# Algorithms

## Lecture 14: Linear Programming (Part 4)

**Anxiao (Andrew) Jiang**

# CH 29.5 Initial Basic Feasible Solution

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

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# CH 29.5 Initial Basic Feasible Solution

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\le 2 \\ x_1 - 5x_2 &\le -4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Is this LP feasible?

That is, does this LP have a solution to $x_1$ and $x_2$ that satisfies all the constraints?

Note that feasibility has nothing to do with the objective function.

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Slack-form LP:

$$z = 2x_1 - x_2$$

$$x_3 = 2 - 2x_1 + x_2$$

$$x_4 = -4 - x_1 + 5x_2$$

The slack-form LP does not have a feasible basic solution.

Does it mean the LP is infeasible? Not at all.

Basic solution: $x_1 = 0,\ x_2 = 0,\ x_3 = 2,\ x_4 = -4$

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\begin{aligned}
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Is this LP feasible?

Whether this LP is feasible or not,

we can always add a "helper" to make it feasible.

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Is this LP feasible?

Whether this LP is feasible or not,

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Auxiliary LP:

maximize [redacted]

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\leq 2 \\ x_1 - 5x_2 - x_0 &\leq -4 \\ x_0, x_1, x_2 &\geq 0 \end{aligned}$$

$x_0$ is our "helper".

This auxiliary LP is ALWAYS feasible. Why?

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$$

$x_0$ is our "helper".

This auxiliary LP is ALWAYS feasible. Why?

Because we can always make $x_0$ sufficiently large to satisfy all constraints.

For example, here we can make $x_0 = 4$, and a feasible solution is

$$x_0 = 4,\ x_1 = 0,\ x_2 = 0$$

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Is this LP feasible?

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we can always add a “helper” to make it feasible.

Auxiliary LP:

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s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\le 2 \\ x_1 - 5x_2 - x_0 &\le -4 \\ x_0, x_1, x_2 &\ge 0 \end{aligned}$$

$x_0$ is our “helper”. This auxiliary LP is ALWAYS feasible.

But how large does $x_0$ have to be to make this auxiliary LP feasible?

If $x_0$ can be 0, then no “help” is needed, which means the original LP is feasible.

If $x_0$ has to be greater than 0, then the original LP is not feasible.

Which case is it? Let’s find out.

# CH 29.5 Initial Basic Feasible Solution

Original LP:

$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & \\ & 2x_1 - x_2 \le 2 \\ & x_1 - 5x_2 \le -4 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Is this LP feasible?

Auxiliary LP:

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{s.t.} \quad & \\ & 2x_1 - x_2 - x_0 \le 2 \\ & x_1 - 5x_2 - x_0 \le -4 \\ & x_0, x_1, x_2 \ge 0 \end{aligned}$$

How small can $x_0$ be?

In particular, can $x_0$ be as small as 0?

How to solve an LP:

1) Decide if LP is feasible
2) If LP is feasible, turn it into a slack form whose basic solution is feasible.
3) Run SIMPLEX algorithm.

# CH 29.5 Initial Basic Feasible Solution

Theorem: Let L be an LP

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Let $L_{aux}$ be its auxiliary LP

maximize $-x_0$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n - x_0 \le b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n - x_0 \le b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n - x_0 \le b_m$$
$$x_0,\ x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Then L is feasible if and only if the optimal objective value for $L_{aux}$ is 0.

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&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\le b_m \\
x_1, x_2, \cdots, x_n &\ge 0
\end{aligned}
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Let $L_{aux}$ be its auxiliary LP

maximize $-x_0$

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\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n - x_0 &\le b_1 \\
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x_0, x_1, x_2, \cdots, x_n &\ge 0
\end{aligned}
$$

Then L is feasible if and only if the optimal objective value for $L_{aux}$ is 0.

Proof: $\Rightarrow$ If L has a feasible solution $(x_1, x_2, \cdots, x_n)$,

then $L_{aux}$ has an optimal solution $(x_0 = 0, x_1, x_2, \cdots, x_n)$.

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Theorem: Let L be an LP

maximize $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \le b_1$$
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$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \le b_m$$
$$x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Let $L_{aux}$ be its auxiliary LP

maximize $-x_0$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n - x_0 \le b_1$$
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$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n - x_0 \le b_m$$
$$x_0,\ x_1,\ x_2,\ \cdots,\ x_n \ge 0$$

Then L is feasible if and only if the optimal objective value for $L_{aux}$ is 0.

Proof: $\Leftarrow$ If $L_{aux}$ has an optimal solution $(x_0 = 0,\ x_1,\ x_2,\ \cdots,\ x_n)$,

then L has a feasible solution $(x_1,\ x_2,\ \cdots,\ x_n)$ .

CH 29.5 Initial Basic Feasible Solution

Original LP:

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\le 2 \\ x_1 - 5x_2 &\le -4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Is this LP feasible?

Let's solve the Auxiliary LP to find out.

Auxiliary LP:

maximize $-x_0$

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\le 2 \\ x_1 - 5x_2 - x_0 &\le -4 \\ x_0, x_1, x_2 &\ge 0 \end{aligned}$$

How small can $x_0$ be?

In particular, can $x_0$ be as small as 0?

Auxiliary LP:

maximize $\boxed{-x_0}$

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\le 2 \\ x_1 - 5x_2 - x_0 &\le -4 \\ x_0,\ x_1,\ x_2 &\ge 0 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} z &= \quad - x_0 \\ x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

Auxiliary LP:

maximize $-x_0$

s.t.

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\begin{aligned}
2x_1 - x_2 - x_0 &\le 2 \\
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x_0, x_1, x_2 &\ge 0
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$\Rightarrow$

$$
\begin{aligned}
z &= -x_0 \\
x_3 &= 2 + x_0 - 2x_1 + x_2 \\
x_4 &= -4 + x_0 - x_1 + 5x_2
\end{aligned}
$$

Do a pivot:

1) Move $x_0$ to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

maximize $-x_0$

s.t.

$$2x_1 - x_2 - x_0 \leq 2$$
$$x_1 - 5x_2 - x_0 \leq -4$$
$$x_0,\ x_1,\ x_2 \geq 0$$

$$z = \ -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$x_4 = -4 + x_0 - x_1 + 5x_2$$

Most negative

Do a pivot:

1) Move $x_0$ to left.

