

Exercise 1

1

Let $x \in M$, then we can find a neighbourhood U of x and a chart $x : U \rightarrow \mathbb{R}^n$, and $\{\frac{\partial}{\partial x_i}\}$ be the induced basis on TU . Then locally, we can write:

$$X = \sum x_i \frac{\partial}{\partial x_i}; \quad Y = \sum y_j \frac{\partial}{\partial x_j}$$

Then a simple calculation yields:

$$XYf = \sum x_i \frac{\partial y_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \sum x_i y_j \frac{\partial^2 f}{\partial x_j \partial x_i}; \quad YXf = \sum y_j \frac{\partial x_i}{\partial x_j} \frac{\partial f}{\partial x_i} + \sum x_i y_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$$[X, Y] = \sum \left(x_i \frac{\partial y_j}{\partial x_i} - y_j \frac{\partial x_i}{\partial x_j} \right) \frac{\partial f}{\partial x_j}$$

Clearly, $B_x(X, Y)$ is symmetric if and only if $[X, Y]_x = 0$. Since by definition of B ,

$$B \cap U = \left\{ x \in U \mid \frac{\partial f}{\partial x_j}(x) = 0 \right\}$$

We get that if $x \in B$ then:

$$(XYf)_x = \sum x_i(x) y_j(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x); \quad (YXf)_x = \sum x_i(x) y_j(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x); \quad [X, Y]_x = 0 \quad (1)$$

Hence the symmetry of $B_x(X, Y)$ is proven and from (1) is clear that it only depends on $x_i(x)$ and $y_i(x)$, that is: $B_x(X, Y)$ only depends on X_x and Y_x .

Finally, if B is a submanifold, then $X \in TB$, X can be integrated to obtain a curve $x(t)$ contained in B so that $X_{x(t)} = x'(t)$.

Since $x(t) \in B$ we have $\frac{\partial f}{\partial x_i}(x(t)) = 0$, we can differentiate to obtain:

$$0 = \sum_j x'_j(t) \frac{\partial^2 f}{\partial x_i \partial x_j}, \quad \forall i \in \{1, \dots, n\}$$

Since $x'_i(0) = x_i(x)$, by the above calculations we have that:

$$(XYf)_x = \sum_{i,j=1}^n x_i(x) y_j(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \sum_{j=1}^n \left\{ y_j(x) \sum_{i=1}^n x_i(x) \frac{\partial^2 f}{\partial x_j \partial x_i} \right\} = 0$$

This proves the case $X_x \in T_x B$. The case $Y_x \in T_x B$ is completely analogous, since for $x \in B$, $(XYf)_x = (YXf)_x$

2

Let $H_x(f)(u, v) \stackrel{\text{def}}{=} (XYf)_x$, where we choose X, Y so that $u = X_x$ and $v = Y_x$.

We have to check that the choice of X, Y does not alter the value of H , however this was proven in the previous section where we checked that $(XYf)_x$ only depended on X_x and Y_x .

Finally, since:

$$\ker H_x(f) = \{X \in TM \mid H_x(f)(X, Y) = 0, \forall Y\} = \{X \in TM \mid (YXf)_x = 0, \forall Y\} = TB \quad (2)$$

we deduce that the hessian of f is non-degenerate on $N_{B/M, x}$.

Finally, we can choose U and a local frame $\mathcal{B} = \frac{\partial}{\partial x_j}$ on U so that $H_x(f)$ has an associated symmetric matrix M so that

$$H_x(f) \left(\sum x_i \frac{\partial}{\partial x_i}, \sum y_j \frac{\partial}{\partial x_j} \right) = \sum_{i,j} x_i y_j M_{ij}$$

Then if we change the frame $\frac{\partial}{\partial x_j}$ to another frame $\frac{\partial}{\partial y_j}$ we have a matrix P a linear transformation on TM so that $P \frac{\partial}{\partial y_j} = \frac{\partial}{\partial x_j}$.

Then the matrix M' associated to the Hessian by the new matrix is given by:

$$M' = P^T M P$$

By Sylvester's inertia theorem, we can find an invertible matrix S , so that $S^T M S$ is diagonal.

By the above discussion, it's enough to choose $\{S^{-1} \frac{\partial}{\partial x_i}\}$, as in this frame the Hessian has the form of (0.5).

It also clear, that since each connected component C_j of B is a submanifold where $df|_{C_j} = 0$ then $f|_{C_j} = \text{const}$.

a)

By the usual Poincaré lemma, we can find a form γ so that $\gamma|_W = 0$ and $d\gamma = \alpha$.

Since the action is trivial on W we can find a G -invariants neighbourhood of W that we shall call U . Then we apply the standard trick of averaging γ over U . Let β we defined as

$$\beta = \int_U g^* \gamma d\mu$$

Where we chose μ to be a Haar measure, as it is well-know that compact Lie groups admits such a measure. And we could normalize in the usual way by requiring $\int_U d\mu = 1$.

Then since the action is trivial on W , $\beta|_W = 0$ and since d commutes with the integral sign and with the pullback by the naturality condition we have that:

$$\begin{aligned} d\beta &= d \int_U g^* \gamma d\mu = \int_U g^* (d\gamma) d\mu \\ &= \int_U g^* \alpha d\mu \stackrel{(*)}{=} \int_U \alpha d\mu = \alpha \int_U d\mu \\ &= \alpha \end{aligned}$$

Where $(*)$ follows because α is invariant by hypothesis.

