Introductory concepts

1.1. Ordered complex and canonical form

Definition 1.1 ($\mathbb{R}$-Filtered complex)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a complex. A $\mathbb{R}$-filtration on $\{C_k\}_{k=0}^{\infty}$ is an increasing sequence of real numbers, $\{r_i\}_{i=0}^{n}$ so that for each r_i there as associated $F_{\leq r_i}C_k \subset C_k$ for every k that satisfies:

$$\{0\} \subset F_{\leq r_0}C_k \subset F_{\leq r_k}C_k \subset \cdots \subset F_{\leq r_n}C_k = C_k$$

As we well see, there are a lot of natural circumstances on which a filtration might arise. For example, the singular chain comples of a CW-complex is naturally filtered by its skeleton.

There are more structures we can put on a complex:

Definition 1.2 (Complex with ordered generators)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a chain complex, with some basis $\{e_k^i\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ has ordered generators when we fix the order $e_k^i < e_l^j$ if $k < l$ or $k = l$ and $i < j$.

Remark. *An ordered complex is naturally filtrated.*

Definition 1.3 (Canonical form)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a chain complex, with some basis $\{e_k^i\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ is in it simplest form if, ∂e_k^i is either 0 or another generator.

Remark. *The canonical form is equivalent to saying that we can find a basis S of $\{C_k\}_{k=0}^{\infty}$ so that S can be separated into:*

1. *S_H: Generators of the homology of the complex.*
2. *S_{birth}: Births, that is, elements whose boundary is 0, but get killed in homology by an element of higher degree.*
3. *S_{death}: Deaths, elements whose boundary is another generator.*

That is:

$$S = S_{birth} \sqcup S_{death} \sqcup S_H$$

Theorem 1. *Every chain complex with ordered generators can be reduced to one in canonical form by an upper-triangular change of basis.*

Furthermore, the canonical form is unique.

Proof. First we prove that we prove the first statement. Let $C_{k_{k=0}}^{\infty}$ and a basis $\{e_i^j\}_{i,j}$ with $e_i^j \in C_i, \forall i$.

Let $i+1$ be the smallest natural so that $C_{\leq n}$ is not in canonical form. Obviously, $i+1 > 0$, since $\partial|_{C_0} = 0$.

Let e_{i+1}^j be the generator with smallest j so that ∂e_j^i is neither 0 nor another generator. Clearly, $i+j > 0$, since $\partial|_{C_0} = 0$.

Let $\partial e_{i+1}^j = \sum \alpha_k e_k^{j-1}$, then on the LHS, we have terms which are exact (that is, on S_{death}), and those which are not. We move all the exact terms in the LHS of the form $e_k^{j-1} = \partial e_q^j$, with $q \leq i$.

$$\partial \left(e_{i+1}^j - \sum_{q=1}^{i} e_q^j \alpha_{k(q)} \right) = \sum \beta_k e_k^{j-1}$$

Here we separate two cases, if the LHS is 0, then we simply define

$$f_{i+1}^j = e_{i+1}^j - \sum_{q=1}^{i} e_q^j \alpha_{k(q)}$$

In this case, clearly the linear map $Ae_{i+1}^j = f_{i+1}^j$ and the identity on the rest has an upper triangular matrix and $\partial f_{i+1}^j = 0$.

In the case we can find $\beta_{k_0} \neq 0$ we pick such k_0 so it is maximal. Then:

$$\partial \frac{1}{\beta_{k_0}} \left(e_{i+1}^j - \sum_{q=1}^{i} e_q^j \alpha_{k(q)} \right) = e_{k_0}^{j-1} + \sum_{k<k_0} \frac{\beta_k}{\beta_{k_0}} e_k^{j-1}$$

In this case, we define:

$$f_{i+1}^j = \frac{1}{\beta_{k_0}} \left(e_{i+1}^j - \sum_{q=1}^{i} e_q^j \alpha_{k(q)} \right); \quad f_{k_0}^{j-1} = e_{k_0}^{j-1} + \sum_{k<k_0} \frac{\beta_k}{\beta_{k_0}} e_k^{j-1}$$

Then clearly $\partial f_{i+1}^j = f_{k_0}^{j-1}$ and the obvious change of variable has the desired form. This violates our minimality condition, therefore the first statement is proved.

Now we prove the uniqueness part: Suppose we have two bases in canonical form $\{e_i^j\}$ and $\{f_i^j\}$. Then choose i and j so that for all $p < j$ and any m or $p = j$ and $m < i$ then $e_i^j = f_i^j$.

We can find coefficients α_k and β_k so that:

$$f_i^j = \sum_{k=1}^{i} \alpha_k e_k^j, \quad \partial f_i^j = f_l^{j-1} = \sum_{n=1}^{l} \beta_k e_k^j$$

Hence

$$\partial e_i^j = \sum_{n=1}^{l} e_n^{j-1} \frac{\beta_n}{\alpha_i} - \sum_{k=1}^{l} \partial e_k^{j-1} \frac{\alpha_k}{\alpha_i}$$

This proves the uniqueness of the canonical form since: $\partial e_i^j = e_m^{j-1}, m > l$ and $e_m^{j-1} \neq \partial a_k^j, \forall k \neq i$

QED

1.2. Some examples of complexes

Definition 1.4 (Čech Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define the Čech-complex of level ϵ as the following simplicial set:

$$\tilde{C}_\epsilon(X) = \{\sigma \subset X : \bigcap_{y \in \sigma} B(y, \epsilon) \neq \emptyset\}$$

However, in practice the Čech complex can get too large to handle, notice that when ϵ is large enough, the Čech complex equals the power set of X. One solution to this is using the Vietories-Rips complex.

Definition 1.5 (Vietoris-Rips Complex)**.**

1.3. Motivation