

A Catalog of Mass Models for Gravitational Lensing

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ABSTRACT

Many different families of mass models are used in modern applications of strong gravitational lensing. I review a wide range of popular models, with two points of emphasis: (1) a discussion of strategies for building models suited to a particular lensing problem; and (2) a summary of technical results for a canonical set of models. All of the models reviewed here are included in publicly-available lensing software called *gravlens*.

1. Introduction

As applications of strong gravitational lensing have become more sophisticated, the variety and complexity of mass distributions used for lensing studies have increased. Gone are the days when the singular isothermal sphere was all you needed to know. Now we have softened power law ellipsoids, pseudo-Jaffe models, generalized NFW models, exponential disks embedded in isothermal halos, and so on. Plus, all of these models are likely to be perturbed by other galaxies, groups, or clusters near the lens galaxy or along the line of sight.

I have developed software called *gravlens* for a variety of lensing and lens modeling applications.² In the course of writing the code I have collected most of the mass models used for lensing studies, and this paper reviews those models. The outline is as follows. Section 2 discusses circular and elliptical symmetry in lensing mass distributions, and argues that a canonical set of circular and elliptical models provides a useful basis set for building much more complex composite models. Section 3 gives some suggestions (propaganda, really) about selecting models appropriate for your application. Section 4 presents the general equations that describe the lensing properties of a given mass distribution. Finally, Section 5 presents a detailed catalog of results for a number of models. The mass models discussed in this paper (and included in the *gravlens* software) are listed in Table 1.

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²The software is discussed in a separate paper (Keeton 2001), and is available to the community via the web site of the CfA/Arizona Space Telescope Lens Survey, at <http://cfa-www.harvard.edu/castles>.

2. Circular, Elliptical, and Composite Lens Models

The lensing properties of any mass distribution can be written in terms of two-dimensional integrals over the surface mass density (see §4). In general the integrals cannot be evaluated analytically, but many lensing applications offer simplifications due to symmetry. Applications such as microlensing may permit the use of a lens with circular symmetry, in which case the lensing properties can usually be found analytically. In other applications, such as lensing by galaxies, it may be reasonable to assume elliptical symmetry, which allows the lensing properties to be written as a set of one-dimensional integrals (see §4); the integrals can sometimes be evaluated analytically and are always amenable to fast numerical techniques.

The geometric simplifications may not hold in increasingly sophisticated lensing applications. For example, companion stars or planets can break circular symmetry in microlensing (e.g., Mao & Paczyński 1991; Gould & Loeb 1992); neighboring galaxies (e.g., Young et al. 1981; Schechter et al. 1997; Koopmans & Fassnacht 1999) or the lens galaxy’s internal structure (e.g., Maller, Flores & Primack 1997; Bernstein & Fischer 1999; Keeton et al. 2000) may break elliptical symmetry in lensing by galaxies; and individual cluster members can ruin the symmetry of the smooth background potential in lensing by clusters (e.g., Bartelmann & Steinmetz 1995; Tyson, Kochanski & Dell’Antonio 1998). Nevertheless, in all of these examples the total mass distribution can be written as a combination of circular or elliptical components, either placed at different positions to represent different objects, or combined at the same position to mimic an object with more complicated internal structure (also see Schneider & Weiss 1991). Because the Poisson equation is linear, the lensing potential of such composite models is simply the sum of the component potentials. In other words, arbitrary combinations of circular and elliptical models yield a wide range of complex mass distributions whose lensing properties are nevertheless easy to compute.

While composite models provide a great deal of freedom and complexity, they are not completely general. This limitation is eliminated in the elegant algorithm by Saha & Williams (1997; also Williams & Saha 2000) for finding non-parametric lens models. The approach is to introduce a set of mass pixels and construct a large linear programming problem for determining their masses. The problem is severely underconstrained, but by requiring positive-definite masses and imposing some smoothness criteria it is possible to find a wide but finite range of non-parametric models consistent with the data.

A limitation of the Saha & Williams algorithm is that the constraint of a positive-definite surface density is weaker than the constraint of a positive 3-d mass density — or, better yet, a positive-definite quasi-equilibrium distribution function. In other words, while parametric models may provide too little freedom, the Saha & Williams non-parametric models provide too much. For example, their method does not determine whether models are consistent with stellar dynamics, and many of the models found by the method probably are not consistent. The *gravlens* software includes only parametric models that have physical motivations apart from lensing, but it offers the ability to combine them in arbitrary ways to achieve extensive and reasonable complexity.

3. Selecting Models

Selecting the class of models to use for a particular application is a key part of lens modeling. When modeling lenses produced by galaxies, a simple and useful place to start is an isothermal model, i.e. a model with density $\rho \propto r^{-2}$ and a flat rotation curve. Spiral galaxy rotation curves (e.g., Rubin, Ford & Thonnard 1978, 1980), stellar dynamics of elliptical galaxies (e.g., Rix et al. 1997), X-ray halos of elliptical galaxies (Fabbiano 1989), models of some individual lenses (e.g., Kochanek 1995; Cohn et al. 2001), joint lensing and dynamical analyses (e.g., Treu & Koopmans 2002; Koopmans & Treu 2002), and lens statistics (e.g., Maoz & Rix 1993; Kochanek 1993, 1996) are all consistent with roughly isothermal profiles.