Furthermore, β is invariant by construction. This proves the theorem.

b)

We follow the proof of the standard Darboux's lemma, with a slight twist:

Let $\omega_t = (1-t)\omega_0 + t\omega_1$. Then clearly, $\omega_t|_W = \omega_0|_W$. Then let X_t be an arbitrary differential vector field, and let ϕ_t be it's flow. Then:

$$\begin{aligned} \frac{d}{dt} \phi_t^* \omega_t &= \phi_t^* (L_{X_t} \omega_t + \omega_1 - \omega_0) = 0 \\ \iff L_{X_t} \omega_t + \omega_1 - \omega_0 &= 0 \end{aligned} \tag{3}$$

Then, because the action is trivial on W we can find a neighbourhood U of W so that the action of G is well defined. That is $g \cdot U \subset U, \forall g \in G$.

By part a) we can, shrinking U if necessary find a G -invariant form β so that $d\beta = \omega_1 - \omega_0$ and $\beta_W = 0$, since $\omega_1 - \omega_0$ is in the hypothesis of a).

Then, using Cartan's formula, since ω_t is closed we get: $L_{X_t} \omega_t = di_{X_t} \omega$.

So (3) turns into:

$$i_{X_t} \omega_t + \beta = 0 \tag{4}$$

Because ω_t is not degenerate, because it is the convex combination of two symplectic forms, we can solve (4) to get a vector field X_t that solves (3), since $\frac{d}{dt} \phi_t^* \omega_t = 0 \implies \phi_1^* \omega_1 = \omega_0$

If we could show that ϕ_t is G -invariant we would be done. We are going to show that if we define $\hat{\phi}_t = g \cdot \phi_t(g^{-1} \cdot x)$. By the uniqueness guaranteed by the Picard-Lindelöf theorem we have to show that:

$$\begin{cases} \frac{\partial}{\partial t} \hat{\phi}_t = X_t \hat{\phi}_t \\ \hat{\phi}_0(x) = x \end{cases}$$

For the first part:

$$\begin{aligned} \frac{\partial}{\partial t} \hat{\phi}_t &= dg X_t \phi_t(g^{-1} \cdot x) = dg X_t g^{-1} \cdot g \phi_t(g^{-1} \cdot x) \\ &= dg X_t g^{-1} \hat{\phi}_t(x) = X_t \hat{\phi}_t(x) \end{aligned} \tag{5}$$

Where last equality comes from the G -invariance of X_t which in turn comes from the G -invariance of ω_t .

For the second part: $\hat{\phi}_0 = g \cdot \phi_0(g^{-1} \cdot x) = g \cdot g^{-1} \cdot x = x$

And so the invariant Darboux theorem follows.

Exercise 2

1

We have that ω is invariant by hypothesis, furthermore, since both M and \mathbb{T} are compact we can find $\langle \cdot, \cdot \rangle$ a bi-invariant scalar product by the standard averaging procedure.

Then there is a unique matrix A so that:

$$\omega(X, Y) = \langle AX, Y \rangle, \quad \forall X, Y \in TM \quad (6)$$

Since both ω and $\langle \cdot, \cdot \rangle$ are invariant, we have that by uniqueness, so is A , that is because $g^{-1}Ag$ also verifies (6) as we show. Let $g \in G$, then:

$$\langle AX, Y \rangle \stackrel{(6)}{=} \omega(X, Y) \stackrel{*}{=} \omega(dg(X), dg(JY)) = \langle Adg(X), dg(Y) \rangle \stackrel{*}{=} \langle dg^{-1}Adg(X), Y \rangle$$

Where $*$ follows by the invariance of ω and $\langle \cdot, \cdot \rangle$

By the skew-symmetry of ω we have that $A = -A^T$, therefore $-A^2 = AA^T$ is symmetric and we can take it's square root. (By diagonalising $-A^2 = P \text{diag}(\lambda_1, \dots, \lambda_n) P^{-1}$, then $\sqrt{-A^2} = P \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}) P^{-1}$)

Then $J = (\sqrt{-A^2})^{-1} A$ is invariant, because A is and is the composition of G -invariant homomorphism.

Clearly, $J^2 = -A^{-2}A^2 = -Id$ and J is compatible with ω because:

$$\begin{aligned} \omega(X, JX) &= \langle AX, JX \rangle \\ &= \left\langle AX, \left(\sqrt{-A^2} \right)^{-1} AX \right\rangle = \left\langle X, \left(\sqrt{-A^2} \right)^{-1} X \right\rangle > 0 \end{aligned}$$

Where the last inequality follows because $\sqrt{-A^2}^{-1}$ is definite positive.

J is therefore the almost complex structure we were looking for.

2

Since g^{TM} is G -invariant, G acts by isometry, and takes geodesics to geodesics. Let $\gamma(t, x_0, v_0)$ be the unique geodesic starting in x_0 with derivative in $0 v_0$. Then $\exp_{x_0}(v_0) = \gamma(1, x_0, v_0)$

Therefore, if $x \in M^{\mathbb{T}}$, let's consider $\alpha(t) = g \cdot \gamma(t, x_0, v_0)$. By the above discussion, $\alpha(t)$ is also a geodesic.

Since $\alpha(0) = g \cdot x_0 = x_0$ and $\alpha'(0) = dg_{x_0}(\gamma'(0, x_0, v_0)) = dg_{x_0}(v_0)$

By uniqueness: $\alpha(t) = \gamma(t, x_0, dg_{x_0}(v_0))$. In particular, for $t=1$ we get:

$$\exp_{x_0}(dg_{x_0}(v_0)) = g \cdot \exp_{x_0}(v_0) \quad (7)$$

3

Let $x_0 \in M^{\mathbb{T}}$, then we can restrict the U in the previous part to a \mathbb{T} -invariant neighbourhood. In this way the \mathbb{T} -action on U is well-defined. Furthermore, since in the previous part we used an arbitrary $g \in \mathbb{T}$ we have that the exponential map is equivariant.

