Thursday, November 4, 2021 9:30 PM

Homework 9

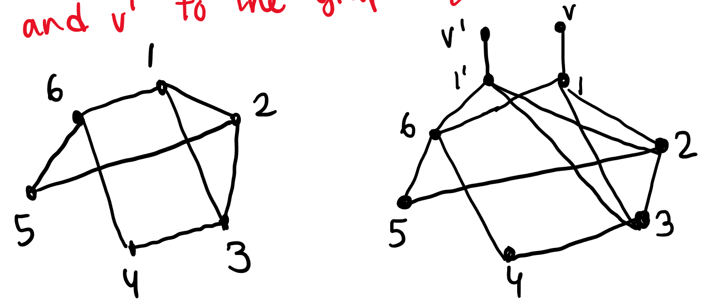
① Textbook page 1066, Exercise 34.2-6

A hamiltonian path in a graph is a simple path that visits every vertex exactly once. Show that the language HAM-path = $\{\langle G, u, v\rangle$: there is a hamiltonian path from u to v in graph G$\}$ belongs to NP.

Answer:

A hamiltonian path is a simple open path that contains each vertex in a graph exactly once. The hamiltonian path problem is the problem to determine whether a given graph contains a Hamiltonian path. To show that this problem is NP-complete we first need to show that it actually belongs to the class NP and then find a known NP-complete problem that can be reduced to an hamiltonian path. For a given graph G we can solve the Hamiltonian Path by randomly choosing edges from G that are to be included in the path. Then we traverse the path & make sure that we visit vertex EXACTLY ONCE. This obviously can be done in polynomial time, and hence, the problem belongs to NP. Now we have to find an NP-complete problem that can be reduced to a hamiltonian path. A closely related problem is the problem to determine whether a graph contains a hamiltonian cycle, that is, a hamiltonian path that begins & ends in the same vertex. Moreover, we know that the hamiltonian cycle is NP-complete, so we may try to reduce this problem to a hamiltonian path.

Given a graph $G = \langle V, E\rangle$ we construct a graph G' such that G contains a hamiltonian cycle if and only if G' contains a hamiltonian path. This is done by choosing an arbitrary vertex u in G and adding a copy, u', of it together with all its edges. Then add vertices v and v' to the graph & connect v with u and v' with u';



G G'

Suppose first that G contains a hamiltonian cycle. Then we get a hamiltonian path in G' if we start in v, follow the cycle that we got from G back to u' instead of u and finally end in v'. For example, consider the left graph, G, which contains 1, 2, 5, 6, 4, 3, 1. In G' this corresponds to the path v, 1, 2, 5, 6, 4, 3, 1', v'. Conversely, suppose G' contains a hamiltonian path. In that case, the path must have endpoints in v and v'. This path can be transformed to a cycle in G. If we disregard v and v', the path must have endpoints in u and u' and if we remove u' we get a cycle in G if we close the path back to u instead of u'.

The construction won't work when G is a single edge, so this has to be taken care of as a special case. Hence, we have shown that G contains a hamitonian cycle if and only if G' contains a hamiltonian path, which shows that the Hamiltonian path is NP Complete.

② Textbook page 1077, Exercise 34.3-2

Show that the $\le_p$ relation is a transitive relation on languages. That is, show that if $L_1 \le_P L_2$ and $L_2 \le_P L_3$, then $L_1 \le_P L_3$.

answer: Let $L_1 \le_P L_2$ and $L_2 \le_P L_3$ also known as there exists polynomial-time computable reduction functions: $f_1 : \{0,1\}^* \rightarrow \{0,1\}^*$ and $f_2 : \{0,1\}^* \rightarrow \{0,1\}^*$ such that

$x \in L_1 \Leftrightarrow f_1(x) \in L_2$

$x \in L_2 \Leftrightarrow f_2(x) \in L_3$

Define $f_3 = f_1 \circ f_2$ then L_3 is a polynomial time computable function: $\{0,1\}^* \rightarrow \{0,1\}^*$ and, $x \in L_1 \Leftrightarrow f_3(x) \in L_3$ holds.

Therefore $L_1 \le_P L_3$.