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Auxiliary LP:

maximize $-x_0$

s.t.

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\begin{aligned}
2x_1 - x_2 - x_0 &\le 2 \\
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\begin{aligned}
z &= -x_0 \\
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$$x_1 - 5x_2 - x_0 \le -4$$
$$x_0, x_1, x_2 \ge 0$$

$$z = -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
$$x_4 = -4 + x_0 - x_1 + 5x_2$$

$$z = -4 - x_1 + 5x_2 - x_4$$
$$x_3 = 6 - x_1 - 4x_2 + x_4$$
$$x_0 = 4 + x_1 - 5x_2 + x_4$$

Do a pivot:

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Auxiliary LP:

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$$x_0 = 4 + x_1 - 5x_2 + x_4$$

Positive

Do a pivot:

1) Move $x_0$ to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

maximize $-x_0$

s.t.

$$2x_1 - x_2 - x_0 \leq 2$$
$$x_1 - 5x_2 - x_0 \leq -4$$
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$$z = -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
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$$z = -4 - x_1 + 5x_2 - x_4$$
$$x_3 = 6 - x_1 - 4x_2 + x_4$$
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Where does this 6 come from?

2 + 4 = 6

Do a pivot:

1) Move $x_0$ to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

maximize $-x_0$

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\le 2 \\ x_1 - 5x_2 - x_0 &\le -4 \\ x_0, x_1, x_2 &\ge 0 \end{aligned}$$

What if

$$\begin{aligned} z &= -x_0 \\ x_3 &= -2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$$\begin{aligned} z &= -4 - x_1 + 5x_2 - x_4 \\ x_3 &= 6 - x_1 - 4x_2 + x_4 \\ x_0 &= 4 + x_1 - 5x_2 + x_4 \end{aligned}$$

-2 + 4 = 2

Still non-negative!

Do a pivot:

1) Move $x_0$ to left.

2) Pick a basic variable whose constant term is the most negative, and move it to right.

Auxiliary LP:

$$\begin{aligned}&\text{maximize} \quad -x_0\\&\text{s.t.}\\&\quad 2x_1 - x_2 - x_0 \le 2\\&\quad x_1 - 5x_2 - x_0 \le -4\\&\quad x_0, x_1, x_2 \ge 0\end{aligned}$$

$$\begin{aligned} z &= -\boxed{x_0}\\ x_3 &= 2 + x_0 - 2x_1 + x_2\\ \boxed{x_4} &= -4 + x_0 - x_1 + 5x_2\end{aligned}$$

$$\begin{aligned} z &= -4 - x_1 + 5x_2 - x_4\\ x_3 &= 6 - x_1 - 4x_2 + x_4\\ x_0 &= 4 + x_1 - 5x_2 + x_4\end{aligned}$$

All constant terms are non-negative.

Its basic solution is feasible,
So we can run the SIMPLEX Algorithm now!

Do a pivot:

1) Move $x_0$ to left.

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Auxiliary LP:

maximize $-x_0$

s.t.

$$2x_1 - x_2 - x_0 \leq 2$$
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$$x_0, x_1, x_2 \geq 0$$

$$z = -x_0$$
$$x_3 = 2 + x_0 - 2x_1 + x_2$$
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$$z = -4 - x_1 + 5x_2 - x_4$$
$$x_3 = 6 - x_1 - 4x_2 + x_4$$
$$x_0 = 4 + x_1 - 5x_2 + x_4$$

Basic solution:

$x_0 = 4,\ x_1 = 0,\ x_2 = 0,\ x_3 = 6,\ x_4 = 0$

Auxiliary LP:

maximize $-x_0$

s.t.

$$2x_1 - x_2 - x_0 \le 2$$
$$x_1 - 5x_2 - x_0 \le -4$$
$$x_0,\ x_1,\ x_2 \ge 0$$

$$z = -x_0$$
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$$z = -4 - x_1 + 5x_2 - x_4$$
$$x_3 = 6 - x_1 - 4x_2 + x_4$$
$$x_0 = 4 + x_1 - 5x_2 + x_4$$

Run SIMPLEX Algorithm

$$z = -x_0$$
$$x_2 = \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4$$
$$x_3 = \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4$$

Auxiliary LP:

maximize $-x_0$

s.t.

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$$\begin{aligned} z &= -x_0 \\ x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

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Run SIMPLEX Algorithm

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Optimal Objective Value = 0

Original LP:

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\leq 2 \\ x_1 - 5x_2 &\leq -4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

It is feasible!

Auxiliary LP:

maximize $-x_0$

s.t.

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$$\begin{aligned} z &= -4 - x_1 + 5\boxed{x_2} - x_4 \\ x_3 &= 6 - x_1 - 4x_2 + x_4 \\ \boxed{x_0} &= 4 + x_1 - 5x_2 + x_4 \end{aligned}$$

Run SIMPLEX Algorithm

Now we need to solve the Original LP. But how?

$$\begin{aligned} z &= -x_0 \\ x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$

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$$\begin{aligned} z &= -x_0 \\ x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

[redacted]

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$$z = -x_0$$

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Now we need to solve the Original LP. But how?

$$z = -x_0$$

$$\begin{aligned} x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$

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Auxiliary LP:

maximize $-x_0$

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\le 2 \\ x_1 - 5x_2 - x_0 &\le -4 \\ x_0, x_1, x_2 &\ge 0 \end{aligned}$$

$\Rightarrow$

$$z = -x_0$$

$$\begin{aligned} x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$\Rightarrow$

$$z = -x_0$$

$$\begin{aligned} x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$

Original LP:

maximize $2x_1 - x_2$

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$$\begin{aligned} 2x_1 - x_2 &\le 2 \\ x_1 - 5x_2 &\le -4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Auxiliary LP:

