Withholding Verifiable Information

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Abstract

We study a class of finite-action disclosure games where the sender's preferences are state independent and the receiver's optimal action depends solely on the expected state. In such games, the receiver's preferred equilibria feature full revelation of the state, but other equilibria are less well-understood. We show that a sender's preferred equilibrium balances inducing sender-favorable actions and deterring deviations by pooling nonadjacent states in a particular way. Leveraging this observation, we characterize other equilibria of the game and the sender's equilibrium payoff set, and identify conditions under which the sender does not benefit from commitment power. We apply these insights to study influencing voters and selling with quality disclosure.

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1 Introduction

In many economic interactions involving communication between a sender (who possesses private information) and a receiver (who takes an action), the sender's messages are *verifiable* in the sense that they can be vague but never false. Examples include sellers disclosing and highlighting certain features of a product to consumers, political experts organizing and simplifying poll results for politicians, and advisors condensing and distilling market research for their managers. In each of these examples, if the sender releases inherently false information, severe consequences ensue: sellers risk lawsuits for misrepresentation, while experts or advisors could be dismissed for dishonesty. Additionally, upon receiving the sender's message, the receiver often chooses from a *finite* number of possible actions: consumers choose from a selection of similar products, politicians decide between one of several agendas, and managers select from a handful of investment opportunities.

This paper studies *disclosure games*, which are communication games where the sender's messages are verifiable, with *finitely many* receiver actions. In the classical analysis of disclosure games, the sender fully reveals her private information in every equilibrium (Grossman, 1981; Milgrom, 1981). The intuition is that the sender benefits whenever her private information is perceived even slightly more favorably, rendering "pooling" unsustainable because the sender can prove her private information and become strictly better off. Previous studies have observed that the finiteness of the receiver's actions creates pooling opportunities (e.g., Giovannoni and Seidmann, 2007; Titova and Zhang, 2025), as the sender does not always strictly benefit from the receiver's more favorable perception of her private information. Consequently, although the receiver's most preferred equilibrium still involves full revelation, other equilibria may also emerge. This leads to several pertinent questions: what is the sender's most preferred equilibrium? (In other words, what is the most beneficial way for the sender to send pooling messages?) How much can the sender benefit from verifiable communication? What are the potential payoffs the sender may achieve in equilibrium?

To answer these questions, we study a model that differs only from the classic models of Milgrom (1981, 2008) in one aspect: there is a finite number, rather than a continuum, of actions. More specifically, there is a one-dimensional state of the world that is payoff-relevant to the receiver and only observable to the sender; thus, the state is the sender's private information. The sender sends a message to the receiver, who then chooses an action from a finite set. The actions are *ordered* in the sense that the receiver would like to use a higher action when the expected state is higher, and the sender strictly prefers a higher action regardless of the state. Verifiability requires that the sender's messages must contain the true state. Finally, the sender's payoff depends only on the receiver's action (in jargon, the sender's preferences are state-independent), and the receiver's optimal choice of action depends solely on the expected state.

We start by restricting attention to a natural and simple class of equilibria, which we term *obedient recommendation equilibria (ORE)*, in which both the sender and the receiver play pure strategies, and every on-path message of the sender serves as an action recommendation that the receiver deems optimal to follow. Every ORE can be identified by a partition of the state space such that in the states contained in the same partitional element, the same action is recommended. Furthermore, these partitional elements can be taken as the sender's on-path messages. In fact, any partition can be sustained as the set of on-path messages of an ORE if and only if it is *obedient* (the receiver is willing to follow the action recommendation) and *incentive compatible* (the sender prefers to send the partitional element that contains the state rather than fully reveal the state). Consequently, finding an ORE simplifies to finding a partition of the state space satisfying these two properties.

A key insight is that pooling nonadjacent states can maximize the frequencies of inducing desirable actions without creating incentives for deviations. We show that a sender's preferred equilibrium is an ORE where the on-path messages pool nonadjacent states through a simple "laminar" structure (Candogan and Strack, 2023). In this equilibrium, the sender's payoff is strictly higher than that in a fully revealing equilibrium. This is because the sender pools low states with high states in her message until the expected state following the message is just high enough for the receiver to take the recommended action. Moreover, any sender's payoff below that in a sender's preferred equilibrium and above that in a fully revealing equilibrium can be sustained in an ORE in which the on-path messages have the laminar structure. Put differently, there is a continuum of equilibria in pure strategies with distinct payoffs and simple messages.

Section 4 evaluates the sender's gains from verifiable communication. We start by observing that a natural upper bound of the sender's payoff in her preferred equilibria is the value of the corresponding information design problem (Kamenica and Gentzkow, 2011), where the sender can commit to any information structure that reveals information about the state. According to Kleiner, Moldovanu, and Strack (2021) and Arieli, Babichenko, Smorodinsky, and Yamashita (2023), every information design problem admits a salient solution, often referred to as a *bi-pooling solution*. Although it is well-known that a bi-pooling solution can be identified by an obedient partition of the state space, we show that every bi-pooling solution admits an obedient partition that exhibits the highest resistance to deviating to full revelation, which is a special case of a laminar partition. Consequently, a bi-pooling solution can be induced by an equilibrium if and only if the aforementioned partition is incentive compatible.

