Exercise 1

1

Let $x \in M$, then we can find a neighbourhood U of x and a chart $x : U \to \mathbb{R}^n$, and $\{\frac{\partial}{\partial x_i}\}$ be the induced basis on TU.

Then locally, we can write:

$$X = \sum x_i \frac{\partial}{\partial x_i}; \quad Y = \sum y_j \frac{\partial}{\partial x_j}$$

Then a simple calculation yields:

$$XYf = \sum x_i \frac{\partial y_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \sum x_i y_j \frac{\partial^2 f}{\partial x_j \partial x_i}; \quad YXf = \sum y_j \frac{\partial x_i}{\partial x_j}\frac{\partial f}{\partial x_i} + \sum x_i y_j \frac{\partial^2 f}{\partial x_j \partial x_i}$$

$$[X,Y] = \sum \left(x_i \frac{\partial y_j}{\partial x_i} - y_j \frac{\partial x_i}{\partial x_i} \right) \frac{\partial f}{\partial x_j}$$

Clearly, $B_x(X,Y)$ is symmetric if and only if $[X,Y]_x = 0$. Since by definition of B,

$$B \cap U = \left\{ x \in U \,\middle|\, \frac{\partial f}{\partial x_j}(x) = 0 \right\}$$

We get that if $x \in B$ then:

$$(XYf)_x = \sum x_i(x) y_j(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x); \quad (YXf)_x = \sum x_i(x) y_j(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x); \quad [X,Y]_x = 0 \tag{1}$$

Hence the symmetry of $B_x(X,Y)$ is proven and from (1) is clear that is only depends on $x_i(x)$ and $y_i(x)$, that is: $B_x(X,Y)$ only depends on X_x and Y_x.

Finally, if B is a submanifold, then $X \in TB$, X can be integrated to obtain a curve $x(t)$ contained in B so that $X_{x(t)} = x'(t)$.

Since $x(t) \in B$ we have $\frac{\partial f}{\partial x_i}(x(t)) = 0$, we can differentiate to obtain:

$$0 = \sum_j x_j'(t) \frac{\partial^2 f}{\partial x_i x_j} \quad , \forall i \in \{1, \dots, n\}$$

Since $x_i'(0) = x_i(x)$, by the above calculations we have that:

$$(XYf)_x = \sum_{i,j=1}^n x_i(x) y_j(x) \frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \sum_{j=1}^n \left\{ y_j(x) \sum_{i=1}^n x_i(x) \frac{\partial^2 f}{\partial x_j \partial x_i} \right\} = 0$$

This proves the case $X_x \in T_x B$. The case $Y_x \in T_x B$ is completely analogous, since for $x \in B, (XYf)_x = (YXf)_x$

2

Let $H_x(f)(u,v) \overset{\text{def}}{=} (XYf)_x$, where we choose X, Y so that $u = X_x$ and $v = Y_x$.

We have to check that the choice of X, Y does not alter the value of H, however this was proven in the previous section where we checked that $(XYf)_x$ only depended on X_x and Y_x.

Finally, since:

$$KerH_x(f) = \{X \in TM | H_x(f)(X,Y) = 0, \forall Y\} = \{X \in TM | (YXf)_x = 0 \forall Y\} = TB \tag{2}$$

we deduce that the hessian of f is non-degenerate on $N_{B/M,x}$.

Finally, we can choose U and a local frame $\mathcal{B} = \frac{\partial}{\partial x_j}$ on U so that $H_x(f)$ has an associated symmetric matrix M so that

$$H_x(f)\left(\sum x_i \frac{\partial}{\partial x_i}, \sum y_j \frac{\partial}{\partial x_j}\right) = \sum_{i,j} x_i y_j M_{ij}$$

Then if we change the frame $\frac{\partial}{\partial x_j}$ to another frame $\frac{\partial}{\partial y_j}$ we have a matrix P a linear transformation on TM so that $P\frac{\partial}{\partial y_j} = \frac{\partial}{\partial x_j}$.

Then the matrix $M^{'}$ associated to the Hessian by the new matrix is given by:

$$M^{'} = P^T M P$$

By Sylverster's inertia theorem, we can find an invertible matrix S, so that $S^T M S$ is diagonal.

By the above discussion is enough to choose $\{S^{-1}\frac{\partial}{\partial x_i}\}$, as in this frame the Hessian has the form of (0.5).

It also clear, that since each connected component C_j of B is a submanifold where $df|_{C_j} = 0$ then $f|_{C_j} = const.$

3

a)

By the usual Poincaré lemma, we can find a form γ so that $\gamma|_W = 0$ and $d\gamma = \alpha$.

Since the action is trivial on W we can find a $G-invariant$ neighbourhood of W that we shall call U. Then we apply the standard trick of averaging γ over U. Let β we defined as

$$\beta = \int_U g^* \gamma d\mu$$

Where we chose μ to be a Haar measure, as it is well-know that compact Lie groups admits such a measure. And we could normalize in the usual way by requiring $\beta = \int_U d\mu = 1$.

Then since the action is trivial on W, $\beta|_W = 0$ and since d commutes with the integral sign and with the pullback by the naturality condition we have that:

$$\begin{aligned} d\beta &= d\int_U g^*\gamma d\mu = \int_U g^*(d\gamma) d\mu \\ &= \int_U g^*\alpha d\mu = \int_U \alpha d\mu \overset{(*)}{=} \alpha \int_U d\mu \\ &= \alpha \end{aligned}$$

Where $(*)$ follows because α is invariant by hypothesis.

