LFA Presentation

Erdős–Faber–Lovász conjecture

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Overview

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2. The faces of the problem
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Short history of the problem

Short history of the problem
Summary

- In graph theory, the Erdős–Faber–Lovász conjecture is a problem about graph coloring, named after Paul Erdős, Vance Faber, and László Lovász, who formulated it in 1972.
- Combinatorial problems, as Number Theory ones, are easy to understand by everyone. They can even be more direct. Paul Erdös was very fond of combinatorial problems. He had a special gift for posing problems. Today we will discuss the near solution of one of his most famous problems, which he considered among his three favorite problems.
- It says: If k complete graphs, each having exactly k vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be properly colored with k colors.

Short history of the problem

The problem arose in 1972 while Vance Faber, László Lovász and Paul Erdös were having tea. They were trying to find problems that would drive forward the theory of hypergraphs. As first step they propose a conjecture:
The n sets problem. Prove that if $A_1, \ldots, A_n$ are n sets, each of cardinal n and such that each intersection $A_j \cap A_k$, with $j \neq k$, contains at most one element, then there is a coloring of the union set:

$$\bigcup_{k=1}^{n} A_k$$

with n colors so that each A_k has one element of each color.

Short history of the problem

- They thought that the problem was a simple exercise and planned a meeting the next day to write down the best solution. None of them came up the next day with a solution.
- Erdös was in the habit of awarding a cash prize to the first person who solved any of his problems. He set the value of this problem in 50 dollars.
- According to the conjecture we can color the elements with n colors so that each row has one element of each color. If you think of it as a game: the elements in the k-th column are painted in color k. Thus, each set (row) contains one element of each color.

Short history of the problem

A proposed association

Haddad and Tardif in 2004 introduced the problem with a story about seating assignment in committees: suppose that, in a university department, there are k committees, each consisting of faculty members, and that all committees meet in the same room, which has k chairs. Suppose also that at most one person belongs to the intersection of any two committees. Is it possible to assign the committee members to chairs in such a way that each member sits in the same chair for all the different committees to which he or she belongs? In this model of the problem, the faculty members correspond to graph vertices, committees correspond to complete graphs, and chairs correspond to vertex colors.

Short history of the problem

Final solution

The authors are professors Daniela Kühn and Deryk Osthus from the Combinatorics Group of the University of Birmingham, (although they come from German universities) and three young people, Dong Yeap Kang, Tom Kelly and Abhishek Methuku, who are in their early postdoctoral studies, in the same group. Dong Yeap Kang is a Korean who defended his doctoral thesis in 2020 at the Advanced Institute of Science and Technology in Korea; Tom Kelly defended his doctoral thesis at the University of Waterloo in 2019 and Abhishek Methuku defended his doctoral thesis in 2019 in Budapest.

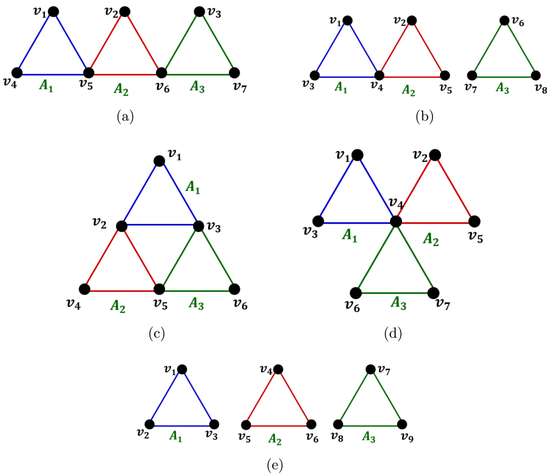

Figure 1: All graphs satisfying the hypothesis of the conjecture for n=3.

