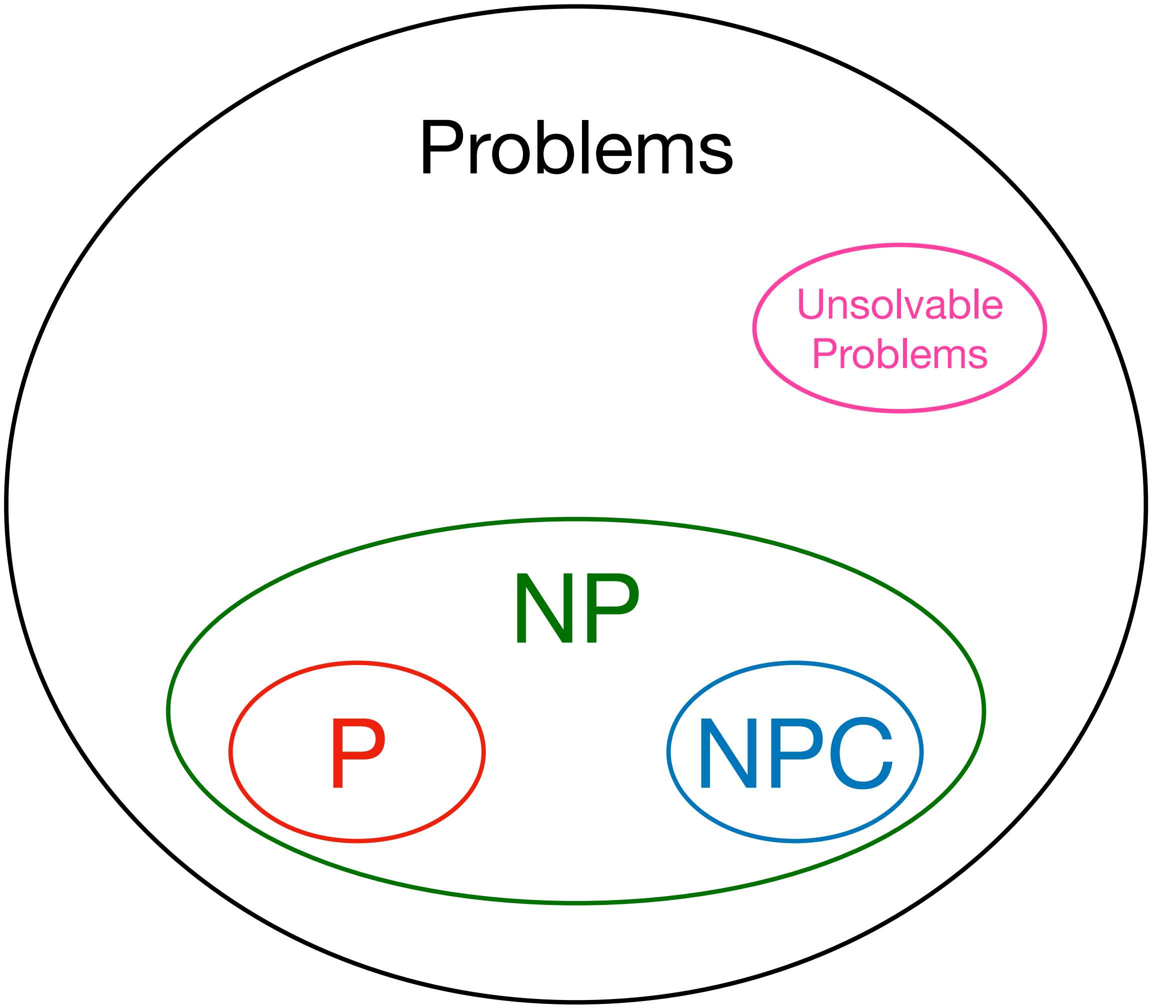
Algorithms

Lecture 15: NP Completeness (Part 1)

Anxiao (Andrew) Jiang

CH 34. NP Completeness



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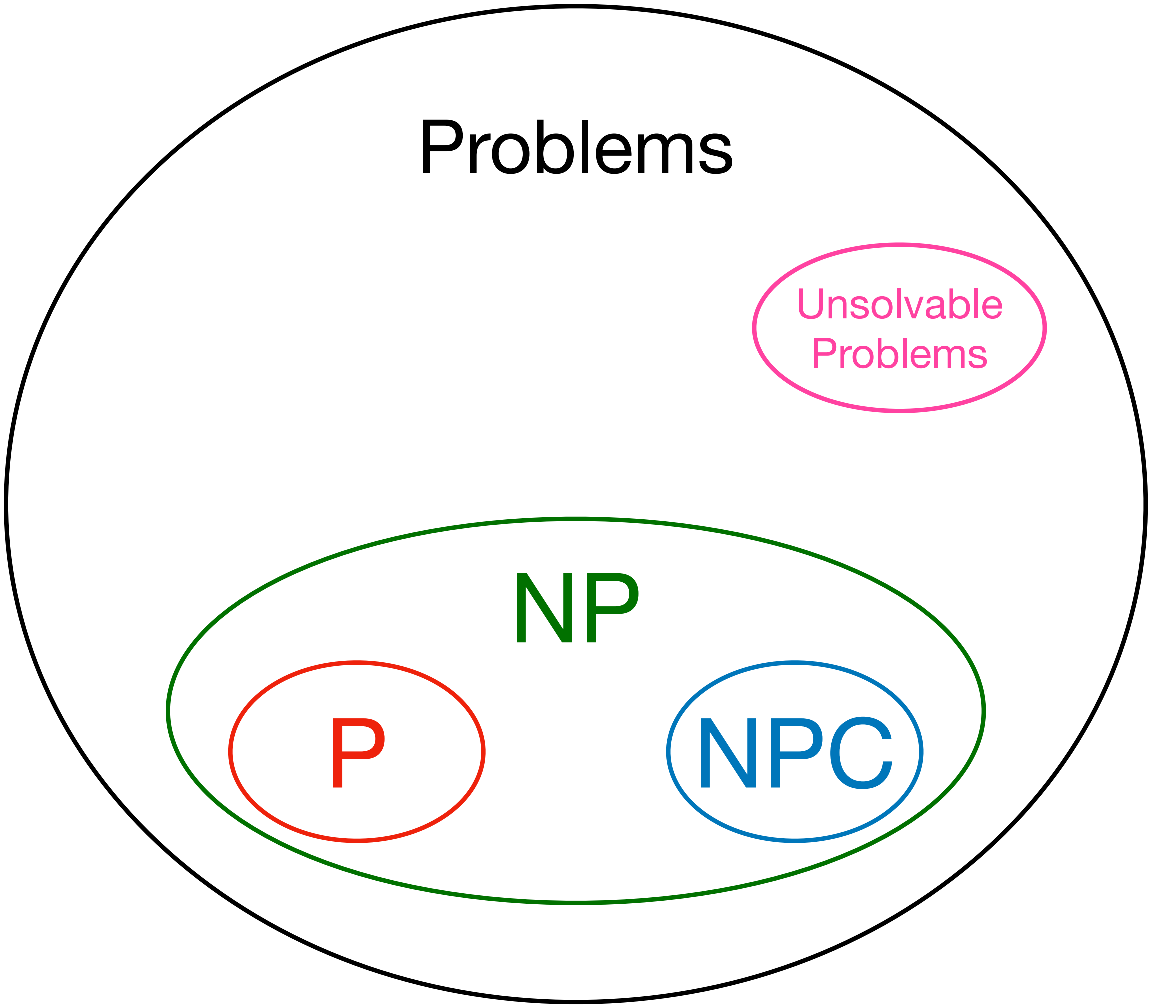

Example of unsolvable problems:

Turing's Halting Problem

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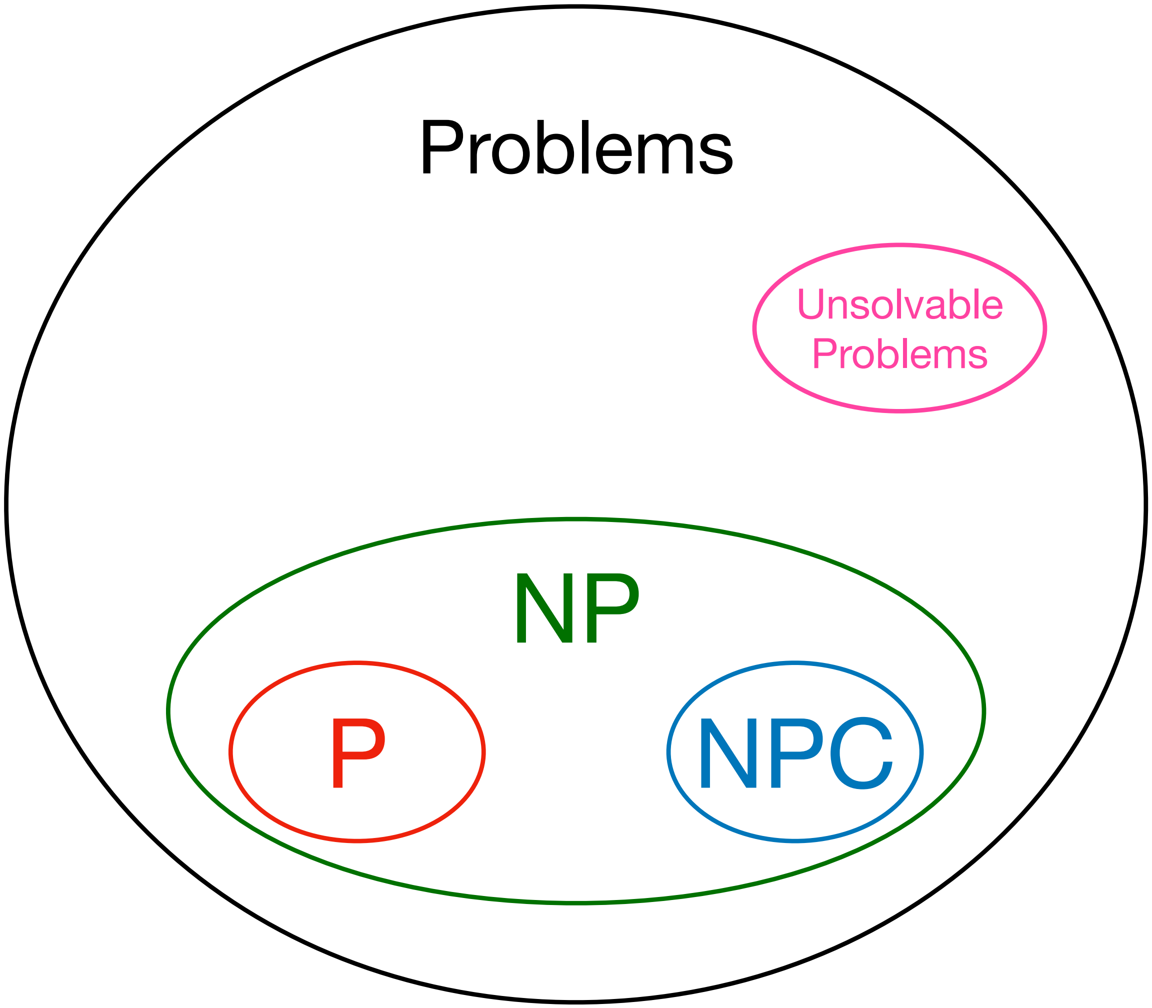

Example of unsolvable problems:

Turing’s Halting Problem

Among solvable problems:

Polynomial-time
v.s.
Super-polynomial time
(Often exponential time)

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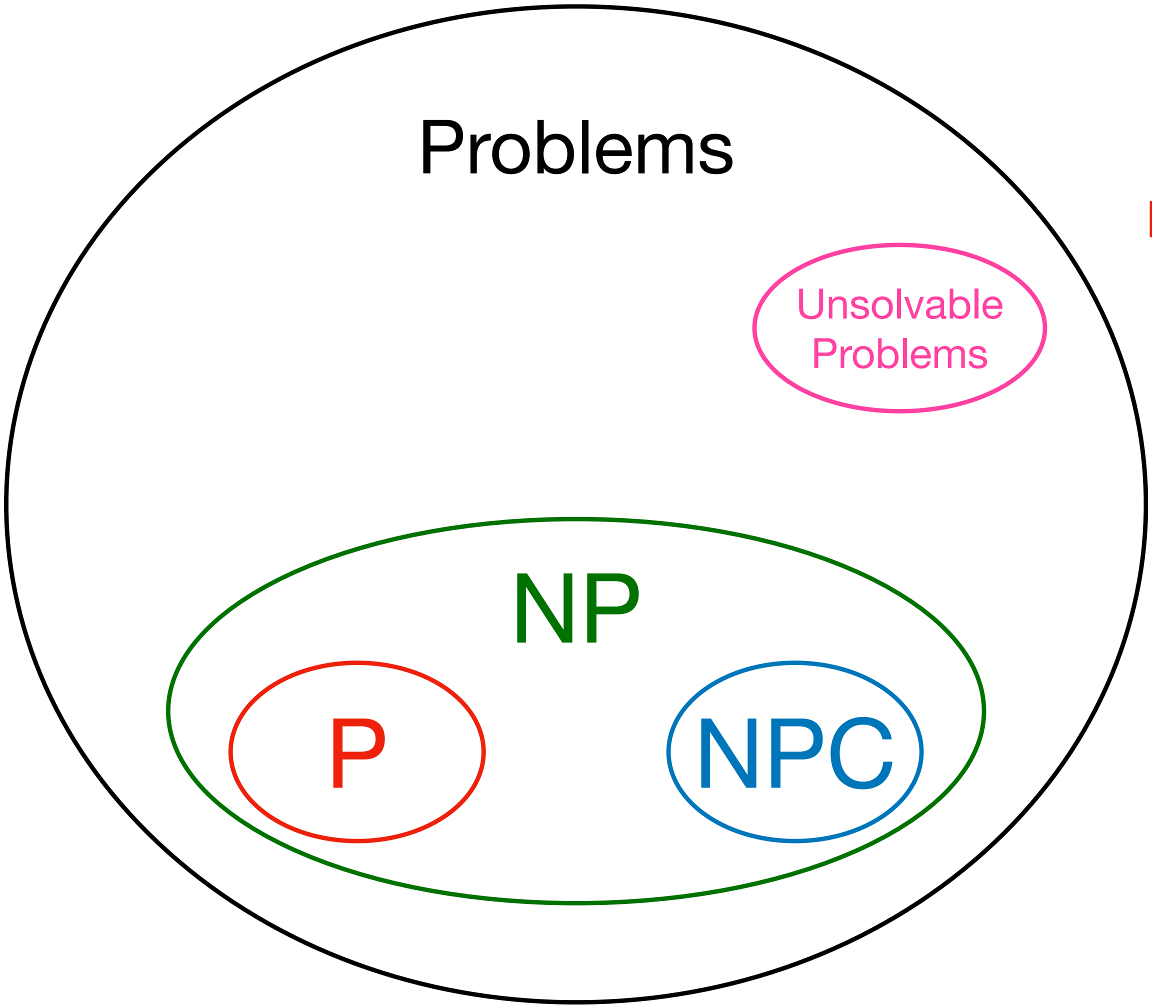

Polynomial-time algorithm

Problem's input size: n bits

Polynomial time: $O(n)$, $O(n^2)$, $O(n^c)$

Polynomial-time solvable

P: the set of all problems that can be solved in polynomial time.

Instance of a problem

Worst-case time complexity

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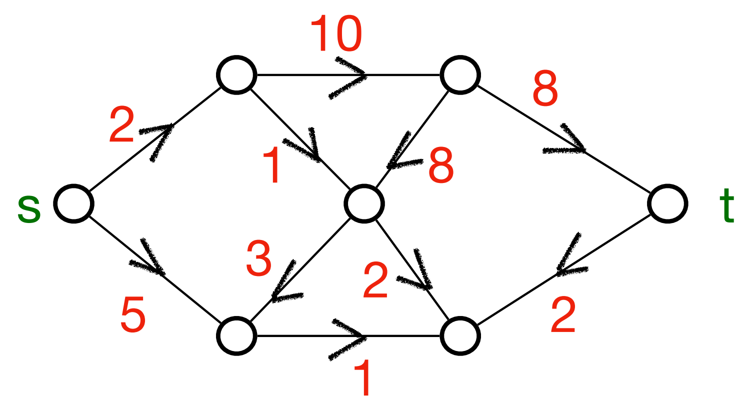
Shortest-Path Problem

Input: A directed graph G=(V,E),
where every edge $e \in E$
has a weight $w(e) > 0.$
Let $s, t \in V$ be two nodes.

Output: A shortest path from s to t.

An instance:



Shortest-Path Problem can be solved in polynomial time. (We have learned it.)

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Simple path: a path without cycles.

