Variant 1

1. Show that the next conditional statement is a tautology without using truth tables: $(\neg q \to \neg p) \to \big((\neg q \to p) \to q\big)$.
   Solution:
$$(\neg q \to \neg p) \to \big((\neg q \to p) \to q\big) = \overline{\bar{\bar{q}} \lor \bar{p}} \lor \big(\overline{\bar{\bar{q}} \lor p} \lor q\big) = \overline{q \lor \bar{p}} \lor (\overline{q \lor p} \lor q)$$
$$= (\bar{q} \land p) \lor \big((\bar{q} \land \bar{p}) \lor q\big) = (\bar{q} \land p) \lor (\bar{q} \land \bar{p}) \lor q$$
$$= \big(\bar{q} \land (p \lor \bar{p})\big) \lor q = (\bar{q} \land T) \lor q = \bar{q} \lor q = T$$

2. Find the sum-of-products expansion (i.e. DNF) of the Boolean function $G(x, y, z)$ that has the value 0 if and only if $x \oplus y = \bar{y} \downarrow z$.
   Solution:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$	$\boldsymbol{x \oplus y}$	$\boldsymbol{\bar{y} \downarrow z}$	$\boldsymbol{G(x, y, z)}$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	1	1	0
0	1	1	1	0	1
1	0	0	1	0	1
1	0	1	1	0	1
1	1	0	0	1	1
1	1	1	0	0	0

$$F(w, x, y, z) = \bar{x}yz + x\bar{y}\bar{z} + x\bar{y}z + xy\bar{z}$$

3. Use Quine-McCluskey method to find a minimal expansion as a Boolean sum of Boolean products of the next function: $(x + \bar{y})\bar{z} + \bar{x}(y|z) + x \downarrow \bar{y}$.
   Solution:
$$(x + \bar{y})\bar{z} + \bar{x}(y|z) + x \downarrow \bar{y} = x\bar{z} + \bar{y}\bar{z} + \bar{x}(\bar{y} + \bar{z}) + \bar{x}y$$
$$= x\bar{z} + \bar{y}\bar{z} + \bar{x}\bar{y} + \bar{x}\bar{z} + \bar{x}y$$
$$= xy\bar{z} + x\bar{y}\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}yz$$
$$+ \bar{x}y\bar{z} = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z} + \bar{x}yz$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x		1	1	
$\bar{x}$	1	1	1	1

$$(x + \bar{y})\bar{z} + \bar{x}(y|z) + x \downarrow \bar{y} = \bar{x} + \bar{z}$$

4. Find a recurrence relation for the number of quaternary strings of length n that do not contain the substring 22.
   Solution:
$$a_n = 3a_{n-1} + 3a_{n-2}.$$

Variant 2

1. Suppose that the domain of the propositional function $A(x,y,z)$ consists of the integers 0 and 1. Write out the next propositions using disjunctions, conjunctions, and negations: a) $\exists x \forall y \exists z A(x,y,z)$; b) $\forall x \neg \exists y \forall z \neg A(x,y,z)$.
Solution:
   a) $\exists x \forall y \exists z A(x,y,z) = \Big(\big(A(0,0,0) \vee A(0,0,1)\big) \wedge \big(A(0,1,0) \vee A(0,1,1)\big)\Big) \vee \Big(\big(A(1,0,0) \vee A(1,0,1)\big) \wedge \big(A(1,1,0) \vee A(1,1,1)\big)\Big).$
   b) $\forall x \neg \exists y \forall z \neg A(x,y,z) = \forall x \forall y \exists z A(x,y,z) = \big(A(0,0,0) \vee A(0,0,1)\big) \wedge \big(A(0,1,0) \vee A(0,1,1)\big) \wedge \big(A(1,0,0) \vee A(1,0,1)\big) \wedge \big(A(1,1,0) \vee A(1,1,1)\big).$

