LEMPo-RMHD

Linear Eigenvalue Modular solver for Pressure- and current-driven modes (Resistive MHD) in tokamaks

Documentation

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September 12, 2025

Abstract

LEMPo-RMHD is a modular eigensolver written in *MATLAB* for studying the linear stability of internal magnetohydrodynamics (MHD) modes, in a tokamak plasma. It computes linear growth rates and perturbations associated with moderate- to long-wavelength internal MHD instabilities, including **tearing modes**, **resistive interchange** modes, and **infernal modes** (in a torus). The solver (presented in Ref. (*1*)) is based on a unified set of global equations for long-wavelength, pressure- or current-driven MHD modes, incorporating resistivity and compressibility in an inverse aspect ratio expansion (see Ref. (*2*)).

The solver is named after Lempo: both a Kuiper Belt object (discovered in 1999) and a demon from Finnish mythology. This reflects its focus on *infernal* modes, and its ultimate integration within the VENUS-MHD framework. (See credits in REF)

Contents

1 Overview

Based on an inverse aspect ratio expansion, LEMPo-RMHD computes linear growth rates and eigenmodes for moderate- to long-wavelength internal MHD instabilities, including tearing modes (in a torus), resistive interchange modes, and infernal modes. High n ballooning modes are not captured, but weak ballooning associated with toroidal effects inherent in internal kink modes and infernal modes is captured.

The inputs to the model are the q profile, B_0, and the pressure profile. Focusing on a single helicity (m, n), we then solve the dispersion relation (turned into a generalised eigenvalue problem) over the whole plasma extent, with the boundary condition that the perturbation must vanish at the plasma edge (internal instabilities). We obtain the linear growth rate γ/ω_A, the radial displacement ξ, and the resistive and compressibility corrections χ and $\Delta\xi_\Gamma$. See Section 2.5 to relate these variables to more commonly used physical quantities (*e.g.* $\boldsymbol{\delta B}$, δP ...) Except for the interchange versions of the solver, the $(m+1)$ and $(m-1)$ neighbours are also explicitely included in the process, and we obtain the associated plasma displacement.

There are four different models which can be run:

- **The resistive full model $\mathbf{LEMPo_{res}}$**: `LEMPo_res.m` file. This solves the full set of equation in Section 2: (2) for the main harmonic, (5) for the sidebands, Ohm's laws for the main harmonic (6) and for the sidebands (7), and the definition of the compressibility variable (8). It recovers interchange, infernal and Bussac-like internal kink modes as well as their resistive counterparts. Tearing modes in a torus are also retrieved.

- **The ideal full model $\mathbf{LEMPo_{ideal}}$**: `LEMPo_ideal.m` file. This is the ideal MHD limit of the full model, obtained by neglecting resistive effects (*i.e.* $\chi(r)$, $\Delta\xi_\Gamma(r) \to 0, \forall r$). It solves equation **??** for the main harmonic $\xi^{(m)}$, and equation **??** for the sidebands $\xi^{(m\pm1)}$. This model is enough to describe ideal unstable modes when important rational surfaces are avoided, or in high Lundquist number regimes, or in cases with very large pressure gradients. It recovers ideal internal modes, ideal interchange and ideal infernal modes.

- **The resistive interchange model $\mathbf{LEMPo^{IM}_{res}}$**: `IM_LEMPo_res.m` file. The interchange models **IM** are limit solutions of the full model **FM**. They substitute analytically approximated solutions for the sidebands displacements $\xi^{(m\pm1)}$, so they will not appear explicitly in the equations. The model thus consists only of Eq. (11) for $\xi^{(m)}$, and Ohm's law (6) for χ. This model is sufficient to retrieve resistive interchange including compression effects and tearing modes in a torus, but misses destabilising infernal effects (linked to higher order coupling to sidebands) that are included in the full model.

- **Interchange model $\mathbf{LEMPo^{IM}_{ideal}}$**: `IM_LEMPo_ideal.m` file. The ideal version of the interchange model neglects resistive effects and compressibility corrections, and requires only one equation: Eq. 12 for $\xi^{(m)}$.

2 Equations behind the solver

2.1 Notation

For clarity, we summarize the main symbols and variables used throughout the stability calculations:

- r — Radial flux surface label (SFL coordinate), see Eq. (**??**).