Furthermore, β is invariant by construction.

b)

We follow the proof of the standard Darboux's lemma, with a slight twist:

Let $\omega_t = (1-t)\omega_0 + t\omega_1$. Then clearly, $\omega_t|_W = \omega_0|_W$. Then let X_t be an arbitrary differential vector field, and let ϕ_t be it's flow. Then:

$$\begin{aligned} \frac{d}{dt}\phi_t^*\omega_t = \phi^*(L_{X_t}\omega_t + \omega_1 - \omega_0) = 0 \\ \iff L_{X_t}\omega_t + \omega_1 - \omega_0 = 0 \end{aligned} \tag{3}$$

Then, because the action is trivial on W we can find a neighbourhood U of W so that the action of G is well defined. That is $g \cdot U \subset U, \forall g \in G$.

By part a) we can, shrinking U if necessary find a G-invariant form β so that $d\beta = \omega_1 - \omega_0$ and $\beta_W = 0$, since $\omega_1 - \omega_0$ is in the hypothesis of a).

Then, using Cartan's formula, since ω_t is closed we get: $L_{X_t}\omega_t = di_{X_t}\omega$.

So (3) turns into:

$$i_{X_t}\omega_t + \beta = 0 \tag{4}$$

Because ω_t is not degenerate, because it is the convex combination of two symplectic forms, we can solve (4) to get a vector field X_t that solves (3), since $\frac{d}{dt}\phi_t^*\omega_t = 0 \implies \phi_1^*\omega_1 = \omega_0$

If we could show that ϕ_t is G-invariant we would be done. We are going to show that if we define $\hat{\phi}_t = g \cdot \phi_t(g^{-1} \cdot x)$. By the uniqueness guaranteed by the Picard-Lindelöf theorem we have to show that:

$$\begin{cases} \frac{\partial}{\partial t}\hat{\phi}_t = X_t\hat{\phi}_t \\ \hat{\phi}_0(x) = x \end{cases}$$

For the first part:

$$\begin{aligned} \frac{\partial}{\partial t}\hat{\phi}_t = dgX_t\phi_t(g^{-1} \cdot x) &= dgX_tg^{-1} \cdot g\phi_t(g^{-1} \cdot x) \\ &= dgX_tg^{-1}\hat{\phi}_t(x) = X_t\hat{\phi}_t(x) \end{aligned} \tag{5}$$

Where last equality comes from the G-invariance of X_t which in turn comes from the G-invariance of ω_t.

For the second part: $\hat{\phi}_0 = g \cdot \phi_0(g^{-1} \cdot x) = g \cdot g^{-1} \cdot x = x$

And so the invariant Darboux theorem follows.

Exercise 2

1

We have that ω is invariant by hypothesis, furthermore, since both M and $\mathbb{T}$ are compact we can find $\langle \cdot, \cdot \rangle$ a bi-invariant scalar product by the standard averaging procedure.

Then there is a unique matrix A so that:

$$\omega(X,Y) = \langle AX, Y \rangle, \quad \forall X, Y \in TM \tag{6}$$

Since both ω and $< \cdot, \cdot >$ are invariant, we have that by uniqueness, so is A, that is because $g^{-1}Ag$ also verifies (6) as we show. Let $g \in G$, then:

$$\langle AX, Y \rangle \overset{(6)}{=} \omega(X,Y) \overset{*}{=} \omega(dg(X), dg(JY)) = \langle Adg(X), dg(Y) \rangle \overset{*}{=} \langle dg^{-1}Adg(X), Y \rangle$$

Where $*$ follows by the invariance of ω and $\langle \cdot, \cdot \rangle$

By the skew-symmetry of ω we have that $A = -A^T$, therefore $-A^2 = AA^T$ is symmetric and we can take it's square root. (By diagonalising $-A^2 = Pdiag(\lambda_1, \ldots, \lambda_n)P^{-1}$, then $\sqrt{-A^2} = Pdiag(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})P^{-1}$)

Then $J = \left(\sqrt{-A^2}\right)^{-1} A$ is invariant, because A is and is the composition of G-invariant homomorphism.

Clearly, $J^2 = -A^{-2}A^2 = -Id$ and J is compatible with ω because:

$$\begin{aligned} \omega(X, JX) &= \langle AX, JX \rangle \\ &= \left\langle AX, \left(\sqrt{-A^2}\right)^{-1} AX \right\rangle = \left\langle X, \left(\sqrt{-A^2}\right)^{-1} X \right\rangle > 0 \end{aligned}$$

Where the last inequality follows because $\sqrt{-A^2}^{-1}$ is definite positive.

J is therefore the almost complex structure we were looking for.

2

Since g^{TM} is G-invariant, G acts by isometry, and takes geodesics to geodesics. Let $\gamma(t, x_0, v_0)$ be the unique geodesic starting in x_0 with derivative in 0 v_0. Then $exp_{x_0}(v_0) = \gamma(1, x_0, v_0)$

Therefore, if $x \in M^{\mathbb{T}}$, let's consider $\alpha(t) = g \cdot \gamma(t, x_0, v_0)$. By the above discussion, $\alpha(t)$ is also a geodesic.

Since $\alpha(0)) = g \cdot x_0 = x_0$ and $\alpha'(0) = dg_{x_0}(\gamma'(0, x_0, v_0)) = dg_{x_0}(v_0)$

By uniqueness: $\alpha(t) = \gamma(t, x_0, dg_{x_0}(v_0))$. In particular, for t=1 we get:

$$exp_{x_0}(dg_{x_0}(v_0)) = g \cdot exp_{x_0}(v_0) \tag{7}$$

3

Let $x_0 \in M^{\mathbb{T}}$, then we can restrict the U in the previous part to a $\mathbb{T}$-invariant neighbourhood. In this way the $\mathbb{T}$-action on U is well-defined. Furthermore, since in the previous part we used an arbitrary $g \in \mathbb{T}$ we have that the exponential map is equivariant.