We notice that if $\exp_{x_0}(v) \in M^{\mathbb{T}}$ then:

$$\exp_{x_0}(v) = g \cdot \exp_{x_0}(v) = \exp_{x_0}(dg_{x_0}v), \quad \forall g \in \mathbb{T} \quad (8)$$

This means we can identify fixed points of the action in U with fixed point of the differential of the action on $T_{x_0}M$, since for $v \in T_{x_0}M, g \in \mathbb{T}, v = dg_{x_0}v$ because of (8) and \exp being an isomorphism.

That is, the exponential map send the eigenspace of $T_{x_0}M$ of eigenvalue 1 to $M^{\mathbb{T}} \cap U$. Because the exponential map is a diffeomorphism it sends submanifolds to submanifolds, in particular, it sends linear subspaces to submanifolds.

In summary: Every point in $M^{\mathbb{T}}$ admits a neighbourhood that is parametrised by the exponential map.

And so $M^{\mathbb{T}}$ is a submanifold.

4

Because x_0 is a fixed point, the action of \mathbb{T} on M lift to $T_{x_0}M$ via the differential, in particular, given $g, h \in \mathbb{T}$:

$$d(gh)_{x_0} = dg_{x_0}dh_{x_0}$$

By the chain rule. (We reiterate x_0 is a fixed point). Furthermore, since the action acts by symplectomorphism (by definition), we have that because x_0 is a fixed point if we choose $g \in \mathbb{T}$:

$$\omega_{x_0}(X, Y) = \omega_{g \cdot x_0}(dg_{x_0}X, dg_{x_0}Y) = \omega_{x_0}(dg_{x_0}X, dg_{x_0}Y)$$

This proves that the action on $T_{x_0}M$ is symplectic. It only remains to prove that the action commutes with J , however this is the definition for invariance of J which we already prove.

5

At this point we suspect that φ might be given by the exponential map with the natural identification of $B(x_0, \delta)$ with \mathbb{R}^{2n} , that is, under identification: $\varphi = \exp_{x_0}^{-1}$. This is obviously a diffeomorphism and it is \mathbb{T} -equivariant because we have defined precisely the action as the induced by this diffeomorphism.

Furthermore, by the section on compact Lie groups we know that there is a natural identification of \mathbb{T}^k with $(\mathbb{S}^1)^k$ given by:

$$\exp(\tau) = (e^{i\tau_1}, \dots, e^{i\tau_k})$$

And so the action induced on V must be of the above form, since it's linear and acts by elements of $U(n)$.

6

We define $A_g \stackrel{\text{def}}{=} \ker(\mathbb{1} - dg_{x_0}g)$ to ease notation.

Let $v \in A_g$ we have that because of the previous parts of the exercise:

$$v = dg_{x_0}(J_{x_0}v) = J_{x_0}dg_{x_0}(v) = J_{x_0}v \quad (9)$$

And so $J_{x_0}A_g = A_g$, because by (9) we have proved the \subset inclusion and the other inclusion comes from the fact that J is an isomorphism since $-J$ is its inverse.

From which we conclude that:

$$T_{x_0}(M^{\mathbb{T}}) = (T_{x_0}M)^{\mathbb{T}} = \bigcap_{g \in \mathbb{T}} A_g$$

If we showed that each of the A_g was symplectic, then their intersection would be as well and the result would follow. That is the proof follows from this two claims:

Claim: A_g is symplectic: **Claim:** The intersection of symplectic spaces is symplectic.

Proof. (First claim)

By way of contradiction. Let g be an arbitrary element in \mathbb{T} and $v \in A_g$ so that $i_v\omega_{x_0} = 0$.

Because of $J_{x_0}A_g = A_g$ we know we can find w so that $J_{x_0}w = v$. Then:

$$0 = \omega_{x_0}(J_{x_0}w, w) = -g_{x_0}^{TM}(w, w) > 0$$

Which is a contradiction.

QED

Proof. (Second claim) A_g is symplectic if and only if $A_g \cap A_g^{\perp\omega} = \{0\}$.

From this we get:

$$\begin{aligned} \left(\bigcap_{g \in \mathbb{T}} A_g \right) \cap \left(\bigcap_{g \in \mathbb{T}} A_g \right)^{\perp\omega} &= \left(\bigcap_{g \in \mathbb{T}} A_g \right) \cap \left(\bigcup_{g \in \mathbb{T}} A_g^{\perp\omega} \right) \\ &= \bigcap_{h \in \mathbb{T}} \left(\left(\bigcap_{g \in \mathbb{T}} A_g \right) \cap \left(\bigcup_{h \in \mathbb{T}} A_g^{\perp\omega} \right) \right) \\ &= \{0\} \end{aligned}$$

And we are done.

QED

In summary we have proven that $T_{x_0}(M^{\mathbb{T}})$ is symplectic.

Since we have already shown that the exponential map is a symplectomorphism and that it maps $T_{x_0}(M^{\mathbb{T}})$ to $(T_{x_0}M)^{\mathbb{T}}$ we conclude $M^{\mathbb{T}}$ is a symplectic submanifold.

7

We have to prove two things:

1. $d(\mu, \tau) = i_{\tau}^{\mathbb{C}^n} \omega_{st}$.