However, an isolated isothermal ellipsoid rarely yields a good quantitative fit to observed lenses (e.g., Keeton et al. 1997, 1998; Witt & Mao 1997). In general, adding parameters to the radial profile of the galaxy fails to produce a good fit, but adding parameters to the angular structure of the potential dramatically improves the fit (e.g., Keeton & Kochanek 1997; Keeton et al. 1997). The additional angular structure comes from the tidal perturbations of objects near the main lens galaxy or along the line of sight. In other words, the fact that few galaxies are truly isolated means that lens models generically require two independent sources of angular structure: an ellipsoidal galaxy plus external perturbations. The combination of angular terms can make it difficult to disentangle the shape of the galaxy and the nature of the external perturbations, and it is extremely important to understand any degeneracies between the two sources of angular structure before drawing conclusions from the models (see Keeton et al. 1997).

To move beyond isothermal models and explore other radial profiles, softened power law lens models have traditionally been very popular. However, these models have flat cores, while many early-type galaxies have cuspy luminosity distributions (e.g., Faber et al. 1997), and dark matter halos in cosmological simulations have cuspy mass distributions (e.g., Navarro, Frenk & White 1996, 1997; Moore et al. 1998, 1999). The lack of central or “odd” images in most observed galaxy lenses also limits the extent to which galaxies can have flat cores (e.g., Wallington & Narayan 1993; Rusin & Ma 2001). Cohn et al. (2001) thus argue that softened power law models are outdated and should be replaced with families of cuspy lens models. There are three traditional families of models in which the profile of the cusp is fixed: NFW (Navarro, Frenk & White 1996, 1997), Hernquist (1990), and de Vaucouleurs (1948) models. In addition, there are three families of models in which the cusp is taken to be an arbitrary power law: Nuker law models (Lauer et al. 1995; Byun 1996; Keeton 2002), and two variants of NFW models (see §5 for definitions; Jing & Suto 2000; Keeton & Madau 2001; Muñoz, Kochanek & Keeton 2001; Wyithe, Turner & Spergel 2001).

An important reason to study families of cuspy lens models is to test the prediction from the Cold Dark Matter paradigm that halos at a wide range of masses are consistent with a unified family of halo models (e.g., Navarro et al. 1996, 1997). The distribution of image separations among the known lenses rules out the hypothesis that galaxies and clusters can be described by identical lens models (Keeton 1998; Porciani & Madau 2000; Kochanek & White 2001). However, it may be

possible to resolve the conflict with a model in which galaxies and clusters start out with similar halo profiles, but baryonic processes such as cooling modify the inner profiles of galaxies. Kochanek & White (2001) show that such a model can match the observed image separation distribution, but it remains to be seen whether the model agrees with the lack of central or “odd” images in observed lenses, or with detailed models of individual lenses.

4. General Equations

For a mass distribution with surface mass density $\kappa(\mathbf{x}) = \Sigma(\mathbf{x})/\Sigma_{\text{cr}}$ in units of the critical surface density for lensing, the two-dimensional lensing potential is (e.g., Schneider, Ehlers & Falco 1992)

$$\phi(\mathbf{x}) = \frac{1}{\pi} \int \ln |\mathbf{x} - \mathbf{y}| \kappa(\mathbf{y}) d\mathbf{y}. \quad (1)$$

The other lensing properties can be derived from the potential. The deflection angle $\nabla\phi$ determines the positions of images via the lens equation,

$$\mathbf{u} = \mathbf{x} - \nabla\phi(\mathbf{x}), \quad (2)$$

where \mathbf{u} is the source position. The magnification tensor,

$$\mu \equiv \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right)^{-1} = \begin{bmatrix} 1 - \phi_{xx} & -\phi_{xy} \\ -\phi_{xy} & 1 - \phi_{yy} \end{bmatrix}^{-1}, \quad (3)$$

determines the distortions and brightnesses of images. (Subscripts denote partial differentiation, $\phi_{ij} \equiv \partial^2\phi/\partial x_i\partial x_j$.) Many lensing applications involve only the locations and brightnesses of the images, and thus require only the deflection and magnification components. Applications that involve the time delays, such as lensing measurements of the Hubble constant H_0 , also require the potential.

If the mass distribution has circular symmetry, the deflection vector is purely radial and has an amplitude given by the 1-d integral

$$\phi_r(r) = \frac{2}{r} \int_0^r u \kappa(u) du = \frac{1}{\pi\Sigma_{\text{cr}}} \frac{M_{\text{cyl}}(r)}{r}, \quad (4)$$

where $M_{\text{cyl}}(r)$ is the mass enclosed by a cylinder of radius r (the projected mass), which is often easily evaluated. The potential and magnification components can be obtained by integrating or differentiating ϕ_r .

