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Quasi-geometrical optics approximation in gravitational lensing⋆

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Abstract. The gravitational lensing of gravitational waves should be treated in wave optics instead of geometrical optics when the wave length λ of the gravitational waves is larger than the Schwarzschild radius of the lens mass M. Wave optics is based on the diffraction integral which represents the amplification of the wave amplitude by lensing. We study the asymptotic expansion of the diffraction integral in the powers of the wave length λ. The first term, arising from the short wavelength limit $\lambda \to 0$, corresponds to the geometrical optics limit. The second term, being of the order of λ/M, is the leading correction term arising from the diffraction effect. Analysing this correction term, we find that (1) the lensing magnification μ is modified to $\mu\,(1+\delta)$, where δ is of the order of $(\lambda/M)^2$, and (2) if the lens has a cuspy (or singular) density profile at center $\rho(r) \propto r^{-\alpha}$ $(0 < \alpha \leq 2)$, the diffracted image is formed at the lens center with magnification $\mu \sim (\lambda/M)^{3-\alpha}$.

Key words. gravitational lensing – gravitational waves

1. Introduction

The gravitational lensing of light is usually treated in the geometrical optics approximation, which is valid in almost all observational situations since the wavelength of light is much smaller than typical scales of astrophysical lens objects. However, for the gravitational lensing of gravitational waves the wavelength is long so that the geometrical optics approximation is not valid in some cases. As shown by several authors (Ohanian 1983; Bliokh & Minakov 1975; Bontz & Haugan 1981; Thorne 1983; Deguchi & Watson 1986), if the wavelength λ is larger than the Schwarzschild radius of the lens mass M, the diffraction effect becomes important. Thus, the diffraction effect is important for a lens mass smaller than $10^8\ M_\odot\ (\lambda/1\mathrm{AU})$, where 1 AU is the wavelength for the planned laser interferometer space antenna (LISA: Bender et al. 2000).

From the above discussion, for $\lambda \gtrsim M$ the diffraction effect is important and the magnification is small (the wavelength is so long that the wave does not feel the existence of the lens), and for $\lambda \ll M$ the geometrical optics approximation is valid. In this paper, we consider the case for $\lambda \lesssim M$, i.e. the quasi-geometrical optics approximation which is geometrical optics including corrections arising from the effects of the finite wavelength. We can obtain these correction terms by an asymptotic expansion of the diffraction integral in powers of the wavelength λ[1]. The diffraction integral represents the amplification of the wave amplitude by lensing in wave optics. It is important to derive the correction terms for the following two reasons: (1) the calculations in wave optics are based on the diffraction integral, but it is time-consuming to numerically calculate this integral, especially for high frequency (see e.g., Ulmer & Goodman 1995). Hence, it is a great saving of computing time to use the analytical expressions. (2) We can understand clearly the difference between wave optics and geometrical optics (i.e. the diffraction effect).

This paper is organized as follows: in Sect. 2 we briefly discuss wave optics in gravitational lensing of gravitational waves. In Sect. 3 we show that in the short wavelength limit $\lambda \to 0$, wave optics is reduced to the geometrical optics limit. In Sect. 4 we expand the diffraction integral in powers of the wavelength λ, and derive the leading correction terms arising from the effect of finite wavelength. In Sect. 5 we apply the quasi-geometrical optics approximation to simple lens models (the point mass lens, SIS lens, isothermal sphere with a finite core lens, and the NFW lens). Section 6 is devoted to a summary and discussions. We use units of $c = G = 1$.

2. Wave optics in gravitational lensing

We consider gravitational waves propagating near a lens object under the thin lens approximation in which the gravitational wave is scattered only on the thin lens plane. The gravitational wave amplitude is magnified by an amplification factor F which is given by the diffraction integral as (Schneider et al. 1992)

$$F(\omega,\boldsymbol{\eta}) = \frac{D_\mathrm{S}}{D_\mathrm{L} D_\mathrm{LS}} \frac{\omega}{2\pi \mathrm{i}} \int \mathrm{d}^2\xi\ \exp\left[\mathrm{i}\omega t_\mathrm{d}(\boldsymbol{\xi},\boldsymbol{\eta})\right], \tag{1}$$

⋆ Appendices A and B are only available in electronic form at http://www.edpsciences.org

[1] The asymptotic expansion of the diffraction integral has been studied in optics. See the following text books for a detailed discussion: Kline & Kay (1965), Chap. XII; Mandel & Wolf (1995), Chap. 3.3; Born & Wolf (1997), Appendix III, and references therein.

where ω is the frequency of the wave, $\boldsymbol{\xi}$ is the impact parameter in the lens plane, $\boldsymbol{\eta}$ is the source position in the source plane. $D_{\rm L}, D_{\rm S}$ and $D_{\rm LS}$ are the distances to the lens, to the source and from the source to the lens, respectively. The time delay $t_{\rm d}$ from the source position $\boldsymbol{\eta}$ through $\boldsymbol{\xi}$ is given by

$$t_{\rm d}(\boldsymbol{\xi},\boldsymbol{\eta}) = \frac{D_{\rm L}D_{\rm S}}{2D_{\rm LS}}\left(\frac{\boldsymbol{\xi}}{D_{\rm L}}-\frac{\boldsymbol{\eta}}{D_{\rm S}}\right)^2 - \hat{\psi}(\boldsymbol{\xi}) + \hat{\phi}_{\rm m}(\boldsymbol{\eta}). \quad (2)$$

The deflection potential $\hat{\psi}(\boldsymbol{\xi})$ is determined by $\nabla^2_{\xi}\hat{\psi} = 8\pi\Sigma$, where $\Sigma(\boldsymbol{\xi})$ is the surface mass density of the lens. $\hat{\phi}_{\rm m}$ is the additional phase in F, and we choose $\hat{\phi}_{\rm m}$ such that the minimum value of the time delay is zero. F is normalized such that $|F| = 1$ in the no lens limit ($\hat{\psi} = 0$).

