LEMPo-RMHD

Linear Eigenvalue Modular solver for Pressure- and current-driven modes (Resistive MHD) in tokamaks

Documentation

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Abstract

LEMPo-RMHD is a modular eigensolver written in *MATLAB* for studying the linear stability of internal magnetohydrodynamics (MHD) modes, in a tokamak plasma. It computes linear growth rates and perturbations associated with moderate- to long-wavelength internal MHD instabilities, including **tearing modes**, **resistive interchange** modes, and **infernal modes** (in a torus). The solver (presented in Ref. (*1*)) is based on a unified set of global equations for long-wavelength, pressure- or current-driven MHD modes, incorporating resistivity and compressibility in an inverse aspect ratio expansion (see Ref. (*2*)).

The solver is named after Lempo: both a Kuiper Belt object (discovered in 1999) and a demon from Finnish mythology. This reflects its focus on *infernal* modes, and its ultimate integration within the VENUS-MHD framework[1].

[1] See 7 for image credits

Contents

1 Overview

Based on an inverse aspect ratio expansion, LEMPo-RMHD computes linear growth rates and eigenmodes for moderate- to long-wavelength internal MHD instabilities, including tearing modes (in a torus), resistive interchange modes, and infernal modes. A static equilibrium is assumed. High n ballooning modes are not captured, but weak ballooning associated with toroidal effects inherent in internal kink modes and infernal modes is captured.

The inputs to the model are the q profile, B_0, and the pressure profile. Focusing on a single helicity (m, n), we then solve the dispersion relation (turned into a generalised eigenvalue problem) over the whole plasma extent, with the boundary condition that the perturbation must vanish at the plasma edge (internal instabilities). We obtain the linear growth rate γ/ω_A, the radial displacement ξ, and the resistive and compressibility corrections χ and $\Delta\xi_\Gamma$. See Section 3.5 to relate these variables to more commonly used physical quantities (*e.g.* $\boldsymbol{\delta B}$, δP ...) Except for the interchange versions of the solver, the $(m + 1)$ and $(m - 1)$ neighbours are also explicitely included in the process, and we obtain the associated plasma displacement.

There are four different models which can be run:

- **The resistive full model $\mathbf{LEMPo_{res}}$**: `LEMPo_res.m` file. This solves the full set of equation in Section 3: (3) for the main harmonic, (6) for the sidebands, Ohm's laws for the main harmonic (7) and for the sidebands (8), and the definition of the compressibility variable (9). It recovers interchange, infernal and Bussac-like internal kink modes as well as their resistive counterparts. Tearing modes in a torus are also retrieved.

- **The ideal full model $\mathbf{LEMPo_{ideal}}$**: `LEMPo_ideal.m` file. This is the ideal MHD limit of the full model, obtained by neglecting resistive effects (*i.e.* $\chi(r)$, $\Delta\xi_\Gamma(r) \to 0, \forall r$). It solves equation 10 for the main harmonic $\xi^{(m)}$, and equation **??** for the sidebands $\xi^{(m\pm1)}$. This model is enough to describe ideal unstable modes when important rational surfaces are avoided, or in high Lundquist number regimes, or in cases with very large pressure gradients. It recovers ideal internal modes, ideal interchange and ideal infernal modes.

- **The resistive interchange model $\mathbf{LEMPo^{IM}_{res}}$**: `IM_LEMPo_res.m` file. The interchange models **IM** are limit solutions of the full model **FM**. They substitute analytically approximated solutions for the sidebands displacements $\xi^{(m\pm1)}$, so they will not appear explicitly in the equations. The model thus consists only of Eq. (12) for $\xi^{(m)}$, and Ohm's law (7) for χ. This model is sufficient to retrieve resistive interchange including compression effects and tearing modes in a torus, but misses destabilising infernal effects (linked to higher order coupling to sidebands) that are included in the full model.

- **Interchange model $\mathbf{LEMPo^{IM}_{ideal}}$**: `IM_LEMPo_ideal.m` file. The ideal version of the interchange model neglects resistive effects and compressibility corrections, and requires only one equation: Eq. 13 for $\xi^{(m)}$.

2 Inputs, Options, and Outputs

2.1 Inputs

The inputs required to run the 4 versions of the solver (`IM_LEMPo_ideal.m`, `LEMPo_ideal.m`, `IM_LEMPo_res.m`, and `LEMPo_res.m`) are the same:

- m, n — Poloidal and toroidal mode numbers of the main harmonic.

- N — Number of points in the radial grid.

- `ev_guess` — Initial guess for the eigenvalue $\gamma/_A$ (should be well above the expected value, a typical number is 0.01).
- `toPlot` — Boolean flag to enable output plots
- `profiles` — Struct containing equilibrium profiles and other necessary input quantities, see Section 2.2.
- `opts` — Struct containing solver options, see Section 2.3; can be set to empty `[]` to use defaults.

2.2 Profiles

The profiles structure **must** contained certain fields, and other are optional depending on the configuration. Please refer to the list below, or to the examples scripts.

