Contents

Preliminares

1.1 Morse theory: A quick review

Notation. *We will consider all complexes to be over a field $\mathbb{K}$ that will either be $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Unless otherwise stated.*

Given a manifold we can build a nice function on it, f, so that it's sublevels $M^t = f^{-1}(-\infty, t]$ give us a natural way to deduce a CW-structure for the manifold.
In fact, these same function give us in a natural way the Morse-Smale complex, which allows us to compute the homology of the manifold.

Definition 1.1 (Morse function)**.** Let M be a manifold, and $f \in C^\infty(M, \mathbb{R})$. We say that f is a morse function if the set of critical points of f is isolated.
That is, if: $Crit(f) = \{x \in M : df|_x = 0\}$ is isolated.

We say that a critical point of f has index k, if the hessian of f has index k at the critical point.
We denote the critical points of index k as $Crit_k(f)$.
It is well known that for any given manifold there are plenty of Morse functions, in particular we have the following:

Theorem 1.1. *[5] Let $g \in C^\infty(M, \mathbb{R})$, and let $\epsilon > 0$ then there exists $f \in C^\infty(M, \mathbb{R})$ so that f is a Morse function and f and g are C^2-close.*

So Morse functions are dense in the ring $C^\infty(M, \mathbb{R})$, and a fortiori, Morse functions always exits.
Morse functions give quite a bit of topological information on the manifold, specially in the compact case.
One important source of complexes is the Morse-Smale complex, which arises in a natural way.

1.1.1 Fundamental theorems of Morse theory

Theorem 1.2 (Morse lemma)**.** *[5] Let M^n be a manifold, and f a Morse function on M. Let x_0 be critical point of f of index k.*
Then there is a neighbourhood V of x_0 and a chart $y : U \to V \subset M$. so that:

$$f \circ y(y_1, \ldots, y_n) = -y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2 + C$$

Where C is a constant.

Let f be a Morse function on M, We notice that in the compact case f has a minimum, m and a maximum M achieved at points x_m and x_M.
Then $M^{x_m} = \emptyset$ and $M^{x_M} = M$. This hints that we can restruct M from how the sets M^t change as t moves through the real line.
In particular, fix some ε which will be assumed to be as small as needed, we wonder if there is any relation between M^t and $M^{t+\varepsilon}$. It turns out, as Morse [7] showed we have that if f does not have any critical point in the interval $(t, t+\varepsilon)$ then M^t and $M^{t+\varepsilon}$ are difeomorphic. The proof is in fact quite elementary.

A more subtle thing Morse proved is what happens when M^t crosses over a critical point. In this cased he showed that, if choose ε so that $\mathrm{Crit}(f) \cap (t, t+\varepsilon)$ has exactly one point (which we can always do by hypothesis) then $M^{t+\varepsilon}$ can be obtained by adjoining a k-cell to M^t where k happens to be the index of the unique critical point in $(t, t+\varepsilon)$.
In summary

Theorem 1.3. *[7] Let $\varepsilon > 0$ so that*

$$Crit(f) \cap (t, t+\varepsilon) = \emptyset$$

Then M^t and $M^{t+\varepsilon}$ are difeomorphic.

Theorem 1.4. *[7] Let $\varepsilon > 0$ so that*

$$Crit(f) \cap (t, t+\varepsilon) = \{x_0\}$$

Let k be the index of x_0. Then
Then $M^{t+\varepsilon}$ is homotopy equivalent to $M^t \cup_\phi (D^{n-k} \times D^k)$ where D^k is the k-disk and ϕ is and adjunction map.

Example 1.1. *One classical example of the above theorems is how to use a Morse function to obtain the decomposition of the Torus:*

Corollary 1.1. *[8] Any compact manifold of dimension n is homotopy equivalent to a space such as the following:*

$$D^n \cup D^{n-k_1} \times D^{k_1} \cup \cdots \cup D^{n-k_m} \times D^{k_m}$$

Theorem 1.5 (Morse inequalities)**.** *Let M^n be a compact manifold, and b_i it's betti numbers. Let f be a Morse function on M. And let λ_k be the number of k-index critical points of f. Then we have that the so called weak Morse inequality holds:*

$$\lambda_i \geq b_i, \quad \forall i \in \mathbb{N}$$

As well as the so called strong Morse inequality:

$$\sum_{i=0}^{j} (-1)^i \lambda_i \geq \sum_{i=0}^{j} (-1)^i b_i, \quad 0 \leq j \leq n$$

In the particular where $i = n(= dim(M))$ we have equality:

$$\mathcal{X}(M) = \sum_{i=0}^{n} (-1)^i \lambda_i = \sum_{i=0}^{n} (-1)^i b_i$$

Where $\mathcal{X}(M)$ is the Euler-Poincaré characteristic.

1.1.2 An application of Morse theory

In this subsection we describe how to use Morse theory to classify closed 1-manifolds. In particular, we show the well-know theorem:

Theorem 1.6. *A closed connected 1-manifold is diffeomorphic to $\mathbb{S}^1$.*

We briefly mention that it can be used to derive the usual classification of 2-manifolds.

1.1.3 The Morse-Smale complex

Definition 1.2 (Differential complex)**.** We say that a collection of indexed abelian groups $\{C_i\}_{i=0}^{\infty}$ is a differential complex if it has a linear operator: $\delta_i : C_{i+1} \to C_i$ so that $\delta_i \circ \delta_{i+1} = 0$.
Usually, we shall consider $\delta : C \to C$ where $C = \oplus C_i$ and δ is defined as the unique linear operator that verifies: $\delta|_{C_i} = \delta_i$.
And so, we may write that $\{C_i\}_{i=0}^{\infty}$ is a differential complex if and only if $\delta^2 = 0$.
Finally, given C and D two differential compexes, then we say that $f : C \to D$ is a chain map if $f \circ \delta_C = \delta_D \circ f$.