It is useful to rewrite the amplification factor F in terms of dimensionless quantities. We introduce ξ_0 as the normalization constant of the length in the lens plane. The impact parameter $\boldsymbol{\xi}$, the source position $\boldsymbol{\eta}$, the frequency ω, and the time delay $t_{\rm d}$ are rewritten in dimensionless form,

$$\boldsymbol{x} = \frac{\boldsymbol{\xi}}{\xi_0}, \ \ \boldsymbol{y} = \frac{D_{\rm L}}{\xi_0 D_{\rm S}}\boldsymbol{\eta}, \ w = \frac{D_{\rm S}}{D_{\rm LS}D_{\rm L}}\xi_0^2\,\omega, \ T = \frac{D_{\rm L}D_{\rm LS}}{D_{\rm S}}\xi_0^{-2}t_{\rm d}. \quad (3)$$

We use the Einstein radius ($\sim(MD)^{1/2}$) as the arbitrary scale length ξ_0 for convenience. Then the dimensionless frequency is $w \sim M/\lambda$ from Eq. (3). The dimensionless time delay is rewritten from Eqs. (2) and (3) as

$$T(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}\left|\boldsymbol{x}-\boldsymbol{y}\right|^2 - \psi(\boldsymbol{x}) + \phi_{\rm m}(\boldsymbol{y}), \quad (4)$$

where $\psi(\boldsymbol{x})$ and $\phi_{\rm m}(\boldsymbol{y})$ correspond to $\hat{\psi}(\boldsymbol{\xi})$ and $\hat{\phi}_{\rm m}(\boldsymbol{\eta})$ in Eq. (2): $(\psi,\phi_{\rm m}) = D_{\rm L}D_{\rm LS}/(D_{\rm S}\xi_0^2)\ (\hat{\psi},\hat{\phi}_{\rm m})$. Using the above dimensionless quantities, the amplification factor is rewritten as

$$F(w,\boldsymbol{y}) = \frac{w}{2\pi \rm i}\int {\rm d}^2x \exp\left[{\rm i}wT(\boldsymbol{x},\boldsymbol{y})\right]. \quad (5)$$

For axially symmetric lens models, the deflection potential $\psi(\boldsymbol{x})$ depends only on $x = |\boldsymbol{x}|$ and F is rewritten as

$$F(w,y) = -{\rm i}w{\rm e}^{{\rm i}wy^2/2} \times \int_0^\infty {\rm d}x\ x\ J_0(wxy)\exp\left[{\rm i}w\left(\frac{1}{2}x^2-\psi(x)+\phi_{\rm m}(y)\right)\right], \quad (6)$$

where J_0 is the Bessel function of zeroth order. It takes a long time to calculate F numerically in Eqs. (5) and (6), because the integrand is a rapidly oscillating function especially for large w. In Appendix A we present a method for numerical computation to shorten the computing time.

3. Geometrical optics approximation

In the geometrical optics limit ($w \gg 1$), the stationary points of $T(\boldsymbol{x},\boldsymbol{y})$ contribute to the integral of Eq. (5) so that the image positions $\boldsymbol{x}_j$ are determined by the lens equation, $\nabla_x T(\boldsymbol{x},\boldsymbol{y}) = 0$ or $\boldsymbol{y} = \boldsymbol{x} - \nabla_x\psi(\boldsymbol{x})$. This is just Fermat's principle. We expand $T(\boldsymbol{x},\boldsymbol{y})$ around the jth image position $\boldsymbol{x}_j$ as

$$T(\boldsymbol{x},\boldsymbol{y}) = T_j + \frac{1}{2}\sum_{a,b}\partial_a\partial_b T(\boldsymbol{x}_j,\boldsymbol{y})\tilde{x}_a\tilde{x}_b + O\left(\tilde{x}^3\right) \quad (7)$$

where $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_j$, $T_j = T(\boldsymbol{x}_j,\boldsymbol{y})$, and the indices a, b,... run from 1 to 2. Inserting Eq. (7) into Eq. (5), we obtain the amplification factor in the geometrical optics limit as (Schneider et al. 1992; Nakamura & Deguchi 1999)

$$F_{\rm geo}(w,\boldsymbol{y}) = \sum_j \left|\mu_j\right|^{1/2}\ {\rm e}^{{\rm i}wT_j - {\rm i}\pi n_j}, \quad (8)$$

where the magnification of the jth image is $\mu_j = 1/\det\left(\partial\boldsymbol{y}/\partial\boldsymbol{x}_j\right)$ and $n_j = 0, 1/2, 1$ when $\boldsymbol{x}_j$ is a minimum, saddle, or maximum point of $T(\boldsymbol{x},\boldsymbol{y})$, respectively.

4. Quasi-geometrical optics approximation

4.1. Effect on the magnification of the Image

We expand the amplification factor $F(w,\boldsymbol{y})$ in powers of $1/w$ ($\ll 1$) and discuss the behavior of the term of the order of $1/w$. Here, we only consider axially symmetric lens models because the basic formulae are relatively simple, while the case of the non-axially symmetric lens models is discussed in Appendix B. We expand $T(\boldsymbol{x},\boldsymbol{y})$ in Eq. (7) up to the fourth order of $\tilde{x}$ as

$$T(\boldsymbol{x},\boldsymbol{y}) = T_j + \alpha_j\tilde{x}_1^2 + \beta_j\tilde{x}_2^2 + \frac{1}{6}\sum_{a,b,c}\partial_a\partial_b\partial_c T(\boldsymbol{x}_j,\boldsymbol{y})\tilde{x}_a\tilde{x}_b\tilde{x}_c + \frac{1}{24}\sum_{a,b,c,d}\partial_a\partial_b\partial_c\partial_d T(\boldsymbol{x}_j,\boldsymbol{y})\tilde{x}_a\tilde{x}_b\tilde{x}_c\tilde{x}_d + O\left(\tilde{x}^5\right), \quad (9)$$

where α_j and β_j are defined by

$$\alpha_j = \frac{1}{2}\left(1-\psi_j''\right), \ \ \beta_j = \frac{1}{2}\left(1-\frac{\psi_j'}{|x_j|}\right) \quad (10)$$

where $\psi_j^{(n)} = {\rm d}^n\psi(|\boldsymbol{x}_j|)/{\rm d}x^n$. Inserting Eq. (9) into Eq. (5) and expanding Eq. (5) in powers of $1/w$, we obtain

