Algorithms

Lecture 12: Linear Programming (Part 2)

Anxiao (Andrew) Jiang

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$

SIMPLEX Algorithm

1) Turn the LP into Standard Form

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$

Already in Standard Form

SIMPLEX Algorithm

1) Turn the LP into Standard Form

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$

- 1) Turn the LP into Standard Form
- 2) Turn the LP into Slack Form

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$



$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

- 1) Turn the LP into Standard Form
- 2) Turn the LP into Slack Form

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

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$$x_1, x_2, x_3 \ge 0$$



$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

- 1) Turn the LP into Standard Form
- 2) Turn the LP into Slack Form
- 3) Get a basic solution: set all non-basic variables to 0

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$



$$z = 3x_1 + x_2 + 2x_3$$

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$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Non-basic variables = 0

SIMPLEX Algorithm

- 1) Turn the LP into Standard Form
- 2) Turn the LP into Slack Form
- 3) Get a basic solution: set all non-basic variables to 0

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$



$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 30, x_5 = 24, x_6 = 36$$

SIMPLEX Algorithm

- 1) Turn the LP into Standard Form
- 2) Turn the LP into Slack Form
- 3) Get a basic solution: set all non-basic variables to 0

Basic variables

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$



$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

- 1) Turn the LP into Standard Form
- 2) Turn the LP into Slack Form
- 3) Get a basic solution: set all non-basic variables to 0
- 4) If the basic solution if feasible (i.e., all variables are non-negative), continue.

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$



$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 30, x_5 = 24, x_6 = 36$$

SIMPLEX Algorithm

- 1) Turn the LP into Standard Form
- 2) Turn the LP into Slack Form
- 3) Get a basic solution: set all non-basic variables to 0
- 4) If the basic solution if feasible (i.e., all variables are non-negative), continue.

Lucky day: this basic solution is feasible. (If not, we will discuss how to handle it later.)

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$



Objective value = 0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 + 2x_2 + 5x_3 \le 24$$
$$4x_1 + x_2 + 2x_3 \le 36$$
$$x_1, x_2, x_3 \ge 0$$



Objective value = 0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

Here the non-basic variables

$$x_1, x_2, x_3$$

all have positive coefficients in the objective function.

So we can pick any of them.

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 + 2x_2 + 5x_3 \le 24$$
$$4x_1 + x_2 + 2x_3 \le 36$$
$$x_1, x_2, x_3 \ge 0$$



Objective value = 0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).



Why a non-basic variable?

The objective value is a function of non-basic variables.

So if we want to greedily increase the objective value, we want to adjust the value of a non-basic variable.

Note: we see not only the objective value, but also the Basic Variables, as functions of the non-basic variables.

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 + 2x_2 + 5x_3 \le 24$$
$$4x_1 + x_2 + 2x_3 \le 36$$
$$x_1, x_2, x_3 \ge 0$$



Objective value = 0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).



Why does the non-basic variable need to have a positive coefficient in the objective function?

In the current solution (the basic solution),
all non-basic variables have value 0.
So we cannot decrease its value (to keep the solution feasible).
So we can only increase its value (from 0 to something bigger).
To increase the objective value at the same time,
the coefficient needs to be positive.

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$



Objective value = 0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).



Why do we care about this?

Because we need to keep the solution feasible.

Note that when we increase the value of the above non-basic variable:

- 1) The other non-basic variables are still 0.
- The objective value increases its value.
- 3) The basic variables will change their values.

maximize $3x_1 + x_2 + 2x_3$ s.t.

$$x_1 + x_2 + 3x_3 \le 30$$
$$2x_1 + 2x_2 + 5x_3 \le 24$$
$$4x_1 + x_2 + 2x_3 \le 36$$
$$x_1, x_2, x_3 \ge 0$$



Objective value = 0

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

Here the non-basic variables

$$x_1, x_2, x_3$$

all have positive coefficients in the objective function.