Morse functions give a natural complex which computes the homology of the manifold.
We define the chain groups to be:

$$C_k(f) \stackrel{\text{def}}{=} \left\{ \sum_{c \in \mathrm{Crit}_k(f)} ac : a \in \mathbb{Z}_2 \right\}$$

And we define the differential operator ∂ to be:

$$\partial a = \sum_{b \in \mathrm{Crit}_{k-1}(f)} n(a, b) b$$

Where $n(a, b)$ is the number of trajectories flowing from a to b through the gradient flow. This number is finite, and we have that $\partial^2 = 0$.

1.1.4 Discrete Morse Theory

We can consider Morse theory on simplicial complexes or simplicial sets, in this case, instead of having critical points we have critical faces instead of critical points.
This idea is as was previously, we consider the growing sets $f^{-1}(\infty, t]$ and we observe for which values of t the type of homotopy changes.

Definition 1.3 (Discrete Morse functions)**.** Let $\mathcal{K}$ be a simplicial complex.
Then we denote $\alpha_i^{(p)} \in \mathcal{K}$ a p-dimensional face of $\mathcal{K}$ and we denote $\beta < \alpha$ is β is subface of α.
Then we say that $f : \mathcal{K} \to \mathbb{R}$ is that a discrete Morse function if and only if f assigns higher values to higher dimensional simplices with few exceptions. Formally, if f satifies both of the following definitions:

$$\begin{cases} \#\{\beta^{(p+1)} > \alpha | f(\beta) \leq f(\alpha)\} & \leq 1 \\ \#\{\gamma^{(p-1)} < \alpha | f(\gamma) \geq f(\alpha)\} & \leq 1 \end{cases}$$

Finally, we say that $\alpha^{(p)} \in \mathcal{K}$ is a critical simplex if and only if:

$$\begin{cases} \#\{\beta^{(p+1)} > \alpha | f(\beta) \leq f(\alpha)\} & = 0 \\ \#\{\gamma^{(p-1)} < \alpha | f(\gamma) \geq f(\alpha)\} & = 0 \end{cases}$$

In general, we will have that always one of the above sets will be empty, that is:

Lemma 1.1. *Let $\mathcal{K}$ be a simplicial complex, and let $\alpha^{(p)} \in \mathcal{K}$ be a non-critical compolex, then either*

$$\#\{\beta^{(p+1)} > \alpha | f(\beta) \leq f(\alpha)\} = 0$$

or

$$\#\{\gamma^{(p-1)} < \alpha | f(\gamma) \geq f(\alpha)\} = 0$$

And we have a similar result as in the smooth case, that is the discrete Morse function describes the homotopy type of the complex:

Theorem 1.7. *Let $\mathcal{K}$ be a simplicial complex and f a discrete Morse function on it. Then $\mathcal{K}$ is homotopy equivalent to another simplicial complex which has exactly one complex for every critical complex of $\mathcal{K}$.*

Another important similarity with the smooth case is that the standart Morse inequalities hold:

Theorem 1.8 (Morse inequalities)**.** *Let $\mathcal{K}$ be a simplicial complex, and b_i it's betti numbers. Let f be a Morse function on M. And let λ_k be the number of k-dimensional critical simplices. Then everything holds exactly as in theorem 1.5.*
In particular, all of the following holds:

$$\lambda_i \geq b_i, \quad \forall i \in \mathbb{N}$$

$$\sum_{i=0}^{j} (-1)^i \lambda_i \geq \sum_{i=0}^{j} (-1)^i b_i, \quad 0 \leq j \leq n$$

$$\mathcal{X}(M) = \sum_{i=0}^{n} (-1)^i \lambda_i = \sum_{i=0}^{n} (-1)^i b_i$$

Where in this case n is the dimension of $\mathcal{K}$, which is the maximum dimension of the simplices of $\mathcal{K}$.

Add handle decomposition theorem

Theorem 1.9 (Handle decomposition)**.**

1.1.5 Stability of Morse-Smale complexes

1.2 Kernel estimators and density functions

1.2.1 Kernel regresssion

Definition 1.4 (Kernel functino)**.** Let X be a probability space, then we say that $K : X \to [0, \infty)$ is a density function if:

1. K is integrable.
2. $\int_{\mathbb{R}} K = 1$.
3. $K(x) = K(-x), \forall x \in \mathbb{R}$.

It's common to refer to the maxima of kernel regression as their modes.

There are several commonly used kernel functions, we list some of them below:

Example 1.2.

Name	$K(x)$
Epanechnikov	$\frac{3}{4}(1 - x^2)1_{[-1,1]}$
Gaussian	$\frac{1}{\sqrt{2}} e^{-\frac{1}{2}x^2}$
Sigmoid	$\frac{2}{\pi} \frac{1}{e^x - e^{-x}}$

1.3 Brief review of statistical tests

Canonical form

2.1 Ordered complex and canonical form

Definition 2.1 ($\mathbb{R}$-Filtered complex)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a complex. A $\mathbb{R}$-filtration (sometimes we will simply say a filtration), on $\{C_k\}_{k=0}^{\infty}$ is an increasing sequence of real numbers, $\{r_i\}_{i=0}^{n}$ so that for each r_i there as associated $F_{\leq r_i}C_k \subset C_k$ for every k that satisfies:

$$\{0\} \subset F_{\leq r_0}C_k \subset F_{\leq r_k}C_k \subset \cdots \subset F_{\leq r_n}C_k = C_k$$

As we well see, there are a lot of natural circumstances on which a filtration might arise. For example, the singular chain comples of a CW-complex are naturally filtered by its skeleton.
In general, we may filter a complex by any partial ordered set, $\mathbb{P}$, by simply requiring $F_{\leq r_0}C_k \subset F_{\leq r_1}C_k$ for any $r_0 \leq r_1 \in \mathbb{P}$.
One example for this might be the following: Suppose that we have a complex that has two different $\mathbb{R}$-filtrations, then we can combine them into a single $\mathbb{R}^2$-filtration.
There are more structures we can put on a complex:

Definition 2.2 (Complex with ordered generators)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a filtered chain complex, with some basis $\{e_j^{(i)}\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ has ordered generators if the generatros are ordered as such: $e_k^{(i)} < e_l^{(j)}$ if $k = l$ or if $k = l$ and $i < j$.
Notice we do not compare generators that live on different chain groups.

