Contents

Morse theory: A quick review

Notation. *We will consider all complexes to be over a field $\mathbb{K}$ that will either be $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Unless otherwise stated.*

Given a manifold we can build a nice function on it, f, so that it's sublevels $M^t = f^{-1}(-\infty, t]$ give us a natural way to deduce a CW-structure for the manifold.
In fact, these same function give us in a natural way the Morse-Smale complex, which allows us to compute the homology of the manifold.

Definition 1.1 (Morse function)**.** Let M be a manifold, and $f \in C^\infty(M, \mathbb{R})$. We say that f is a morse function if the set of critical points of f is isolated.
That is, if: $Crit(f) = \{x \in M : df|_x = 0\}$ is isolated.

We say that a critical point of f has index k, if the hessian of f has index k at the critical point.
We denote the critical points of index k as $Crit_k(f)$.
It is well known that for any given manifold there are plenty of Morse functions, in particular we have the following:

Theorem 1.1. *[3] Let $g \in C^\infty(M, \mathbb{R})$, and let $\epsilon > 0$ then there exists $f \in C^\infty(M, \mathbb{R})$ so that f is a Morse function and f and g are C^2-close.*

So Morse functions are dense in the ring $C^\infty(M, \mathbb{R})$, and a fortiori, Morse functions always exits.
Morse functions give quite a bit of topological information on the manifold, specially in the compact case.
One important source of complexes is the Morse-Smale complex, which arises in a natural way.

1.1 Fundamental theorems of Morse theory

Theorem 1.2 (Morse lemma)**.** *[3] Let M^n be a manifold, and f a Morse function on M. Let x_0 be critical point of f of index k.*
Then there is a neighbourhood V of x_0 and a chart $y : U \to V \subset M$. so that:

$$f \circ y(y_1, \dots, y_n) = -y_1^2 - \dots - y_k^2 + y_{k+1}^2 + \dots + y_n^2 + C$$

Where C is a constant.

Let f be a Morse function on M, We notice that in the compact case f has a minimum, m and a maximum M achieved at points x_m and x_M.
Then $M^{x_m} = \emptyset$ and $M^{x_M} = M$. This hints that we can restruct M from how the sets M^t change as t moves through the real line.
In particular, fix some ε which will be assumed to be as small as needed, we wonder if there is any relation between M^t and $M^{t+\varepsilon}$. It turns out, as Morse [4] showed we have that if f does not have any critical point in the interval $(t, t+\varepsilon)$ then M^t and $M^{t+\varepsilon}$ are difeomorphic. The proof is in fact quite elementary.
A more subtle thing Morse proved is what happens when M^t crosses over a critical point. In this cased he showed that, if choose ε so that $\text{Crit}(f) \cap (t, t+\varepsilon)$ has exactly one point (which we can always do by

hypothesis) then $M^{t+\varepsilon}$ can be obtained by adjoining a k-cell to M^t where k happens to be the index of the unique critical point in $(t, t+\varepsilon)$.
In summary

Theorem 1.3. *[4] Let $\varepsilon > 0$ so that*

$$Crit(f) \cap (t, t+\varepsilon) = \emptyset$$

Then M^t and $M^{t+\varepsilon}$ are difeomorphic.

Theorem 1.4. *[4] Let $\varepsilon > 0$ so that*

$$Crit(f) \cap (t, t+\varepsilon) = \{x_0\}$$

Let k be the index of x_0. Then
Then $M^{t+\varepsilon}$ is homotopy equivalent to $M^t \cup_\phi (D^{n-k} \times D^k)$ where D^k is the k-disk and ϕ is and adjunction map.