- θ — Poloidal angle in straight field line (SFL) coordinates.

- ϕ — Toroidal angle.

- R_0 — Major radius at the magnetic axis.
- $\epsilon = r/R_0$ — Inverse aspect ratio.
- $\alpha(r)$ — MHD ballooning parameter, measuring pressure drive:

$$\boxed{\alpha(r) = -\frac{2\mu_0 q^2 R_0}{B_0^2}\frac{\mathrm{d}P}{\mathrm{d}r}\ .} \tag{1}$$

- $\Delta(r)$ — Shafranov shift of flux surfaces: $\Delta'(r) = \frac{q(r)^2}{r^3}\left(\int_0^r \frac{\tilde{r}^3}{q(\tilde{r})^2 R_0} + \frac{\tilde{r}^2\alpha(\tilde{r})}{q(\tilde{r})^2}\mathrm{d}\tilde{r}\right)$.
- $q(r)$ — Safety factor (constant on each flux surface in SFL coordinates)
- $\boldsymbol{\xi}$ — Radial plasma displacement, where the vector of the plasma displacement is defined as: $\boldsymbol{v} \approx \partial\boldsymbol{\xi}/\partial t$, with $\boldsymbol{v}$ the perturbed velocity.
- $\Delta\xi_\Gamma^r$ — Variable encompassing compressibility effects, defined in Eq. (**??**)
- χ — Resistive correction to the displacement
- m, n — Poloidal and toroidal mode numbers
- $\xi^{(m)}$ — Radial displacement corresponding to poloidal harmonic m.
- $\xi^{(m\pm1)}$ — Radial displacement corresponding to the sidebands, with poloidal harmonic $m \pm 1$.

2.2 Full resistive model

The full resistive system $\mathbf{LEMPo_{res}}$ consists of seven coupled equations for the variables $(\xi^{(m)}, \xi^{(m+1)}, \xi^{(m-1)}, \chi^{(m)}, \chi^{(m+1)}, \chi^{(m-1)}, \Delta\xi_\Gamma)$ (there are no equations for the compressibility corrections on the sidebands because they are eliminated analytically), where we did not count the "dummy" variables (see Section REF).

Main harmonic equation The version for $m \neq 1$ version is presented below:

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi^{(m)}}{dr}\right]-(m^2-1)\xi^{(m)}\right\}+\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{(m)}}{dr}\right]-(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi^{(m)}\\
&+\frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d\chi}{dr}-\frac{r^3}{q_s}\chi\frac{d}{dr}\left(\frac{1}{q}\right)\right]-(m^2-1)\frac{1}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\chi\\
&+\frac{s^2}{q_s^2}D_I(r)\,\xi^{(m)}\\
&+\frac{\alpha\epsilon}{q_s^2}\left(\frac{1}{q_s^2}-1\right)\Delta\xi_\Gamma-\frac{\alpha^2}{2q_s^2}\left(\xi^{(m)}+\Delta\xi_\Gamma\right)\\
&+\frac{2+m}{2(1+m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[(1+m)\epsilon+\alpha+m\Delta'\right]\right)\overline{\xi}^{(m+1)}\\
&+\frac{2-m}{2(1-m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[(1-m)\epsilon+\alpha-m\Delta'\right]\right)\overline{\xi}^{(m-1)}\\
&+\frac{r}{2(1+m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[2(1+m)\epsilon+(2+m)\alpha-(1+2m)\Delta'\right]\right)\overline{\xi}^{(m+1)\prime}-\frac{\Delta q}{q_s^3}\frac{r^2}{(1+m)}\Delta'\overline{\xi}^{(m+1)\prime\prime}\\
&+\frac{r}{2(1-m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[2(1-m)\epsilon+(2-m)\alpha-(1-2m)\Delta'\right]\right)\overline{\xi}^{(m-1)\prime}-\frac{\Delta q}{q_s^3}\frac{r^2}{(1-m)}\Delta'\overline{\xi}^{(m-1)\prime\prime}\\
&+\chi\left(-\frac{1}{q_s^2}\left[4\epsilon^2(2-\frac{1}{q_s^2})+\epsilon\alpha(3+\frac{1}{q_s^2})+\alpha^2+\Delta'(12\Delta'-7\alpha+r\alpha'-6\epsilon)\right]\right)\\
&+\xi^{(m)}\left(\frac{\Delta q}{q_s^3}\left[4\epsilon^2(2-\frac{1}{q_s^2})+\epsilon\alpha(5-\frac{2}{q_s^2})+2\alpha^2+\Delta'(12\Delta'-7\alpha+r\alpha'-6\epsilon)\right]\right)\\
&+\frac{r}{q_s^2}\alpha\Delta'\left(\frac{\Delta q}{q_s}\xi^{(m)\prime}-\chi'\right)\\
&+\Delta\xi_\Gamma\left(\frac{\Delta q}{q_s^3}\left[\epsilon\alpha\left(2-\frac{3}{q_s^2}\right)+\alpha^2\right]-\frac{q'}{q_s^3}r\Delta'\alpha\right)\,,
\end{aligned}
\tag{2}
$$