$$\begin{aligned}F(w,y) = \ & \frac{1}{2\pi \rm i}\sum_j {\rm e}^{{\rm i}wT_j}\int {\rm d}^2\boldsymbol{x}'{\rm e}^{{\rm i}\left(\alpha_j x_1'^2+\beta_j x_2'^2\right)} \\ & \times\left[1+\frac{\rm i}{6\sqrt{w}}\sum_{a,b,c}\partial_a\partial_b\partial_c T(\boldsymbol{x}_j,\boldsymbol{y})x_a'x_b'x_c' \right. \\ & + \frac{1}{w}\left\{-\frac{1}{72}\left(\sum_{a,b,c}\partial_a\partial_b\partial_c T(\boldsymbol{x}_j,\boldsymbol{y})x_a'x_b'x_c'\right)^2\right. \\ & \left.\left. + \frac{\rm i}{24}\sum_{a,b,c,d}\partial_a\partial_b\partial_c\partial_d T(\boldsymbol{x}_j,\boldsymbol{y})x_a'x_b'x_c'x_d'\right\} + O\left(w^{-3/2}\right)\right], \quad (11)\end{aligned}$$

where $\boldsymbol{x}' = \sqrt{w}\tilde{\boldsymbol{x}}$. The first term of the above Eq. (11) is the amplification factor in the geometrical optics limit $F_{\rm geo}$ in Eq. (8). The integral in the second term vanishes because the integrand is an odd function of x_a'. The third term is the leading correction term, being proportional to $1/w$, arising from the diffraction effect. Thus the deviation from geometrical optics is of the order of $1/w \sim \lambda/M$. Inserting $T(\boldsymbol{x},\boldsymbol{y})$ in Eq. (4) into Eq. (11), we obtain F as

$$F(w,y) = \sum_j \left|\mu_j\right|^{1/2}\left(1+\frac{\rm i}{w}\Delta_j\right){\rm e}^{{\rm i}wT_j-{\rm i}\pi n_j} + O\left(w^{-2}\right), \quad (12)$$

where

$$\Delta_j = \frac{1}{16}\left[\frac{1}{2\alpha_j^2}\psi_j^{(4)} + \frac{5}{12\alpha_j^3}\psi_j^{(3)2} + \frac{1}{\alpha_j^2}\frac{\psi_j^{(3)}}{|x_j|} + \frac{\alpha_j-\beta_j}{\alpha_j\beta_j}\frac{1}{|x_j|^2}\right], \quad (13)$$

and Δ_j is a real number. We denote $\mathrm{d}F_\mathrm{m}$ as the second term of Eq. (12),

$$\mathrm{d}F_\mathrm{m}(w,y) \equiv \frac{\mathrm{i}}{w}\sum_j \Delta_j \left|\mu_j\right|^{1/2} \mathrm{e}^{\mathrm{i}wT_j - \mathrm{i}\pi n_j}. \quad (14)$$

Since $\mathrm{d}F_\mathrm{m}$ is the correction term arising near the image positions, this term represents the corrections to the properties of these images such as their magnifications and the time delays. We rewrite F in Eq. (12) as, $F(w,y) = \sum_j |\,\mu_j[1 + (\Delta_j/w)^2]|^{1/2}\mathrm{e}^{\mathrm{i}wT_j+i\delta\varphi_j-\mathrm{i}\pi n_j} + O(w^{-2})$, where $\delta\varphi_j = \arctan(\Delta_j/w)$. Thus in the quasi-geometrical optics approximation, the magnification μ_j is modified to $\mu_j[1+(\Delta_j/w)^2]$, where $(\Delta_j/w)^2$ is of the order of $(\lambda/M)^2$. That is, the magnification is slightly larger than in the geometrical optics limit. The phase is also changed by $\delta\varphi_j$, which is of the order of λ/M.

4.2. Contributions from the non-stationary points

In the previous section, we showed that the contributions to the diffraction integral F arise from the stationary points (or image positions). In this section, we discuss the contributions from the non-stationary points. We denote $\boldsymbol{x}_\mathrm{ns}$ as the non-stationary point, at which the condition $|\nabla_x T| \neq 0$ is satisfied. If $T(\boldsymbol{x},\boldsymbol{y})$ can be expanded at $\boldsymbol{x}_\mathrm{ns}$, we obtain the series of T similar to Eq. (7) as

$$T(\boldsymbol{x},\boldsymbol{y}) = T_\mathrm{ns} + T_1'\tilde{x}_1 + T_2'\tilde{x}_2 + \alpha_\mathrm{ns}\tilde{x}_1^2 + \beta_\mathrm{ns}\tilde{x}_2^2 + O\left(\tilde{x}^3\right), \quad (15)$$

where $\tilde{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_\mathrm{ns}$, $T_\mathrm{ns} = T(\boldsymbol{x}_\mathrm{ns},\boldsymbol{y})$ and $T_{1,2}' = \partial_{1,2}T(\boldsymbol{x}_\mathrm{ns},\boldsymbol{y})$. Note that either T_1' or T_2' does not vanish because of $|\nabla T| \neq 0$ at $\boldsymbol{x}_\mathrm{ns}$. α_ns, and β_ns are defined the same way as in Eq. (10), but at the non-stationary point $\boldsymbol{x}_\mathrm{ns}$.

Inserting Eq. (15) into Eq. (5) and expanding the integrand in powers of $1/w$, we obtain

$$F(w,y) = 2\pi\frac{\mathrm{e}^{\mathrm{i}wT_\mathrm{ns}}}{\mathrm{i}w}\Bigg[\delta\left(T_1'\right)\delta\left(T_2'\right) - \frac{\mathrm{i}}{w}\left(\alpha_\mathrm{ns}\frac{\partial^2}{\partial T_1'^2} + \beta_\mathrm{ns}\frac{\partial^2}{\partial T_2'^2}\right)\delta\left(T_1'\right)\delta\left(T_2'\right) + O\left(w^{-2}\right)\Bigg]. \quad (16)$$

Thus, if both $T_1' \neq 0$ and $T_2' \neq 0$, the above equation vanishes. Even if either $T_1' = 0$ or $T_2' = 0$, it is easy to show that F vanishes. Thus, the contributions to the amplification factor F at the non-stationary points are negligible.

In the above discussion, we assume that $T(\boldsymbol{x},\boldsymbol{y})$ has derivatives at the non-stationary point $\boldsymbol{x}_\mathrm{ns}$ in Eq. (15). But, if the derivatives of T are not defined at $\boldsymbol{x}_\mathrm{ns}$, the result in Eq. (16) should be reconsidered. If the lens has a cuspy (or singular) density profile at the center, the derivatives of T are not defined at the lens center. We will discuss this case in the next section.