- `q` (mandatory) — Safety factor profile as a function handle of the radial coordinate (SFL).
  - If missing, a default ultra-flat resonant q profile is used.
- `qp` (optional, needed for discontinuous q profiles) — Radial derivative of the safety factor, as a function handle of the radial coordinate (SFL).
  - Needed for discontinuous q profiles.
  - If provided, must also include`qpp` function handle (second derivative).
  - If missing, derivatives are computed automatically from `q` (for continuous profile).
- `RS` (optional) — Double array, main rational surface(s) location(s).
  - Needed for discontinuous and non-resonant q profiles.
  - For non-resonant q profiles, you should supply the end of low shear zone radius (approximatively).
  - If missing, it is computed by solving $q(x) = m/n$. Multiple rational surfaces are handled correctly.
- `RSp`, `RSm` (optional) — Double, rational surface locations for the $m+1$ and $m-1$ sidebands.
  - Needed for discontinuous q profiles.
  - If missing, they are computed automatically (for continuous profiles).
- `alpha` (mandatory, unless `beta` is provided) — Double, target value for α at a given radius (see Eq. 1). It will set a parabolic pressure profile.
  - If used, can be combined with `rSetAlpha` (optional profile fields) to enforce α at a specific radius. Otherwise, value is enforced at `profiles.RS`, or at the lowest rational surface computed.
  - if you provide `alpha`< 0, a hollow pressure profile will be set.
- `beta` (alternative to `alpha`) — Beta profile as a function handle of x (so you can set custom pressure profiles): $\beta = P/(B^2/2\mu_0)$, and $\alpha = -q^2 R_0 \beta'$.
  - If provided, you should also include `betap` and `betapp` fields in case of discontinuous q profiles. Otherwise they are derived automatically.
- `elong`, `triangByR` (mandatory only if `opts.addShaping = 1`) — Shape parameters of the flux surfaces.

  - `elong` is the flux surface elongation, considered constant with r. Corresponds to κ in the equations in Section 3.
  - `triangByR` corresponds to δ/r in the equations, triangularity over radius is considered constant.

- `SL` (optional) — Sets the Lundquist number (defined in (2)) for resistive simulations.
  - Alternatively, η can be provided instead of `SL`, through the field `resistivity`.

2.3 Options

The fields which can be supplied to the `opts` structure, as well as their default values, are listed in the tables 1 and 2. If there is a ✓symbol, the option is available for the model specified, otherwise it is not. The `opts` structure can also be left empty using `opts = []`, in which case all the values will be left to default.

Table 1: Available options in `opts` structure: 1/2.

Field name in `opts`	**Description**	**LEMPo$^{\mathbf{IM}}_{\mathbf{ideal}}$**	**LEMPo$^{\mathbf{IM}}_{\mathbf{res}}$**	**LEMPo$_{\mathbf{ideal}}$**	**LEMPo$_{\mathbf{res}}$**
`B0`	Toroidal magnetic field at axis [T]. Default = 3.				✓
`R0`	Major radius [m]. Default = 3.	✓		✓	✓
`additionalPlot`	Boolean flag to produce additional plots of q, β, and grid configuration (only if `toPlot=1`). Default = 0.	✓		✓	✓
`addShaping`	Boolean flag to enable flux surface shaping. Requires `profiles.elong` and `profiles.triangByR`. Default = 0 (circular).			✓	✓
`higherOscModes`	Boolean flag to be able to pick up eigenmodes with higher radial oscillations (lower growth rates). Default = 0, in which case the solver will try higher growth rates if encountering a higher oscillating modes.	✓		✓	✓
`N_tries`	Maximum number of iterations of eigenvalue solver (eigs routine) before giving up. If you increase you can typically reach lower growth rates with same guess. Default = 10.	✓		✓	✓
`upperBound`	Sets radial boundary for the simulation domain (e.g. 0.8). If lower than upper sideband rational surface, Neumann BC is applied for $\xi^{(m-1)}$. Default = 1.			✓	✓
`reverse_qprofile`	Boolean flag improve how the solution from eigs solver is selected in case of reverse q profile. Default = 0.	✓		✓	✓

Table 2: Available options in `opts` structure: 2/2.

Field name in `opts`	**Description**	$\textbf{LEMPo}^{\textbf{IM}}_{\textbf{ideal}}$	$\textbf{LEMPo}^{\textbf{IM}}_{\textbf{res}}$	$\textbf{LEMPo}_{\textbf{ideal}}$	$\textbf{LEMPo}_{\textbf{res}}$
`resOnlyOnMainRatSurf`	Boolean flag to apply resistivity only on the main rational surface. Default = 0.				✓
`neglectDeltaBParallel`	Boolean flag to neglect parallel magnetic field fluctuations ($\delta A_\perp \to 0$, giving $\delta B_\| \approx 0$). Default = 0.				✓
`noCompr`	Boolean flag to remove plasma compression effects. Default = 0.				✓

