example: Stack Operations



Size of Stack: $|S| = 1$

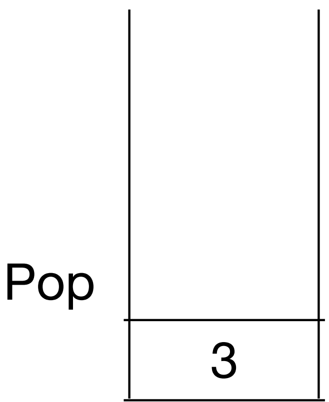
Operations:

1) PUSH: push a number into stack

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

Technique 1: Aggregate Analysis

Example: Stack Operations



Size of Stack: $|S| = 1$

Operations:

1) PUSH: push a number into stack
Cost: 1

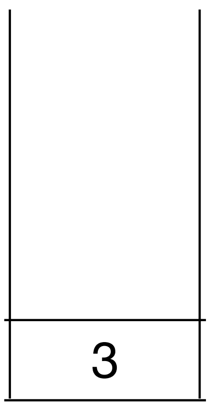
2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

Cost: $\min\{k, |S|\}$

Technique 1: Aggregate Analysis

Example: Stack Operations

Pop



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Cost: 1

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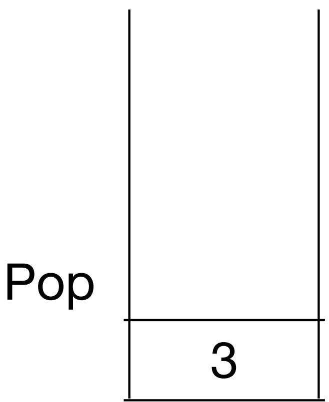
Cost: $\min\{k, |S|\}$

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Technique 1: Aggregate Analysis

Example: Stack Operations



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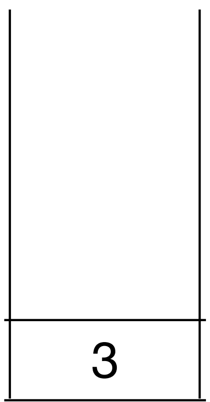
What is a tight upper bound on its total cost ?

Is it $O(n^2)$?

Technique 1: Aggregate Analysis

Example: Stack Operations

Pop



Size of Stack: $|S| = 1$

Operations:

1) PUSH: push a number into stack
   Cost: 1

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

   Cost: $\min\{k, |S|\}$

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Is it $O(n^2)$?

It is actually $O(n)$

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Observation: The stack is initially empty. To pop out a number,
we first need to push it in to the stack.

So the cost of POP operations can never be more than the cost of PUSH operations.
(No matter what k is in POP(k).)

Consider a sequence of n stack operations.
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Total cost of PUSH operations $\leq n$

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

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So the cost of POP operations can never be more than the cost of PUSH operations.
(No matter what k is in POP(k).)

Total cost of PUSH operations $\leq n$

Total cost of POP operations $\leq n$

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Observation: The stack is initially empty. To pop out a number, we first need to push it in to the stack.

So the cost of POP operations can never be more than the cost of PUSH operations.
(No matter what k is in POP(k).)

Total cost of PUSH operations $\leq n$

Total cost of POP operations $\leq n$

Total cost of n operations: $\leq 2n,$ or $O(n)$

Technique 1: Aggregate Analysis

Example: Counter Incrementation

Counter: 0, 1, 2, 3, ……

Large Binary Counter

0 0 0 0 0 0 0 0 0 0 0

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

cost: 1

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

cost: 1

cost: 2

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	
5	0 0 0 0 0 0 0 1 0 1	
6	0 0 0 0 0 0 0 1 1 0	
7	0 0 0 0 0 0 0 1 1 1	
8	0 0 0 0 0 0 1 0 0 0	
9	0 0 0 0 0 0 1 0 0 1	
10	0 0 0 0 0 0 1 0 1 0	
11	0 0 0 0 0 0 1 0 1 1	

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0 0 0 0 0 0 0 0 0 0 0

cost: 1

1 0 0 0 0 0 0 0 0 0 1

cost: 2

2 0 0 0 0 0 0 0 0 1 0

cost: 1

3 0 0 0 0 0 0 0 0 1 1

cost: 3

4 0 0 0 0 0 0 0 1 0 0

5 0 0 0 0 0 0 0 1 0 1

6 0 0 0 0 0 0 0 1 1 0

7 0 0 0 0 0 0 0 1 1 1

8 0 0 0 0 0 0 1 0 0 0

9 0 0 0 0 0 0 1 0 0 1

10 0 0 0 0 0 0 1 0 1 0

11 0 0 0 0 0 0 1 0 1 1

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	
7	0 0 0 0 0 0 0 1 1 1	
8	0 0 0 0 0 0 1 0 0 0	
9	0 0 0 0 0 0 1 0 0 1	
10	0 0 0 0 0 0 1 0 1 0	
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Cost of incrementing counter:
Number of bits that are changed.

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0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
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3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	
8	0 0 0 0 0 0 1 0 0 0	
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Cost of incrementing counter:
Number of bits that are changed.