Proof. We begin by seeing who is $\tau^{\mathbb{C}^n}$. Let $z = x + iy \in \mathbb{C}$ then:

$$\begin{aligned}\tau^{\mathbb{C}^n}|_z &= \frac{d}{ds} \exp(s\tau) \cdot z = (-i\langle w_1, \tau \rangle z_1, \dots, -i\langle w_n, \tau \rangle z_n) \\ &= (\langle w_1, \tau \rangle y_1, \langle w_1, \tau \rangle x_1, \dots, \langle w_1, \tau \rangle y_n, \langle w_1, \tau \rangle x_n) \\ &= \sum_{j=1}^n \langle w_j, \tau \rangle \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right)\end{aligned}$$

Again, by computation we get:

$$i_{y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}} (dx_j \wedge dy_j) = x_j dx_j + y_j dy_j$$

And gluing all together:

$$i_{\tau^{\mathbb{C}^n}} \omega_{st} = \sum_{j=1}^n \langle w_j, \tau \rangle (x_j dx_j + y_j dy_j) \quad (10)$$

Now we go to the LHS, let's see who is $d(\mu, \tau)$. Let $z \in \mathbb{C}$:

$$d(\mu_0(z), \tau) = d\left(\frac{1}{2} \sum_{j=1}^n (x_j^2 + y_j^2)(w_j, \tau)\right) = \sum_{j=1}^n (x_j dx_j + y_j dy_j)(w_j, \tau) \quad (11)$$

Joining (10) with (11) we get the result.

QED

2. ω_{st} is equivariant, i.e $\mu_0(g \cdot z) = Ad_g^* \mu_0(z)$.

Proof. Firstly, we notice that since \mathbb{T} is abelian, so is its action, in the sense that Ad_g acts by identity. So the RHS is just $\mu(z)$. For the LHS, we notice that since \mathbb{T} is compact, it is generated as a group by $\{\exp(\tau), \tau \in \mathfrak{g}\}$, so if we show that $\mu_0(\exp(\tau) \cdot z) = \mu_0(z)$ this would show that the LHS is equal to $\mu_0(z)$ for any $z \in \mathbb{T}$.

We show the final claim:

$$\mu_0(\exp(\tau) \cdot z) = \frac{1}{2} \sum_{j=1}^n |e^{-i\langle w_j, \tau \rangle} z_j|^2 w_j = \frac{1}{2} \sum_{j=1}^n |z_j|^2 w_j = \mu_0(z)$$

QED

8

By definition of moment map we have:

$$d(\mu, \tau) = i_{\tau^M} \omega$$

And unraveling the identifications in μ_0 we have that we actually have $\varphi^* \mu_0$, where φ is the diffeomorphism above.

$$d(\varphi^* \mu_0, \tau) = \varphi^* i_{\tau^{\mathbb{C}^n}} \omega_{st} = i_{\varphi^* \tau^{\mathbb{C}^n}} \varphi^* \omega_{st} = i_{\varphi^* \tau^{\mathbb{C}^n}} \omega$$

By the non-degeneracy of ω we have that

$$d(\mu - \mu_0) = 0 \iff i_{\tau^M} \omega - i_{\varphi^* \tau^{\mathbb{C}^n}} \omega \iff \tau^M = \varphi^* \tau^{\mathbb{C}^n}$$

We can show that the last equality holds by direct computation:

$$\begin{aligned}\varphi^* \tau^{\mathbb{C}^n}|_x &= d\varphi^{-1} \left(\frac{d}{ds} \varphi \exp(s\tau) \cdot x \Big|_{s=0} \right) \\ &= d\varphi^{-1} \left(d\varphi \left(\frac{d}{ds} \exp(s\tau) \cdot x \Big|_{s=0} \right) \right) \\ &= \frac{d}{ds} \exp(s\tau) \cdot x \Big|_{s=0} = \tau^M|_x\end{aligned}$$

Thus $d(\mu - \mu_0) = 0$, which implies that $(\mu, \tau) - (\mu_0, \tau)$ is locally constant for any τ . Since we are working on a connected neighbourhood, we deduce that $(\mu, \tau) = (\mu(x_0), \tau) + (\mu_0, \tau)$.

Finally, since τ is arbitrary we conclude $\mu = \mu(x_0) + \mu_0$ which is what we wanted. (Notice that with the identification of μ , we have that $\mu(0) = \mu(x_0)$).

9

By definition $Crit(\mu_X) = \{x \in M : d\mu_x = 0\}$ Then by the definition of momentum map we get:

$$\begin{aligned} d(\mu_X)_x &= 0 \\ \iff d(\mu_X)_x(Y) &= 0, \quad \forall Y \in C^\infty(M, TM) \\ \iff \omega(X_x^M, Y) &= 0, \quad \forall Y \in C^\infty(M, TM) \\ \iff X_x^M &= 0 \end{aligned} \tag{12}$$

The last equality follows from the non-degeneracy of ω .

We notice that $X_x^M = 0$ if and only if x is a fixed point of the action, that is:

$$Crit(\mu_x) = M^{\mathbb{T}}$$

And so $Crit(\mu_x)$ is a symplectic submanifold because we have already proven it in (6),

10

It follows from the previous section that $Crit(\mu_x) = M^{\mathbb{T}}$ and by (6) $Crit(\mu_x)$ is symplectic submanifold.

The Hessian can be calculated as follows: $H_x(\mu_X)(Y, Z) = (ZY\mu_X)_x$

Then:

$$\begin{aligned} Y\mu_X &= d\mu_X(Y) = \omega(X^M, Y) \\ ZY\mu_X &= d(\omega(X^M, Y)(Z)) = (i_{[Y, X^M]}\omega)(Z) = \omega([Y, X^M], Z) = \omega(L_Y(X^M), Z) \end{aligned}$$

It follows that: $H_x(\mu_X)(Y, Z) = \omega_x([Y, X^M], Z)$.