2.4 Outputs

All versions of the solver have two outputs:

$$\texttt{ev} \quad \& \quad \texttt{result}\,.$$

`ev` corresponds to the linear growth rate of the mode γ/ω_A (complex or real double). `result` is a structure array containing the following fields:

- `rgrid` — Radial grid points used for the discretization.
- `RS` — Rational surface(s) of the main harmonic, or value given in `profiles.RS` for non resonant profiles.
- `RSp` — Rational surface(s) of the upper sideband.
- `RSm` — Rational surface(s) of the lower sideband (if $m \neq 1$).
- `alphaRS` — Value of α at RS.
- `s` — Magnetic shear at RS.
- `q` — Vector, q-profile evaluated on the radial grid.
- `deltap` — Vector, radial derivative of the Shafranov shift evaluated on `rgrid`.
- `elong` — Elongation κ of the flux surfaces (if shaping included).
- `triangByR` — Triangularity parameter δ/r of flux surfaces (if shaping included).
- `Xi` — Vector, radial plasma displacement of the main harmonic, evaluated on `rgrid`.
- `Xip` — Vector, radial plasma displacement of the upper sideband, evaluated on `rgrid`.
- `Xim` — Vector, radial plasma displacement of the lower sideband (if $m \neq 1$), evaluated on `rgrid`.

3 Equations behind the solver

3.1 Notation

For clarity, we summarize the main symbols and variables used throughout the stability calculations:

- r — Radial flux surface label (SFL coordinate), see Eq. (**??**).
- θ — Poloidal angle in straight field line (SFL) coordinates.
- ϕ — Toroidal angle.
- $q(r)$ — Safety factor (constant on each flux surface in SFL coordinates)
- R_0 — Major radius at the magnetic axis.
- $\epsilon = r/R_0$ — Inverse aspect ratio.
- r_s — Location of the rational surface, defined by $q(r_s) = m/n$.

- $\alpha(r)$ — MHD ballooning parameter, measuring pressure drive:

$$\boxed{\alpha(r) = -\frac{2\mu_0 q^2 R_0}{B_0^2}\frac{\mathrm{d}P}{\mathrm{d}r}\,.} \tag{1}$$

- S_L — Lundquist number, characterizing the ratio between resistive and Alfvén timescales:

$$S_L = \tau_R/\tau_A = \frac{\mu_0 r_s^2}{\eta}\omega_A(r_s)\,. \tag{2}$$

- $\Delta(r)$ — Shafranov shift of flux surfaces: $\Delta'(r) = \frac{q(r)^2}{r^3}\left(\int_0^r \frac{\tilde{r}^3}{q(\tilde{r})^2 R_0} + \frac{\tilde{r}^2\alpha(\tilde{r})}{q(\tilde{r})^2}\mathrm{d}\tilde{r}\right)$.
- $\boldsymbol{\xi}$ — Radial plasma displacement, where the vector of the plasma displacement is defined as: $\boldsymbol{v} \approx \partial\boldsymbol{\xi}/\partial t$, with $\boldsymbol{v}$ the perturbed velocity.
- $\Delta\xi_\Gamma^r$ — Variable encompassing compressibility effects, defined in Eq. (**??**)
- χ — Resistive correction to the displacement
- m, n — Poloidal and toroidal mode numbers
- $\xi^{(m)}$ — Radial displacement corresponding to poloidal harmonic m.
- $\xi^{(m\pm1)}$ — Radial displacement corresponding to the sidebands, with poloidal harmonic $m \pm 1$.

3.2 Full resistive model

The full resistive system **LEMPo$_{\mathbf{res}}$** consists of seven coupled equations for the variables $(\xi^{(m)}, \xi^{(m+1)}, \xi^{(m-1)}, \chi^{(m)}, \chi^{(m+1)}, \chi^{(m-1)}, \Delta\xi_\Gamma)$ (there are no equations for the compressibility corrections on the sidebands because they are eliminated analytically), where we did not count the "dummy" variables (see Section REF).

Main harmonic equation The version for $m \neq 1$ version is presented below:

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi^{(m)}}{dr}\right]-(m^2-1)\xi^{(m)}\right\}+\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{(m)}}{dr}\right]-(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi^{(m)}\\
&+\frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d\chi}{dr}-\frac{r^3}{q_s}\chi\frac{d}{dr}\left(\frac{1}{q}\right)\right]-(m^2-1)\frac{1}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\chi\\
&+\frac{s^2}{q_s^2}D_I(r)\,\xi^{(m)}\\
&+\frac{\alpha\epsilon}{q_s^2}\left(\frac{1}{q_s^2}-1\right)\Delta\xi_\Gamma-\frac{\alpha^2}{2q_s^2}\left(\xi^{(m)}+\Delta\xi_\Gamma\right)\\
&+\frac{2+m}{2(1+m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[(1+m)\epsilon+\alpha+m\Delta'\right]\right)\overline{\xi}^{(m+1)}\\
&+\frac{2-m}{2(1-m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[(1-m)\epsilon+\alpha-m\Delta'\right]\right)\overline{\xi}^{(m-1)}\\
&+\frac{r}{2(1+m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[2(1+m)\epsilon+(2+m)\alpha-(1+2m)\Delta'\right]\right)\overline{\xi}^{(m+1)\prime}-\frac{\Delta q}{q_s^3}\frac{r^2}{(1+m)}\Delta'\overline{\xi}^{(m+1)\prime\prime}\\
&+\frac{r}{2(1-m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[2(1-m)\epsilon+(2-m)\alpha-(1-2m)\Delta'\right]\right)\overline{\xi}^{(m-1)\prime}-\frac{\Delta q}{q_s^3}\frac{r^2}{(1-m)}\Delta'\overline{\xi}^{(m-1)\prime\prime}\\
&+\chi\left(-\frac{1}{q_s^2}\left[4\epsilon^2(2-\frac{1}{q_s^2})+\epsilon\alpha(3+\frac{1}{q_s^2})+\alpha^2+\Delta'(12\Delta'-7\alpha+r\alpha'-6\epsilon)\right]\right)\\
&+\xi^{(m)}\left(\frac{\Delta q}{q_s^3}\left[4\epsilon^2(2-\frac{1}{q_s^2})+\epsilon\alpha(5-\frac{2}{q_s^2})+2\alpha^2+\Delta'(12\Delta'-7\alpha+r\alpha'-6\epsilon)\right]\right)\\
&+\frac{r}{q_s^2}\alpha\Delta'\left(\frac{\Delta q}{q_s}\xi^{(m)\prime}-\chi'\right)\\
&+\Delta\xi_\Gamma\left(\frac{\Delta q}{q_s^3}\left[\epsilon\alpha\left(2-\frac{3}{q_s^2}\right)+\alpha^2\right]-\frac{q'}{q_s^3}r\Delta'\alpha\right)\ ,
\end{aligned}
\tag{3}
$$

where the Mercier term $s^2D_I(r)$ is given in below:

$$
s^2D_I(r)=\ \alpha\epsilon\left[\frac{1}{q_s^2}-1\ +\ \frac{3}{4}(\kappa-1)\left(1-2\frac{\delta}{\epsilon}\right)\right]\ . \tag{4}
$$

We consider flux surfaces with constant elongation κ along the minor radius r, and triangularity following $\delta(r)=\delta_1\,r/r_1$ for $r<r_1$, where r_1 marks the end of the low shear region. This is exact in a large aspect ratio, low beta tokamak, for a shearless q profile and more generally $\delta(r)=\delta_a\,r/a$ provides a good approximation. But it remains an approximation, and since we also neglected the quasicylindrical shaping corrections, **values of κ and δ_a should not be pushed too high.**

The $m = 1$ version of the governing equation for the main harmonic reads:

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi^{(m)}}{dr}\right]\right\} + \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{(m)}}{dr}\right] \\
&+ \frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d\chi}{dr} - \frac{r^3}{q_s}\chi\frac{d}{dr}\left(\frac{1}{q}\right)\right] \\
&+ \frac{\alpha}{q_s^2}\left[\epsilon\left(\frac{1}{q_s^2}-1+\frac{\Delta q}{2q_s}\right)-\frac{\alpha}{4}\right]\left(\xi^{(m)}+\Delta\xi_\Gamma\right) + \frac{\alpha\epsilon}{q_s^2}\left[\frac{3}{4}(\kappa-1)\left(1-2\frac{\delta}{\epsilon}\right)\right]\xi^{(m)} \\
&- \frac{r\alpha\Delta'}{2q_s^2}\left(2\chi' + \frac{\Delta q'}{q_s}\Delta\xi_\Gamma + \frac{\Delta q}{q_s}\left(\Delta\xi_\Gamma' - \xi^{(m)\prime}\right)\right) \\
&+ \frac{2+m}{2(1+m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[(1+m)\epsilon+\alpha+m\Delta'\right]\right)\overline{\xi}^{(m+1)} \\
&+ \frac{r}{2(1+m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[2(1+m)\epsilon+(2+m)\alpha-(1+2m)\Delta'\right]\right)\overline{\xi}^{(m+1)\prime} - \frac{\Delta q}{q_s^3}\frac{r^2}{(1+m)}\Delta'\overline{\xi}^{(m+1)\prime\prime} \\
&+ \chi\left(-\frac{1}{q_s^2}\left[4\epsilon^2(2-\frac{1}{q_s^2})+\epsilon\alpha(\frac{7}{2}+\frac{1}{q_s^2})+\frac{3}{2}\alpha^2+\Delta'(12\Delta'-7\alpha+r\alpha'-6\epsilon)\right]\right) \\
&+ \xi^{(m)}\left(\frac{\Delta q}{q_s^3}\left[4\epsilon^2(2-\frac{1}{q_s^2})+\epsilon\alpha(5-\frac{2}{q_s^2})+\frac{3}{2}\alpha^2+\Delta'(12\Delta'-\frac{13}{2}\alpha+\frac{1}{2}r\alpha'-6\epsilon)\right]\right) \\
&+ \frac{\Delta q}{q_s^3}\Delta\xi_\Gamma\left(3\epsilon\alpha\left(\frac{1}{2}-\frac{1}{q_s^2}\right)+\frac{1}{2}\Delta'(\alpha-r\alpha')\right) - \frac{\Delta q'}{2q_s^3}\Delta\xi_\Gamma r\Delta'\alpha\,.
\end{aligned}
\tag{5}
$$