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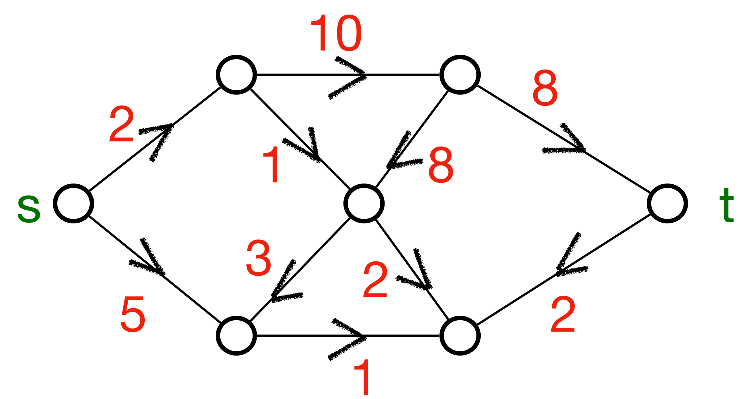

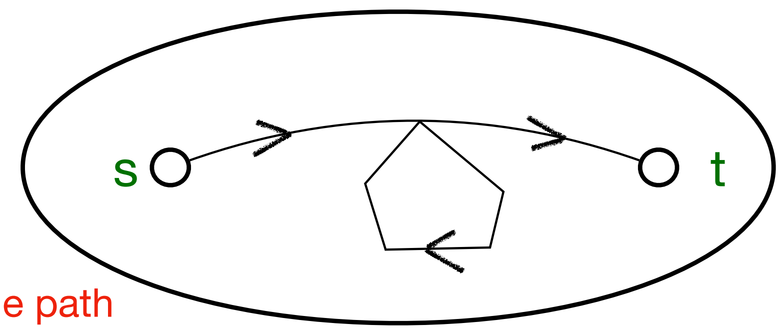

Not simple path

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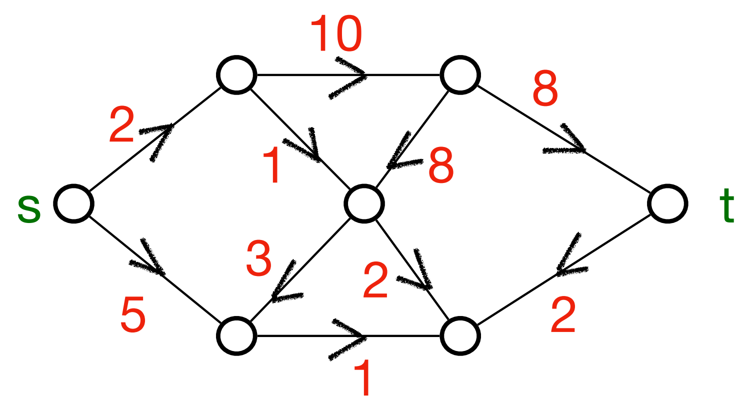

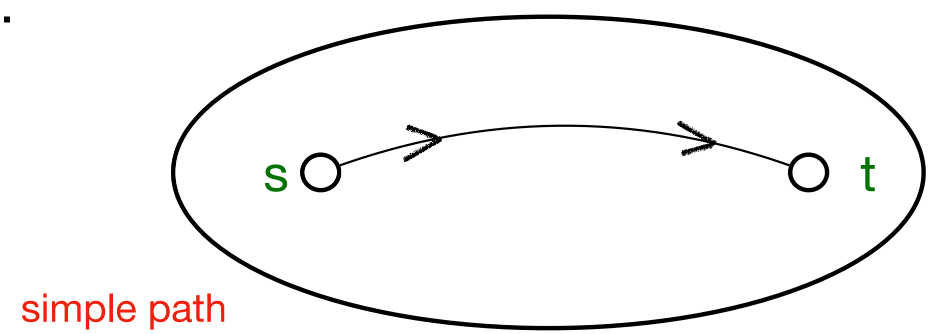

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Longest
~~Shortest~~-Path Problem

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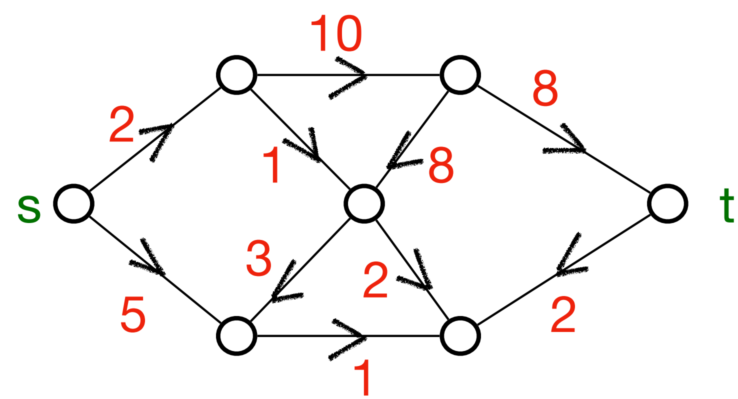

Output: A ~~shortest~~ simple path from s to t.
longest

Can the Longest Path Problem be solved in polynomial time?

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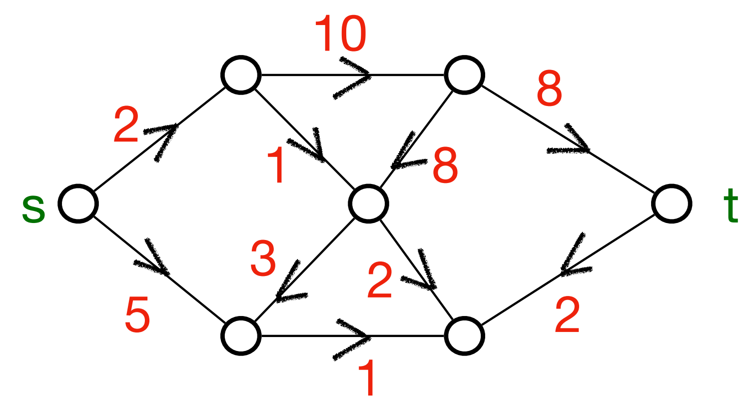


Output: A ~~shortest~~ simple path from s to t.

longest

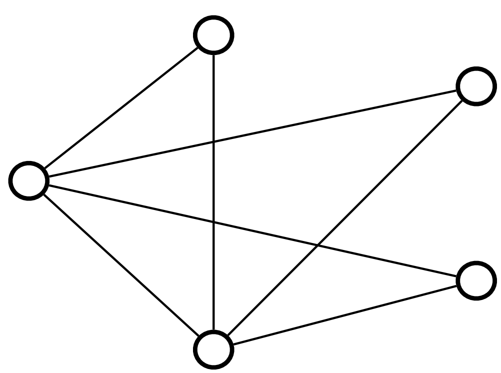
Can the Longest Path Problem be solved in polynomial time?

No one knows. But if you do ……

CH 34. NP Completeness

Euler Tour:

Given a graph G=(V,E), an Euler Tour is a cycle that passes every edge of G exactly once.



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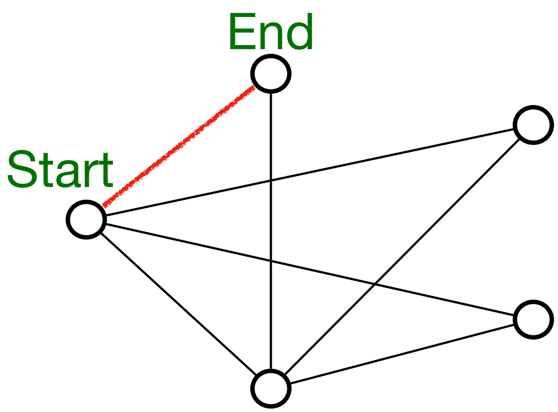

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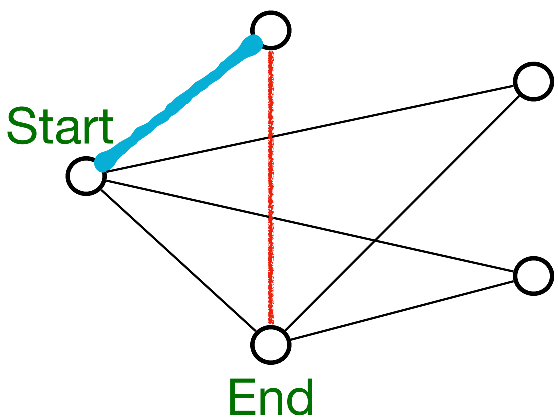

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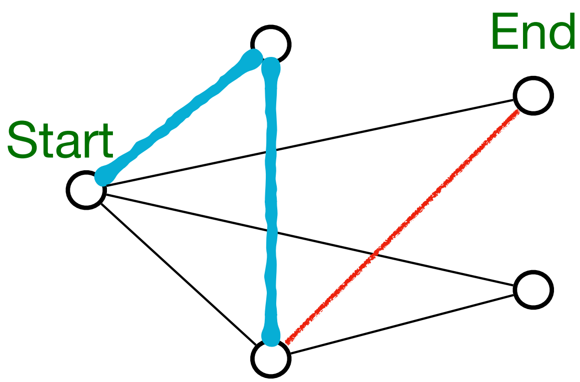

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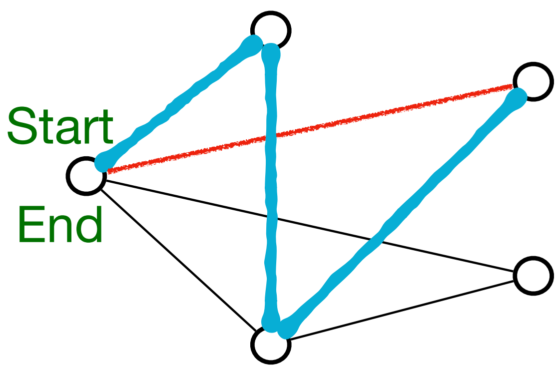

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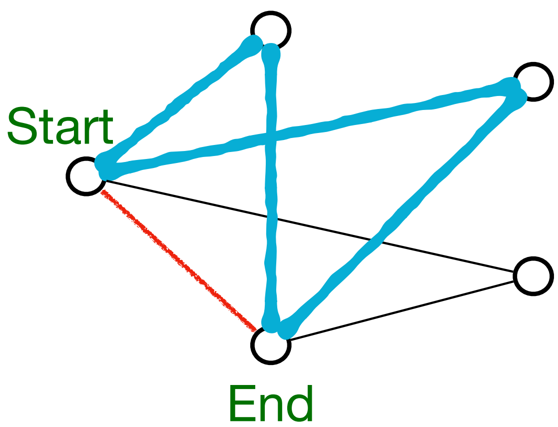