4.3. Central cusp of the lens

We consider the correction terms in the amplification factor F arising at the central cusp of the lens. For the inner density profile of the lens $\rho \propto r^{-\alpha}$ $(0 < \alpha \leq 2)$[2], the surface density and the deflection potential at small radius are given by

$$\begin{aligned}\Sigma(\boldsymbol{\xi}) &\propto \xi^{-\alpha+1} \quad \text{for } \alpha \neq 1,\\ &\propto \ln\xi \quad \text{for } \alpha = 1,\end{aligned} \quad (17)$$

$$\begin{aligned}\psi(\boldsymbol{x}) &\propto x^{-\alpha+3} \quad \text{for } \alpha \neq 1,\\ &\propto x^2\ln x \quad \text{for } \alpha = 1.\end{aligned} \quad (18)$$

We note that the Taylor series of $\psi(\boldsymbol{x})$ around $\boldsymbol{x} = 0$ like Eq. (15) cannot be obtained from Eq. (18). For example, in the case of $\alpha = 2$ the deflection potential is $\psi \propto x$, but the derivative of $|\boldsymbol{x}|$ is discontinuous at $\boldsymbol{x} = 0$. Hence we use ψ in Eq. (18) directly, not the Taylor series. Let us calculate the correction terms in the amplification factor arising from the lens center for the following cases; $\alpha = 2, 1$, and the others.

Case of $\alpha = 2$. If the inner density profile is $\rho \propto r^{-2}$ (e.g. singular isothermal sphere model), the deflection potential is given by $\psi(\boldsymbol{x}) = \psi_0 x$ (ψ_0 is a constant) from Eq. (18). Inserting this potential ψ into Eq. (5), we obtain

$$F(w,y) = \frac{\mathrm{e}^{\mathrm{i}w\left[y^2/2+\phi_\mathrm{m}(y)\right]}}{2\pi\mathrm{i}w}\int \mathrm{d}x'^2 \times\exp\left[-\mathrm{i}\left\{yx_1' + \psi_0\sqrt{x_1'^2+x_2'^2} + O(1/w)\right\}\right], \quad (19)$$

where $\boldsymbol{x}' = w\boldsymbol{x}$. Denoting $\mathrm{d}F_\mathrm{c}(w,y)$ as the leading term of the above integral which is proportional to $1/w$, we obtain $\mathrm{d}F_\mathrm{c}$ as

$$\mathrm{d}F_\mathrm{c}(w,y) = \frac{\mathrm{e}^{\mathrm{i}w\left[y^2/2+\phi_\mathrm{m}(y)\right]}}{w}\frac{1}{\left(y^2-\psi_0^2\right)^{3/2}} \quad \text{for } |y| > |\psi_0|, \quad (20)$$

$$= \frac{\mathrm{e}^{\mathrm{i}w\left[y^2/2+\phi_\mathrm{m}(y)\right]}}{w}\frac{\mathrm{i}}{\left(\psi_0^2-y^2\right)^{3/2}} \quad \text{for } |y| < |\psi_0|. \quad (21)$$

Thus, the contribution to the amplification factor F at the lens center is of the order of $1/w \sim \lambda/M$. This is because of the singularity in the density profile at the lens center. The correction terms $\mathrm{d}F_\mathrm{c}$ in Eqs. (20) and (21) represent a diffracted image which is formed at the lens center by the diffraction effect. The magnification of this image is of the order of $\sim\lambda/M$.

Case of $\alpha = 1$. If the inner density profile is $\rho \propto r^{-1}$ (e.g. the Navarro Frenk White model), the deflection potential is given by $\psi(\boldsymbol{x}) = \psi_0 x^2 \ln x$ (ψ_0 is a constant) from Eq. (18). Inserting this ψ into Eq. (6), we have

$$F(w,y) = -\frac{\mathrm{i}}{w}\,\mathrm{e}^{\mathrm{i}w[y^2/2+\phi_\mathrm{m}(y)]} \times\int_0^\infty \mathrm{d}x'\,x'\,J_0\left(yx'\right)\,\mathrm{e}^{i/(2w)x'^2\left[1-2\psi_0\ln(x'/w)\right]}, \quad (22)$$

[2] We do not consider the steeper profile $\alpha > 2$, since the mass at the lens center is infinite.

where $x' = wx$. By expanding the integrand of Eq. (22) in powers of $1/w$, the above equation can be integrated analytically. Denoting $\mathrm{d}F_{\mathrm{c}}(w, y)$ as the leading term, we obtain

$$\mathrm{d}F_{\mathrm{c}}(w, y) = \frac{-4\psi_0}{(wy^2)^2} \, \mathrm{e}^{\mathrm{i}w[y^2/2+\phi_{\mathrm{m}}(y)]}. \tag{23}$$

Thus, the diffracted image is formed at the lens center with the magnification $\sim(\lambda/M)^2$.

The other cases ($0 < \alpha < 1$, $1 < \alpha < 2$). The deflection potential is $\psi(\boldsymbol{x}) = \psi_0 \, x^{-\alpha+3}$ (ψ_0 is a constant) from Eq. (18). Inserting ψ into Eq. (6), we obtain $\mathrm{d}F_{\mathrm{c}}(w, y)$ as

$$\mathrm{d}F_{\mathrm{c}}(w, y) = -\frac{1}{2}\left(\frac{y}{2}\right)^{\alpha-5} \psi_0 w^{\alpha-3} \, \mathrm{e}^{\mathrm{i}w[y^2/2+\phi_{\mathrm{m}}(y)]} \, \frac{\Gamma((5-\alpha)/2)}{\Gamma((\alpha-3)/2)}, \tag{24}$$

where Γ is the gamma function.

From the above discussion for the three cases, the diffracted image is always formed at the lens center for an inner density profile $\rho \propto r^{-\alpha}$ ($0 < \alpha \leq 2$). The magnification of this central image is roughly given by $\mu \sim (\lambda/M)^{3-\alpha}$.