Since $[Y, X^M] = L_Y X^M = \frac{d}{dt}\big|_{t=0} \exp(-tY)_* X^M$. Which means by (12) that:

$$TCrit(\mu_X) = \{Y \in TM : L_Y X^M = 0\} = \bigcap_{Y \in TM} \ker H_x(\mu_X)(Y)$$

This proves the Hessian is non-degenerate on the set of critical points. Bringing this together with the fact that $Crit(\mu_X)$ is a submanifold we conclude that μ_X is a Morse-Bott function.

11

By (0.10) in the exam, we have that the hessian of μ as identified in \mathbb{C}^n and then identified again in \mathbb{R}^{2n} is $\frac{1}{2} \sum_{j=1}^n (x_j^2 + y_j^2) \omega_j(X)$.

Where by the \mathbb{C}^n to \mathbb{R}^{2n} identification we mean identifying $z = x + iy$ with (x, y) .

This means the matrix of the Hessian is as follows:

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

Where $\lambda_i \stackrel{\text{def}}{=} (w_i, X)$. That is every eigenvalue appears twice due to the identification. So there must be an even number of them that are negative (maybe 0, but even nonetheless). Which proves the result.

Exercise 3

1

By non-degeneracy of ω it is enough to show that for a given $X \in TM$ we have:

$$\omega(\mu^M, X) = \omega\left(-\frac{1}{2}J(d\mathcal{H})^*), X\right) \tag{13}$$

On the LHS we have:

$$\omega(\mu^M, X) = i_{\mu^M} \omega(X) = d(\langle \mu, \mu(x) \rangle_{\mathfrak{g}})(X) \tag{14}$$

On the RHS we have:

$$\begin{aligned} \omega\left(-\frac{1}{2}J(d\mathcal{H})^*), X\right) &= \frac{1}{2}g^{TM}(X, (d\mathcal{H})^*) \\ &= \frac{1}{2}(d\mathcal{H})(X) \\ &= d(\langle \mu, \mu(x) \rangle_{\mathfrak{g}})(X) \end{aligned} \tag{15}$$

From (13) we get that $\mu_x^M = 0 \iff d\mathcal{H}_x = 0$ and so we get:

$$\text{Crit}(\mathcal{H}) = \{x \in M : \mu_x^M = 0\}$$

As we wanted.

2

First we clarify the way μ_T is defined, which is by the following diagram:

$$\mu_T : M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{h} \mathfrak{g} \xrightarrow{i^*} \mathfrak{t}^* \xrightarrow{h^{-1}|_{\mathfrak{t}^*}} \mathfrak{t}$$

Where i is the natural inclusion: $\mathfrak{t} \xrightarrow{i} \mathfrak{g}$ and h is the natural pairing $\mathfrak{g} \xrightarrow{h} \mathfrak{g}^*$ given by $h(u)(v) = \langle u, v \rangle_{\mathfrak{g}}$.

Equivariance of μ_T is trivial since if $\mu(g \cdot x) = \text{Ad}_g^* \mu(x)$ is true for any $g \in G$ then it certainly true for any $g \in T \subset G$.

By the same logic, if $X \in T$ then X^M coincides if seen as an element of T or as an element of G because it is defined in the same exact manner, and because $i^*X = X$ we have that $(\mu, X) = (\mu_T, X)$ so we have that:

$$i_{X^M} \omega = d(\mu_T, X)$$

Which proves μ_T is a moment map.

Let $\mathcal{B} = \{t_1, \dots, t_n, x_1, \dots, x_m\}$ be an orthonormal basis of \mathfrak{g} and $\mathcal{B}^* = \{t_1^*, \dots, t_n^*, x_1^*, \dots, x_m^*\}$ it's dual basis of \mathfrak{g}^* . Notice that because of the orthonormality condition the dual basis coincides in the usual sense and in the scalar product sense.

Then given $x \in M$, we have:

$$\mu(x) = \sum_{i=1}^n u_i t_i + \sum_{j=1}^m v_j x_j$$

Then because $i^*x_j = x_j|_{\mathfrak{t}} = 0$ we have:

$$\mu_T(x) = \sum_{i=1}^n u_i t_i = P^{\mathfrak{t}} \left(\sum_{i=1}^n u_i t_i + \sum_{j=1}^m v_j x_j \right) = P^{\mathfrak{t}} \mu(x)$$

3

This is a direct consequence of Theorem 1. Since Theorem 1 guarantees that $\mu_T(M^T)$ are the vertices of a convex polytope, which by definition of polytope means it must be finite.

4

This is a direct consequence of a classic theorem in Functional Analysis, however here we give a slightly simplified proof for the case of finite dimension and the case where the point exterior to the set is the origin.

Firstofall, if $0 \in A$ we are done. If not, let:

$$\delta = \text{dist}(0, A) = \inf_{x \in A} |x| > 0$$

Where the strict inequality comes from the fact that A is closed.

We have to show 2 thing:

1. There exists $x \in A$ so that $|x| = \delta$.
2. There is only one such $x \in A$.

Proof. (Existence)

Because δ is defined as an infimum, we can find a sequence of points x_n so that $|x_n| \rightarrow \delta$. By picking a subsequence, we may assume x_n that

$$x \leq |x_n| \leq x + \frac{1}{n+1}$$

Let $n, m \in \mathbb{N}$, then by hypothesis $d \leq \frac{1}{2}(y_n + y_m) \in A$.

