Introductory concepts

1.1 Ordered complex and canonical form

Definition 1.1 ($\mathbb{R}$-Filtered complex)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a complex. A $\mathbb{R}$-filtration (sometimes we will simply say a filtration), on $\{C_k\}_{k=0}^{\infty}$ is an increasing sequence of real numbers, $\{r_i\}_{i=0}^{n}$ so that for each r_i there as associated $F_{\leq r_i}C_k \subset C_k$ for every k that satisfies:

$$\{0\} \subset F_{\leq r_0}C_k \subset F_{\leq r_k}C_k \subset \cdots \subset F_{\leq r_n}C_k = C_k$$

As we well see, there are a lot of natural circumstances on which a filtration might arise. For example, the singular chain comples of a CW-complex is naturally filtered by its skeleton.

There are more structures we can put on a complex:

Remark 1. *A filtrated complex has a natural order on the generators compatible with the filtration. Furthermore, if this filtration increases the dimension of each chain group by dimension* 1 *(or* 0*) on each step, then there is natural correspondence between the filtration and the ordering.*

Definition 1.2 (Complex with ordered generators)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a filtered chain complex, with some basis $\{e_j^{(i)}\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ has ordered generators when we fix the order $e_k^{(i)} < elj$ if $k = l$ and $i < j$.

Notice we do not compare generators that live on different chain groups.

Definition 1.3 (Canonical form)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a chain complex, with some basis $\{e_k^{(i)}\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ is in canonical form if,

1. $\partial e_k^{(i)}$ is either 0 or another generator.
2. If $\partial e_k^{(i)} = \partial e_l^{(i)} \neq 0$, then $e_k^{(i)} = e_l^{(i)}$.

Remark 2. *One consequence of the canonical form is that we can find a basis S of $\{C_k\}_{k=0}^{\infty}$ so that S can be separated into:*

1. *S_H: Generators of the homology of the complex.*
2. *S_{birth}: Births, that is, elements whose boundary is* 0*, but get killed in homology by an element of higher degree.*
3. *S_{death}: Deaths, elements whose boundary is another generator.*
4. *∂ is a bijection between S_{death} and S_{birth}.*

That is:

$$S = S_{birth} \sqcup S_{death} \sqcup S_H$$

Theorem 1. *[1] Every filtered chain complex can be reduced to one in canonical form by an upper-triangular change of basis which preserves the filtration.*

Furthermore, the canonical form is unique.

Corollary 1.1. *Category of filtered complexes is semi-simple Any filtered complex, (or one with an ordered basis) can be expressed as a direct sum of 1 dimensional complexes with trivial differential and 2 dimensional complexes with trivial homology.*

Proof. Because any complex can we brought to canonical form, we can find a basis of the complex in the form of Remark 2. Then $\text{Span}(S_H)$ splits as a sum of 1 dimensional complexes with trivial differential, since the differential is trivial on S_H. Similarly, the birth and death pairs splits into 2 dimensional complexes with one generator killing the other in homology.

QED

1.2 Representations of the canonical form

Definition 1.4 (Persistence diagrams)**.** We call a persistence diagram to a finite family of half lines, indexed by $0, 1, ...$ so that there is a discrete collection of points on each line and there are segments connecting points on a line of index i to a line of index $i - 1$.

Furthermore, this segments always connect point of higher value with points of lower value.