So we can pick any of them.

Let's pick x_1

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3 - x_1 \le 30$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

$$z = 3x_1 + x_2 + 2x_3$$

$$\sqrt{x_4} = 30 - x_1 - x_2 - 3x_3 - x_1 \le 30$$

$$\sqrt{x_5} = 24 - 2x_1 - 2x_2 - 5x_3 - x_1 \le 12$$

$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

$$z = 3x_1 + x_2 + 2x_3$$

$$\sqrt{x_4} = 30 - x_1 - x_2 - 3x_3 - x_1 \le 30$$

$$\sqrt{x_5} = 24 - 2x_1 - 2x_2 - 5x_3 - x_1 \le 12$$

$$\sqrt{x_6} = 36 - 4x_1 - x_2 - 2x_3 - x_1 \le 9$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

$$z = 3x_1 + x_2 + 2x_3$$
30
21 $\sqrt{x_4} = 30 - x_1 - x_2 - 3x_3$

$$x_1 \le 30$$
24
6 $\sqrt{x_5} = 24 - 2x_1 - 2x_2 - 5x_3$

$$x_1 \le 12$$
36
0 $\sqrt{x_6} = 36 - 4x_1 - x_2 - 2x_3$
Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

Objective value = 0

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

By how much can we increase x_1 ?

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SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

$$z = 3x_1 + x_2 + 2x_3$$

$$x_4 = 30 - x_1 - x_2 - 3x_3$$

$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3$$

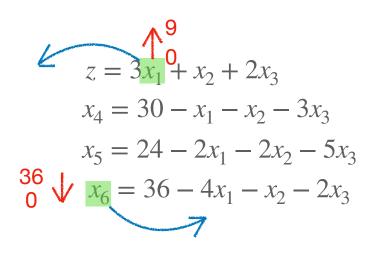
$$x_6 = 36 - 4x_1 - x_2 - 2x_3$$

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).



Pivot

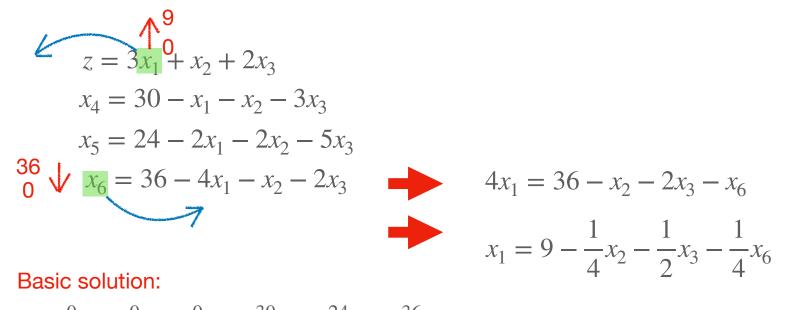
(So that the new solution will be the basic solution for the new LP.)

Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

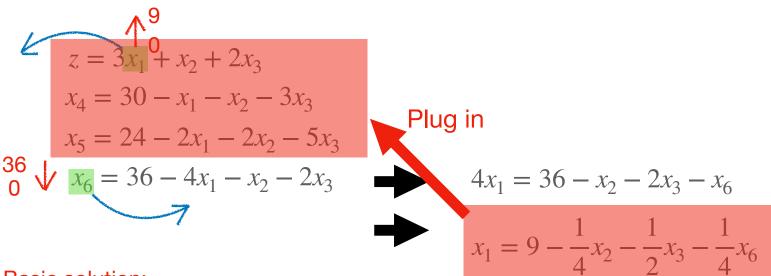
Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).



