

# Algorithms

## Lecture 13: Linear Programming (Part 3)

Anxiao (Andrew) Jiang

## CH 29.3 SIMPLEX Algorithm

A generic step:

$$z = 27 + \frac{1}{4}x_2 + \frac{1}{2}x_3 - \frac{3}{4}x_6$$

$$x_1 = 9 - \frac{1}{4}x_2 - \frac{1}{2}x_3 - \frac{1}{4}x_6$$

$$x_4 = 21 - \frac{3}{4}x_2 - \frac{5}{2}x_3 + \frac{1}{4}x_6$$

$$x_5 = 6 - \frac{3}{2}x_2 - 4x_3 + \frac{1}{2}x_6$$

Basic solution:

$$x_1 = 9, x_2 = 0, x_3 = 0, x_4 = 21, x_5 = 6, x_6 = 0$$

Objective value = 27

SIMPLEX Algorithm:

Slack-Form LP 1

## CH 29.3 SIMPLEX Algorithm

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SIMPLEX Algorithm:

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Feasible  
Basic Solution

## CH 29.3 SIMPLEX Algorithm

A generic step:

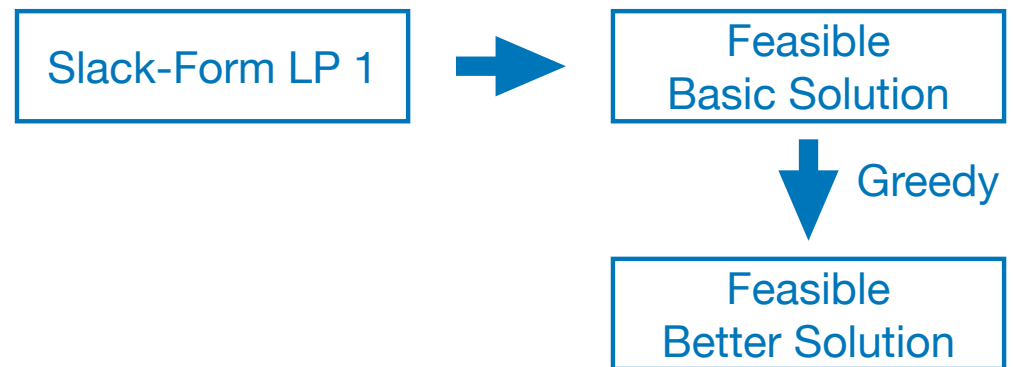
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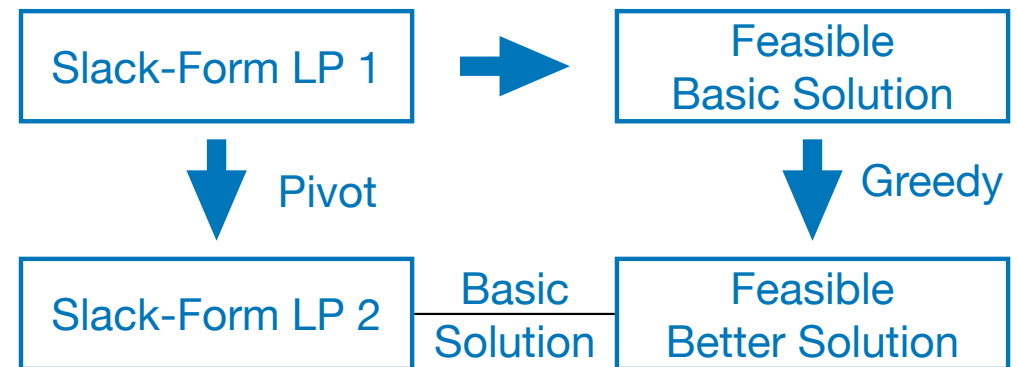
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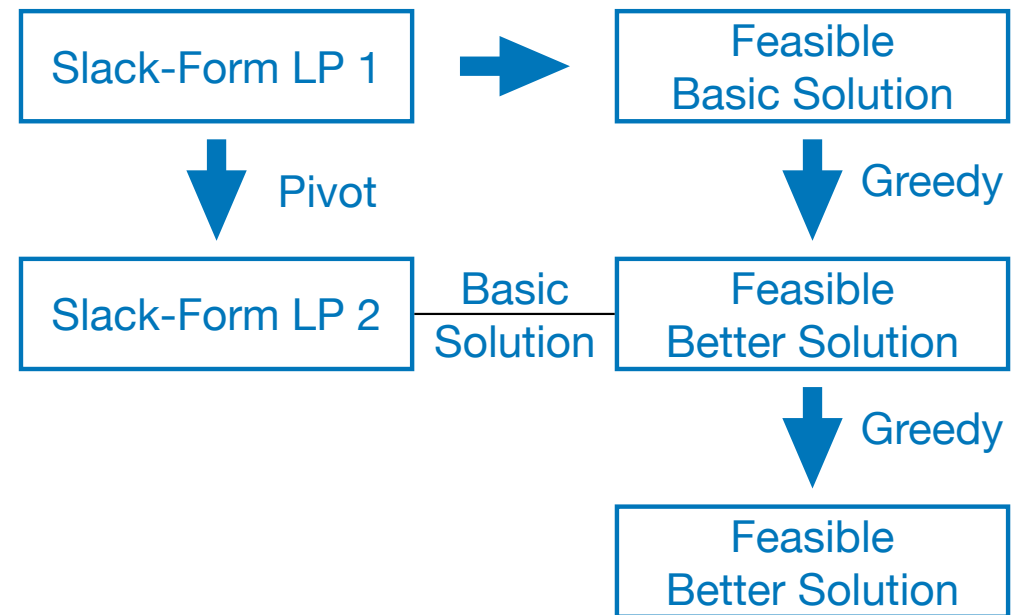
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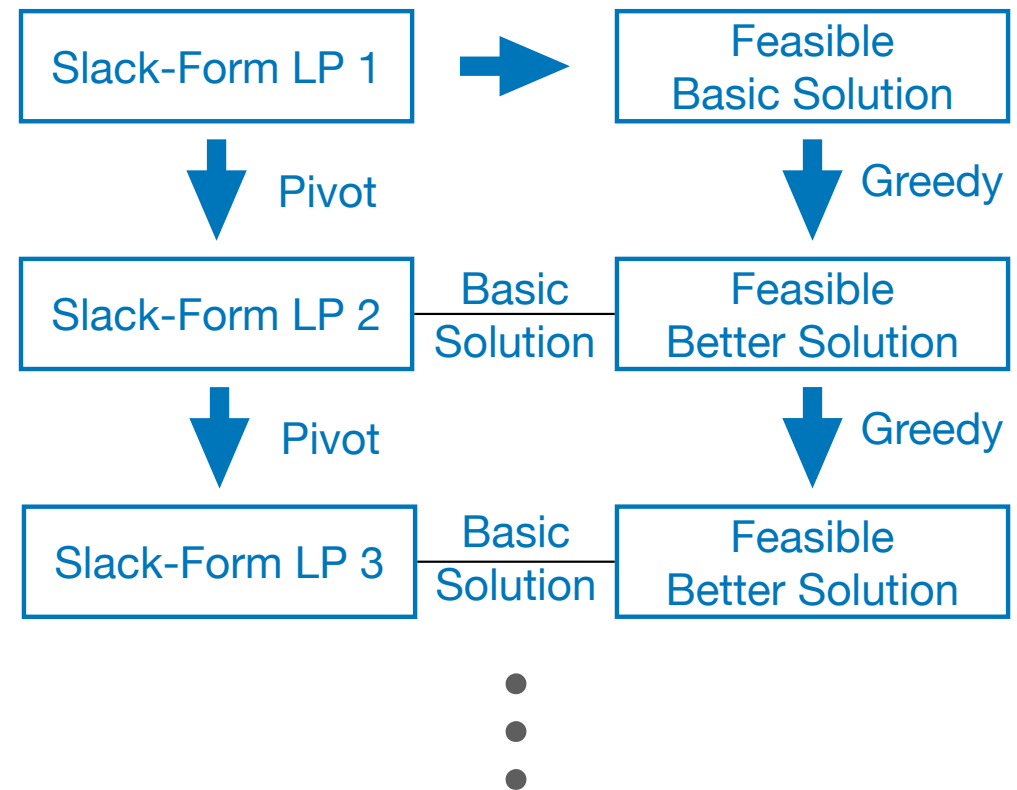
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SIMPLEX Algorithm:



## CH 29.4 Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

### Standard-Form LP (Primal LP)

maximize  $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m$$

$$x_1, x_2, \cdots, x_n \geq 0$$



## CH 29.4 Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

### Primal LP

maximize  $c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t.