By direct computation we have:

$$\begin{aligned} \delta^2 &\leq \left| \frac{1}{2}y_n + \frac{1}{2}y_m \right|^2 \stackrel{(1)}{=} \frac{1}{2} \left(|y_n|^2 + |y_m|^2 \right) - \frac{1}{4} |y_n + y_m|^2 \\ &\leq \delta^2 + \frac{1}{2} \left(\left(\frac{1}{n+1} \right)^2 + 2d \left(\frac{1}{n+1} + \frac{1}{m+1} \right) + \left(\frac{1}{m+1} \right)^2 \right) - \frac{1}{4} |x_n - x_m|^2 \end{aligned} \tag{16}$$

From which we get:

$$|x_n - x_m|^2 \leq 2 \left(\left(\frac{1}{n+1} \right)^2 + 2d \left(\frac{1}{n+1} + \frac{1}{m+1} \right) + \left(\frac{1}{m+1} \right)^2 \right)$$

$$|x_n - x_m|^2 \xrightarrow{n, m \rightarrow \infty} 0$$

Which means $\{x_n\}$ is a Cauchy sequence and therefore convergent, because A is closed we get that the limit is in A and obviously the limit has norm δ by the continuity of the norm.

QED

Proof. (Uniqueness) Suppose $x, x' \in A$, $x \neq x'$, so that $|x| = |x'| = \delta$

Let $y = \frac{1}{2}x + (1 - \frac{1}{2})x'$. By hypothesis $y \in A$ and so $|y| \geq \delta$

And so by use of the parallelogram identity:

$$\delta^2 \leq \frac{1}{2} \leq \left| \frac{1}{2}x + (1 - \frac{1}{2})x' \right|^2 = 2 \left(\left| \frac{1}{2}x \right|^2 + \left| \frac{1}{2}x' \right|^2 \right) - \frac{1}{2}|x - x'|^2 = \delta^2 - \frac{1}{2}|x - x'|^2 < \delta^2$$

Where the last inequality comes because $|x - x'|^2 > 0$.

QED

5

By definition each element b of \mathcal{B} is associated to a subset of $\mathcal{P}(\mathcal{A})$ which is finite because \mathcal{A} is finite. In particular, $\#\mathcal{P}(\mathcal{A}) = 2^{\#\mathcal{A}}$ so:

$$\#\mathcal{P}(\mathcal{A}) < \infty \iff \#\mathcal{A} < \infty$$

Since this association of \mathcal{B} with subsets of \mathcal{A} gives an injection we have:

$$\#\mathcal{B} \leq \#\mathcal{P}(\mathcal{A}) < \infty$$

Which is what we wanted.

6

We already know by the previous part of the exercise that $\text{Crit}(\mu_\beta) = \{x \in M \mid \beta_x^M = 0\}$.

The set $Z_\beta = \{x \in M \mid \beta_x^M = 0, \mu_\beta(x) \neq 0\}$ is clearly closed in $\text{Crit}(\mu_\beta)$ by continuity of $|\cdot|^2$. Furthermore, since μ_β must be locally constant along the connected components of $\text{Crit}(\mu_\beta)$ we conclude Z_β must be relatively open in $\text{Crit}(\mu_\beta)$.

This means Z_β is clopen in $\text{Crit}(\mu_\beta)$ and so is the union of open connected components of $\text{Crit}(\mu_\beta)$ (we know they are open because there are only a finite amount of them due to M being compact).

In particular, this means Z_β is the disjoint union of symplectic manifolds. Since being symplectic is a local condition (it is a condition on the differential, which is local and on the tangent space on a point, which obviously local) we conclude that Z_β must be symplectic.

7

Since for $\mu(x) \in \mathfrak{t}^*$ $\mu(x) = \mu_T(x)$ by (2)

From this we have:

$$x \in \text{Crit}(\mathcal{H}) \iff x \in \text{Crit}(|\mu|^2) \iff x \in \text{Crit}(|\mu_T|^2)$$

We now prove the following: **Claim:** Given $x \in M$ with $\mu_T(x) = \beta \in \mathfrak{t}$. Then we have that $x \in \text{Crit}(\mathcal{H})$ if and only if $x \in Z_\beta$. And when this happens β is the element of minimal norm in $\mu_t(Z_\beta)$ which is obviously a convex set.

Proof. We have the following:

$$x \in \text{Crit}(\mathcal{H}) \iff \mu^M|_x = 0 \iff \beta_X^M = 0$$

Where the last double implication comes from exercise 2 part 9 (and the definition).

And so the first implication (\Rightarrow) follows by the definition of Z_β .

For the \Leftarrow implication we know by Theorem 1 that $\mu_T(Z_\beta) = \text{convex hull}(\mu_T(Z_\beta^T))$

By the previous part there exists a element of minimal norm in $\mu_T(Z_\beta)$. We only have to show that this point is β . Let's argue by way of contradiction, let γ be the point of minimal norm, in particular, $|\gamma| < |\beta|$ and we can find $x \in Z_\beta$ so that $\gamma = \mu(x) = \mu_T(x)$, where the last equality holds because $\mu_t(Z_\beta) \subset \mathfrak{t}$.

We have that by definition of Z_β (and by the identification of \mathfrak{t} with \mathfrak{t}^*) that $\langle \mu(y), \beta \rangle = \langle \beta, \beta \rangle$

from which trivially we have:

$$\langle \mu(y) - \beta, \beta \rangle = 0$$

$$|\gamma|^2 = |\mu_T(y)|^2 = \langle \mu(y), \mu(y) \rangle \stackrel{\pm\beta}{=} \langle \mu(y) - \beta, \mu(y) - \beta \rangle + \langle \beta, \beta \rangle = |\mu(y) - \beta|^2 + |\beta|^2 < |\beta|^2$$

Which is a contradiction and so we have proven the claim above.