The faces of the problem

The faces of the problem

Some introductory notions

- A graph G=(V,E) is made up of a finite set V of vertices and a family E of edges that join them. We can think of edges as subsets $\{a, b\} \subset$V of two elements. The structure of the vertices can be ignored, V should be thought of as a set of points.
- This structure was extended to a hypergraph.A hypergraph H=(V,E) is also composed of a finite set of vertices V and, but in this case, E is a family of subsets $A \subset$V of V that we will call edges of the hypergraph.
- For example, if you consider "family," a relation connecting two or more people. If each person is a vertex, a family hyperedge connects the father, the mother, and all of their children. So G = (people, family) is a hypergraph. Contrast this with the binary relation "married to," which connects a man and a woman, or "child of," which is directed from a child to his or her father or mother

The faces of the problem

Some introductory notions

- A hypergraph is k-uniform if all edges have the same cardinal $|A| = k$ (given a set A, we designate by $|A|$ its cardinal). The 2-uniform hypergraphs are ordinary graphs.
- A hypergraph is linear if every two different edges A and B intersect at most at one point, that is, $|A \cap B| \leq 1$. As with the lines of the plane, edges are either parallel or intersect at one point.

The faces of the problem
Some introductory notions

- In a graph G=(V,E) a set of vertices $Q \subset$V is a clique if each pair (a$\neq$b) $\in$Q of elements of Q form an edge ($\{a, b\}$) $\in$E) of G.
- If Q is a finite set we denote by C(Q) = (Q, $Q^{(2)}$) the complete graph that has Q as a set of vertices and its set of edges $Q^{(2)}$ is maximal, that is, it is made up of all pairs {a, b } with $a \neq$b elements of Q.
- Let's consider a social network, where the graph's vertices represent people, and the graph's edges represent mutual acquaintance. Then a clique represents a subset of people who all know each other, and algorithms for finding cliques can be used to discover these groups of mutual friends.

The faces of the problem

Some introductory notions

- A proper coloring of a graph is an assignment of colors to vertices such that adjacent vertices receive different colors. The chromatic number of G, denoted $\chi(G)$, is the least number of colors in a proper coloring of G.
- A strong coloring of a hypergraph is an assignment of colors to vertices such that no edge contains two vertices of the same color (the usual definition of proper coloring requires only that no edge be monochromatic).
- The strong chromatic number of a hypergraph is the least number of colors in a proper coloring. A proper edge-coloring of is an assignment of colors to edges such that intersecting edges have different colors. The chromatic index of H, denoted $\chi'(H)$, is the least number of colors in a proper edge-coloring of H .

The faces of the problem

Three equivalent conjectures that are equivalent to the Erdös-Faber-Lovász conjecture

Conjecture 1. Let V =

$$\bigcup_{j=1}^{n} A_j$$

be a finite set, union of n subsets A_j each of cardinal $|A| =$n and such that for $j \neq k$ we have $|$A$_j \cap$A$_k$ $| \leq$1, then there is a function f:V→$\{1, 2, \ldots, n\}$ such that the image f(A_j)=1,2,…,n for all j.

In the language of hypergraphs: Every linear and n-uniform hypergraph H=(V,E), admits a coloring of the vertices such that each color appears on each edge.

The faces of the problem

Three equivalent conjectures that are equivalent to the Erdös-Faber-Lovász conjecture

Conjecture 2. Let G=(V,E) be a graph such that there are n cliques $Q_1, \ldots, Q_n$ so that $|Q_j| \leq n$ for each j, and such that for $j \neq k$, we have $|Q_j \cap Q_k| \leq 1$ and finally E = $\bigcup_{j=1}^{n} Q_j^{(2)}$. Then we can color the vertices of V with at most n colors and such that each edge $\{a, b\} \in E$ has vertices of different colors.

The faces of the problem

Three equivalent conjectures that are equivalent to the Erdös-Faber-Lovász conjecture

- **Conjecture 3.** Let V be a set of cardinal n and E a family of subsets of V such that $A \neq B$ in E implies $|A \cap B| \leq 1$. Then there exists a coloration f:V→ $\{1, 2, \ldots, n\}$ such that $|A \cap B| = 1$ implies $f(A) \neq f(B)$.
- In other words: Every linear hypergraph H with n vertices admits an edge coloring with no more than n colors. Of course, edge coloring requires that if $|A \cap B| = 1$, then $f(A) \neq f(B)$.
- The first conjecture is the n sets problem. It is not difficult to see that the three conjectures are equivalent. This what is known as the **Erdös-Faber-Lovász conjecture**.

