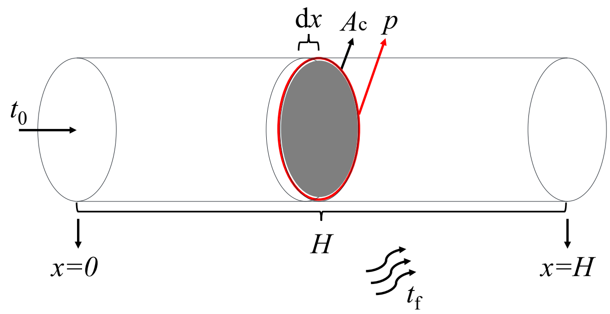
Consider a cylinder fin exposed to a surrounding fluid of temperature t_f=16°C. The wall is of temperature t_0=260°C and the pure aluminum made fin is of length H=150mm. It is assumed that the heat transfer coefficient for the fin has the value h=15W/(m^2·K) and the end of the fin is adiabatic.

Question: (1) Calculate the heat loss by convection. (2) If the length of the fin is extended to H'=2H=300mm, can the heat loss by convection double? **Try to calculate analytical solution and numerical simulation result respectively.**



Method A: Analytical method

We can make an assumption that the temperature is unchangeable along the cross-section so that the question is simplified to one-dimension heat transfer. Then the heat conduction equation is

$$\frac{d^2t}{dx^2}+\frac{\dot{\Phi}}{\lambda}=0$$

According to the Fourier's law

$$\Phi_s=(pdx)h(t-t_f) \quad [\mathrm{W}]$$

$$\dot{\Phi}=-\frac{\Phi_s}{A_c dx}=-\frac{hp(t-t_f)}{A_c} \quad [\mathrm{W/m^3}]$$

Where

Cross-sectional perimeter: $p=0.025\times\pi=0.0785\mathrm{m}$

Cross-sectional area: $A_c=0.0125^2\times\pi=0.00049\mathrm{m}^2$

We can obtain the solvable differential equation

$$\frac{d^2t}{dx^2}=\frac{hp(t-t_f)}{\lambda A_c}, x=0, t=t_0; x=H, \frac{dt}{dx}=0.$$

Thus

$$\theta=\theta_0\frac{\mathrm{ch}\left[m(x-H)\right]}{\mathrm{ch}(mH)}, \quad \frac{d\theta}{dx}=-m\theta_0\frac{\mathrm{sh}(mH)}{\mathrm{ch}(mH)}$$

Where

Excess temperature: $\theta=t-t_f$

Thermal conductivity of pure aluminum: $\lambda = 238\text{W}/(\text{m}\cdot\text{K})$

Excess temperature at x=0: $\theta_0 = t_0 - t_\text{f} = 260 - 16 = 244°\text{C}$

Constant: $m = \sqrt{\dfrac{hp}{\lambda A_c}} = \sqrt{\dfrac{15\times\pi\times0.025}{238\times\pi\times0.0125^2}} = 3.1755$

Because the heat lost by convection equals to the heat transfer rate at x=0, thus

$$\Phi = -\lambda A_\text{c}\left(\frac{dt}{dx}\right)_{x=0} = \lambda A_c\theta_0 mth(mH) = 238\times\pi\times0.0125^2\times244\times3.1755\times0.443 = 40.1\text{W}$$

We can also obtain the heat lost value when H'=2H through the method mentioned above

$$\Phi' = 66.9\text{W} < 2\Phi = 80.2\text{W}$$

In conclusion, when the length of the fin increase to 2H, **the heat loss by convection cannot double.**

Method B: Numerical method

In this question, I choose *COMSOL Multiphysics 5.3* for numerical simulation.

(1) Modeling

Step 1

Set two parameters ‘h_ext’ and ‘T_ext’ in ‘Global defintions’. Then set varible ‘Q’ and integration ‘intop1’ in ‘compoent’.

Where

h_ext =15[W/((m^2)*K)]

T_ext=16[degC]

Q=h_ext*intop1(T-T_ext)

intop1 is the integration on Boundary 2

Step 2

Build a cylinder model shown in Fig. 1 with **radius 12.5mm** and **height 150mm**.

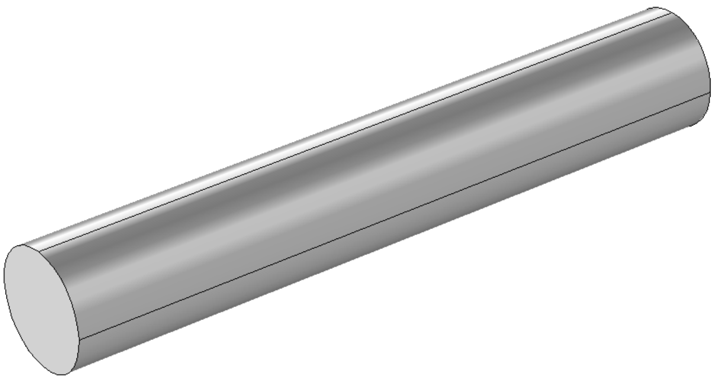


Fig. 1

Step 3
Set the material as **Aluminum**.