We notice that if $exp_{x_0}(v) \in M^{\mathbb{T}}$ then:

$$exp_{x_0}(v) = g \cdot exp_{x_0}(v) = exp_{x_0}(dg_{x_0}v), \quad \forall g \in \mathbb{T} \tag{8}$$

This mens we can identify fixed points of the action in U with fixed point of the differential of the action on $T_{x_0}M$, since for $v \in T_{x_0}M, g \in \mathbb{T}$, $v = dg_{x_0}v$ because of (8) and exp being an isomorphism.

That is, the exponential map send the eigenspace of $T_{x_0}M$ of eigenvalue 1 to $M^{\mathbb{T}} \cap U$. Because the exponential map is a diffeomorphism it sends submanifolds to submanifolds, in particular, it sends linear subspaces to submanifolds.

In summary: Every point in $M^{\mathbb{T}}$ admits a neighbourhood that is parametrised by the exponential map.

And so $M^{\mathbb{T}}$ is a submanifold.

4

Because x_0 is a fixed point, the action of $\mathbb{T}$ on M lift to $T_{x_0}M$ via the differential, in particular, given $g, h \in \mathbb{T}$:

$$d(gh)_{x_0} = dg_{x_0} dh_{x_0}$$

By the chain rule. (We reiterate x_0 is a fixed point). Furthermore, since the action acts by symplectomorphism (by defintion), we have that because x_0 is a fixed point if we choose $g \in \mathbb{T}$:

$$\omega_{x_0}(X, Y) = \omega_{g \cdot x_0}(dg_{x_0}X, dg_{x_0}Y) = \omega_{x_0}(dg_{x_0}X, dg_{x_0}Y)$$

This proves that the action on $T_{x_0}M$ is symplectic. It only remains to prove that the action commutes with J, however this is the definition for invariance of J which we already prove.

5

At this point we suspect that φ might be given by the exponential map with the natural identification of $B(x_0, \delta)$ with $\mathbb{R}^{2n}$, that is, under indentification: $\varphi = exp_{x_0}^{-1}$. This is obviously a diffeomorphism and it is $\mathbb{T}$-equivariant because we have defined precisely the action as the induced by this diffeomorphism.

Furthermore, by the section on compact Lie groups we know that there is a natural identification of $\mathbb{T}^k$ with $(\mathbb{S}^1)^k$ given by:

$$exp(\tau) = (e^{i\tau_1}, \ldots, e^{i\tau_k})$$

And so the action induced on V must be of the above form, since it's linear and unitary.

6

We define $A_g \stackrel{\text{def}}{=} \ker(\mathbb{1} - dg_{x_0}g)$ to ease notation.

Let $v \in A_g$ we have that because of the previous parts of the exercise:

$$v = dg_{x_0}(J_{x_0}v) = J_{x_0}dg_{x_0}(v) = J_{x_0}v \tag{9}$$

And so $J_{x_0}A_g = A_g$, because by (9) we have proved the $\subset$ inclusion and the other inclusion comes from the fact that J is an isomorphism since $-J$ is it's inverse.

From which we conclude that:

$$T_{x_0}(M^{\mathbb{T}}) = (T_{x_0}M)^{\mathbb{T}} = \bigcap_{g\in\mathbb{T}} A_g$$

If we showed that each of the A_g was symplectic, then theirs intersection would be as well and the result would follow. That is the proof follows from this two claims:

Claim: A_g is symplectic: **Claim**: The intersection of symplectic spaces is symplectic.

Proof. (First claim)

By way of contradiction. Let g be an arbitrary element in $\mathbb{T}$ and $v \in A_g$ so that $i_v\omega_{x_0} = 0$.

Because of $J_{x_0}A_g = A_g$ we know we can find w so that $J_{x_0}w = v$. Then:

$$0 = \omega_{x_0}(J_{x_0}w, w) = -g^{TM}_{x_0}(w, w) > 0$$

Which is a contradiction. *QED*

Proof. (Second claim) A_g is symplectic if and only if $A_g \cap A_g^{\perp\omega} = \{0\}$.

From this we get:

$$\begin{aligned}
\left(\bigcap_{g\in\mathbb{T}} A_g\right) \cap \left(\bigcap_{g\in\mathbb{T}} A_g\right)^{\perp\omega} &= \left(\bigcap_{g\in\mathbb{T}} A_g\right) \cap \left(\bigcup_{g\in\mathbb{T}} A_g^{\perp\omega}\right) \\
&= \bigcap_{h\in\mathbb{T}} \left(\left(\bigcap_{g\in\mathbb{T}} A_g\right) \bigcap \left(\bigcup_{h\in\mathbb{T}} A_g^{\perp\omega}\right)\right) \\
&= \{0\}
\end{aligned}$$

And we are done.

QED

In summary we have proven that $T_{x_0}(M^{\mathbb{T}})$ is symplectic.

Since we have already shown that the exponential map is a symplectomorphism and that it maps $T_{x_0}(M^{\mathbb{T}})$ to $(T_{x_0}M)^{\mathbb{T}}$ we conclude $M^{\mathbb{T}}$ is a symplectic submanifold.

7

We have to prove two things:

1. $d(\mu, \tau) = i_\tau^{\mathbb{C}^n}\omega_{st}$.

