Numerical computation of the Stokes constant for the breakdown of homoclinic orbits to L_3 in the RPC3BP

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This document is devoted to summarize the information necessary to compute the Stokes constant obtained in [BGG22] (see https://arxiv.org/abs/2107.09942).

1 Inner equation

We consider the following Hamiltonian given in [BGG22, Proposition 2.5] in

$$\mathcal{H}(U, W, X, Y) = W + XY + \mathcal{K}(U, W, X, Y), \tag{1.1}$$

with

$$\mathcal{K}(U, W, X, Y) = -\frac{3}{4} U^{\frac{2}{3}} W^2 - \frac{1}{3U^{\frac{2}{3}}} \left(\frac{1}{\sqrt{1 + \mathcal{J}(U, W, X, Y)}} - 1 \right) \tag{1.2}$$

and

$$\begin{aligned} \mathcal{J}(U, W, X, Y) = &\frac{4W^2}{9U^{\frac{2}{3}}} - \frac{16W}{27U^{\frac{4}{3}}} + \frac{16}{81U^2} + \frac{4(X+Y)}{9U} \left(W - \frac{2}{3U^{\frac{2}{3}}} \right) \\ &- \frac{4i(X-Y)}{3U^{\frac{2}{3}}} - \frac{X^2+Y^2}{3U^{\frac{4}{3}}} + \frac{10XY}{9U^{\frac{4}{3}}}. \end{aligned} \tag{1.3}$$

We study two special solutions of the inner equation given by the Hamiltonian $\mathcal{H}$ in (1.1). We introduce $Z = (W, X, Y)$ and the matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}. \tag{1.4}$$

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Then, the equation associated to the Hamiltonian $\mathcal{H}$ can be written as

$$\begin{cases} \dot{U} = 1 + g(U, Z), \\ \dot{Z} = \mathcal{A}Z + f(U, Z), \end{cases} \tag{1.5}$$

where $f = (-\partial_U \mathcal{K}, i\partial_Y \mathcal{K}, -i\partial_X \mathcal{K})^T$ and $g = \partial_W \mathcal{K}$.

We look for solutions of this equation parametrized as graphs with respect to U, namely we look for functions

$$Z_0^\diamond(U) = \big(W_0^\diamond(U), X_0^\diamond(U), Y_0^\diamond(U)\big)^T, \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s},$$

satisfying the invariance condition given by (1.5), that is

$$\partial_U Z_0^\diamond = \mathcal{A}Z_0^\diamond + \mathcal{R}[Z_0^\diamond], \qquad \text{for } \diamond = \mathrm{u}, \mathrm{s}, \tag{1.6}$$

where

$$\mathcal{R}[\varphi](U) = \frac{f(U, \varphi) - g(U, \varphi)\mathcal{A}\varphi}{1 + g(U, \varphi)}. \tag{1.7}$$

In order to "select" the solutions we are interested in (see [BGG22, Section 2.4]), we look for $Z_0^\diamond$ satisfying the asymptotic conditions

$$\lim_{\operatorname{Re} U \to -\infty} Z_0^{\mathrm{u}}(U) = 0, \qquad \lim_{\operatorname{Re} U \to +\infty} Z_0^{\mathrm{s}}(U) = 0. \tag{1.8}$$

In fact, for a fixed $\beta_0 \in \big(0, \frac{\pi}{2}\big)$, we look for functions Z_0^{u} and Z_0^{s} satisfying (1.6), (1.8) defined in the domains

$$\mathcal{D}_\kappa^{\mathrm{u}} = \{U \in \mathbb{C} : |\operatorname{Im} U| \geq \tan\beta_0 \operatorname{Re} U + \kappa\}, \qquad \mathcal{D}_\kappa^{\mathrm{s}} = -\mathcal{D}_\kappa^{\mathrm{u}}, \tag{1.9}$$

respectively, for some $\kappa > 0$ big enough (see Figure 1).