Remark 1. *A filtrated complex has a natural order on the generators compatible with the filtration.*

Definition 2.3 (Canonical form)**.** Let $\{C_k\}_{k=0}^{\infty}$ be a chain complex, with some basis $\{e_k^{(i)}\}$. Then we say that $\{C_k\}_{k=0}^{\infty}$ is in canonical form if,

1. $\partial e_k^{(i)}$ is either 0 or another generator.
2. If $\partial e_k^{(i)} = \partial e_l^{(i)}$, then $e_k^{(i)} = e_l^{(i)}$.

Remark 2. *An equivalent formulation of the canonical form is that we can find a basis S of $\{C_k\}_{k=0}^{\infty}$ so that S can be separated into:*

1. *S_H: Generators of the homology of the complex.*
2. *S_{birth}: Births, that is, elements whose boundary is 0, but get killed in homology by an element of higher degree.*
3. *S_{death}: Deaths, elements whose boundary is another generator.*
4. *∂ is a bijection between S_{death} and S_{birth}.*

That is:

$$S = S_{birth} \sqcup S_{death} \sqcup S_H$$

Theorem 2.1. *[1] Every filtered chain complex can be reduced to one in canonical form by an upper-triangular change of basis which preserves the filtration.*
Furthermore, the canonical form is unique.

Canonical form allows us to classify graded differential complexes. In particular, by will say two complexes are equivalent if there is a bijective chain map preserving the filtration between the two complexes. More precisely:

Definition 2.4 (Equivalent complexes)**.** Let C and D be two filtered complex as such:

$$\{0\} \subset F_{\leq r_0} C_k \subset F_{\leq r_k} C_k \subset \cdots \subset F_{\leq r_n} C_k = C_k$$
$$\{0\} \subset F_{\leq r_0} D_k \subset F_{\leq r_k} D_k \subset \cdots \subset F_{\leq r_n} D_k = D_k$$

Then we say that they are equivalent if and only if there is chain map $f : C \to D$ so that

$$f\left(F_{\leq r_k} C_k\right) \subset F_{\leq r_k} D_k, \quad \forall k, r_k$$

Corollary 2.1. *Two filtered complexes are equivalent if and only if they have the same canonical form.*

Corollary 2.2. *Category of filtered complexes is semi-simple. That is, any filtered complex, (or one with an ordered basis) can be expressed as a direct sum of 1 dimensional complexes with trivial differential and 2 dimensional complexes with trivial homology.*

Proof. Because any complex can we brought to canonical form, we can find a basis of the complex in the form of Remark 2. Then $\mathrm{Span}(S_H)$ splits as a sum of 1 dimensional complexes with trivial differential, since the differential is trivial on S_H. Similarly, the birth and death pairs splits into 2 dimensional complexes with one generator killing the other in homology.

QED

2.2 Persistence complexes

Definition 2.5 (Persistent complex)**.** A persistence complex is a sequence of vector spaces $\{V_i\}_{0 \leq i \leq n}$ and homomorphisms $f_i : V_i \to V_{i+1}$. That is, a diagram as follows:

$$V_0 \xrightarrow{f_0} V_1 \longrightarrow \cdots \xrightarrow{f_{n-1}} V_n$$

We denote $f_i^j = f_j \circ \cdots f_{i+1} \circ f_i$ whith $i < j$. Persistence diagrams may be of arbitrary cardinal, as long as the indexed is set is well-ordered.
But we shall only considered them when they are finite or countable.
We further define it's graded Betti number as:

$$b_i^j = \begin{cases} \dim(V_i) & \text{if} \quad i = j \\ \mathrm{rank}(f_i^j) & \text{if} \quad i < j \\ 0 & \text{if} \quad i > j \end{cases}$$

It is worth mentioning that we usually consider the maps f_i as inclusions and the spaces V_i as chain groups. As such, we can sometimes drop the vector space assumption and take modules over a PID. For example, it is common to consider $\mathbb{Z}$ instead of $\mathbb{K}$. We call this constructions persistent modules. Precisely we define:

Definition 2.6. (Persistence module) A persistence complex is a sequence of modules over a PID $\{M_i\}_{0 \leq i \leq n}$ and homomophisms $f_i : M_i \to M_{i+1}$. With the rest of the definitions as above.

We give examples of useful persistence diagrams in 2.2.3.

Example 2.1 (Interval module)**.** *We define the interval module for $i < j$ to be the module where all maps are either the trivial ones, being the identity whenever possible and the zero map otherwise and the V_k to be:*

$$V_k = \begin{cases} \mathbb{K} & \text{if } i \leq k \leq j \\ 0 & \text{otherwise} \end{cases}$$

We denote it by $m(i,j)$ and we define it's associated interval to be $[i,j)$.

Proposition 2.1. *[4] The category of countable persistence complex is equivalent to the one of graded modules over $\mathbb{K}[t]$*

Proposition 2.2. *[4] Finite persistence diagrams are classified by their graded Betti numbers.*

Proposition 2.3. *[4] Every persistence module is the direct sum of interval modules. Furthermore, the multiplicity of each interval module is an invariant of the complex.*
This intervals are called the persistence intervals of a given persistence module.

