Algorithms

Lecture 20: Approximation Algorithms (Part 2)

Anxiao (Andrew) Jiang

CH 35. Approximation Algorithms

Traveling Salesman Problem (TSP)

Input: An undirected complete graph G=(V,E), where every edge $(u, v) \in E$ has a non-negative integer weight $w(u, v)$.

Output: A Hamiltonian cycle of minimum weight.

TSP with Triangle Inequality

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The edge weights satisfy the triangle inequality.

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Does this general TSP have any approximation algorithm?

TSP with Triangle Inequality

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The edge weights satisfy the triangle inequality.

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The TSP with triangle inequality has a polynomial-time 2-approximation algorithm.

Theorem: If $P \neq NP,$ then for any constant $\rho \geq 1,$ there is NO polynomial-time ρ-approximation algorithm for the general TSP.

Even if $\rho = 999^{999^{999^{999}}}$ (or any other huge number),
there still cannot exist a polynomial-time ρ-approximation algorithm for TSP,
unless $P \neq NP$ (which most people consider to be unlike).

Theorem: If $P \neq NP$, then for any constant $\rho \geq 1$, there is NO polynomial-time ρ-approximation algorithm for the general TSP.

How to prove it? A hint: if we let $\rho = 1$,

the theorem becomes:

If $P \neq NP$, then TSP has no polynomial-time 1-approximation algorithm.

$\Updownarrow$

If $P \neq NP$, then TSP is not polynomial-time solvable.

$\Uparrow$

$TSP \in NPC$.

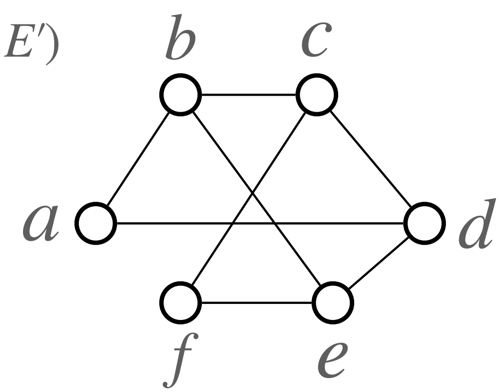
So the above theorem can actually be a generalization of proving TSP to be NP-Complete.

Theorem: If $P \neq NP$, then for **any constant** $\rho \geq 1$, there is NO polynomial-time ρ-approximation algorithm for the general TSP.

Proof:

Hamiltonian Cycle Problem:

$G' = (V, E')$



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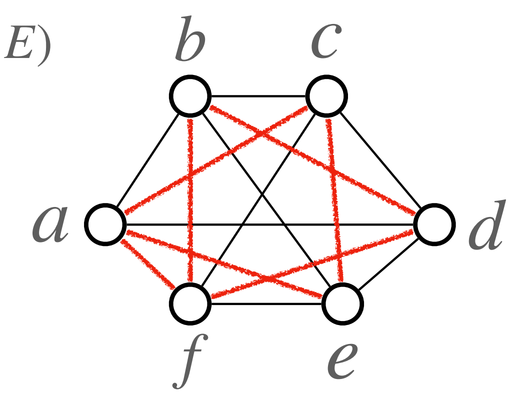

Black edges: weight 1

Red edges: weight $\rho |V| + 1$

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Proof: By contradiction: assume the ρ-approximation algorithm exists.

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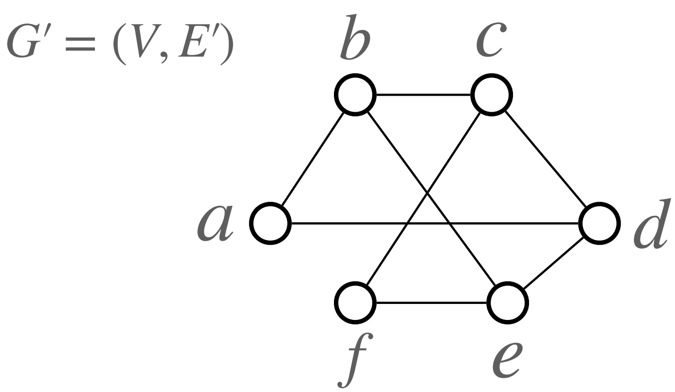