We also provide sufficient conditions on model primitives under which commitment has no value; that is, the commitment payoff can be achieved in an equilibrium. Vaguely, these conditions require that for any two consecutive actions, the benefit of inducing the higher action over the lower one must be sufficiently higher for the sender. This ensures that incentive compatibility cannot be violated. Under any of these conditions, a sender's preferred equilibrium of a disclosure game, as well as the sender's equilibrium payoff set, can be obtained by solving the corresponding information design problem. Moreover, for any information design problem that satisfies one of these conditions, the commitment assumption can be dispensed with.

Implications. The results discussed above have several implications. In the existing literature, pooling nonadjacent states is typically attributed to the presence of the receiver's private information.[1] The laminar structure provides an alternative rationale for pooling nonadjacent states, suggesting it could be a more general and robust feature that does not depend solely on the interaction of multiple information sources. Moreover, the large multiplicity of equilibria with distinct payoffs and simple on-path messages may explain the plethora of disclosure policies observed in practice. This also suggests that focusing exclusively on the fully revealing outcomes in policy debates need not always be appropriate. Finally, this paper contributes to the ongoing debates over whether disclosure should be mandated. Our results identify communication environments where the sender can gain significantly by pooling a substantial portion of low states with high states. In such contexts, from the receiver's perspective, the advantages of mandatory disclosure may outweigh the shortcomings.

[1] This is documented in various contexts, such as signaling games (e.g., Feltovich, Harbaugh, and To, 2002), disclosure games (e.g., Harbaugh and To, 2020), information design (e.g., Guo and Shmaya, 2019), and in empirical studies (e.g., Bederson, Jin, Leslie, Quinn, and Zou, 2018).

Applications. In Section 5, we apply these insights to study selling with quality disclosure and influencing voters. In the selling setting, Milgrom (1981) shows that when the buyer can buy any fraction of the product, every equilibrium features full revelation. If the buyer is restricted to buying integer units; however, we show that the seller may be able to achieve the same outcome as if she has commitment power. Following this, we consider an expert who discloses verifiable information to a group of voters in an amendment voting setting. Here, voters choose from three alternatives: the amended bill, the (unamended) bill, and no bill (maintaining the status quo). Interestingly, the expert can be hurt even if, all else equal, all voters are more inclined toward the expert's most preferred alternative.

Related literature. The study of disclosure games is initiated from Grossman and Hart (1980), Grossman (1981), and Milgrom (1981); for surveys of this literature, see Milgrom (2008) and Dranove and Jin (2010).

This paper belongs to the growing literature on whether, and to what extent, the sender can benefit from verifiable communication. Mezzetti (2023) studies a disclosure game in which the sender's bias is uncertain to the receiver. He shows that almost all sender types can influence the receiver to select actions more aligned with their preferences than those based on complete information, and a positive measure of types obtain their most preferred action. Celik and Drugov (2022) examine the disclosure strategy of a monopolist who sells a product with multiple quality attributes, identifying an equilibrium that is most preferred by the monopolist in which only the average quality is revealed. Glode, Opp, and Zhang (2018), Pram (2021), and Ali, Lewis, and Vasserman (2023) show that when trading with a monopolist, having access to sufficiently rich verifiable messages can improve consumer welfare. In these papers, pooling messages can also be sustained in equilibrium because the sender does not strictly benefit from the receiver's more favorable perception of her private information. The more specific reason; however, is different from this paper: the sender (the consumer) in their settings never strictly prefers fully revealing the state to sending any other message, which is not the case here.

The most closely related paper is Titova and Zhang (2025), which pioneers the study of disclosure games with finitely many actions. Studying a similar model with some different assumptions, Titova and Zhang show that every equilibrium is "outcome equivalent" to that characterized by a partition of the state space where the receiver takes the same action within each partitional element. Furthermore, if there is a partition that identifies an information design solution, then it can be sustained in an equilibrium if and only if the sender obtains no less than her complete information payoff in every state.[2] Departing from Titova and Zhang (2025), we characterize the sender's preferred equilibrium regardless of whether the commitment payoff can be attained in an equilibrium or not, as well as the sender's equilibrium payoff set. We also provide sufficient conditions on the model primitives for the sender to attain her commitment payoff. Consequently, the main insights and implications derived from these results are new. [**KZ:** This needs to be revised; we should make the comparison sharper and shorter.]

This work also connects to the literature on the possibility (or impossibility) of attaining commitment outcomes in various communication environments without commitment. Lipnowski (2020) shows that in a cheap talk game with finite parameters, if the sender's value function is continuous, she can attain the same payoff as she would with communication under commitment. Several papers, including Best and Quigley (2023), Kuvalekar, Lipnowski, and Ramos (2022), Mathevet, Pearce, and Stacchetti (2022), and Pei (2023), study whether commitment can be restored in differing settings through repeated cheap-talk communications. This paper, instead, focuses on one-shot verifiable communication.