More general is the case of elliptical symmetry, in which the surface mass density has the form

$$\kappa = \kappa(\xi), \quad \text{where} \quad \xi^2 = x^2 + y^2/q^2, \quad (5)$$

where q is the projected axis ratio and ξ is an ellipse coordinate. This is the functional form in a coordinate system with the ellipse centered on the origin and aligned along the x -axis; other

coordinate systems can be reached by suitable translation and rotation. With elliptical symmetry the lensing properties can be written as a set of 1-d integrals (see Schramm 1990, although I have changed variables in the integrals),

$$\phi(x, y) = \frac{q}{2} I(x, y) \quad (6)$$

$$\phi_x(x, y) = q x J_0(x, y) \quad (7)$$

$$\phi_y(x, y) = q y J_1(x, y) \quad (8)$$

$$\phi_{xx}(x, y) = 2 q x^2 K_0(x, y) + q J_0(x, y) \quad (9)$$

$$\phi_{yy}(x, y) = 2 q y^2 K_2(x, y) + q J_1(x, y) \quad (10)$$

$$\phi_{xy}(x, y) = 2 q x y K_1(x, y) \quad (11)$$

where the integrals are

$$I(x, y) = \int_0^1 \frac{\xi(u)}{u} \frac{\phi_r(\xi(u))}{[1 - (1 - q^2)u]^{1/2}} du \quad (12)$$

$$J_n(x, y) = \int_0^1 \frac{\kappa(\xi(u)^2)}{[1 - (1 - q^2)u]^{n+1/2}} du \quad (13)$$

$$K_n(x, y) = \int_0^1 \frac{u \kappa'(\xi(u)^2)}{[1 - (1 - q^2)u]^{n+1/2}} du \quad (14)$$

$$\text{where } \xi(u)^2 = u \left(x^2 + \frac{y^2}{1 - (1 - q^2)u} \right) \quad (15)$$

and $\kappa'(\xi^2) = d\kappa(\xi^2)/d(\xi^2)$. Note from eq. (12) that the potential can be written as an integral over the circular deflection function ϕ_r from eq. (4), but ϕ_r must be evaluated at the appropriate ellipse coordinate $\xi(u)$.

All of the previous expressions assume that the surface density κ is known. Some models have 3-d density distributions for which the projection integral cannot be evaluated analytically. In this case even a spherical lens model requires computationally expensive double integrals (the projection integral followed by the lensing integral). However, the double integrals can be rewritten as follows so the projection integral is replaced by the enclosed mass $M(r)$, which can often be computed analytically. (The mass $M(r)$ is the mass in spheres, which is different from $M_{\text{cyl}}(r)$ in eq. 4.) Writing $\kappa(r)$ as an integral over $\rho(r)$ — the projection integral — and substituting $\rho(r) = M'(r)/(4\pi r^2)$ where $M'(r) = dM/dr$, we find

$$\kappa(r) = \frac{1}{2\pi\Sigma_{\text{cr}}} \int_r^\infty du \frac{M'(u)}{u\sqrt{u^2 - r^2}}, \quad (16)$$

$$= \frac{1}{2\pi\Sigma_{\text{cr}}r} \int_0^1 dy \frac{1}{1+y^2} \left[M' \left(r\sqrt{1+y^2} \right) + M' \left(r\sqrt{1+y^{-2}} \right) \right], \quad (17)$$

where the second line represents a change of variables so the integral has a finite range, which is convenient for numerical integration. Combining eqs. (4) and (16) gives the circular deflection as

$$\phi_r = \frac{1}{\pi\Sigma_{\text{cr}}r} \int_0^r du u \int_u^\infty dv \frac{M'(v)}{v\sqrt{v^2 - u^2}}, \quad (18)$$

$$= -\frac{1}{\pi\Sigma_{\text{cr}}r} \int_0^r du u \frac{\partial}{\partial u} \int_u^\infty dv M'(v) \frac{\sqrt{v^2 - u^2}}{v}, \quad (19)$$

$$= \frac{1}{\pi\Sigma_{\text{cr}}r} \left[M_\infty - \int_r^\infty dv M'(v) \frac{\sqrt{v^2 - u^2}}{v} \right], \quad (20)$$

where M_∞ is the total mass. Integrating by parts then yields

$$\phi_r = \frac{r}{\pi\Sigma_{\text{cr}}} \int_r^\infty dv \frac{M(v)}{v^2 \sqrt{v^2 - r^2}}, \quad (21)$$

$$= \frac{1}{\pi\Sigma_{\text{cr}}r} \int_0^1 dy \frac{1}{(1+y^2)^{3/2}} \left[M\left(r\sqrt{1+y^2}\right) + y M\left(r\sqrt{1+y^2}\right) \right], \quad (22)$$

where again the second line represents a change of variables for numerical integration. Eqs. (16) and (21), or eqs. (17) and (22) for numerical integration, represent the desired 1-d integrals for the surface density and deflection. The magnification also requires ϕ_{rr} , which could be computed by differentiating ϕ_r ; however, it is easier to compute κ and ϕ_r and then use the identity $r^{-1}\phi_r + \phi_{rr} = 2\kappa$ to determine ϕ_{rr} .

5. The Catalog

Table 1 lists a wide variety of popular lens models, all of which are available in the *gravlens* software. This section summarizes what is known about the mass distributions and lensing properties of the models. Analytic results are given where available, which includes all but one of the circular models and some of the elliptical models. Note that if the potential ϕ is regular at the origin, it is normalized to have $\phi(0) = 0$. Lensing is insensitive to an arbitrary constant added to the potential.

I have tried to include relevant references. *If you use results given here, please cite the original references rather than this catalog.* As for the unreferenced results, some of them are new, while others are derived easily enough that references seem unnecessary. Use your own judgement about citing such material.