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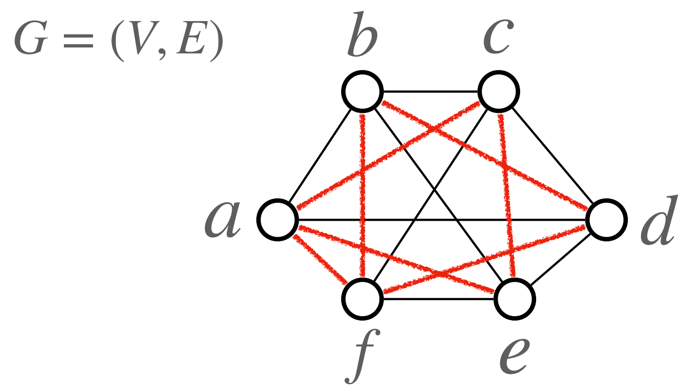

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G' has a Hamiltonian cycle → G has a Hamiltonian cycle of weight |V| → Using the ρ-approximation algorithm, we can find a Hamiltonian cycle of weight $\leq \rho |V|$ in polynomial time.

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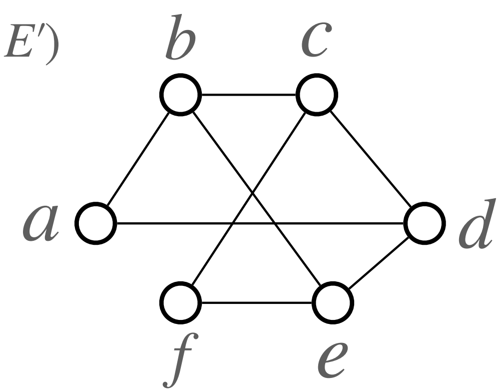

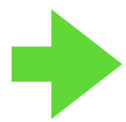

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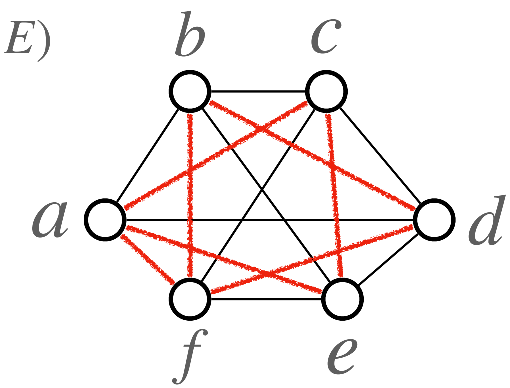

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G' has a Hamiltonian cycle ➡ G has a Hamiltonian cycle of weight |V| ➡ Using the ρ-approximation algorithm, we can find a Hamiltonian cycle of weight $\leq \rho |V|$ in G in polynomial time.

G' has no Hamiltonian cycle ➡ Any Hamiltonian cycle in G needs to use at least one red edge ➡ Any Hamiltonian cycle in G has weight $\geq \rho |V| + 1 + |V| - 1 = (\rho + 1)|V|$ ➡ Using the ρ-approximation algorithm, we find a Hamiltonian cycle of weight $\geq (\rho + 1)|V|$ in G in polynomial time.

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We can solve the Hamiltonian Cycle Problem "exactly" in polynomial time by solving TSP approximately using the polynomial-time ρ-approximation algorithm:

1) If we find a Hamiltonian cycle of weight $\leq \rho |V|$ in G → G' has a Hamiltonian cycle

2) If we find a Hamiltonian cycle of weight $\geq (\rho + 1)|V|$ in G → G' has no Hamiltonian cycle

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Hamiltonian Cycle Problem: $G' = (V, E')$ $\Rightarrow$ Traveling Salesman Problem: $G = (V, E)$

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Proof: By contradiction: assume the ρ-approximation algorithm exists.

We can solve the Hamiltonian Cycle Problem "exactly" in polynomial time.

But we know the Hamiltonian Cycle Problem in NP-complete.

So it has to be

$$P = NP$$

which is a contradiction.

