Exercise 1

1

Let $x \in M$, then we can find a neighbourhood U of x and a chart $x : U \to \mathbb{R}^n$, and $\{\frac{\partial}{\partial x_i}\}$ be the induced basis on TU.

Then locally, we can write:

$$X = \sum x_i \frac{\partial}{\partial x_i}; \quad Y = \sum y_j \frac{\partial}{\partial x_j}$$

Then a simple calculation yields:

$$XYf = \sum x_i \frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum x_i y_j \frac{\partial^2 f}{\partial x_j \partial x_i}; \quad YXf = \sum y_j \frac{\partial x_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \sum x_i y_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$$[X, Y] = \sum \left(x_i \frac{\partial y_j}{\partial x_i} - y_j \frac{\partial x_i}{\partial x_i} \right) \frac{\partial f}{\partial x_j}$$

Clearly, $B_x(X, Y)$ is symmetric if and only if $[X, Y]_x = 0$. Since by definition of B,

$$B \cap U = \left\{ x \in U \,\middle|\, \frac{\partial f}{\partial x_j}(x) = 0 \right\}$$

We get that if $x \in B$ then:

$$(XYf)_x = \sum x_i(x) y_j(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x); \quad (YXf)_x = \sum x_i(x) y_j(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x); \quad [X, Y]_x = 0 \tag{1}$$

Hence the symmetry of $B_x(X, Y)$ is proven and from (1) is clear that is only depends on $x_i(x)$ and $y_i(x)$, that is: $B_x(X, Y)$ only depends on X_x and Y_x.

Finally, if B is a submanifold, then $X \in TB$, X can be integrated to obtain a curve $x(t)$ contained in B so that $X_{x(t)} = x'(t)$.

Since $x(t) \in B$ we have $\frac{\partial f}{\partial x_i}(x(t)) = 0$, we can differentiate to obtain:

$$0 = \sum_j x_j'(t) \frac{\partial^2 f}{\partial x_i x_j} \quad , \forall i \in \{1, \dots, n\}$$

Since $x_i'(0) = x_i(x)$, by the above calculations we have that:

$$(XYf)_x = \sum_{i,j=1}^{n} x_i(x) y_j(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \sum_{j=1}^{n} \left\{ y_j(x) \sum_{i=1}^{n} x_i(x) \frac{\partial^2 f}{\partial x_j \partial x_i} \right\} = 0$$

This proves the case $X_x \in T_x B$. The case $Y_x \in T_x B$ is completely analogous, since for $x \in B, (XYf)_x = (YXf)_x$

2

Let $H_x(f)(u, v) \overset{\text{def}}{=} (XYf)_x$, where we choose X, Y so that $u = X_x$ and $v = Y_x$.

We have to check that the choice of X, Y does not alter the value of H, however this was proven in the previous section where we checked that $(XYf)_x$ only depended on X_x and Y_x.

Finally, since:

$$KerH_x(f) = \{X \in TM | H_x(f)(X, Y) = 0, \forall Y\} = \{X \in TM | (YXf)_x = 0 \forall Y\} = TB \tag{2}$$

we deduce that the hessian of f is non-degenerate on $N_{B/M,x}$.

Finally, we can choose U and a local frame $\mathcal{B} = \frac{\partial}{\partial x_j}$ on U so that $H_x(f)$ has an associated symmetric matrix M so that

$$H_x(f)\left(\sum x_i \frac{\partial}{\partial x_i}, \sum y_j \frac{\partial}{\partial x_j}\right) = \sum_{i,j} x_i y_j M_{ij}$$

Then if we change the frame $\frac{\partial}{\partial x_j}$ to another frame $\frac{\partial}{\partial y_j}$ we have a matrix P a linear transformation on TM so that $P\frac{\partial}{\partial y_j} = \frac{\partial}{\partial x_j}$.

Then the matrix $M^{'}$ associated to the Hessian by the new matrix is given by:

$$M^{'} = P^T M P$$

By Sylverster's inertia theorem, we can find an invertible matrix S, so that $S^T M S$ is diagonal.