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{s.t.} \quad & 2x_1 - x_2 - x_0 \leq 2 \\ & x_1 - 5x_2 - x_0 \leq -4 \\ & x_0, x_1, x_2 \geq 0 \end{aligned}$$

$$z = -x_0$$

$$\begin{aligned} x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$$z = -x_0$$

$$\begin{aligned} x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$

Original LP:

$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \leq 2 \\ & x_1 - 5x_2 \leq -4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$z = 2x_1 - x_2$$

$$\begin{aligned} x_3 &= 2 - 2x_1 + x_2 \\ x_4 &= -4 - x_1 + 5x_2 \end{aligned}$$

Auxiliary LP:

maximize $-x_0$

s.t.

$$
\begin{aligned}
2x_1 - x_2 - x_0 &\le 2 \\
x_1 - 5x_2 - x_0 &\le -4 \\
x_0, x_1, x_2 &\ge 0
\end{aligned}
$$

$$z = -x_0$$

$$
\boxed{
\begin{aligned}
x_3 &= 2 + x_0 - 2x_1 + x_2 \\
x_4 &= -4 + x_0 - x_1 + 5x_2
\end{aligned}
}
$$

$$z = -x_0$$

$$
\boxed{
\begin{aligned}
x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\
x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4
\end{aligned}
}
$$

The constraints are equivalent when $x_0 = 0$

Original LP:

maximize $2x_1 - x_2$

s.t.

$$
\begin{aligned}
2x_1 - x_2 &\le 2 \\
x_1 - 5x_2 &\le -4 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

$$z = 2x_1 - x_2$$

$$
\boxed{
\begin{aligned}
x_3 &= 2 - 2x_1 + x_2 \\
x_4 &= -4 - x_1 + 5x_2
\end{aligned}
}
$$

Auxiliary LP:

$$
\begin{aligned}
&\text{maximize} \quad -x_0 \\
&\text{s.t.} \\
&\quad 2x_1 - x_2 - x_0 \le 2 \\
&\quad x_1 - 5x_2 - x_0 \le -4 \\
&\quad x_0, x_1, x_2 \ge 0
\end{aligned}
$$

$$z = -x_0$$

$$
\begin{aligned}
x_3 &= 2 + x_0 - 2x_1 + x_2 \\
x_4 &= -4 + x_0 - x_1 + 5x_2
\end{aligned}
$$

$$z = -x_0$$

$$
\begin{aligned}
x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\
x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4
\end{aligned}
$$

The constraints are equivalent when $x_0 = 0$

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Original LP:

$$
\begin{aligned}
&\text{maximize} \quad 2x_1 - x_2 \\
&\text{s.t.} \\
&\quad 2x_1 - x_2 \le 2 \\
&\quad x_1 - 5x_2 \le -4 \\
&\quad x_1, x_2 \ge 0
\end{aligned}
$$

$$z = 2x_1 - x_2$$

$$
\begin{aligned}
x_3 &= 2 - 2x_1 + x_2 \\
x_4 &= -4 - x_1 + 5x_2
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\
x_3 &= \frac{14}{5} - \frac{9}{5}x_1 + \frac{1}{5}x_4
\end{aligned}
$$

Original LP:

maximize $2x_1 - x_2$

s.t.

$$\begin{aligned} 2x_1 - x_2 &\le 2 \\ x_1 - 5x_2 &\le -4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$z = 2x_1 - x_2$$

$$\boxed{\begin{aligned} x_3 &= 2 - 2x_1 + x_2 \\ x_4 &= -4 - x_1 + 5x_2 \end{aligned}}$$

$$\begin{aligned} z &= 2x_1 - x_2 \\ &= 2x_1 - \left(\frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4\right) \\ &= -\frac{4}{5} + \frac{9}{5}x_1 - \frac{1}{5}x_4 \end{aligned}$$

$$\boxed{\begin{aligned} x_2 &= \frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}}$$

This slack-form LP
has a feasible basic solution.

Now we can solve it using
the SIMPLEX Algorithm.

Original LP:

$$
\begin{array}{ll}
\text{maximize} & 2x_1 - x_2 \\
\text{s.t.} & \\
& 2x_1 - x_2 \leq 2 \\
& x_1 - 5x_2 \leq -4 \\
& x_1, x_2 \geq 0
\end{array}
$$

$$z = 2x_1 - x_2$$

$$
\boxed{
\begin{aligned}
x_3 &= 2 - 2x_1 + x_2 \\
x_4 &= -4 - x_1 + 5x_2
\end{aligned}
}
$$

$$z = -\frac{4}{5} + \frac{9}{5}x_1 - \frac{1}{5}x_4$$

$$
\boxed{
\begin{aligned}
x_2 &= \frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\
x_3 &= \frac{14}{5} - \frac{9}{5}x_1 + \frac{1}{5}x_4
\end{aligned}
}
$$

Auxiliary LP:

maximize $-x_0$

s.t.

$$\begin{aligned} 2x_1 - x_2 - x_0 &\le 2 \\ x_1 - 5x_2 - x_0 &\le -4 \\ x_0, x_1, x_2 &\ge 0 \end{aligned}$$

$$z = -x_0$$

$$\begin{aligned} x_3 &= 2 + x_0 - 2x_1 + x_2 \\ x_4 &= -4 + x_0 - x_1 + 5x_2 \end{aligned}$$

$$z = -x_0$$

$$\begin{aligned} x_2 &= \frac{4}{5} - \frac{1}{5}x_0 + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} + \frac{4}{5}x_0 - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$

Last question: What if $x_0$ are a basic variable in these constraint equations?

Original LP:

maximize $2x_1 - x_2$

s.t.