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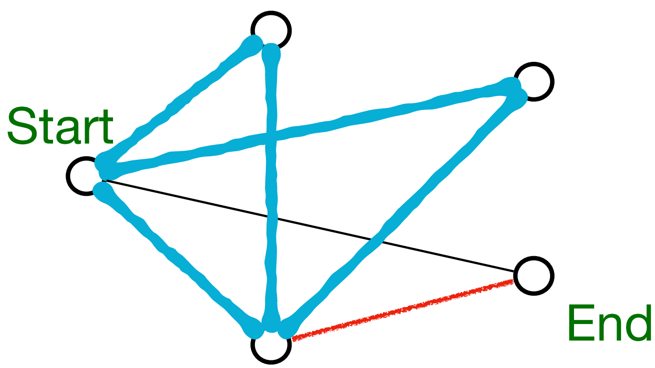

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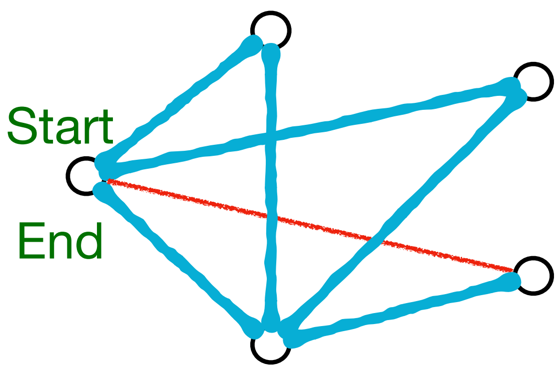

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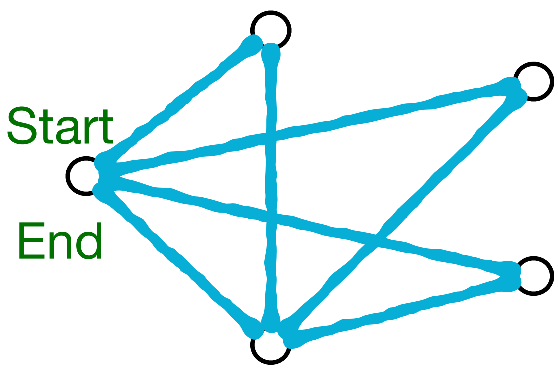

Euler Tour Problem:

Input: A graph G=(V,E).

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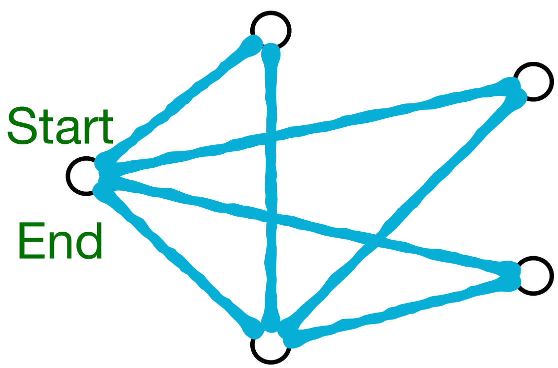

Euler Tour Problem:

Input: A graph G=(V,E).

Question: Does G have an Euler Tour?

Theorem: G has an Euler Tour if and only if G is connected and all its nodes have even degrees.

So the Euler Tour Problem is polynomial-time solvable.

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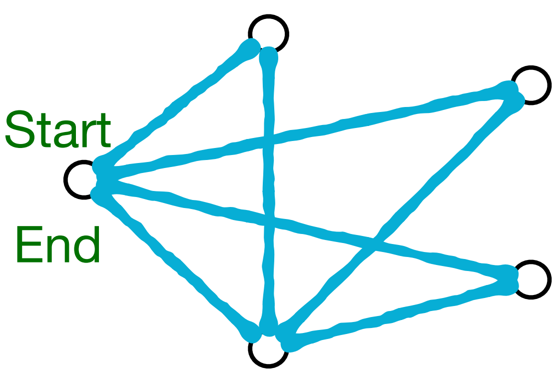
How about we change "edge" to "node" below"

Hamiltonian cycle:

~~Euler Tour~~:

Hamiltonian cycle node

Given a graph G=(V,E), an ~~Euler Tour~~ is a cycle that passes every ~~edge~~ of G exactly once.



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The Euler Tour Problem is polynomial-time solvable.

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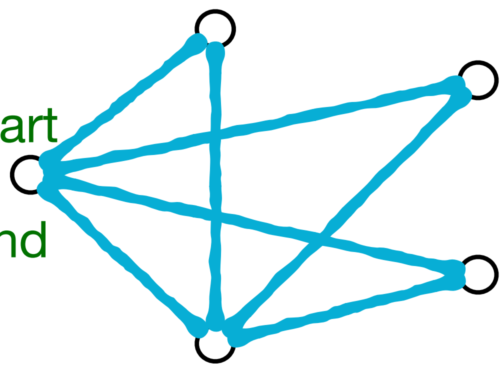

Hamiltonian cycle

~~Euler Tour~~ Problem:

Input: A graph G=(V,E).

Question: Does G have an ~~Euler Tour~~?

Hamiltonian cycle

The Euler Tour Problem is polynomial-time solvable.

Is the Hamiltonian cycle Problem polynomial-time solvable?

No one knows.

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Boolean logic: AND operation : $0 \wedge 0 = 0, \quad 0 \wedge 1 = 0, \quad 1 \wedge 0 = 0, \quad 1 \wedge 1 = 1$

OR operation : $0 \vee 0 = 0, \quad 0 \vee 1 = 1, \quad 1 \vee 0 = 1, \quad 1 \vee 1 = 1$

NOT operation : $\bar{0} = 1, \quad \bar{1} = 0$

Boolean variables: $x_1, x_2, \cdots, x_n \in \{0,1\}$

Boolean literal: $x_i, \bar{x}_i$

Boolean formula: $(x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2 \vee x_4) \wedge (x_1 \vee \bar{x}_4 \vee x_5) \wedge (x_2 \vee \bar{x}_3 \vee \bar{x}_5)$

clause clause clause clause

CNF: Conjunctive Normal Form

2-CNF SAT Problem:

Input: A CNF formula with n variables and k clauses,
where each clause is the "OR" of 2 literals.

Question: Does there exist a solution to the variables that make the formula be true?

Instance: $(x_1 \vee x_3) \wedge (\bar{x_2} \vee x_4) \wedge (x_1 \vee x_5) \wedge (\bar{x_3} \vee \bar{x_5})$

n=5 variables k=4 clauses

2-CNF SAT Problem:

Input: A CNF formula with n variables and k clauses, where each clause is the "OR" of 2 literals.

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A solution: $x_1 = 1,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ x_5 = 0$

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$$\begin{array}{cccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{array}$$

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The 2-CNF SAT Problem can be solved in polynomial time.

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n=5 variables k=4 clauses

3-CNF SAT Problem:

Input: A CNF formula with n variables and k clauses, where each clause is the "OR" of 3 literals.

Question: Does there exist a solution to the variables that make the formula be true?

The 2-CNF SAT Problem can be solved in polynomial time.

How about 3-CNF SAT Problem? No one knows. But if you do …

Instance: $\underbrace{(x_1 \vee x_2 \vee x_3)}_{\text{clause}} \wedge \underbrace{(\bar{x}_1 \vee \bar{x}_2 \vee x_4)}_{\text{clause}} \wedge \underbrace{(x_1 \vee \bar{x}_4 \vee x_5)}_{\text{clause}} \wedge \underbrace{(x_2 \vee \bar{x}_3 \vee \bar{x}_5)}_{\text{clause}}$

