Introductory concepts

1.1. Ordered complex and canonical form

Definition 1.1 ($\mathbb{R}$-Filtered complex)**.**

Definition 1.2 (Complex with ordered generators)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a chain complex, with some basis $\{e_k^i\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ has ordered generators when we fix the order $e_k^i < e_l^j$ if $k < l$ or $k = l$ and $i < j$.

Remark. *An ordered complex is naturally filtrated.*

Definition 1.3 (Canonical form)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a chain complex, with some basis $\{e_k^i\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ is in it simplest form if, ∂e_k^i is either 0 or another generator.

Remark. *The canonical form is equivalent to saying that we can find a basis S of $\{C_k\}_{k=0}^{\infty}$ so that S can be separated into:*

1. *S_H: Generators of the homology of the complex.*
2. *S_{birth}: Births, that is, elements whose boundary is 0, but get killed in homology by an element of higher degree.*
3. *S_{death}: Deaths, elements whose boundary is another generator.*

That is:

$$S = S_{birth} \sqcup S_{death} \sqcup S_H$$

Theorem 1. *Every chain complex with ordered generators can be reduced to one in canonical form by an upper-triangular change of basis.*
Furthermore, the canonical form is unique.

Proof. First we prove that we prove the first statement. Let $C_{k\,k=0}^{\,\infty}$ and a basis $\{e_i^j\}_{i,j}$ with $e_i^j \in C_i, \forall i$.

Let $i+1$ be the smallest natural so that $C_{\leq n}$ is not in canonical form. Obviously, $i+1 > 0$, since $\partial|_{C_0} = 0$.

Let e_{i+1}^j be the generator with smallest j so that ∂e_j^i is neither 0 nor another generator. Clearly, $i+j > 0$, since $\partial|_{C_0} = 0$.

Let $\partial e_{i+1}^j = \sum \alpha_k e_k^{j-1}$, then on the LHS, we have terms which are exact (that is, on S_{death}), and those which are not. We move all the exact terms in the LHS of the form $e_k^{j-1} = \partial e_q^j$, with $q \leq i$.

$$\partial\left(e_{i+1}^j - \sum_{q=1}^{i} e_q^j \alpha_{k(q)}\right) = \sum \beta_k e_k^{j-1}$$

Here we separate two cases, if the LHS is 0, then we simply define

$$f_{i+1}^j = e_{i+1}^j - \sum_{q=1}^{i} e_q^j \alpha_{k(q)}$$

In this case, clearly the linear map $Ae^j_{i+1} = f^j_{i+1}$ and the identity on the rest has an upper triangular matrix and $\partial f^j_{i+1} = 0$.

In the case we can find $\beta_{k_0} \neq 0$ we pick such k_0 so it is maximal. Then:

$$\partial \frac{1}{\beta_{k_0}} \left(e^j_{i+1} - \sum_{q=1}^{i} e^j_q \alpha_{k(q)} \right) = e^{j-1}_{k_0} + \sum_{k<k_0} \frac{\beta_k}{\beta_{k_0}} e^{j-1}_k$$

In this case, we define:

$$f^j_{i+1} = \frac{1}{\beta_{k_0}} \left(e^j_{i+1} - \sum_{q=1}^{i} e^j_q \alpha_{k(q)} \right); \quad f^{j-1}_{k_0} = e^{j-1}_{k_0} + \sum_{k<k_0} \frac{\beta_k}{\beta_{k_0}} e^{j-1}_k$$

Then clearly $\partial f^j_{i+1} = f^{j-1}_{k_0}$ and the obvious change of variable has the desired form. This violates our minimality condition, therefore the first statement is proved.

Now we prove the uniqueness part: Suppose we have two bases in canonical form $\{e^j_i\}$ and $\{f^j_i\}$. Then choose i and j so that for all $p < j$ and any m or $p = j$ and $m < i$ then $e^j_i = f^j_i$.

We can find coefficients α_k and β_k so that:

$$f^j_i = \sum_{k=1}^{i} \alpha_k e^j_k, \quad \partial f^j_i = f^{j-1}_l = \sum_{n=1}^{l} \beta_k e^j_k$$

Hence

$$\partial e^j_i = \sum_{n=1}^{l} e^{j-1}_n \frac{\beta_n}{\alpha_i} - \sum_{k=1}^{l} \partial e^{j-1}_k \frac{\alpha_k}{\alpha_i}$$

This proves the uniqueness of the canonical form since: $\partial e^j_i = e^{j-1}_m, m > l$ and $e^{j-1}_m \neq \partial a^j_k, \forall k \neq i$

QED

Definition 1.4 (Cech Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define:

$$X_d = \{x \in \mathbb{R}^n : d(x, X) \leq d\}$$

Definition 1.5 (Vietoris-Rips Complex)**.**

1.2. Motivation

Canonical form=Persistence diagrams