Example 1.1. *One classical example of the above theorems is how to use a Morse function to obtain the decomposition of the Torus:*

Corollary 1.1. *[5] Any compact manifold of dimension n is homotopy equivalent to a space such as the following:*

$$D^n \cup D^{n-k_1} \times D^{k_1} \cup \cdots \cup D^{n-k_m} \times D^{k_m}$$

Theorem 1.5 (Morse inequalities)**.** *Let M^n be a compact manifold, and b_i it's betti numbers. Let f be a Morse function on M. And let λ_k be the number of k-index critical points of f. Then we have that the so called weak Morse inequality holds:*

$$\lambda_i \geq b_i, \quad \forall i \in \mathbb{N}$$

As well as the so called strong Morse inequality:

$$\sum_{i=0}^{j} (-1)^i \lambda_i \geq \sum_{i=0}^{j} (-1)^i b_i, \quad 0 \leq j \leq n$$

In the particular where $i = n(= dim(M))$ we have equality:

$$\mathcal{X}(M) = \sum_{i=0}^{n} (-1)^i \lambda_i = \sum_{i=0}^{n} (-1)^i b_i$$

Where $\mathcal{X}(M)$ is the Euler-Poincaré characteristic.

1.2 An application of Morse theory

In this section we describe how to use Morse theory to classify closed 1-manifolds. In particular, we show the well-know theorem:

Theorem 1.6. *A closed connected 1-manifold is diffeomorphic to $\mathbb{S}^1$.*

We briefly mention that it can be used to derive the usual classification of 2-manifolds.

1.3 The Morse-Smale complex

Definition 1.2 (Differential complex)**.** We say that a collection of indexed abelian groups $\{C_i\}_{i=0}^{\infty}$ is a differential complex if it has a linear operator: $\delta_i : C_{i+1} \to C_i$ so that $\delta_i \circ \delta_{i+1} = 0$.
Usually, we shall consider $\delta : C \to C$ where $C = \oplus C_i$ and δ is defined as the unique linear operator that verifies: $\delta|_{C_i} = \delta_i$.
And so, we may write that $\{C_i\}_{i=0}^{\infty}$ is a differential complex if and only if $\delta^2 = 0$.
Finally, given C and D two differential compexes, then we say that $f : C \to D$ is a chain map if $f \circ \delta_C = \delta_D \circ f$.

Morse functions give a natural complex which computes the homology of the manifold.
We define the chain groups to be:

$$C_k(f) \overset{\text{def}}{=} \left\{ \sum_{c \in \mathrm{Crit}_k(f)} ac : a \in \mathbb{Z}_2 \right\}$$

And we define the differential operator ∂ to be:

$$\partial a = \sum_{b \in \mathrm{Crit}_{k-1}(f)} n(a,b)b$$

Where $n(a,b)$ is the number of trajectories flowing from a to b through the gradient flow. This number is finite, and we have that $\partial^2 = 0$.

1.4 Discrete Morse Theory

We can consider Morse theory on simplicial complexes or simplicial sets, in this case, instead of having critical points we have critical faces instead of critical points.
This idea is as was previously, we consider the growing sets $f^{-1}(\infty, t]$ and we observe for which values of t the type of homotopy changes.

Definition 1.3 (Discrete Morse functions)**.** Let $\mathcal{K}$ be a simplicial complex.
Then we denote $\alpha_i^{(p)} \in \mathcal{K}$ a p-dimensional face of $\mathcal{K}$ and we denote $\beta < \alpha$ is β is subface of α.
Then we say that $f : \mathcal{K} \to \mathbb{R}$ is that a discrete Morse function if and only if f assigns higher values to higher dimensional simplices with few exceptions. Formally, if f satifies both of the following definitions:

$$\begin{cases} \#\{\beta^{(p+1)} > \alpha | f(\beta) \leq f(\alpha)\} & \leq 1 \\ \#\{\gamma^{(p-1)} < \alpha | f(\gamma) \geq f(\alpha)\} & \leq 1 \end{cases}$$

Finally, we say that $\alpha^{(p)} \in \mathcal{K}$ is a critical simplex if and only if:

$$\begin{cases} \#\{\beta^{(p+1)} > \alpha | f(\beta) \leq f(\alpha)\} & = 0 \\ \#\{\gamma^{(p-1)} < \alpha | f(\gamma) \geq f(\alpha)\} & = 0 \end{cases}$$

In general, we will have that always one of the above sets will be empty, that is:

Lemma 1.1. *Let $\mathcal{K}$ be a simplicial complex, and let $\alpha^{(p)} \in \mathcal{K}$ be a non-critical compolex, then either*

$$\#\{\beta^{(p+1)} > \alpha | f(\beta) \leq f(\alpha)\} = 0$$

or

$$\#\{\gamma^{(p-1)} < \alpha | f(\gamma) \geq f(\alpha)\} = 0$$

And we have a similar result as in the smooth case, that is the discrete Morse function describes the homotopy type of the complex:

Theorem 1.7. *Let $\mathcal{K}$ be a simplicial complex and f a discrete Morse function on it. Then $\mathcal{K}$ is homotopy equivalent to another simplicial complex which has exactly one complex for every critical complex of $\mathcal{K}$.*

Another important similarity with the smooth case is that the standart Morse inequalities hold:

Theorem 1.8 (Morse inequalities)**.** *Let $\mathcal{K}$ be a simplicial complex, and b_i it's betti numbers. Let f be a Morse function on M. And let λ_k be the number of k-dimensional critical simplices. Then everything holds exactly as in theorem 1.5.*
In particular, all of the following holds:

$$\lambda_i \geq b_i, \quad \forall i \in \mathbb{N}$$

$$\sum_{i=0}^{j} (-1)^i \lambda_i \geq \sum_{i=0}^{j} (-1)^i b_i, \quad 0 \leq j \leq n$$

$$\mathcal{X}(M) = \sum_{i=0}^{n} (-1)^i \lambda_i = \sum_{i=0}^{n} (-1)^i b_i$$

Where in this case n is the dimension of $\mathcal{K}$, which is the maximum dimension of the simplices of $\mathcal{K}$.

Canonical form

2.1 Ordered complex and canonical form

Definition 2.1 ($\mathbb{R}$-Filtered complex)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a complex. A $\mathbb{R}$-filtration (sometimes we will simply say a filtration), on $\{C_k\}_{k=0}^{\infty}$ is an increasing sequence of real numbers, $\{r_i\}_{i=0}^{n}$ so that for each r_i there as associated $F_{\leq r_i}C_k \subset C_k$ for every k that satisfies:

$$\{0\} \subset F_{\leq r_0}C_k \subset F_{\leq r_k}C_k \subset \cdots \subset F_{\leq r_n}C_k = C_k$$

As we well see, there are a lot of natural circumstances on which a filtration might arise. For example, the singular chain comples of a CW-complex are naturally filtered by its skeleton.
In general, we may filter a complex by any partial ordered set, $\mathbb{P}$, by simply requiring $F_{\leq r_0}C_k \subset F_{\leq r_1}C_k$ for any $r_0 \leq r_1 \in \mathbb{P}$.
One example for this might be the following: Suppose that we have a complex that has two different $\mathbb{R}$-filtrations, then we can combine them into a single $\mathbb{R}^2$-filtration.
There are more structures we can put on a complex:

Definition 2.2 (Complex with ordered generators)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a filtered chain complex, with some basis $\{e_j^{(i)}\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ has ordered generators if the generatros are ordered as such: $e_k^{(i)} < e_l^{(j)}$ if $k = l$ or if $k = l$ and $i < j$.
Notice we do not compare generators that live on different chain groups.

Remark 1. *A filtrated complex has a natural order on the generators compatible with the filtration.*

Definition 2.3 (Canonical form)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a chain complex, with some basis $\{e_k^{(i)}\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ is in canonical form if,

1. $\partial e_k^{(i)}$ is either 0 or another generator.
2. If $\partial e_k^{(i)} = \partial e_l^{(i)}$, then $e_k^{(i)} = e_l^{(i)}$.