QED

It follows immediately from the equivariance of μ that $g \cdot \text{Crit}(\mathcal{H}) = \text{Crit}(\mathcal{H})$.
Putting all together we have that:

$$g \cdot x \in \text{Crit}(\mathcal{H}) \iff g \cdot z \in Z_\beta \iff x \in C_\beta$$

Which shows: $\text{Crit}(\mathcal{H}) = \cup_{\beta \in \mathcal{B}} C_\beta$

It only remains to show that C_β are disjoint. Let $\beta \in \mathcal{B}$, then:

$$\mu(C_\beta) = \{\mu(g \cdot x) | x \in Z_\beta \cap \mu^{-1}(\beta), g \in G\} \{Ad_g(\mu(x)) | x \in Z_\beta, \mu(x) = \beta, g \in G\} = Ad_G(\beta) = \mathcal{O}_\beta$$

And by a theorem of the course we know that each coadjoint orbit intersects each positive Weyl chamber number in precisely one point, this directly implies that C_β are disjoint since otherwise we could use the coadjoint action to bring a point in the intersection to a point in a positive Weyl chamber, which will mean two different coadjoint orbits intersect, which is imposible.

8

We just showed: $\mu(C_\beta) = Ad_G(\beta)$

We can actually say more:

$$\begin{aligned} C_\beta &= \{g \cdot x | g \in G, x \in Z_\beta, \mu(x) = \beta\} = \{g \cdot x | g \in G, x \in \text{Crit}(\mathcal{H}), \mu(x) = \beta\} \\ &= \{g \cdot x | g \in G, g \cdot x \in \text{Crit}(\mathcal{H}), \mu(x) = \beta\} = \{x \in \text{Crit}(\mathcal{H}) | \exists g \in G, \mu(x) = Ad_g(\beta)\} \end{aligned}$$

As we have already seen $x \in C_\beta \implies x \in Z_\beta$ and $\mathcal{H}(x) = |\mu(x)|^2 = |\beta|^2$ for any x in C_β , where again we have omitted the dual identification.

So we have proven: $\mathcal{H}(C_\beta) = |\beta|^2$

Exercise 4

1

Let $\alpha, \beta \in \Omega^{0,k}(M)$ and $\langle \cdot, \cdot \rangle$ be the scalar product we have seen in class for $(0, k)$ forms.

We have to show that: $\langle D_T \alpha, \beta \rangle = \langle \alpha, D_T \beta \rangle$ We compute:

$$\langle D_T \alpha, \beta \rangle = \langle D + iTc(\mu^M) \alpha, \beta \rangle = \langle D \alpha, \beta \rangle + \langle iTc(\mu^M) \alpha, \beta \rangle$$

By the lectures we know that D is self-adjoint so $\langle D \alpha, \beta \rangle = \langle \alpha, D \beta \rangle$. It is therefore enough to prove that $iTc(\mu^M)$ is self-adjoint. Because $c(\mu^M)^* = -c(\mu^M)$, and because $\langle \cdot, \cdot \rangle$ is hermitian it follows that:

$$\langle iTc(\mu^M) \alpha, \beta \rangle = \langle \alpha, -iT(-c(\mu^M)) \beta \rangle = \langle \alpha, iTc(\mu^M) \beta \rangle$$

2

Obviously $\ker D_T \subset \ker D_T^2$. Let $x \in \ker D_T^2$, then, because D_T^2 is self-adjoint:

$$0 = \langle D_T^2 x, x \rangle = \langle D_T x, D_T x \rangle$$

So we conclude $D_T x = 0$ and so $\ker D_T^2 \subset \ker D_T$ and $\ker D_T = \ker D_T^2$.

The last part is clear since $\ker D_{+,T} = \ker D_T|_{\Omega^{0,\text{even}}(M,L)} = \ker(D_T) \cap \Omega^{0,\text{even}}(M,L) \subset \Omega^{0,\text{even}}$ and in a completely analogous manner $\ker D_{-,T} = \ker D_T|_{\Omega^{0,\text{odd}}(M,L)} = \ker(D_T) \cap \Omega^{0,\text{odd}}(M,L) \subset \Omega^{0,\text{odd}}$

3

The first equality is trivial from the previous part since $\ker D_T^2 = \ker D_T$. The last equality would follow (as we will show) if D_T were G -equivariant. On the one hand D is G -equivariant as has been shown in the lectures. μ^M is also G -equivariant as we have shown in the previous exercise. So D_T must be G -equivariant as well, and a fortiori $D_{\pm,T}$ must be invariant by G . And so:

$$\begin{aligned} \ker D_T|_{\Omega^{0,\text{even}}(M,L)^G} &= \ker D_T \cap \Omega^{0,\text{even}}(M,L)^G = \ker D_T^2 \cap \Omega^{0,\text{even}}(M,L)^G \\ &\stackrel{(*)}{=} (\ker D_{+,T} \cap \oplus \ker D_{-,T}) \cap \Omega^{0,\text{even}}(M,L)^G \\ &= \ker D_{+,T} \cap \Omega^{0,\text{even}}(M,L)^G = \ker D_{+,T}^G \end{aligned}$$

Where $(*)$ follows because D_T decomposes as $D_T = D_{+,T} + D_{-,T}$ And the case for $D_{-,T}$ is completely analogous (except replacing $\Omega^{0,\text{even}}(M,L)^G$ with $\Omega^{0,\text{odd}}(M,L)^G$).