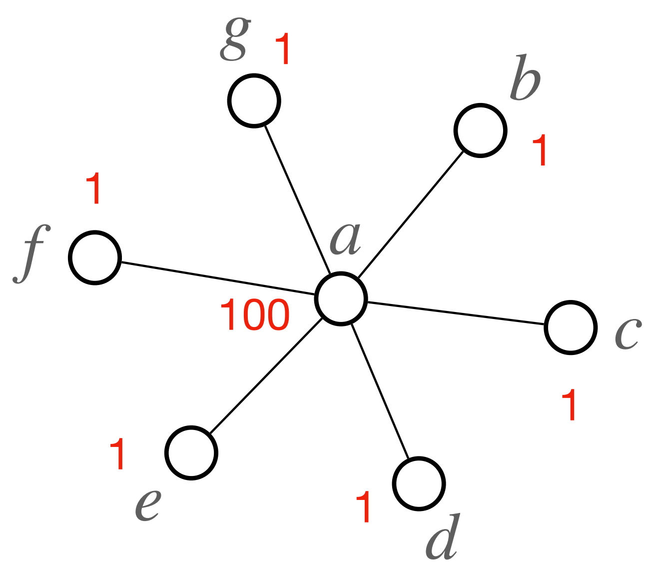
Q.E.D.

Technique: Linear Programming and Rounding for approximation

Weighted Vertex Cover Problem:

Input: An undirected graph G=(V,E), where every vertex $v \in V$ has a weight $w(v) > 0$.

Output: A vertex cover of minimum total weight.



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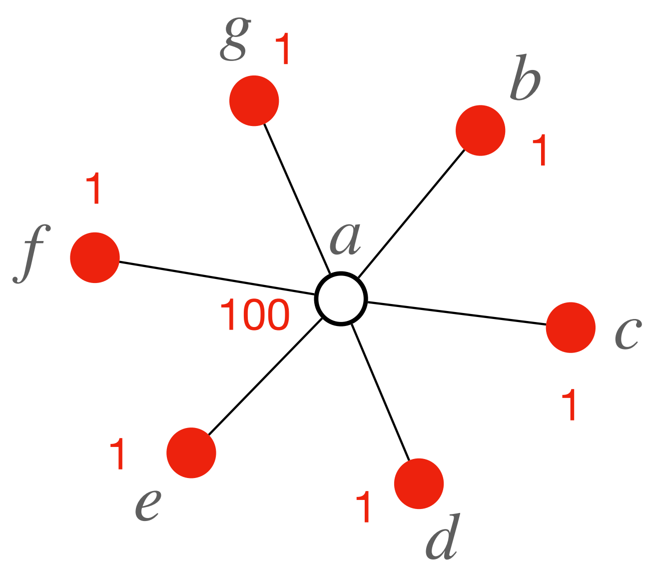
Weight of vertex cover: 6



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Our technique:

1. Formulate the problem as an integer programming problem.

Define variables:

For every node $v \in V,$ define a variable $x(v) = \begin{cases} 1 & \text{if } v \text{ is in vertex cover} \\ 0 & \text{otherwise} \end{cases}$

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Integer Programming Problem:

minimize $\sum_{v \in V} w(v)x(v)$

s.t. for every edge $(u, v) \in E,\ \ x(u) + x(v) \geq 1$

for every node $v \in V,\ \ x(v) \in \{0,1\}$

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2. Relax condition to turn it into an LP.

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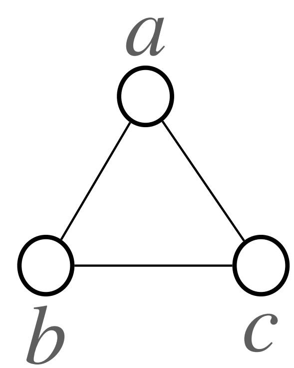

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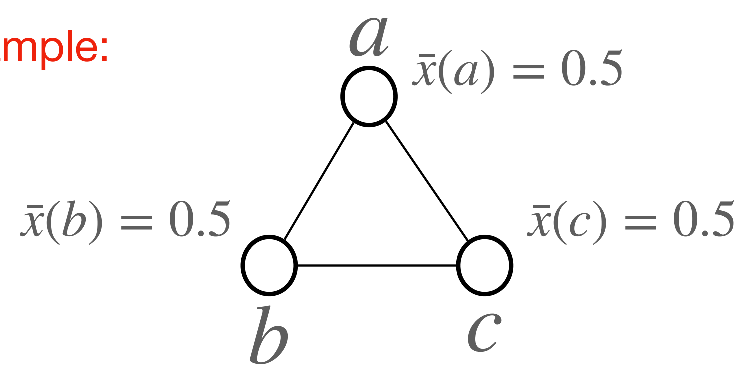