Remark 2. *An equivalent formulation of the canonical form is that we can find a basis S of $\{C_k\}_{k=0}^{\infty}$ so that S can be separated into:*

1. *S_H: Generators of the homology of the complex.*
2. *S_{birth}: Births, that is, elements whose boundary is 0, but get killed in homology by an element of higher degree.*
3. *S_{death}: Deaths, elements whose boundary is another generator.*
4. *∂ is a bijection between S_{death} and S_{birth}.*

That is:

$$S = S_{birth} \sqcup S_{death} \sqcup S_H$$

Theorem 2.1. *[1] Every filtered chain complex can be reduced to one in canonical form by an upper-triangular change of basis which preserves the filtration.*
Furthermore, the canonical form is unique.

Canonical form allows us to classify graded differential complexes. In particular, by will say two complexes are equivalent if there is a bijective chain map preserving the filtration between the two complexes. More precisely:

Definition 2.4 (Equivalent complexes)**.** Let C and D be two filtered complex as such:

$$\{0\} \subset F_{\leq r_0}C_k \subset F_{\leq r_k}C_k \subset \cdots \subset F_{\leq r_n}C_k = C_k$$
$$\{0\} \subset F_{\leq r_0}D_k \subset F_{\leq r_k}D_k \subset \cdots \subset F_{\leq r_n}D_k = D_k$$

Then we say that they are equivalent if and only if there is chain map $f : C \to D$ so that

$$f\left(F_{\leq r_k}C_k\right) \subset F_{\leq r_k}D_k, \quad \forall k, r_k$$

Corollary 2.1. *Two filtered complexes are equivalent if and only if they have the same canonical form.*

Corollary 2.2. *Category of filtered complexes is semi-simple. That is, any filtered complex, (or one with an ordered basis) can be expressed as a direct sum of 1 dimensional complexes with trivial differential and 2 dimensional complexes with trivial homology.*

Proof. Because any complex can we brought to canonical form, we can find a basis of the complex in the form of Remark 2. Then Span(S_H) splits as a sum of 1 dimensional complexes with trivial differential, since the differential is trivial on S_H. Similarly, the birth and death pairs splits into 2 dimensional complexes with one generator killing the other in homology.

QED

2.2 Persistence complexes

Definition 2.5 (Persistent complex)**.** A persistence complex is a sequence of vector spaces $\{V_i\}_{0\leq i\leq n}$ and homomorphisms $f_i : V_i \to V_{i+1}$. That is, a diagram as follows:

$$V_0 \xrightarrow{f_0} V_1 \longrightarrow \cdots \xrightarrow{f_{n-1}} V_n$$

We denote $f_i^j = f_j \circ \cdots f_{i+1} \circ f_i$ whith $i < j$. Persistence diagrams may be of arbitrary cardinal, as long as the indexed is set is well-ordered.
But we shall only considered them when they are finite or countable.
We further define it's graded Betti number as:

$$b_i^j = \begin{cases} \dim(V_i) & \text{if} \quad i = j \\ \operatorname{rank}(f_i^j) & \text{if} \quad i < j \\ 0 & \text{if} \quad i > j \end{cases}$$

It is worth mentioning that we usually consider the maps f_i as inclusions and the spaces V_i as chain groups. As such, we can sometimes drop the vector space assumption and take modules over a PID. For example, it is common to consider $\mathbb{Z}$ instead of $\mathbb{K}$. We call this constructions persistent modules. Precisely we define:

Definition 2.6. (Persistence module) A persistence complex is a sequence of modules over a PID $\{M_i\}_{0\leq i\leq n}$ and homomophisms $f_i : M_i \to M_{i+1}$. With the rest of the definitions as above.

We give examples of useful persistence diagrams in 2.2.2.

Example 2.1 (Interval module). *We define the interval module for $i < j$ to be the module where all maps are either the trivial ones, being the identity whenever possible and the zero map otherwise and the V_k to be:*

$$V_k = \begin{cases} \mathbb{K} & \text{if } i \leq k \leq j \\ 0 & \text{otherwise} \end{cases}$$

We denote it by $m(i, j)$ and we define it's associated interval to be $[i, j)$.

Proposition 2.1. *[2] The category of countable persistence complex is equivalent to the one of graded modules over $\mathbb{K}[t]$*

Proposition 2.2. *[2] Finite persistence diagrams are classified by their graded Betti numbers.*

Proposition 2.3. *[2] Every persistence module is the direct sum of interval modules. Furthermore, the multiplicity of each interval module is an invariant of the complex.*
This intervals are called the persistence intervals of a given persistence module.