Point mass: This model is inherently circular. A point mass M produces a lensing potential

$$\phi = R_E^2 \ln r, \quad (23)$$

where the Einstein radius is (in angular units)

$$R_E = \sqrt{\frac{4GM}{c^2} \frac{D_{\text{ls}}}{D_{\text{ol}}D_{\text{os}}}}. \quad (24)$$

Softened power law potential: Sometimes for analytic simplicity it is convenient to put the elliptical symmetry in the potential rather than in the density (e.g., Blandford & Kochanek

1987; Witt & Mao 1997, 2000; Evans & Hunter 2002),

$$\phi = b (s^2 + x^2 + y^2/q^2)^{\alpha/2} - b s^\alpha. \quad (25)$$

The deflection and magnification are given by simple derivatives.

Softened power law ellipsoid: This model has a projected surface density

$$\kappa(\xi) = \frac{1}{2} \frac{b^{2-\alpha}}{(s^2 + \xi^2)^{1-\alpha/2}}, \quad (26)$$

which represents a flat core with scale radius s , and then a power law decline with exponent α defined such that the mass grows as $M_{\text{cyl}}(r) \propto r^\alpha$ asymptotically. The core radius can be zero if $\alpha > 0$. The model gives a softened isothermal model for $\alpha = 1$, a modified Hubble model for $\alpha = 0$, and a Plummer model for $\alpha = -2$ (see Binney & Tremaine 1987). The circular model has lensing properties

$$\begin{aligned} \phi &= \frac{1}{\alpha^2} b^{2-\alpha} r^\alpha {}_2F_1 \left[-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1 - \frac{\alpha}{2}; -\frac{s^2}{r^2} \right] - \frac{1}{\alpha} b^{2-\alpha} s^\alpha \ln \left(\frac{r}{s} \right) \\ &\quad - \frac{1}{2\alpha} b^{2-\alpha} s^\alpha \left[\gamma_E - \Psi \left(-\frac{\alpha}{2} \right) \right] \end{aligned} \quad (27)$$

$$= \frac{1}{\alpha^2} b^{2-\alpha} r^\alpha \quad (\alpha > 0, s = 0) \quad (28)$$

$$\phi_r = \frac{b^{2-\alpha}}{\alpha r} \left[(s^2 + r^2)^{\alpha/2} - s^\alpha \right] \quad (\alpha \neq 0) \quad (29)$$

$$= \frac{b^2}{r} \ln \left(1 + \frac{r^2}{s^2} \right) \quad (\alpha = 0) = a^{**2}/x * \ln (1 + x^{**2}/b^{**2}) \quad (30)$$

In the potential, ${}_2F_1[a, b; c; x]$ is a hypergeometric function, which can be written as or transformed into a quickly converging series (see Press et al. 1992; Gradshteyn & Ryzhik 1994, §9.1). Also, $\gamma_E = 0.577216\dots$ is Euler's constant, and $\Psi(x) = d[\ln \Gamma(x)]/dx$ is the digamma function, or the logarithmic derivative of the factorial function $\Gamma(x)$. Analytic solutions for the elliptical model are possible for $\alpha = 0, \pm 1$, and two of these are given below. Barkana (1998) gives a fast numerical algorithm for general softened power law ellipsoid models.

Isothermal ellipsoid, $\alpha = 1$: This model describes mass distributions with flat rotation curves (outside the core). Its lensing properties are:

$$\text{circular: } \phi = r \phi_r - b s \ln \left(\frac{s + \sqrt{s^2 + r^2}}{2s} \right) \quad (31)$$

$$\phi_r = \frac{b}{r} \left(\sqrt{s^2 + r^2} - s \right) \quad (32)$$

$$\begin{aligned} \text{elliptical: } \phi &= x \phi_x + y \phi_y - b q s \ln [(\psi + s)^2 + (1 - q^2)x^2]^{1/2} \\ &\quad + b q s \ln [(1 + q)s] \end{aligned} \quad (33)$$

$$\phi_x = \frac{bq}{\sqrt{1-q^2}} \tan^{-1} \left[\frac{\sqrt{1-q^2} x}{\psi + s} \right] \quad (34)$$

$$\phi_y = \frac{bq}{\sqrt{1-q^2}} \tanh^{-1} \left[\frac{\sqrt{1-q^2} y}{\psi + q^2 s} \right] \quad (35)$$

where $\psi^2 = q^2(s^2 + x^2) + y^2$. The elliptical solutions have been given by Kassiola & Kovner (1993), Kormann, Schneider & Bartelmann (1994), and in the simple form quoted here by Keeton & Kochanek (1998). In **the limit of a singular ($s = 0$) and spherical ($q = 1$) model**, b is the Einstein radius of the model and is related to the 1-d velocity dispersion σ by

$$b = 4\pi \left(\frac{\sigma}{c} \right)^2 \frac{D_{\text{ls}}}{D_{\text{os}}} \quad (36)$$

(in angular units).