5. Results for specific lens models

We apply the quasi-geometrical optics approximation to simple lens models. We consider the following axially symmetric lens models: point mass lens, singular isothermal sphere (SIS) lens, isothermal sphere lens with a finite core, and Navarro Frenk White (NFW) lens. We derive the amplification factor F, that in the geometrical optics limit F_{geo}, and its correction terms $\mathrm{d}F_{\mathrm{m}}$ (in Sect. 4.1) and $\mathrm{d}F_{\mathrm{c}}$ (in Sect. 4.3) for the above lens models. We define $\mathrm{d}F$ for convenience as the sum of $\mathrm{d}F_{\mathrm{m}}$ and $\mathrm{d}F_{\mathrm{c}}$: $\mathrm{d}F(w, y) \equiv \mathrm{d}F_{\mathrm{m}}(w, y) + \mathrm{d}F_{\mathrm{c}}(w, y)$.

5.1. Point mass lens

The surface mass density is expressed as $\Sigma(\boldsymbol{\xi}) = M\delta^2(\boldsymbol{\xi})$, where M is the lens mass, and the deflection potential is $\psi(\boldsymbol{x}) = \ln x$. This lens model is most frequently used in wave optics in gravitational lensing of gravitational waves (Nakamura 1998; Ruffa 1999; De Paolis et al. 2001, 2002; Zakharov & Baryshev 2002; Takahashi & Nakamura 2003; Yamamoto 2003; Varvella et al. 2003; Seto 2004). The amplification factor F of Eq. (6) is analytically integrated as (Peters 1974; Deguchi & Watson 1986)

$$\begin{aligned} F(w, y) \; = \; & \exp\left[\frac{\pi w}{4} + \mathrm{i}\frac{w}{2}\left\{\ln\left(\frac{w}{2}\right) - 2\phi_{\mathrm{m}}(y)\right\}\right] \\ & \times \Gamma\left(1 - \frac{\mathrm{i}}{2}w\right) {}_1F_1\left(\frac{\mathrm{i}}{2}w, 1; \frac{\mathrm{i}}{2}wy^2\right), \end{aligned} \tag{25}$$

where ${}_1F_1$ is the confluent hypergeometric function and $\phi_{\mathrm{m}}(y) = (x_{\mathrm{m}} - y)^2/2 - \ln x_{\mathrm{m}}$ with $x_{\mathrm{m}} = (y + \sqrt{y^2+4})/2$. In the geometrical optics limit ($w \gg 1$) from Eq. (8) we have

$$F_{\mathrm{geo}}(w, y) = |\mu_+|^{1/2} - \mathrm{i}\,|\mu_-|^{1/2} \, \mathrm{e}^{\mathrm{i}w\Delta T}, \tag{26}$$

where the magnification of each image is $\mu_\pm = 1/2 \pm (y^2+2)/(2y\sqrt{y^2+4})$ and the time delay between double images is $\Delta T = y\sqrt{y^2+4}/2 + \ln((\sqrt{y^2+4}+y)/(\sqrt{y^2+4}-y))$. In the quasi-geometrical optics approximation, $\mathrm{d}F_{\mathrm{c}} = 0$ and $\mathrm{d}F(= \mathrm{d}F_{\mathrm{m}})$ is given from Eq. (14) by

$$\begin{aligned} \mathrm{d}F(w, y) \; = \; & \frac{\mathrm{i}}{3w} \frac{4x_+^2 - 1}{\left(x_+^2+1\right)^3 \left(x_+^2-1\right)} |\mu_+|^{1/2} \\ & + \frac{1}{3w} \frac{4x_-^2 - 1}{\left(x_-^2+1\right)^3 \left(x_-^2-1\right)} |\mu_-|^{1/2} \, \mathrm{e}^{\mathrm{i}w\Delta T} \end{aligned} \tag{27}$$

where $x_\pm = (y \pm \sqrt{y^2+4})/2$ is the position of each image. The first and second terms in Eq. (27) are correction terms for the magnifications of the two images as discussed in Sect. 4.1.

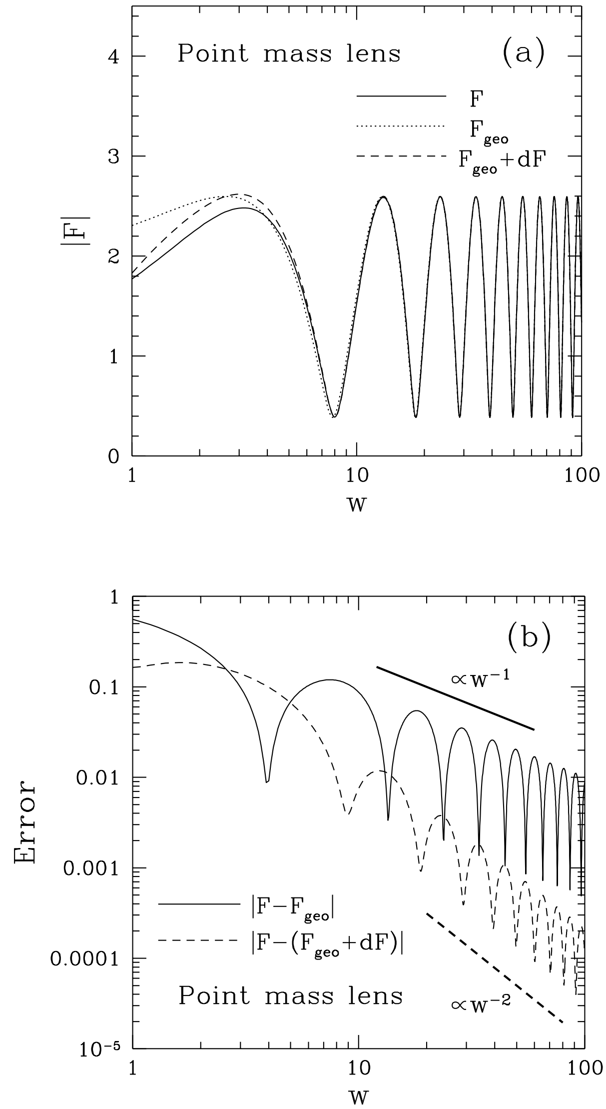

Fig. 1. a) The amplification factor $|F(w, y)|$ for a point mass lens as a function of w with a fixed source position $y = 0.3$. The solid line is the full result F; the dotted line is the geometrical optics approximation $\mathrm{d}F_{\mathrm{geo}}$; the dashed line is the quasi-geometrical optics approximation $F_{\mathrm{geo}} + \mathrm{d}F$. **b)** The differences between F, F_{geo} and $F_{\mathrm{geo}} + \mathrm{d}F$ for a point mass lens as a function of w with $y = 0.3$. The thin solid line is $|F - F_{\mathrm{geo}}|$, and the thin dashed line is $|F - (F_{\mathrm{geo}} + \mathrm{d}F)|$. The thick solid (dashed) line is the power of $1/w$ ($1/w^2$).