2. Find the product-of-sums expansion (i.e. CNF) of the Boolean function: $\big((xy) \downarrow \bar{y}\big) + (\bar{x} \oplus z)$.
Solution:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$	$\boldsymbol{xy}$	$(\boldsymbol{xy}) \downarrow \bar{\boldsymbol{y}}$	$\bar{\boldsymbol{x}} \oplus \boldsymbol{z}$	$\big((\boldsymbol{xy}) \downarrow \bar{\boldsymbol{y}}\big) + (\bar{\boldsymbol{x}} \oplus \boldsymbol{z})$
0	0	0	0	0	1	1
0	0	1	0	0	0	0
0	1	0	0	1	1	1
0	1	1	0	1	0	1
1	0	0	0	0	0	0
1	0	1	0	0	1	1
1	1	0	1	0	0	0
1	1	1	1	0	1	1

$$\big((xy) \downarrow \bar{y}\big) + (\bar{x} \oplus z) = (x + y + \bar{z})(\bar{x} + y + z)(\bar{x} + \bar{y} + z)$$

3. Use K-map method to simplify the next sum-of-products expansion: $xy\bar{z} + \big(\overline{x|\bar{y}}\big)z + w\overline{xy}$.
Solution:

$$\begin{aligned} xy\bar{z} + \big(\overline{x|\bar{y}}\big)z + w\overline{xy} &= xy\bar{z} + x\bar{y}z + w(\bar{x} + \bar{y}) \\ &= wxy\bar{z} + \bar{w}xy\bar{z} + wx\bar{y}z + \bar{w}x\bar{y}z + w\bar{x} + w\bar{y} \\ &= wxy\bar{z} + \bar{w}xy\bar{z} + wx\bar{y}z + \bar{w}x\bar{y}z + w\bar{x}yz + w\bar{x}y\bar{z} + w\bar{x}\bar{y}\bar{z} \\ &+ w\bar{x}\bar{y}z + wx\bar{y}z + wx\bar{y}\bar{z} + w\bar{x}\bar{y}z + w\bar{x}\bar{y}\bar{z} \\ &= wxy\bar{z} + \bar{w}xy\bar{z} + wx\bar{y}z + \bar{w}x\bar{y}z + w\bar{x}yz + w\bar{x}y\bar{z} + w\bar{x}\bar{y}\bar{z} \\ &+ w\bar{x}\bar{y}z + wx\bar{y}\bar{z} \end{aligned}$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx		1	1	..1..
$w\bar{x}$	1	1	1	1
$\bar{w}\bar{x}$				
$\bar{w}x$		1		..1..

$$xy\bar{z} + \left(\overline{x|\bar{y}}\right)z + w\overline{xy} = w\bar{x} + w\bar{y} + xy\bar{z} + x\bar{y}z$$

4. What is the solution of the recurrence relation $a_n = 5a_{n-2} - 7a_{n-1}$ with $a_0 = 1$ and $a_1 = 2$?
   Solution:
   The characteristic equation of the recurrence relation is $r^2 + 7r - 5 = 0$. Its roots are $r = \frac{-7-\sqrt{69}}{2}$ and $r = \frac{-7+\sqrt{69}}{2}$. Hence $a_n = \alpha_1 \left(\frac{-7-\sqrt{69}}{2}\right)^n +$ $\alpha_2 \left(\frac{-7+\sqrt{69}}{2}\right)^n$. From the initial conditions, it follow that
   $$\begin{cases} a_0 = 1 = \alpha_1 + \alpha_2, \\ a_1 = 2 = \alpha_1 \left(\dfrac{-7-\sqrt{69}}{2}\right) + \alpha_2 \left(\dfrac{-7+\sqrt{69}}{2}\right). \end{cases}$$
   Then $\alpha_1 = \frac{\sqrt{69}-11}{2\sqrt{69}}$ and $\alpha_2 = \frac{\sqrt{69}+11}{2\sqrt{69}}$. Consequently,
   $$a_n = \left(\frac{\sqrt{69}-11}{2\sqrt{69}}\right) \cdot \left(\frac{-7-\sqrt{69}}{2}\right)^n + \left(\frac{\sqrt{69}+11}{2\sqrt{69}}\right) \cdot \left(\frac{-7+\sqrt{69}}{2}\right)^n.$$