   Proof. We begin by seeing who is $\tau^{\mathbb{C}^n}$. Let $z = x + iy \in \mathbb{C}$ then:

$$\begin{aligned}
\tau^{\mathbb{C}^n}|_z = \frac{d}{ds}exp(s\tau)\cdot z &= (-i\langle w_1, \tau\rangle z_1, \ldots, -i\langle w_n, \tau\rangle z_n) \\
&= (\langle w_1, \tau\rangle y_1, \langle w_1, \tau\rangle x_1, \ldots, \langle w_1, \tau\rangle y_n, \langle w_1, \tau\rangle x_n,) \\
&= \sum_{j=1}^{n} \langle w_j, \tau\rangle \left(y_j\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial y_j}\right)
\end{aligned}$$

Again, by computation we get:

$$i_{y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}}(dx_j \wedge dy_j) = x_j dx_j + y_j dy_j$$

And gluing all together:

$$i_\tau^{\mathbb{C}^n} \omega_{st} = \sum_{j=1}^n \langle w_j, \tau \rangle (x_j dx_j + y_j dy_j) \tag{10}$$

Now be go the LHS, let's see who is $d(\mu, \tau)$. Let $z \in \mathbb{C}$:

$$d(\mu_0(z), \tau) = d\left(\frac{1}{2}\sum_{j=1}^n (x_j^2 + y_j^2)(w_j, \tau)\right) = \sum_{j=1}^n (x_j dx_j + y_j dy_j)(w_j, \tau) \tag{11}$$

Joining (10) with (11) we get the result.

QED

2. ω_{st} is equivariant, i.e $\mu_0(g \cdot z) = Ad_g^* \mu_0(z)$.

   Proof. Firstly, we notice that since $\mathbb{T}$ is abelian, so is it action, in the sense that Ad_g acts by identity. So the RHS is just $\mu(z)$.

   For the LHS, we notice that since $\mathbb{T}$ is compact, it is generated as a group by $\{exp(\tau), \tau \in \mathfrak{g}\}$, so if we show that $\mu_0(\exp(\tau) \cdot z) = \mu_0(z)$ this would that the LHS is also equal to $\mu_0(z)$.

   We show the final claim:

$$\mu_0(\exp(\tau) \cdot z) = \frac{1}{2}\sum_{j=1}^n |e^{-i\langle w_j, \tau \rangle} z_j|^2 w_j = \frac{1}{2}\sum_{j=1}^n |z_j|^2 w_j = \mu_0(z)$$

QED

8

By definition of moment map we have:

$$d(\mu, \tau) = i_{\tau^M} \omega$$

And unraveling the identifications in μ_0 we have that we actually have $\varphi^* \mu_0$, where φ is the diffeomorphism above.

$$d(\varphi^* \mu_0, \tau) = \varphi^* i_{\tau^{\mathbb{C}^n}} \omega_{st} = i_{\varphi^* \tau^{\mathbb{C}^n}} \varphi^* \omega_{st} = i_{\varphi^* \tau^{\mathbb{C}^n}} \omega$$

By the non-degeneracy of ω we have that

$$d(\mu - \mu_0) = 0 \iff i_{\tau^M} \omega - i_{\varphi^* \tau^{\mathbb{C}^n}} \omega \iff \tau^M = \varphi^* \tau^{\mathbb{C}^n}$$

We can show that the last equality holds by direct computation:

$$\begin{aligned}
\varphi^* \tau^{\mathbb{C}^n}|_x &= d\varphi^{-1}\left(\frac{d}{ds} \varphi exp(s\tau) \cdot x \Big|_{s=0}\right) \\
&= d\varphi^{-1}\left(d\varphi\left(\frac{d}{ds} exp(s\tau) \cdot x \Big|_{s=0}\right)\right) \\
&= \frac{d}{ds} exp(s\tau) \cdot x \Big|_{s=0} = \tau^M|_x
\end{aligned}$$

Thus $d(\mu - \mu_0) = 0$, which implies that $(\mu, \tau) - (\mu_0, \tau)$ is locally constant for any τ. Since are working on a connected neighbourhood, we deduce that $(\mu, \tau) = (\mu(x_0), \tau) + (\mu_0, \tau)$.

Finally, since τ is arbitrary we conclude $\mu = \mu(x_0) + \mu_0$ which is what we wanted. (Notice that with the identification of μ, we have that $\mu(0) = \mu(x_0)$).

9

By definition $Crit(\mu_X) = \{x \in M : d\mu_x = 0\}$ Then by the definition of momentum map we get:

$$\begin{aligned} d(\mu_X)_x &= 0 \\ &\iff d(\mu_X)_x(Y) = 0, \quad \forall Y \in C^\infty(M, TM) \\ &\iff \omega(X_x^M, Y) = 0, \quad \forall Y \in C^\infty(M, TM) \\ &\iff X_x^M = 0 \end{aligned} \tag{12}$$

The last equality follows from the non-degeneracy of ω.

We notice that $X_x^M = 0$ if and only if x is a fixed point of the action, that is:

$$Crit(\mu_x) = M^{\mathbb{T}}$$

And so $Crit(\mu_x)$ is a symplectic submanifold because we have already proven it in (6),

10

It follows from the previous section that $Crit(\mu_x) = M^{\mathbb{T}}$ and by (6) $Crit(\mu_x)$ is symplectic submanifold.

The Hessian can be calculated as follows: $H_x(\mu_X)(Y, Z) = (ZY\mu_X)_x$

Then:

$$\begin{aligned} &Y\mu_X = d\mu_X(Y) = \omega(X^M, Y) \\ &ZY\mu_X = d(\omega(X^M, Y)(Z)) = (i_{[Y,X^M]}\omega)(Z) = \omega([Y, X^M], Z) = \omega(L_Y(X^M), Z) \end{aligned}$$

It follows that: $H_x(\mu_X)(Y, Z) = \omega_x([Y, X^M], Z)$.

