Algorithms

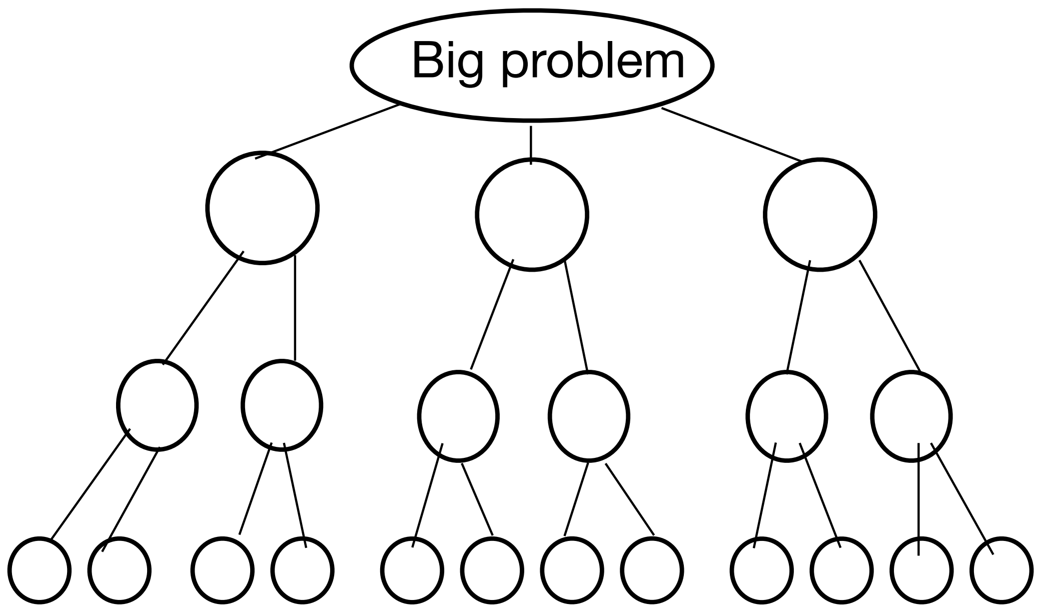
Lecture 1: Dynamic Programming

Anxiao (Andrew) Jiang

CH 15. Dynamic Programming

But first, let's recall "Divide and Conquer"