n=5 variables k=4 clauses

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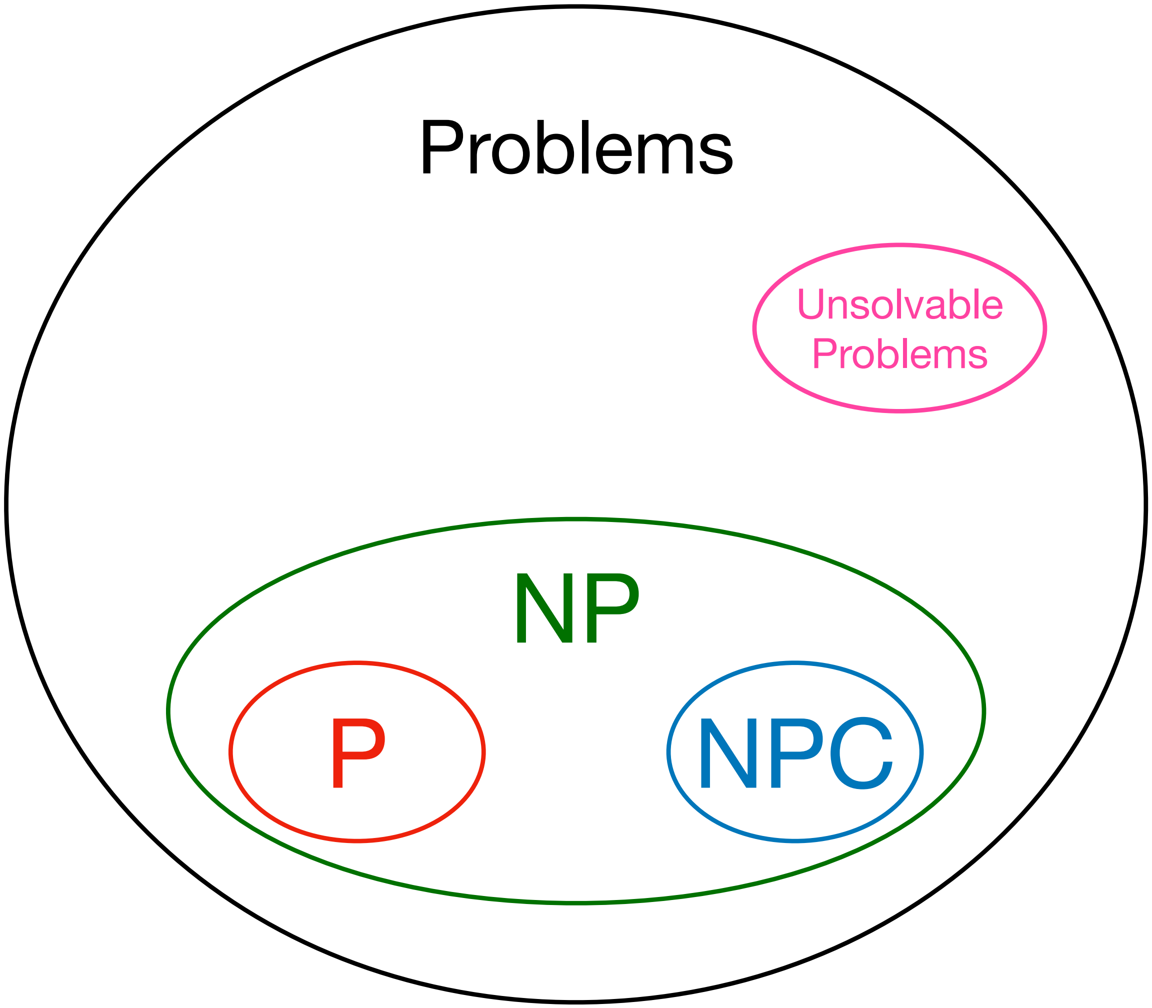

P: the set of all problems that can be solved in polynomial time.

NP: the set of all problems with this property: it takes polynomial time to verify the correctness of the solution.

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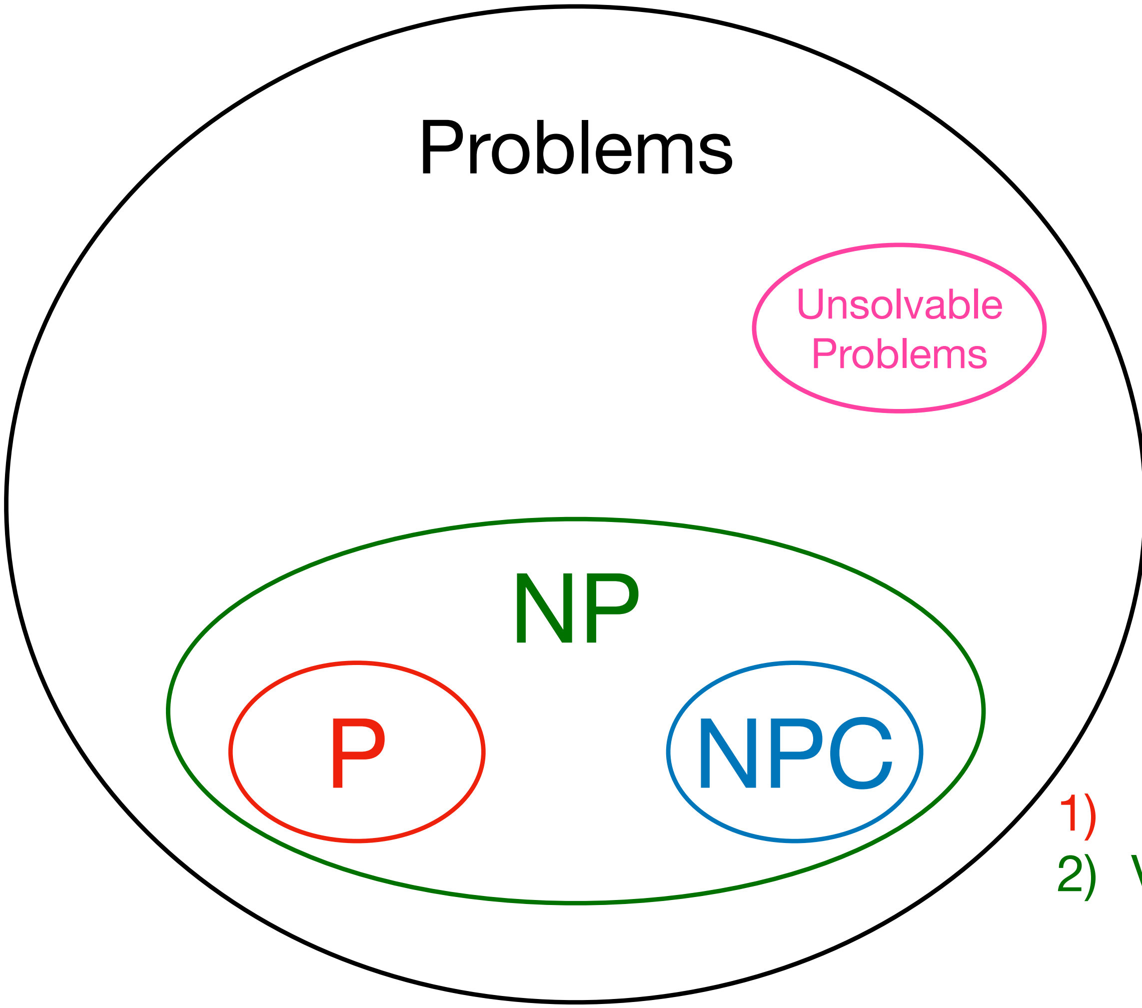

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Two elements of an algorithm:

1) The process of finding a solution.
2) Verify the correctness of the solution (so it knows when to end).

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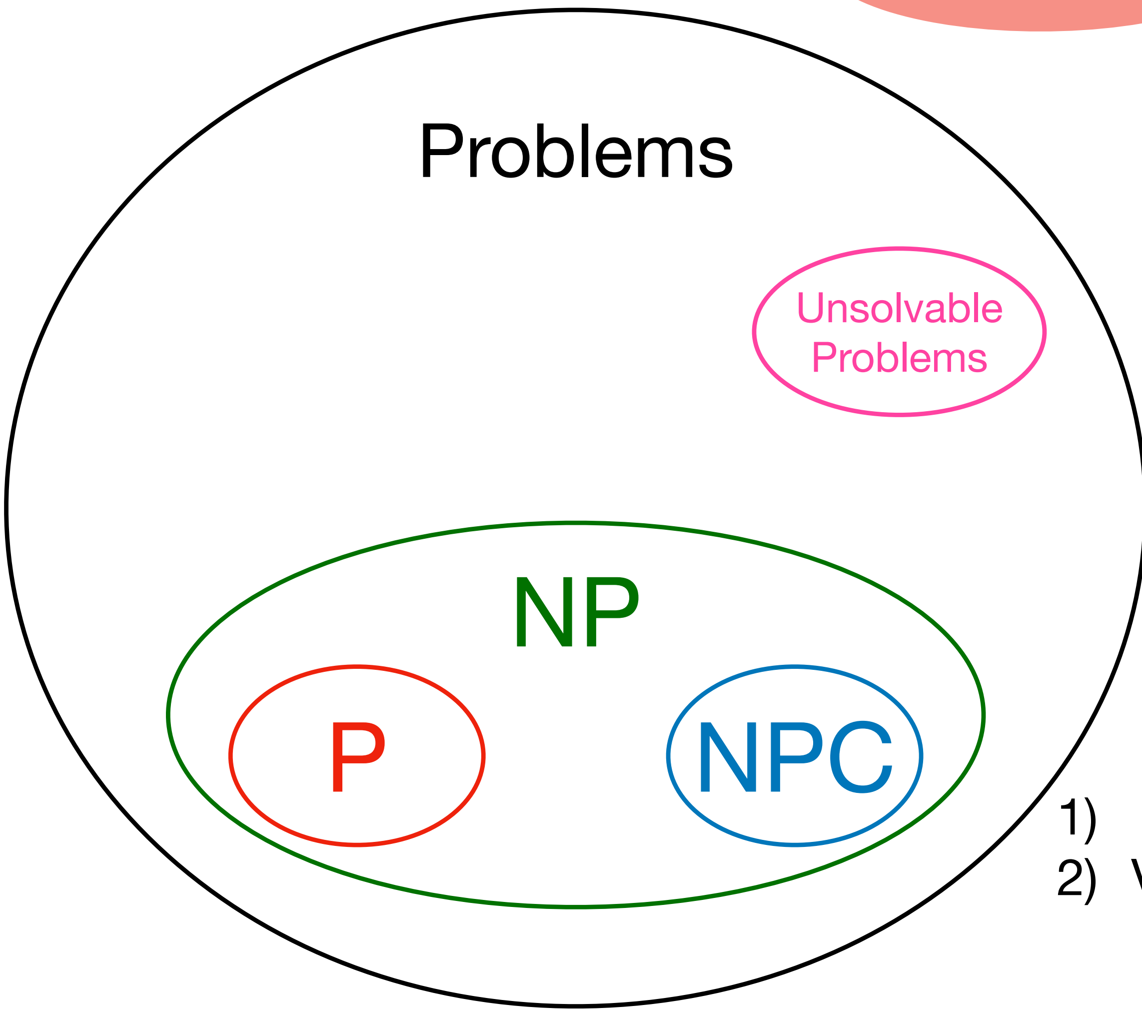
$P \subseteq NP$