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0	0 0 0 0 0 0 0 0 0 0	cost: 1
1	0 0 0 0 0 0 0 0 0 1	cost: 2
2	0 0 0 0 0 0 0 0 1 0	cost: 1
3	0 0 0 0 0 0 0 0 1 1	cost: 3
4	0 0 0 0 0 0 0 1 0 0	cost: 1
5	0 0 0 0 0 0 0 1 0 1	cost: 2
6	0 0 0 0 0 0 0 1 1 0	cost: 1
7	0 0 0 0 0 0 0 1 1 1	
8	0 0 0 0 0 0 1 0 0 0	
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Number of bits that are changed.

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0	0 0 0 0 0 0 0 0 0 0	cost: 1
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3	0 0 0 0 0 0 0 0 1 1	cost: 3
4	0 0 0 0 0 0 0 1 0 0	cost: 1
5	0 0 0 0 0 0 0 1 0 1	cost: 2
6	0 0 0 0 0 0 0 1 1 0	cost: 1
7	0 0 0 0 0 0 0 1 1 1	cost: 4
8	0 0 0 0 0 0 1 0 0 0	
9	0 0 0 0 0 0 1 0 0 1	
10	0 0 0 0 0 0 1 0 1 0	
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Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0	
		cost: 1
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2	0 0 0 0 0 0 0 0 1 0	
		cost: 1
3	0 0 0 0 0 0 0 0 1 1	
		cost: 3
4	0 0 0 0 0 0 0 1 0 0	
		cost: 1
5	0 0 0 0 0 0 0 1 0 1	
		cost: 2
6	0 0 0 0 0 0 0 1 1 0	
		cost: 1
7	0 0 0 0 0 0 0 1 1 1	
		cost: 4
8	0 0 0 0 0 0 1 0 0 0	
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0	0 0 0 0 0 0 0 0 0 0
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	cost: 1
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	cost: 3
4	0 0 0 0 0 0 0 1 0 0
	cost: 1
5	0 0 0 0 0 0 0 1 0 1
	cost: 2
6	0 0 0 0 0 0 0 1 1 0
	cost: 1
7	0 0 0 0 0 0 0 1 1 1
	cost: 4
8	0 0 0 0 0 0 1 0 0 0
	cost: 1
9	0 0 0 0 0 0 1 0 0 1
	cost: 2
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
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Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0	
		cost: 1
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2	0 0 0 0 0 0 0 0 1 0	
		cost: 1
3	0 0 0 0 0 0 0 0 1 1	
		cost: 3
4	0 0 0 0 0 0 0 1 0 0	
		cost: 1
5	0 0 0 0 0 0 0 1 0 1	
		cost: 2
6	0 0 0 0 0 0 0 1 1 0	
		cost: 1
7	0 0 0 0 0 0 0 1 1 1	
		cost: 4
8	0 0 0 0 0 0 1 0 0 0	
		cost: 1
9	0 0 0 0 0 0 1 0 0 1	
		cost: 2
10	0 0 0 0 0 0 1 0 1 0	
		cost: 1
11	0 0 0 0 0 0 1 0 1 1	

What is the total cost of n increments?

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0 — 0 0 0 0 0 0 0 0 0 0

cost: 1

1 — 0 0 0 0 0 0 0 0 0 1

cost: 2

2 — 0 0 0 0 0 0 0 0 1 0

cost: 1

3 — 0 0 0 0 0 0 0 0 1 1

cost: 3

4 — 0 0 0 0 0 0 0 1 0 0

cost: 1

5 — 0 0 0 0 0 0 0 1 0 1

cost: 2

6 — 0 0 0 0 0 0 0 1 1 0

cost: 1

7 — 0 0 0 0 0 0 0 1 1 1

cost: 4

8 — 0 0 0 0 0 0 1 0 0 0

cost: 1

9 — 0 0 0 0 0 0 1 0 0 1

cost: 2

10 — 0 0 0 0 0 0 1 0 1 0

cost: 1

11 — 0 0 0 0 0 0 1 0 1 1

Cost of incrementing counter:
Number of bits that are changed.

What is the total cost of n increments?

$O(n^2)$?

$O(n \log n)$?

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0	
		cost: 1
1	0 0 0 0 0 0 0 0 0 1	
		cost: 2
2	0 0 0 0 0 0 0 0 1 0	
		cost: 1
3	0 0 0 0 0 0 0 0 1 1	
		cost: 3
4	0 0 0 0 0 0 0 1 0 0	
		cost: 1
5	0 0 0 0 0 0 0 1 0 1	
		cost: 2
6	0 0 0 0 0 0 0 1 1 0	
		cost: 1
7	0 0 0 0 0 0 0 1 1 1	
		cost: 4
8	0 0 0 0 0 0 1 0 0 0	
		cost: 1
9	0 0 0 0 0 0 1 0 0 1	
		cost: 2
10	0 0 0 0 0 0 1 0 1 0	
		cost: 1
11	0 0 0 0 0 0 1 0 1 1	

What is the total cost of n increments?