When we use persistence diagrams to represent filtered complexes in canonical form, we will represent the filtration on the complex by the height of the point. Meaning that the points of height at most r_i will represent the subcomplex $F_{\leq r_i} C_*$

We clarify this definition with an example:

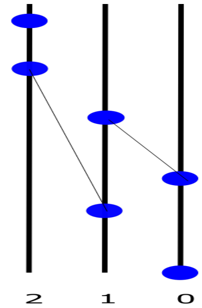
Example 1.1. *The diagram below is a persistence diagram, notice that we can associate a natural complex in canonical form in the following manner:*

1. *To a line of index i we assign it the complex C_k.*
2. *Assign a generator to every point on every line a generator of C_k*
3. *We define the differential operator ∂ in the following way. If there is no segment coming out of the point, we define its differential to be zero. If two points are joined by a line, we define the differential of one as the other.*

For example, in this particular case the associated complex would be:

1. $C_0 = \{e_0^{(0)}, e_1^{(0)}\}$
2. $C_1 = \{e_0^{(1)}, e_1^{(1)}\}$
3. $C_0 = \{e_0^{(2)}, e_1^{(2)}\}$
4. $\partial e_1^{(1)} = e_0^{(1)}$, $\partial e_0^{(2)} = e_0^{(1)}$ *and* 0 *on the rest of the generators.*

And obviosuly $S_{birth} = \{e_0^{(1)}, e_0^{(1)}\}$, $S_{death} = \{e_1^{(1)}, e_0^{(2)}\}$, $S_H = \{e_0^{(0)}, e_1^{(2)}\}$



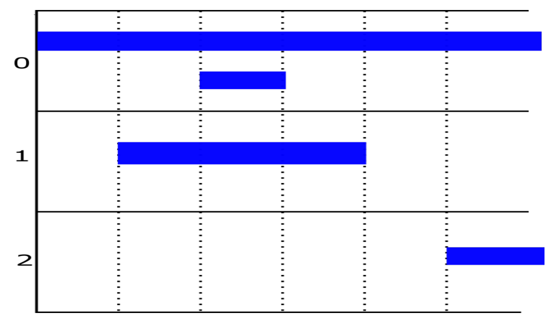
Clearly, from any complex in canonical form we can construct an associated persistent diagram by reversing the above process. In the above example we ignored the filtration of the complex, but we consider that the diagram is filtered by height. Therefore:

Proposition 1 (Canonical form = Persistence diagrams)**.** *There is a natural equivalence between persistence diagrams and canonical forms.*

Persistence diagrams are not the only way we can represent a canonical form. Another important example are codebars.

Definition 1.5 (Codebars)**.** By codebar we will mean a representation of similar to one below, that a collection of bars, each one index (although index might be repeated), where finite bars represent births and deaths and infinite bars represent elements that generate the homology of the complex.

Example 1.2. *This example represents the barcode associated to the complex of the previous example.*



1.3 Some important examples of complexes

Definition 1.6 (Čech Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define the Čechcomplex of level ϵ as the following simplicial set:

$$\tilde{C}_\epsilon(X) = \{\sigma \subset X : \bigcap_{y \in \sigma} B(y, \epsilon) \neq \emptyset\}$$

The Čech complex is defined as $\{\tilde{C}_\epsilon\}_{\epsilon > 0}$

However, in practice the Čech complex can get too large to handle, notice that when ϵ is large enough, the Čech complex equals the power set of X. One solution to this is using the Vietories-Rips complex.

Definition 1.7 (Vietoris-Rips Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define the Čechcomplex of level ϵ as the following simplicial set:

$$\tilde{R}_\epsilon(X) = \{\sigma \subset X : \mathrm{diam}(\sigma) \leq \epsilon\}$$

The Vietoris-Rips complex is defined as $\{\tilde{R}_\epsilon\}_{\epsilon > 0}$

Remark 3. *Notice than if both the Čech and Vietories-Rips complex, the complex grows in a discrete manner meaning that there are only finitely many ϵ values for which the complex changes if ϵ is pertubed slightly.*

Furthermore, this complex are naturally filtrated and thus, they can be brought into canonical form.

1.4 Motivation

Another way in which filtered complexes arise is through Morse functions. For example, let's consider the following manifold with the morse function being the height function:

1.4.1 Arnold's extension problem

Bibliography

[1] Serguei Barannikov. The framed morse complex and its invariants. 1994.