One important observation about the previous proposition, is that a complex in canonical form gives a clear idea of what the persistence diagrams decomposition is. In particular, the beginning of a interval indicates a birth and the end of an interval indicates a death. And so, an element represent a non-trivial cohomology class if and only if it belongs to an interval of the form $[\alpha, \infty)$

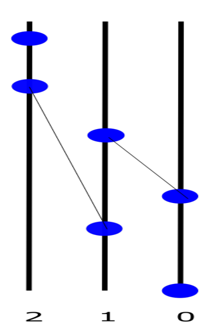
2.2.1 Representations of the persistence complexes

Firstofall, we can represent canonical form as in the following example:
Clearly, from any complex in canonical form we can construct an associated persistent diagram by reversing the above process. In the above example we ignored the filtration of the complex, but we consider that the diagram is filtered by height. Therefore:
We have two natural ways to represent the persistent homology of a filtered complex, one is persistent barcodes and the other is persistent

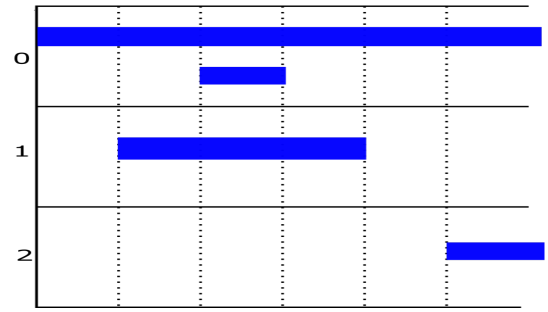
Example 2.2. *Consider the following complex:*

1. $C_0 = \{e_0^{(0)}, e_1^{(0)}\}$
2. $C_1 = \{e_0^{(1)}, e_1^{(1)}\}$
3. $C_0 = \{e_0^{(2)}, e_1^{(2)}\}$
4. $\partial e_1^{(1)} = e_0^{(1)}$, $\partial e_0^{(2)} = e_0^{(1)}$ *and* 0 *on the rest of the generators.*

And obviosuly $S_{birth} = \{e_0^{(1)}, e_0^{(1)}\}$, $S_{death} = \{e_1^{(1)}, e_0^{(2)}\}$, $S_H = \{e_0^{(0)}, e_1^{(2)}\}$ *Then it's persistence diagram would be the following:*



And we have that it's persistence barcode would be:



Notice that the barcode gives the an obvious idea of what the interval decomposition is.
We will explore further representations in chapter 3.

2.2.2 Some important examples of persistent complexes

Definition 2.7 (Discrete morse complex)**.** Let $f : M \to \mathbb{R}$ be a morse function from a compact manifold with finitely many critical points $t_0 < \cdots < t_n$. Then the following diagram is a persistence complex:

$$H^{dR}(f^{-1}(-\infty, t_0)) \xrightarrow{\subseteq} H^{dR}(f^{-1}(-\infty, t_1)) \xrightarrow{\subseteq} \cdots \xrightarrow{\subseteq} H^{dR}(f^{-1}(-\infty, t_n))$$

Where H^{dR} is the de Rham complex and $\subseteq$ is homomorphism induced by the natural inclusion.

Definition 2.8 (Čech Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define the Čech complex of level ϵ as the following simplicial set:

$$\tilde{C}_\epsilon(X) = \{\sigma \subset X : \bigcap_{y \in \sigma} B(y, \epsilon) \neq \emptyset\}$$

Then we define the Čech complex at level ε to be $\tilde{C}_\epsilon$.

The Čech has one key advantage, it's homotopy type is easy to predict. In particular, given a compact manifold M of $\mathbb{R}$ we can pick a discrete subset X and an appropiate ε so that $\tilde{C}_\varepsilon(X)$ has the homotopy type of M.
This a consequence of the following theorem (and the equivalence between Čech cohomology and De Rham cohomology on manifolds).