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### Dual LP

minimize  $b_1y_1 + b_2y_2 + \cdots + b_my_m$

s.t.

$$a_{1,1}y_1 + a_{2,1}y_2 + \cdots + a_{m,1}y_m \geq c_1$$

$$a_{1,2}y_1 + a_{2,2}y_2 + \cdots + a_{m,2}y_m \geq c_2$$

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$$a_{1,n}y_1 + a_{2,n}y_2 + \cdots + a_{m,n}y_m \geq c_n$$

$$y_1, y_2, \cdots, y_m \geq 0$$

## CH 29.4 Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

### Primal LP

$$\begin{aligned} &\text{maximize} && c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ &\text{s.t.} && \\ &&& a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \leq b_1 \\ &&& a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \leq b_2 \\ &&& \vdots \\ &&& a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \leq b_m \\ &&& x_1, x_2, \cdots, x_n \geq 0 \end{aligned}$$

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### Primal LP

$$\begin{aligned} &\text{maximize} && 3x_1 + x_2 + 2x_3 \\ &\text{s.t.} && \\ &&& x_1 + x_2 + 3x_3 \leq 30 \\ &&& 2x_1 + 2x_2 + 5x_3 \leq 24 \\ &&& 4x_1 + x_2 + 2x_3 \leq 36 \\ &&& x_1, x_2, x_3 \geq 0 \end{aligned}$$

### Dual LP

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### Dual LP

$$\begin{aligned} &\text{minimize} && 30y_1 + 24y_2 + 36y_3 \\ &\text{s.t.} && \\ &&& y_1 + 2y_2 + 4y_3 \geq 3 \\ &&& y_1 + 2y_2 + y_3 \geq 1 \\ &&& 3y_1 + 5y_2 + 2y_3 \geq 2 \\ &&& y_1, y_2, y_3 \geq 0 \end{aligned}$$

## CH 29.4 Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

### Primal LP

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^n c_j x_j \\ &\text{s.t.} && \\ &&& \sum_{j=1}^n a_{i,j} x_j \leq b_i, \text{ for } i = 1, 2, \dots, m \\ &&& x_j \geq 0, \text{ for } j = 1, 2, \dots, n \end{aligned}$$

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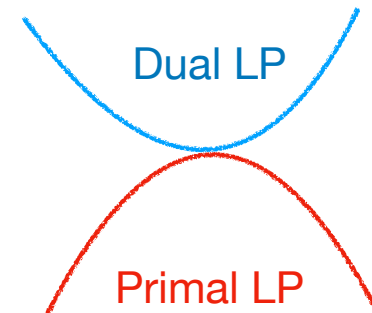
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**Theorem:** Let  $(x_1, x_2, \dots, x_n)$  be any feasible solution to Primal LP.  
Let  $(y_1, y_2, \dots, y_m)$  be any feasible solution to Dual LP.

Then

$$\sum_{j=1}^n c_j x_j \leq \sum_{i=1}^m b_i y_i$$



## CH 29.4 Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

### Primal LP

$$\begin{array}{ll}\text{maximize} & \sum_{j=1}^n c_j x_j \\ \text{s.t.} & \\ & \sum_{j=1}^n a_{i,j} x_j \leq b_i, \text{ for } i = 1, 2, \dots, m \\ & x_j \geq 0, \text{ for } j = 1, 2, \dots, n\end{array}$$

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Proof: 
$$\sum_{j=1}^n c_j x_j \leq \sum_{j=1}^n \left( \sum_{i=1}^m a_{i,j} y_i \right) x_j$$

**Theorem:** Let  $(x_1, x_2, \dots, x_n)$  be any feasible solution to Primal LP.

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## CH 29.4 Duality

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Proof:

$$\begin{aligned} &\sum_{j=1}^n c_j x_j \\ &\leq \sum_{j=1}^n \left( \sum_{i=1}^m a_{i,j} y_i \right) x_j \\ &= \sum_{i=1}^m \left( \sum_{j=1}^n a_{i,j} x_j \right) y_i \end{aligned}$$

**Theorem:** Let  $(x_1, x_2, \dots, x_n)$  be any feasible solution to Primal LP.

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Proof:

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Weak Duality

## CH 29.4 Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

### Primal LP

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### Dual LP

$$\begin{aligned} &\text{minimize} && \sum_{i=1}^m b_i y_i \\ &\text{s.t.} && \sum_{i=1}^m a_{i,j} y_i \geq c_j, \text{ for } j = 1, 2, \dots, n \\ &&& y_i \geq 0, \text{ for } i = 1, 2, \dots, m \end{aligned}$$

Proof:

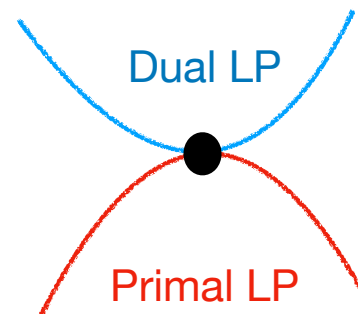
$$\begin{aligned} &\sum_{j=1}^n c_j x_j \\ &\leq \sum_{j=1}^n \left( \sum_{i=1}^m a_{i,j} y_i \right) x_j \\ &= \sum_{i=1}^m \left( \sum_{j=1}^n a_{i,j} x_j \right) y_i \\ &\leq \sum_{i=1}^m b_i y_i \end{aligned}$$

**Theorem:** Let  $(x_1, x_2, \dots, x_n)$  be any feasible solution to Primal LP.

Let  $(y_1, y_2, \dots, y_m)$  be any feasible solution to Dual LP.

Then

$$\sum_{j=1}^n c_j x_j \leq \sum_{i=1}^m b_i y_i$$



Weak Duality

Strong Duality

## CH 29.4 Duality

How to prove the SIMPLEX Algorithm returns an optimal solution?

### Primal LP

$$\begin{aligned} &\text{maximize} \quad \sum_{j=1}^n c_j x_j \\ &\text{s.t.} \\ &\quad \sum_{j=1}^n a_{i,j} x_j \leq b_i, \text{ for } i = 1, 2, \dots, m \\ &\quad x_j \geq 0, \text{ for } j = 1, 2, \dots, n \end{aligned}$$

### Dual LP

$$\begin{aligned} &\text{minimize} \quad \sum_{i=1}^m b_i y_i \\ &\text{s.t.} \\ &\quad \sum_{i=1}^m a_{i,j} y_i \geq c_j, \text{ for } j = 1, 2, \dots, n \\ &\quad y_i \geq 0, \text{ for } i = 1, 2, \dots, m \end{aligned}$$

Proof:

$$\begin{aligned} &\sum_{j=1}^n c_j x_j \\ &\leq \sum_{j=1}^n \left( \sum_{i=1}^m a_{i,j} y_i \right) x_j \\ &= \sum_{i=1}^m \left( \sum_{j=1}^n a_{i,j} x_j \right) y_i \\ &\leq \sum_{i=1}^m b_i y_i \end{aligned}$$

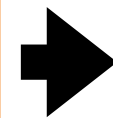
**Theorem:** Let  $(x_1, x_2, \dots, x_n)$  be any feasible solution to Primal LP.

Let  $(y_1, y_2, \dots, y_m)$  be any feasible solution to Dual LP.

Then

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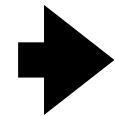
If  $\sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$ , then  $(x_1, x_2, \dots, x_n)$  is an optimal solution to the Primal LP,  
and  $(y_1, y_2, \dots, y_m)$  is an optimal solution to the Dual LP.



Optimality of  
SIMPLEX  
Algorithm.

## CH 29.4 Duality

If  $\sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$ , then  $(x_1, x_2, \dots, x_n)$  is an optimal solution to the Primal LP,  
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Optimality of  
SIMPLEX  
Algorithm.

Let's use an example to illustrate how to find such optimal solutions. (For rigorous proof, see textbook.)

## CH 29.4 Duality

### Primal LP

maximize  $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$

### Dual LP

minimize  $30y_1 + 24y_2 + 36y_3$

s.t.

$$y_1 + 2y_2 + 4y_3 \geq 3$$

$$y_1 + 2y_2 + y_3 \geq 1$$

$$3y_1 + 5y_2 + 2y_3 \geq 2$$

$$y_1, y_2, y_3 \geq 0$$

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### Primal LP

maximize  $3x_1 + x_2 + 2x_3$

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$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$



Final solution  
by SIMPLEX Algorithm:

$$x_1 = 8, x_2 = 4, x_3 = 0$$

### Dual LP

minimize  $30y_1 + 24y_2 + 36y_3$

s.t.

$$y_1 + 2y_2 + 4y_3 \geq 3$$

$$y_1 + 2y_2 + y_3 \geq 1$$

$$3y_1 + 5y_2 + 2y_3 \geq 2$$

$$y_1, y_2, y_3 \geq 0$$

## CH 29.4 Duality

### Primal LP

maximize  $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$



Final solution  
by SIMPLEX Algorithm:

$$x_1 = 8, x_2 = 4, x_3 = 0$$

Direct  
derivation



### Dual LP

minimize  $30y_1 + 24y_2 + 36y_3$

s.t.

$$y_1 + 2y_2 + 4y_3 \geq 3$$

$$y_1 + 2y_2 + y_3 \geq 1$$

$$3y_1 + 5y_2 + 2y_3 \geq 2$$

$$y_1, y_2, y_3 \geq 0$$

Corresponding solution:

$$y_1 = 0, y_2 = \frac{1}{6}, y_3 = \frac{2}{3}$$



## CH 29.4 Duality

### Primal LP

maximize  $3x_1 + x_2 + 2x_3$

s.t.