$O(n^2)$?

$O(n \log n)$?

It is actually: $O(n)$

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

Cost of the last bit: n

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

Cost of the last bit: n

Cost of 2nd last bit: $\leq \frac{n}{2}$

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

Cost of the last bit: n

Cost of 2nd last bit: $\leq \frac{n}{2}$

Cost of 3rd last bit: $\leq \frac{n}{4}$

Cost of incrementing counter:
Number of bits that are changed.

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

Cost of incrementing counter:
Number of bits that are changed.

Cost of the last bit: n

Cost of 2nd last bit: $\leq \frac{n}{2}$

Cost of 3rd last bit: $\leq \frac{n}{4}$

Cost of 4th last bit: $\leq \frac{n}{8}$

......

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0
1	0 0 0 0 0 0 0 0 0 1
2	0 0 0 0 0 0 0 0 1 0
3	0 0 0 0 0 0 0 0 1 1
4	0 0 0 0 0 0 0 1 0 0
5	0 0 0 0 0 0 0 1 0 1
6	0 0 0 0 0 0 0 1 1 0
7	0 0 0 0 0 0 0 1 1 1
8	0 0 0 0 0 0 1 0 0 0
9	0 0 0 0 0 0 1 0 0 1
10	0 0 0 0 0 0 1 0 1 0
11	0 0 0 0 0 0 1 0 1 1

Cost of incrementing counter:
Number of bits that are changed.

Cost of the last bit: n

Cost of 2nd last bit: $\leq \frac{n}{2}$

Cost of 3rd last bit: $\leq \frac{n}{4}$

Cost of 4th last bit: $\leq \frac{n}{8}$

......

Total cost $\leq n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \frac{n}{16} \cdots \leq 2n$

O(n)

Technique 2: Accounting Method

Consider n operations.

For $i = 1, 2, \cdots, n$,

let C_i be the real cost of the i-th operation,

let $\hat{C}_i$ be the amortized cost of the i-th operation.

such that

$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Technique 2: Accounting Method

Consider n operations.
For $i = 1, 2, \cdots, n$,
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let $\hat{C}_i$ be the amortized cost of the i-th operation.

such that

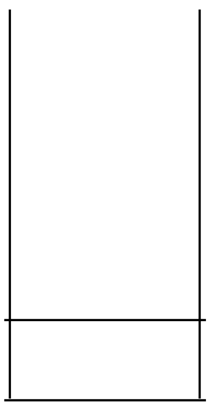
$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

If we get an upper bound for the amortized cost $\sum_{i=1}^{n} \hat{C}_i$,

we also get an upper bound for the real cost $\sum_{i=1}^{n} C_i$

Technique 2: Accounting Method

Example: Stack Operations



Size of Stack: $|S| = 0$

Operations:

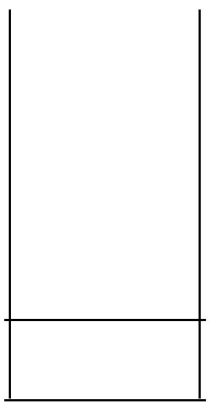
1) PUSH: push a number into stack
   Cost: 1

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.
   Cost: $\min\{k, |S|\}$

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Technique 2: Accounting Method

Example: Stack Operations



Size of Stack: $|S| = 0$

Operations:

1) PUSH: push a number into stack

Cost: 1 Amortized Cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

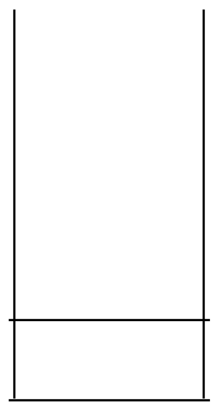
Cost: $\min\{k, |S|\}$

Amortized Cost: 0

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Technique 2: Accounting Method

Example: Stack Operations



Size of Stack: $|S| = 0$

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Operations:

1) PUSH: push a number into stack

Cost: 1 Amortized Cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

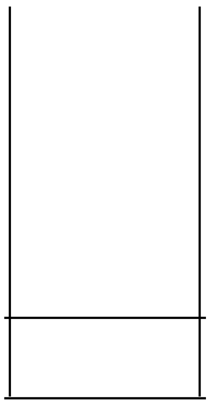
Cost: $\min\{k, |S|\}$

Amortized Cost: 0

Is it true that $\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$?

Technique 2: Accounting Method

Example: Stack Operations



Size of Stack: $|S| = 0$

Total real cost: 0

Total amortized cost: 0

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Operations:

1) PUSH: push a number into stack

Cost: 1 Amortized Cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

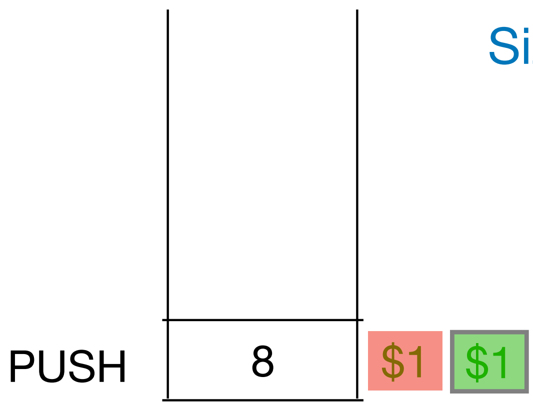
Cost: $\min\{k, |S|\}$

Amortized Cost: 0

Is it true that $\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$?