Sideband equations The sidebands variables in the code are denoted $\overline{\xi}^{(m\pm1)}$ in the current document and in related articles. Indeed these are modified sidebands, linked to the actual sideband displacement via:

$$
\xi^{(m\pm1)} = \overline{\xi}^{(m\pm1)} + \frac{1\pm m}{r^{2\pm m}}\int_0^r l^{2\pm m}\Delta'\frac{d}{dl}\left(\frac{\Delta q}{q_s}\xi^{(m)} - \chi\right)\mathrm{d}l\,,
$$

the code's final output is the correct $\xi^{(m\pm1)}$, the reverse change of variable is done after the eigenvalue problem is solved.

The governing equation for the modified sidebands $\overline{\xi}^{(m\pm1)}$ is given below, with $q_{m\pm1} \equiv$

$(m \pm 1)/n$:

$$
\begin{aligned}
0 = & \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{n}{m\pm 1}\right)^2\frac{d\overline{\xi}^{r(m\pm1)}}{dr}\right] - m(m\pm 2)\left(\frac{1}{q}-\frac{n}{(m\pm 1)}\right)^2\overline{\xi}^{r(m\pm1)} \\
& + \frac{\gamma^2(1+2q_{m\pm1})^2)}{(m\pm1)^2\omega_A^2}\left\{\left(\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\overline{\xi}^{r(m\pm1)}}{dr}\right] - m(m\pm 2)\overline{\xi}^{r(m\pm1)}\right)\right\} \\
& + \frac{1}{r}\frac{d}{dr}\left[r^3\frac{n}{(m\pm1)}\left(\frac{1}{q}-\frac{n}{(m\pm1)}\right)\frac{d\chi^{(m\pm1)}}{dr}\right] - \frac{1}{r}\frac{d}{dr}\left[r^3\frac{n}{(m\pm1)}\left(\frac{1}{q}\right)'\chi^{(m\pm1)}\right] \\
& - m(m\pm2)\frac{1}{q_{m\pm1}}\left(\frac{1}{q}-\frac{n}{(m\pm1)}\right)\chi^{(m\pm1)} \\
& - \frac{r^{1\pm m}}{2q_s^2(1\pm m)}\frac{d}{dr}\left(\frac{\alpha}{r^{\pm m}}(\xi^{(m)}+\Delta\xi_\Gamma)\right) \\
& + \frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left(\left[(1\pm 2m)\epsilon+(1\pm m)(\alpha-3\Delta')\right]\frac{1}{r^{\pm m}}\left(\frac{\Delta q}{q_s}\xi^{(m)}-\chi^{(m)}\right)\right) \\
& + \frac{1}{q_s^2}(2\pm m)(\epsilon+\alpha-4\Delta')\left(\frac{\Delta q}{q_s}\xi^{(m)}-\chi^{(m)}\right) \\
& + \frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left[\frac{\alpha}{r^{\pm m}}\left(\chi^{(m)}+\frac{\Delta q}{q_s}\Delta\xi_\Gamma\right)\right].
\end{aligned}
\tag{6}
$$

This is valid for $\overline{\xi}^{(m\pm1)}$ for $m>1$ and $\overline{\xi}^{(m+1)}$ for $m=1$.

Additional relations needed to close the system The equations cited above involve the variables χ and $\Delta\xi_\Gamma$, whose relationship with ξ^r are needed to close the system. We have "Ohm's law":

$$
\begin{aligned}
\frac{\gamma}{\omega_A}r^3\chi = \frac{r_s^2}{S_L}\Bigg[& \frac{d}{dr}\left(r^3\frac{d}{dr}\left(\frac{m}{n}\left(\frac{1}{q}-\frac{n}{m}\right)\xi+\chi\right)\right) \\
& + r(1-m^2)\left(\frac{m}{n}\left(\frac{1}{q}-\frac{n}{m}\right)\xi+\chi\right)\Bigg],
\end{aligned}
\tag{7}
$$

If one wishes to include resistivity on the sidebands, the sideband harmonics of χ which appear in (6) follow a similar Ohm's law:

$$
\begin{aligned}
\frac{\gamma}{\omega_A}r^3\chi^{(m\pm1)} = \frac{r_{m\pm1}^2}{S_L}\Bigg[& \frac{d}{dr}\left(r^3\frac{d}{dr}\left(\frac{(m\pm1)}{n}\left\{\frac{1}{q}-\frac{n}{(m\pm1)}\right\}\overline{\xi}^{(m\pm1)}+\chi^{(m\pm1)}\right)\right) \\
& + r(1-(m\pm1)^2)\left(\frac{(m\pm1)}{n}\left\{\frac{1}{q}-\frac{n}{(m\pm1)}\right\}\overline{\xi}^{(m\pm1)}+\chi^{(m\pm1)}\right)\Bigg],
\end{aligned}
\tag{8}
$$

where $r_{m\pm1}$ is defined as $q(r_{m\pm1}) = (m\pm1)/n$.

