Introductory concepts

1.1 Ordered complex and canonical form

Definition 1.1 ($\mathbb{R}$-Filtered complex)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a complex. A $\mathbb{R}$-filtration on $\{C_k\}_{k=0}^{\infty}$ is an increasing sequence of real numbers, $\{r_i\}_{i=0}^{n}$ so that for each r_i there as associated $F_{\leq r_i}C_k \subset C_k$ for every k that satisfies:

$$\{0\} \subset F_{\leq r_0}C_k \subset F_{\leq r_k}C_k \subset \cdots \subset F_{\leq r_n}C_k = C_k$$

As we well see, there are a lot of natural circumstances on which a filtration might arise. For example, the singular chain comples of a CW-complex is naturally filtered by its skeleton.

There are more structures we can put on a complex:

Definition 1.2 (Complex with ordered generators)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a chain complex, with some basis $\{e_k^i\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ has ordered generators when we fix the order $e_k^i < e_l^j$ if $k < l$ or $k = l$ and $i < j$.

Remark. *An ordered complex is naturally filtrated.*

Definition 1.3 (Canonical form)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a chain complex, with some basis $\{e_k^i\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ is in it simplest form if, ∂e_k^i is either 0 or another generator.

Remark. *The canonical form is equivalent to saying that we can find a basis S of $\{C_k\}_{k=0}^{\infty}$ so that S can be separated into:*

1. *S_H: Generators of the homology of the complex.*
2. *S_{birth}: Births, that is, elements whose boundary is 0, but get killed in homology by an element of higher degree.*
3. *S_{death}: Deaths, elements whose boundary is another generator.*

That is:

$$S = S_{birth} \sqcup S_{death} \sqcup S_H$$

Theorem 1. *[1] Every filtered chain complex can be reduced to one in canonical form by an upper-triangular change of basis.*

Furthermore, the canonical form is unique.

Proof. First we prove that we prove the first statement. Let $C_{k\,k=0}^{\ \infty}$ and a basis $\{e_i^{(j)}\}_{i,j}$ with $e_i^{(j)} \in C_i, \forall i$.

Let $i+1$ be the smallest natural so that $C_{\leq n}$ is not in canonical form. Obviously, $i+1 > 0$, since $\partial|_{C_0} = 0$.

Let $e_{i+1}^{(j)}$ be the generator with smallest j so that ∂e_j^i is neither 0 nor another generator. Clearly, $i+j > 0$, since $\partial|_{C_0} = 0$.

Let $\partial e_{i+1}^{(j)} = \sum \alpha_k e_k^{(j-1)}$, then on the LHS, we have terms which are exact (that is, on S_{death}), and those which are not. We move all the exact terms in the LHS of the form $e_k^{(j-1)} = \partial e_q^{(j)}$, with $q \leq i$.

$$\partial \left(e_{i+1}^{(j)} - \sum_{q=1}^{i} e_q^j \alpha_{k(q)} \right) = \sum \beta_k e_k^{(j-1)}$$

Here we separate two cases, if the LHS is 0, then we simply define

$$f_j^{(i+1)} = e_{i+1}^{(j)} - \sum_{q=1}^{i} e_j^{(q)} \alpha_{k(q)}$$

In this case, clearly the linear map $Ae_{i+1}^{(j)} = f_{i+1}^{(j)}$ and the identity on the rest has an upper triangular matrix and $\partial fi + 1j = 0$.

In the case we can find $\beta_{k_0} \neq 0$ we pick such k_0 so it is maximal. Then:

$$\partial \frac{1}{\beta_{k_0}} \left(e_{i+1}^{(j)} - \sum_{q=1}^{i} e_j^{(q)} \alpha_{k(q)} \right) = e_{k_0}^{(j-1)} + \sum_{k<k_0} \frac{\beta_k}{\beta_{k_0}} e_k^{(j-1)}$$

In this case, we define:

$$f_{i+1}^{(j)} = \frac{1}{\beta_{k_0}} \left(e_{i+1}^{(j)} - \sum_{q=1}^{i} e_q^{(j)} \alpha_{k(q)} \right); \quad f_{k_0}^{(j-1)} = e_{k_0}^{(j-1)} + \sum_{k<k_0} \frac{\beta_k}{\beta_{k_0}} e_k^{(j-1)}$$

Then clearly $\partial f_{i+1}^{(j)} = f_{k_0}^{(j-1)}$ and the obvious change of variable has the desired form. This violates our minimality condition, therefore the first statement is proved.

Now we prove the uniqueness part: Suppose we have two bases in canonical form $\{e_i^{(j)}\}$ and $\{f_i^{(j)}\}$. Then choose i and j so that for all $p < j$ and any m or $p = j$ and $m < i$ then $e_i^{(j)} = f_i^{(j)}$.