We analyze the the difference $\Delta Z_0 = Z_0^{\mathrm{u}} - Z_0^{\mathrm{s}}$ in the overlapping domain

$$\mathcal{E}_\kappa = \mathcal{D}_\kappa^{\mathrm{u}} \cap \mathcal{D}_\kappa^{\mathrm{s}} \cap \{U \in \mathbb{C} \,:\, \operatorname{Im} U < 0\}. \tag{1.10}$$

The proof of next theorem can be found in [BGG22, Theorem 2.7].

Theorem 1.1. *There exist $\kappa_0, b_1 > 0$ such that for any $\kappa \geq \kappa_0$, the equation (1.6) has analytic solutions $Z_0^\diamond(U) = (W_0^\diamond(U), X_0^\diamond(U), Y_0^\diamond(U))^T$, for $U \in \mathcal{D}_\kappa^\diamond$, $\diamond = \mathrm{u}, \mathrm{s}$, satisfying*

$$|U^{\frac{8}{3}} W_0^\diamond(U)| \leq b_1, \qquad |U^{\frac{4}{3}} X_0^\diamond(U)| \leq b_1, \qquad |U^{\frac{4}{3}} Y_0^\diamond(U)| \leq b_1. \tag{1.11}$$

In addition, there exist $\Theta \in \mathbb{C}$ and $b_2 > 0$ independent of κ, and a function $\chi = (\chi_1, \chi_2, \chi_3)^T$ such that

$$\Delta Z_0(U) = Z_0^{\mathrm{u}}(U) - Z_0^{\mathrm{s}}(U) = \Theta e^{-iU}\Big((0, 0, 1)^T + \chi(U)\Big), \tag{1.12}$$

and, for $U \in \mathcal{E}_\kappa$,

$$|U^{\frac{7}{3}} \chi_1(U)| \leq b_2, \qquad |U^2 \chi_2(U)| \leq b_2, \qquad |U \chi_3(U)| \leq b_2.$$

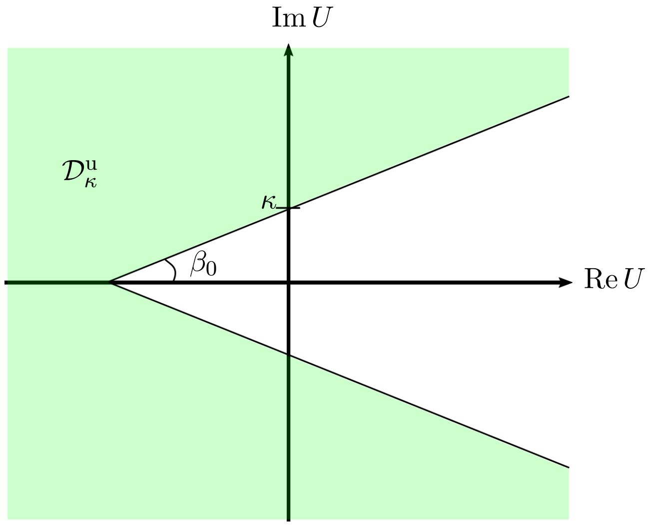

Figure 1: The inner domain, $\mathcal{D}^{\mathrm{u}}_{\kappa}$, for the unstable case.

2 Numerical computation Θ

Theorem 1.1 implies that $\Theta = \lim_{\operatorname{Im} U \to -\infty} \Delta Y_0(U) e^{iU}$. In particular, Θ corresponds to a Stokes constant that depends on the full jet of the Hamiltonian and, as a result, it does not have a closed formula.

However, we can obtain a numerical approximation of the constant Θ. Indeed, for $\rho > \kappa_0$, we can define

$$\Theta_\rho = |\Delta Y_0(-i\rho)|\, e^{\rho}, \tag{2.1}$$

which, for ρ big enough, satisfies $\Theta_\rho \approx |\Theta|$.