In Fig. 1a, the amplification factor $|F|$ is shown as a function of w with a fixed source position $y = 0.3$. The solid line is the result F in Eq. (25); the dotted line is the geometrical optics approximation $F_{\rm geo}$ in Eq. (26); the dashed line is the quasi-geometrical optics approximation $F_{\rm geo} + {\rm d}F$ in Eqs. (26) and (27). For the high frequency limit $w \gg 1$, $|F|$ converge to the geometrical optics limit in Eq. (26),

$$\left|F_{\rm geo}\right|^2 = |\mu_+| + |\mu_-| + 2\,|\mu_+\mu_-|^{1/2}\sin(w\Delta T). \tag{28}$$

The first and second terms in Eq. (28), $|\mu| = |\mu_+| + |\mu_-|$, represent the total magnification in geometrical optics. The third term expresses the interference between the double images. The oscillatory behavior (in Fig. 1a) is due to this interference. The amplitude and the period of this oscillation are approximately equal to $2|\mu_+\mu_-|^{1/2}$ and $w\Delta T$ in the third term of Eq. (28), respectively. For large w ($\gtrsim$10), these three lines converge asymptotically.

In Fig. 1b, the differences between F, $F_{\rm geo}$ and $F_{\rm geo} + {\rm d}F$ are shown as a function of w with $y = 0.3$. The thin solid line is $|F - F_{\rm geo}|$, and the thin dashed line is $|F - (F_{\rm geo} + {\rm d}F)|$. The thick solid (dashed) line is the power of $w^{-1}(w^{-2})$. From this figure, for larger w ($\gg$1) F converges to $F_{\rm geo}$ with an error of $O(1/w)$, and converges to $F_{\rm geo} + {\rm d}F$ with an error of $O(1/w^2)$. These results are consistent with the analytical calculations in Sects. 3 and 4.

5.2. Singular isothermal sphere

The surface density of the SIS (Singular Isothermal Sphere) is characterized by the velocity dispersion v as $\Sigma(\boldsymbol{\xi}) = v^2/(2\xi)$. and the deflection potential is $\psi(\boldsymbol{x}) = x$. The SIS model is used for more realistic lens objects than the point mass lens, such as galaxies, star clusters and dark halos (Takahashi & Nakamura 2003). In this model F is numerically computed in Eq. (6). In the geometrical optics limit $w \gg 1$, $F_{\rm geo}$ is given by

$$\begin{aligned} F_{\rm geo}(w,y) &= |\mu_+|^{1/2} - {\rm i}\,|\mu_-|^{1/2}\,{\rm e}^{{\rm i}w\Delta T} && \text{for } y < 1, \\ &= |\mu_+|^{1/2} && \text{for } y \geq 1, \end{aligned} \tag{29}$$

where $\mu_\pm = \pm 1 + 1/y$ and $\Delta T = 2y$. For $y < 1$ double images are formed, while for $y \geq 1$ a single image is formed. In the quasi-geometrical optics approximation, ${\rm d}F$ is given by

$$\begin{aligned} {\rm d}F(w,y) = {} & \frac{\rm i}{8w}\frac{|\mu_+|^{1/2}}{y(y+1)^2} - \frac{1}{8w}\frac{|\mu_-|^{1/2}}{y(1-y)^2}\,{\rm e}^{{\rm i}w\Delta T} \\ & + \frac{\rm i}{w}\frac{1}{(1-y^2)^{3/2}}\,{\rm e}^{{\rm i}w[y^2/2+\phi_{\rm m}(y)]} \qquad \text{for } y < 1, \end{aligned} \tag{30}$$

$$= \frac{\rm i}{8w}\frac{|\mu_+|^{1/2}}{y(y+1)^2} + \frac{1}{w}\frac{1}{(y^2-1)^{3/2}}\,{\rm e}^{{\rm i}w[y^2/2+\phi_{\rm m}(y)]} \qquad \text{for } y \geq 1, \tag{31}$$

where $\phi_{\rm m}(y) = y + 1/2$. For $y < 1$, the first and the second term in Eq. (30) are correction terms for the magnifications of the images formed in the geometrical optics (i.e. ${\rm d}F_{\rm m}$ in Sect. 4.1), and the third term corresponds to the diffracted image at the lens center (i.e. ${\rm d}F_{\rm c}$ in Sect. 4.3). For $y \geq 1$, the first term of Eq. (31) is the correction term for the magnification ${\rm d}F_{\rm m}$, and the second term corresponds to the diffracted image at the lens center ${\rm d}F_{\rm c}$. Thus, in the quasi-geometrical optics approximation, for $y < 1$ three images are formed, while for $y \geq 1$ double images are formed in the SIS model.

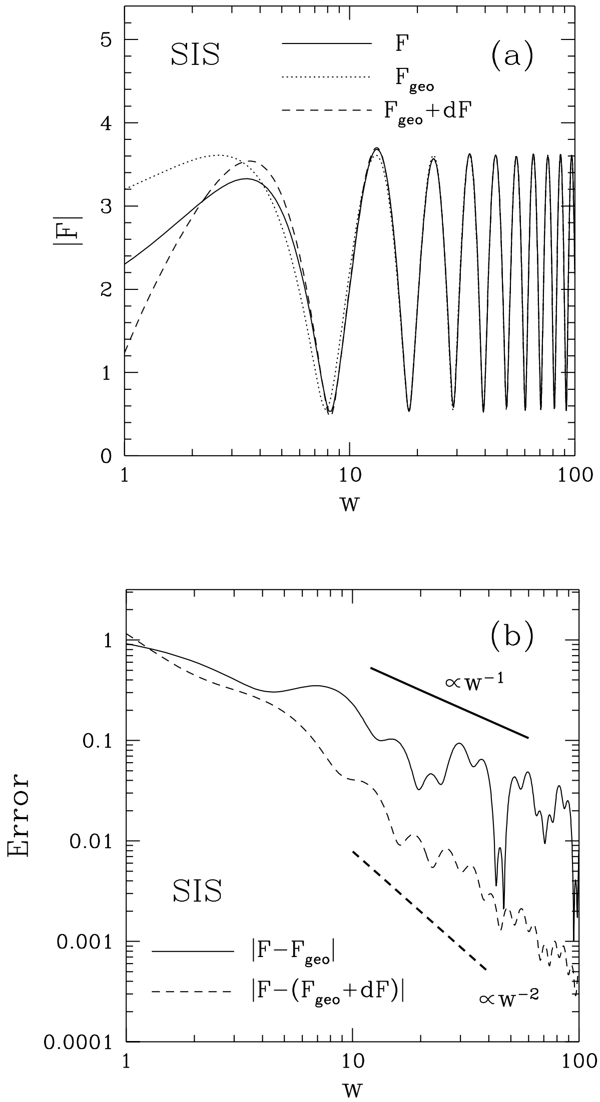

Fig. 2. Same as Fig. 1, but for SIS lens with a source position $y = 0.3$.