where the Mercier term $s^2D_I(r)$ is given in below:

$$
s^2D_I(r) = \alpha\epsilon\left[\frac{1}{q_s^2}-1+\frac{3}{4}(\kappa-1)\left(1-2\frac{\delta}{\epsilon}\right)\right]\,. \tag{3}
$$

We consider flux surfaces with constant elongation κ along the minor radius r, and triangularity following $\delta(r) = \delta_1\, r/r_1$ for $r < r_1$, where r_1 marks the end of the low shear region. This is exact in a large aspect ratio, low beta tokamak, for a shearless q profile and more generally $\delta(r) = \delta_a\, r/a$ provides a good approximation. But it remains an approximation, and since we also neglected the quasicylindrical shaping corrections, **values of κ and δ_a should not be pushed too high.**

The $m = 1$ version of the governing equation for the main harmonic reads:

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi^{(m)}}{dr}\right]\right\} + \frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{(m)}}{dr}\right] \\
&+ \frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d\chi}{dr} - \frac{r^3}{q_s}\chi\frac{d}{dr}\left(\frac{1}{q}\right)\right] \\
&+ \frac{\alpha}{q_s^2}\left[\epsilon\left(\frac{1}{q_s^2}-1+\frac{\Delta q}{2q_s}\right)-\frac{\alpha}{4}\right]\left(\xi^{(m)}+\Delta\xi_\Gamma\right) + \frac{\alpha\epsilon}{q_s^2}\left[\frac{3}{4}(\kappa-1)\left(1-2\frac{\delta}{\epsilon}\right)\right]\xi^{(m)} \\
&- \frac{r\alpha\Delta'}{2q_s^2}\left(2\chi' + \frac{\Delta q'}{q_s}\Delta\xi_\Gamma + \frac{\Delta q}{q_s}\left(\Delta\xi_\Gamma' - \xi^{(m)\prime}\right)\right) \\
&+ \frac{2+m}{2(1+m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[(1+m)\epsilon + \alpha + m\Delta'\right]\right)\overline{\xi}^{(m+1)} \\
&+ \frac{r}{2(1+m)q_s^2}\left(\alpha - 2\frac{\Delta q}{q_s}\left[2(1+m)\epsilon + (2+m)\alpha - (1+2m)\Delta'\right]\right)\overline{\xi}^{(m+1)\prime} - \frac{\Delta q}{q_s^3}\frac{r^2}{(1+m)}\Delta'\overline{\xi}^{(m+1)\prime\prime} \\
&+ \chi\left(-\frac{1}{q_s^2}\left[4\epsilon^2(2-\frac{1}{q_s^2}) + \epsilon\alpha(\frac{7}{2}+\frac{1}{q_s^2}) + \frac{3}{2}\alpha^2 + \Delta'(12\Delta' - 7\alpha + r\alpha' - 6\epsilon)\right]\right) \\
&+ \xi^{(m)}\left(\frac{\Delta q}{q_s^3}\left[4\epsilon^2(2-\frac{1}{q_s^2}) + \epsilon\alpha(5-\frac{2}{q_s^2}) + \frac{3}{2}\alpha^2 + \Delta'(12\Delta' - \frac{13}{2}\alpha + \frac{1}{2}r\alpha' - 6\epsilon)\right]\right) \\
&+ \frac{\Delta q}{q_s^3}\Delta\xi_\Gamma\left(3\epsilon\alpha\left(\frac{1}{2}-\frac{1}{q_s^2}\right) + \frac{1}{2}\Delta'(\alpha - r\alpha')\right) - \frac{\Delta q'}{2q_s^3}\Delta\xi_\Gamma r\Delta'\alpha\,.
\end{aligned}
\tag{4}
$$