$\alpha = -1$ **ellipsoid**: This model corresponds to an unnamed density profile with $\Sigma \propto r^{-3}$ ($\rho \propto r^{-4}$) asymptotically. Its lensing properties are:

$$\text{circular: } \phi = \frac{b^3}{s} \ln \left(\frac{s + \sqrt{s^2 + r^2}}{2s} \right) \quad (37)$$

$$\phi_r = \frac{b^3}{sr} \left[1 - \frac{s}{\sqrt{s^2 + r^2}} \right] \quad (38)$$

$$\text{elliptical: } \phi = \frac{b^3 q}{s} \ln [(\psi + s)^2 + (1 - q^2)x^2]^{1/2} - \frac{b^3 q}{s} \ln [(1 + q)s] \quad (39)$$

$$\phi_x = \frac{b^3 qx}{s\psi} \frac{\psi + q^2 s}{(\psi + s)^2 + (1 - q^2)x^2} \quad (40)$$

$$\phi_y = \frac{b^3 qy}{s\psi} \frac{\psi + s}{(\psi + s)^2 + (1 - q^2)x^2} \quad (41)$$

where $\psi^2 = q^2(s^2 + x^2) + y^2$. The elliptical solutions are given by Keeton & Kochanek (1998).

Pseudo-Jaffe ellipsoid: A standard Jaffe (1983) model has a 3-d density distribution $\rho \propto r^{-2}(r + a)^{-2}$ where a is the break radius. For lensing it is useful to modify this model and write $\rho \propto (r^2 + s^2)^{-1}(r^2 + a^2)^{-1}$, where a is again the break radius and we have added a core radius $s < a$. The projected surface density of the elliptical model has the form

$$\kappa(\xi) = \frac{b}{2} \left[\frac{1}{\sqrt{s^2 + \xi^2}} - \frac{1}{\sqrt{a^2 + \xi^2}} \right], \quad (42)$$

which is constant inside s , falls as R^{-1} between s and a , and falls as R^{-3} outside a ; the total mass is $M = \pi \Sigma_{\text{cr}} q b (a - s)$. Eq. (42) defines the pseudo-Jaffe ellipsoid. In the limit $a \rightarrow \infty$ it reduces to the isothermal ellipsoid ($\alpha = 1$). In the limit $a \rightarrow s$ it reduces to the $\alpha = -1$ ellipsoid, although the limit must be taken in ρ rather than in κ (i.e., the limit must be taken before the projection integral is evaluated). The pseudo-Jaffe model is equivalent to a combination of two softened

isothermal ellipsoids, so its lensing properties can be computed with appropriate combinations of eqs. (33)–(35).

King model: The King model can be approximated as a combination of two softened isothermal models (see Young et al. 1980; Barkana et al. 1999),

$$\kappa(\xi) = \frac{2.12 b}{\sqrt{0.75 r_s^2 + \xi^2}} - \frac{1.75 b}{\sqrt{2.99 r_s^2 + \xi^2}} . \quad (43)$$

It has a single scale radius r_s . This approximation is convenient because it is written as the difference of two softened isothermal ellipsoids, so its lensing properties can be computed with appropriate combinations of eqs. (33)–(35).

De Vaucouleurs model: This is the prototypical constant mass-to-light ratio lens model (de Vaucouleurs 1948), with surface mass density

$$\kappa(\xi) = \kappa_0 \exp \left[-k(\xi/R_e)^{1/4} \right] , \quad (44)$$

where $k = 7.66925001$ and R_e is the major-axis effective (or half-mass) radius. The circular deflection is (Maoz & Rix 1993)

$$\begin{aligned} \phi_r = & \kappa_0 \frac{40320}{k^8} \frac{R_e^2}{r} \left[1 - \right. \\ & \left. e^{-\zeta} \left(1 + \zeta \left(1 + \frac{\zeta}{2} \left(1 + \frac{\zeta}{3} \left(1 + \frac{\zeta}{4} \left(1 + \frac{\zeta}{5} \left(1 + \frac{\zeta}{6} \left(1 + \frac{\zeta}{7} \right) \right) \right) \right) \right) \right) \right) \right] , \end{aligned} \quad (45)$$

where $\zeta = k(r/R_e)^{1/4}$. The elliptical model can be computed numerically with eqs. (6)–(11).

Hernquist model: The Hernquist (1990) model is a 3-d density distribution with a projected distribution that mimics the luminosity distribution of early-type galaxies. It has the form

$$\rho = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^3} , \quad (46)$$

where r_s is a scale length and ρ_s is a characteristic density. The projected surface mass density has the form

$$\kappa(r) = \frac{\kappa_s}{(x^2 - 1)^2} \left[-3 + (2 + x^2)\mathcal{F}(x) \right] , \quad (47)$$

where $x = r/r_s$, $\kappa_s = \rho_s r_s / \Sigma_{\text{cr}}$, and $\mathcal{F}(x)$ is the function

$$\mathcal{F}(x) = \begin{cases} \frac{1}{\sqrt{x^2 - 1}} \tan^{-1} \sqrt{x^2 - 1} & (x > 1) \\ \frac{1}{\sqrt{1 - x^2}} \tanh^{-1} \sqrt{1 - x^2} & (x < 1) \\ 1 & (x = 1) \end{cases} \quad (48)$$

A useful technical result is the derivative of this function,

$$\mathcal{F}'(x) = \frac{1 - x^2 \mathcal{F}(x)}{x(x^2 - 1)} . \quad (49)$$

The circular potential and deflection are

$$\phi = \kappa_s r_s^2 \left[\ln \frac{x^2}{4} + 2\mathcal{F}(x) \right], \quad (50)$$

$$\phi_r = 2\kappa_s r_s \frac{x[1 - \mathcal{F}(x)]}{x^2 - 1}, \quad (51)$$

where again $x = r/r_s$. The elliptical model $\kappa(\xi)$ can be computed numerically with eqs. (6)–(11).