[**KZ:** Maybe take out this paragraph as otherwise we may need to talk about KS once again...] Finally, this paper is related to the literature on relaxing the commitment assumption in information design problems. Lipnowski, Ravid, and Shishkin (2022) and Min (2021) allow the sender to change her message costlessly with some exogenously given probability after observing the state. Nguyen and Tan (2021) studies a model where altering the messages generated by the chosen information structure incurs costs for the sender. Lin and Liu (2024) consider a sender who can covertly switch to alternative information structures, provided the message distribution remains consistent with the announced information structure. Our model differs in that the only constraint that the sender faces is that her messages have to be verifiable.

2 The Model

There are two players, Sender and Receiver. The state space is $\Omega = [0, 1]$, and Sender and Receiver share a common prior F, which admits a strictly positive density f. Let μ_F denote the probability measure induced by F. We assume that Receiver's optimal action only

[2] The difference between this result and the related result in this paper is discussed after Proposition 4.3.

depends on the *expected state*, denoted by x. Receiver has $1 < n < \infty$ actions, and hence her action space is $\mathcal{A} = \{a_0, a_1, \ldots, a_{n-1}\}$. There exist cutoffs $0 = \gamma_0 \le \gamma_1 \le \cdots \le \gamma_n = 1$ such that a_i is optimal if and only if $x \in A_i := [\gamma_i, \gamma_{i+1}]$.[3] Sender's utility $v : \mathcal{A} \to \mathbb{R}$ only depends on Receiver's action; we normalize $v(a_0) = 0$, and assume that $v(a_k) > v(a_l)$ whenever $k > l$.

Sender observes the state and sends a message to Receiver. For every $\omega \in [0, 1]$, Sender's message space is $\mathcal{M}(\omega) = \{m \in \mathcal{C} : \omega \in m\}$, where $\mathcal{C}$ denotes the collection of nonempty closed subsets of $[0, 1]$. Observing Sender's message, Receiver chooses an action $a \in \mathcal{A}$. Sender and Receiver play a **disclosure game**, first studied by Grossman and Hart (1980), Grossman (1981), and Milgrom (1981); Sender's message space in our problem is also identical to the aforementioned papers.[4] Any message is *verifiable* in the sense that it contains the true state. For each state $\omega \in [0, 1]$, the assumption on message space allows Sender to fully reveal the state, namely sending the **fully revealing message** $m = \{\omega\}$, or to disclose nothing, that is sending the **fully concealing message** $m = [0, 1]$.

Formally, the disclosure game is defined as follows. Sender's strategy set is the set of functions $\sigma : [0, 1] \to \Delta(\mathcal{C})$ such that $\sigma(\omega)$ is supported on $\mathcal{M}(\omega)$ for each $\omega \in [0, 1]$.[5] That is, if Sender uses message m with positive probability in state ω, then it must be that m is feasible in this state. Receiver's strategy set is the set of functions $\tau : \mathcal{C} \to \Delta(\mathcal{A})$. A belief system for Receiver, $p : \mathcal{C} \to \Delta([0, 1])$, yielding Receiver's beliefs about the state as a function of the observed message.

Say that (σ, τ, p) is a (perfect Bayesian) **equilibrium** if the following conditions hold. First, $\sigma(\omega)$ is supported on $\arg\max_{m' \in \mathcal{M}(\omega)} v(\tau(m'))$. Second, for every $m \in \mathcal{C}$ and $a_i \in \mathcal{A}$, $\tau(a_i \,|\, m) > 0$ implies that Receiver's posterior expected state (*posterior mean* henceforth) $\mathbb{E}[\omega \mid \omega \in m] \in [\gamma_i, \gamma_{i+1}]$. Third, for every $\omega \in [0, 1]$ and $m \in \mathcal{C}$, $\omega \notin \operatorname{supp}(p(m))$ if $m \notin \mathcal{M}(\omega)$.[6] That is, Receiver must deem any state in which the observed message m cannot be sent impossible. Finally, p is obtained from the prior F given σ using Bayes' rule; that is, μ is a regular conditional probability. An **outcome** of this disclosure game is

[3]One can think of Receiver wanting to "match" the state and the action, but constrained by the number of available actions. Such cutoff structure can be obtained by, for example, imposing the further restriction of finitely many feasible actions on the class of Receiver preferences studied in Crawford and Sobel (1982): Receiver's utility function $r(a, \omega)$ is twice continuously differentiable with $\frac{\partial^2}{\partial a^2} r(a, \omega) < 0$ and $\frac{\partial^2}{\partial a \partial \omega} r(a, \omega) > 0$. A special case is the classic "quadratic loss" utility: $r(a, \omega) = -(a - \omega)^2$.

[4]The assumption on Sender's message space is further discussed in Section 6.1.

[5]For a compact metric space Y, let $\Delta(Y)$ denote the set of all probability measures on the Borel subsets of Y. Endowing $\mathcal{C}$ with the Hausdorff distance, it is a compact metric space.