Sideband equations The sidebands variables in the code are denoted $\overline{\xi}^{(m\pm1)}$ in the current document and in related articles. Indeed these are modified sidebands, linked to the actual sideband displacement via:

$$\xi^{(m\pm1)} = \overline{\xi}^{(m\pm1)} + \frac{1\pm m}{r^{2\pm m}}\int_0^r l^{2\pm m}\Delta'\frac{d}{dl}\left(\frac{\Delta q}{q_s}\xi^{(m)} - \chi\right)\mathrm{d}l\,,$$

the code's final output is the correct $\xi^{(m\pm1)}$, the reverse change of variable is done after the eigenvalue problem is solved.

The governing equation for the modified sidebands $\overline{\xi}^{(m\pm1)}$ is given below, with $q_{m\pm1} \equiv$

$(m \pm 1)/n$:

$$
\begin{aligned}
0 = &\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q} - \frac{n}{m\pm 1}\right)^2 \frac{d\overline{\xi}^{r(m\pm1)}}{dr}\right] - m(m\pm 2)\left(\frac{1}{q} - \frac{n}{(m\pm 1)}\right)^2 \overline{\xi}^{r(m\pm1)} \\
&+ \frac{\gamma^2(1+2q_{m\pm1})^2)}{(m\pm 1)^2\omega_A^2}\left\{\left(\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\overline{\xi}^{r(m\pm1)}}{dr}\right] - m(m\pm 2)\overline{\xi}^{r(m\pm1)}\right)\right\} \\
&+ \frac{1}{r}\frac{d}{dr}\left[r^3\frac{n}{(m\pm1)}\left(\frac{1}{q} - \frac{n}{(m\pm 1)}\right)\frac{d\chi^{(m\pm1)}}{dr}\right] - \frac{1}{r}\frac{d}{dr}\left[r^3\frac{n}{(m\pm1)}\left(\frac{1}{q}\right)'\chi^{(m\pm1)}\right] \\
&- m(m\pm 2)\frac{1}{q_{m\pm1}}\left(\frac{1}{q} - \frac{n}{(m\pm 1)}\right)\chi^{(m\pm1)} \\
&- \frac{r^{1\pm m}}{2q_s^2(1\pm m)}\frac{d}{dr}\left(\frac{\alpha}{r^{\pm m}}(\xi^{(m)} + \Delta\xi_\Gamma)\right) \\
&+ \frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left(\left[(1\pm 2m)\epsilon + (1\pm m)(\alpha - 3\Delta')\right]\frac{1}{r^{\pm m}}\left(\frac{\Delta q}{q_s}\xi^{(m)} - \chi^{(m)}\right)\right) \\
&+ \frac{1}{q_s^2}(2\pm m)(\epsilon + \alpha - 4\Delta')\left(\frac{\Delta q}{q_s}\xi^{(m)} - \chi^{(m)}\right) \\
&+ \frac{r^{1\pm m}}{q_s^2(1\pm m)}\frac{d}{dr}\left[\frac{\alpha}{r^{\pm m}}\left(\chi^{(m)} + \frac{\Delta q}{q_s}\Delta\xi_\Gamma\right)\right].
\end{aligned} \tag{5}
$$

This is valid for $\overline{\xi}^{(m\pm1)}$ for $m > 1$ and $\overline{\xi}^{(m+1)}$ for $m = 1$.