Divide and Conquer



smaller problems

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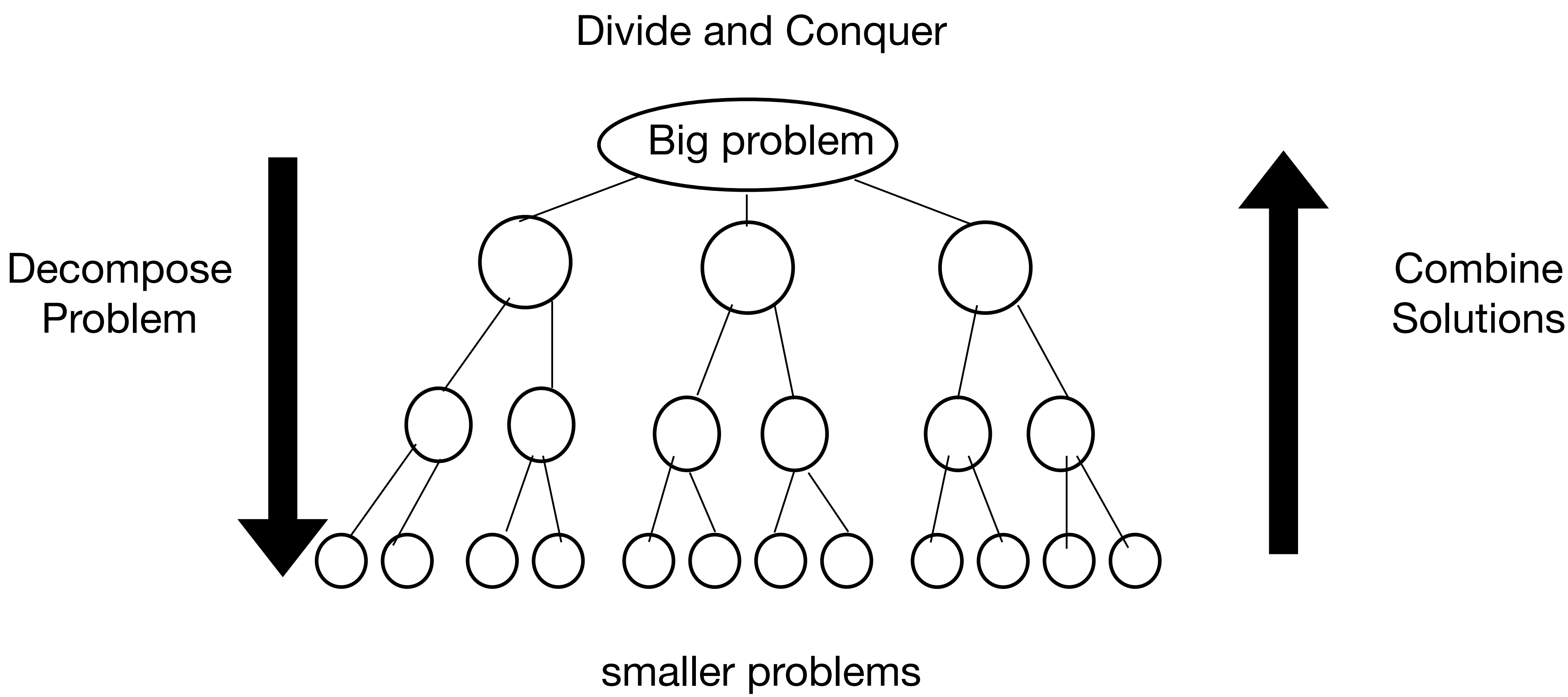
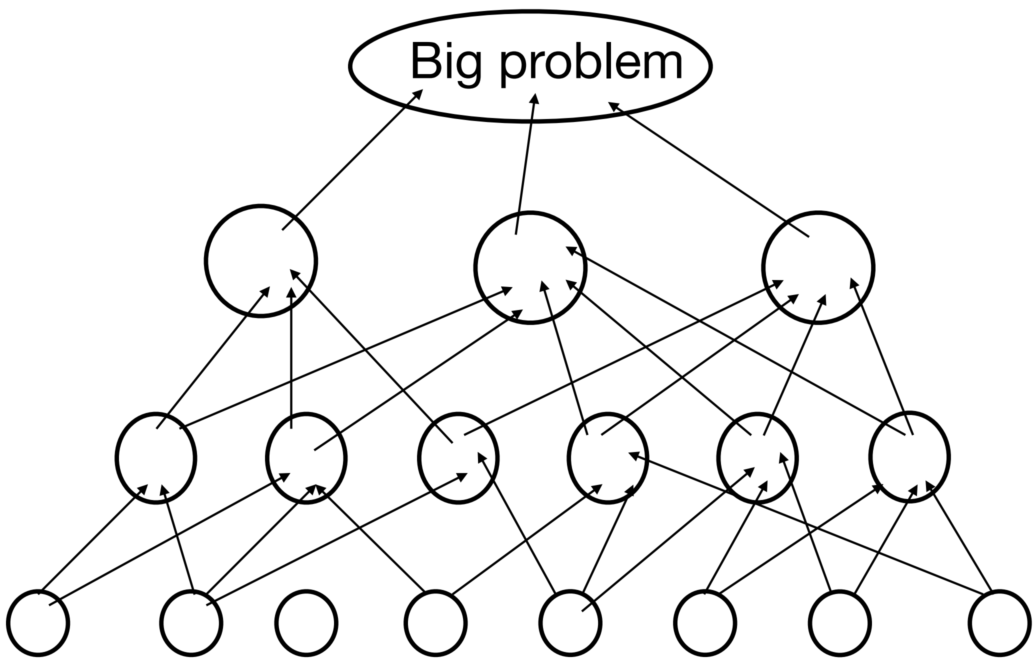


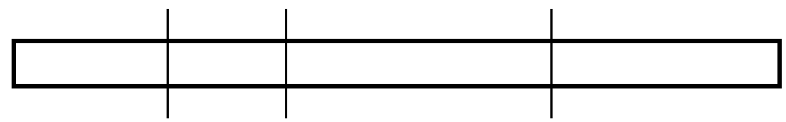
Dynamic Programming

Difference: the solution to a smaller problem can be used more than once by bigger problems.
As a result, dynamic programming can be more efficient than “divide and conquer”.



smaller problems

15.1 Rod Cutting Problem



Input: A rod of length n.