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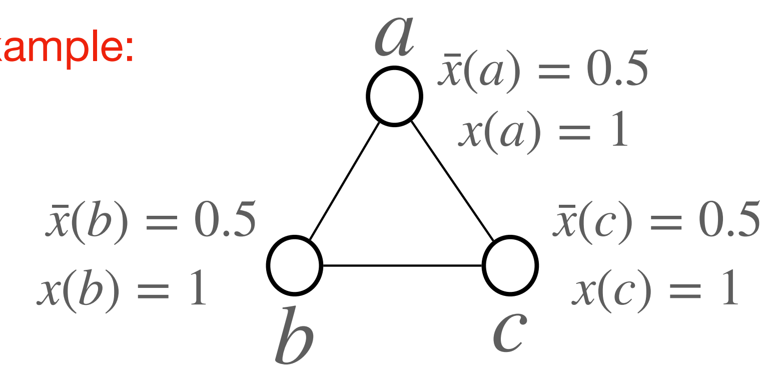

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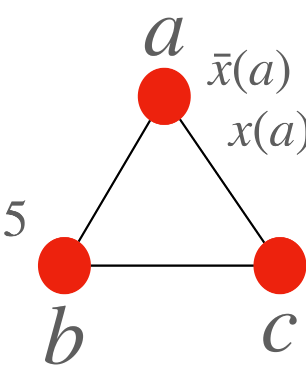

$\bar{x}(a) = 0.5$, $x(a) = 1$

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Why is this indeed a vertex cover?

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Theorem: The above algorithm is a polynomial-time 2-approximation algorithm.

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Proof: Weight of LP solution

$$\bar{C} = \sum_{v \in V} w(v)\bar{x}(v)$$

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Theorem: The above algorithm is a polynomial-time 2-approximation algorithm.

Proof: Weight of LP solution

$$\bar{C}=\sum_{v\in V} w(v)\bar{x}(v)$$

Weight of optimal vertex cover: C^*

Integer Programming Problem:

$$\begin{array}{ll} \text{minimize} & \sum_{v\in V} w(v)x(v) \\ \text{s.t.} & \forall (u,v)\in E,\ \ x(u)+x(v)\geq 1 \\ & \forall v\in V,\ \ x(v)\in\{0,1\} \end{array}$$

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Why?

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$\Rightarrow$

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Vertex Cover Problem: 2-approximation.

TSP with triangle inequality: 2-approximation.

General TSP: no constant ratio approximation unless P=NP.

Set Covering Problem: $\rho(n)$ -approximation

$\rho(n) \to \infty$ as $n \to \infty$

Subset Sum Problem: $(1+\epsilon)$ -approximation

Time complexity $\text{poly}(n, \frac{1}{\epsilon})$

CH. 35.4 Randomized Algorithm

Consider a maximization problem.

Let $C^* > 0$ be the cost of an optimal solution.

Let $C > 0$ be the expected cost of the solution of a randomized algorithm.

If for all instances, we have $\dfrac{C^*}{C} \leq \rho,$

then the randomized algorithm is called a

ρ-approximation randomized algorithm.

CH. 35.4 Randomized Algorithm

Consider a minimization problem.

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Max 3-CNF SAT Problem

Input: A 3-CNF Boolean formula of n variables and k clauses,
Where every clause is the OR of 3 literals.
The 3 variables involved in each clause are distinct.

Output: A solution to the variables that maximizes the number of satisfied clauses.

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Example of a clause

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Probability that the clause is satisfied = $\frac{7}{8}$

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Number of satisfied clauses for an optimal solution $C^* \leq k$

$$\frac{C^*}{C} \leq \frac{k}{\frac{7}{8} \cdot k} = \frac{8}{7} \approx 1.14$$