Technique 2: Accounting Method

Example: Stack Operations



Size of Stack: $|S| = 0$

Total real cost: 1
Total amortized cost: 2

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Operations:

1) PUSH: push a number into stack

Cost: 1 Amortized Cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

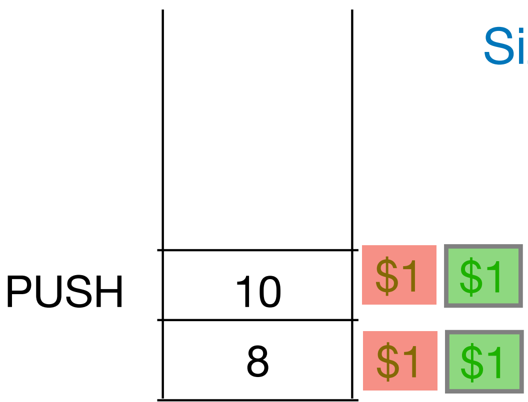
Cost: $\min\{k, |S|\}$

Amortized Cost: 0

Is it true that $\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$?

Technique 2: Accounting Method

Example: Stack Operations



Size of Stack: $|S| = 0$

Total real cost: 2

Total amortized cost: 4

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Operations:

1) PUSH: push a number into stack

Cost: 1 Amortized Cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

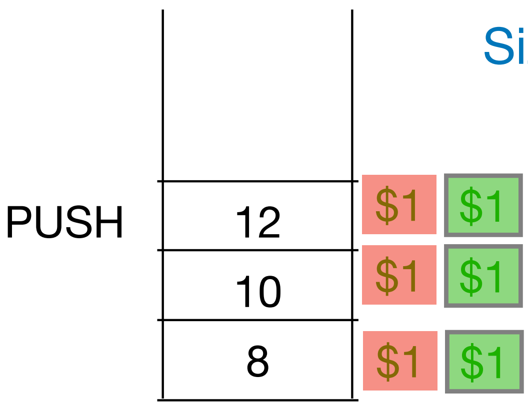
Cost: $\min\{k, |S|\}$

Amortized Cost: 0

Is it true that $\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$?

Technique 2: Accounting Method

Example: Stack Operations



Size of Stack: $|S| = 0$

Total real cost: 3

Total amortized cost: 6

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Operations:

1) PUSH: push a number into stack

Cost: 1 Amortized Cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

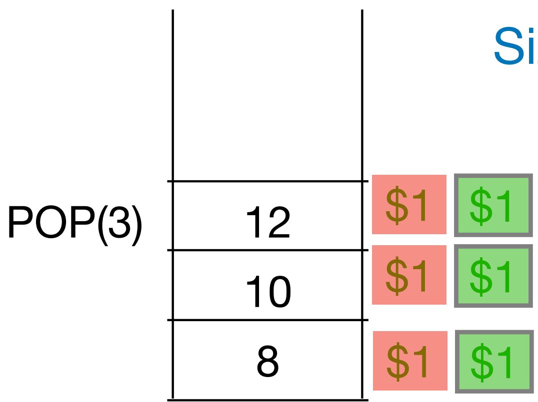
Cost: $\min\{k, |S|\}$

Amortized Cost: 0

Is it true that $\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$?

Technique 2: Accounting Method

Example: Stack Operations



Size of Stack: $|S| = 0$

How to cover the real cost?
Use the over-paid dollars.

Total real cost: 6
Total amortized cost: 6

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Operations:

1) PUSH: push a number into stack

Cost: 1 Amortized Cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

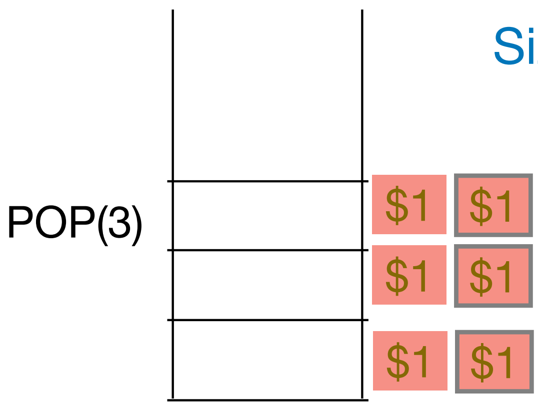
Cost: $\min\{k, |S|\}$

Amortized Cost: 0

Is it true that $\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$?

Technique 2: Accounting Method

Example: Stack Operations



Size of Stack: $|S| = 0$

How to cover the real cost?
Use the over-paid dollars.

Idea: Pay In Advance.

Total real cost: 6
Total amortized cost: 6

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ?