Variant 3

1. Show that the next conditional statement is a tautology without using truth tables: $(p \to q) \to \big((q \to r) \to (p \to r)\big)$.
   Solution:
   $$\begin{aligned}(p \to q) \to \big((q \to r) \to (p \to r)\big) &= \overline{\bar{p} \vee q} \vee \big(\overline{\bar{q} \vee r} \vee (\bar{p} \vee r)\big)\\ &= (p \wedge \bar{q}) \vee \big((q \wedge \bar{r}) \vee \bar{p} \vee r\big) = (p \wedge \bar{q}) \vee \bar{p} \vee (q \wedge \bar{r}) \vee r\\ &= (\bar{q} \vee \bar{p}) \vee (q \vee r) = (\bar{q} \vee q) \vee \bar{p} \vee r = T \vee \bar{p} \vee r = T\end{aligned}$$

2. Find the sum-of-products expansion (i.e. DNF) of the Boolean function $G(x, y, z)$ that has the value 0 if and only if $x \oplus y = x\bar{z}$.
   Solution:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$	$\boldsymbol{x\bar{z}}$	$\boldsymbol{x \oplus y}$	$\boldsymbol{G(x, y, z)}$
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	1	1
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	0	0	0

$$F(x, y, z) = \bar{x}y\bar{z} + \bar{x}yz + x\bar{y}z + xy\bar{z}$$

3. Use Quine-McCluskey method to find a minimal expansion as a Boolean sum of Boolean products of the next function: $x|(\bar{y}z) + (x \downarrow y)\bar{z} + \overline{xy}$.
   Solution:
   $$\begin{aligned}x|(\bar{y}z) + (x \downarrow y)\bar{z} + \overline{xy} &= \bar{x} + \overline{\bar{y}z} + (\bar{x}\bar{y})\bar{z} + (\bar{x} + \bar{y})\\ &= \bar{x} + y + \bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x} + \bar{y} = \bar{x} + y + \bar{z} + \bar{y} + \bar{x}\bar{y}\bar{z}\\ &= \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + xyz + xy\bar{z} + \bar{x}yz + \bar{x}y\bar{z} + xy\bar{z}\\ &+ x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}\bar{z}\\ &= \bar{x}yz + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + xyz + xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z}\end{aligned}$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	1
$\bar{x}$	1	1	1	1

$$x|(\bar{y}z) + (x \downarrow y)\bar{z} + \overline{xy} = 1$$

4. Find a recurrence relation for the number of bit strings of length n that do not contain the substring 0000.
   Solution:
   $$a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$$

Variant 4

1. Suppose that the domain of the propositional function $B(x, y, z)$ consists of the integers 3 and 4. Write out the next propositions using disjunctions, conjunctions, and negations: a) $\forall x \forall y \exists z B(x, y, z)$; b) $\neg \forall x \exists y \neg \exists z B(x, y, z)$.
Solution:
a) $\forall x \forall y \exists z B(x, y, z) = \big(B(3,3,3) \vee B(3,3,4)\big) \wedge \big(B(3,4,3) \vee B(3,4,4)\big) \wedge \big(B(4,3,3) \vee B(4,3,4)\big) \wedge \big(B(4,4,3) \vee B(4,4,4)\big)$
b) $\neg \forall x \exists y \neg \exists z B(x, y, z) = \exists x \forall y \exists z B(x, y, z) = \Big(\big(B(3,3,3) \vee B(3,3,4)\big) \wedge \big(B(3,4,3) \vee B(3,4,4)\big)\Big) \vee \Big(\big(B(4,3,3) \vee B(4,3,4)\big) \wedge \big(B(4,4,3) \vee B(4,4,4)\big)\Big)$