For the compressibility variable, we use its definition:

$$
\Delta\xi_\Gamma^r = -\chi\left(\frac{q}{q-q_s}\right)\left(\frac{\omega_s^2(nq-m)^2}{\omega_s^2(nq-m)^2+\gamma^2q^2}\right). \tag{9}
$$

3.3 Full ideal model

This is the ideal MHD limit of the full model, obtained by neglecting resistive effects (*i.e.* $\chi(r)$, $\Delta\xi_\Gamma(r)\to 0, \forall r$ in all equations above). The full ideal system $\mathbf{LEMPo_{ideal}}$ will thus consists of three coupled equations for the variables $(\xi^{(m)}, \xi^{(m+1)}, \xi^{(m-1)})$. There are no "dummy" variables for the ideal model.

The version for $m \neq 1$ version is presented below:

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi^{(m)}}{dr}\right] - (m^2-1)\xi^{(m)}\right\} + \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{(m)}}{dr}\right] - (m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi^{(m)} \\
&+ \frac{s^2}{q_s^2}D_I(r)\,\xi^{(m)} - \frac{\alpha^2}{2q_s^2}\xi^{(m)} \\
&+ \frac{2+m}{2(1+m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[(1+m)\epsilon + \alpha + m\Delta'\right]\right)\overline{\xi}^{(m+1)} \\
&+ \frac{2-m}{2(1-m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[(1-m)\epsilon + \alpha - m\Delta'\right]\right)\overline{\xi}^{(m-1)} \\
&+ \frac{r}{2(1+m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[2(1+m)\epsilon + (2+m)\alpha - (1+2m)\Delta'\right]\right)\overline{\xi}^{(m+1)\prime} - \frac{\Delta q}{q_s^3}\frac{r^2}{(1+m)}\Delta'\overline{\xi}^{(m+1)\prime\prime} \\
&+ \frac{r}{2(1-m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[2(1-m)\epsilon + (2-m)\alpha - (1-2m)\Delta'\right]\right)\overline{\xi}^{(m-1)\prime} - \frac{\Delta q}{q_s^3}\frac{r^2}{(1-m)}\Delta'\overline{\xi}^{(m-1)\prime\prime} \\
&+ \xi^{(m)}\left(\frac{\Delta q}{q_s^3}\left[4\epsilon^2(2-\frac{1}{q_s^2}) + \epsilon\alpha(5-\frac{2}{q_s^2}) + 2\alpha^2 + \Delta'(12\Delta' - 7\alpha + r\alpha' - 6\epsilon)\right]\right) \\
&+ \frac{r}{q_s^2}\alpha\Delta'\frac{\Delta q}{q_s}\xi^{(m)\prime},
\end{aligned}
\tag{10}
$$

where the Mercier term $s^2 D_I(r)$ is given in Eq. 4.

The $m = 1$ version of the governing equation for the main harmonic reads:

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi^{(m)}}{dr}\right]\right\} + \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{(m)}}{dr}\right] \\
&+ \frac{\alpha}{q_s^2}\left[\epsilon\left(\frac{1}{q_s^2} - 1 + \frac{\Delta q}{2q_s}\right) - \frac{\alpha}{4}\right]\xi^{(m)} + \frac{\alpha\epsilon}{q_s^2}\left[\frac{3}{4}(\kappa-1)\left(1 - 2\frac{\delta}{\epsilon}\right)\right]\xi^{(m)} \\
&+ \frac{r\alpha\Delta'}{2q_s^2}\frac{\Delta q}{q_s}\xi^{(m)\prime} \\
&+ \frac{2+m}{2(1+m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[(1+m)\epsilon + \alpha + m\Delta'\right]\right)\overline{\xi}^{(m+1)} \\
&+ \frac{r}{2(1+m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[2(1+m)\epsilon + (2+m)\alpha - (1+2m)\Delta'\right]\right)\overline{\xi}^{(m+1)\prime} - \frac{\Delta q}{q_s^3}\frac{r^2}{(1+m)}\Delta'\overline{\xi}^{(m+1)\prime\prime} \\
&+ \xi^{(m)}\left(\frac{\Delta q}{q_s^3}\left[4\epsilon^2(2-\frac{1}{q_s^2}) + \epsilon\alpha(5-\frac{2}{q_s^2}) + \frac{3}{2}\alpha^2 + \Delta'(12\Delta' - \frac{13}{2}\alpha + \frac{1}{2}r\alpha' - 6\epsilon)\right]\right).
\end{aligned}
\tag{11}
$$

3.4 Interchange models

The interchange models $\mathbf{LEMPo}^{\mathbf{IM}}_{\mathbf{ideal}}$ and $\mathbf{LEMPo}^{\mathbf{IM}}_{\mathbf{res}}$ are limit solutions of the full models $\mathbf{LEMPo}_{\mathbf{ideal}}$ and $\mathbf{LEMPo}_{\mathbf{res}}$. They substitute analytically approximated solutions for the sidebands displacements $\xi^{(m\pm1)}$, so they will not appear explicitly in the following equations.