Operations:

1) PUSH: push a number into stack

Cost: 1 Amortized Cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

Cost: $\min\{k, |S|\}$

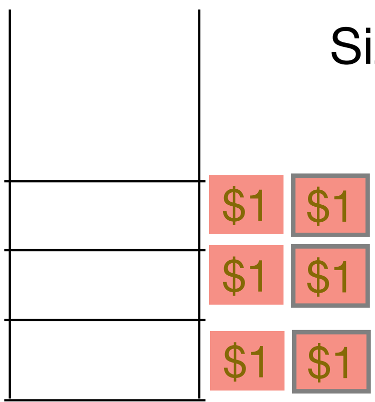
Amortized Cost: 0

Is it true that $\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$?

Technique 2: Accounting Method

Example: Stack Operations

POP(3)



Size of Stack: $|S| = 0$

How to cover the real cost?
Use the over-paid dollars.

Idea: Pay In Advance.

Total real cost: 6
Total amortized cost: 6

Consider a sequence of n stack operations.
What is a tight upper bound on its total cost ? $O(n)$

Operations:

1) PUSH: push a number into stack
Cost: 1 Amortized Cost: 2

2) POP(k): pop out the top k numbers from stack. If the stack has fewer than k numbers, we pop out all numbers in stack.

Cost: $\min\{k, |S|\}$

Amortized Cost: 0

2n >= $\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$

Technique 2: Accounting Method

Large Binary Counter

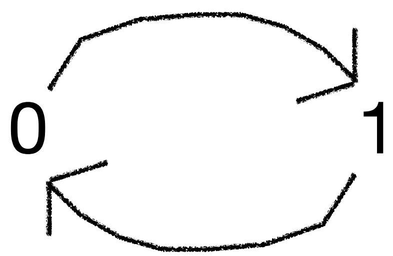
0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
11	0 0 0 0 0 0 1 0 1 1	cost: 1

Cost of incrementing counter:
Number of bits that are changed.

What is the total cost of n increments?

Technique 2: Accounting Method

Lifecycle of a bit:



Large Binary Counter

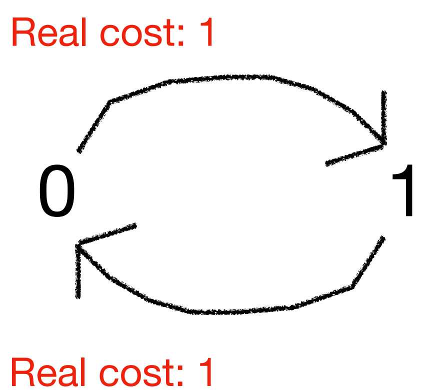
0	0 0 0 0 0 0 0 0 0 0	cost: 1
1	0 0 0 0 0 0 0 0 0 1	cost: 2
2	0 0 0 0 0 0 0 0 1 0	cost: 1
3	0 0 0 0 0 0 0 0 1 1	cost: 3
4	0 0 0 0 0 0 0 1 0 0	cost: 1
5	0 0 0 0 0 0 0 1 0 1	cost: 2
6	0 0 0 0 0 0 0 1 1 0	cost: 1
7	0 0 0 0 0 0 0 1 1 1	cost: 4
8	0 0 0 0 0 0 1 0 0 0	cost: 1
9	0 0 0 0 0 0 1 0 0 1	cost: 2
10	0 0 0 0 0 0 1 0 1 0	cost: 1
11	0 0 0 0 0 0 1 0 1 1	

Cost of incrementing counter:
Number of bits that are changed.

What is the total cost of n increments?

Technique 2: Accounting Method

Lifecycle of a bit:



Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
11	0 0 0 0 0 0 1 0 1 1	cost: 1

Cost of incrementing counter:
Number of bits that are changed.

What is the total cost of n increments?

Technique 2: Accounting Method

Lifecycle of a bit:

Real cost: 1 Amortized cost: 2

Real cost: 1 Amortized cost: 0

Since the amortized cost "pre-pays" the total cost of every lifecycle,

$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Large Binary Counter

0	0 0 0 0 0 0 0 0 0 0	
		cost: 1
1	0 0 0 0 0 0 0 0 0 1	
		cost: 2
2	0 0 0 0 0 0 0 0 1 0	
		cost: 1
3	0 0 0 0 0 0 0 0 1 1	
		cost: 3
4	0 0 0 0 0 0 0 1 0 0	
		cost: 1
5	0 0 0 0 0 0 0 1 0 1	
		cost: 2
6	0 0 0 0 0 0 0 1 1 0	
		cost: 1
7	0 0 0 0 0 0 0 1 1 1	
		cost: 4
8	0 0 0 0 0 0 1 0 0 0	
		cost: 1
9	0 0 0 0 0 0 1 0 0 1	
		cost: 2
10	0 0 0 0 0 0 1 0 1 0	
		cost: 1
11	0 0 0 0 0 0 1 0 1 1	

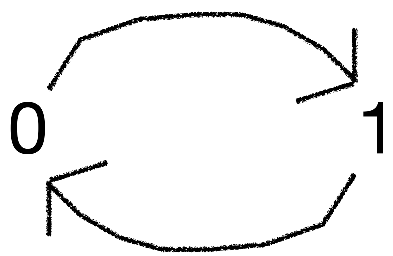
Cost of incrementing counter:
Number of bits that are changed.