One important observation about the previous proposition, is that a complex in canonical form gives a clear idea of what the persistence diagrams decomposition is. In particular, the beginning of a interval indicates a birth and the end of an interval indicates a death. And so, an element represent a non-trivial cohomology class if and only if it belongs to an interval of the form $[\alpha, \infty)$

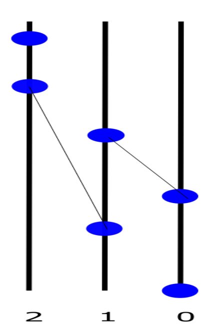
2.2.1 Representations of the persistence complexes

Firstofall, we can represent canonical form as in the following example:
Clearly, from any complex in canonical form we can construct an associated persistent diagram by reversing the above process. In the above example we ignored the filtration of the complex, but we consider that the diagram is filtered by height. Therefore:
We have two natural ways to represent the persistent homology of a filtered complex, one is persistent barcodes and the other is persistent

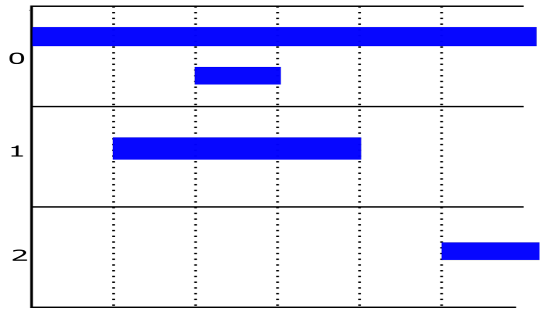
Example 2.2. *Consider the following complex:*

1. $C_0 = \{e_0^{(0)}, e_1^{(0)}\}$
2. $C_1 = \{e_0^{(1)}, e_1^{(1)}\}$
3. $C_0 = \{e_0^{(2)}, e_1^{(2)}\}$
4. $\partial e_1^{(1)} = e_0^{(1)}$, $\partial e_0^{(2)} = e_0^{(1)}$ *and* 0 *on the rest of the generators.*

And obviosuly $S_{birth} = \{e_0^{(1)}, e_0^{(1)}\}$, $S_{death} = \{e_1^{(1)}, e_0^{(2)}\}$, $S_H = \{e_0^{(0)}, e_1^{(2)}\}$ Then it's persistence diagram would be the following:



And we have that it's persistence barcode would be:



Notice that the barcode gives the an obvious idea of what the interval decomposition is.
We will explore further representations in chapter 3.

2.2.2 Comparing persistence complexes

Bottleneck distance

Definition 2.7 (Bottleneck distance)**.**

2.2.3 Some important examples of persistent complexes

Definition 2.8 (Discrete morse complex)**.** Let $f : M \to \mathbb{R}$ be a morse function from a compact manifold with finitely many critical points $t_0 < \cdots < t_n$. Then the following diagram is a persistence complex:

$$H^{dR}(f^{-1}(-\infty, t_0)) \xrightarrow{\subseteq} H^{dR}(f^{-1}(-\infty, t_1)) \xrightarrow{\subseteq} \cdots \xrightarrow{\subseteq} H^{dR}(f^{-1}(-\infty, t_n))$$

Where H^{dR} is the de Rham complex and $\subseteq$ is homomorphism induced by the natural inclusion.

Definition 2.9 (Čech Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define the Čech complex of level ϵ as the following simplicial set:

$$\tilde{C}_\epsilon(X) = \{\sigma \subset X : \bigcap_{y \in \sigma} B(y, \epsilon) \neq \emptyset\}$$

Then we define the Čech complex at level ε to be $\tilde{C}_\epsilon$.

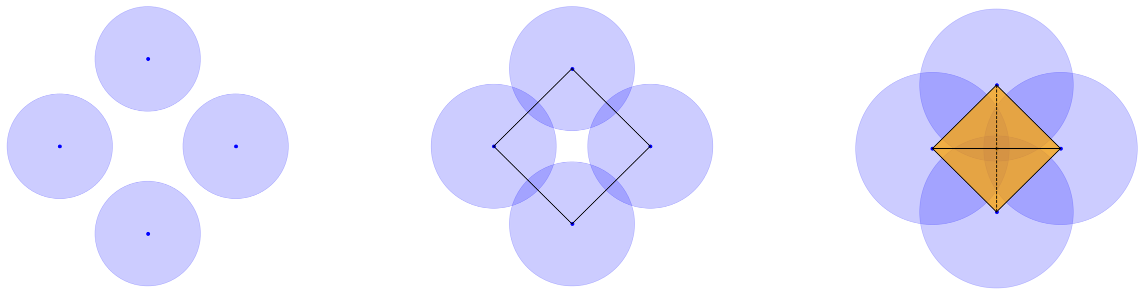
The Čech has one key advantage, it's homotopy type is easy to predict. In particular, given a compact manifold M of $\mathbb{R}$ we can pick a discrete subset X and an appropiate ε so that $\tilde{C}_\varepsilon(X)$ has the homotopy type of M.
This a consequence of the following theorem (and the equivalence between Čech cohomology and De Rham cohomology on manifolds).

Theorem 2.2. *Let X and $\varepsilon > 0$, then $\tilde{C}_\varepsilon(X)$ is homotopy equivalent to the union of the ε-balls centered around the point of the complex. Explicitly:*

$$\tilde{C}_\varepsilon(X) \overset{h\text{-}eq}{=\!=} \bigcup_{x \text{ vertex in } \tilde{C}_\varepsilon(X)} B(x, \varepsilon)$$

Example 2.3. *In the following example we illustrate the evolution of the Čech complex as ε grows:*

However, in practice the Čech complex is very hard to calculate, notice that as the dimension of $\mathbb{R}^n$ increases it would be necessary to calculate the intersection of $n-1$-dimensional spheres, which is it a computationally complex problem.
One solution to this is using the Vietoris-Rips complex.

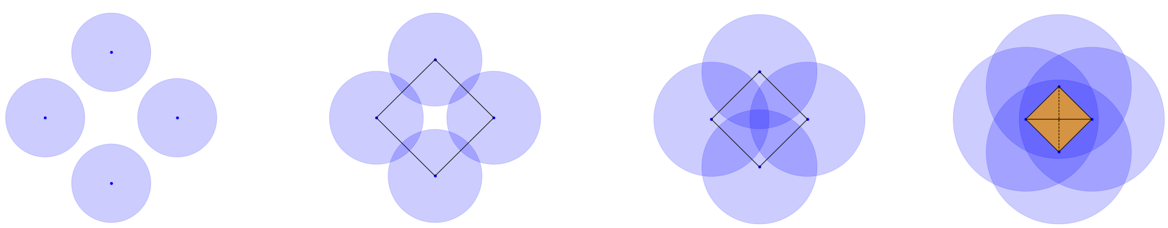


Čech complex for $\varepsilon < \frac{\sqrt{2}}{2}$ Čech complex for $\frac{\sqrt{2}}{2} < \varepsilon < 1$ Čech complex for $\varepsilon > 1$