$P \overset{?}{=} NP$



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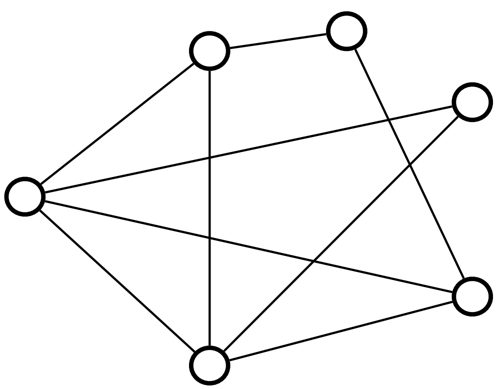
Input: A graph G=(V,E).

Question: Does G have a Hamiltonian cycle?



Finding a Hamiltonian cycle might be hard ……

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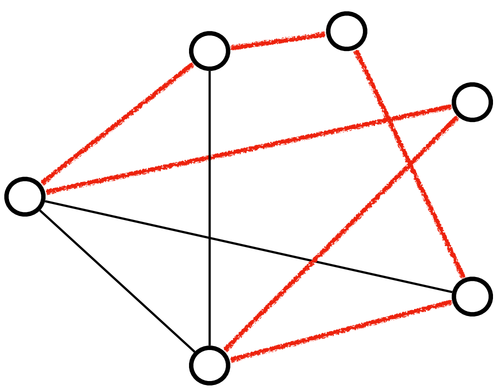

Finding a Hamiltonian cycle might be hard ……

But given a Hamiltonian cycle, it is easy to verify it is indeed A Hamiltonian cycle.

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Optimization Problem: A problem where we look for a solution whose objective value is maximized or minimized.

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Input: A directed graph G=(V,E),
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Let $s, t \in V$ be two nodes.
Let k be a real number.

Question: Is there a path from s to t whose length is at most k?

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Shortest-Path Problem

Input: A directed graph G=(V,E), where every edge $e \in E$ has a weight $w(e)$. Let $s, t \in V$ be two nodes.

Output: A shortest path from s to t.

Find the shortest path.

Decision Problem

Is there a sufficiently short path?

Shortest-Path Problem

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Find the best solution.

Decision Problem

Is there a sufficiently good solution?

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Longest-Path Problem

Input: A directed graph G=(V,E), where every edge $e \in E$ has a weight $w(e)$. Let $s, t \in V$ be two nodes.

Output: A longest simple path from s to t.

Decision Problem

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If an optimization problem is “easy", then the corresponding decision problem is also “easy”.

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If an optimization problem is “easy", then the corresponding decision problem is also “easy”.

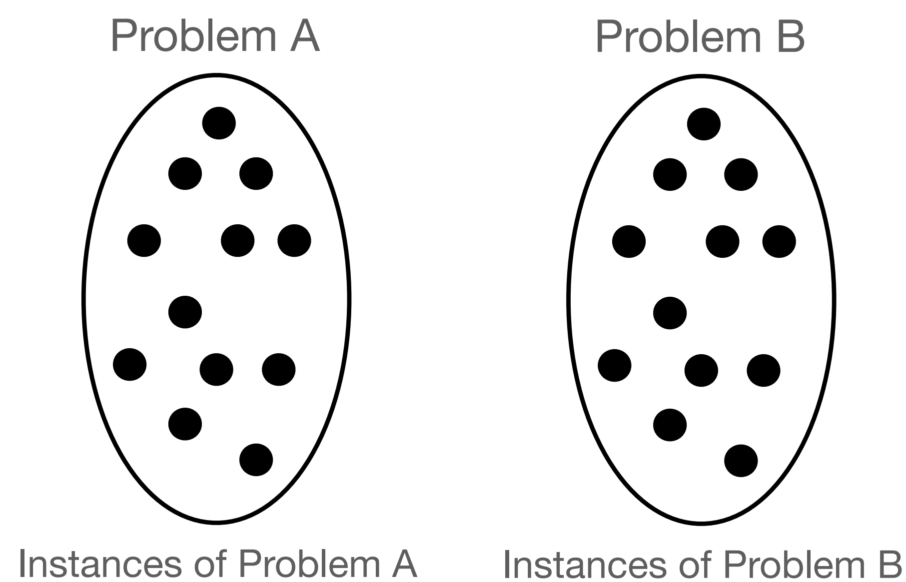
If a decision problem is “hard”, then the corresponding optimization problem is also “hard”.

From now on, we focus on “Decision Problems” only.

CH 34. NP Completeness

Decision Problem: A problem that asks a Yes/No question.

Reduction



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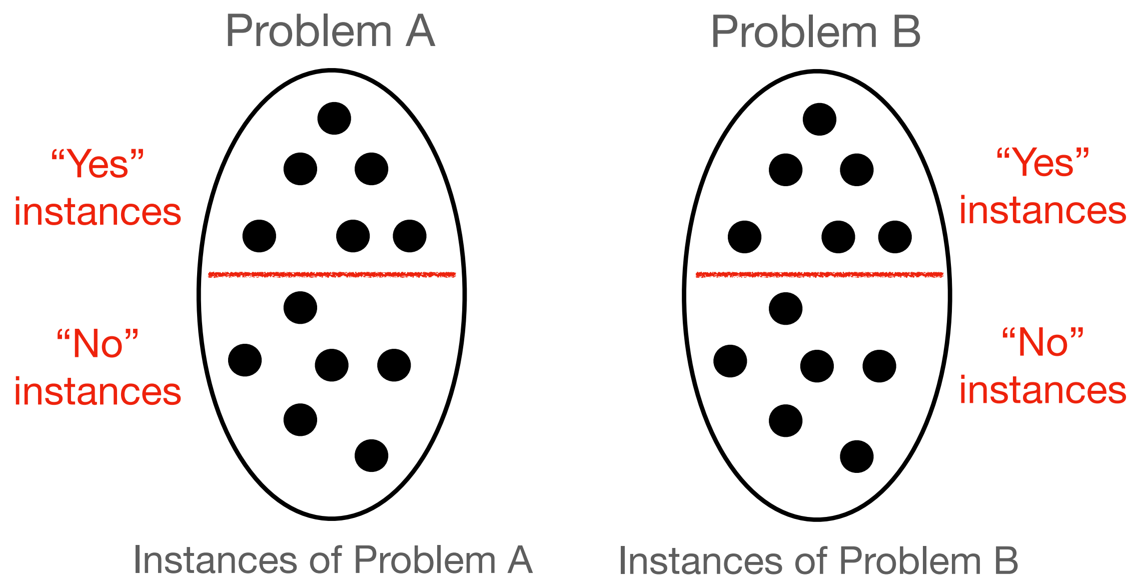

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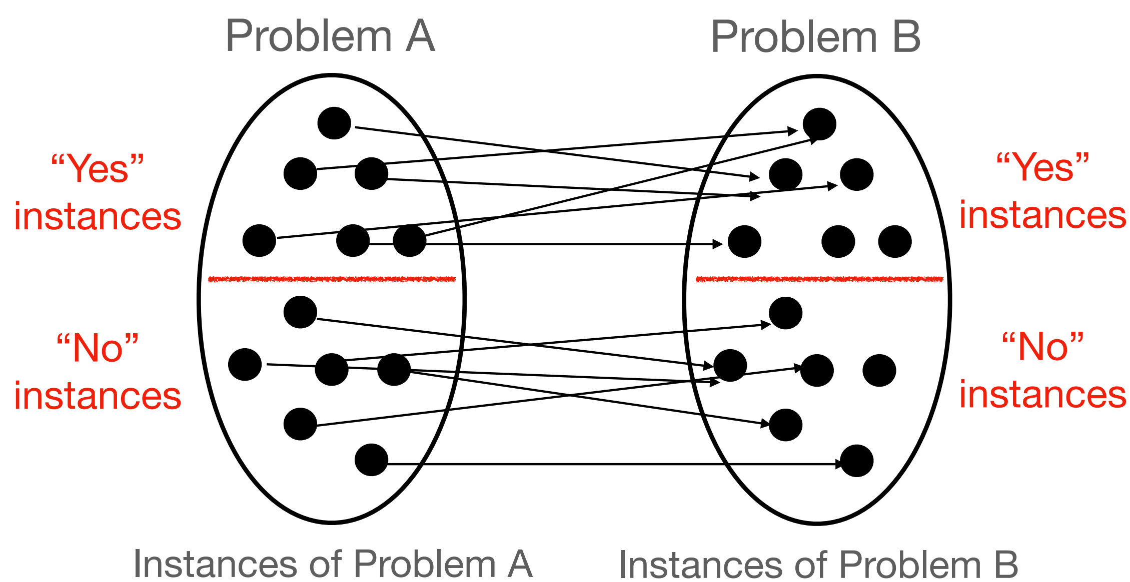
Decision Problem: A problem that asks a Yes/No question.