[6]The support of a distribution, denoted by $\operatorname{supp}(\cdot)$, is the smallest closed set that has probability one.

a pair $(G, R) \in \Delta([0,1]) \times \mathbb{R}$, where G is the induced distribution over Receiver's posterior means, and R is Sender's (ex ante) payoff. An outcome is an **equilibrium outcome** if it is induced by an equilibrium.

An important part of this paper concerns Sender's preferred equilibria of the disclosure game, which are the equilibria that attain the highest possible Sender's payoff. For this purpose, it is useful to introduce Sender's value function $u : [0,1] \to \mathbb{R}$, which maps every expected state x to the highest attainable Sender payoff, conditional on the expected state being x and Receiver choosing her optimal action. By definition, $u(x)$ is constant on (γ_i, γ_{i+1}), has discontinuity at γ_i for each $i = 1, 2, \dots, n-1$, and is upper semicontinuous.

To understand how much can Sender benefit from verifiable communication, we also consider the information design problem based on this communication environment as a benchmark. In this problem, Sender can commit to any experiment to reveal information about the state. An **experiment** (χ, S) consists of a **signal space** S and a mapping $\chi : [0,1] \to \Delta(S)$. For each state $\omega \in [0,1]$, a signal $s \in S$ realizes according to $\chi(\omega)$. Because Receiver's optimal action only depends on the expected state, it is without loss to restrict attention on the class of experiments where $S = [0,1]$, and each $s \in S$ equals the induced posterior mean: $s = \mathbb{E}[\omega \mid s]$. Thus, every experiment induces a distribution of posterior means.

It is well known that a distribution of posterior means G is induced by an experiment if and only if G is a mean-preserving contraction (MPC) of F.[7] Consequently, Sender's problem is to choose a distribution of posterior means that solves

$$\max_{G \in \mathrm{MPC}(F)} \int_0^1 u(x)\, \mathrm{d}G(x), \tag{1}$$

where $\mathrm{MPC}(F)$ is the set of mean-preserving contractions of F. We call any solution to problem (1) an **information design solution**, and call the value of problem (1) the **commitment payoff**. The commitment payoff is an upper bound of Sender's equilibrium payoff in the disclosure game.

Say that a pair $(G, R) \in \Delta([0,1]) \times \mathbb{R}$ is a **commitment outcome** if G is an information design solution and R is Sender's commitment payoff. A commitment outcome is **implementable** by verifiable messages, or just implementable for short, if it is an equilibrium outcome of the disclosure game based on the same communication environment. We

[7] See, for example, Gentzkow and Kamenica (2016) and Kolotilin (2018). A distribution $G \in \Delta([0,1])$ is a mean-preserving contraction of F if $\int_0^x G(s)\mathrm{d}s \le \int_0^x F(s)\mathrm{d}s$ for all $x \in [0,1]$, where the inequality holds with equality at $x = 1$.

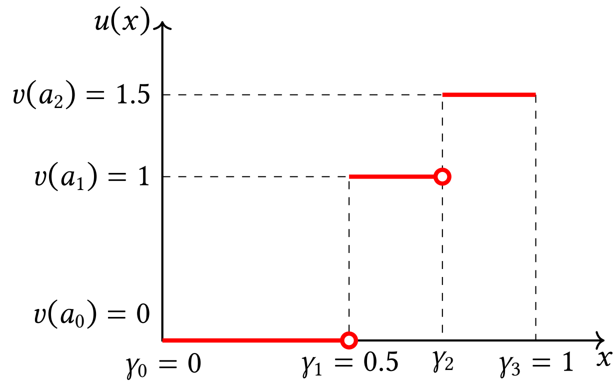

Figure 1: Sender's value function $u(x)$ in the example in Section 2.1.

sometimes abuse notation and say that an information design solution G is implementable when the commitment outcome it induces is implementable.

2.1 An Example

Consider a seller (Sender) promoting a product and its add-on to a buyer (Receiver), who chooses from buying nothing (a_0), buying the product only (a_1), and buying both the product and its add-on (a_2). The buyer's payoffs depend on the product's quality ω, which is uniformly distributed on $[0, 1]$. If the buyer buys the product only, her payoff is $\omega - 0.5$, and the seller's payoff is 1. If the buyer buys both the product and the add-on, her payoff is $2\omega - 1.25$, and the seller's payoff is 1.5. If the buyer does not buy, both players get zero.

Under complete information, if $\omega \in [0.5, 0.75]$, it is optimal for the buyer to purchase the product alone; and if $\omega \in [0.75, 1]$, it is optimal for the buyer to buy both the product and the add-on. Otherwise, the buyer should not make a purchase. Therefore, $\gamma_1 = 0.5$ and $\gamma_2 = 0.75$; the seller's value function is shown in Figure 1. Upon observing the product's quality, the seller communicates with the buyer using verifiable, or "truthful," messages. These messages are closed subsets of $[0, 1]$, each containing the true quality.