NFW model: Cosmological N -body simulations (e.g., Navarro et al. 1996, 1997) suggest that dark matter halos can be described by a “universal” density profile with the form

$$\rho = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}. \quad (52)$$

For the spherical NFW model, the projected surface mass density, potential, and deflection are (Bartelmann 1996; Meneghetti, Bartelmann & Moscardini 2001, 2002; Golse & Kneib 2001)

$$\kappa(r) = 2\kappa_s \frac{1 - \mathcal{F}(x)}{x^2 - 1}, \quad (53)$$

$$\phi = 2\kappa_s r_s^2 \left[\ln^2 \frac{x}{2} - \operatorname{arctanh}^2 \sqrt{1 - x^2} \right], \quad (54)$$

$$\phi_r = 4\kappa_s r_s \frac{\ln(x/2) + \mathcal{F}(x)}{x}, \quad (55)$$

where $x = r/r_s$, $\kappa_s = \rho_s r_s / \Sigma_{\text{cr}}$, and the function $\mathcal{F}(x)$ is the same as in the Hernquist model. The elliptical model $\kappa(\xi)$ can be computed numerically with eqs. (6)–(11).

Cuspy NFW model: Moore et al. (1998, 1999) have suggested that the inner cusp of the NFW profile is too shallow, so Jing & Suto (2000), Keeton & Madau (2001), and Wyithe et al. (2001) have studied a generalized NFW-type profile of the form

$$\rho = \frac{\rho_s}{(r/r_s)^\gamma (1 + r/r_s)^{3-\gamma}}, \quad (56)$$

so the central cusp has $\rho \propto r^{-\gamma}$. The projected surface density cannot be computed analytically even for a spherical halo. For a spherical model, eqs. (16) and (21) allow the surface density and deflection to be written as

$$\kappa(r) = 2\kappa_s x^{1-\gamma} \left[(1+x)^{\gamma-3} + (3-\gamma) \int_0^1 dy (y+x)^{\gamma-4} \left(1 - \sqrt{1-y^2} \right) \right], \quad (57)$$

$$\phi_r = 4\kappa_s r_s x^{2-\gamma} \times \left\{ \frac{1}{3-\gamma} {}_2F_1[3-\gamma, 3-\gamma; 4-\gamma; -x] + \int_0^1 dy (y+x)^{\gamma-3} \frac{1 - \sqrt{1-y^2}}{y} \right\}, \quad (58)$$

where $x = r/r_s$, $\kappa_s = \rho_s r_s / \Sigma_{\text{cr}}$, and ${}_2F_1$ is the hypergeometric function. The integrals can be evaluated for several values of γ to obtain simple analytic expressions. For $\gamma = 1$ the model reduces

to the standard NFW model (see eqs. 53 and 55). Two other analytic cases are as follows:

$$\gamma = 2 : \quad \kappa(r) = \kappa_s \left[\frac{\pi}{x} - 2\mathcal{F}(x) \right], \quad (59)$$

$$\phi_r = 4\kappa_s r_s \left[\frac{\pi}{2} + \frac{1}{x} \ln \left(\frac{x}{2} \right) + \frac{1-x^2}{x} \mathcal{F}(x) \right], \quad (60)$$

$$\gamma = 0 : \quad \kappa(r) = \kappa_s \frac{1+x^2[2-3\mathcal{F}(x)]}{(x^2-1)^2}, \quad (61)$$

$$\phi_r = \frac{2\kappa_s r_s}{x} \left[2 \ln \left(\frac{x}{2} \right) + \frac{x^2 + (2-3x^2)\mathcal{F}(x)}{1-x^2} \right]. \quad (62)$$

Elliptical NFW-cusp models $\kappa(\xi)$ can be computed numerically with eqs. (6)–(11). In general they are slow to compute because they require double integrals.

Cuspy halo models: To obtain a general cuspy model that is more amenable to lensing, Muñoz et al. (2001) introduce a model with a profile of the form

$$\rho = \frac{\rho_s}{(r/r_s)^\gamma [1 + (r/r_s)^2]^{(n-\gamma)/2}}, \quad (63)$$

where again r_s is a scale length, and γ and n are the logarithmic slopes at small and large radii, respectively. This model is a subset of the models whose physical properties were studied by Zhao (1996). The central cusp must have $\gamma < 3$ for the mass to be finite. For $(\gamma, n) = (1, 4)$ this is a pseudo-Hernquist model, for $(1, 3)$ it is a pseudo-NFW model, and for $(2, 4)$ it is a singular pseudo-Jaffe model. Compared with eq. (56), replacing $(1 + r/r_s)$ with $\sqrt{1 + (r/r_s)^2}$ does not greatly change the profile shape but does make it possible to solve the spherical model analytically (Muñoz et al. 2001),

