Algorithms

Lecture 16: NP Completeness (Part 2)

Anxiao (Andrew) Jiang

CH 34. NP Completeness

We focus on “Decision Problems”.

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there exists an algorithm that can determine if the answer is YES or NO in polynomial time.

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Two elements of an algorithm:

1) The process of finding a solution.
2) Verify the correctness of the solution (so it knows when to end).

Sub-case 1: answer is “YES”
Sub-case 2: answer is “NO”

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1) ~~The process of finding a solution.~~
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The constraint for NP is weaker:

Sub-case 1: answer is “YES”

~~Sub-case 2: answer is “NO”~~

$$P \subseteq NP$$

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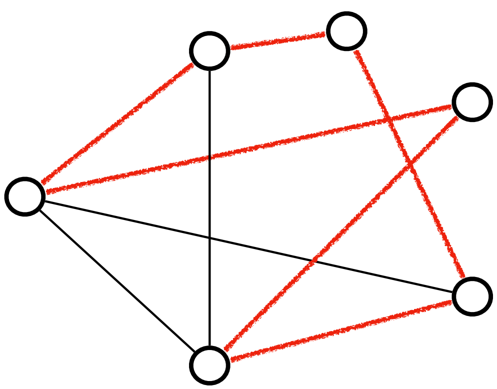
Hamiltonian cycle Problem:

Input: A graph G=(V,E).

Question: Does G have a Hamiltonian cycle?



Finding a Hamiltonian cycle might be hard ……

But given a Hamiltonian cycle, it is easy to verify it is indeed A Hamiltonian cycle.

Hamiltonian cycle Problem $\in NP$

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NP: the set of decision problems with this property: for every instance of the problem, If the answer is “YES”, there exists some data with which an algorithm can verify that the answer is indeed “YES” in polynomial time.

Co-NP: the set of decision problems with this property: for every instance of the problem, If the answer is “NO”, there exists some data with which an algorithm can verify that the answer is indeed “NO” in polynomial time.

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$P \subseteq \text{Co-NP}$

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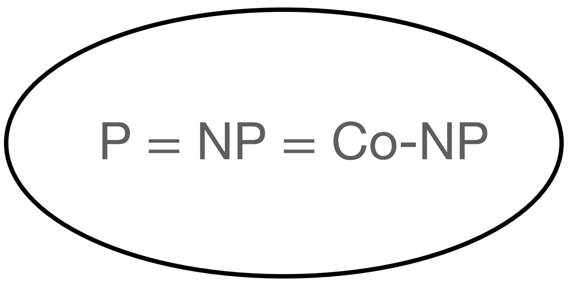
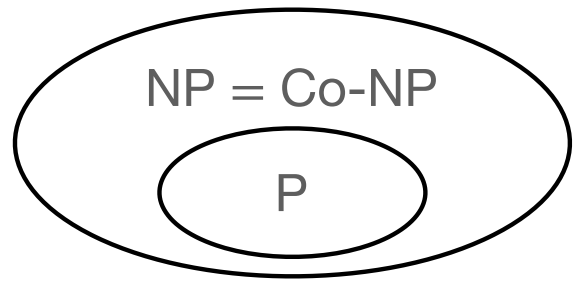
$P \subseteq NP$ $\qquad$ $P \subseteq \text{Co-NP}$

Four possible cases:

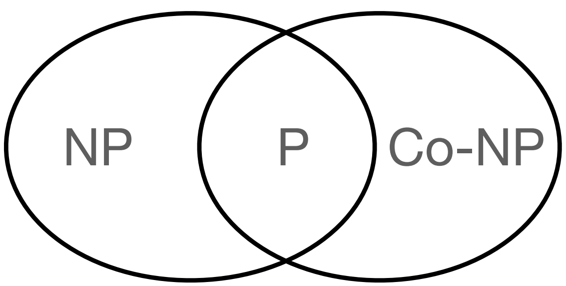
(a)



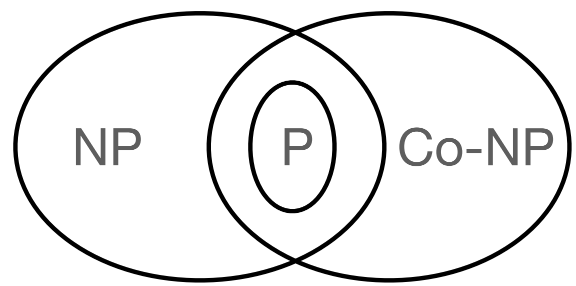
(b)