Theorem 2.2. *Let X and $\varepsilon > 0$, then $\tilde{C}_\varepsilon(X)$ is homotopy equivalent to the union of the ε-balls centered around the point of the complex. Explicitly:*

$$\tilde{C}_\varepsilon(X) \quad \stackrel{h\text{-}eq}{=\!=} \quad \bigcup_{x \text{ vertex in } \tilde{C}_\varepsilon(X)} \quad B(x, \varepsilon)$$

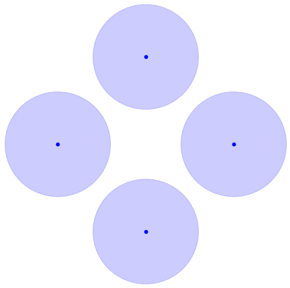
Example 2.3. *In the following example we illustrate the evolution of the Čech complex as ε grows:*

However, in practice the Čech complex is very hard to calculate, notice that as the dimension of $\mathbb{R}^n$ increases it would be necessary to calculate the intersection of $n-1$-dimensional spheres, which is it a computationally complex problem.
One solution to this is using the Vietoris-Rips complex.

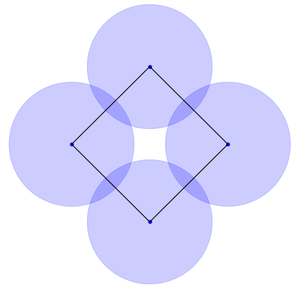
Definition 2.9 (Vietoris-Rips Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define the Čechcomplex of level ϵ as the following simplicial set:

$$VR_\epsilon(X) = \{\sigma \subset X : \operatorname{diam}(\sigma) \leq \epsilon\}$$

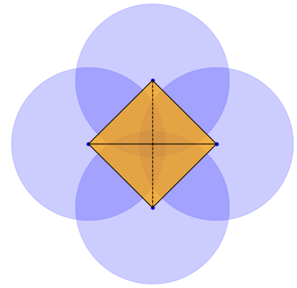
Then we define the Vietoris-Rips complex at level ε to be VR_ϵ.
There are many softwares that can be used to compute the Čech and Vietoris-Rips complexes, most of them rely on bringing the complex to canonical form.
The following example shows how the Čech and Vietories-Rips complex might differ: (compare to example 2.3)



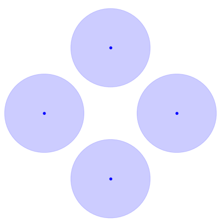
Čech complex for $\varepsilon < \frac{\sqrt{2}}{2}$



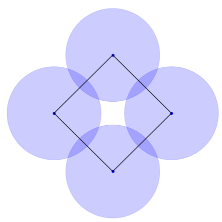
Čech complex for $\frac{\sqrt{2}}{2} < \varepsilon < 1$



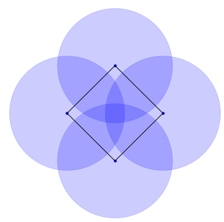
Čech complex for $\varepsilon > 1$



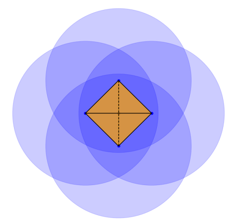
Vietoris-Rips complex for $\varepsilon < \frac{\sqrt{2}}{2}$



Vietoris-Rips complex for $\frac{\sqrt{2}}{2} < \varepsilon < 1$



Vietoris-Rips complex for $\varepsilon > 1$



Vietoris-Rips complex for $\varepsilon > 2$

Example 2.4.