Figure 2 illustrates an equilibrium in which the seller attains her highest possible surplus. In this equilibrium, if the quality is less than 0.191, the seller communicates using message $B_0 = [0, 0.191]$. This message advises the buyer against making a purchase. If the quality is between 0.25 and 0.75, the seller sends message $B_1 = [0.25, 0.75]$, suggesting that buying the product alone is the optimal choice. For qualities within $[0.191, 0.25] \cup [0.75, 1]$, the seller sends $B_2 = [0.191, 0.25] \cup [0.75, 1]$, which recommends purchasing both the product and the add-on.

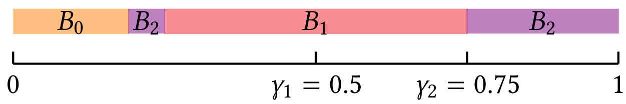

Figure 2: A Sender's preferred equilibrium when there are three actions.

Consequently, when the quality falls within an intermediate range, the buyer purchases only the product, and she buys both the product and the add-on when the quality is *either high or low*. To prevent deviation, it is necessary that $\max B_1 \leq \gamma_2 = 0.75$. If not, for any quality $\omega \in B_1$ where $\omega > 0.75$, the seller would strictly prefer to fully reveal the quality rather than pool it under B_1, as full revelation would lead the buyer to purchase both the product and the add-on.[8] Therefore, *grouping the qualities together* at which the buyer only purchases the product maximizes the probability of such purchases, while simultaneously avoiding incentives for the seller to deviate. On the other hand, *pooling the extreme qualities* under B_2 allows for the highest chance of selling both the product and the add-on without compromising the opportunity to sell the product alone.[9]

In any equilibrium where the seller fully reveals the quality, her surplus is 0.625; in the equilibrium described above, the seller's surplus is 0.9635. This shows that the seller's gain from verifiable communication can be sizable. It turns out that any seller surplus between 0.625 and 0.9635 can be attained in an equilibrium where the seller's messages take a simple form similar to that of the sender-surplus-maximizing equilibrium.

If the seller *were* able to provide information by committing to an experiment, standard arguments reveal that it is optimal for the seller to provide no information. Without further information, the buyer believes that the mean quality is 0.5, and hence buys the product alone with probability 1. Equivalently, the seller sends the message $B_1 = [0, 1]$ no matter what the quality is, recommending the buyer to only purchase the product. In this benchmark, the seller's surplus is $1 > 0.9635$.

The commitment outcome is *not* an equilibrium outcome of the disclosure game due to the seller's incentive constraints: $B_1 = [0, 1]$ is not an optimal message when the quality exceeds $\gamma_2 = 0.75$. In other words, the seller recommends purchasing only the product too often under commitment. This would not occur if the seller's payoff from selling both the product and the add-on were sufficiently high. In fact, if this payoff exceeds two (i.e.,

[8] It turns out that deviation to full revelation is the only kind that needs to be considered.

[9] It is easy to verify that $\mathbb{E}[\omega \mid \omega \in B_1] = 0.5 = \gamma_1$ and $\mathbb{E}[\omega \mid \omega \in B_2] = 0.75 = \gamma_2$; that is, the seller pools low qualities with high ones until the expected qualities are *just enough* for recommending buying the product alone and buying both the product and the add-on, respectively.

$v(a_2) > 2$), every commitment outcome is implementable.

3 Equilibrium Analysis

This section analyzes the equilibria of the disclosure game. The first main result, Theorem 3.3, characterizes a Sender's preferred equilibrium. In this equilibrium, every on-path message serves as an action recommendation that Receiver finds optimal to follow, and the messages have a simple "laminar" structure (Candogan and Strack, 2023). This structure solves the trade-off between inducing desirable actions with sufficiently high probabilities and suppressing incentives to deviate. The second main result, Proposition 3.5, characterizes Sender's equilibrium payoffs: any payoff ranging from Sender's payoff in a fully revealing equilibrium to that in a Sender's preferred equilibrium can be attained in an equilibrium with simple message structure. This result might elucidate the diverse disclosure policies observed in practice.

3.1 Obedient Recommendation Equilibria

[**MT:** I think this would be the part that we've subsumed] [**KZ:** In a sense, yes. But for completeness of this paper, my sense is that we can omit some details but it's hard to get rid of this at its entirety.] It is useful to first introduce a class of simple equilibria defined by a partition of the state space, which we call obedient recommendation equilibria.

Definition 1. A collection of closed subsets of the state space $\Omega = [0, 1]$, $\{B_i\}_{i=0}^{n-1}$, is a **partition** if $\cup_{i=0}^{n-1} B_i = [0, 1]$, and for any $i, j = 0, 1, \dots, n-1$, $\mu_F\left(B_i \cap B_j\right) = 0$.