Reduction



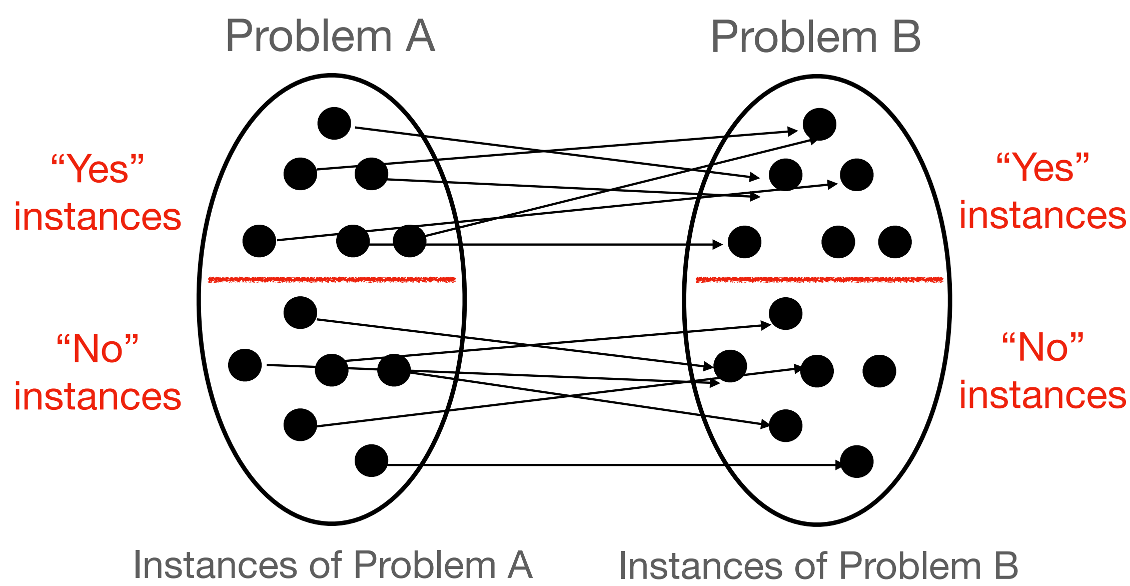
The mapping is from A to B (not B to A)

The mapping preserves the "Yes/No" answer.

CH 34. NP Completeness

Decision Problem: A problem that asks a Yes/No question.

Reduction: If there is a mapping from the instances of Problem A to the instances of Problem B, such that every "Yes" instance of Problem A is mapped to a "Yes" instance of Problem B, and every "No" instance of Problem A is mapped to a "No" instance of Problem B, then the mapping is called a "Reduction" from Problem A to Problem B. We can also say "Problem A is reduced to Problem B".



The mapping is from A to B (not B to A)

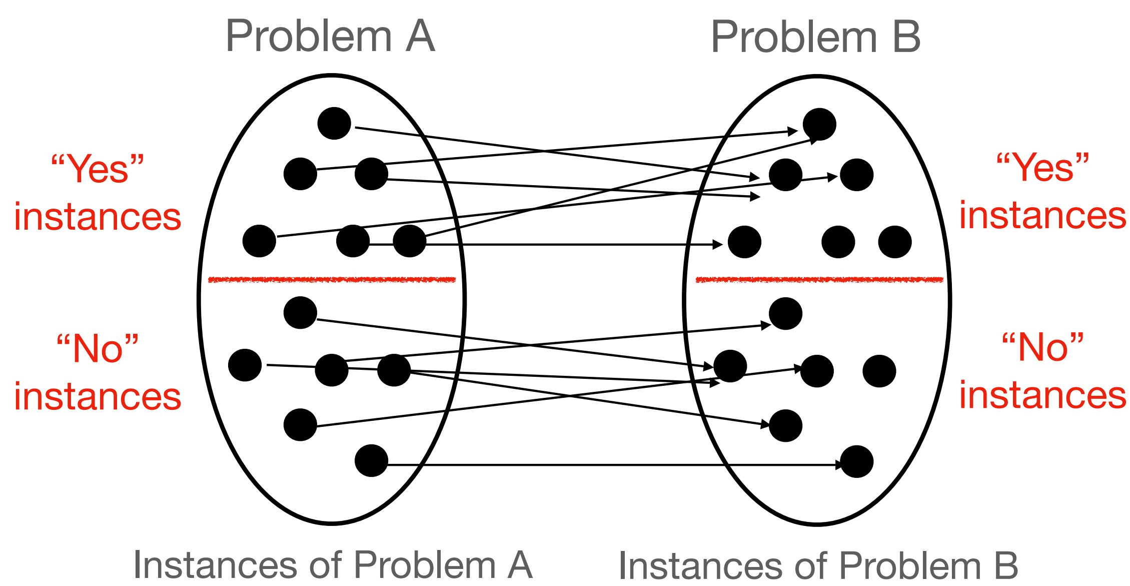
The mapping preserves the "Yes/No" answer.

CH 34. NP Completeness

Example:

Problem A: Does $ax + b = 0$ have an integer solution?

Problem B: Does $ax^2 + bx + c = 0$ have an integer solution?



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CH 34. NP Completeness

Example:

Problem A: Does $ax + b = 0$

have an integer solution?

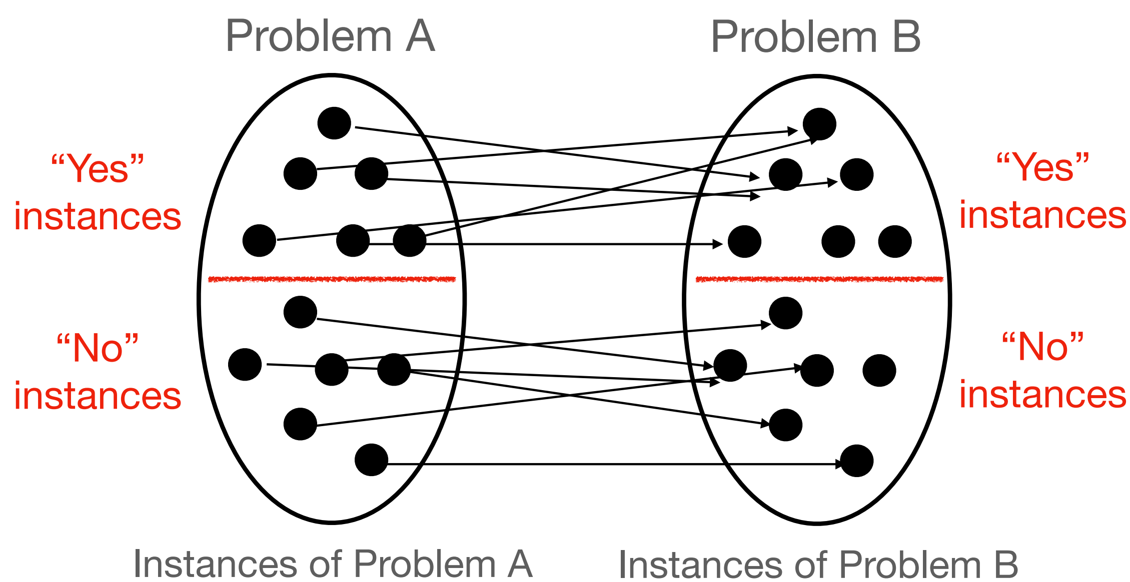
Problem B: Does $ax^2 + bx + c = 0$

have an integer solution?