$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

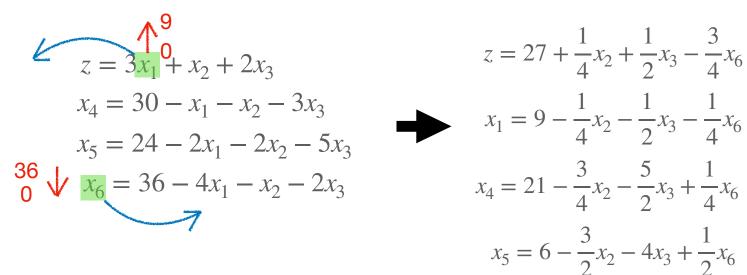


Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).

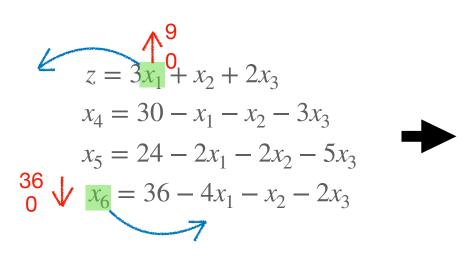


Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

SIMPLEX Algorithm now takes a greedy approach:

Pick a non-basic variable that has a positive coefficient in the objective function, increase its value as much as possible (without making any basic variable negative).



Basic solution:

$$x_1 = 0$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 30$, $x_5 = 24$, $x_6 = 36$

Objective value = 0

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

Basic solution:

$$x_1 = 9$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 21$, $x_5 = 6$, $x_6 = 0$

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

We can increase either x_2 or x_3 .

Let's increase χ_3 .

Basic solution:

$$x_1 = 9$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 21$, $x_5 = 6$, $x_6 = 0$

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

Basic solution:

$$x_1 = 9$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 21$, $x_5 = 6$, $x_6 = 0$

Objective value = 27

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

Basic solution:

$$x_1 = 9$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 21$, $x_5 = 6$, $x_6 = 0$

Objective value = 27

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_3 \le 18$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

Basic solution:

$$x_1 = 9$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 21$, $x_5 = 6$, $x_6 = 0$

Objective value = 27

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_3 \le 18$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

$$x_3 \le \frac{3}{2}$$

Basic solution:

$$x_1 = 9$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 21$, $x_5 = 6$, $x_6 = 0$

Objective value = 27

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$\sqrt{x_1} = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$\sqrt{x_4} = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_3 \le \frac{42}{5}$$

$$\sqrt{x_5} = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

$$x_3 \le \frac{3}{2}$$

Basic solution:

$$x_1 = 9$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 21$, $x_5 = 6$, $x_6 = 0$

Objective value = 27

By how much can we increase x_3 ? $\frac{3}{2} = 1.5$

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6 \qquad x_3 \le 18$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6 \qquad x_3 \le \frac{42}{5}$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6 \qquad x_3 \le \frac{3}{2}$$

Basic solution:

$$x_1 = 9$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 21$, $x_5 = 6$, $x_6 = 0$

Objective value = 27

Pivot

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

Basic solution:

$$x_1 = 9$$
, $x_2 = 0$, $x_3 = 0$, $x_4 = 21$, $x_5 = 6$, $x_6 = 0$

$$z = \frac{111}{4} + \frac{1}{16}x_2 - \frac{1}{8}x_5 - \frac{11}{6}x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16}x_2 + \frac{1}{8}x_5 - \frac{5}{16}x_6$$

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6$$

$$x_4 = \frac{69}{4} + \frac{3}{16}x_2 + \frac{5}{8}x_5 - \frac{1}{16}x_6$$

Basic solution:

$$x_1 = \frac{33}{4}$$
, $x_2 = 0$, $x_3 = \frac{3}{2}$, $x_4 = \frac{69}{4}$, $x_5 = 0$, $x_6 = 0$