Proof

Proof

Tight cases

- **Projective planes.** For each finite field with q elements. The projective plane has q^2+q+1 points and there are precisely q^2+q+1 lines each with q+1 points. We can thus form a hypergraph where the vertices are the points of the projective plane and the edges are the lines. Two different lines always intersect at one point, so the graph is linear. In a coloration, all the lines must have different colors since any two intersect. Therefore, the chromatic index is precisely the number of vertices q^2+q+1.
- **Degenerated plane.** Another extreme case occurs in the hypergraph H_n=(V=1,2,…,n,E=1,2,1,3,…,1,n,2,3,…,n. Obviously n colors are needed to color it.
- **Complete graph.** The third and last extreme case is the complete graph K_n with n vertices and the $\binom{n}{2}$ possible edges.

Proof

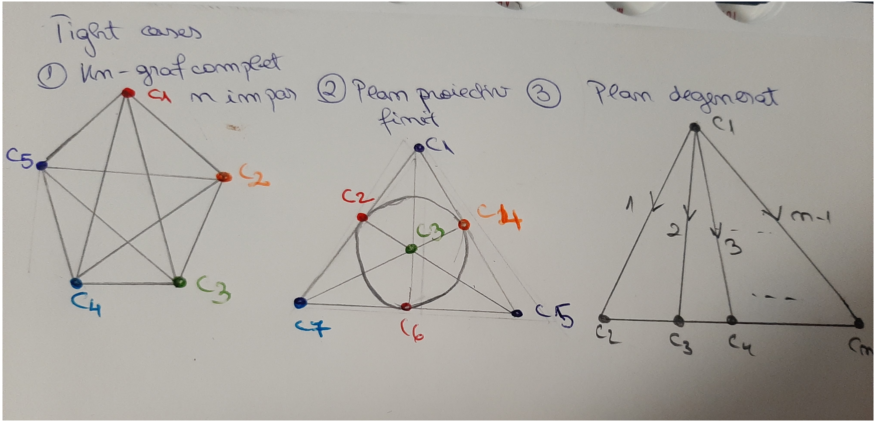

Figure 2: Examples of the tight cases

Proof

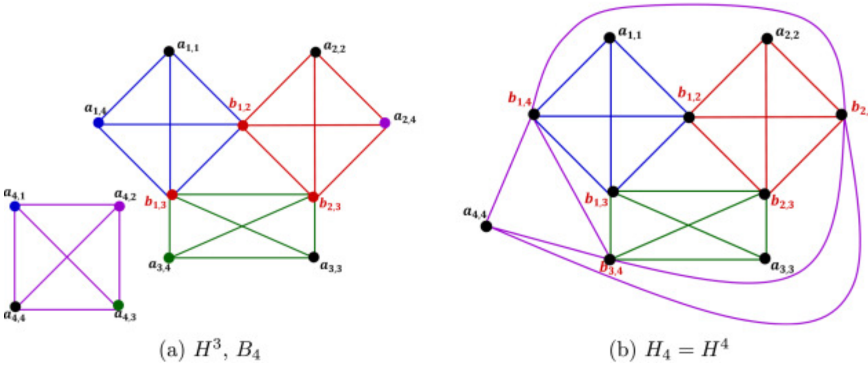

Figure 3: Construction of H^4 from H^3, B_4.

Proof

Let be a graph satisfying the hypothesis of the conjecture.Then b_X be the new labeling to the vertices v of clique degree greater than 1 in G , where x= i : vertex v is in A_i . Define N_i = b_X: cardinal of X=i for i=1,...,n. Then the graph is constructed from as given below:

Step 1: For every common vertex $b_{i,j}$ in H_i which is not in N_2, split the vertex $b_{i,j}$ into two vertices v_{ij} and v_{ji} such that vertex $v_{i,j}$ is adjacent only to the vertices of B_i and the vertex $v_{i,j}$ is adjacent only to the vertices of B_j in H_n .

Step 2: For every vertex b_X in N_i where i=3,4,...,n merge the vertices $v_{l_k l_j}$ into a single vertex v_x in H_n where l_k is in X and $l_k < l_j$ for k<j .