Additional relations needed to close the system The equations cited above involve the variables χ and $\Delta\xi_\Gamma$, whose relationship with ξ^r are needed to close the system. We have "Ohm's law":

$$
\begin{aligned}
\frac{\gamma}{\omega_A}r^3\chi = \frac{r_s^2}{S_L}&\left[\frac{d}{dr}\left(r^3\frac{d}{dr}\left(\frac{m}{n}\left(\frac{1}{q} - \frac{n}{m}\right)\xi + \chi\right)\right)\right. \\
&\left. + r(1-m^2)\left(\frac{m}{n}\left(\frac{1}{q} - \frac{n}{m}\right)\xi + \chi\right)\right],
\end{aligned} \tag{6}
$$

If one wishes to include resistivity on the sidebands, the sideband harmonics of χ which appear in (5) follow a similar Ohm's law:

$$
\begin{aligned}
\frac{\gamma}{\omega_A}r^3\chi^{(m\pm1)} = \frac{r_{m\pm1}^2}{S_L}&\left[\frac{d}{dr}\left(r^3\frac{d}{dr}\left(\frac{(m\pm1)}{n}\left\{\frac{1}{q} - \frac{n}{(m\pm 1)}\right\}\overline{\xi}^{(m\pm1)} + \chi^{(m\pm1)}\right)\right)\right. \\
&\left. + r(1-(m\pm1)^2)\left(\frac{(m\pm1)}{n}\left\{\frac{1}{q} - \frac{n}{(m\pm 1)}\right\}\overline{\xi}^{(m\pm1)} + \chi^{(m\pm1)}\right)\right],
\end{aligned} \tag{7}
$$

where $r_{m\pm1}$ is defined as $q(r_{m\pm1}) = (m\pm1)/n$.

For the compressibility variable, we use its definition:

$$
\Delta\xi_\Gamma^r = -\chi\left(\frac{q}{q-q_s}\right)\left(\frac{\omega_s^2(nq-m)^2}{\omega_s^2(nq-m)^2 + \gamma^2 q^2}\right). \tag{8}
$$

2.3 Full ideal model

This is the ideal MHD limit of the full model, obtained by neglecting resistive effects (*i.e.* $\chi(r)$, $\Delta\xi_\Gamma(r) \to 0, \forall r$ in all equations above). The full ideal system $\mathbf{LEMPo_{ideal}}$ will thus consists of three coupled equations for the variables $(\xi^{(m)}, \xi^{(m+1)}, \xi^{(m-1)})$. There are no "dummy" variables for the ideal model.

The version for $m \neq 1$ version is presented below:

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi^{(m)}}{dr}\right]-(m^2-1)\xi^{(m)}\right\}+\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{(m)}}{dr}\right]-(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi^{(m)}\\
&+\frac{s^2}{q_s^2}D_I(r)\,\xi^{(m)}-\frac{\alpha^2}{2q_s^2}\xi^{(m)}\\
&+\frac{2+m}{2(1+m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[(1+m)\epsilon+\alpha+m\Delta'\right]\right)\overline{\xi}^{(m+1)}\\
&+\frac{2-m}{2(1-m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[(1-m)\epsilon+\alpha-m\Delta'\right]\right)\overline{\xi}^{(m-1)}\\
&+\frac{r}{2(1+m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[2(1+m)\epsilon+(2+m)\alpha-(1+2m)\Delta'\right]\right)\overline{\xi}^{(m+1)\prime}-\frac{\Delta q}{q_s^3}\frac{r^2}{(1+m)}\Delta'\overline{\xi}^{(m+1)\prime\prime}\\
&+\frac{r}{2(1-m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[2(1-m)\epsilon+(2-m)\alpha-(1-2m)\Delta'\right]\right)\overline{\xi}^{(m-1)\prime}-\frac{\Delta q}{q_s^3}\frac{r^2}{(1-m)}\Delta'\overline{\xi}^{(m-1)\prime\prime}\\
&+\xi^{(m)}\left(\frac{\Delta q}{q_s^3}\left[4\epsilon^2(2-\frac{1}{q_s^2})+\epsilon\alpha(5-\frac{2}{q_s^2})+2\alpha^2+\Delta'(12\Delta'-7\alpha+r\alpha'-6\epsilon)\right]\right)\\
&+\frac{r}{q_s^2}\alpha\Delta'\frac{\Delta q}{q_s}\xi^{(m)\prime},
\end{aligned}
\tag{9}
$$

where the Mercier term $s^2D_I(r)$ is given in Eq. 3.