What is the total cost of n increments?

Technique 2: Accounting Method

Lifecycle of a bit:

Real cost: 1 Amortized cost: 2



Real cost: 1 Amortized cost: 0

Since the amortized cost "pre-pays" the total cost of every lifecycle,

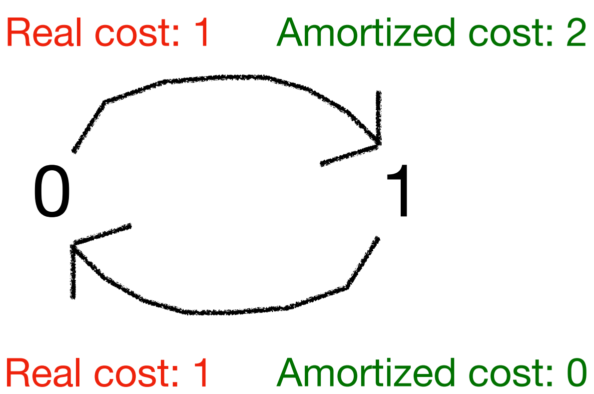
$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
11	0 0 0 0 0 0 1 0 1 1	cost: 1

Technique 2: Accounting Method

Lifecycle of a bit:



Since the amortized cost "pre-pays" the total cost of every lifecycle,

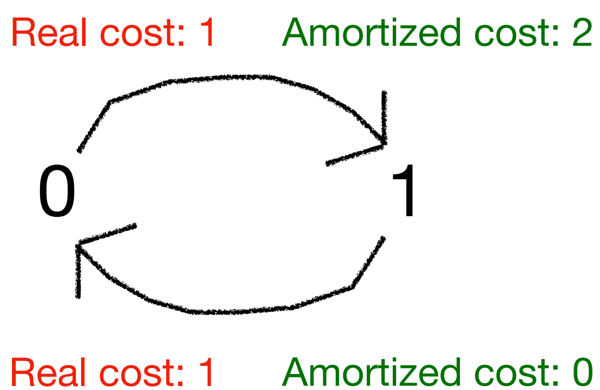
$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	cost: 1
1	0 0 0 0 0 0 0 0 0 1	cost: 2
2	0 0 0 0 0 0 0 0 1 0	cost: 1
3	0 0 0 0 0 0 0 0 1 1	cost: 3
4	0 0 0 0 0 0 0 1 0 0	cost: 1
5	0 0 0 0 0 0 0 1 0 1	cost: 2
6	0 0 0 0 0 0 0 1 1 0	cost: 1
7	0 0 0 0 0 0 0 1 1 1	cost: 4
8	0 0 0 0 0 0 1 0 0 0	cost: 1
9	0 0 0 0 0 0 1 0 0 1	cost: 2
10	0 0 0 0 0 0 1 0 1 0	cost: 1
11	0 0 0 0 0 0 1 0 1 1	

Technique 2: Accounting Method

Lifecycle of a bit:



Since the amortized cost “pre-pays” the total cost of every lifecycle,

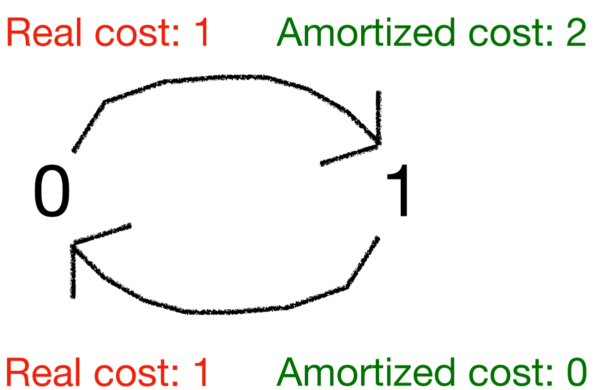
$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
		cost: 1
1	0 0 0 0 0 0 0 0 0 1	
		cost: 2
2	0 0 0 0 0 0 0 0 1 0	
		cost: 1
3	0 0 0 0 0 0 0 0 1 1	
		cost: 3
4	0 0 0 0 0 0 0 1 0 0	
		cost: 1
5	0 0 0 0 0 0 0 1 0 1	
		cost: 2
6	0 0 0 0 0 0 0 1 1 0	
		cost: 1
7	0 0 0 0 0 0 0 1 1 1	
		cost: 4
8	0 0 0 0 0 0 1 0 0 0	
		cost: 1
9	0 0 0 0 0 0 1 0 0 1	
		cost: 2
10	0 0 0 0 0 0 1 0 1 0	
		cost: 1
11	0 0 0 0 0 0 1 0 1 1	

Technique 2: Accounting Method

Lifecycle of a bit:



Since the amortized cost "pre-pays" the total cost of every lifecycle,

$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
		cost: 1
1	0 0 0 0 0 0 0 0 0 1	
		cost: 2
2	0 0 0 0 0 0 0 0 1 0	
		cost: 1
3	0 0 0 0 0 0 0 0 1 1	
		cost: 3
4	0 0 0 0 0 0 0 1 0 0	
		cost: 1
5	0 0 0 0 0 0 0 1 0 1	
		cost: 2
6	0 0 0 0 0 0 0 1 1 0	
		cost: 1
7	0 0 0 0 0 0 0 1 1 1	
		cost: 4
8	0 0 0 0 0 0 1 0 0 0	
		cost: 1
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		cost: 2
10	0 0 0 0 0 0 1 0 1 0	
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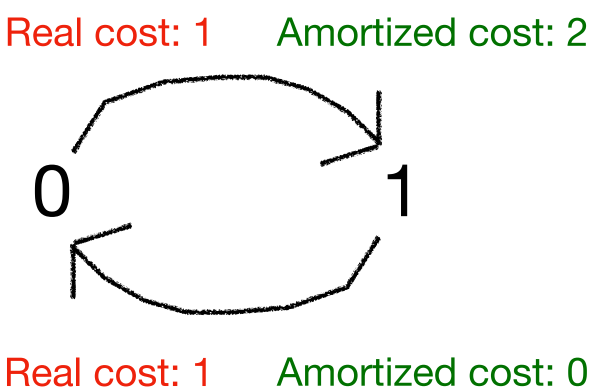
Technique 2: Accounting Method

Lifecycle of a bit:



Since the amortized cost "pre-pays" the total cost of every lifecycle,

$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
11	0 0 0 0 0 0 1 0 1 1	cost: 1

Technique 2: Accounting Method

Lifecycle of a bit:

Real cost: 1 Amortized cost: 2

Real cost: 1 Amortized cost: 0

Since the amortized cost “pre-pays” the total cost of every lifecycle,

$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

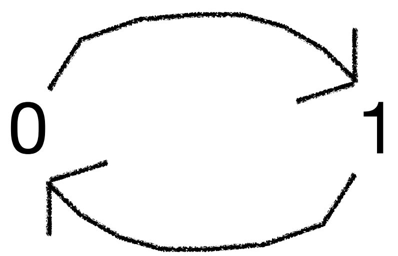
Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
		cost: 1
1	0 0 0 0 0 0 0 0 0 1	
		cost: 2
2	0 0 0 0 0 0 0 0 1 0	
		cost: 1
3	0 0 0 0 0 0 0 0 1 1	
		cost: 3
4	0 0 0 0 0 0 0 1 0 0	
		cost: 1
5	0 0 0 0 0 0 0 1 0 1	
		cost: 2
6	0 0 0 0 0 0 0 1 1 0	
		cost: 1
7	0 0 0 0 0 0 0 1 1 1	
		cost: 4
8	0 0 0 0 0 0 1 0 0 0	
		cost: 1
9	0 0 0 0 0 0 1 0 0 1	
		cost: 2
10	0 0 0 0 0 0 1 0 1 0	
		cost: 1
11	0 0 0 0 0 0 1 0 1 1	

Technique 2: Accounting Method

Lifecycle of a bit:

Real cost: 1 Amortized cost: 2



Real cost: 1 Amortized cost: 0

Since the amortized cost “pre-pays” the total cost of every lifecycle,

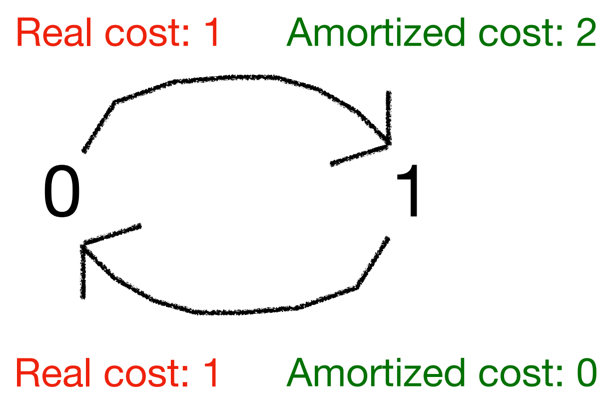
$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
		cost: 1
1	0 0 0 0 0 0 0 0 0 1	
		cost: 2
2	0 0 0 0 0 0 0 0 1 0	
		cost: 1
3	0 0 0 0 0 0 0 0 1 1	
		cost: 3
4	0 0 0 0 0 0 0 1 0 0	
		cost: 1
5	0 0 0 0 0 0 0 1 0 1	
		cost: 2
6	0 0 0 0 0 0 0 1 1 0	
		cost: 1
7	0 0 0 0 0 0 0 1 1 1	
		cost: 4
8	0 0 0 0 0 0 1 0 0 0	
		cost: 1
9	0 0 0 0 0 0 1 0 0 1	
		cost: 2
10	0 0 0 0 0 0 1 0 1 0	
		cost: 1
11	0 0 0 0 0 0 1 0 1 1	