By the above discussion is enough to choose $\{S^{-1}\frac{\partial}{\partial x_i}\}$, as in this frame the Hessian has the form of (0.5).

It also clear, that since each connected component C_j of B is a submanifold where $df|_{C_j} = 0$ then $f|_{C_j} = const.$

3

a)

By the usual Poincaré lemma, we can find a form γ so that $\gamma|_W = 0$ and $d\gamma = \alpha$.

Since the action is trivial on W we can find a $G - invariant$ neighbourhood of W that we shall call U. Then we apply the standard trick of averaging γ over U. Let β we defined as

$$\beta = \int_U g^* \gamma d\mu$$

Where we chose μ to be a Haar measure, as it is well-know that compact Lie groups admits such a measure. And we could normalize in the usual way by requiring $\beta = \int_U d\mu = 1$.

Then since the action is trivial on W, $\beta|_W = 0$ and since d commutes with the integral sign and with the pullback by the naturality condition we have that:

$$\begin{aligned}
d\beta &= d\int_U g^*\gamma d\mu = \int_U g^*(d\gamma) d\mu \\
&= \int_U g^*\alpha d\mu = \int_U \alpha d\mu \overset{(*)}{=} \alpha \int_U d\mu \\
&= \alpha
\end{aligned}$$

Where $(*)$ follows because α is invariant by hypothesis.

Furthermore, β is invariant by construction.

b)

We follow the proof of the standard Darboux's lemma, with a slight twist:

Let $\omega_t = (1-t)\omega_0 + t\omega_1$. Then clearly, $\omega_t|_W = \omega_0|_W$. Then let X_t be an arbitrary differential vector field, and let ϕ_t be it's flow. Then:

$$\begin{aligned}
\frac{d}{dt}\phi_t^*\omega_t &= \phi^*(L_{X_t}\omega_t + \omega_1 - \omega_0) = 0 \\
&\iff L_{X_t}\omega_t + \omega_1 - \omega_0 = 0
\end{aligned} \tag{3}$$

Then, because the action is trivial on W we can find a neighbourhood U of W so that the action of G is well defined. That is $g \cdot U \subset U, \forall g \in G$.

By part a) we can, shrinking U if necessary find a G-invariant form β so that $d\beta = \omega_1 - \omega_0$ and $\beta_W = 0$, since $\omega_1 - \omega_0$ is in the hypothesis of a).

Then, using Cartan's formula, since ω_t is closed we get: $L_{X_t}\omega_t = di_{X_t}\omega$.

So (3) turns into:

$$i_{X_t}\omega_t + \beta = 0 \tag{4}$$

Because ω_t is not degenerate, because it is the convex combination of two symplectic forms, we can solve (4) to get a vector field X_t that solves (3), since $\frac{d}{dt}\phi_t^*\omega_t = 0 \implies \phi_1^*\omega_1 = \omega_0$

If we could show that ϕ_t is G-invariant we would be done. We are going to show that if we define $\hat{\phi}_t = g \cdot \phi_t(g^{-1} \cdot x)$. By the uniqueness guaranteed by the Picard-Lindelöf theorem we have to show that:

$$\begin{cases} \frac{\partial}{\partial t}\hat{\phi}_t = X_t\hat{\phi}_t \\ \hat{\phi}_0(x) = x \end{cases}$$

For the first part:

$$\begin{aligned} \frac{\partial}{\partial t}\hat{\phi}_t = dgX_t\phi_t(g^{-1} \cdot x) &= dgX_tg^{-1} \cdot g\phi_t(g^{-1} \cdot x) \\ &= dgX_tg^{-1}\hat{\phi}_t(x) = X_t\hat{\phi}_t(x) \end{aligned} \tag{5}$$

Where last equality comes from the G-invariance of X_t which in turn comes from the G-invariance of ω_t.