$$\kappa(r) = \kappa_s B \left(\frac{n-1}{2}, \frac{1}{2} \right) (1+x^2)^{(1-n)/2} {}_2F_1 \left[\frac{n-1}{2}, \frac{\gamma}{2}; \frac{n}{2}; \frac{1}{1+x^2} \right], \quad (64)$$

$$\begin{aligned} n \neq 3 : \quad \phi_r = & 2 \frac{\kappa_s r_s}{x} \left\{ B \left(\frac{n-3}{2}, \frac{3-\gamma}{2} \right) \right. \\ & \left. - B \left(\frac{n-3}{2}, \frac{3}{2} \right) (1+x^2)^{(3-n)/2} {}_2F_1 \left[\frac{n-3}{2}, \frac{\gamma}{2}; \frac{n}{2}; \frac{1}{1+x^2} \right] \right\}, \end{aligned} \quad (65)$$

where $x = r/r_s$, $\kappa_s = \rho_s r_s / \Sigma_{\text{cr}}$, ${}_2F_1$ is the hypergeometric function, $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the Euler beta function. For $n = 3$ eq. (65) is not valid (it has singularities of the form $0/0$), so it must be replaced with one of the two following expressions:

$$n = 3 : \quad \phi_r = 2\kappa_s r_s \left\{ \frac{1}{x} \ln(1+x^2) - G \left[\frac{\gamma}{2}, \frac{\gamma-1}{2}; \frac{x^2}{1+x^2} \right] \right. \quad (66)$$

$$\begin{aligned} & \left. - x^{2-\gamma} (1+x^2)^{(\gamma-3)/2} B \left(\frac{\gamma-3}{2}, \frac{3}{2} \right) {}_2F_1 \left[\frac{3}{2}, \frac{3-\gamma}{2}; \frac{5-\gamma}{2}; \frac{x^2}{1+x^2} \right] \right\} \\ = & 2 \frac{\kappa_s r_s}{x} \left\{ \ln(1+x^2) - G \left[\frac{\gamma}{2}, \frac{3}{2}; \frac{1}{1+x^2} \right] + \Psi \left(\frac{3}{2} \right) - \Psi \left(\frac{3-\gamma}{2} \right) \right\} \end{aligned} \quad (67)$$

where $\Psi(x)$ is the digamma function (see eq. 27), and $G(b, c; z)$ is the function

$$G(b, c; z) \equiv \lim_{a \rightarrow 0} \frac{{}_2F_1[a, b, c; z] - 1}{a} \quad (68)$$

$$= \frac{b}{c} z + \frac{b(b+1)}{c(c+1)} \frac{z^2}{2} + \frac{b(b+1)(b+2)}{c(c+1)(c+2)} \frac{z^3}{3} + \dots \quad (69)$$

$$= \sum_{j=1}^{\infty} \frac{b(b+1) \dots (b+j-1)}{c(c+1) \dots (c+j-1)} \frac{z^j}{j} \quad (70)$$

Note that eqs. (66) and (67) are equivalent and can be transformed into each other using identities for the hypergeometric function (see Gradshteyn & Rhyzik 1994, §9.131). It is convenient to use eq. (66) for $x < 1$ and eq. (67) for $x > 1$ in order to make the series for the G function converge rapidly.

Elliptical cusp models $\kappa(\xi)$ can be computed numerically with eqs. (6)–(11), or with a Fourier series solution (Chae 2002).

Nuker law: Many early-type galaxies have surface brightness profiles that can be modeled as a Nuker law (e.g., Lauer et al. 1995; Byun et al. 1996),

$$I(r) = 2^{(\beta-\gamma)/\alpha} I_b \left(\frac{r}{r_b} \right)^{-\gamma} \left[1 + \left(\frac{r}{r_b} \right)^{\alpha} \right]^{(\gamma-\beta)/\alpha}, \quad (71)$$

where γ and β are inner and outer power law indices, respectively, r_b is the radius where the break in the power law occurs, α gives the sharpness of the break, and I_b is the surface brightness at the break radius. If the luminosity distribution has circular symmetry and the mass-to-light ratio is Υ , the lensing deflection is (Keeton 2002)

$$\phi_r = \frac{2^{1+(\beta-\gamma)/\alpha}}{2-\gamma} \kappa_b r_b \left(\frac{r}{r_b} \right)^{1-\gamma} {}_2F_1 \left[\frac{2-\gamma}{\alpha}, \frac{\beta-\gamma}{\alpha}; 1 + \frac{2-\gamma}{\alpha}; - \left(\frac{r}{r_b} \right)^{\alpha} \right], \quad (72)$$

where $\kappa_b = \Upsilon I_b / \Sigma_{\text{cr}}$ is the surface mass density at the break radius in units of the critical density for lensing, and ${}_2F_1$ is the hypergeometric function. Ellipsoidal models can be computed numerically with eqs. (6)–(11).

Exponential disk: The projected surface density is

$$\kappa(\xi) = q^{-1} \kappa_0 \exp[-\xi/R_d], \quad (73)$$

which represents a thin, circular disk with intrinsic central density κ_0 and scale length R_d , seen in projection with axis ratio $q = |\cos i|$ where i is the inclination angle (such that $i = 0^\circ$ is face-on and $i = 90^\circ$ is edge-on). The circular deflection is

$$\phi_r = 2 \kappa_0 \frac{R_d^2}{r} \left[1 - \left(1 + \frac{r}{R_d} \right) e^{-r/R_d} \right]. \quad (74)$$

The elliptical model $\kappa(\xi)$ can be computed with eqs. (6)–(11), or it can be approximated with one or more Kuzmin disks (see Keeton & Kochanek 1998).