Definition 2.10 (Vietoris-Rips Complex)**.** Let $X \subset \mathbb{R}^n$ be a discrete subset. Then, for $d > 0$, we define the Čechcomplex of level ϵ as the following simplicial set:

$$VR_\epsilon(X) = \{\sigma \subset X : \mathrm{diam}(\sigma) \leq \epsilon\}$$

Then we define the Vietoris-Rips complex at level ε to be VR_ϵ.
There are many softwares that can be used to compute the Čech and Vietoris-Rips complexes, most of them rely on bringing the complex to canonical form.
The following example shows how the Čech and Vietories-Rips complex might differ: (compare to example 2.3)



Vietoris-Rips complex for $\varepsilon < \frac{\sqrt{2}}{2}$ Vietoris-Rips complex for $\frac{\sqrt{2}}{2} < \varepsilon < 1$ Vietoris-Rips complex for $\varepsilon > 1$ Vietoris-Rips complex for $\varepsilon > 2$

Example 2.4.

Definition 2.11. Let (X, d) be a metric space, we say it is augmented if it comes equiped with a function $f : X \to \mathbb{R}$.

We will say that an augmented metric space is injective if f is. And we will be interested in the case where X is finite.
Notice that in an augmented metric space, we can combine the Vietoris-Rips (or Čech) complex with a natural filtration, given by f, to give a bifiltrated complex:

Definition 2.12 (Bifiltrated Vietoris-Rips complex)**.** Let $\mathcal{X} = (X, d, f)$ be an augmented metric space, let $X_\sigma = f^{-1}(-\infty, \sigma]$ we define the bifiltrated Vietoris-Rips complex as the complex $\{VR^\sigma_\epsilon\}_{\epsilon>0,\sigma}$ with:

$$VR^\sigma_\epsilon = VR_\epsilon(X_\sigma)$$

2.3 Comparison of the Čech and Vietoris-Rips complex

One great advantage of the Čech complex is that it's homotopy type is highly predictable. In particular, we have the classical Čech nerve theorem:
This means we can always approximate the homology of a compact manifold with the Čech complex using a relatively low amount of points. In the case of the Vietoris-Rips complex things are different.
However, there is a natural relation between these two complexes:

Proposition 2.4. *The Čech complex is always a subcomplex of the Rips complex. We also have a partial converse:* $VR_{\varepsilon/2}(X)$ *is a subcomplex of* $\check{C}_{\varepsilon}(X)$
More precisely:

$$\check{C}_{\varepsilon/2}(X) \hookrightarrow VR_{\varepsilon/2}(X) \hookrightarrow \check{C}_{\varepsilon}(X) \hookrightarrow VR_{\varepsilon}(X)$$

In general, two complexes can be compared using the so-called something distance. This is very useful, because it allows to define a notion of convergence on persistence diagrams.

2.4 Motivation

2.5 Example of canonical form

The proof of theorem 2.1 gives an algorithmic way to bring a complex into canonical form, we now expose how this methods works with an example. The general case should be clear afterwards.

Elder-rule Staircodes

In this chapter we will introduce the so-called Elder-rule staircodes, this is technique that will allows to confidently identify $H_0(M)$ combining the information given by a Morse function f and a metric function d_X.
Combining f and d_X in a barcode-like summary. Only that in this case we will have a bigraduation.
The function d_X need not be the euclidean distance, it could be the diffusion distance induced by a graph and f need not be the standard height function. We could also take it to be, for example, the discrete Ricci curvature.

3.1 Staircodes

We define the ϵ-Vietories-Rips complex, $VR^{\sigma}_{\epsilon}(X)$ as we did in the previous chapter.
By taking a line $\mathcal{L}$ with strictly positive slope: $\mathcal{L} : ax + b, a > 0$ we can reduced the bifiltration of $VR^{\sigma}_{\epsilon}(X)$ to a single filtrated complex by setting $F_\delta(X) = VR^{a\delta+b}_{\delta}(X)$.
We call the barcode of this complex, $F_\delta(X)$, the k-th fibered complexk-th fibered complex barcode. And we shall denote it by $M|_L$.
We will need another definition:

Definition 3.1 (ϵ-chain)**.** A sequence $(x_i)_1^l$ is an ϵ-chain if $d(x_i, x_{i+1}) \leq \epsilon, \forall 1 \leq \epsilon < l$
We define $[x]^{\epsilon}_{\sigma}$ to be the points of X_σ that can be reached by an ϵ-chain starting from x.
Furthermore, we say that x is older than y if and only if $f(x) \leq f(x+)$. Clearly, this is an order relationship.
Finally, we call the staircode of x to be:

$$I_x = \{\sigma, \epsilon \in \mathbb{R} | x \in X_\sigma \text{ x is the oldest in } X_\sigma\}$$

Notice this defintion might not we well defined in the case f is not injective. In the non-injective case we simply choose an order that is compatible with the older-order.
The set $\mathcal{I} = \{I_x\}_{x \in X}$ the elder rule staircode, or ER-staircode for short.

Insert some images here
When we instersect a line $\mathcal{L} : ax + b, a > 0$ with each I_x we get a barcode, which turns out coincides with the 0-homology of the space X:

Theorem 3.1. *[2] Let $M = H_0(VR^{\cdot}_{\cdot}(X))$ and $\mathcal{I}_X$ it's ER-staircode. And let $\mathcal{L}$ be the line defined by:*
$\mathcal{L} : ax + b, a > 0$.
Then the barcodes of $M|_L$ are precisely $\{\{L \cap I_x | x \in X\}\}$
Where by te use of $\{\{\}\}$ we mean we count occurences of $L \cap I_x$ with multiplicity.

Morse functions on point clouds

TO-DO: It seems that there are some properties about Cazals' algorithm that could be proved but hasn't been done so far.

In this chapter we introduce a quite succesful technique introduced by: F. Cazals, A.Roth, C. Robert, M. Christian in [3].