For the second part: $\hat{\phi}_0 = g \cdot \phi_0(g^{-1} \cdot x) = g \cdot g^{-1} \cdot x = x$

And so the invariant Darboux theorem follows.

Exercise 2

1

We have that ω is invariant by hypothesis, furthermore, since both M and $\mathbb{T}$ are compact we can find $\langle \cdot, \cdot \rangle$ a bi-invariant scalar product by the standard averaging procedure.

Then there is a unique matrix A so that:

$$\omega(X, Y) = \langle AX, Y \rangle, \quad \forall X, Y \in TM \tag{6}$$

Since both ω and $< \cdot, \cdot >$ are invariant, we have that by uniqueness, so is A, that is because $g^{-1}Ag$ also verifies (6) as we show. Let $g \in G$, then:

$$\langle AX, Y \rangle \overset{(6)}{=} \omega(X, Y) \overset{*}{=} \omega(dg(X), dg(JY)) = \langle Adg(X), dg(Y) \rangle \overset{*}{=} \langle dg^{-1}Adg(X), Y \rangle$$

Where $*$ follows by the invariance of ω and $\langle \cdot, \cdot \rangle$

By the skew-symmetry of ω we have that $A = -A^T$, therefore $-A^2 = AA^T$ is symmetric and we can take it's square root. (By diagonalising $-A^2 = Pdiag(\lambda_1, \dots, \lambda_n)P^{-1}$, then $\sqrt{-A^2} = Pdiag(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})P^{-1}$)

Then $J = \left(\sqrt{-A^2}\right)^{-1} A$ is invariant, because A is and is the composition of G-invariant homomorphism. Clearly, $J^2 = -A^{-2}A^2 = -Id$ and J is compatible with ω because:

$$\begin{aligned} \omega(X, JX) &= \langle AX, JX \rangle \\ &= \left\langle AX, \left(\sqrt{-A^2}\right)^{-1} AX \right\rangle = \left\langle X, \left(\sqrt{-A^2}\right)^{-1} X \right\rangle > 0 \end{aligned}$$

Where the last inequality follows because $\sqrt{-A^2}^{-1}$ is definite positive.

J is therefore the almost complex structure we were looking for.

2

Since g^{TM} is G-invariant, G acts by isometry, and takes geodesics to geodesics. Let $\gamma(t, x_0, v_0)$ be the unique geodesic starting in x_0 with derivative in 0 v_0. Then $exp_{x_0}(v_0) = \gamma(1, x_0, v_0)$

Therefore, if $x \in M^{\mathbb{T}}$, let's consider $\alpha(t) = g \cdot \gamma(t, x_0, v_0)$. By the above discussion, $\alpha(t)$ is also a geodesic.

Since $\alpha(0)) = g \cdot x_0 = x_0$ and $\alpha'(0) = dg_{x_0}(\gamma'(0, x_0, v_0)) = dg_{x_0}(v_0)$

By uniqueness: $\alpha(t) = \gamma(t, x_0, dg_{x_0}(v_0))$. In particular, for t=1 we get:

$$exp_{x_0}(dg_{x_0}(v_0)) = g \cdot exp_{x_0}(v_0) \tag{7}$$

3

Let $x_0 \in M^{\mathbb{T}}$, then we can restrict the U in the previous part to a $\mathbb{T}$-invariant neighbourhood. In this way the $\mathbb{T}$-action on U is well-defined. Furthermore, since in the previous part we used an arbitrary $g \in \mathbb{T}$ we have that the exponential map is equivariant.

We notice that if $exp_{x_0}(v) \in M^{\mathbb{T}}$ then:

$$exp_{x_0}(v) = g \cdot exp_{x_0}(v) = exp_{x_0}(dg_{x_0}v), \quad \forall g \in \mathbb{T} \tag{8}$$

This mens we can identify fixed points of the action in U with fixed point of the differential of the action on $T_{x_0}M$, since for $v \in T_{x_0}M, g \in \mathbb{T}$, $v = dg_{x_0}v$ because of (8) and exp being an isomorphism.