For i = 1,2,3,…,n, a rod of length i has price $p_i \geq 0$

Output: How to cut the rod to maximize the total price?

15.1 Rod Cutting Problem (Example)

n=4

i	1	2	3	4
p_i	1	5	8	9

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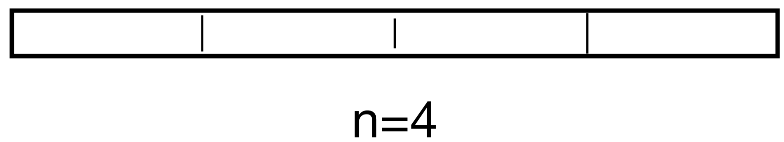

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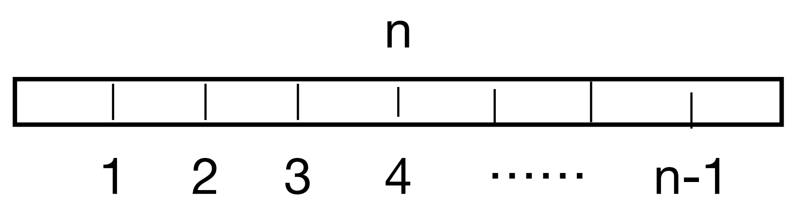

Should we use exhaustive search?

Time complexity is too high: 2^{n-1}

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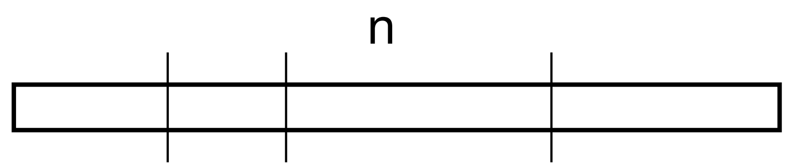

For $i = 0, 1, 2, \cdots, n$

define r_i as the maximum price for cutting a rod of length i

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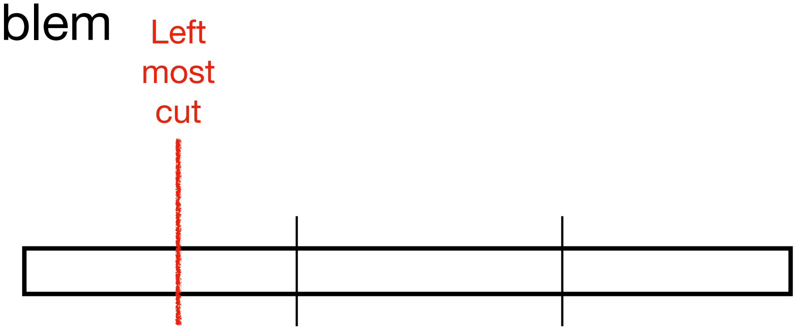

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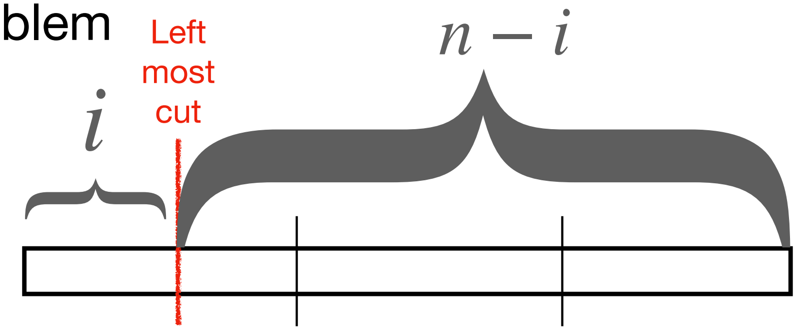