(c)



(d)



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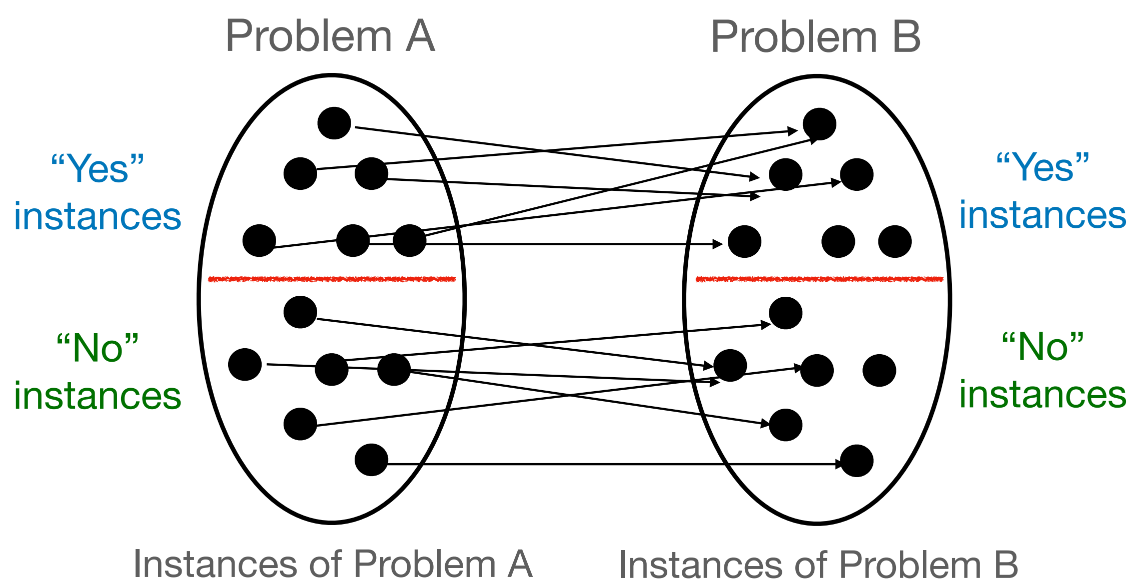
Polynomial-time reduction
from Problem A to Problem B:

$$A \leq_p B$$

Easier Harder

If Problem A is known to be "hard",
then we can use the above relation
to show that Problem B is also "hard".

Polynomial-time reduction:



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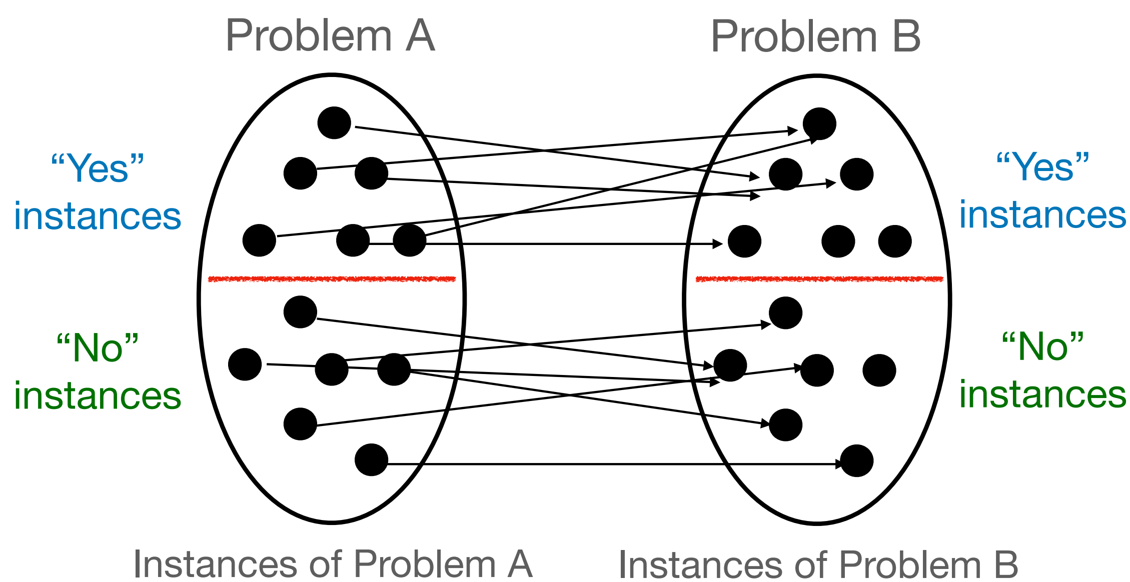
Lemma: If $A \leq_p B$, then $B \in P$ implies $A \in P$.

