Variant 1

1. Express the operations $\cap, \backslash$ in terms of: $\oplus$, $\cup$. Moreover, prove your answers.

Solution:

$$A \cap B = (A \cup B) \oplus A \oplus B$$
$$A \backslash B = (A \cup B) \oplus B$$

2. Find $dom(R), range(R), R^{-1}, R \circ R, R \circ R^{-1}, R^{-1} \circ R$ for the following relation: $R = \{(x, y) |\ x, y \in \mathbb{N}\ and\ 3x = 4y\}$.

Solution:

$$dom(R) = \{x \mid x\ is\ divisible\ by\ 4\}$$
$$range(R) = \{y\ |y\ is\ divisible\ by\ 3\}$$
$$R^{-1} = \{(x, y) |\ x, y \in \mathbb{N}\ and\ 3y = 4x\}$$
$$R \circ R = \{(x, y) |\ x, y \in \mathbb{N}\ and\ 9x = 16y\}$$
$$R \circ R^{-1} = \{(x, y) |\ x, y \in \mathbb{N}\ and\ x = y\}$$
$$R^{-1} \circ R = \{(x, y) |\ x, y \in \mathbb{N}\ and\ x = y\}$$

3. Let us define a relation S on the set $\mathbb{N} \times \mathbb{N}$ as follows: $\big((a, b), (c, d)\big) \in S \Leftrightarrow [\big((a \cdot d = b \cdot c), b \neq 0\ and\ d \neq 0\big) or\ (a = c, b = 0\ and\ d = 0)]$. Prove that S is an equivalence relation.

Solution:

$$\big((a, b), (a, b)\big) \in S$$
$$\big((a, b), (c, d)\big) \in S \Rightarrow\ \big((c, d), (a, b)\big) \in S$$
$$\big((a, b), (c, d)\big) \in S\ and\ \big((c, d), (e, f)\big) \in S \Rightarrow\ \big((a, b), (e, f)\big) \in S$$

4. Determine whether $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto, one-to-one if $f(x, y) = x^2 + y$. Justify your answers.

Solution:

f is onto, since for any $n \in \mathbb{Z}$ there exist x and y such that $f(x, y) = n$ (for instance $x = 0\ and\ y = n$).

f is not one-to-one, since there exist $(x_1, y_1) \neq (x_2, y_2)$ such that $f(x_1, y_1) = f(x_2, y_2)$ (for instance $(1, 0) \neq (-1, 0)$ but $f(1, 0) = f(-1, 0) = 1$).

Variant 2

1. Given sets A, B, C such that $B \subseteq A \subseteq C$. Find a set X for which the system of equations $\begin{cases} A \cap X = B, \\ A \cup X = C, \end{cases}$ hold. Prove your answer.

Solution:

$$X = (C \setminus A) \cup B$$

2. Find $dom(R), range(R), R^{-1}, R \circ R, R \circ R^{-1}, R^{-1} \circ R$ for the following relation: $R = \{(x, y) | \ x, y \in [0, \pi] \ and \ x \geq \cos(y)\}$.

Solution:

$$dom(R) = \{x \mid x \in [0, \pi]\} = [0, \pi]$$
$$range(R) = \{y \mid y \in [0, \pi]\} = [0, \pi]$$
$$R^{-1} = \{(x, y) | \ x, y \in [0, \pi] \ and \ y \geq \cos(x)\}$$
$$R \circ R = \{(x, y) | \ x, y \in [0, \pi]\} = [0, \pi] \times [0, \pi]$$
$$R \circ R^{-1} = \{(x, y) | \ x, y \in [0, \pi]\} = [0, \pi] \times [0, \pi]$$
$$R^{-1} \circ R = \{(x, y) | \ x, y \in [0, \pi]\} = [0, \pi] \times [0, \pi]$$

3. Construct a binary relation, which is reflexive and transitive, but not symmetric. Prove your answer.

Solution:

$$\{(x, y) | \ x, y \in \mathbb{R} \ and \ x \leq y\}$$

4. Determine whether $f: \mathbb{N} \to \mathbb{N}$ is onto, one-to-one if $f(x) = x^2 - 4x + 5$. And find its inverse. Justify your answers.

Solution:

$$f(x) = (x - 2)^2 + 1$$

f is not onto, since there is no $x \in \mathbb{N}$ such that $(x - 2)^2 + 1 = 3$.
f is not one-to-one, since there exist $(x_1, y_1) \neq (x_2, y_2)$ such that $f(x_1, y_1) = f(x_2, y_2)$ (for instance $(1, 0) \neq (3, 0)$ but $f(1, 0) = f(3, 0) = 2$).
There is no inverse, since $f(x)$ is not 1-1 correspondence.