That is, the exponential map send the eigenspace of $T_{x_0}M$ of eigenvalue 1 to $M^{\mathbb{T}} \cap U$. Because the exponential map is a diffeomorphism it sends submanifolds to submanifolds, in particular, it sends linear subspaces to submanifolds.

In summary: Every point in $M^{\mathbb{T}}$ admits a neighbourhood that is parametrised by the exponential map.

And so $M^{\mathbb{T}}$ is a submanifold.

4

Because x_0 is a fixed point, the action of $\mathbb{T}$ on M lift to $T_{x_0}M$ via the differential, in particular, given $g, h \in \mathbb{T}$:

$$d(gh)_{x_0} = dg_{x_0}dh_{x_0}$$

By the chain rule. (We reiterate x_0 is a fixed point). Furthermore, since the action acts by symplectomorphism (by defintion), we have that because x_0 is a fixed point if we choose $g \in \mathbb{T}$:

$$\omega_{x_0}(X, Y) = \omega_{g \cdot x_0}(dg_{x_0}X, dg_{x_0}Y) = \omega_{x_0}(dg_{x_0}X, dg_{x_0}Y)$$

This proves that the action on $T_{x_0}M$ is symplectic. It only remains to prove that the action commutes with J, however this is the definition for invariance of J which we already prove.

5

6

We define $A_g \stackrel{\text{def}}{=} \ker(\mathbb{1} - dg_{x_0}g)$ to ease notation.

Let $v \in A_g$ we have that because of the previous parts of the exercise:

$$v = dg_{x_0}(J_{x_0}v) = J_{x_0}dg_{x_0}(v) = J_{x_0}v \tag{9}$$

And so $J_{x_0}A_g = A_g$, because by (9) we have proved the $\subset$ inclusion and the other inclusion comes from the fact that J is an isomorphism since $-J$ is it's inverse.

From which we conclude that:

$$T_{x_0}(M^{\mathbb{T}}) = (T_{x_0}M)^{\mathbb{T}} = \bigcap_{g \in \mathbb{T}} A_g$$

If we showed that each of the A_g was symplectic, then theirs intersection would be as well and the result would follow. That is the proof follows from this two claims:

Claim: A_g is symplectic: **Claim**: The intersection of symplectic spaces is symplectic.

Proof. (First claim)

By way of contradiction. Let g be an arbitrary element in $\mathbb{T}$ and $v \in A_g$ so that $i_v \omega_{x_0} = 0$.

Because of $J_{x_0} A_g = A_g$ we know we can find w so that $J_{x_0} w = v$. Then:

$$0 = \omega_{x_0}(J_{x_0} w, w) = -g_{x_0}^{TM}(w, w) > 0$$

Which is a contradiction. *QED*

Proof. (Second claim) A_g is symplectic if and only if $A_g \cap A_g^{\perp\omega} = \{0\}$.

From this we get:

$$\begin{aligned}
\left(\bigcap_{g\in\mathbb{T}} A_g\right) \cap \left(\bigcap_{g\in\mathbb{T}} A_g\right)^{\perp\omega} &= \left(\bigcap_{g\in\mathbb{T}} A_g\right) \cap \left(\bigcup_{g\in\mathbb{T}} A_g^{\perp\omega}\right) \\
&= \bigcap_{h\in\mathbb{T}} \left(\left(\bigcap_{g\in\mathbb{T}} A_g\right) \bigcap \left(\bigcup_{h\in\mathbb{T}} A_g^{\perp\omega}\right)\right) \\
&= \{0\}
\end{aligned}$$

And we are done.

QED

In summary we have proven that $T_{x_0}(M^{\mathbb{T}})$ is symplectic.

Since we have already shown that the exponential map is a symplectomorphism and that it maps $T_{x_0}(M^{\mathbb{T}})$ to $(T_{x_0}M)^{\mathbb{T}}$ we conclude $M^{\mathbb{T}}$ is a symplectic submanifold.