The $m=1$ version of the governing equation for the main harmonic reads:

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi^{(m)}}{dr}\right]\right\}+\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi^{(m)}}{dr}\right]\\
&+\frac{\alpha}{q_s^2}\left[\epsilon\left(\frac{1}{q_s^2}-1+\frac{\Delta q}{2q_s}\right)-\frac{\alpha}{4}\right]\xi^{(m)}+\frac{\alpha\epsilon}{q_s^2}\left[\frac{3}{4}(\kappa-1)\left(1-2\frac{\delta}{\epsilon}\right)\right]\xi^{(m)}\\
&+\frac{r\alpha\Delta'}{2q_s^2}\frac{\Delta q}{q_s}\xi^{(m)\prime}\\
&+\frac{2+m}{2(1+m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[(1+m)\epsilon+\alpha+m\Delta'\right]\right)\overline{\xi}^{(m+1)}\\
&+\frac{r}{2(1+m)q_s^2}\left(\alpha-2\frac{\Delta q}{q_s}\left[2(1+m)\epsilon+(2+m)\alpha-(1+2m)\Delta'\right]\right)\overline{\xi}^{(m+1)\prime}-\frac{\Delta q}{q_s^3}\frac{r^2}{(1+m)}\Delta'\overline{\xi}^{(m+1)\prime\prime}\\
&+\xi^{(m)}\left(\frac{\Delta q}{q_s^3}\left[4\epsilon^2(2-\frac{1}{q_s^2})+\epsilon\alpha(5-\frac{2}{q_s^2})+\frac{3}{2}\alpha^2+\Delta'(12\Delta'-\frac{13}{2}\alpha+\frac{1}{2}r\alpha'-6\epsilon)\right]\right).
\end{aligned}
\tag{10}
$$

2.4 Interchange models

The interchange models $\mathbf{LEMPo}^{\mathbf{IM}}_{\mathbf{ideal}}$ and $\mathbf{LEMPo}^{\mathbf{IM}}_{\mathbf{res}}$ are limit solutions of the full models $\mathbf{LEMPo}_{\mathbf{ideal}}$ and $\mathbf{LEMPo}_{\mathbf{res}}$. They substitute analytically approximated solutions for the sidebands displacements $\xi^{(m\pm1)}$, so they will not appear explicitly in the following equations.

Resistive interchange model The $\mathbf{LEMPo}^{\mathbf{IM}}_{\mathbf{res}}$ model thus consists only of Eq. (11) for ξ, and Ohm's law (6) for χ (to lighten the notation, ξ will stand for $\xi^{(m)}$ when no confusion is possible). For the "dummy" variables see Section REF.

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi}{dr}\right]-(m^2-1)\xi\right\}\\
&+\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi}{dr}\right]-(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi\\
&+\frac{1}{r}\frac{d}{dr}\left[\frac{r^3}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\frac{d\chi}{dr}-\frac{r^3}{q_s}\chi\frac{d}{dr}\left(\frac{1}{q}\right)\right]\\
&-(m^2-1)\frac{1}{q_s}\left(\frac{1}{q}-\frac{1}{q_s}\right)\chi+\frac{\epsilon\alpha}{q_s^2}\left(\frac{1}{q_s^2}-1\right)(\xi+\Delta\xi_\Gamma)\\
&-\frac{\alpha}{q_s^2}\Delta' r\frac{d}{dr}\left(\chi+\frac{q-q_s}{q}\Delta\xi_\Gamma\right).
\end{aligned}
\tag{11}
$$

This model is sufficient to retrieve resistive interchange including compression effects and tearing modes in a torus, but misses destabilising infernal effects (linked to higher order coupling to sidebands) that are included in the full model.

Ideal interchange model The ideal version of the interchange model $\mathbf{LEMPo}^{\mathbf{IM}}_{\mathbf{ideal}}$ neglects resistive effects and compressibility corrections, and requires only one equation:

$$
\begin{aligned}
0 =& \frac{\gamma^2(1+2q_s^2)}{m^2\omega_A^2}\left\{\frac{1}{r}\frac{d}{dr}\left[r^3\frac{d\xi}{dr}\right]-(m^2-1)\xi\right\}\\
&+\frac{1}{r}\frac{d}{dr}\left[r^3\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\frac{d\xi}{dr}\right]-(m^2-1)\left(\frac{1}{q}-\frac{1}{q_s}\right)^2\xi\\
&+\frac{\epsilon\alpha}{q_s^2}\left(\frac{1}{q_s^2}-1\right)\xi,
\end{aligned}
\tag{12}
$$

for the eigenvector ξ.