Since $[Y, X^M] = L_Y X^M = \frac{d}{dt}\big|_{t=0} exp(-tY)_* X^M$. Which means by (12) that:

$$TCrit(\mu_X) = \{Y \in TM : L_Y X^M = 0\} = \bigcap_{Y \in TM} \ker H_x(\mu_X)(Y)$$

This proves the Hessian is non-degenerate on the set of critical points. Bringing this together with the fact that $Crit(\mu_X)$ is a submanifold we conclude that μ_X is a Morse-Bott function.

11

By (0.10) in the exam, we have that he hessian of μ as identified in $\mathbb{C}^n$ and then identified again in $\mathbb{R}^{2n}$ is $\frac{1}{2}\sum_{j=1}^n (x_j^2 + y_j^2)\omega_j(X)$.

Where by the $\mathbb{C}^n$ to $\mathbb{R}^{2n}$ identification we mean identifying $z = x + iy$ with (x, y).

This means the matrix of the Hessian is as follows:

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda_n & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix}$$

Where $\lambda_i \stackrel{\text{def}}{=} (w_i, X)$. That is every eigenvalue appears twice due to the identification. So there must a even number of them that are negative (maybe 0, but even nonetheless). Which proves the result.

Exercise 3

1

By non-degeneracy of ω it is enough to show that for a given $X \in TM$ we have:

$$\omega\left(\mu^M, X\right) = \omega\left(-\frac{1}{2}J(d\mathcal{H})^*)), X\right) \tag{13}$$

On the LHS we have:

$$\omega\left(\mu^M, X\right) = i_{\mu^M}\omega(X) = d\left(\langle \mu, \mu(x)\rangle_{\mathfrak{g}}\right)(X) \tag{14}$$

On the RHS we have:

$$\begin{aligned} \omega\left(-\frac{1}{2}J(d\mathcal{H})^*)), X\right) &= \frac{1}{2}g^{TM}\left(X, (d\mathcal{H})^*\right) \\ &= \frac{1}{2}(d\mathcal{H})(X) \\ &= d(\langle \mu, \mu(x)\rangle_{\mathfrak{g}})(X) \end{aligned} \tag{15}$$

From (13) we get that $\mu^M_x = 0 \iff d\mathcal{H}_x = 0$ and so we get:

$$Crit(\mathcal{H}) = \{x \in M : \mu^M_x = 0\}$$

As we wanted.

2

First we clarify the way μ_T is defined, which is by the following diagram:

$$\mu_T : \; M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{h} \mathfrak{g} \xrightarrow{i^*} \mathfrak{t}^* \xrightarrow{h^{-1}|_{\mathfrak{t}^*}} \mathfrak{t}$$

Where i is the natural inclusion: $\mathfrak{t} \xrightarrow{i} \mathfrak{g}$ and h is the natural pairing $\mathfrak{g} \xrightarrow{h} \mathfrak{g}^*$ given by $h(u)(v) = \langle u, v\rangle_{\mathfrak{g}}$.

Equivariance of μ_T is trivial since if $\mu(g \cdot x) = Ad^*_g\mu(x)$ is true for any $g \in G$ then it certainly true for any $g \in T \subset G$.

By the same logic, if $X \in T$ then X^M coincides if seen as an element of T or as an element of G because it is defined in the same exact manner, and because $i^*X = X$ we have that $(\mu, X) = (\mu_T, X)$ so we have that:

$$i_{X^M}\omega = d(\mu_T, X)$$

Which proves μ_T is a moment map.

Let $\mathcal{B} = \{t_1, \ldots, t_n, x_1, \ldots, x_m\}$ be an orthonormal basis of $\mathfrak{g}$ and $\mathcal{B}^* = \{t_1^*, \ldots, t_n^*, x_1^*, \ldots, x_m^*\}$ it's dual basis of $\mathfrak{g}^*$. Notice that because of the orthonormality condition the dual basis coincides in the usual sense and in the scalar product sense.

Then given $x \in M$, we have:

$$\mu(x) = \sum_{i=1}^n u_i t_i + \sum_{j=1}^m v_j x_j$$

Then because $i^*x_j = x_j|_{\mathfrak{t}} = 0$ we have:

$$\mu_T(x) = \sum_{i=1}^n u_i t_i = P^{\mathfrak{t}}\left(\sum_{i=1}^n u_i t_i + \sum_{j=1}^m v_i x_i\right) = P^{\mathfrak{t}}\mu(x)$$

3

This is a direct consequence of Theorem 1. Since Theorem 1 guarantees that $\mu_T(M^T)$ are the vertices of a convex polytope, which by definition of polytope means it must be finite.

4

This is a direct consequence of a classic theorem in Functional Analysis, however here we give a slightly simplified proof for the case of finite dimension and the case where the point exterior to the set is the origin.

Firstofall, if $0 \in A$ we are done. If not, let:

$$\delta = dist(0, A) = \inf_{x \in A} |x| > 0$$

Where the strict inequality comes from the fact that A is closed.

We have to show 2 thing:

1. There exists $x \in A$ so that $|x| = \delta$.
2. There is only one such $x \in A$.

Proof. (Existence)

Because δ is defined as an infimum, we can find a sequence of points x_n so that $|x_n| \to \delta$. By picking a subsequence, we may assume x_n that

$$x \leq |x_n| \leq x + \frac{1}{n+1}$$

Let $n, m \in \mathbb{N}$, then by hypothesis $d \leq \frac{1}{2}(y_n + y_m) \in A$.