7

We have to prove two things:

1. $d(\mu, \tau) = i_{\tau}^{\mathbb{C}^n} \omega_{st}$.

   Proof. We begin by seeing who is $\tau^{\mathbb{C}^n}$. Let $z = x + iy \in \mathbb{C}$ then:

$$\begin{aligned}
\tau^{\mathbb{C}^n}|_z = \frac{d}{ds} exp(s\tau) \cdot z &= (-i\langle w_1, \tau\rangle z_1, \ldots, -i\langle w_n, \tau\rangle z_n) \\
&= (\langle w_1, \tau\rangle y_1, \langle w_1, \tau\rangle x_1, \ldots, \langle w_1, \tau\rangle y_n, \langle w_1, \tau\rangle x_n,) \\
&= \sum_{j=1}^{n} \langle w_j, \tau\rangle \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right)
\end{aligned}$$

   Again, by computation we get:

$$i_{y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}} (dx_j \wedge dy_j) = x_j dx_j + y_j dy_j$$

   And gluing all together:

$$i_{\tau}^{\mathbb{C}^n} \omega_{st} = \sum_{j=1}^{n} \langle w_j, \tau\rangle (x_j dx_j + y_j dy_j) \tag{10}$$

   Now be go the LHS, let's see who is $d(\mu, \tau)$. Let $z \in \mathbb{C}$:

$$d(\mu_0(z),\tau) = d\left(\frac{1}{2}\sum_{j=1}^{n}(x_j^2+y_j^2)(w_j,\tau)\right) = \sum_{j=1}^{n}(x_j dx_j + y_j dy_j)(w_j,\tau) \tag{11}$$

Joining (10) with (11) we get the result.

QED

2. ω_{st} is equivariant, i.e $\mu_0(g\cdot z) = Ad_g^*\mu_0(z)$.

   Proof. Firstly, we notice that since $\mathbb{T}$ is abelian, so is it action, in the sense that Ad_g acts by identity. So the RHS is just $\mu(z)$.

   For the LHS, we notice that since $\mathbb{T}$ is compact, it is generated as a group by $\{exp(\tau), \tau \in \mathfrak{g}\}$, so if we show that $\mu_0(\exp(\tau)\cdot z) = \mu_0(z)$ this would that the LHS is also equal to $\mu_0(z)$.

   We show the final claim:

$$\mu_0(\exp(\tau)\cdot z) = \frac{1}{2}\sum_{j=1}^{n}|e^{-i\langle w_j,\tau\rangle}z_j|^2 w_j = \frac{1}{2}\sum_{j=1}^{n}|z_j|^2 w_j = \mu_0(z)$$

QED

8

By definition of moment map we have:

$$d(\mu,\tau) = i_{\tau^M}\omega$$

And unraveling the identifications in μ_0 we have that we actually have $\varphi^*\mu_0$, where φ is the diffeomorphism above.

$$d(\varphi^*\mu_0,\tau) = \varphi^* i_{\tau^{\mathbb{C}^n}}\omega_{st} = i_{\varphi^*\tau^{\mathbb{C}^n}}\varphi^*\omega_{st} = i_{\varphi^*\tau^{\mathbb{C}^n}}\omega$$

By the non-degeneracy of ω we have that

$$d(\mu-\mu_0) = 0 \iff i_{\tau^M}\omega - i_{\varphi^*\tau^{\mathbb{C}^n}}\omega \iff \tau^M = \varphi^*\tau^{\mathbb{C}^n}$$

We can show that the last equality holds by direct computation:

$$\begin{aligned}
\varphi^*\tau^{\mathbb{C}^n}|_x &= d\varphi^{-1}\left(\frac{d}{ds}\varphi exp(s\tau)\cdot x\Big|_{s=0}\right) \\
&= d\varphi^{-1}\left(d\varphi\left(\frac{d}{ds}exp(s\tau)\cdot x\Big|_{s=0}\right)\right) \\
&= \frac{d}{ds}exp(s\tau)\cdot x\Big|_{s=0} = \tau^M|_x
\end{aligned}$$

Thus $d(\mu-\mu_0)=0$, which implies that $(\mu,\tau)-(\mu_0,\tau)$ is locally constant for any τ. Since are working on a connected neighbourhood, we deduce that $(\mu,\tau) = (\mu(x_0),\tau) + (\mu_0,\tau)$.