Kuzmin disk: The $\alpha = -1$ ellipsoid can be re-interpreted as the projection of a thin disk, in which case it corresponds to a Kuzmin (1956) or Toomre (1962) Model I disk; see Keeton & Kochanek (1998). Its projected surface density is

$$\kappa(\xi) = q^{-1} \kappa_0 r_s^3 (r_s^2 + \xi^2)^{-3/2}, \quad (75)$$

where κ_0 is the intrinsic central surface density of the disk, and $q = |\cos i|$ is again the projected axis ratio of the inclined disk. The only difference between the Kuzmin disk and the $\alpha = -1$ ellipsoid is the normalization.

External perturbations: Objects near the main lens galaxy or along the line of sight often perturb the lensing potential. If the perturbation is weak it may be sufficient to expand the perturbing potential as a Taylor series and keep only a few terms. In a coordinate system centered on the lens galaxy, the expansion to 3rd order can be written as (see Kochanek 1991; Bernstein & Fischer 1999)

$$\phi \approx \phi_0 + \mathbf{b} \cdot \mathbf{x} + \frac{r^2}{2} [\kappa - \gamma \cos 2(\theta - \theta_\gamma)] + \frac{r^3}{3} [\delta \cos(\theta - \theta_\delta) - \varepsilon \cos 3(\theta - \theta_\varepsilon)] + \dots \quad (76)$$

The 0th order term ϕ_0 represents an unobservable zero point of the potential and can be dropped. The 1st order term $\mathbf{b} \cdot \mathbf{x}$ represents an unobservable uniform deflection and can also be dropped. The 2nd order term κ represents the convergence from the perturbing mass and is equivalent to a uniform mass sheet with density $\Sigma/\Sigma_{\text{cr}} = \kappa$. The only observable effect of this term is to rescale the time delay(s) by $1 - \kappa$, which leads to the “mass sheet degeneracy” (e.g., Falco, Gorenstein & Shapiro 1985); hence this term is often omitted from lens models and introduced *a posteriori* using independent mass constraints (see, e.g., Bernstein & Fischer 1999). The 2nd order term γ represents an external tidal shear with strength γ and direction θ_γ . The 3rd order term δ arises from the gradient of the surface density $\kappa(\mathbf{x})$ of the perturber; it has an amplitude $\delta = (3/4)|\nabla\kappa|$ and a direction equal to the direction of $\nabla\kappa$. The 3rd order term ε arises from the $m = 3$ multipole moment of the perturbing mass. The constant coefficients ($\kappa, \gamma, \delta, \varepsilon$) are all evaluated at the position of the lens galaxy, and the corresponding direction angles are written here as theory angles measured counter-clockwise from the x -axis.

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Table 1. Mass Models for Lensing

Model	N_r	Density $\rho(r)$	Surface Density $\kappa(r)$
Point mass	0	$\delta(\mathbf{x})$	$\delta(\mathbf{x})$
Power law or α -models	2	$(s^2 + r^2)^{(\alpha-3)/2}$	$(s^2 + r^2)^{(\alpha-2)/2}$
Isothermal ($\alpha = 1$)	1	$(s^2 + r^2)^{-1}$	$(s^2 + r^2)^{-1/2}$
$\alpha = -1$	1	$(s^2 + r^2)^{-2}$	$(s^2 + r^2)^{-3/2}$
Pseudo-Jaffe	2	$(s^2 + r^2)^{-1} (a^2 + r^2)^{-1}$	$(s^2 + r^2)^{-1/2} - (a^2 + r^2)^{-1/2}$
King (approximate)	1	\dots	$2.12 (0.75r_s^2 + r^2)^{-1/2} - 1.75 (2.99r_s^2 + r^2)^{-1/2}$
de Vaucouleurs	1	\dots	$\exp[-7.67(r/R_e)^{1/4}]$
Hernquist	1	$r^{-1} (r_s + r)^{-3}$	see eq. (47)
NFW	1	$r^{-1} (r_s + r)^{-2}$	see eq. (53)
Cuspy NFW	2	$r^{-\gamma} (r_s + r)^{\gamma-3}$	see eq. (57)
Cusp	3	$r^{-\gamma} (r_s^2 + r^2)^{(\gamma-n)/2}$	see eq. (64)
Nuker	4	\dots	see eq. (71)
Exponential disk	1	\dots	$\exp[-r/R_d]$
Kuzmin disk	1	\dots	$(r_s^2 + r^2)^{-3/2}$

Note. — Density profiles for lensing mass models; see §5 for detailed definitions, including normalizations. Three-dimensional density profiles are not given for the King, de Vaucouleurs, Nuker, exponential disk, and Kuzmin disk models because these models are defined by their surface densities. Column 2 indicates the number of parameters associated with the radial profile alone; each model would also have parameters for the position and the mass scale, and elliptical models would have parameters for the ellipticity and orientation. The profiles are given for spherical models; elliptical models are defined by $\kappa(\xi)$ where ξ is an ellipse coordinate (see eq. 5).