Proof

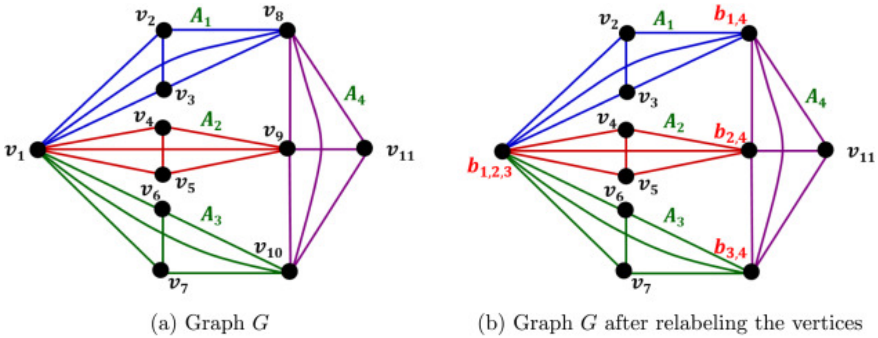

Figure 4: Graph G, before and after relabeling the vertices.

Proof

A help in understanding the problem

Let H_n be the graph defined as above. Let M_n be an matrix in which an entry m_{ij}=b_{ij}, be a vertex of H_n, belongs to both B_i, B_j for $i \neq j$ and m_{ii}=a_{ii} be the vertex of H_n which belongs to B_i.

$$\begin{pmatrix} a_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & a_{22} & b_{23} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & a_{nn} \end{pmatrix}$$

Clearly M is a symmetric matrix. We know that, for every n there is a Symmetric Latin Square of order nxn.

Proof

A help in understanding the problem

Let $v_1, v_2, ..., v_n$ be the vertices of K_n and e_{ij} be the edge joining the vertices v_i and v_j of K_n, where $i < j$, then arrange the edges of K_n in the matrix form A=$[a_{ij}]$, a_{ij}=a_{ji}=e_{ij} for $i < j$ and $a_{ii} = 0$.

$$\begin{pmatrix} 0 & e_{12} & e_{13} & \dots & e_{1n} \\ e_{12} & 0 & e_{23} & \dots & e_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & e_{n3} & \dots & 0 \end{pmatrix}$$

Then we will also consider V a matrix which has on the primary diagonal the values $v_1, v_2, ..., v_n$.

Proof

A help in understanding the problem

Then, define a matrix A' as A+V $\begin{pmatrix} \mathrm{v}_1 & e_{12} & e_{13} & \dots & e_{1n} \\ e_{12} & \mathrm{v}_2 & e_{23} & \dots & e_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{n1} & e_{n2} & e_{n3} & \dots & \mathrm{v}_n \end{pmatrix}$

If you consider a coloring matrix for A', where c_{ij}=colour of e_{ij} and c_{ii}=v_i, you'll get the Symmetric Latin Square. As the elements of M are the vertices of H_n, one can assign the colors to the vertices of H_n from the color matrix C.

Proof

A help in understanding the problem

- The ideal proof would be an algorithm that, given the linear hypergraph, determines the coloring of the edges. However, it is difficult to imagine such an algorithm that would work in all cases. Rather, the proof consists on splitting the hypergraph into pieces, applying different algorithms to each piece, then trusting that the pieces might fit, and in case they don't fit, modify something in the last part so that everything goes well.
- If we imagine the problem solved, we will have the edges colored with maximum n colors. Edges with the same color will be disjoint and this is what is called a matching. The reason for this searching is that if we find a partition of the edges into n or less matchings, we will have the desired coloring.
- One of the authors, Abhishek, gave a lecture on February 16 in 2021 at the seminar of the Alfréd Rényi Institute of Mathematics on the solution to the Conjecture. Among the attendees was László Lovász who congratulated Abhishek for the proof.

Proof

In the proof they use the following hierarchy of constants:

$$0 < 1/n_0 \ll 1/r_0 \ll \xi \ll 1/r_1 \ll \beta \ll \kappa \ll$$
$$\gamma_1 \ll \varepsilon_1 \ll \rho_1 \ll \sigma \ll \delta \ll \gamma_2 \ll \rho_2 \ll \varepsilon_2 \ll 1$$

This is exactly like saying: there is some $0<\epsilon_2<1$, and given this ϵ_2 there exists $\rho_2 < \epsilon_2$. Then there is $\gamma_2<\rho_2$ and, once all these constants are given, there is $\delta < \gamma_2 \ldots$. The process continues until there is a natural number n_0< such that if H=(V,E) is a linear hypergraph with $n \geq n_0$ vertices, then we can color the edges E of H with no more than n colors.