Variant 3

1. Express the operations ∪,∩ in terms of: \, ⊕. Moreover, prove your answers.

Solution:

$$A \cup B = (A \oplus B) \oplus (A \backslash (A \backslash B))$$
$$A \cap B = A \backslash (A \backslash B)$$

2. Find $dom(R), range(R), R^{-1}, R \circ R, R \circ R^{-1}, R^{-1} \circ R$ for the following relation: $R = \{(x,y) |\ x, y \in \mathbb{R}\ and\ 2x + 1 \geq y^2\}$.

Solution:

$$dom(R) = \{x \mid x \geq -1/2\} = [-\frac{1}{2}; +\infty]$$
$$range(R) = \{y \mid y \in \mathbb{R}\} = \mathbb{R}$$
$$R^{-1} = \{(x,y) |\ x, y \in \mathbb{R}\ and\ 2y + 1 \geq x^2\}$$
$$R \circ R = \{(x,y) |\ x, y \in \mathbb{R}\ and\ 2x + 1 \geq (y^2 - 1)^2/4\}$$
$$R \circ R^{-1} = \{(x,y) |\ x, y \geq -1/2\} = [-\frac{1}{2}; +\infty] \times [-\frac{1}{2}; +\infty]$$
$$R^{-1} \circ R = \{(x,y) |\ x, y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R}$$

3. Let us define a relation Q on the set $\mathbb{N} \times \mathbb{N}$ as follows: $\big((a,b),(c,d)\big) \in Q \Leftrightarrow 2a + 3d = 3b + 2c$. Prove that Q is an equivalence relation.

Solution:

$$\big((a,b),(a,b)\big) \in Q$$
$$\big((a,b),(c,d)\big) \in S \Rightarrow \big((c,d),(a,b)\big) \in Q$$
$$\big((a,b),(c,d)\big) \in Q\ and\ \big((c,d),(e,f)\big) \in Q \Rightarrow \big((a,b),(e,f)\big) \in Q$$

4. Determine whether $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is onto, one-to-one if $f(x,y) = \sin(x) + y - 1$. Justify your answers.

Solution:

f is onto, since for any $z \in \mathbb{R}$ there exist x and y such that $f(x,y) = z$ (for instance $x = 0\ and\ y = z + 1$).

f is not one-to-one, since there exist $(x_1, y_1) \neq (x_2, y_2)$ such that $f(x_1, y_1) = f(x_2, y_2)$ (for instance $(0,0) \neq (\pi, 0)$ but $f(0,0) = f(\pi, 0) = -1$).

Variant 4

1. Given sets A, B, C such that $B \subseteq A$ and $A \cap C = \emptyset$. Find a set X for which the system of equations $\begin{cases} A \backslash X = B, \\ X \backslash A = C, \end{cases}$ hold. Prove your answer.

Solution:

$$X = (A \backslash B) \cup C$$

2. Find $dom(R), range(R), R^{-1}, R \circ R, R \circ R^{-1}, R^{-1} \circ R$ for the following relation: $R = \{(x, y) | \; x, y \in \mathbb{Z} \; and \; x + 2y \; is \; divisible \; by \; 3\}$.

Solution:

$$dom(R) = \{x \mid x \in \mathbb{Z}\} = \mathbb{Z}$$
$$range(R) = \{y \mid y \in \mathbb{Z}\} = \mathbb{Z}$$
$$R^{-1} = \{(x, y) | \; x, y \in \mathbb{Z} \; and \; y + 2x \; is \; divisible \; by \; 3\}$$
$$R \circ R = \{(x, y) | \; x, y \in \mathbb{Z} \; and \; x + 2y \; is \; divisible \; by \; 3\} = R$$
$$R \circ R^{-1} = \{(x, y) | \; x \; and \; y \; have \; same \; remainders \; after \; dividing \; on \; 3\}$$
$$R^{-1} \circ R = \{(x, y) | \; x \; and \; y \; have \; same \; remainders \; after \; dividing \; on \; 3\}$$

3. Construct a binary relation, which is antisymmetric and transitive, but not reflexive. Prove your answer.

Solution:

$$\{(x, y) | \; x, y \in \mathbb{R} \; and \; x = y = 0\}$$

4. Determine whether $f: \mathbb{R} \to \mathbb{R}$ is onto, one-to-one if $f(x) = x^3 - 1$. And find its inverse. Justify your answers.

Solution:

f is onto, since for any $z \in \mathbb{R}$ there exists x such that $f(x) = z$ (for instance $x = \sqrt[3]{z + 1}$).

f is one-to-one, since for any $x_1 \neq x_2$ we have $f(x_1) \neq f(x_2)$.

$f^{-1}(x) = \sqrt[3]{x + 1}$.