Proof: We can use the reduction, to solve A through solving B, all in polynomial time.

Polynomial-time reduction:



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NP-complete definition:

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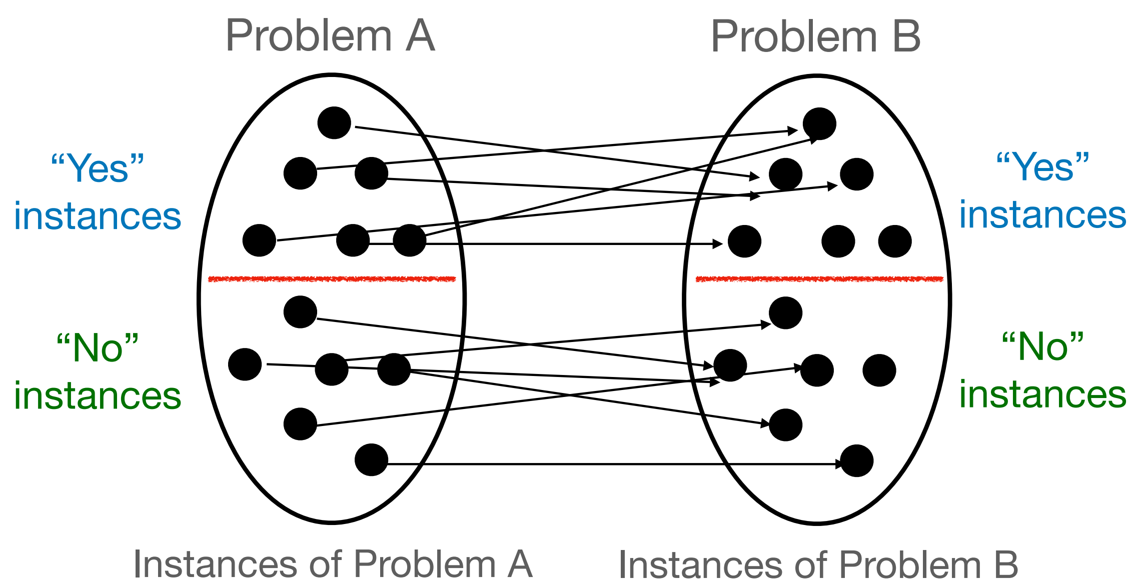

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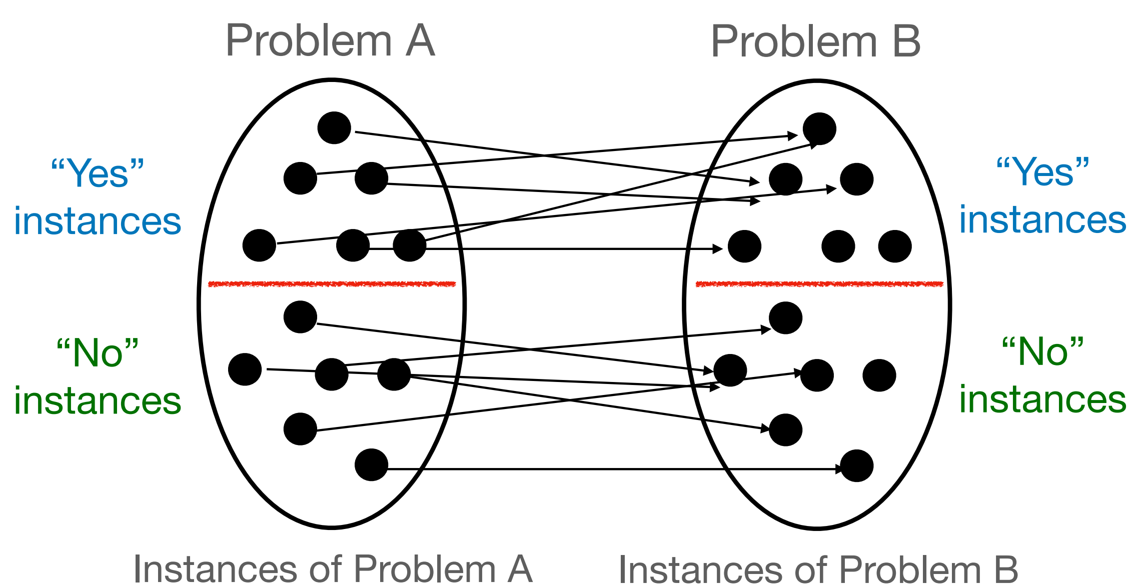