Objective value =
$$\frac{111}{4}$$
 = 27.75

$$z = \frac{111}{4} + \frac{1}{16}x_2 - \frac{1}{8}x_5 - \frac{11}{6}x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16}x_2 + \frac{1}{8}x_5 - \frac{5}{16}x_6$$

$$x_3 = \frac{3}{2} - \frac{3}{8}x_2 - \frac{1}{4}x_5 + \frac{1}{8}x_6$$

$$x_4 = \frac{69}{4} + \frac{3}{16}x_2 + \frac{5}{8}x_5 - \frac{1}{16}x_6$$

Basic solution:

$$x_1 = \frac{33}{4}, x_2 = 0, x_3 = \frac{3}{2}, x_4 = \frac{69}{4}, x_5 = 0, x_6 = 0$$

Objective value =
$$\frac{111}{4}$$
 = 27.75

We can increase χ_{γ}

By how much?

$$z = \frac{111}{4} + \frac{1}{16} x_2 - \frac{1}{8} x_5 - \frac{11}{6} x_6$$

$$\downarrow x_1 = \frac{33}{4} - \frac{1}{16} x_2 + \frac{1}{8} x_5 - \frac{5}{16} x_6$$

$$x_3 = \frac{3}{2} - \frac{3}{8} x_2 - \frac{1}{4} x_5 + \frac{1}{8} x_6$$

$$x_4 = \frac{69}{4} + \frac{3}{16} x_2 + \frac{5}{8} x_5 - \frac{1}{16} x_6$$

Basic solution:

$$x_1 = \frac{33}{4}$$
, $x_2 = 0$, $x_3 = \frac{3}{2}$, $x_4 = \frac{69}{4}$, $x_5 = 0$, $x_6 = 0$

Objective value =
$$\frac{111}{4}$$
 = 27.75

$$z = \frac{111}{4} + \frac{1}{16} x_2 - \frac{1}{8} x_5 - \frac{11}{6} x_6$$

$$v_1 = \frac{33}{4} - \frac{1}{16} x_2 + \frac{1}{8} x_5 - \frac{5}{16} x_6$$

$$v_2 \le 132$$

$$v_3 = \frac{3}{2} - \frac{3}{8} x_2 - \frac{1}{4} x_5 + \frac{1}{8} x_6$$

$$v_4 = \frac{69}{4} + \frac{3}{16} x_2 + \frac{5}{8} x_5 - \frac{1}{16} x_6$$

Basic solution:

$$x_1 = \frac{33}{4}$$
, $x_2 = 0$, $x_3 = \frac{3}{2}$, $x_4 = \frac{69}{4}$, $x_5 = 0$, $x_6 = 0$

Objective value =
$$\frac{111}{4}$$
 = 27.75

$$z = \frac{111}{4} + \frac{1}{16} x_2 - \frac{1}{8} x_5 - \frac{11}{6} x_6$$

$$v_1 = \frac{33}{4} - \frac{1}{16} x_2 + \frac{1}{8} x_5 - \frac{5}{16} x_6$$

$$v_2 \le 132$$

$$v_3 = \frac{3}{2} - \frac{3}{8} x_2 - \frac{1}{4} x_5 + \frac{1}{8} x_6$$

$$v_4 = \frac{69}{4} + \frac{3}{16} x_2 + \frac{5}{8} x_5 - \frac{1}{16} x_6$$

$$v_2 \le \infty$$

Basic solution:

$$x_1 = \frac{33}{4}$$
, $x_2 = 0$, $x_3 = \frac{3}{2}$, $x_4 = \frac{69}{4}$, $x_5 = 0$, $x_6 = 0$

Objective value =
$$\frac{111}{4}$$
 = 27.75

$$z = \frac{111}{4} + \frac{1}{16} x_2 - \frac{1}{8} x_5 - \frac{11}{6} x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16} x_2 + \frac{1}{8} x_5 - \frac{5}{16} x_6$$

$$1.5 \quad x_3 = \frac{3}{2} - \frac{3}{8} x_2 - \frac{1}{4} x_5 + \frac{1}{8} x_6$$

$$x_4 = \frac{69}{4} + \frac{7}{16} x_2 + \frac{5}{8} x_5 - \frac{1}{16} x_6$$

Basic solution:

$$x_1 = \frac{33}{4}$$
, $x_2 = 0$, $x_3 = \frac{3}{2}$, $x_4 = \frac{69}{4}$, $x_5 = 0$, $x_6 = 0$