To compute $\Delta Y_0(-i\rho) = Y_0^{\mathrm{u}}(-i\rho) - Y_0^{\mathrm{s}}(-i\rho)$, we first look for good approximations of $Z_0^{\mathrm{u}}(U)$ for $\operatorname{Re} U \ll -1$ and of $Z_0^{\mathrm{s}}(U)$ for $\operatorname{Re} U \gg 1$, as power series in $U^{-\frac{1}{3}}$. One can easily check that $Z_0^{\mathrm{u}}(U)$ as $\operatorname{Re} U \to -\infty$ and $Z_0^{\mathrm{s}}(U)$ as $\operatorname{Re} U \to +\infty$ have the same asymptotics expansion:

$$\begin{aligned}
W_0^{\diamond}(U) &= \frac{4}{243\, U^{\frac{8}{3}}} - \frac{172}{2187\, U^{\frac{14}{3}}} + \mathcal{O}\left(U^{-\frac{20}{3}}\right), \\
X_0^{\diamond}(U) &= -\frac{2i}{9\, U^{\frac{4}{3}}} + \frac{28}{81\, U^{\frac{7}{3}}} + \frac{20i}{27\, U^{\frac{10}{3}}} - \frac{16424}{6561\, U^{\frac{13}{3}}} + \mathcal{O}\left(U^{-\frac{16}{3}}\right), \\
Y_0^{\diamond}(U) &= \frac{2i}{9\, U^{\frac{4}{3}}} + \frac{28}{81\, U^{\frac{7}{3}}} - \frac{20i}{27\, U^{\frac{10}{3}}} - \frac{16424}{6561\, U^{\frac{13}{3}}} + \mathcal{O}\left(U^{-\frac{16}{3}}\right).
\end{aligned}$$

We use these expressions to set up the initial conditions for the numerical integration for computing $\Delta Y_0(-i\rho)$. We take as initial points the value of the truncated power series at order $U^{-\frac{13}{3}}$ at $U = 1000 - i\rho$ (for $\diamond = \mathrm{s}$) and $U = -1000 - i\rho$ (for $\diamond = \mathrm{u}$). (See Table 1). We perform the numerical integration for different values of $\rho \leq 23$ and an integration solver with tolerance 10^{-12}.

ρ	$\lvert\Delta Y_0(-i\rho)\rvert$	e^ρ	Θ_ρ
13	$3.7\cdot 10^{-6}$	$4.4\cdot 10^{5}$	1.6373
14	$1.4\cdot 10^{-6}$	$1.2\cdot 10^{6}$	1.6361
15	$5.0\cdot 10^{-7}$	$3.3\cdot 10^{6}$	1.6351
16	$1.8\cdot 10^{-7}$	$8.9\cdot 10^{6}$	1.6341
17	$3.7\cdot 10^{-8}$	$2.4\cdot 10^{7}$	1.6333
18	$6.8\cdot 10^{-8}$	$6.6\cdot 10^{7}$	1.6326
19	$9.1\cdot 10^{-9}$	$1.8\cdot 10^{8}$	1.6320
20	$3.4\cdot 10^{-9}$	$4.9\cdot 10^{8}$	1.6315
21	$1.2\cdot 10^{-9}$	$1.3\cdot 10^{9}$	1.6312
22	$4.6\cdot 10^{-10}$	$3.6\cdot 10^{9}$	1.6313
23	$1.7\cdot 10^{-10}$	$9.7\cdot 10^{9}$	1.6323

Table 1: Computation of Θ_ρ, as defined in (2.1), for different values of $\rho \leq 23$.

Table 1 shows that the constant Θ is approximately 1.63 which indicates that it is not zero. We expect that, by means of a computer assisted proof, it would be possible to obtain rigorous estimates and verify that $|\Theta| \neq 0$.

References

[BGG22] I. Baldomá, M. Giralt, and M. Guardia. Breakdown of homoclinic orbits to L3 in the RPC3BP (I). Complex singularities and the inner equation. *To be published in Advances in Mathematics*, 2022.