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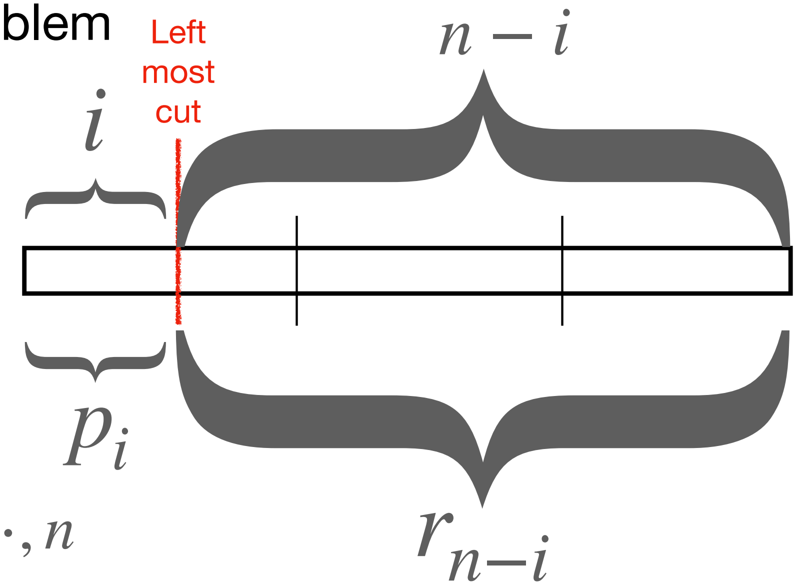

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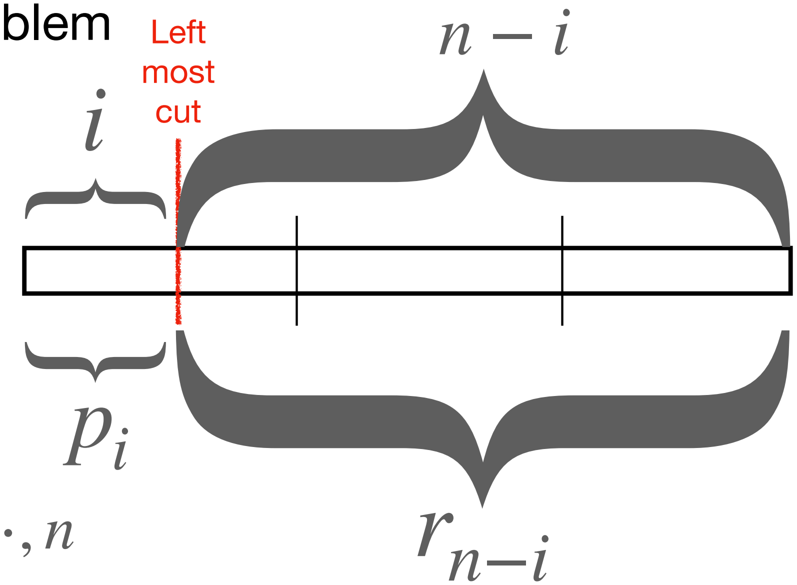

For $i = 0,1,2,\cdots,n$

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$$r_n = p_i + r_{n-i}$$

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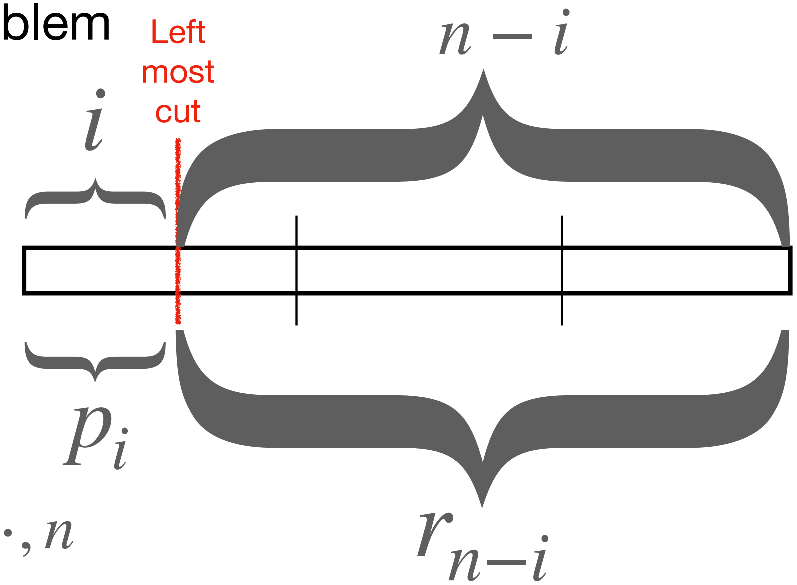