Finally, since τ is arbitrary we conclude $\mu = \mu(x_0) + \mu_0$ which is what we wanted. (Notice that with the identification of μ, we have that $\mu(0) = \mu(x_0)$).

9

By definition $Crit(\mu_X) = \{x \in M : d\mu_x = 0\}$ Then by the definition of momentum map we get:

$$\begin{aligned}
d(\mu_X)_x = 0 & \\
&\iff d(\mu_X)_x(Y) = 0, \quad \forall Y \in C^\infty(M, TM) \\
&\iff \omega(X_x^M, Y) = 0, \quad \forall Y \in C^\infty(M, TM) \\
&\iff X_x^M = 0
\end{aligned}$$

The last equality follows from the non-degeneracy of ω.
We notice that $X_x^M = 0$ if and only if x is a fixed point of the action, that is:

$$Crit(\mu_x) = M^{\mathbb{T}}$$

And so $Crit(\mu_x)$ is a symplectic submanifold because we have already proven it in (6),

10

It follows from the previous section that, since M is compact, $Crit(\mu_X)$ has finitely many connected components, so they are open, and therefore submanifolds themselves.
The Hessian can be calculated as follows: $H_x(\mu_X)(Y, Z) = (ZY\mu_X)_x$
Then:

$$\begin{aligned}
Y\mu_X &= d\mu_X(Y) = \omega(X^M, Y) \\
ZY\mu_X &= d(\omega(X^M, Y)(Z)) = (i_{[Y,X^M]}\omega)(Z) = \omega([Y, X^M], Z)
\end{aligned}$$

It follows that: $H_x(\mu_X)(Y, Z) = \omega_x([Y, X^M], Z)$.
Since $[Y, X^M] = L_{X^M}Y = \frac{d}{dt}\big|_{t=0} exp(-tX)_*Y$

11

Exercise 3

1

By non-degeneracy of ω it is enough to show that for a given $X \in TM$ we have:

$$\omega\left(\mu^M, X\right) = \omega\left(-\frac{1}{2}J(d\mathcal{H})^*)), X\right) \tag{12}$$

On the LHS we have:

$$\omega\left(\mu^M, X\right) = i_{\mu^M}\omega(X) = d\left(\langle \mu, \mu(x)\rangle_{\mathfrak{g}}\right)(X) \tag{13}$$

On the RHS we have:

$$\begin{aligned}
\omega\left(-\frac{1}{2}J(d\mathcal{H})^*)), X\right) &= \frac{1}{2}g^{TM}\left(X, (d\mathcal{H})^*\right) \\
&= \frac{1}{2}(d\mathcal{H})(X) \\
&= d(\langle \mu, \mu(x)\rangle_{\mathfrak{g}})(X)
\end{aligned} \tag{14}$$

From (12) we get that $\mu_x^M = 0 \iff d\mathcal{H}_x = 0$ and so we get:

$$Crit(\mathcal{H}) = \{x \in M : \mu_x^M = 0\}$$

As we wanted.

2

First we clarify the way μ_T is defined, which is by the following diagram:

$$\mu_T: \; M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{h} \mathfrak{g} \xrightarrow{i^*} \mathfrak{t}^* \xrightarrow{h^{-1}} \mathfrak{t}$$

Where i is the natural inclusion: $\mathfrak{t} \xrightarrow{i} \mathfrak{g}$ and h is the natural pairing $\mathfrak{g} \xrightarrow{h} \mathfrak{g}^*$ given by $h(u)(v) = \langle u, v\rangle_{\mathfrak{g}}$.

