Part 1

First, if ρ is an exact solution, then it's mass is preserved, meaning $m(t) := \int \rho(t,x)dx = \int \rho_0(x)dx = \text{cst}$. Then conservation follows from the boundary conditions:

$$m'(t) = \int_{\mathbb{R}} \partial_t \rho(t,x)dx, = \int_{\mathbb{R}} \partial_{xx}\rho(t,x)dx, = \partial_x \rho(t,\infty) - \partial_x \rho(t,-\infty), 0$$

Now we center on the simpler problem:

$$\begin{array}{ll} \partial_t \rho - \varepsilon^2 \Delta \rho & = 0 \text{ on } (0,1)\times(0,1) \\ \rho(0,x) & = \rho_0(x),\ \forall x \in [0,1] \\ \partial_x \rho(0,t) & = \partial_x \rho(1,t) = 0,\ \forall t \in [0,1] \text{ Neumann boundary conditions.} \end{array}$$

To solve it we implement the Crank-Nicolson method, although we note that an explicit Euler, or implicit Euler in space would work as well. Also, order one boundary conditions work just fine.

$$\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} = \frac{\varepsilon^2}{2(\Delta x)^2}\left[\left(\rho_{i-1}^{n+1} - 2\rho_i^{n+1} + \rho_{i+1}^{n+1}\right) + \left(\rho_{i-1}^{n} - 2\rho_i^{n} + \rho_{i+1}^{n}\right)\right] \tag{1}$$

Here $\rho(t_n, x_i) = \rho_i^n$. Which in matrix form is:

$$L\rho^{n+1} = R\rho^n \tag{2}$$

Where:

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & \cdots & 0 \\ -\alpha & 1+2\alpha & -\alpha & 0 & \cdots & \cdots & 0 \\ 0 & -\alpha & 1+2\alpha & -\alpha & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & -\alpha & 1+2\alpha & -\alpha \\ 0 & \cdots & \cdots & 0 & 0 & -1 & 1 \end{pmatrix} \tag{3}$$

$$R = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \alpha & 1-2\alpha & \alpha & \cdots & \cdots & \cdots & 0 \\ 0 & \alpha & 1-2\alpha & \alpha & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \cdots & \alpha & 1-2\alpha & \alpha \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix} \tag{4}$$

It is easy to check that L is invertible, since it's strict diagonal dominant if $|\alpha| < 1$.

To check that the mass is conserved by the method, we observe that the mass: $m(\rho) = \sum_{i=1}^{M-1} \rho_i$ is preserved by the application of one iteration of the method. Furthermore, by linearity, it enough to check on a basis (of the space of vectors which satisfies the boundary condition, the method does not preserve the mass for any vector, this is clear from the observation $Le_1 = 0$): Then we observe that $m(Ax) = m(x) \iff m(x) = m(A^{-1}x)$ in the case where A is invertible. So we have to check:

$$\begin{aligned} m(L(e_3)) = \cdots = m(L(e_{n-2})) &= 1 \\ m(R(e_3)) = \cdots = m(R(e_{n-2})) &= 1 \\ m(R(e_1+e_2)) = m(L(e_1+e_2)) &= 1 \\ m(R(e_{n-2}+e_{n-1})) = m(L(e_{n-1}+e_{n-1})) &= 1 \end{aligned}$$

Some other conditions Which are trivial to check.

1.1 Solution mantains positivity

No idea.

1.2 Godunov

Choosing $A = L^{-1}R$ and $B(v_i)$ adecuately, it is clear that the system can we written as:

$$\rho^{n+1} = (A + B(v_i))\rho^n$$

Part 2: Optimisation

2.1 Energy

The energy is defined as:

$$E_{\Delta t,\Delta x} = \frac{\Delta t}{2}\sum_{n=1}^{N-1}\langle \rho^n, q(v^n)\rangle + \langle \rho^N, V\rangle$$

For the first inequality, the first two terms follow by direct computation, the last term comes from using transposition and expanding as a telescopic sum:

$$\langle {\rho'}^N, V'\rangle - \langle \rho^N, V\rangle = \langle {\rho'}^N, \phi'\rangle - \langle \rho^N, \phi\rangle = \sum_{i=1}^{N-1}\langle {\rho'}^{i+1} - \rho^{i+1}, {\phi'}^i\rangle - \langle {\rho'}^i - \rho^i, {\phi'}^i\rangle$$

Where in the last equation we use that ${\rho'}^0 = \rho^0$.

The second part follows similarly:

For the last part:

2.2 Find v'