Definition 2. An equilibrium (σ, τ, p) of the disclosure game is an **obedient recommendation equilibrium** (ORE) if there exist a partition $\{B_i\}_{i=0}^{n-1}$ such that

(a) for each $i = 0, 1, \dots, n-1$, $\omega \in B_i \setminus \cup_{j>i} B_j$ implies $\sigma(B_i \mid \omega) = 1$;

(b) for each $i = 0, 1, \dots, n-1$, $\tau(a_i \mid B_i) = 1$.

In an ORE, in any state a message $m \in \{B_i\}_{i=0}^{n-1}$ is sent with probability one; upon receiving message B_i, Receiver plays action a_i with probability one. Therefore, an on-path message B_i can be interpreted as a recommendation of action a_i, and Receiver finds it optimal to follow. Consequently, in such an equilibrium, the state space is partitioned such that for any state $\omega \in B_i \setminus \cup_{j>i} B_j$, Receiver takes action a_i.

Definition 3. A partition $\{B_i\}_{i=0}^{n-1}$ is

- **obedient** if $\mathbb{E}[\omega \mid \omega \in B_i] \in A_i = [\gamma_{i-1}, \gamma_i]$ for each $i = 0, 1, \dots, n-1$;
- **incentive compatible** if deviating to fully reveal the state is never profitable:

$$A_i \subseteq \bigcup_{j=i}^{n-1} B_j \tag{IC}$$

for all $i = 1, \dots, n-1$.[10]

Obedience requires that, for each i, it is optimal for Receiver to play a_i when she believes that the state is contained in B_i. To understand Condition (IC), consider any state $\omega \in A_i$. In this state, Receiver would take action a_i under complete information, that is, when the state is fully revealed. $A_i \subseteq \cup_{j \geq i} B_j$ then implies that in state ω the partition recommends an action at least as preferred as a_i, and under obedience Receiver obeys this recommendation. Consequently, deviating to fully reveal the state is never profitable.

Lemma 3.1 characterizes the set of partitions that are associated with an ORE. This result highlights an appealing property of this class of equilibria: identifying an equilibrium can be reduced to finding a partition of the state space that satisfies two properties. [**KZ:** I think what we can do is to make "weaker" statements here and give all credits to our (PVI) paper.]

Lemma 3.1. *A partition $\{B_i\}_{i=0}^{n-1}$ is associated with an ORE if and only if it is both obedient and incentive compatible.*

Lemma 3.1 essentially follows from Theorem 1 in Titova and Zhang (2025); for completeness, we provide some intuition below, and the proof is relegated to the Supplementary Appendix.

The "only if" direction of Lemma 3.1 follows directly from the definition of ORE. For the other direction, obedience guarantees that for each $i = 0, \dots, n-1$, using action a_i with probability 1 is a best response to B_i, and hence Receiver does not have a profitable deviation. It remains to show that if a partition $\{B_i\}_{i=0}^{n-1}$ further satisfies (IC), Sender cannot profitably deviate from the strategy σ given by $\sigma(B_i \mid \omega) = 1$ for all $\omega \in B_i \backslash (\cup_{j>i} B_j)$. This is because for any $\omega \in B_i \backslash (\cup_{j>i} B_j)$, B_i is either the lone available on-path message or the most profitable one. For off-path messages, consider the *maximally skeptical beliefs*:

[10] For $i = 0$, (IC) trivially holds.

for any $m \notin \{B_i\}_{i=0}^{n-1}$, $p(\min m \,|\, m) = 1$.[11] In words, Receiver believes that the state is the lowest possible one when she sees an off-path message. Then because each message must contain the true state, any off-path message is no better than the fully-revealing message. Since (IC) ensures that deviating to fully reveal the state is never profitable, it also deters all other deviations.

3.2 Sender's Preferred Equilibria

To study Sender's preferred equilibria, it is useful to introduce the following definition, which describes a special class of partitions. Let $\mathrm{co}(\cdot)$ denote the convex hull and denote by $\mathrm{cl}(\cdot)$ the closure,

Definition 4. A partition $\{B_i\}_{i=0}^{n-1}$ is **laminar** if it satisfies

$$B_i = \mathrm{cl}\left(\mathrm{co}\,(B_i) \Big\backslash \bigcup_{k<i} \mathrm{co}\,(B_k)\right) \tag{2}$$

for each $i \in \{0, 1, \ldots, n-1\}$.[12]

The definition implies that each partitional element of a laminar partition can be identified by the smallest closed interval that contains it (which is its convex hull), and it is obtained by "taking out" the convex hulls of the elements with a lower index number from the aforementioned closed interval.

If $\{B_i\}_{i=0}^{n-1}$ is a laminar partition, for $i, j \in \{0, 1, \ldots, n-1\}$ with $i < j$, either $\mathrm{int}\,(\mathrm{co}\,(B_i)) \cap \mathrm{int}\,\left(\mathrm{co}\,\left(B_j\right)\right) = \emptyset$, or $\mathrm{co}\,(B_i) \subseteq \mathrm{co}\,\left(B_j\right)$.[13] In words, for any two elements of the laminar partition, either the interiors of their respective convex hulls do not intersect, or the convex hulls are "nested" in that the convex hull of the element with a higher index contains the one with a lower index. Figure 3 illustrates the restrictions imposed by the laminar structure.