Resistive interchange model The $\mathbf{LEMPo}^{\mathbf{IM}}_{\mathbf{res}}$ model thus consists only of Eq. (12) for ξ, and Ohm's law (7) for χ (to lighten the notation, ξ will stand for $\xi^{(m)}$ when no confusion is possible). For the "dummy" variables see Section REF.

$$
\begin{aligned}
0 =&\frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi}{dr}\right]-(m^2-1)\xi\right\} \\
&+\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi}{dr}\right]-(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi \\
&+\frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d\chi}{dr}-\frac{r^3}{q_s}\chi\frac{d}{dr}\left(\frac{1}{q}\right)\right] \\
&-(m^2-1)\frac{1}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\chi+\frac{\epsilon\alpha}{q_s^2}\left(\frac{1}{q_s^2}-1\right)(\xi+\Delta\xi_\Gamma) \\
&-\frac{\alpha}{q_s^2}\Delta' r\frac{d}{dr}\left(\chi+\frac{q-q_s}{q}\Delta\xi_\Gamma\right).
\end{aligned}
\tag{12}
$$

This model is sufficient to retrieve resistive interchange including compression effects and tearing modes in a torus, but misses destabilising infernal effects (linked to higher order coupling to sidebands) that are included in the full model.

Ideal interchange model The ideal version of the interchange model $\mathbf{LEMPo}^{\mathbf{IM}}_{\mathbf{ideal}}$ neglects resistive effects and compressibility corrections, and requires only one equation:

$$
\begin{aligned}
0 =&\frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi}{dr}\right]-(m^2-1)\xi\right\} \\
&+\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi}{dr}\right]-(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi \\
&+\frac{\epsilon\alpha}{q_s^2}\left(\frac{1}{q_s^2}-1\right)\xi,
\end{aligned}
\tag{13}
$$

for the eigenvector ξ.

3.5 Link to other physical quantities

Perturbations in δB, $\delta\psi$ and δP

For some applications, it is useful to have explicit expressions that relate these variables to more commonly used physical quantities. In this section, we show how to recover the perturbation to the equilibrium magnetic field $\boldsymbol{\delta B}$, the perturbed poloidal magnetic flux $\delta\psi_p$, and the perturbed pressure δP.

We obtain, for the (m,n) helicity, the following contravariant components of $\boldsymbol{\delta B}$ (to leading order in ϵ):

$$
\begin{aligned}
\delta B^r &= \frac{F_0}{R^2}\left[\frac{1}{q}\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\phi}\right]\xi_R^r \\
&= -\frac{B_0}{R_0}in\left[\left(\frac{1}{q}-\frac{n}{m}\right)\xi+\chi\right],
\end{aligned}
\tag{14}
$$

where we drop the superscript r for the ideal displacement ξ, as in the previous section, since the radial displacement is the primary output of our equations. For the poloidal perturbation, we arrive at:

$$
\begin{aligned}
\delta B^\theta &= -\frac{F_0}{R^2}\left[\frac{\partial}{\partial r}\left(\frac{r\xi_R^r}{q}\right) - \frac{\partial}{\partial \phi}\xi_R^\theta\right] \\
&= -r\frac{B_0}{R_0}\frac{n}{m}\frac{\partial}{\partial r}\left[\frac{m}{n}\left(\frac{1}{q}-\frac{n}{m}\right)\xi + \chi\right] \qquad (15)\\
&= -\frac{B_0}{R_0}\left[\left(\frac{1}{q}-\frac{n}{m}\right)(\xi + r\xi') - \frac{s}{q}\xi + \frac{n}{m}(\chi + r\chi')\right]. \qquad (16)
\end{aligned}
$$

In the ideal limit, the expression (16) reduces to the usual result:

$$
\delta B_I^\theta = -\frac{B_0}{R_0}\frac{\partial}{\partial r}\left[\left(\frac{1}{q}-\frac{n}{m}\right)r\xi^r\right] = -\frac{B_0}{R_0}\left[\left(\frac{1}{q}-\frac{n}{m}\right)(\xi + r\xi') - \frac{s}{q}\xi\right].
$$