Objective value =
$$\frac{111}{4}$$
 = 27.75

Pivot

$$z = \frac{111}{4} + \frac{1}{16} x_2 - \frac{1}{8} x_5 - \frac{11}{6} x_6$$

$$x_1 = \frac{33}{4} - \frac{1}{16} x_2 + \frac{1}{8} x_5 - \frac{5}{16} x_6$$

$$1.5 \quad x_3 = \frac{3}{2} - \frac{3}{8} x_2 - \frac{1}{4} x_5 + \frac{1}{8} x_6$$

$$x_4 = \frac{69}{4} + \frac{3}{16} x_2 + \frac{5}{8} x_5 - \frac{1}{16} x_6$$

$$z = 28 - \frac{1}{6}x_3 - \frac{1}{6}x_5 - \frac{2}{3}x_6$$

$$x_1 = 8 + \frac{1}{6}x_3 + \frac{1}{6}x_5 - \frac{1}{3}x_6$$

$$x_2 = 4 - \frac{8}{3}x_3 - \frac{2}{3}x_5 + \frac{1}{3}x_6$$

$$x_4 = 18 - \frac{1}{2}x_3 + \frac{1}{2}x_5$$

Basic solution:

$$x_1 = \frac{33}{4}$$
, $x_2 = 0$, $x_3 = \frac{3}{2}$, $x_4 = \frac{69}{4}$, $x_5 = 0$, $x_6 = 0$

Objective value = $\frac{111}{4}$ = 27.75

Basic solution:

$$x_1 = 8$$
, $x_2 = 4$, $x_3 = 0$, $x_4 = 18$, $x_5 = 0$, $x_6 = 0$

Objective value = 28

$$z = 28 - \frac{1}{6}x_3 - \frac{1}{6}x_5 - \frac{2}{3}x_6$$

$$x_1 = 8 + \frac{1}{6}x_3 + \frac{1}{6}x_5 - \frac{1}{3}x_6$$

$$x_2 = 4 - \frac{8}{3}x_3 - \frac{2}{3}x_5 + \frac{1}{3}x_6$$

$$x_4 = 18 - \frac{1}{2}x_3 + \frac{1}{2}x_5$$

This is the end of the SIMPLEX Algorithm!

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

Basic solution:

$$x_1 = 8$$
, $x_2 = 4$, $x_3 = 0$, $x_4 = 18$, $x_5 = 0$, $x_6 = 0$

Objective value = 28

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait ... if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait ... if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait ... if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

$$z = x_1^0 + x_2 + x_3$$

$$x_4 = 8 - x_1 - x_2$$

$$x_5 = x_2 - x_3$$

$$x_1 \le \infty$$

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

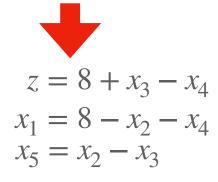
But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait ... if the objective value increments monotonically each time, how could it possibly loop forever?

$$z = x_1^{8} + x_2 + x_3$$

$$x_1 = 8 - x_1 - x_2$$

$$x_2 = x_2 - x_3$$



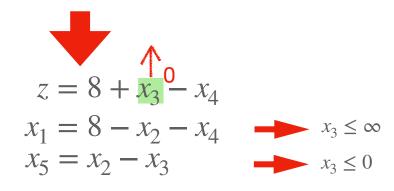
What if the objective value increments by 0?

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait ... if the objective value increments monotonically each time, how could it possibly loop forever?

$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$

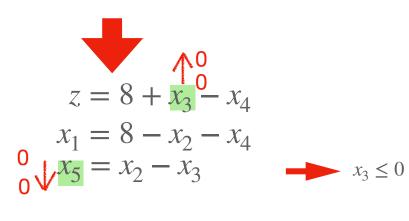


What if the objective value increments by 0?