Figure 2 is the same as Fig. 1, but for a SIS lens with a source position $y = 0.3$. In Fig. 2a, the behavior is similar to that in the point mass lens (in Fig. 1a). The oscillation of $|F_{\rm geo}|$ is due to the interference between the double images, while the oscillation of $|F_{\rm geo} + {\rm d}F|$ is due to the interference among the three images. As shown in Fig. 2b, the errors decrease as w increases, and the results are consistent with the analytical calculations.

5.3. Isothermal sphere with a finite core

We investigate the effect of a finite core in the lens center on the amplification factor. We consider an isothermal sphere having a finite core. The deflection potential is $\psi(x) = (x^2 + x_{\rm c}^2)^{1/2}$ where $x_{\rm c}$ is the dimensionless core radius. In this model, the central core of the lens contributes the integral of F in Eq. (5). Denoting $\mathrm{d}F_\mathrm{c}$ as the contribution of F at the lens center, we obtain,

$$\mathrm{d}F_\mathrm{c}(w,y) = \frac{\mathrm{e}^{\mathrm{i}wy^2/2}}{2\pi \mathrm{i}w}\int \mathrm{d}\boldsymbol{x}'^2 \times \exp\left[-\mathrm{i}\left\{yx_1' + \sqrt{x_1'^2+x_2'^2+(wx_\mathrm{c})^2} + O(1/w)\right\}\right]. \quad (32)$$

For $wx_\mathrm{c} \lesssim 1$ the above equation is the same as $\mathrm{d}F_\mathrm{c}$ in the SIS model (see Eq. (19)) and $\mathrm{d}F_\mathrm{c} \propto 1/w$, but for $wx_\mathrm{c} \gtrsim 1$, $\mathrm{d}F_\mathrm{c}$ decreases exponentially as w increases. Thus even if the lens has a finite core at the center, the wave does not feel the existence of the small core $x_\mathrm{c} \lesssim 1/w$, and the diffracted image is formed similarly to that of lens without the core.

5.4. NFW lens

The NFW profile was proposed from numerical simulations of cold dark matter (CDM) halos by Navarro, Frenk & White (1997). They showed that the density profile of dark halos has the "universal" form, $\rho(r) = \rho_\mathrm{s}(r/r_\mathrm{s})^{-1}(r/r_\mathrm{s}+1)^{-2}$, where r_s is a scale length and ρ_s is a characteristic density. The NFW lens is used for lensing by galactic halos and clusters of galaxies. The deflection potential is written as (Bartelmann 1996),

$$\begin{aligned}\psi(x) &= \frac{\kappa_\mathrm{s}}{2}\left[\left(\ln\frac{x}{2}\right)^2 - \left(\mathrm{arctanh}\sqrt{1-x^2}\right)^2\right] \quad \text{for } x \le 1,\\ &= \frac{\kappa_\mathrm{s}}{2}\left[\left(\ln\frac{x}{2}\right)^2 + \left(\arctan\sqrt{x^2-1}\right)^2\right] \quad \text{for } x \ge 1, \quad (33)\end{aligned}$$

where $\kappa_\mathrm{s} = 16\pi\rho_\mathrm{s}(D_\mathrm{L}D_\mathrm{LS}/D_\mathrm{S})r_\mathrm{s}$ is the dimensionless surface density and we adopt the scale radius, not the Einstein radius, as the normalization length: $\xi_0 = r_\mathrm{s}$. The image positions x_j, magnifications μ_j and time delays T_j are numerically obtained from the lens equation, $y = x - \psi'(x)$. In the quasi-geometrical optics approximation, $\mathrm{d}F$ is given by,

$$\mathrm{d}F(w,y) = \frac{\mathrm{i}}{w}\sum_j \Delta_j \left|\mu_j\right|^{1/2} \mathrm{e}^{\mathrm{i}wT_j - \mathrm{i}\pi n_j} + \frac{\kappa_s}{(wy^2)^2}\,\mathrm{e}^{\mathrm{i}w\left[y^2/2+\phi_\mathrm{m}(y)\right]}. \quad (34)$$

The first term in Eq. (34) is the correction for the magnification of the images (i.e. $\mathrm{d}F_\mathrm{m}$ in Sect. 4.1). The second term corresponds to the diffracted image at the lens center (i.e. $\mathrm{d}F_\mathrm{c}$ in Sect. 4.3).

6. Conclusions

We studied gravitational lensing in the quasi-geometrical optics approximation which is geometrical optics including the corrections arising from the effect of a finite wavelength. These correction terms can be obtained analytically by asymptotic expansion of the diffraction integral in powers of the wavelength λ. The first term, arising from the short wavelength limit $\lambda \to 0$, corresponds to the geometrical optics limit. The second term, being of the order of λ/M (M is the Schwarzschild radius of the lens), is the first correction term arising from the diffraction effect. By analyzing this correction term, we obtain the following results: (1) the lensing magnification μ is modified to $\mu\,(1+\delta)$, where δ is of the order of $(\lambda/M)^2$. (2) If the lens has a cuspy (or singular) density profile at the center $\rho(r) \propto r^{-\alpha}$ ($0 < \alpha \le 2$) the diffracted image is formed at the lens center with magnification $\mu \sim (\lambda/M)^{3-\alpha}$. Thus if we observe this diffracted image in various wavelengths, the slope α can be determined.