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Theorem: If any NP-complete (NPC) problem can be solved in polynomial time, then P=NP.

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Theorem: If any NP-complete (NPC) problem can be solved in polynomial time, then P=NP.

If $P \neq NP$, then no NPC problem can be solved in polynomial time.

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If only condition 2 is satisfied,
L is called "NP-hard".

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How to prove a problem L is NP-complete ?

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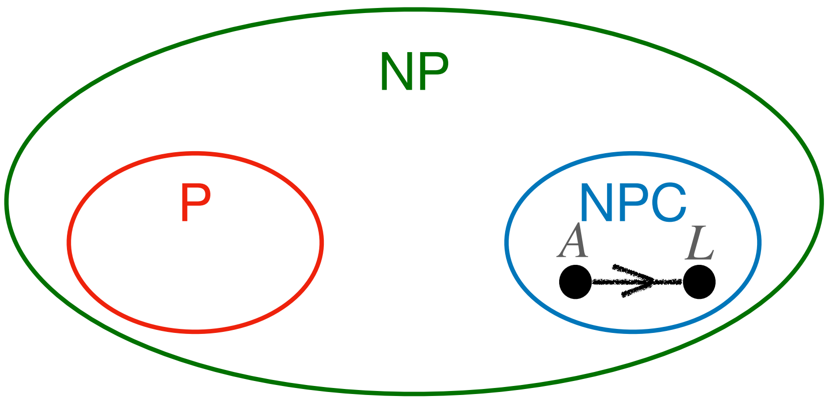

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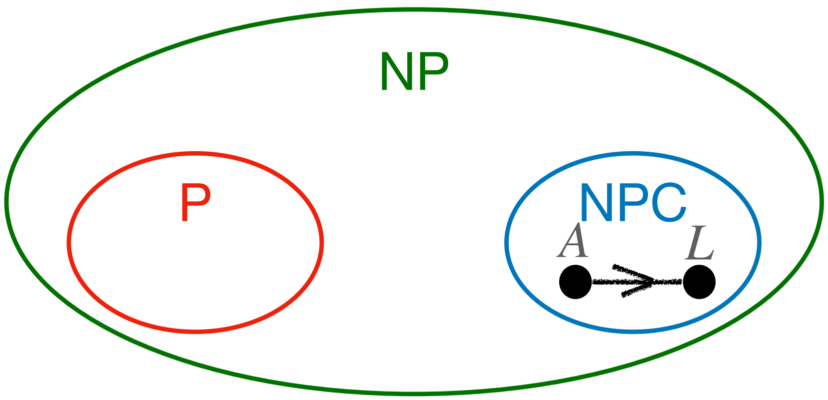