If B_j is the lowest partitional element with nonempty interior,[14] by definition, it must

[11] The maximally skeptically beliefs are widely used in the literature in disclosure games. Our results; however, do not hinge on this particular choice of off-path beliefs: they are used for simplicity.

[12] The term "laminar" is borrowed from Candogan and Strack (2023). In fact, if $\{B_i\}_{i=0}^{n-1}$ is a laminar partition according to our definition, and for each $i = 0, \ldots, n-1$, let $J_i = B_i \backslash \cup_{k<j} B_k$, then $\{J_i\}_{i=0}^{n-1}$ is a laminar partition according to Definition 2 in Candogan and Strack (2023). However, our definition is less permissive than theirs.

[13] For a subset S of $[0, 1]$, $\mathrm{int}\,(S)$ denotes the interior of S.

[14] "Lowest partitional element" refers to the partitional element with the lowest index.

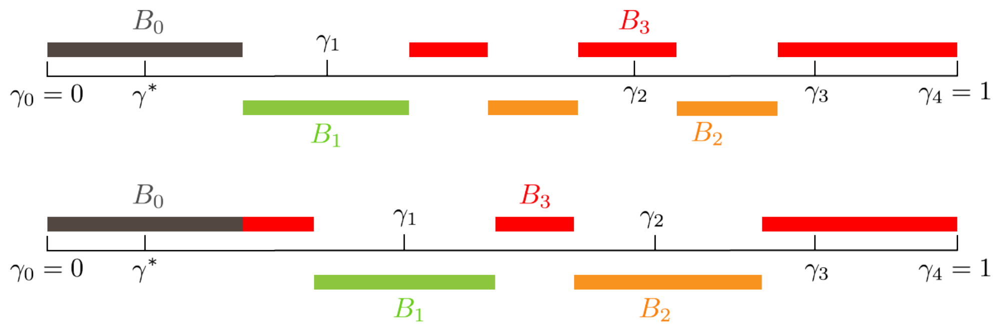

Figure 3: The partition in the upper panel is not laminar since (2) fails to hold for B_3. The one in the lower panel is a laminar partition: (2) holds for each $i = 0, 1, 2, 3$; and for any pair of its elements, the convex hulls are either nested or have disjoint interiors.

be that $B_j = \mathrm{co}\,(B_j)$, and hence B_j must be an interval. In fact, this observation holds more generally:

Observation 3.2. *Let $\{B_i\}_{i=0}^{n-1}$ be a laminar partition. For each $i = 0, \dots, n-1$, B_i is the union of at most $i+1$ closed intervals.*

Because $\mathrm{co}\,(B_k)$ is an interval for any $k = 0, 1, \dots, n-1$, $\cup_{k<i}\,\mathrm{co}\,(B_k)$ is the union of at most i intervals. Thus, by taking out $\cup_{k<i}\,\mathrm{co}\,(B_k)$ from $\mathrm{co}\,(B_i)$, at most i intervals are removed, and hence the remainder, namely $\mathrm{co}\,(B_i)\setminus \cup_{k<i}\,\mathrm{co}\,(B_k)$, must be the union of at most $i+1$ intervals. Then by (2), B_i is also the union of at most $i+1$ intervals.

For a laminar partition $\{B_i\}_{i=0}^{n-1}$, Observation 3.2 allows us to identify B_i by $\mathrm{cl}\,(\mathrm{int}(B_i))$, where $\mathrm{int}\,B_i$ denotes the interior of B_i, for each $i = 0, 1, \dots, n-1$. For example, if $B_i = \{0\} \cup [1/4, 1/2] \cup \{3/4\}$, we identify B_i by $[1/4, 1/2]$. This helps to avoid unnecessary discussion on zero-measure intervals and simplifies the analysis.

The first main result of the paper characterizes a Sender's preferred equilibrium.

Theorem 3.3. *There exists a Sender's preferred equilibrium that is an ORE such that*

(a) the associated partition $\{B_i\}_{i=0}^{n-1}$ is laminar; and for each $k = 0, \dots, n-1$, B_k is the union of at most $\max\{k, 1\}$ *closed intervals.*

(b) Sender's payoff is strictly higher than that in any fully revealing equilibrium.

(c) Sender maximally exploits Receiver's obedience: for all but the lowest nonempty partitional element, $\mathbb{E}[\omega \mid \omega \in B_k] = \gamma_k$.

Part (a) of Theorem 3.3 states that there exists a Sender's preferred equilibrium that is an ORE associated with a laminar partition. In this equilibrium, every on-path message is the union of at most $n-1$ disjoint intervals; in many cases, the number of intervals needed is much smaller. This implies that Sender can achieve the highest equilibrium payoff through messages with a relatively simple structure. Specifically, complex messages, such as those comprising a disjoint union of countably many subsets of states, or mixed strategies are unnecessary. However, for reasons that will become clear after Proposition 3.4, it may be necessary for Sender to pool nonadjacent states; that is, Sender may need to use messages that are the unions of disjoint closed intervals to achieve her maximal payoff.