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.



$$z = x_1 + x_2 + x_3$$
$$x_4 = 8 - x_1 - x_2$$
$$x_5 = x_2 - x_3$$



Should we pivot?

Yes, even though here the objective value increments by 0 (i.e., it does not change).

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

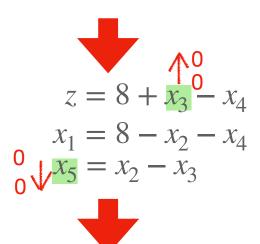
Wait ... if the objective value increments monotonically each time, how could it possibly loop forever?

What if the objective value increments by 0?

$$z = x_1 + x_2 + x_3$$

$$x_4 = 8 - x_1 - x_2$$

$$x_5 = x_2 - x_3$$



$$z = 8 + x_2 - x_4 - x_5$$

$$x_1 = 8 - x_2 - x_4$$

$$x_3 = x_2 - x_5$$

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

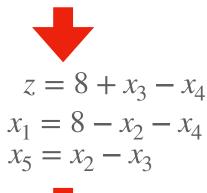
Wait ... if the objective value increments monotonically each time, how could it possibly loop forever?

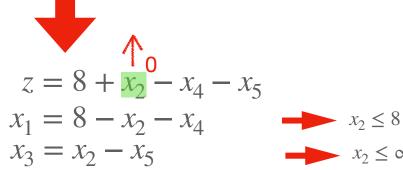
What if the objective value increments by 0?

$$z = x_1 + x_2 + x_3$$

$$x_4 = 8 - x_1 - x_2$$

$$x_5 = x_2 - x_3$$





The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait ... if the objective value increments monotonically each time, how could it possibly loop forever?

Really ... can it happen?

$$z = x_1 + x_2 + x_3$$

$$x_4 = 8 - x_1 - x_2$$

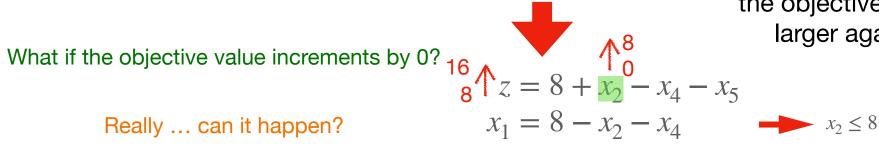
$$x_5 = x_2 - x_3$$



$$z = 8 + x_3 - x_4$$

$$x_1 = 8 - x_2 - x_4$$

$$x_5 = x_2 - x_3$$



$$x_3 = x_2 - x_5$$

So we can make the objective value larger again.

$$x_2 \leq 8$$

$$x_2 \leq \infty$$

The SIMPLEX Algorithm ends when in the objective function, the coefficients of the non-basic variables are all <= 0.

But can we guarantee SIMPLEX Algorithm will end (instead of looping forever)?

Wait ... if the objective value increments monotonically each time, how could it possibly loop forever?

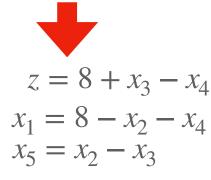
Really ... can it happen?

$$z = x_1 + x_2 + x_3$$

$$x_4 = 8 - x_1 - x_2$$

$$x_5 = x_2 - x_3$$

This example shows that the objective value can indeed increment by 0.



If this happens continuously, we get into an infinite loop.

Can we avoid this?

$$x_2 \le 8$$

$$x_2 \le \infty$$

Bland's Rule:

- 1) When we choose a non-basic variable for incrementing its value, if there is a tie, choose the variable of the smallest index.
- 2) When we choose a basic variable for pivoting, if there is a tie, choose the variable of the smallest index.

Bland's Rule guarantees the SIMPLEX algorithm will end.