Notice that we can combine the Vietoris-Rips (or Čech) complex with an already existing filtration to give a bifiltrated complex:

Definition 2.10 (Bifiltrated Vietoris-Rips complex)**.** Let $\mathcal{X} = (X, d, f)$ be an augmented metric space, let $X_\sigma = f^{-1}(-\infty, \sigma]$ we define the bifiltrated Vietoris-Rips complex as the complex $\{VR^\sigma_\epsilon\}_{\epsilon>0,\sigma}$ with:

$$VR^\sigma_\epsilon = VR_\epsilon(X_\sigma)$$

2.3 Comparison of the Čech and Vietoris-Rips complex

One great advantage of the Čech complex is that it's homotopy type is highly predictable. In particular, we have the classical Čech nerve theorem:
This means we can always approximate the homology of a compact manifold with the Čech complex using a relatively low amount of points. In the case of the Vietoris-Rips complex things are different.
However, there is a natural relation between these two complexes:

Proposition 2.4. *The Čech complex is always a subcomplex of the Rips complex. We also have a partial converse: $VR_{\varepsilon/2}(X)$ is a subcomplex of $\tilde{C}_\varepsilon(X)$*
More precisely:

$$\tilde{C}_{\varepsilon/2}(X) \hookrightarrow VR_{\varepsilon/2}(X) \hookrightarrow \tilde{C}_\varepsilon(X) \hookrightarrow VR_\varepsilon(X)$$

2.4 Motivation

2.5 Example of canonical form

The proof of theorem 2.1 gives an algorithmic way to bring a complex into canonical form, we now expose how this methods works with an example. The general case should be clear afterwards.

Elder-rule Staircodes

3.1 Augmented metric spaces

Definition 3.1. Let (X, d) be a metric space, we say it is augmented if it comes equiped with a function $f : X \to \mathbb{R}$.

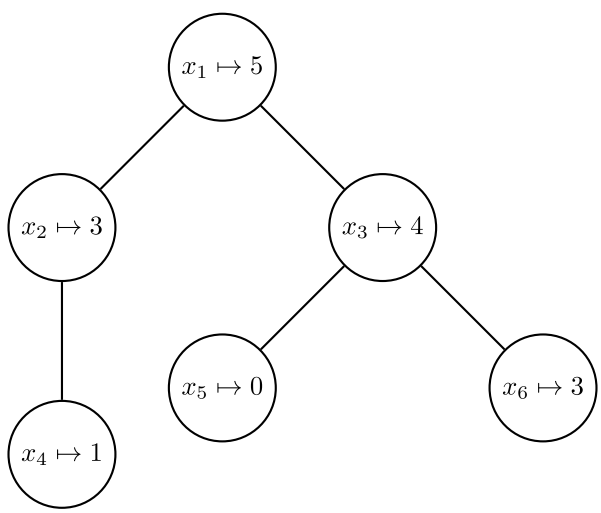
We will say that an augmented metric space is injective if f is. And we will be interested in the case where X is finite.

Morse functions on point clouds

We now focus on discrete augmented metric spaces. In particular, suppose we have a compact smooth manifold M equipped with a Morse function $f : M \to \mathbb{R}$.
From this manifold M we sample a finite amount of points $X = \{x_i\}_{i=0}^n$ and we record the value of f on X.
We investigate what sort of topological information we can recover from this discrete sample.
The first thing we would like to do is recover the gradient of the function. For this purpose we create a graph using X as its base set and we define a flow on the graph by moving from one point to the point connected with it where the morse function f takes the smallest value.

Definition 4.1 (NNG graph)**.** Given X a discrete set, we define $G^-(X)$ it's descending pseudo-gradient graph as the nearest neighbour graph of X.
That is the graph where the edges $\{x_i, x_j\}$ are precisely where either $d(x_i, x_j) = d(x_i, X)$ or $d(x_i, x_j) = d(x_j, X)$. Notice that the binary relation $x_i \sim x_j$ if and only if x_j is the closest vertex to x_i is not symmetric.
We denote as $\omega^- : G \to G$ the application that maps $p \in X$ to the result of iteratively following the point with that its joined to p and that minimises the value of f.
We define α^- in an analogous manner, by following the direction of maximum growth of f.

Example 4.1. *For example, let's consider the following graph, where we denote the value of f by $x_i : f(x_i)$:*



Then $\omega^-(x_1) = x_4$. So ω maps to local minimums, but not necessarily to absolute ones, even when restricting to the same component.

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