This technique focuses on discrete augmented metric spaces. In particular, suppose we have a compact smooth manifold M equipped with a Morse function $f : M \to \mathbb{R}$.

Usually this Morse function f will be the height function.

From this manifold M we sample a finite amount of points $X = \{x_i\}_{i=0}^{n}$ and we record the value of f on X.

As usual, we investigate what sort of topological information we can recover from this discrete sample.

The basic idea is the following: Suppose we are in dimension n, then we iterate n times. On each iteration we build a graph around with vertices in X. Our hope is that as we build this graph it will be natural to identify the critical points of f as well as their indices.

4.1 Pseudo-gradient graphs

The first thing we would like to do is recover the gradient of the function f. For this purpose we create a graph using X as its base set and we define a flow on the graph by moving from one point to the point connected with it where the morse function f takes the smallest value.

Definition 4.1 (NNG graph)**.** Given X a discrete set, we define $G^{-}(X)$ it's descending pseudo-gradient graph as the nearest neighbour graph of X.

That is the graph where the edges $\{x_i, x_j\}$ are precisely where either $d(x_i, x_j) = d(x_i, X)$ or $d(x_i, x_j) = d(x_j, X)$.

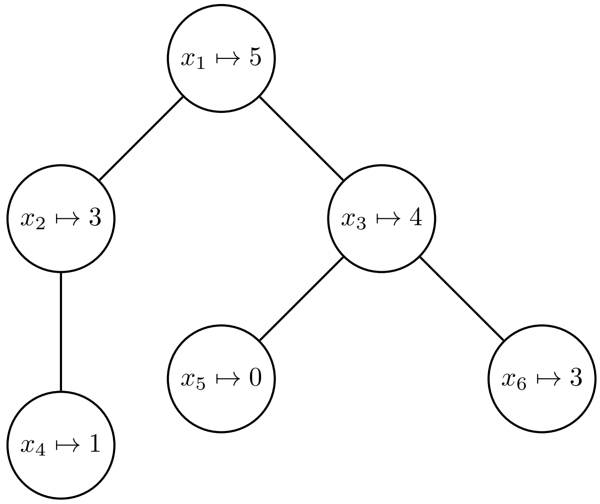
Notice that the binary relation $x_i \sim x_j$ if and only if x_j is the closest vertex to x_i is not symmetric.

We denote as $\omega^{-} : G \to G$ the application that maps $p \in X$ to the result of iteratively following the point with that its joined to p and that minimises the value of f.

We define α^{+} in an analogous manner, by following the direction of maximum growth of f.

We call $\omega^{-}(p)$ the ω-limit of p. Analogous for α-limits.

Example 4.1. *For example, let's consider the following graph, where we denote the value of f in x_i by : $f(x_i)$:*



Then $\omega^-(x_1) = x_4$. So ω maps to local minimums, but not necessarily to absolute ones, even when restricting to the same connected component.

4.2 Brief description of the algorithm

Mising examples

With this in mind we can describe what the algorithm is. Each iteration of the algorithm is separated in 6 steps.

As mentioned at the beginning of the chapter we try to progressively build a graph around some discrete set $X \subset M$. We start by setting $T[0]$ to be the NNG graph of X.

We also set the critical points of index 0 (that is the minimums) σ_0^k to be the image under ω^- of $T[0]$. We will need one final definition:

Definition 4.2 (Bifurcating lower link)**.** Let X be a finite set with a Morse function f, and G a graph on X. Given $x \in X$, we call lower link of x and denote $L^-|_G$ to the set:

$$\{y \in X : \{x, y\} \text{ is and edge of } G \text{ and } f(y) \leq f(x)\}$$

We call the bifurcating lower link of x and denote $L^{-\rightsquigarrow}|_G$

Now we described how the algorithm works:

If it were to be done in practice, before starting we would need to compute the NNG, the pseudo-gradient graph and the local minima and maxima of the psedu-gradient function.

As mentioned before, if n is the dimension of M, we iterate n times. From now on k is the iteration count. At the beginning of iteration k we may assume that the following are known:

- The critical points of index $k-1$.
- The global thickening of order $k-1$: $T[k-1]$.
- The set $T[k-1]$ has been sorted by increasing value of the function f.

On each iteration we do the following:

- **Step 1** In the first step we idenitify two important sets:
  - **Bifurcating samples**: $B[k] = \{x \in X :$ Vertices of $T[k-1]$ so that every two differents elements from the lower link flow to different critical points $\}$

– **Confluence samples**: $B[k] = \{x \in X :$ Vertices of $T[k-1]$ so that every element from the lower link flow to different critical points $\}$
Confluence samples usually indicate that we have reduced several flow lines to a single point. That is, confluence samples can be eliminated by increasing the number of samples in a neighbourhood of the confluence sample.
For example in example 4.1 both x_1 and x_3 are confluence samples.

- **Step 2** In this step we use the bifurcating samples to enlarge $T[k-1]$

Definition 4.3 (Anchor and Hooks)**.** We call a k-hook any set of critical points of index $k-1$.

Given $p \in X$ we call the anchor of p the maximal hook that includes $\omega^-(p)$ and $\omega^-(L^{-\rightsquigarrow}(p))$

Definition 4.4 (Local thickenings)**.** We define $T(A^{(k)})$ as the set of all elements from $B[k]$ having the same anchor and we call it the local thickening of the anchor $A^{(k)}$.

The lower boundary of $T(A^{(k)})$ is defined as the elemenits of $T(A^{(k)})$ that have a lower link lying in $T[k-1]$. We denote the lower boundary by $\partial_l T(A^{(k)})$.

Local thickening have an associated thickness:

Definition 4.5 (Open local thickening)**.** Let $p \in X$ is called an interior node if every other point connected to it by $B[k]$ has the same anchor as p.

A local thickening is open if it has no interior node.

- **Step 3** In this step we further split $B[k]$ into flow lines.