Equivariance of μ_T is trivial since if $\mu(g \cdot x) = Ad_g^* \mu(x)$ is true for any $g \in G$ then it certainly true for any $g \in T \subset G$.

By the same logic, if $X \in T$ then X^M coincides if seen as an element of T or as an element of G because it is defined in the same exact manner, and because $i^*X = X$ we have that $(\mu, X) = (\mu_T, X)$ so we have that:

$$i_{X^M}\omega = d(\mu_T, X)$$

Which proves μ_T is a moment map.

Let $\mathcal{B} = \{t_1, \dots, t_n, x_1, \dots, x_m\}$ be an orthonormal basis of $\mathfrak{g}$ and $\mathcal{B}^* = \{t_1^*, \dots, t_n^*, x_1^*, \dots, x_m^*\}$ it's dual basis of $\mathfrak{g}^*$. Notice that because of the orthonormality condition the dual basis coincides in the usual sense and in the scalar product sense.

Then given $x \in M$, we have:

$$\mu(x) = \sum_{i=1}^{n} u_i t_i + \sum_{j=1}^{m} v_j x_j$$

Then because $i^* x_j = x_j|_{\mathfrak{t}} = 0$ we have:

$$\mu_T(x) = \sum_{i=1}^{n} u_i t_i = P^{\mathfrak{t}} \left(\sum_{i=1}^{n} u_i t_i + \sum_{j=1}^{m} v_i x_i \right) = P^{\mathfrak{t}} \mu(x)$$

3

This is a direct consequence of Theorem 1. Since Theorem 1 guarantees that $\mu_T(M^T)$ are the vertices of a convex polytope, which by definition of polytope means it must be finite.

4

This is a direct consequence of a classic theorem in Functional Analysis, however here we give a slightly simplified proof for the case of finite dimension and the case where the point exterior to the set is the origin.

Firstofall, if $0 \in A$ we are done. If not, let:

$$\delta = dist(0, A) = \inf_{x \in A} |x| > 0$$

Where the strict inequality comes from the fact that A is closed.

We have to show 2 thing:

1. There exists $x \in A$ so that $|x| = \delta$.

2. There is only one such $x \in A$.

Proof. (Existence)

Because δ is defined as an infimum, we can find a sequence of points x_n so that $|x_n| \to \delta$. By picking a subsequence, we may assume x_n that

$$x \leq |x_n| \leq x + \frac{1}{n+1}$$

Let $n, m \in \mathbb{N}$, then by hypothesis $d \leq \frac{1}{2}(y_n + y_m) \in A$.

By direct computation we have:

$$\delta^2 \le \left|\frac{1}{2}y_n + \frac{1}{2}y_m\right|^2 \overset{(1)}{=} \frac{1}{2}\left(|y_n|^2 + |y_m|^2\right) - \frac{1}{4}\left|y_n + y_m\right|^2$$
$$\le \delta^2 + \frac{1}{2}\left(\left(\frac{1}{n+1}\right)^2 + 2d\left(\frac{1}{n+1} + \frac{1}{m+1}\right) + \left(\frac{1}{m+1}\right)^2\right) - \frac{1}{4}|x_n - x_m|^2 \tag{15}$$

From which we get:

$$|x_n - x_m|^2 \le 2\left(\left(\tfrac{1}{n+1}\right)^2 + 2d\left(\tfrac{1}{n+1} + \tfrac{1}{m+1}\right) + \left(\tfrac{1}{m+1}\right)^2\right)$$
$$|x_n - x_m|^2 \overset{n,m\to\infty}{\longrightarrow} 0$$

Which means $\{x_n\}$ is a Cauchy sequence and therefore convergent, because A is closed we get that the limit is in A and obviously the limit has norm δ by the continuity of the norm.