We can find coefficients α_k and β_k so that:

$$f_i^{(j)} = \sum_{k=1}^{i} \alpha_k e_k^{(j)}, \quad \partial f_i^{(j)} = f_l^{(j-1)} = \sum_{n=1}^{l} \beta_k e_k^{(j)}$$

Hence

$$\partial e_i^{(j)} = \sum_{n=1}^{l} e_n^{(j-1)} \frac{\beta_n}{\alpha_i} - \sum_{k=1}^{l} \partial e_k^{(j-1)} \frac{\alpha_k}{\alpha_i}$$

This proves the uniqueness of the canonical form since: $\partial e_i^{(j)} = e_m^{(j-1)}, m > l$ and $e_m^{(j-1)} \neq \partial a_k^j, \forall k \neq i$

QED

Corollary 1.1. *Category of filtered complexes is semi-simple Any filtered complex, (or one with an ordered basis) can be expressed as a direct sum of 1 dimensional complexes with trivial differential and 2 dimensional complexes with trivial homology.*

Proof. Because any complex can we brought to canonical form, we can find a basis of the complex in the form of Remark 1.1. Then $\mathrm{Span}(S_H)$ splits as a sum of 1 dimensional complexes with trivial differential, since the differential is trivial on S_H. Similarly, the birth and death pairs splits into 2 dimensional complexes with one generator killing the other in homology.

QED

1.2 Representations of the canonical form

Definition 1.4 (Persistence diagrams)**.** We call a persistence diagram to a finite family on half lines, indexed by $0, 1, ...$ so that there is a discrete collection of points on each line and there are segments connecting points on a line of index i to a line of index $i - 1$.

Furthermore, this segments always connect point of higher value with points of lower value.

We clarify this definition with an example:

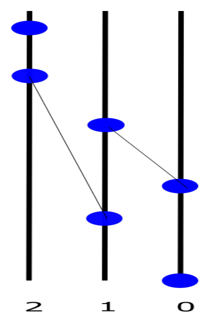
Example 1.1. *The diagram below is a persistence diagram, notice that we can associate a natural complex in canonical form in the following manner:*

1. *To a line of index i we assign it the complex C_k.*
2. *Assign a generator to every point on every line a generator of C_k*
3. *We define the differential operator ∂ in the following way. If there is no segment coming out of the point, we define its differential to be zero. If two points are joined by a line, we define the differential of one as the other.*

For example, in the example previous the associated complex would be:

1. $C_0 = \{e_0^{(0)}, e_1^{(0)}\}$
2. $C_1 = \{e_0^{(1)}, e_1^{(1)}\}$
3. $C_0 = \{e_0^{(2)}, e_1^{(2)}\}$
4. $\partial e_1^{(1)} = e_0^{(1)}$, $\partial e_0^{(2)} = e_0^{(1)}$ *and* 0 *on the rest of the generators.*

And obviosuly $S_{birth} = \{e_0^{(1)}, e_0^{(1)}\}$, $S_{death} = \{e_1^{(1)}, e_0^{(2)}\}$, $S_H = \{e_0^{(0)}, e_1^{(2)}\}$



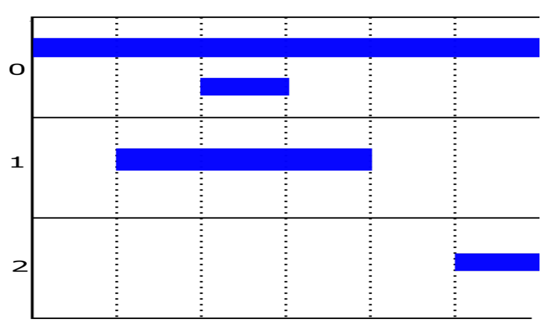
Clearly, from any complex in canonical form we can construct an associated persistent diagram by reversing the above process. In the above example we ignored the filtration of the complex, but we consider that the diagram is filtered by height. Therefore:

Proposition 1 (Canonical form = Persistence diagrams)**.** *There is a natural equivalence between persistence diagrams and canonical forms.*

Persistence diagrams are not the only way we can represent a canonical form. Another important example are codebars.

Definition 1.5 (Codebars)**.** By codebar we will mean a representation of similar to one below, that a collection of bars, each one index (although index might be repeated), where finite bars represent births and deaths and infinite bars represent elements that generate the homology of the complex.

Example 1.2. *This example represents the barcode associated to the complex of the previous example.*



1.3 Some important examples of complexes

Definition 1.6 (Čech Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define the Čechcomplex of level ϵ as the following simplicial set:

$$\tilde{C}_\epsilon(X) = \{\sigma \subset X : \bigcap_{y \in \sigma} B(y, \epsilon) \neq \emptyset\}$$

The Čech complex is defined as $\{\tilde{C}_\epsilon\}_{\epsilon > 0}$

However, in practice the Čech complex can get too large to handle, notice that when ϵ is large enough, the Čech complex equals the power set of X. One solution to this is using the Vietories-Rips complex.

Definition 1.7 (Vietoris-Rips Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define the Čechcomplex of level ϵ as the following simplicial set:

$$\tilde{R}_\epsilon(X) = \{\sigma \subset X : \text{diam}(\sigma) \leq \epsilon\}$$

The Vietoris-Rips complex is defined as $\{\tilde{R}_\epsilon\}_{\epsilon > 0}$

Remark. *Notice than if both the Čech and Vietories-Rips complex, the complex grows in a discrete manner meaning that there are only finitely many ϵ values for which the complex changes if ϵ is pertubed slightly.*

Furthermore, this complex are naturally filtrated and thus, they can be brought into canonical form.

1.4 Motivation

Another way in which filtered complexes arise is through Morse functions.

1.4.1 Arnold's extension problem

Bibliography

[1] Serguei Barannikov. The framed morse complex and its invariants. 1994.