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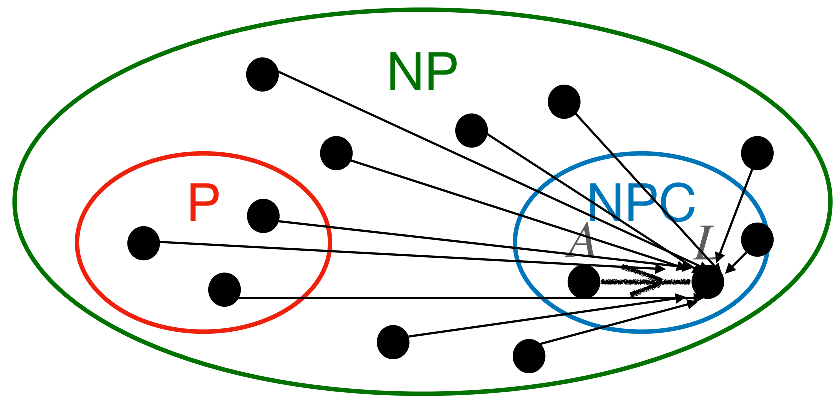

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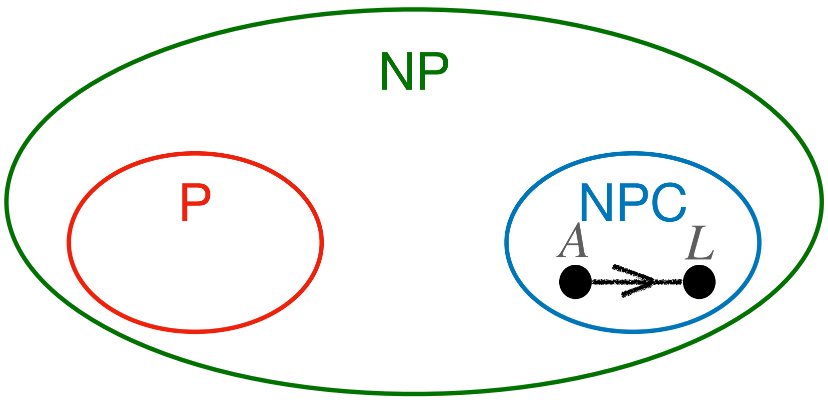

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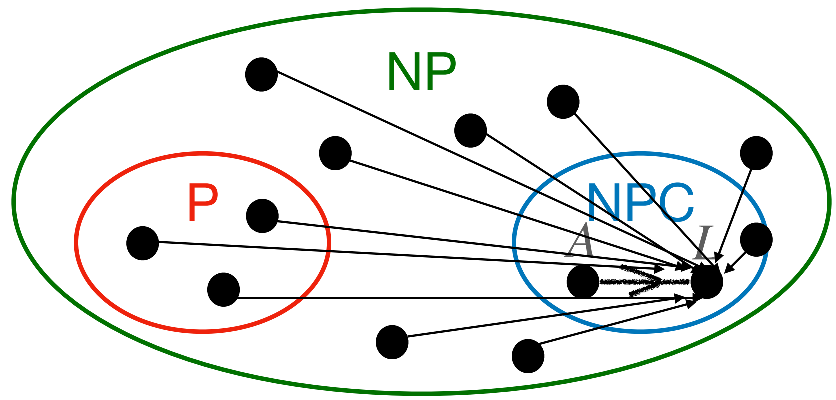

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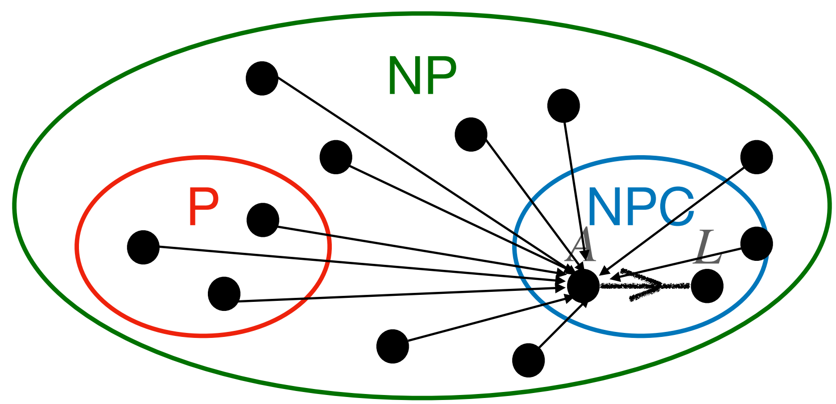
What we do:



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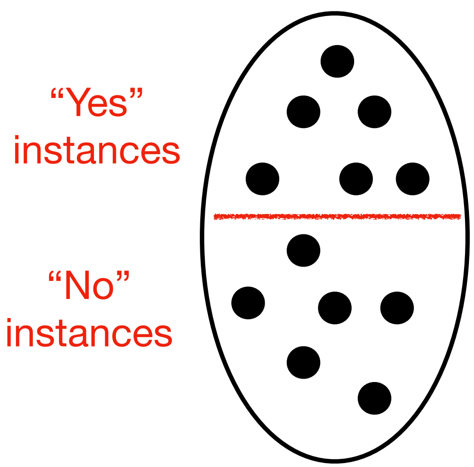
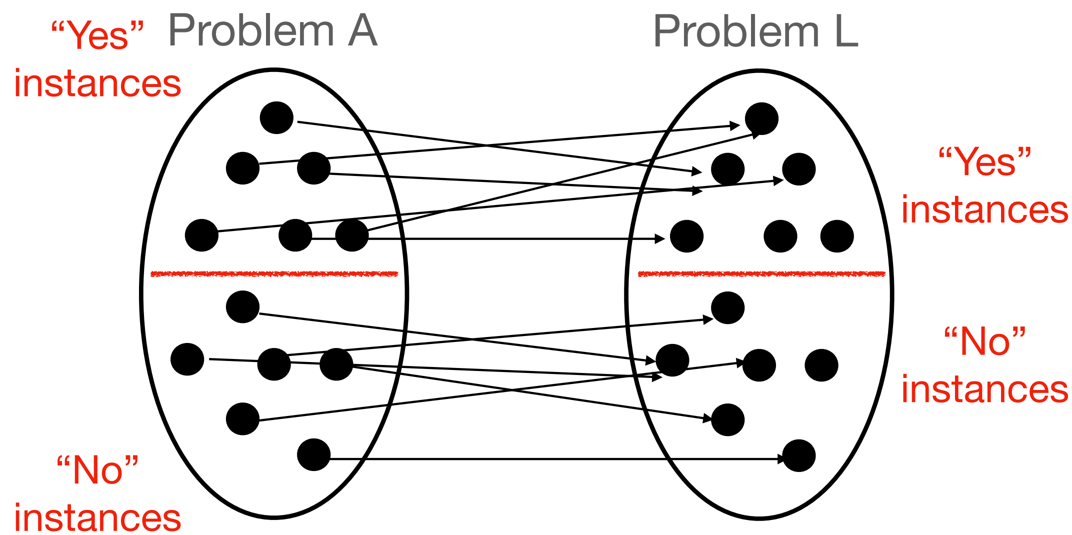
Proof:

Polynomial-time reduction
From A to L:

Any Problem B $\in NP$



Instances of Problem B



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Instances of Problem L

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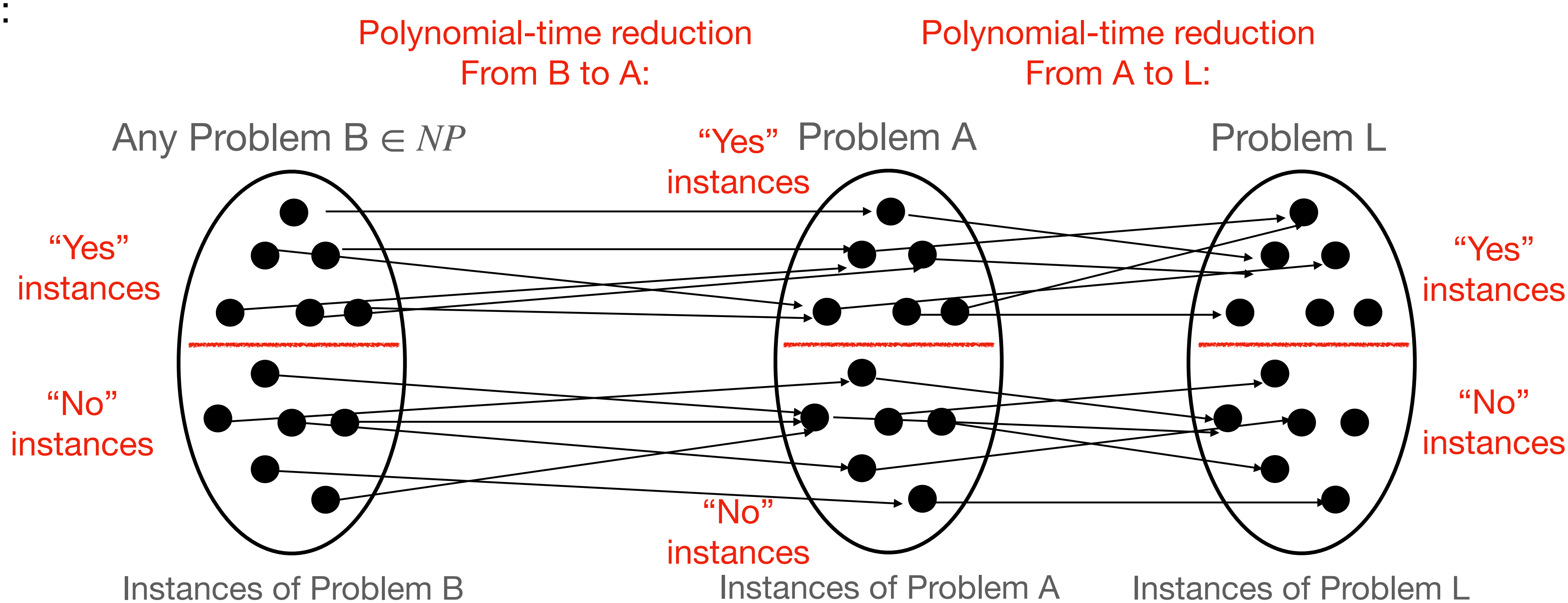

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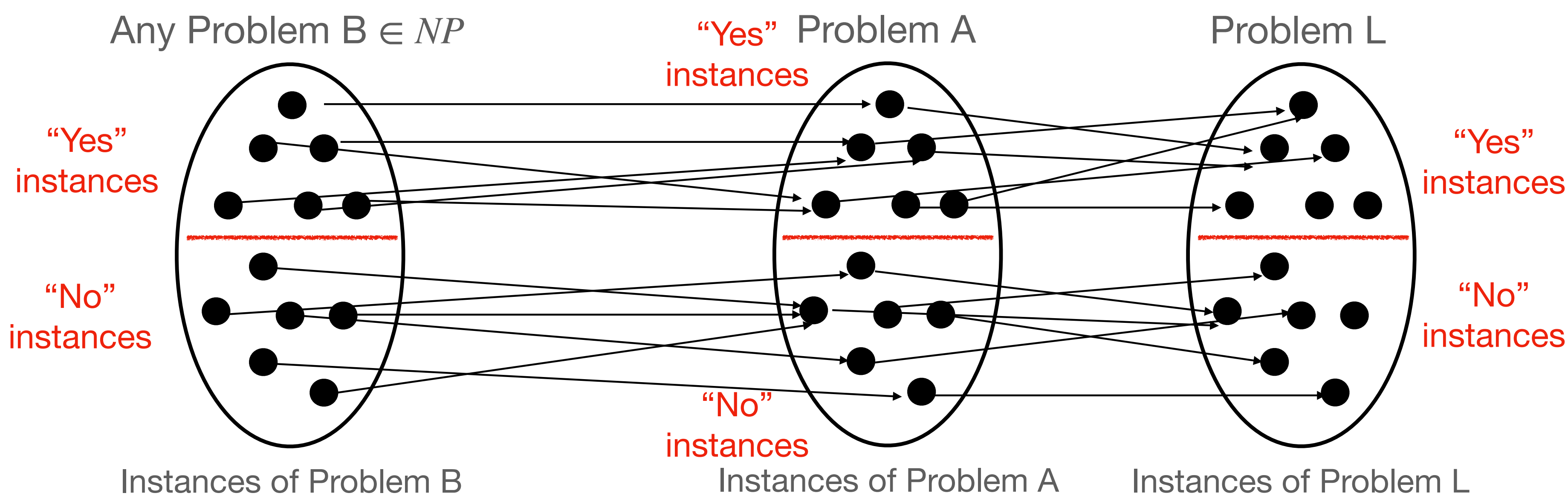
Polynomial-time reduction from B to L:



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Standard technique for proving NP-completeness today

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