By direct computation we have:

$$\begin{aligned} \delta^2 \leq \left| \frac{1}{2} y_n + \frac{1}{2} y_m \right|^2 &\overset{(1)}{=} \frac{1}{2} \left(|y_n|^2 + |y_m|^2 \right) - \frac{1}{4} |y_n + y_m|^2 \\ &\leq \delta^2 + \frac{1}{2} \left(\left(\frac{1}{n+1} \right)^2 + 2d \left(\frac{1}{n+1} + \frac{1}{m+1} \right) + \left(\frac{1}{m+1} \right)^2 \right) - \frac{1}{4} |x_n - x_m|^2 \end{aligned} \tag{16}$$

From which we get:

$$\begin{aligned} &|x_n - x_m|^2 \leq 2 \left(\left(\tfrac{1}{n+1} \right)^2 + 2d \left(\tfrac{1}{n+1} + \tfrac{1}{m+1} \right) + \left(\tfrac{1}{m+1} \right)^2 \right) \\ &|x_n - x_m|^2 \overset{n,m \to \infty}{\longrightarrow} 0 \end{aligned}$$

Which means $\{x_n\}$ is a Cauchy sequence and therefore convergent, because A is closed we get that the limit is in A and obviously the limit has norm δ by the continuity of the norm.

QED

Proof. (Uniqueness) Suppose $x, x' \in A$, $x \neq x'$, so that $|x| = |x'| = \delta$

Let $y = \frac{1}{2}x + (1 - \frac{1}{2}x')$. By hypothesis $y \in A$ and so $|y| \geq \delta$

And so by use of the parallelogram identity:

$$\delta^2 \leq \frac{1}{2} \leq \left| \frac{1}{2} x + (1 - \frac{1}{2}) x' \right|^2 = 2 \left(\left| \frac{1}{2} x \right|^2 + \left| \frac{1}{2} x' \right|^2 \right) - \frac{1}{2} |x - x'|^2 = \delta^2 - \frac{1}{2} |x - x'|^2 < \delta^2$$

Where the last inequality comes because $|x - x'|^2 > 0$.

QED

5

By definition each element b of $\mathcal{B}$ is associated to a subset of $\mathcal{P}(\mathcal{A})$ which is finite because $\mathcal{A}$ is finite. In particular, $\#\mathcal{P}(\mathcal{A}) = 2^{\#\mathcal{A}}$ so:

$$\#\mathcal{P}(\mathcal{A}) < \infty \iff \#\mathcal{A} < \infty$$

Since this association of $\mathcal{B}$ with subsets of $\mathcal{A}$ gives and injection we have:

$$\#\mathcal{B} \leq \mathcal{P}(\mathcal{A}) < \infty$$

Which is what we wanted.

6

We already know by the previous part of the exercise that $Crit(\mu_\beta) = \{x \in M | \beta_x^M = 0\}$.

The set $Z_\beta = \{x \in M | \beta_x^M = 0,\ \mu_\beta(x)|\beta|^2\}$ is clearly closed in $Crit(\mu_\beta)$ by continuity of $|\cdot|^2$. Furthermore, since μ_β must be locally constant along the connected components of $Crit(\mu_\beta)$ we conclude Z_β must be relatively open in $Crit(\mu_\beta)$.

This means Z_β is clopen in $Crit(\mu_\beta)$ and so is the union of open connected components of $Crit(\mu_\beta)$ (we know they are open because there are only a finite amount of them due to M being compact).

In particular, this means Z_β is the disjoint union of symplectic manifolds. Since being symplectic is a local condition (it is a condition on the differential, which is local and on the tangent space on a point, which obviously local) we conclude that Z_β must be symplectic.

7

Since for $\mu(x) \in \mathfrak{t}^*$ $\mu(x) = \mu_T(x)$ by (2)

From this we have:

$$x \in Crit(\mathcal{H}) \iff x \in Crit(|\mu|^2) \iff x \in Crit(|\mu_T|^2)$$

We now prove the following: **Claim**: Given $x \in M$ with $\mu_T(x) = \beta \in \mathfrak{t}$. Then we have that $x \in Crit(\mathcal{H})$ if and only if $x \in Z_\beta$. And when this happens β is the element of minimal norm in $\mu_t(Z_\beta)$ which is obviously a convex set.

Proof. We have the following:

$$x \in Crit(\mathcal{H}) \iff \mu^M\big|_x = 0 \iff \beta_X^M = 0$$

Where the last double implication comes from exercise 2 part 9 (and the definition).

And so the first implication ($\Rightarrow$) follows by the definition of Z_β.

For the $\Leftarrow$ implication we know by Theorem 1 that $\mu_T(Z_\beta) = \text{convex hull}(\mu_T(Z_\beta^T))$

By the previous part there exits a element of minimal norm in $\mu_T(Z_\beta)$. We only have to show that this point is β. Let's argue by way of contradiction, let γ be the point of minimal norm, in particular, $|\gamma| < |\beta|$ and we can find $x \in Z_\beta$ so that $\gamma = \mu(x) = \mu_T(x)$, where the last equality holds because $\mu_t(Z_\beta) \subset \mathfrak{t}$.