Step 4
Set parameters in ‘heat transfer in solids’:
Initial values: *T*=293.15K
Thermal insulation: Boundary 3
Temperature of Boundary 1: T=260degC
Heat flux of Boundary 2: choose ‘**convective heat flux**’, set ‘**heat transfer coefficient**’ and ‘**external temperature**’ as shown in Fig. 2.

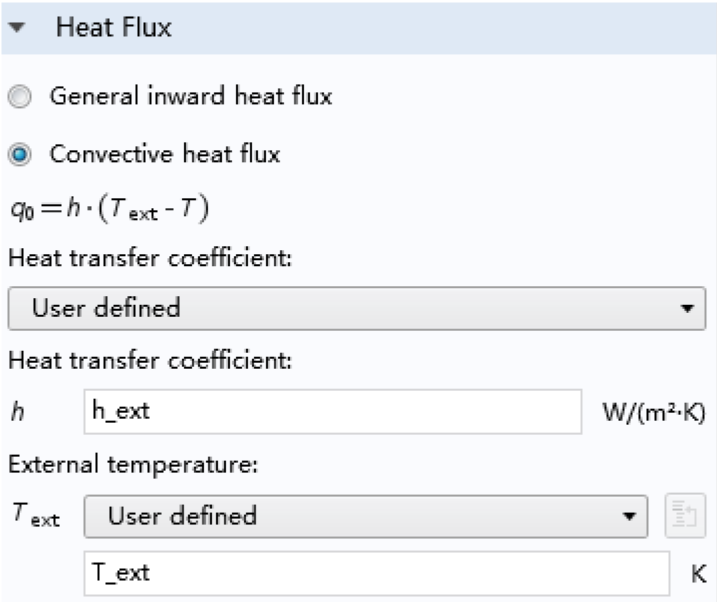


Fig. 2

Step 5
As shown in Fig. 3, set **element size** of mesh as ‘**normal**’.

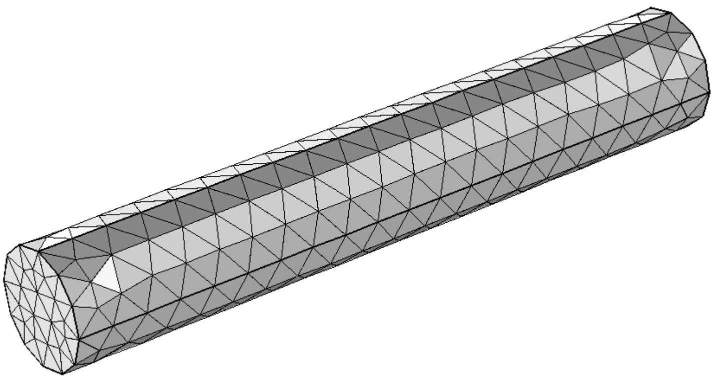


Fig. 3

(2) Results
Fig. 4 indicates the distribution of temperature. The heat flux by convection can be calculated by the following expression (namely ‘Q’ in Step 1):

$$\Phi = \iint_{B2} h\left(t - t_f\right) \mathrm{d}S = 40.118\ \mathrm{W}$$

And

$$\Phi' = 67.059\ \mathrm{W}$$

To sum up

$$\Phi_{\text{analytical}} = 40.1\mathrm{W}, \Phi_{\text{numerical}} = 40.118\mathrm{W}$$

$$\Phi'_{\text{analytical}} = 66.9\mathrm{W}, \Phi'_{\text{numerical}} = 67.059\mathrm{W}$$

Therefore we can obtain similar results by both two methods.