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + 2x_2 + 5x_3 \leq 24$$

$$4x_1 + x_2 + 2x_3 \leq 36$$

$$x_1, x_2, x_3 \geq 0$$



Final solution  
by SIMPLEX Algorithm:

$$x_1 = 8, x_2 = 4, x_3 = 0$$

Direct  
derivation

### Dual LP

minimize  $30y_1 + 24y_2 + 36y_3$

s.t.

$$y_1 + 2y_2 + 4y_3 \geq 3$$

$$y_1 + 2y_2 + y_3 \geq 1$$

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$$y_1, y_2, y_3 \geq 0$$

All constraints  
are satisfied.

Corresponding solution:

$$y_1 = 0, y_2 = \frac{1}{6}, y_3 = \frac{2}{3}$$

Feasible

## CH 29.4 Duality

### Primal LP

$$\begin{aligned} &\text{maximize} && 3x_1 + x_2 + 2x_3 \\ &\text{s.t.} && \\ &&& x_1 + x_2 + 3x_3 \leq 30 \\ &&& 2x_1 + 2x_2 + 5x_3 \leq 24 \\ &&& 4x_1 + x_2 + 2x_3 \leq 36 \\ &&& x_1, x_2, x_3 \geq 0 \end{aligned}$$



Final solution  
by SIMPLEX Algorithm:

$$x_1 = 8, x_2 = 4, x_3 = 0$$

Objective Value:

$$3 \times 8 + 4 + 2 \times 0 = 28$$

Direct  
derivation

Both  
Optimal

### Dual LP

$$\begin{aligned} &\text{minimize} && 30y_1 + 24y_2 + 36y_3 \\ &\text{s.t.} && \end{aligned}$$

$$\begin{aligned} &y_1 + 2y_2 + 4y_3 \geq 3 \\ &y_1 + 2y_2 + y_3 \geq 1 \\ &3y_1 + 5y_2 + 2y_3 \geq 2 \\ &y_1, y_2, y_3 \geq 0 \end{aligned}$$

All constraints  
are satisfied.

Corresponding solution:

$$y_1 = 0, y_2 = \frac{1}{6}, y_3 = \frac{2}{3}$$

Feasible

Objective Value:

$$30 \times 0 + 24 \times \frac{1}{6} + 36 \times \frac{2}{3} = 28$$

## SIMPLEX Algorithm

- 1) Start with a slack-form LP, whose basic solution is feasible.
- 2) Repeatedly transform the slack-form LP into new slack-form LPs, such that the basic solution gives higher and higher objective values.

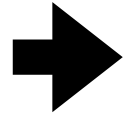
## SIMPLEX Algorithm

What if the basic solution is not feasible?

- 1) Start with a slack-form LP, whose basic solution is feasible.
- 2) Repeatedly transform the slack-form LP into new slack-form LPs, such that the basic solution gives higher and higher objective values.

Example:

$$\begin{array}{ll}\text{maximize} & 2x_1 - x_2 \\ \text{s.t.} & \\ & 2x_1 - x_2 \leq 2 \\ & x_1 - 5x_2 \leq -4 \\ & x_1, x_2 \geq 0\end{array}$$



Slack-form LP:

$$\begin{array}{l}z = 2x_1 - x_2 \\ x_3 = 2 - 2x_1 + x_2 \\ x_4 = -4 - x_1 + 5x_2\end{array}$$

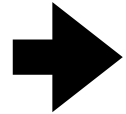
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Slack-form LP:

$$\begin{array}{l}z = 2x_1 - x_2 \\ x_3 = 2 - 2x_1 + x_2 \\ x_4 = -4 - x_1 + 5x_2\end{array}$$

Basic solution:  $x_1 = 0, x_2 = 0, x_3 = 2, x_4 = -4$

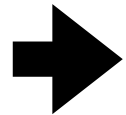
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Basic solution:  $x_1 = 0, x_2 = 0, x_3 = 2, x_4 = -4$

The basic solution is not feasible.

We will study how to handle this case.