Definition 4.6 (Monotonic section)**.** Let α and ω be respectively a local maximum and minimun of the discrete flow on $T(A^{(k)})$. Then the monotonic section, denoted $M^{(\alpha,\omega)}T(A^{(k)})$, to be the sub-graph of $T(A^{(k)})$ induced by the vertices whose α-limit is α and whose ω-limit is ω.(Recall definition 4.1).

One important type of monotonic section are linkers:

Definition 4.7 (Linker)**.** A monotonic section M, of a local thickening T, is called a linker if: If every vertex $p \in M - \partial_l T$ then $p \in C[k]$.

Definition 4.8 (Incidence graph of monotonic sections)**.** We say that two monotonic sections are incident if the union of their vertices in $B[k]$ is connected.

The are interior incident if they share the same anchor, and exterior otherwise.

We define the incidence graph as:

$G^k_M =$ The unique graph that contains one node per monotonic section which is not a linker, and two incident monot

- **Step 4**

In this step the objecitive is to remove some edges from G^k_M. We shall call this refined grap $\overline{G}^k_M$.

To do this we use three different partial order relation.

Definition 4.9. – $\prec_i$ (i for inclusion):

$$M_i(A^k_u) \prec_i M_j(A^k_v) \iff A^k_u \subset A^k_v$$

– $\prec_h$ (h for height):

$$M_i(A^k_u) \prec_h M_j(A^k_v) \iff \exists C \text{ monotonic section with extrema } (\alpha_i, \omega_j) \text{ and } C \subset T[k-1] \text{ is the graph induced by}$$

– $\prec_{ih}$ which is defined in the fashion you would expect:

$$M_i(A^k_u) \prec_{ih} M_j(A^k_v) \iff M_i(A^k_u) \prec_i M_j(A^k_v) \text{ and } M_i(A^k_u) \prec_h M_j(A^k_v)$$

- Given $u, v \in G_M^k$ we say $u||_i v$ if u and v are not comparable in the order $\prec_i$. Analogously for $\prec_h$ and $\prec_{ih}$.

This allows for the classification of the edges G_M^k: Let u, v be two edges of G_M^k, then the pair (u, v) is:

1. Canonical if $u \prec_{ih} v$.
2. Compoiste if $u \prec_i v$ and $v \prec_h u$.
3. H-non-I if $u||_i v$ and $u \prec_h v$.
4. I-non-H if $u||_h v$ and $u \prec_i v$.
5. Non-hierarchical if $u||_i v$ and $u||hv$.

Insert examples of edges

What does it mean for each edge to fall under one specific category? **Composite** edges:

H-non-I edges: **I-non-H** edges: **Non hierarchical** edges:

- **Step 5** Finally in this steps we begin to identify some candidates for critical points. The algorithm takes advantage of the fact that critical points whose index differs by 1 are linked by homogeneous flow manifold. And identifies this lines with anchors.

  Definition 4.10 (Critical point)**.** A critical point is a local minimum of a minimal montonic section of the graph $\overline{G}_M^k$.

  We will define the stable manifold of a critical point σ^k to be:

$$W^s(\sigma^{k-1}) = \{p \in C(B[k]) \cup M_l(A^k)|_{\omega_|^- T[k-1]}(p) = \sigma^{k-1}\}$$

  where $M_l(A^k)$ are linkers and $C(B[k])$ is the complementary of $V[T[k-1]]$.

- **Step 6** In this final step we enlarge $T[k-1]$ to $T[k]$ and get ready for the next iteration. Clearly this step isn't needed in the last iteration.

  $T[k]$ is defined in the following way:

  - $T[k]$ vertex set:
$$V[T[k]] = \cup_j V[M_i(A_j^k)], \text{ with } M_i(A_j^k) \in G_M^k$$
  - $T[k]$ edge set: $E[T[k]] = \cup_j E[M_i(A_j^k)] \cup C$, where C is:
$$C = \{e = (u, v) \in T[k-1] | u \in M_i(A_u^k), v \in M_i(A_v^k), M_i(A_v^k) \prec_h M_i(A_v^k\}$$

4.3 Some observations on the algorithm

To simplify, we give a brief overview of the steps taken by the algorithm:
First we precompute ω and the NNG, as well as the local extrema of ω.

1. Identify $B[k]$ and $C[k]$ using $T[k-1]$ and $\{\sigma^{k-1}\}$.
2. Build an equivalence realtion, and work out it's quotient set: Local thickenings.
3. Decompose each local thickening into monotonic section.
   - Build the graph incidence: G_M^k.
4. Filter G_M^k using the relation $\prec_{ih}$.
5. Identify the set $\{\sigma^k\}$.
   - Compute the stable and unstable manifold of the pairs (σ^k, σ^{k-1})
6. Calculate $T[k]$ so you can keep iterating.

The algorithm makes some unusual (or maybe original) constructions. In this section we explore their raison d'être.

4.3.1 The point of the thickenings

Suppose we had two points in M, p_i, p_j and q, and we have: $W(p, q_i) \cap W(q_j, p) = \{p\}$. However, it is not possible that the stable manifold of a critical point be reduced to a singleton.

4.3.2 An observation about Step 3

Edit this part

- The vertex set of monotonic sections is a covering of th elocal thickening
- The vertex set of G_M^k is a subset of the set of monotonic sections fiund at k-th iteration.
- The edge set of the union of monotonic section is included in the edge set of the local thickening.

4.4 Comparison with the previous algorithms

Statistical Inference Using the Morse-Smale Complex

It is natural to think that be could treat the problem of aproximating the homology of a manifold from a finite sample from a stastical point of view. That is what we will try to do in this chapter. It will be based on the findings of the paper: [9].
The main idea is to use a decomposition similar to the one given in 1.9.
As in the previous sections we assume we have a closed d-manifold $M^d \subset \mathbb{R}^n$ and we have a Morse function on it $f : M^d \to \mathbb{R}$.
And we call the ω-limit of $x \in M$ to: $\omega^-(x) = \lim_{t\to\infty} \phi^t(x)$ where ϕ^t is the flow generated by ∇f.
We define the descendings k-manifolds as the inverse image by ω^- of $Crit_{d-k}(f)$. We recall this are the k-index critical points of f. In this case, use this Morse function usually is a density function. Formally:

$$D_k = (\omega^-)^{-1}(Crit_{d-k})$$

Since $Crit_k$ is finite because M is closed we can index it: $Crit_k = \{c_{k,1}, \dots, c_{k,m_k}\}$ and we can define analogously:

$$D_{k,j} = (\omega^-)^{-1}(C_{d-k,j})$$

Obviously, $D_k = \coprod D_{k,j}$. (By $\coprod$ we mean disjoint union). And similarly $M = \coprod_i D_i$.
We define the ascending d-manifold in the same way except replacing ϕ^t by the flow generated by $-\nabla f$ and we denote them A_k.