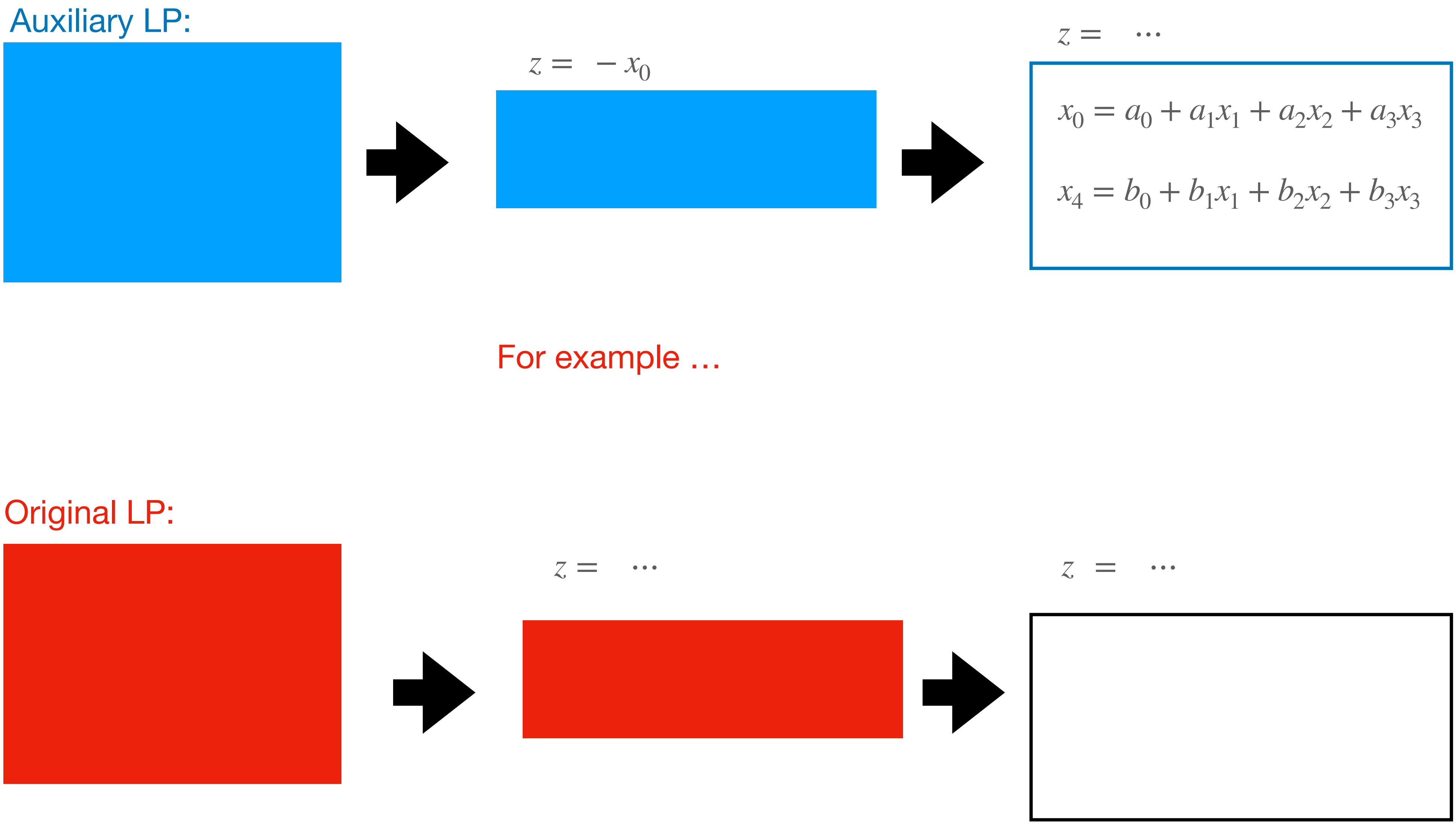
$$\begin{aligned} 2x_1 - x_2 &\le 2 \\ x_1 - 5x_2 &\le -4 \\ x_1, x_2 &\ge 0 \end{aligned}$$

$$z = 2x_1 - x_2$$

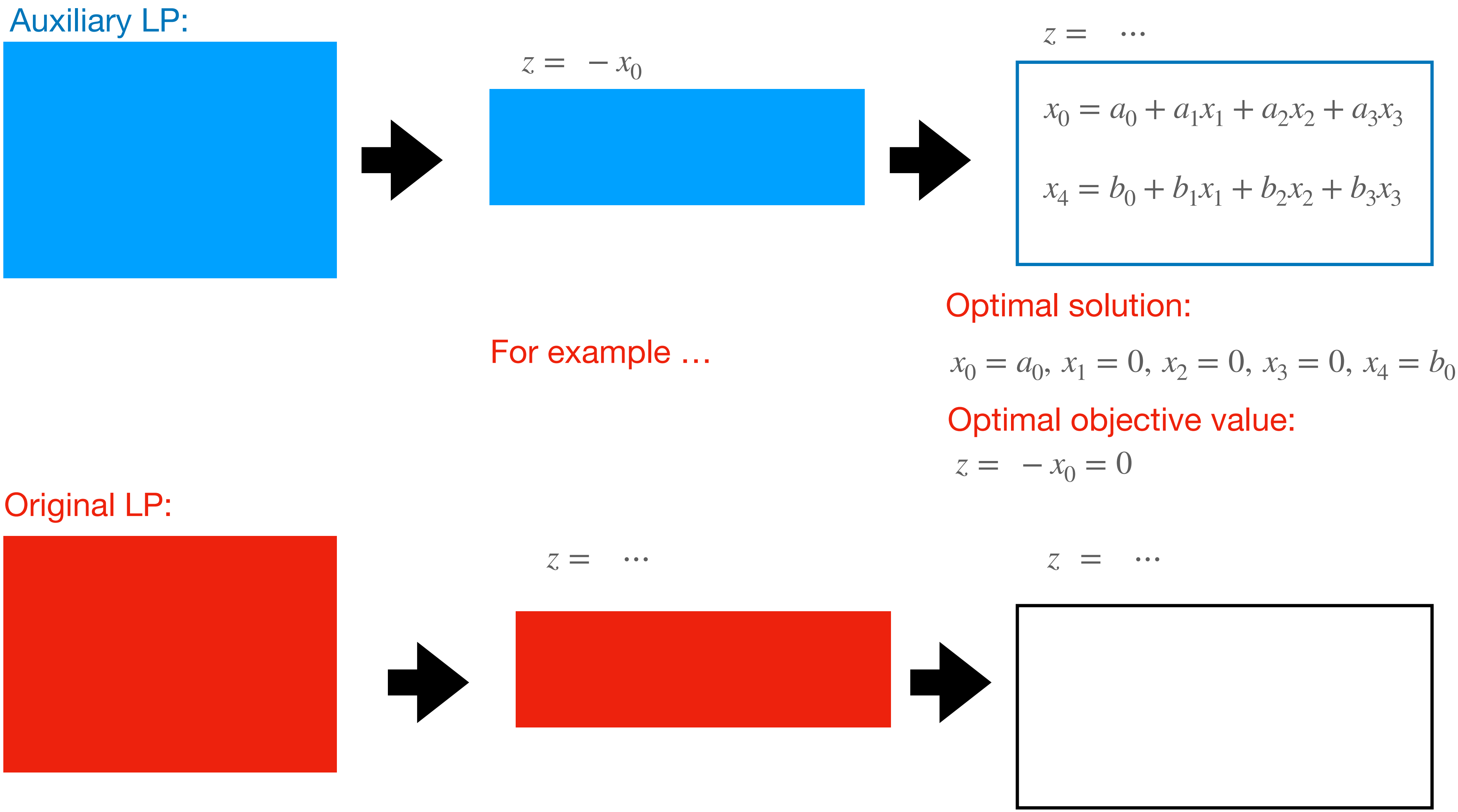
$$\begin{aligned} x_3 &= 2 - 2x_1 + x_2 \\ x_4 &= -4 - x_1 + 5x_2 \end{aligned}$$