For $i = 0, 1, 2, \cdots, n$

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Recursive Function:

$$r_n = \max_{1 \le i \le n} \{p_i + r_{n-i}\}$$

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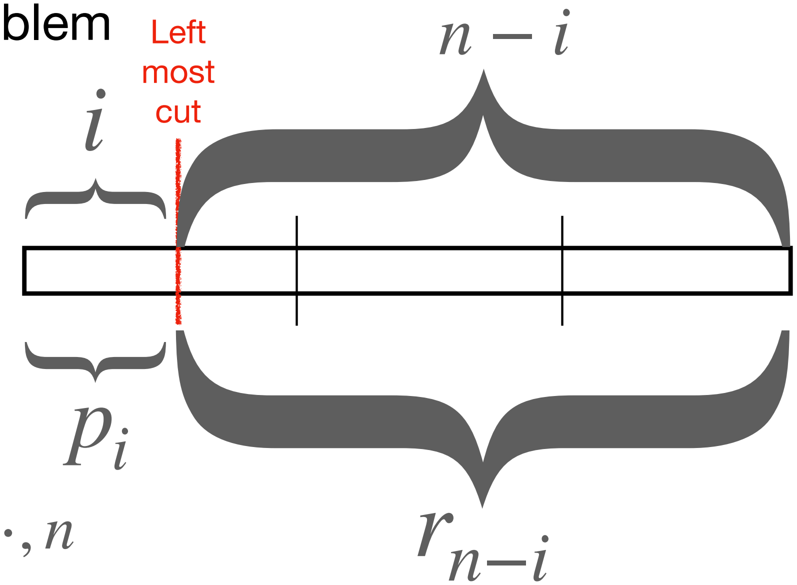

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Recursive Function:

Bigger problem $$r_n = \max_{1 \le i \le n} \{p_i + r_{n-i}\}$$ Smaller problem

The “smaller solution” will be re-used multiple times

Rod Cutting Problem:

Length i	1	2	3	4	······	n
Price p_i	p_1	p_2	p_3	p_4	······	p_n

Recursive Function: $$r_n = \max_{1 \le i \le n} \{p_i + r_{n-i}\}$$

Compute solutions bottom-up (from smaller to bigger) using the recursive function:

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Compute solutions bottom-up (from smaller to bigger) using the recursive function:

$$r_0 = 0$$
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Memorize solutions

Re-use solutions

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Rod Cutting Problem:

Length	i	1	2	3	4	······	n
Price	p_i	p_1	p_2	p_3	p_4	······	p_n

Recursive Function: $$r_n = \max_{1 \le i \le n} \{p_i + r_{n-i}\}$$ Find out where to make optimal cuts

Compute solutions bottom-up (from smaller to bigger) using the recursive function:

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Rod Cutting Problem:

Length i	1	2	3	4	······	n
Price p_i	p_1	p_2	p_3	p_4	······	p_n

Recursive Function: $$r_n = \max_{1 \le i \le n} \{p_i + r_{n-i}\} = p_{i*} + r_{n-i*}$$

First optimal cut is at length $i*$

Compute solutions bottom-up (from smaller to bigger) using the recursive function:

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For example, if n=10, and

$$r_{10} = p_3 + r_7$$

then the left-most piece has length 3

Time Complexity of an Algorithm

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How long it takes to run the algorithm……

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How long it takes to run the algorithm as a function of the input size n ……

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How long it takes to run the algorithm as a function of the input size n in the worst case.

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$$n + 3n^2 \Rightarrow O(n^2)$$

$$n \lg n + n^3 \Rightarrow O(n^3)$$

$$2^n + 100^{99} \cdot n^2 \Rightarrow O(2^n)$$

Time Complexity of the Dynamic-Programming Algorithm for Rod Cutting Problem:

Recursive Function: $$r_n = \max_{1 \le i \le n} \{p_i + r_{n-i}\}$$

Compute solutions bottom-up (from smaller to bigger) using the recursive function:

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Compute solutions bottom-up (from smaller to bigger) using the recursive function:

$$
\begin{array}{ll}
r_0 = 0 & \le O(n) \\
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\end{array}
$$

Time Complexity: $O(n^2)$

4 Essential Steps for Presenting an Algorithm
(Required for all homework and tests)

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1. Explain the idea of your algorithm **(help us understand the main idea)**

2. Pseudo-code of algorithm **(show us exactly how computing is done)**

3. Prove correctness of your algorithm **(prove rigorously it always finds the right solution)**

4. Analyze time complexity of algorithm **(show us it is efficient)**