Remark 3. *We call D_k and A_k descending and ascending manifolds to mantain coherence with the article [9]. But in the literature it usually used a very similar concept called stable and unstable manifolds (for example see [6]). And they are denoted $W^s(p,q)$ and $W^u(p,q)$. They key difference is that stable and unstable manifolds measure the flow between two critical points, not the total incoming (resp outcoming) flow to a critical point. Needless to say a descending (ascending) manifold is the disjoint union of stable (unstable) manifolds. At least in the compact case.*

Definition 5.1 (d-cell)**.** In this chapter we will refer to a d-cell to the intersection of a d-descending manifold and 0-ascending manifolds.

Note that k-descending manifolds for $k < d$ and l-ascending manifolds for $l > 0$ have measure 0. Therefore, M can be decomposed as $M = E_1 \cup \cdots E_K \cup N$ where E_i are d-cells and N is null set.
We refer to this decomposition as the cell decomposition of M.

5.1 Estimating the Morse function

Let's take a finite sample from M: $\mathcal{X} = \{X_1, \dots, X_m\}$. If f is a density function we take the sample $\mathcal{X}$ following the distribution associated to f (that is the one with PDF $p = f/\int_{\mathbb{R}} f$).
However, in previous chapter we have supposed we only know $f|_{\mathcal{X}}$, so we must construct $\hat{f}$ an estimation of f.
This is usually done by the following estimation:

Definition 5.2 (KDE)**.** Given f a Morse function of which only $f|_{\mathcal{X}}$ is known, where $\mathcal{X}$ is a finite sample on a Manifold M. We call the kernel density estimator or KDE and denote $\hat{f}$ to:

$$\hat{f} = \frac{1}{mh^d}\sum_{i=1}^{m} K\left(\frac{x - X_i}{h}\right)$$

We could choose K and h in different ways. We just require that K is a smooth kernel function, as we described in 1.4 and $h > 0$. Since $\hat{f}$ is clearly $\mathcal{C}^\infty$ we can consider $\nabla\hat{f}$ and compute $\hat{\omega}^-$ accordingly.

5.2 Mode clustering

The key point of mode clustering is to use descending d-manifolds to partition M into "clusters".
Let $\mathcal{X}$ be a sample as before and let $\{m_1, \ldots, m_l\}$ be the local maxima (or modes) of $\hat{f}$. Then we partition $\mathcal{X} = \coprod \mathcal{X}_i$ where

$$\mathcal{X}_i = \{X_j \in \mathcal{X} | \omega^-(X_j) = m_i\}$$

5.2.1 Analysis of mode clustering

5.3 Mean Shift Algorithm

For the next sections we need to estimate the cell decomposition of M, this can be done using the mean shift algorithm or the quick-shift algorithm.

5.4 Morse-Smale Regression

Let (X, Y) be a random pair, where $X \in M$ and $Y \in \mathbb{R}$. Then we define the regression function $m(x)$ as:

$$m(x) = \mathbb{E}[Y|X = x]$$

This function is not necessarily easy to compute, specially when the dimension d increases. So we can use simpler function, that estimate $m(x)$. The error will increase but hopefully not too much. The simplest way to do this is to choose a piecewise linear estimator.
There are two versions of the Morse-Smale regression:

- Using $m(x)$ version:

  Let $E_1, \ldots, E_o$ be the d-cells that decompose M using $m(x)$ as a Morse function, then we estimate m as:

  $$\hat{m}(x) = \mu_i + \beta_l^T x, x \in E_i$$

  where we choose μ_i and β_l^T so that they minimise the mean square error:

  $$(\mu_i, \beta_i) = \text{argmin}_{\mu,\beta}\mathbb{E}[(Y - \mu - \beta^X)^2|X \in E_i]$$

- Sample version: Let $(X_1, Y_1), \ldots, (X_m, Y_m)$ be a random sample of values, where X_i are sample from M and $Y_i = \hat{f}(X_i)$. Then we estimate $m(x)$ using the pilot estimator:

  $$\bar{m}(x) = \frac{\sum Y_i K(\frac{x - X_i}{h})}{\sum K(\frac{x - X_i}{h})}$$

  Then again we decompose M into d-cell using $\bar{m}$ as a Morse function:$\hat{E}_1, \ldots, \hat{E}_l$ And we calculate $\hat{m}$ as in the previous point, but changing E_i for $\hat{E}_i$.

5.5 Morse-Smale Signatures

In section 2.2.2 we say a way to compare two persistence diagrams. In this section we explore a related way to compare two finite samples from the manifold M: Morse-Smale singatures. These samples are taken from two different distributions (that is Morse functions).
Suppose we have $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_k$ two samples from M. They are respectively distributed according to the pdfs p_X and p_Y.
We would like to do the study the folowing statistical test:

$$H_0(x) : p_X(x) = p_y(x)$$

In particuar we are interested when we reject the null hypothesis H_0. To do this, from each pdf we construct the KDE as described in , which we denote $\hat{p}_X$ and $\hat{p}_y$ respectively.
Then we consider: $d = \hat{p}_X - \hat{p}_Y$. And we reject H_0 when $d(x) > \lambda$ where we choose $\lambda > 0$. In particular, we define:

$$\Gamma(\lambda) = \{x \in M : |\hat{d}| > \lambda\}$$

So $\Gamma(\lambda)$ is the reject manifold.

5.6 Some remarks

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