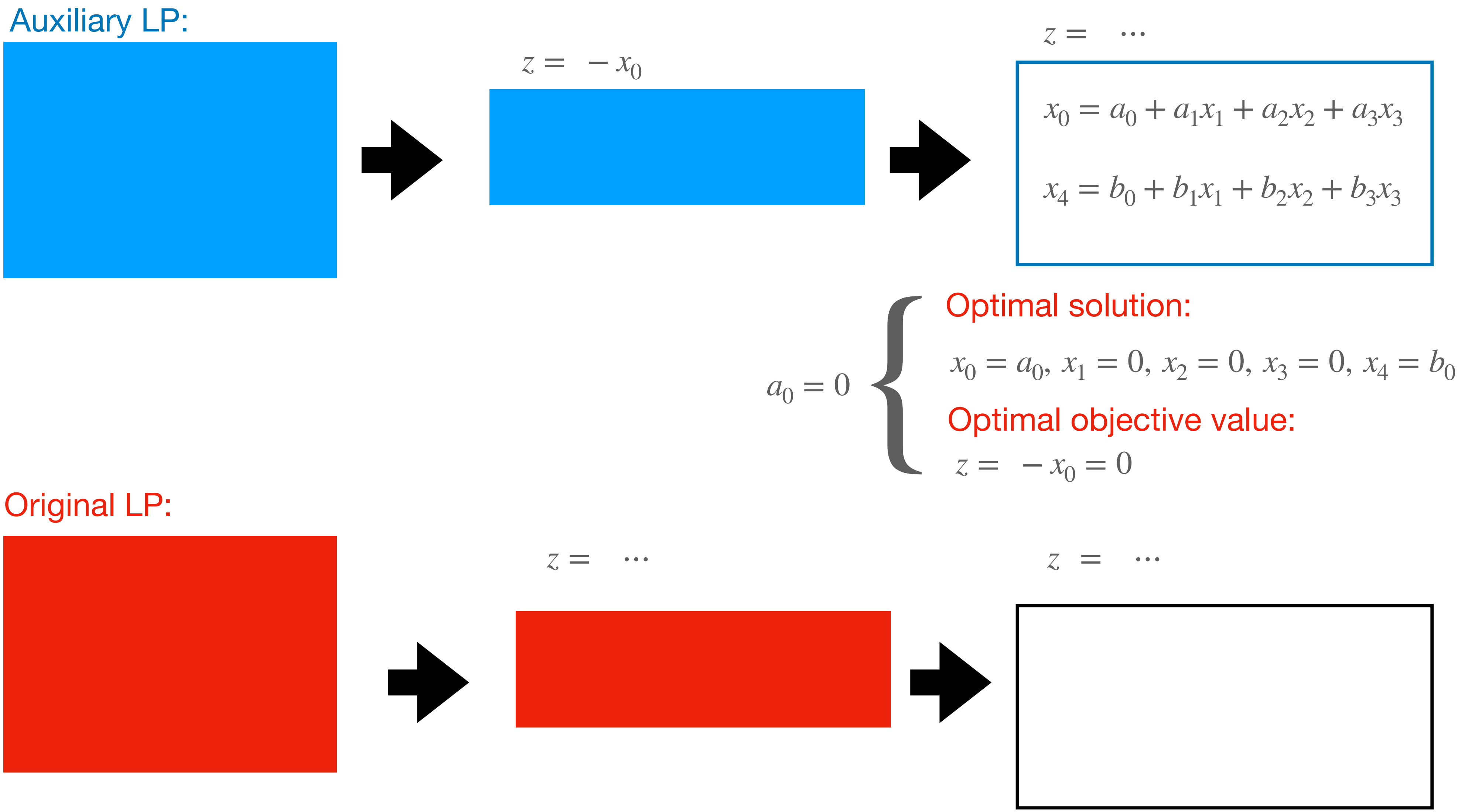
$$z = -\frac{4}{5} + \frac{9}{5}x_1 - \frac{1}{5}x_4$$