2. Find the product-of-sums expansion (i.e. CNF) of the Boolean function: $(x \oplus y) + (x|\bar{y})z$.
Solution:

$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$	$\boldsymbol{x\vert\bar{y}}$	$\boldsymbol{(x\vert\bar{y})z}$	$\boldsymbol{x \oplus y}$	$\boldsymbol{(x \oplus y) + (x\vert\bar{y})z}$
0	0	0	1	0	0	0
0	0	1	1	1	0	1
0	1	0	1	0	1	1
0	1	1	1	1	1	1
1	0	0	0	0	1	1
1	0	1	0	0	1	1
1	1	0	1	0	0	0
1	1	1	1	1	0	1

$$(x \oplus y) + (x|\bar{y})z = (x + y + z)(\bar{x} + \bar{y} + z)$$

3. Use K-map method to simplify the next sum-of-products expansion: $\overline{xy}z + (x \downarrow y)\bar{z} + w\bar{x}(\overline{\bar{y} + z})$.
Solution:

$$\overline{xy}z + (x \downarrow y)\bar{z} + w\bar{x}(\overline{\bar{y} + z}) = (\bar{x} + \bar{y})z + (\bar{x}\bar{y})\bar{z} + w\bar{x}(y\bar{z})$$
$$= \bar{x}z + \bar{y}z + \bar{x}\bar{y}\bar{z} + w\bar{x}y\bar{z}$$
$$= w\bar{x}yz + w\bar{x}\bar{y}z + \bar{w}\bar{x}yz + \bar{w}\bar{x}\bar{y}z + wx\bar{y}z + w\bar{x}\bar{y}z + \bar{w}x\bar{y}z + \bar{w}\bar{x}\bar{y}z + w\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}\bar{y}\bar{z} + w\bar{x}y\bar{z}$$
$$= w\bar{x}yz + w\bar{x}\bar{y}z + \bar{w}\bar{x}yz + \bar{w}\bar{x}\bar{y}z + wx\bar{y}z + \bar{w}x\bar{y}z + w\bar{x}\bar{y}\bar{z} + \bar{w}\bar{x}\bar{y}\bar{z} + w\bar{x}y\bar{z}$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx				1
$w\bar{x}$	..1	1	1..	..1..
$\bar{w}\bar{x}$	..1		1..	..1..
$\bar{w}x$				1

$$\overline{xy}z + (x \downarrow y)\bar{z} + w\bar{x}(\overline{\bar{y} + z}) = w\bar{x} + \bar{y}z + \bar{x}\bar{y} + \bar{x}z$$

4. What is the solution of the recurrence relation $a_n = -5a_{n-1} + 3a_{n-2}$ with $a_0 = 4$ and $a_1 = 7$?
   Solution:
   The characteristic equation of the recurrence relation is $r^2 + 5r - 3 = 0$. Its roots are $r = \frac{-5-\sqrt{37}}{2}$ and $r = \frac{-5+\sqrt{37}}{2}$. Hence $a_n = \alpha_1\left(\frac{-5-\sqrt{37}}{2}\right)^n + \alpha_2\left(\frac{-5+\sqrt{37}}{2}\right)^n$.
   From the initial conditions, it follow that
   $$\begin{cases} a_0 = 4 = \alpha_1 + \alpha_2, \\ a_1 = 7 = \alpha_1\left(\dfrac{-5-\sqrt{37}}{2}\right) + \alpha_2\left(\dfrac{-5+\sqrt{37}}{2}\right). \end{cases}$$
   Then $\alpha_1 = \frac{2\sqrt{37}-17}{\sqrt{37}}$ and $\alpha_2 = \frac{2\sqrt{37}+17}{\sqrt{37}}$. Consequently,

$$a_n = \left(\frac{2\sqrt{37}-17}{\sqrt{37}}\right) \cdot \left(\frac{-5-\sqrt{37}}{2}\right)^n + \left(\frac{2\sqrt{37}+17}{\sqrt{37}}\right) \cdot \left(\frac{-5+\sqrt{37}}{2}\right)^n.$$