The toroidal magnetic field perturbation is given below:

$$
\begin{aligned}
\delta B^\phi &= -\frac{F_0}{rR}\left[\frac{\partial}{\partial \theta}\xi_R^\theta + \frac{\partial}{\partial r}(r\xi_R^r)\right] \\
&= -\frac{B_0}{R}\left[\frac{\alpha}{2q^2}(\xi + \Delta\xi_\Gamma) + \epsilon\frac{n}{mq}\left[\left(\frac{n}{m}+\frac{2}{q}\right)\Delta q\xi_R + \frac{n}{m}r(\Delta q\xi_R)'\right]\right] \\
&= -\frac{B_0}{R}\left[\frac{\alpha}{2q^2}(\xi + \Delta\xi_\Gamma) + \epsilon\frac{n}{mq}\left\{\left(\frac{n}{m}+\frac{2}{q}\right)\Delta q\xi - \left(3 + \frac{n\Delta q}{m}\right)\chi\right\} + \epsilon\left\{s(\xi-\chi) + r\left(\frac{\Delta q}{q}\xi' - \chi'\right)\right\}\right]. \qquad (17)
\end{aligned}
$$

For the perturbed poloidal magnetic flux $\delta\psi_p(r)$, defined as the flux between $r=0$ and r, we have

$$
\delta\psi_p(r) \equiv 2\pi\int_0^r \delta B^\theta(s)\mathrm{d}s\,,
$$

which, using the convenient form (15) for δB^θ, gives:

$$
\delta\psi_p(r) = -2\pi r\frac{B_0}{R_0}\frac{n}{m}\left[\left(\frac{1}{q}-\frac{n}{m}\right)\xi + \chi\right]. \qquad (18)
$$

Finally, for the perturbed pressure, considering isotropic pressure we find:

$$
\delta P = -(\xi + \Delta\xi_\Gamma)\frac{\mathrm{d}P}{\mathrm{d}r}. \qquad (19)
$$

4 Implementation as a generalised eigenvalue solver

4.1 The resistive interchange model $\mathbf{LEMPo}^{\mathbf{IM}}_{\mathbf{res}}$

The system to solve reduces to the dispersion relation (12) together with the Ohm's law (7). These 2 equations still form a non-linear eigenvalue problem. Nevertheless, introducing a "dummy" variable, this can be cast as a linear eigenvalue problem[2], which is easier to solve accurately numerically. First, one can write the equations in terms of differential operators acting on the set of variables $(\xi^{(m)}, \frac{\gamma}{\omega}\xi^{(m)}, \chi)$:

$$
\tfrac{\gamma}{\omega_A}\mathcal{D}(\tfrac{\gamma}{\omega}\xi_0^r) = \mathcal{D}_\xi(\xi_0^r) + \mathcal{D}_\chi(\chi)
$$

$$
\tfrac{\gamma}{\omega_A}\mathcal{O}(\chi) = \mathcal{O}_\xi(\xi_0^r) + \mathcal{O}_\chi(\chi)
$$

[2]Here, "non-linear eigenvalue problem" refers to the fact that (γ/ω_A) appears in the differential equation non-linearly, not that the differential equation has non-linear dependence in its dependent variables $\xi^{(m)}$ and χ.

where the first line reproduces the dispersion relation (12), and the second line the Ohm's law (7). It is then possible to discretize these operators using a finite difference method : the discretized versions of $\mathcal{D}_{\dots}$ and $\mathcal{O}_{\dots}$ are written respectively $D_{\dots}$ and $O_{\dots}$. The problem becomes now :

$$\frac{\gamma}{\omega_A}\begin{pmatrix}\mathbb{1} & & \\ & D & \\ & & O\end{pmatrix}\begin{pmatrix}\xi_0^r \\ \frac{\gamma}{\omega}\xi_0^r \\ \chi\end{pmatrix} = \begin{pmatrix} & \mathbb{1} & \\ D_\xi & & D_\chi \\ O_\xi & & O_\chi\end{pmatrix}\begin{pmatrix}\xi_0^r \\ \frac{\gamma}{\omega}\xi_0^r \\ \chi\end{pmatrix},$$

i.e. a generalized eigenvalue problem which can be solved numerically with high accuracy without overwhelming difficulties.

5 Usage Examples

A COMPLETER, METTRE JUSTE LIEN VERS LES EXEMPLES.

```
% Setup project paths
setup

% Run a sample calculation
LEMPo_ideal('m',1,'n',1,'qProfile','reversed');

% Run simplified interchange model
IM_LEMPo_ideal('m',9,'n',10,'alphaScan',true);
```

6 Folder Structure

```
.
 src/
    LEMPo_ideal.m
    IM_LEMPo_ideal.m
    utils/
 examples/
    ideal_examples/
 setup.m
```

7 Image credits

The symbolic image of LEMPo-RMHD combines the following materials:

- Planet Venus: NASA image (public domain),
- Kuiper Belt object: from Solar System Wiki,
- Demon artwork: from the game Lempo, by One Trick Entertainment.

References

1. M. Coste-Sarguet, J. P. Graves, *Plasma Physics and Controlled Fusion*, ISSN: 0741-3335 (2024).
2. J. P. Graves, M. Coste-Sarguet, C. Wahlberg, *Plasma Physics and Controlled Fusion* **64**, 014001, ISSN: 0741-3335, (https://iopscience.iop.org/article/10.1088/1361-6587/ac3496) (Jan. 2022).