2.5 Link to other physical quantities

Perturbations in δB, $\delta\psi$ and δP

For some applications, it is useful to have explicit expressions that relate these variables to more commonly used physical quantities. In this section, we show how to recover the perturbation to the equilibrium magnetic field $\boldsymbol{\delta B}$, the perturbed poloidal magnetic flux $\delta\psi_p$, and the perturbed pressure δP.

We obtain, for the (m,n) helicity, the following contravariant components of $\boldsymbol{\delta B}$ (to leading order in ϵ):

$$
\begin{aligned}
\delta B^r &= \frac{F_0}{R^2}\left[\frac{1}{q}\frac{\partial}{\partial\theta}+\frac{\partial}{\partial\phi}\right]\xi_R^r\\
&= -\frac{B_0}{R_0}in\left[\left(\frac{1}{q}-\frac{n}{m}\right)\xi+\chi\right],
\end{aligned}
\tag{13}
$$

where we drop the superscript r for the ideal displacement ξ, as in the previous section, since the radial displacement is the primary output of our equations. For the poloidal perturbation, we arrive at:

$$\delta B^\theta = -\frac{F_0}{R^2}\left[\frac{\partial}{\partial r}\left(\frac{r\xi^r_R}{q}\right) - \frac{\partial}{\partial \phi}\xi^\theta_R\right]$$

$$= -r\frac{B_0}{R_0}\frac{n}{m}\frac{\partial}{\partial r}\left[\frac{m}{n}\left(\frac{1}{q}-\frac{n}{m}\right)\xi + \chi\right] \tag{14}$$

$$= -\frac{B_0}{R_0}\left[\left(\frac{1}{q}-\frac{n}{m}\right)(\xi + r\xi') - \frac{s}{q}\xi + \frac{n}{m}\left(\chi + r\chi'\right)\right]. \tag{15}$$

In the ideal limit, the expression (15) reduces to the usual result:

$$\delta B^\theta_I = -\frac{B_0}{R_0}\frac{\partial}{\partial r}\left[\left(\frac{1}{q}-\frac{n}{m}\right)r\xi^r\right] = -\frac{B_0}{R_0}\left[\left(\frac{1}{q}-\frac{n}{m}\right)(\xi + r\xi') - \frac{s}{q}\xi\right].$$

The toroidal magnetic field perturbation is given below:

$$\delta B^\phi = -\frac{F_0}{rR}\left[\frac{\partial}{\partial\theta}\xi^\theta_R + \frac{\partial}{\partial r}\left(r\xi^r_R\right)\right]$$

$$= -\frac{B_0}{R}\left[\frac{\alpha}{2q^2}\left(\xi + \Delta\xi_\Gamma\right) + \epsilon\frac{n}{mq}\left[\left(\frac{n}{m}+\frac{2}{q}\right)\Delta q\xi_R + \frac{n}{m}r\left(\Delta q\xi_R\right)'\right]\right]$$

$$= -\frac{B_0}{R}\left[\frac{\alpha}{2q^2}\left(\xi + \Delta\xi_\Gamma\right) + \epsilon\frac{n}{mq}\left\{\left(\frac{n}{m}+\frac{2}{q}\right)\Delta q\xi - \left(3 + \frac{n\Delta q}{m}\right)\chi\right\} + \epsilon\left\{s\left(\xi - \chi\right) + r\left(\frac{\Delta q}{q}\xi' - \chi'\right)\right\}\right]. \tag{16}$$

For the perturbed poloidal magnetic flux $\delta\psi_p(r)$, defined as the flux between $r = 0$ and r, we have

$$\delta\psi_p(r) \equiv 2\pi\int_0^r \delta B^\theta(s)\mathrm{d}s\,,$$

which, using the convenient form (14) for δB^θ, gives:

$$\delta\psi_p(r) = -2\pi r\frac{B_0}{R_0}\frac{n}{m}\left[\left(\frac{1}{q}-\frac{n}{m}\right)\xi + \chi\right]. \tag{17}$$