We have that by definition of Z_β (and by the identification of $\mathfrak{t}$ with $\mathfrak{t}^*$) that $\langle \mu(y), \beta \rangle = \langle \beta, \beta \rangle$

from which trivially we have:

$$\langle \mu(y) - \beta, \beta \rangle = 0$$
$$|\gamma|^2 = |\mu_T(y)|^2 = \langle \mu(y), \mu(y) \rangle \overset{\pm\beta}{=} \langle \mu(y) - \beta, \mu(y) - \beta \rangle + \langle \beta, \beta \rangle = |\mu(y) - \beta|^2 + |\beta|^2 < |\beta|^2$$

Which is a contradiction and so we have proven the claim above. *QED*

It follows immediately from the equivariance of μ that $g \cdot Crit(\mathcal{H}) = Crit(\mathcal{H})$.

Putting all together we have that:

$$g \cdot x \in Crit(\mathcal{H}) \iff g \cdot z \in Z_\beta \iff x \in C_\beta$$

Which shows: $Crit(\mathcal{H}) = \cup_{\beta \in \mathcal{B}} C_\beta$

It only remains to show that C_β are disjoint. Let $\beta \in \mathcal{B}$, then:

$$\mu(C_\beta) = \{\mu(g \cdot x) | x \in Z_\beta \cap \mu^{-1}(\beta), g \in G\}\{Ad_g(\mu(x)) | x \in Z_\beta, \mu(x) = \beta, g \in G\} = Ad_G(\beta) = \mathcal{O}_\beta$$

And by a theorem of the course we know that each coadjoint orbit intersects each positive Weyl chamber number in precisely one point, this directly implies that C_β are disjoint since otherwise we could use the coadjoint action to bring a point in the intersection to a point in a positive Weyl chamber, which will mean two different coadjoint orbits intersect, which is imposible.

8

We just showed: $\mu(C_\beta) = Ad_G(\beta)$

We can actually say more:

$$\begin{aligned} C_\beta &= \{g \cdot x | g \in G, x \in Z_\beta, \mu(x) = \beta\} = \{g \cdot x | g \in G, x \in Crit(\mathcal{H}), \mu(x) = \beta\} \\ &= \{g \cdot x | g \in G, g \cdot x \in Crit(\mathcal{H}), \mu(x) = \beta\} = \{x \in Crit(\mathcal{H}) | \exists g \in G, \;\; \mu(x) = Ad_g(\beta)\} \end{aligned}$$

As we have already seen $x \in C_\beta \implies x \in Z_\beta$ and $\mathcal{H}(x) = |\mu(x)|^2 = |\beta|^2$ for any x in C_β, where again we have omitted the dual identification.

So we have proven: $\mathcal{H}(C_\beta) = |\beta|^2$

Exercise 4

1

Let $\alpha, \beta \in \Omega^{0,k}(M)$ and $\langle \cdot, \cdot \rangle$ be the scalar product we have seen in class for $(0, k)$ forms.

We have to show that: $\langle D_T \alpha, \beta \rangle = \langle \alpha, D_T \beta \rangle$ We compute:

$$\langle D_T \alpha, \beta \rangle = \langle D + iTc(\mu^M)\alpha, \beta \rangle = \langle D\alpha, \beta \rangle + \langle iTc(\mu^M)\alpha, \beta \rangle$$

By the lectures we know that D is self-adjoint so $\langle D\alpha, \beta \rangle = \langle \alpha, D\beta \rangle$. It is therefore enough to prove that $iTc(\mu^M)$ is self-adjoint. Because $c(\mu^M)^* = -c(\mu^M)$, and because $\langle \cdot, \cdot \rangle$ is hermitian it follows that:

$$\langle iTc(\mu^M)\alpha, \beta \rangle = \langle \alpha, -iT(-c(\mu^M))\beta \rangle = \langle \alpha, iTc(\mu^M)\beta \rangle$$

2

Obviously $\ker D_T \subset \ker D_T^2$. Let $x \in \ker D_T^2$, then, because D_T^2 is self-adjoint:

$$0 = \langle D_T^2 x, x \rangle = \langle D_T x, D_T x \rangle$$

So we conclude $D_T x = 0$ and so $\ker D_T^2 \subset \ker D_T$ and $\ker D_T = \ker D_T^2$.

The last part is clear since $\ker D_{+,T} = \ker D_T|_{\Omega^{0,\text{even}}(M,L)} = \ker(D_T) \cap \Omega^{0,\text{even}}(M, L) \subset \Omega^{0,\text{even}}$ and in a completely analogous manner $\ker D_{-,T} = \ker D_T|_{\Omega^{0,\text{odd}}(M,L)} = \ker(D_T) \cap \Omega^{0,\text{odd}}(M, L) \subset \Omega^{0,\text{odd}}$

3

The first equality is trivial from the previous part since $\ker D_T^2 = \ker D_T$. The last equality would follow (as we will show) if D_T were G-equivariant. On the one hand D is G-equivariant as has been shown in the lectures. μ^M is also G-equivariant as we have shown in the previous exercise. So D_T must be G-equivariant as well, and a fortiori $D_{\pm,T}$ must be invariant by G. And so:

$$\begin{aligned}
\ker D_T|_{\Omega^{0,\text{even}}(M,L)^G} &= \ker D_T \cap \Omega^{0,\text{even}}(M,L)^G = \ker D_T^2 \cap \Omega^{0,\text{even}}(M,L)^G \\
&\overset{(*)}{=} (\ker D_{+,T} \cap \oplus \ker D_{-,T}) \cap \Omega^{0,\text{even}}(M,L)^G \\
&= \ker D_{+,T} \cap \Omega^{0,\text{even}}(M,L)^G = \ker D_{+,T}^G
\end{aligned}$$

Where $(*)$ follows because D_T decomposes as $D_T = D_{+,T} + D_{-,T}$ And the case for $D_{-,T}$ is completely analogous (except replacing $\Omega^{0,\text{even}}(M,L)^G$ with $\Omega^{0,\text{odd}}(M,L)^G$).