We have that:

$$D_T^2 = (D + iTc(\mu^m))(D + iTc(\mu^m)) = D^2 + T^2|\mu^M|^2 + iT(Dc(\mu^M) + c(\mu^M)D)$$

So we have to show that we can find some A_M 0-order differential operator so that:

$$iT(Dc(\mu^M) + c(\mu^M)D) = TA_M - 2iT \sum_{i=1}^{\dim G} \mu_i L_{V_i} \quad (17)$$

Let's see who is $Dc(\mu^M) + c(\mu^M)D$

$$\begin{aligned} Dc(\mu^M) + c(\mu^M)D &= \sum_{j=1}^{2n} c(\mu^M)c(e_j)\nabla_{e_j}^{Cl \otimes L} + c(e_j)\nabla_{e_j}^{Cl \otimes L}c(\mu^M) \\ &\stackrel{(1)}{=} \sum_{j=1}^{2n} c(\mu^M)c(e_j)\nabla_{e_j}^{Cl \otimes L} + c(e_j)c(\mu^M)\nabla_{e_j}^{Cl \otimes L} + c(e_j)c(\nabla_{e_j}^{TM}\mu^M) \\ &= \sum_{j=1}^{2n} (c(\mu^M)c(e_j) + c(e_j))\nabla_{e_j}^{Cl \otimes L} + c(e_j)c(\nabla_{e_j}^{TM}\mu^M) \\ &= -2g^{TM}(\mu^M, e_j)\nabla_{e_j}^{Cl \otimes L} + c(e_j)c(\nabla_{e_j}^{TM}\mu^M) \\ &= -2\nabla_{\mu^M}^{Cl \otimes L} + c(e_j)c(\nabla_{e_j}^{TM}\mu^M) \end{aligned}$$

Where (1) follows from $[\nabla_V^{Cl \otimes L}, c(W)] = c(\nabla_V^{Cl \otimes L}W)$. (This identity holds for ∇_V^{Cl} so it's trivial to check it follows as well for $\nabla^{Cl \otimes L} = \nabla^{Cl} \otimes \mathbb{1} + \mathbb{1} \otimes \nabla^L$).

Let $\alpha \in \Omega^{0,\cdot}(M, L)$. α is locally of the form $\tilde{\alpha} \otimes s$ for $\tilde{\alpha} \in C^\infty(M, \Lambda^{\cdot}T^{(0,1)}M)$ and $v \in C^\infty(M, L)$.

And so if $V \in TM$ we have:

$$L_V\alpha = (L_V\tilde{\alpha}) \otimes s + \tilde{\alpha} \otimes (L_Vs)$$

Where we are abusing the overloading the notation of L_V .

By Kostant's formula we have that:

$$\nabla_{K^M}^L = L_{K^M} + 2\pi i(\mu, K), \quad \forall K \in \mathfrak{g}$$

Then we can express μ^M in the given coordinates: $\mu^M = \sum_{i=1}^{\dim G} \mu_i V_i$ and so:

$$\nabla_{\mu^M}^L s = \sum_{i=1}^{\dim G} \mu_i \nabla_{V_i^M}^L s = \sum_{i=1}^{\dim G} \mu_i (L_{V_i^M} s + 2\pi i(\mu, V_i)) s$$

Finally, notice that although $\nabla_{\mu^M}^L$ is not a linear operator because $\nabla_{\mu^M}^{Cl}(\phi\beta) = (L_{\mu^M}\phi)\beta + \phi\nabla_{\mu^M}^{Cl}\beta$. But this means it is an affine operator, so it's a summand of the Lie derivative plus some linear operator, \tilde{A}_M , so we define: $\tilde{\tilde{A}}_M \stackrel{\text{def}}{=} \tilde{A}_M \otimes \mathbb{1}$

Putting everything together we get:

$$\begin{aligned} D_T^2 &= D^2 + T^2|\mu^M|^2 + iT(Dc(\mu^M) + c(\mu^M)D) \\ &= D^2 + T^2 + T \left(ic(e_j)c(\nabla_{e_j}^{TM}\mu^M) - 2i \sum_{i=1}^{\dim G} (\mu_i(2\pi i(\mu, V_i) + \tilde{\tilde{A}}_M)) \right) - 2iT \left(\sum_{i=1}^{\dim G} \mu_i L_{V_i} \right) \end{aligned}$$

And so defining $A \stackrel{\text{def}}{=} ic(e_j)c(\nabla_{e_j}^{TM}\mu^M) - 2i \sum_{i=1}^{\dim G} (\mu_i(2\pi i(\mu, V_i) + \tilde{\tilde{A}}_M))$ we are done.

5

We know from (0.15) that:

$$\mu^{-1}(0) \subset \text{Crit}(\mathcal{H}) = \{x \in M | \mu^M|_x = 0\} = \emptyset$$

On one hand M is compact and $|\mu^M| > 0$ by hypothesis. This means we can choose T large enough so that $T^2|\mu^M|^2$ is larger than the rest of the terms in (0.24). (Notice some other terms also depend on T , but they depend linearly, so we can dominate them). This means, for this T sufficiently large we have $\langle D_T, D_T \rangle > 0$ and so $\ker D_T = \{0\}$.

Since by definition $\text{ind}(D_+)^G = \dim(\ker D_+)^G - (\ker D_-)^G$

and both $(\ker D_+)^G$ and $(\ker D_-)^G$ are subspaces of $(\ker D_\pm)^G$ we get that their dimension is both 0.

So $\text{ind}(D_+)^G = 0$, which is what we wanted.