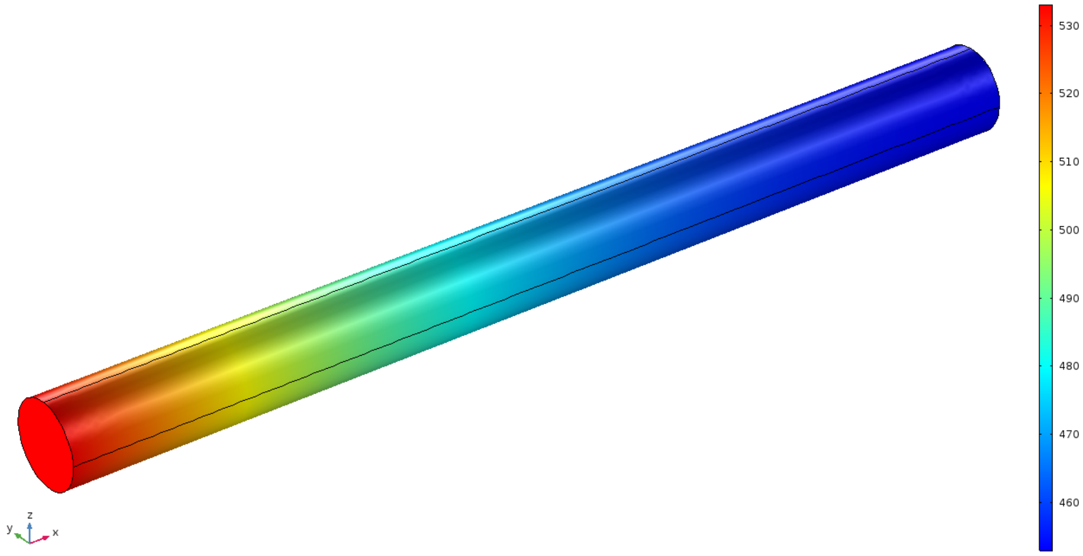


Fig. 4

(3) Discussion

Then we discuss possible errors in analytical and numerical methods.

Firstly, in the analytical method, the one dimension mathematic model is based on the assumption that the temperature of any cross section is equal and even. However, the temperature of inner part of a cross section is actually higher than its boundary. In numerical analysis, we compare the difference of temperature distributions (=$T(K_1) - T(K_2)$) on the two line sections located in the center and boundary along the fin (K_1 and K_2, Fig. 5 and Fig. 6), which was shown in Fig. 7.

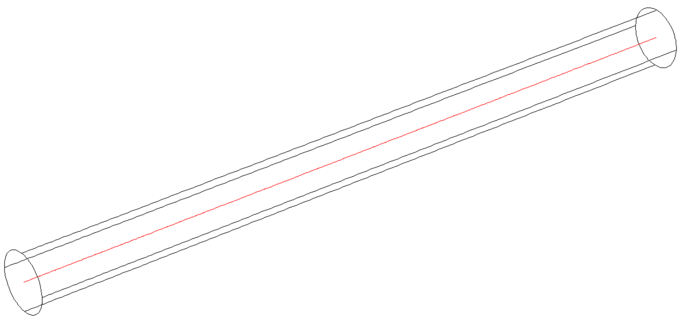


Fig. 5

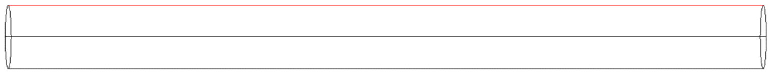


Fig. 6

Fig. 7 demonstrates that for a cross section in the fin, though the difference is minute (Lower than 0.06 K), there is a temperature gradient from the center to boundary.

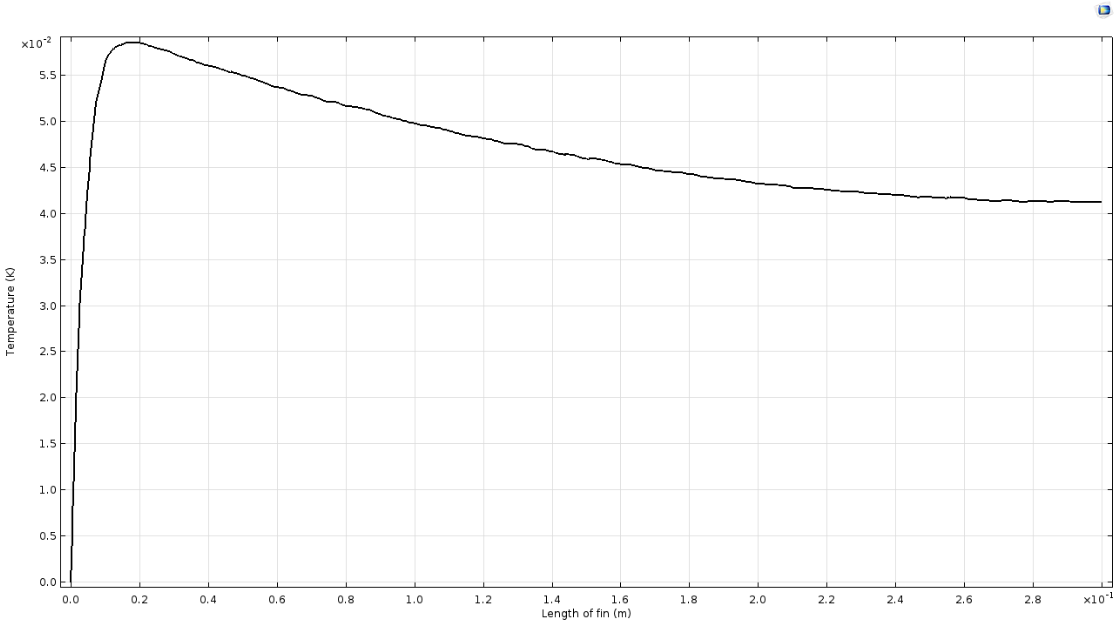


Fig. 7

Secondly, for numerical simulation, the sequence type and element size of mesh have influence on results. Generally when element size is decreased, the calculation accuracy can be enhanced. In this study, for shortening program's run time, 'normal' element size was chosen.

Conclusion

Both analytical and numerical methods show that when the length of the fin increase to $2H$, the heat loss by convection cannot double.