QED

Proof. (Uniqueness) Suppose $x, x' \in A$, $x \neq x'$, so that $|x| = |x'| = \delta$

Let $y = \frac{1}{2}x + (1 - \frac{1}{2}x')$. By hypothesis $y \in A$ and so $|y| \ge \delta$

And so by use of the parallelogram identity:

$$\delta^2 \le \frac{1}{2} \le \left|\frac{1}{2}x + (1 - \frac{1}{2})x'\right|^2 = 2\left(\left|\frac{1}{2}x\right|^2 + \left|\frac{1}{2}x'\right|^2\right) - \frac{1}{2}\left|x - x'\right|^2 = \delta^2 - \frac{1}{2}\left|x - x'\right|^2 < \delta^2$$

Where the last inequality comes because $|x - x'|^2 > 0$.

QED

Exercise 4

1

Let $\alpha, \beta \in \Omega^{0,k}(M)$ and $\langle\cdot,\cdot\rangle$ be the scalar product we have seen in class for $(0,k)$ forms.

We have to show that: $\langle D_T\alpha, \beta\rangle = \langle\alpha, D_T\beta\rangle$ We compute:

$$\langle D_T\alpha, \beta\rangle = \langle D + iTc(\mu^M)\alpha, \beta\rangle = \langle D\alpha, \beta\rangle + \langle iTc(\mu^M)\alpha, \beta\rangle$$

By the lectures we know that D is self-adjoint so $\langle D\alpha, \beta\rangle = \langle\alpha, D\beta\rangle$. It is therefore enough to prove that $iTc(\mu^M$ is self-adjoint. Because $c(\mu^M)^* = -c(\mu^M)$, and because $\langle\cdot,\cdot\rangle$ is hermitian:

$$\langle iTc(\mu^M)\alpha, \beta\rangle = \langle\alpha, -iT(-c(\mu^M))\beta\rangle = \langle\alpha, iTc(\mu^M)\beta\rangle$$

2

Obviously $\ker D_T \subset \ker D_T^2$. Let $x \in \ker D_T^2$, then, because D_T^2 is self-adjoint:

$$0 = \langle D_T^2 x, x\rangle = \langle D_T x, D_T x\rangle$$

So we conclude $D_T x = 0$ and so $\ker D_T^2 \subset \ker D_T$ and $\ker D_T = \ker D_T^2$.

The last part is clear since $\ker D_{+,T} = \ker D_T|_{\Omega^{0,\text{even}}(M,L)} = \ker(D_T) \cap \Omega^{0,\text{even}}(M,L) \subset \Omega^{0,\text{even}}$ and in a completely analogous manner $\ker D_{-,T} = \ker D_T|_{\Omega^{0,\text{odd}}(M,L)} = \ker(D_T) \cap \Omega^{0,\text{odd}}(M,L) \subset \Omega^{0,\text{odd}}$

3

The first equality is trivial from the previous part since $\ker D_T^2 = \ker D_T$. The last equality would follow (as we will show) if D_T were G-equivariant. On the one hand D is G-equivariant as has been shown in the lectures. μ^M is also G-equivariant as we have shown in the previous exercise. So D_T must be G-equivariant as well, and a fortiori $D_{\pm,T}$ must be invariant by G. And so:

$$\begin{aligned}\ker D_T|_{\Omega^{0,\text{even}}(M,L)^G} &= \ker D_T \cap \Omega^{0,\text{even}}(M,L)^G = \ker D_T^2 \cap \Omega^{0,\text{even}}(M,L)^G \\ &\overset{(*)}{=} (\ker D_{+,T} \cap \oplus \ker D_{-,T}) \cap \Omega^{0,\text{even}}(M,L)^G \\ &= \ker D_{+,T} \cap \Omega^{0,\text{even}}(M,L)^G = \ker D_{+,T}^G\end{aligned}$$

Where $(*)$ follows because D_T decomposes as $D_T = D_{+,T} + D_{-,T}$ And the case for $D_{-,T}$ is completely analogous (except replacing $\Omega^{0,\text{even}}(M,L)^G$ with $\Omega^{0,\text{odd}}(M,L)^G$).

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