Finally, for the perturbed pressure, considering isotropic pressure we find:

$$\delta P = -\left(\xi + \Delta\xi_\Gamma\right)\frac{\mathrm{d}P}{\mathrm{d}r}\,. \tag{18}$$

3 Implementation as a generalised eigenvalue solver

3.1 The resistive interchange model $\mathrm{LEMPo}^{\mathrm{IM}}_{\mathrm{res}}$

The system to solve reduces to the dispersion relation (11) together with the Ohm's law (6). These 2 equations still form a non-linear eigenvalue problem. Nevertheless, introducing a "dummy" variable, this can be cast as a linear eigenvalue problem[1], which is easier to solve accurately numerically. First, one can write the equations in terms of differential operators acting on the set of variables $(\xi^{(m)}, \frac{\gamma}{\omega}\xi^{(m)}, \chi)$:

$$\tfrac{\gamma}{\omega_A}\mathcal{D}(\tfrac{\gamma}{\omega}\xi^r_0) = \mathcal{D}_\xi(\xi^r_0) + \mathcal{D}_\chi(\chi)$$

$$\tfrac{\gamma}{\omega_A}\mathcal{O}(\chi) = \mathcal{O}_\xi(\xi^r_0) + \mathcal{O}_\chi(\chi)$$

[1]Here, "non-linear eigenvalue problem" refers to the fact that (γ/ω_A) appears in the differential equation non-linearly, not that the differential equation has non-linear dependence in its dependent variables $\xi^{(m)}$ and χ.

where the first line reproduces the dispersion relation (11), and the second line the Ohm's law (6). It is then possible to discretize these operators using a finite difference method : the discretized versions of $\mathcal{D}_{\dots}$ and $\mathcal{O}_{\dots}$ are written respectively $D_{\dots}$ and $O_{\dots}$. The problem becomes now :

$$\frac{\gamma}{\omega_A}\begin{pmatrix} \mathbb{1} & & \\ & D & \\ & & O \end{pmatrix}\begin{pmatrix} \xi_0^r \\ \frac{\gamma}{\omega}\xi_0^r \\ \chi \end{pmatrix} = \begin{pmatrix} & \mathbb{1} & \\ D_\xi & & D_\chi \\ O_\xi & & O_\chi \end{pmatrix}\begin{pmatrix} \xi_0^r \\ \frac{\gamma}{\omega}\xi_0^r \\ \chi \end{pmatrix},$$

i.e. a generalized eigenvalue problem which can be solved numerically with high accuracy without overwhelming difficulties.

4 Inputs, Options, and Outputs

4.1 Inputs

The inputs required to run the solver `LEMPo_ideal` are:

- m, n — Poloidal and toroidal mode numbers of the main harmonic.
- N — Number of points in the radial grid.
- `ev_guess` — Initial guess for the eigenvalue (should be well above the expected value).
- `toPlot` — Boolean flag to enable output plots
- `profiles` — Struct containing equilibrium profiles and other necessary input quantities. A COMPLETER
- `opts` — Struct containing solver options; can be set to empty `[]` to use defaults.

4.2 Options

Additional options that can be set in the `opts` struct include:

- `additionalPlot` — Boolean flag to enable additional output plots (input profiles, details on the run ...)
- A COMPLETER

4.3 Outputs

- Linear growth rates
- Eigenfunctions (displacement fields)

5 Usage Examples

```
% Setup project paths
setup

% Run a sample calculation
LEMPo_ideal('m',1,'n',1,'qProfile','reversed');

% Run simplified interchange model
IM_LEMPo_ideal('m',9,'n',10,'alphaScan',true);
```

6 Folder Structure

```
.
 src/
    LEMPo_ideal.m
    IM_LEMPo_ideal.m
    utils/
 examples/
    ideal_examples/
 setup.m
```

References

1. M. Coste-Sarguet, J. P. Graves, *Plasma Physics and Controlled Fusion*, ISSN: 0741-3335 (2024).
2. J. P. Graves, M. Coste-Sarguet, C. Wahlberg, *Plasma Physics and Controlled Fusion* **64**, 014001, ISSN: 0741-3335, (https://iopscience.iop.org/article/10.1088/1361-6587/ac3496) (Jan. 2022).