CSCE 411 Design and Analysis of Algorithms
Sketch of Solution to In-class Test 9

Answer to Problem 1

Consider an instance of the 3-dimensional Matching Problem. We map it to a corresponding instance of the SAT Problem by the following steps:

1) For each hyper-edge (r, g, b), we create a corresponding Boolean variable $v_{r,g,b}$.
2) For every node a, say it is the endpoint node of q hyper-edges $\{e_1, e_2, \cdots, e_q\}$. We create a clause
$$v_{e_1} \vee v_{e_2} \vee \cdots \vee v_{e_q}.$$
3) For every node a, say it is the endpoint node of q hyper-edges $\{e_1, e_2, \cdots, e_q\}$. For every pair of indexes (i, j) with $1 \leq i < j \leq q$, we create a clause
$$\bar{v}_{e_i} \vee \bar{v}_{e_j}.$$
4) We create a CNF Boolean formula that connects all the clauses created above by the AND operation.

It is easy to see that the mapping above takes polynomial time. Now let's prove the mapping preserves the Yes/No answer.

First, assume the 3-dimensional Matching Problem has a Yes answer. So there is a set S of hyper-edges that form a matching. For each hyper-edge e in the graph, we set the value of the corresponding Boolean variable v_e as follows: if $e \in S$, we let $v_e = 1$; otherwise, we let $v_e = 0$. Then, in Step 2 of the mapping, the clause there must contain a literal 1 because in a matching, every node must be the endpoint node of exactly one hyper-edge in the matching. For the same reason, the clauses created in Step 3 of the mapping must also be satisfied. So the whole Boolean formula is satisfied. So the SAT Problem has a Yes answer.

Next, assume the SAT Problem has a Yes answer. We create a set S of hyper-edges this way: for each hyper-edge e in the graph, if $v_e = 1$, we add e to S; otherwise, we do not add e to S. Then Step 2 of the mapping ensures that for every node, at least one hyper-edge it is incident to is selected by S. And Step 3 of the mapping ensures that for every node, at most one hyper-edge it is incident to is selected by S. So for every node, exactly one hyper-edge it is incident to is selected by S. So S forms a matching. So the 3-dimensional Matching Problem has a Yes answer.