Mapping:

Instance of Problem A: $ax + b = 0$

Instance of Problem B: $0x^2 + ax + b = 0$



The mapping is from A to B (not B to A)

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CH 34. NP Completeness

Example:

Problem A: Does $ax + b = 0$

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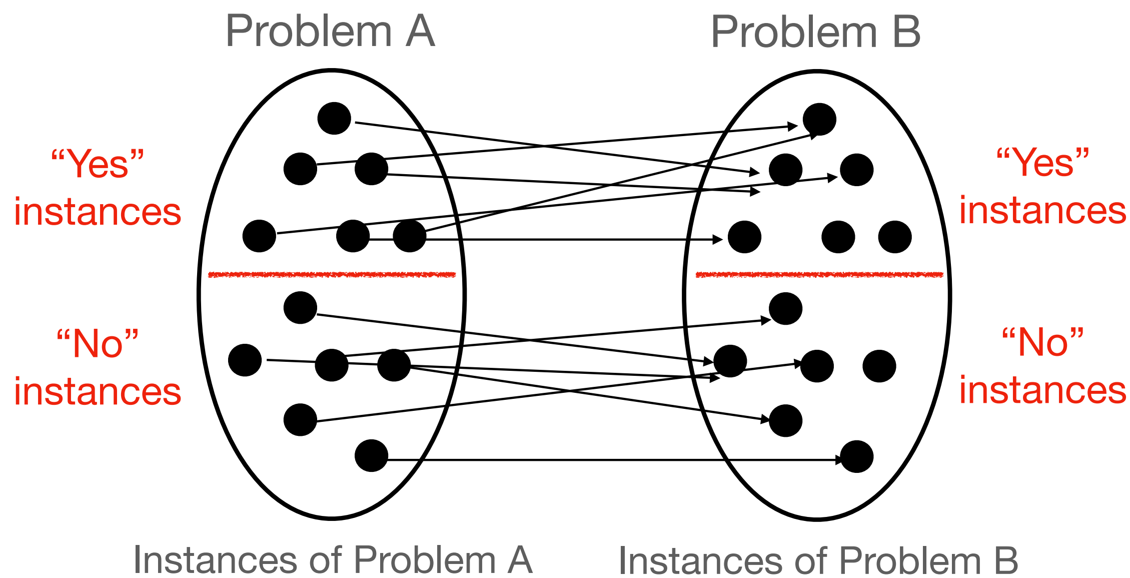
Instance of Problem B: $0x^2 + ax + b = 0$

The mapping is a reduction.



The mapping is from A to B (not B to A)

The mapping preserves the "Yes/No" answer.

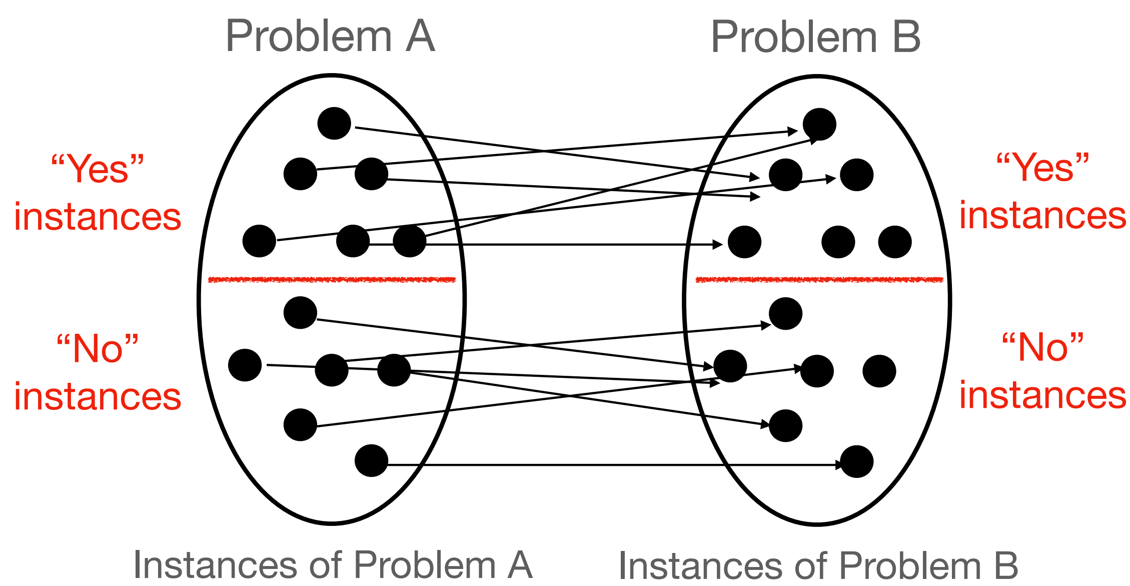
CH 34. NP Completeness

Polynomial-time reduction:
A reduction that takes polynomial time.

Example:

Problem A: Does $ax + b = 0$
have an integer solution?

Problem B: Does $ax^2 + bx + c = 0$
have an integer solution?



Mapping:

Instance of Problem A: $ax + b = 0$

$\Downarrow$

Instance of Problem B: $0x^2 + ax + b = 0$

The mapping is a reduction.

The mapping is from A to B (not B to A)

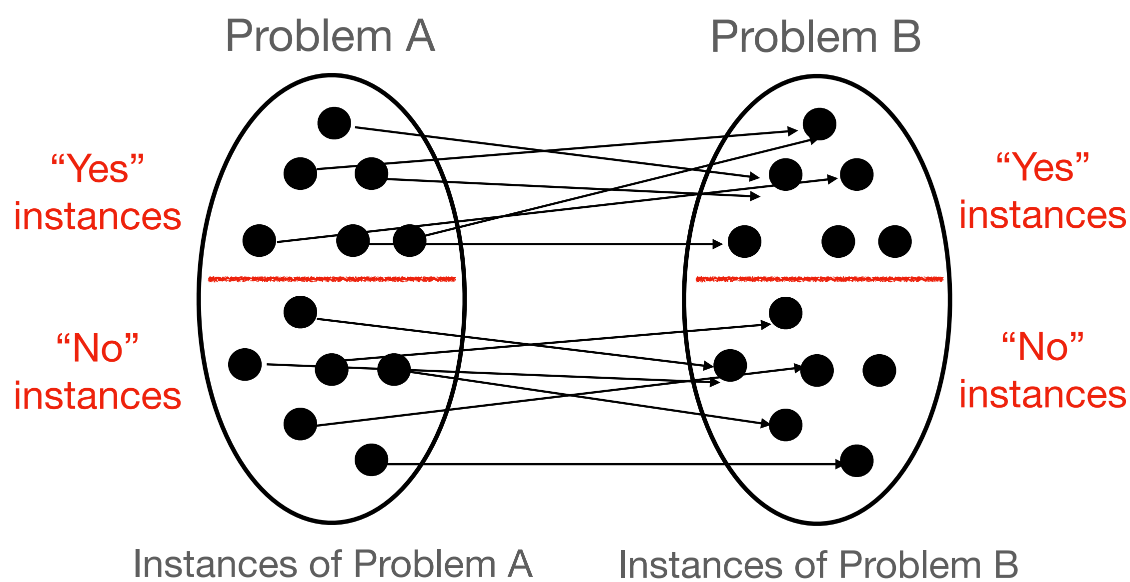
The mapping preserves the "Yes/No" answer.

If B is polynomial-time solvable,

then A is polynomial-time solvable.

Why?

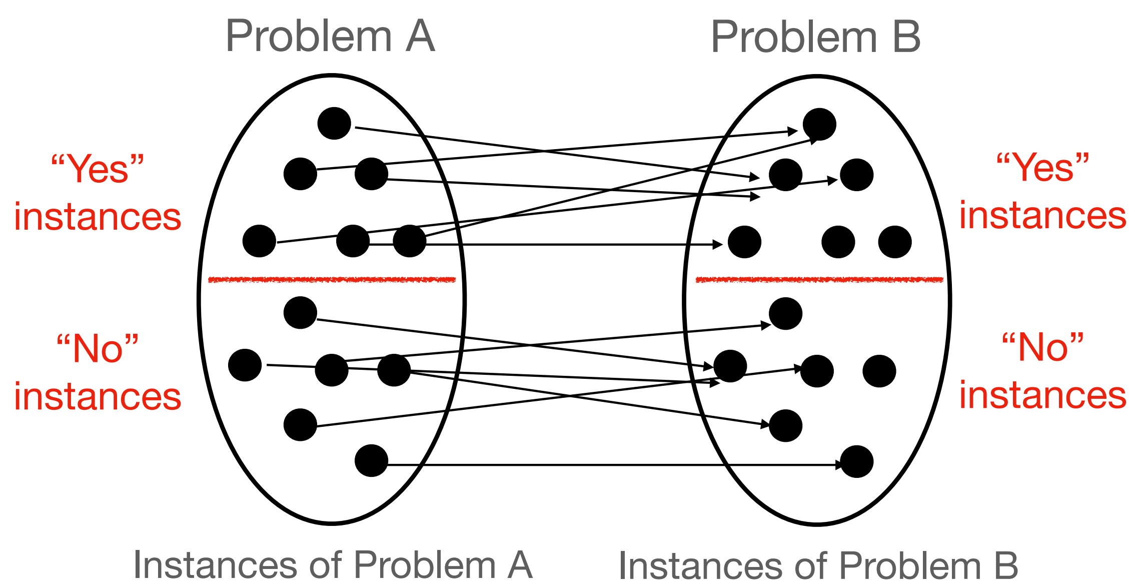
Polynomial-time reduction:



If B is polynomial-time solvable,

then A is polynomial-time solvable.

Polynomial-time reduction:



In this sense,

Problem A is "easier" (or "no harder"),

Problem B is "harder" (or "no easier").

We will use this property to prove the "hardness" of problems.