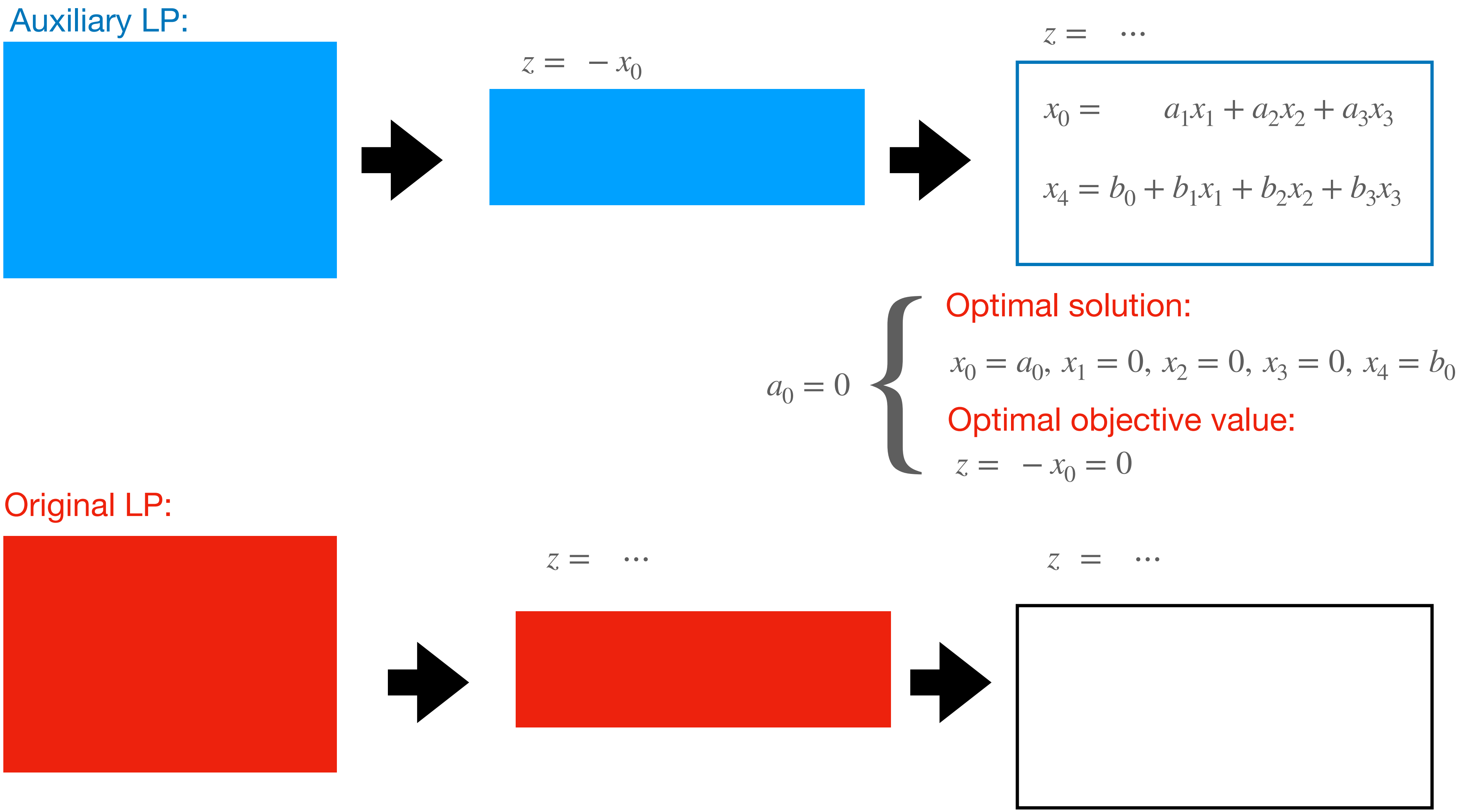
$$\begin{aligned} x_2 &= \frac{4}{5} + \frac{1}{5}x_1 + \frac{1}{5}x_4 \\ x_3 &= \frac{14}{5} - \frac{9}{5}x_1 + \frac{1}{5}x_4 \end{aligned}$$



Auxiliary LP:
$z = -x_0$
$z = \cdots$
$x_0 = a_0 + a_1x_1 + a_2x_2 + a_3x_3$
$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$
For example …
Original LP:
$z = \cdots$
$z = \cdots$


Auxiliary LP:
$z = -x_0$
$z = \cdots$
$x_0 = a_0 + a_1x_1 + a_2x_2 + a_3x_3$
$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$
Optimal solution:
For example …
$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$
Optimal objective value:
$z = -x_0 = 0$
Original LP:
$z = \cdots$
$z = \cdots$


Auxiliary LP:
$z = -x_0$
$z = \cdots$
$x_0 = a_0 + a_1x_1 + a_2x_2 + a_3x_3$
$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$
$a_0 = 0$
Optimal solution:
$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$
Optimal objective value:
$z = -x_0 = 0$
Original LP:
$z = \cdots$
$z = \cdots$



Auxiliary LP:
$z = -x_0$
$z = \cdots$
$x_0 = a_1x_1 + a_2x_2 + a_3x_3$
$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$
$a_0 = 0$
Optimal solution:
$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$
Optimal objective value:
$z = -x_0 = 0$
Original LP:
$z = \cdots$
$z = \cdots$

Auxiliary LP:

$$z = -x_0$$

$$z = \cdots$$

$$x_0 = a_1 x_1 + a_2 x_2 + a_3 x_3$$

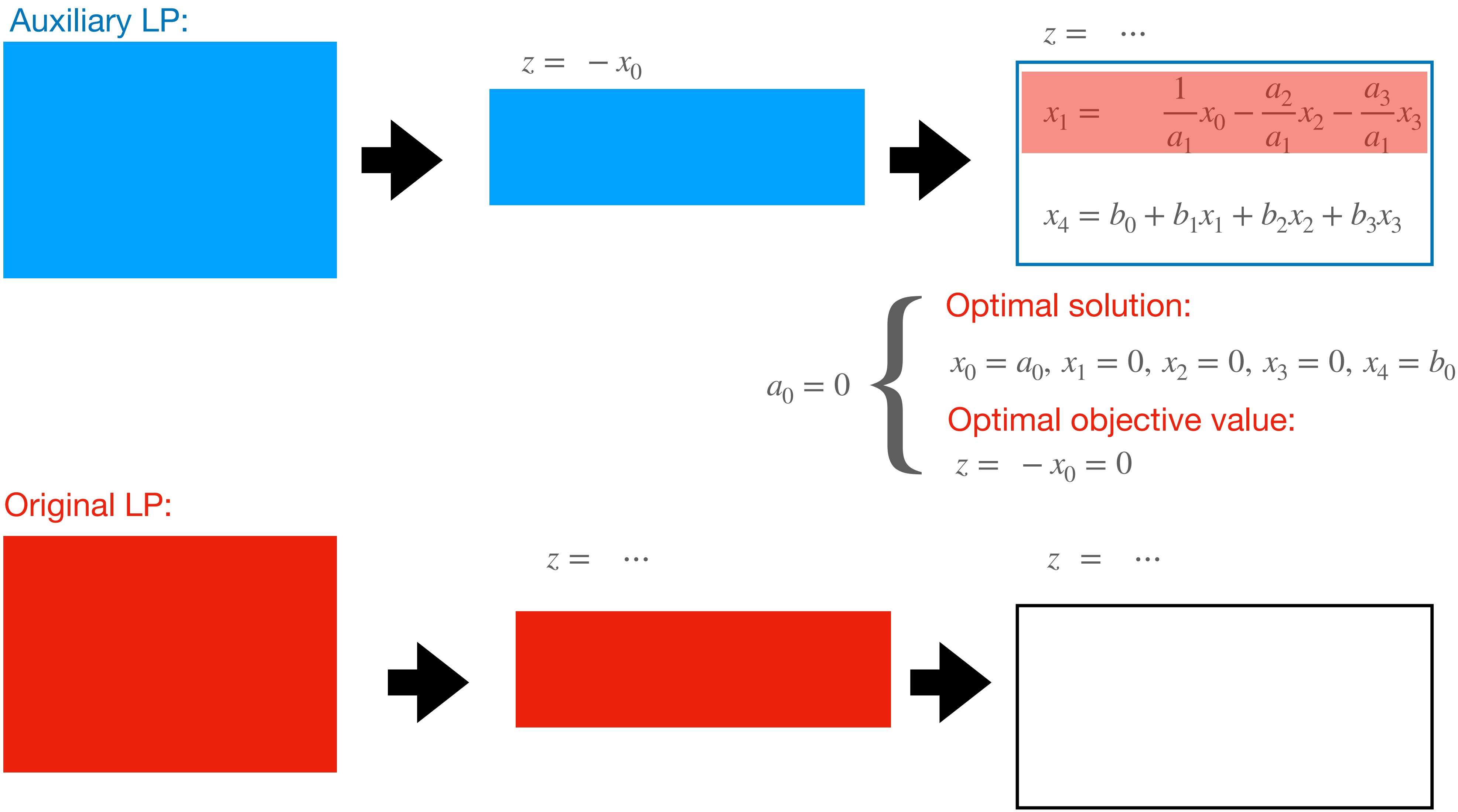
$$x_4 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$

$$a_0 = 0 \quad \begin{cases} \text{Optimal solution:} \\ x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0 \\ \text{Optimal objective value:} \\ z = -x_0 = 0 \end{cases}$$

Original LP:

$$z = \cdots$$

$$z = \cdots$$



Auxiliary LP:
$z = -x_0$
$z = \cdots$
$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$
$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$
$a_0 = 0$
Optimal solution:
$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$
Optimal objective value:
$z = -x_0 = 0$
Original LP:
$z = \cdots$
$z = \cdots$

Auxiliary LP:

$z = \; -x_0$

$z = \; \cdots$

$$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$$

$$x_4 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$$

$a_0 = 0$

Optimal solution:

$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$

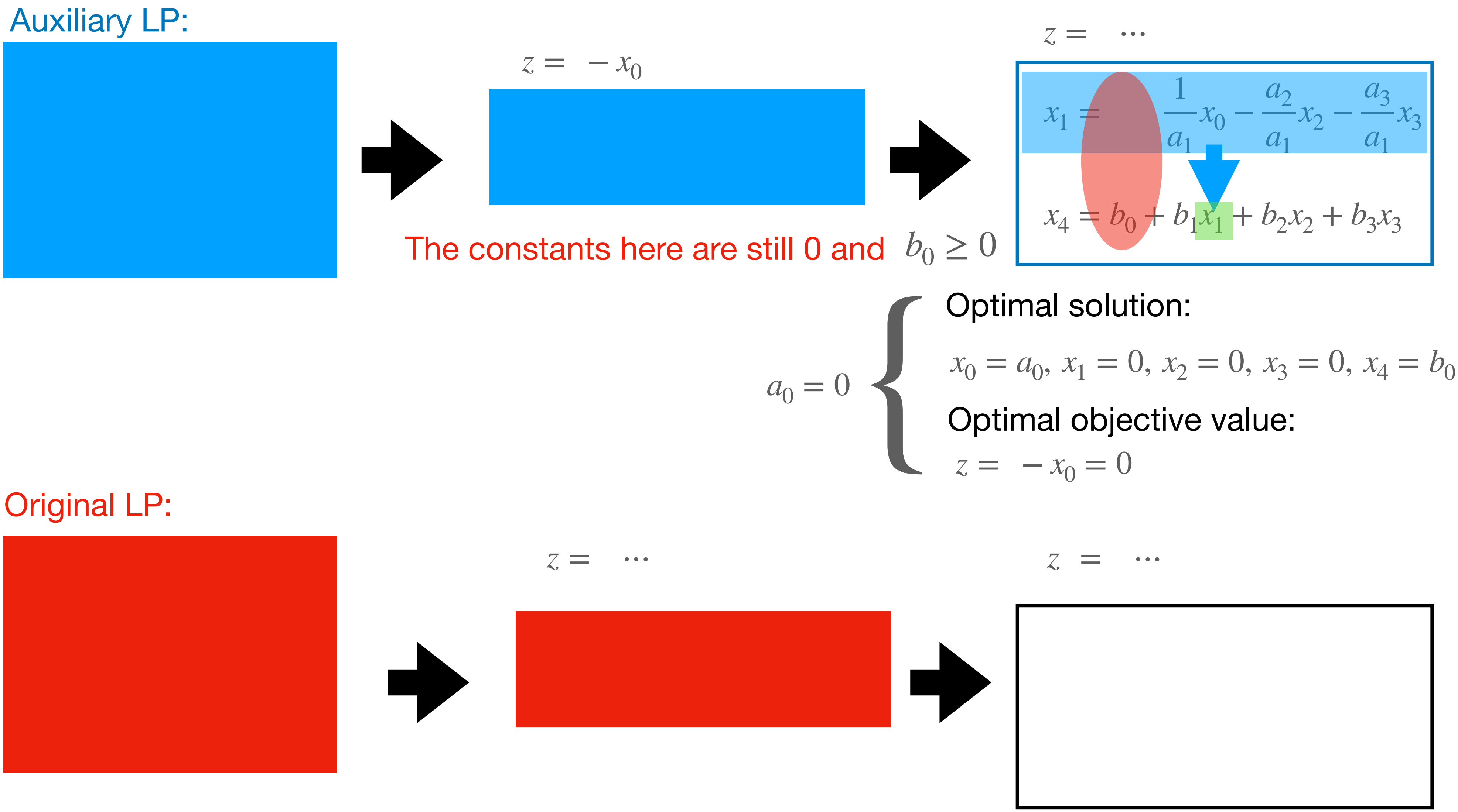
Optimal objective value:

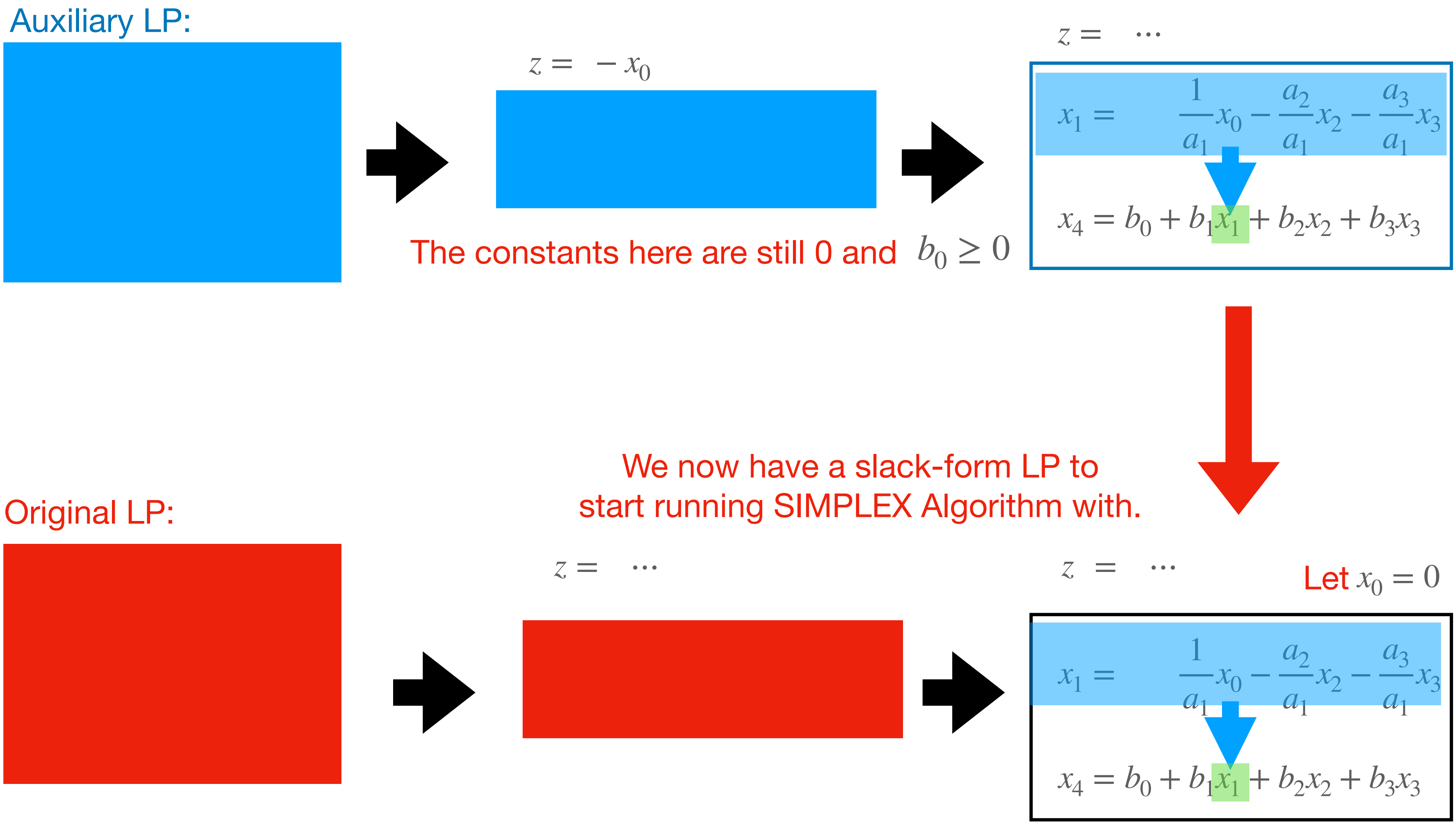
$z = \; -x_0 = 0$

Original LP:

$z = \; \cdots$

$z = \; \cdots$


Auxiliary LP:
$z = -x_0$
$z = \cdots$
$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$
$x_4 = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3$
The constants here are still 0 and $b_0 \geq 0$
$a_0 = 0$
Optimal solution:
$x_0 = a_0,\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = b_0$
Optimal objective value:
$z = -x_0 = 0$
Original LP:
$z = \cdots$
$z = \cdots$



Auxiliary LP:
$z = \ -x_0$
$z = \ \cdots$
$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$
$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$
The constants here are still 0 and $b_0 \geq 0$
We now have a slack-form LP to
start running SIMPLEX Algorithm with.
Original LP:
$z = \ \cdots$
$z = \ \cdots$
Let $x_0 = 0$
$x_1 = \frac{1}{a_1}x_0 - \frac{a_2}{a_1}x_2 - \frac{a_3}{a_1}x_3$
$x_4 = b_0 + b_1x_1 + b_2x_2 + b_3x_3$

Is the SIMPLEX Algorithm a polynomial-time algorithm? No

But in practice it is very efficient. Why?

Can LP be solved in polynomial time? Yes