Technique 2: Accounting Method

Lifecycle of a bit:



Since the amortized cost “pre-pays” the total cost of every lifecycle,

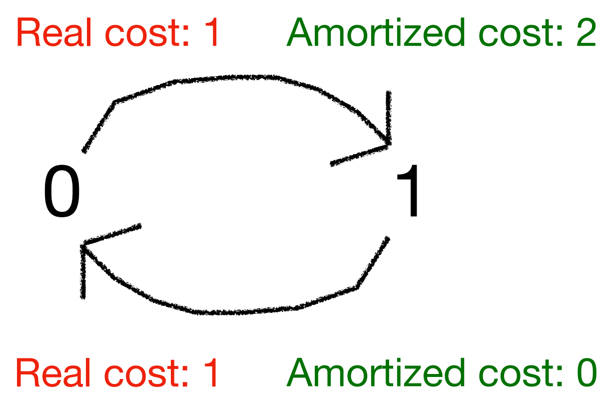
$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
11	0 0 0 0 0 0 1 0 1 1	cost: 1

Technique 2: Accounting Method

Lifecycle of a bit:



Since the amortized cost "pre-pays" the total cost of every lifecycle,

$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

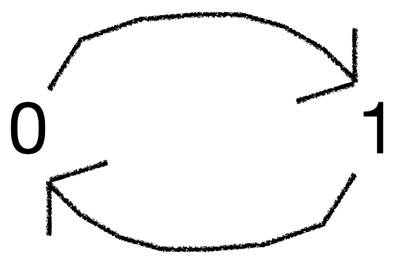
Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
11	0 0 0 0 0 0 1 0 1 1	cost: 1

Technique 2: Accounting Method

Lifecycle of a bit:

Real cost: 1 Amortized cost: 2



Real cost: 1 Amortized cost: 0

Since the amortized cost "pre-pays" the total cost of every lifecycle,

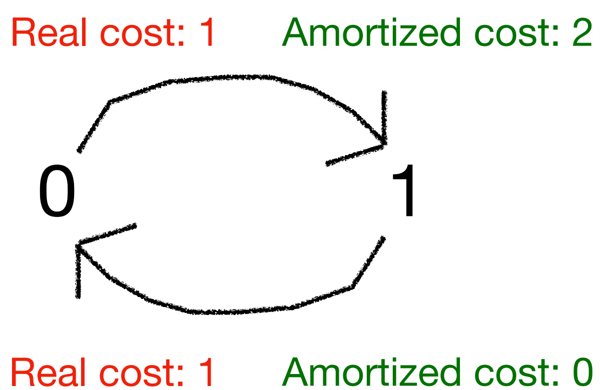
$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
11	0 0 0 0 0 0 1 0 1 1	cost: 1

Technique 2: Accounting Method

Lifecycle of a bit:



Since the amortized cost “pre-pays” the total cost of every lifecycle,

$$\sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
11	0 0 0 0 0 0 1 0 1 1	cost: 1

General pattern of change:

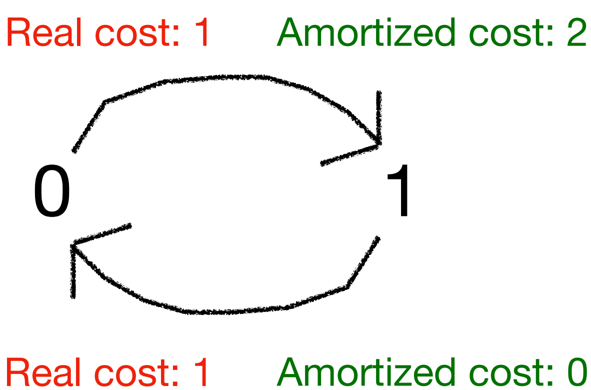
x x x x x 0 1 1 1 1

$\downarrow$

x x x x x 1 0 0 0 0

Technique 2: Accounting Method

Lifecycle of a bit:



Since the amortized cost “pre-pays” the total cost of every lifecycle,

$$2n >= \sum_{i=1}^{n} \hat{C}_i \geq \sum_{i=1}^{n} C_i$$

Amortized cost: 2

0	0 0 0 0 0 0 0 0 0 0	
1	0 0 0 0 0 0 0 0 0 1	cost: 1
2	0 0 0 0 0 0 0 0 1 0	cost: 2
3	0 0 0 0 0 0 0 0 1 1	cost: 1
4	0 0 0 0 0 0 0 1 0 0	cost: 3
5	0 0 0 0 0 0 0 1 0 1	cost: 1
6	0 0 0 0 0 0 0 1 1 0	cost: 2
7	0 0 0 0 0 0 0 1 1 1	cost: 1
8	0 0 0 0 0 0 1 0 0 0	cost: 4
9	0 0 0 0 0 0 1 0 0 1	cost: 1
10	0 0 0 0 0 0 1 0 1 0	cost: 2
11	0 0 0 0 0 0 1 0 1 1	cost: 1

total cost: $O(n)$