4

We have that:

$$D_T^2 = (D + iTc(\mu^m))\,(D + iTc(\mu^m)) = D^2 + T^2|\mu^M|^2 + iT(Dc(\mu^M) + c(\mu^M)D)$$

So we have to show that we can find some A_M 0-order differential operator so that:

$$iT(Dc(\mu^M) + c(\mu^M)D) = TA_M - 2iT\sum_{i=1}^{\dim G} \mu_i L_{V_i} \tag{17}$$

Let's see who is $Dc(\mu^M) + c(\mu^M)D$

$$\begin{aligned}
Dc(\mu^M) + c(\mu^M)D &= \sum_{j=1}^{2n} c(\mu^M)c(e_j)\nabla_{e_j}^{Cl\otimes L} + c(e_j)\nabla_{e_j}^{Cl\otimes L} c(\mu^M) \\
&\overset{(1)}{=} \sum_{j=1}^{2n} c(\mu^M)c(e_j)\nabla_{e_j}^{Cl\otimes L} + c(e_j)c(\mu^M)\nabla_{e_j}^{Cl\otimes L} + c(e_j)c(\nabla_{e_j}^{TM}\mu^M) \\
&= \sum_{j=1}^{2n} \left(c(\mu^M)c(e_j) + c(e_j)\right)\nabla_{e_j}^{Cl\otimes L} + c(e_j)c(\nabla_{e_j}^{TM}\mu^M) \\
&= -2g^{TM}(\mu^M, e_j)\nabla_{e_j}^{Cl\otimes L} + c(e_j)c(\nabla_{e_j}^{TM}\mu^M) \\
&= -2\nabla_{\mu^M}^{Cl\otimes L} + c(e_j)c(\nabla_{e_j}^{TM}\mu^M)
\end{aligned}$$

Where (1) follows from $[\nabla_V^{Cl\otimes L}, c(W)] = c(\nabla_V^{Cl\otimes L} W)$. (This identity holds for ∇_V^{Cl} so it's trivial to check it follows as well for $\nabla^{Cl\otimes L} = \nabla Cl \otimes \mathbb{1} + \mathbb{1}\otimes \nabla^L$).

Let $\alpha \in \Omega^{0,\cdot}(M,L)$. α is locally of the form $\tilde{\alpha}\otimes s$ for $\tilde{\alpha} \in C^\infty(M, \Lambda^{\cdot} T^{(0,1)}M)$ and $v \in C^\infty(M,L)$.

And so if $V \in TM$ we have:

$$L_V\alpha = (L_V\tilde{\alpha})\otimes s + \tilde{\alpha}\otimes(L_V s)$$

.

Where we are abusing the overloading the notation of L_V.

By Kostant's formula we have that:

$$\nabla_{K^M}^L = L_{K^M} + 2\pi i(\mu, K), \quad \forall K \in \mathfrak{g}$$

Then we can express μ^M in the given coordinates: $\mu^M = \sum_{i=1}^{dimG} \mu_i V_i$

and so:

$$\nabla^L_{\mu^M} s = \sum_{i=1}^{dimG} \mu_i \nabla^L_{V_i^M} s = \sum_{i=1}^{dimG} \mu_i \left(L_{V_i^M} s + 2\pi i(\mu, V_i) \right) s$$

Finally, notice that although $\nabla^L_{\mu^M}$ is not a linear operator because $\nabla^{Cl}_{\mu^M}(\phi\beta) = (L_{\mu^M}\phi)\beta + \phi\nabla^{Cl}_{\mu^M}\beta$. But this means it is an affine operator, so it's a summand of the Lie derivative plus some linear operator, $\tilde{A}_M$, so we define: $\tilde{\tilde{A}}_M \overset{\text{def}}{=} \tilde{A_M} \otimes \mathbb{1}$

Putting everything together we get:

$$\begin{aligned} D_T^2 &= D^2 + T^2|\mu^M|^2 + iT(Dc(\mu^M) + c(\mu^M)D) \\ &= D^2 + T^2 + T\left(ic(e_j)c(\nabla^{TM}_{e_j}\mu^M) - 2i\sum_{i=1}^{dimG}(\mu_i(2\pi i(\mu, V_i) + \tilde{\tilde{A}}_M) \right) - 2iT\left(\sum_{i=1}^{dimG} \mu_i L_{V_i} \right) \end{aligned}$$

And so defining $A \overset{\text{def}}{=} ic(e_j)c(\nabla^{TM}_{e_j}\mu^M) - 2i\sum_{i=1}^{dimG}(\mu_i(2\pi i(\mu, V_i) + \tilde{\tilde{A}}_M)$ we are done.

5

We know from (0.15) that:

$$\mu^{-1}(0) \subset Crit(\mathcal{H})) = \{x \in M | \mu^M|_x = 0\} = \emptyset$$

On one hand M is compact and $|\mu^M| > 0$ by hypothesis. This means we can choose T large enough so that $T^2|\mu^M|^2$ is larger than the rest of the terms in (0.24). (Notice some other terms also depend on T, but they depend lineally, so we can dominate them). This means, for this T sufficiently large we have $|D_T, D_T| > 0$ and so $\ker D_T = \{0\}$.

Since by definition $ind(D_+)^G = \dim(\ker D_+)^G - (\ker D_-)^G$

and both $(\ker D_+)^G$ and $(\ker D_-)^G$ are subspaces of $(\ker D_\pm)^G$ we get that their dimension is both 0.

So $ind(D_+)^G = 0$, which is what we wanted.