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Online Material

Appendix A: Numerical computation of the amplification factor

We present the method for the numerical integration of F discussed in Sect. 2. F in Eq. (5) is rewritten as

$$F(w,\boldsymbol{y}) = \frac{w}{2\pi \mathrm{i}} \mathrm{e}^{\mathrm{i}w\left[y^2/2+\phi_{\mathrm{m}}(\boldsymbol{y})\right]} \int_0^\infty \mathrm{d}x\, x\, \mathrm{e}^{\mathrm{i}wx^2/2} \times \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-\mathrm{i}w[xy\cos\theta+\psi(x,\theta)]}, \tag{A.1}$$

where θ is defined as $\boldsymbol{x}\cdot\boldsymbol{y} = xy\cos\theta$. Changing the integral variable from x to $z = x^2/2$ in Eq. (A.1), we obtain

$$F(w,\boldsymbol{y}) = \int_0^\infty \mathrm{d}z f(z:w,\boldsymbol{y})\, \mathrm{e}^{\mathrm{i}wz} \tag{A.2}$$

where the function f is defined by

$$f(z:w,\boldsymbol{y}) \equiv \frac{w}{2\pi\mathrm{i}} \mathrm{e}^{\mathrm{i}w\left[y^2/2+\phi_{\mathrm{m}}(\boldsymbol{y})\right]} \times \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-\mathrm{i}w\left[y\sqrt{2z}\cos\theta+\psi\left(\sqrt{2z},\theta\right)\right]}. \tag{A.3}$$

Thus, we obtain the Fourier integral (A.2). We present the method for computing the Fourier integral in Numerical Recipes (Press et al. 1992). The Eq. (A.2) is rewritten as

$$\begin{aligned} F(w,y) &= \int_0^b \mathrm{d}z f(z:w,y)\, \mathrm{e}^{\mathrm{i}wz} + \int_b^\infty \mathrm{d}z f(z:w,y)\, \mathrm{e}^{\mathrm{i}wz}, \\ &= \int_0^b \mathrm{d}z f(z:w,y)\, \mathrm{e}^{\mathrm{i}wz} - \frac{f(b:w,y)\mathrm{e}^{\mathrm{i}wb}}{\mathrm{i}w} \\ &\quad + \frac{f'(b:w,y)\mathrm{e}^{\mathrm{i}wb}}{(\mathrm{i}w)^2} - \cdots, \end{aligned} \tag{A.4}$$

where $f' = \partial f/\partial b$. The first term of the above Eq. (A.4) is evaluated by numerical integration directly. For the series of the above equation, we use integration by parts. The asymptotic expansion in Eq. (A.4) converges for large b.

Especially for an axially symmetric lens, the function f in Eq. (A.3) is reduced to the simple form as

$$f(z:w,y) = -\mathrm{i}w\mathrm{e}^{\mathrm{i}w\left[y^2/2+\phi_{\mathrm{m}}(y)\right]} J_0\left(wy\sqrt{2z}\right) \mathrm{e}^{\mathrm{i}wz}. \tag{A.5}$$

Appendix B: Asymptotic expansion of the amplification factor for non-axially symmetric lens models

We consider the expansion of F in powers of $1/w$ for the non-axially symmetric lens model. In this case, the expansion of $T(\boldsymbol{x},\boldsymbol{y})$ around the jth image position $\boldsymbol{x}_j$ in Eq. (9) is rewritten as

$$\begin{aligned} T(\boldsymbol{x},\boldsymbol{y}) &= T_j + \frac{1}{2}\sum_{a,b} T_{ab}\tilde{x}_a\tilde{x}_b + \frac{1}{6}\sum_{a,b,c} \partial_a\partial_b\partial_c T(\boldsymbol{x}_j,\boldsymbol{y})\tilde{x}_a\tilde{x}_b\tilde{x}_c \\ &\quad + \frac{1}{24}\sum_{a,b,c,d} \partial_a\partial_b\partial_c\partial_d T(\boldsymbol{x}_j,\boldsymbol{y})\tilde{x}_a\tilde{x}_b\tilde{x}_c\tilde{x}_d + O\left(\tilde{x}^5\right), \end{aligned} \tag{B.1}$$

where $T_{ab} = \partial_a\partial_b T(\boldsymbol{x}_j,\boldsymbol{y})$ is a 2×2 matrix. We change the variable from $\tilde{x}_a$ to $z_a = \sum_b A_{ab}\tilde{x}_b$ in order to diagonalize the matrix T_{ab} in the above Eq. (B.1). Here, A_{ab} satisfies, $\sum_{a,b} T_{ab}A_{ac}A_{bd} = \lambda_c\delta_{cd}$, where λ_c is the eigenvalue of T_{ab}. Using the variable z, the Eq. (B.1) is rewritten as

$$\begin{aligned} T(\boldsymbol{x},\boldsymbol{y}) &= T_j + \frac{1}{2}\left(\lambda_1 z_1^2 + \lambda_2 z_2^2\right) + \sum_{a,b,c} M_{abc}\, z_a z_b z_c \\ &\quad + \sum_{a,b,c,d} N_{abcd}\, z_a z_b z_c z_d + O\left(z^5\right), \end{aligned} \tag{B.2}$$

where M_{abc} and N_{abcd} are defined by

$$M_{abc} \equiv \frac{1}{6}\sum_{d,e,f} \partial_d\partial_e\partial_f T(\boldsymbol{x}_j,\boldsymbol{y})\, A_{ad}A_{be}A_{cf},$$

$$N_{abcd} \equiv \frac{1}{24}\sum_{e,f,g,h} \partial_e\partial_f\partial_g\partial_h T(\boldsymbol{x}_j,\boldsymbol{y})\, A_{ae}A_{bf}A_{cg}A_{dh}.$$

Doing the same calculations in Sect. 4.1 with Eq. (B.2), we obtain the same amplification factor as in Eq. (12), but Δ_j is given by

$$\begin{aligned} \Delta_j &= \frac{15}{2}\left(\frac{M_{111}^2}{\lambda_1|\lambda_1|^2} + 3\frac{M_{111}M_{122}}{|\lambda_1|^2\lambda_2} + 3\frac{M_{112}M_{222}}{\lambda_1|\lambda_2|^2} + \frac{M_{222}^2}{\lambda_2|\lambda_2|^2}\right) \\ &\quad - 3\left(\frac{N_{1111}}{|\lambda_1|^2} + 2\frac{N_{1122}}{\lambda_1\lambda_2} + \frac{N_{2222}}{|\lambda_2|^2}\right). \end{aligned} \tag{B.3}$$