Firstly, one should organize the edges of H into three types, according to the number of edges that they contain: small, medium and large. The constants r_1 and r_0 are the ones that appear in the hierarchy of constants above. The edges from 2 to r_1 = small, from to r_1 to r_0 = medium and from r_0 to n =large.

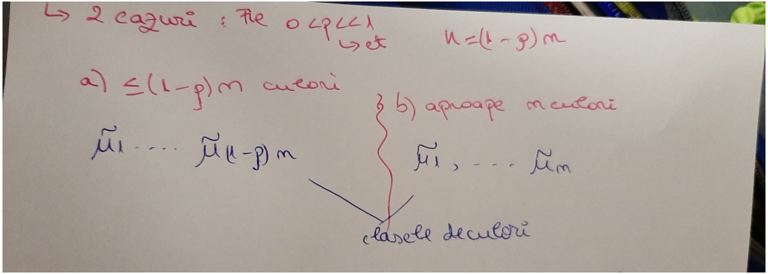

Figure 5: Step 1.

We can see cases (a) and (b) the matchings, with different colors, formed with the large edges in the previous stage. Now we take almost all the small edges, those with *cardinal* ≥ 3, and extend the previous matchings with them.

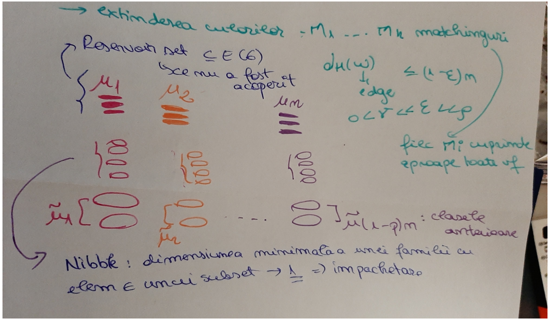

Figure 6: Step 2.

- The authors use absorption as a way of gradually adding edges into a coloring while ensuring along the way that the coloring always maintains the right properties. It's especially useful for coloring the places where many small edges converge on a single vertex, like the special vertex connected to all the others in the third extreme hypergraph. Clusters like these use almost all the available colors and need to be colored carefully.
- To do so, the authors created a reservoir of small edges, pulled from these tricky clusters. Then they applied a random coloring procedure to many of the small edges that remained (basically, rolling a die to decide which color to apply to a given edge). As the coloring proceeded, the authors strategically chose from the unused colors and applied them in a carefully chosen way to the reserved edges, "absorbing" them into the colorings.

- Absorption improves the efficiency of the random coloring procedure. Coloring edges randomly is a nice basis for a very general procedure, but it's also wasteful — if applied to all edges, it's unlikely to produce the optimal configuration of colors. But the recent proof tempers the flexibility of random colorings by complementing it with absorption, which is a more exact method.
- Because of the probabilistic elements, their proof only works for large enough hypergraphs — those with more than a specific number of vertices. This approach is common in combinatorics, and mathematicians consider it a nearly complete proof since it only omits a finite number of hypergraphs.

- Once Step 2 is finished, we already have many colored edges. The remainder is formed by edges of cardinal 2, i.e., what remains is a graph in the ordinary sense and we can apply to it results from Graph Theory. In particular the Vizing theorem:
- **Theorem Vizing**. Every graph G with $\Delta(G) \leq D$ satisfies $\chi'(G) \leq D + 1$. Moreover, if G contains at most two vertices of degree D, then $\chi'(G) \leq D$.
- With this, the edges of the remaining graph can be colored with n colors and in case (a) we have finished, since these colors can be added with the $(1 - \rho)n$ that we used in step 1 to color the entire graph with n colors.

Conclusion

Conclusion

- The Erdős-Faber-Lovász conjecture started as a question that seemed as if it could be asked and answered within the span of a single party. In the years that followed, mathematicians realized the conjecture was not as simple as it sounded, which is maybe what the three mathematicians would have wanted anyway. One of the only things better than solving a math problem over tea is coming up with one that ends up inspiring decades of mathematical innovation on the way to its final resolution.
- "Efforts to prove it forced the discovery of techniques that have more general application," said Kahn. "This is kind of the way Erdős got at mathematics."

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Some references to showcase [allowframebreaks] [1, 5, 3, 2, 4]

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