As long as Receiver's action space is finite, in all but finitely many states, Sender does not strictly benefit from a marginal increase in Receiver's perception of the expected state. This makes messages that pool high and low states together sustainable in equilibrium. Compared to full revelation, such pooling messages do not hurt Sender in high states and strictly benefit Sender in low states, and hence Part (b) of Theorem 3.3 must hold. Part (c) of Theorem 3.3 further shows that in this equilibrium, except from the message recommending the lowest on-path action, Sender pools low states with high states in her message until the expected state following the message is *just high enough* for Receiver to take the recommended action. Put differently, Sender maximally exploits Receiver's obedience.

The next result shows that the laminar structure and Theorem 3.3 (c) together allow for further simplification of incentive compatibility. Fixing a laminar partition $\{B_i\}_{i=0}^{n-1}$, that an action a_i with $i \geq 1$ is **skipped** if $B_i = \emptyset$. That is, from an ex ante perspective, action a_i is recommended with probability zero. For each action a_i that is not skipped, let $\overline{b}_i := \sup B_i$, and $\underline{b}_i := \inf B_i$. Say that action a_j that is not skipped is **immediately below** action a_i if $j = \max\{k < i : B_k \neq \emptyset\}$; that is, a_j is the highest unskipped action "below" a_i.

Proposition 3.4. *Let $\{B_i\}_{i=0}^{n-1}$ be a laminar partition satisfying Part (c) of Theorem 3.3. Then $\{B_i\}_{i=0}^{n-1}$ is incentive compatible if and only if*

(i) for $i \geq 1$, if a_i is skipped, the action a_j immediately below a_i is such that $\overline{b}_j \leq \gamma_i$, and

(ii) if B_k and B_ℓ are such that $\mathrm{co}(B_k) \subseteq \mathrm{co}(B_\ell)$, $\overline{b}_k \leq \gamma_\ell$.

An implication of Proposition 3.4 is that deviating to full revelation is profitable only when an action a_i with $i < n-1$ is induced too frequently within the laminar partition. To check whether a laminar partition satisfies (IC), it suffices to ensure that a weaker condition holds: the highest state in which a lower action is recommended is smaller than the

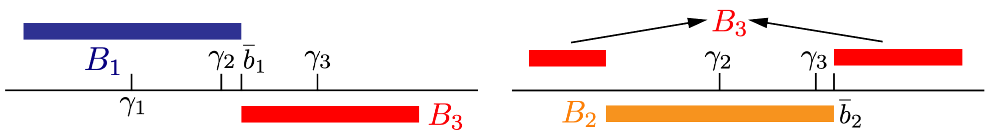

Figure 4: Violation of the conditions in Proposition 3.4: in the left panel condition (i) is violated because action a_2 is skipped, but $\gamma_2 < \overline{b}_1$ where a_1 is a "lower" action that is not skipped; in the right panel condition (ii) is violated since $\mathrm{co}(B_2) \subseteq \mathrm{co}(B_3)$, but $\overline{b}_2 > \gamma_3$.

lowest state in which a higher action is optimal under complete information.[15] Moreover, in finding Sender's preferred equilibrium, this "threat" is only relevant for skipped actions and the actions immediately below them, and for those actions associated with partitional elements whose convex hull contains another partitional element. This is illustrated in Figure 4.

Like in Candogan and Strack (2023), the laminar structure emerges because of the incentive constraints. However, Proposition 3.4 reveals that, unlike in Candogan and Strack (2023) where Receiver is privately informed and the laminar structure is useful in suppressing Receiver's incentives of misrepresenting her private information, here the laminar structure optimally balances the tension between preventing Sender's deviations (fully revealing the state) and inducing desirable actions with sufficiently high probabilities.

Fixing the frequencies of recommending each of the actions, laminar partitions are the most deviation-proof partitions: if a partition is incentive compatible, a corresponding laminar partition that also satisfies (IC) must exist. The laminar structure not only makes incentive compatibility easier to satisfy (as indicated by Proposition 3.4), but also ensures that states above γ_i pooled into B_i are those *closest* to γ_i, thereby minimizing the chance of violating $\overline{b}_i \leq \gamma_k$ for $k \geq i$. This is particularly clear when B_i is an interval, where all states above γ_i in B_i are "immediately above" γ_i.

Furthermore, pooling nonadjacent states is useful because it grants more flexibility in increasing the probability of inducing certain actions without violating incentive compatibility. To see this, suppose Sender wants to recommend both a_1 and a_2 with high probabilities, and γ_1 and γ_2 are relatively close to each other. As shown in the left panel of Figure 5, by pooling adjacent states via interval messages, while a_2 can be recommended with high probability, it largely "squeezes out" the chance of inducing a_1. Employing a message that is the union of two intervals allows for a large increase in the frequency

[15] We use "smaller," "increasing," and "greater" in the weak sense: "strictly" will be added whenever needed.

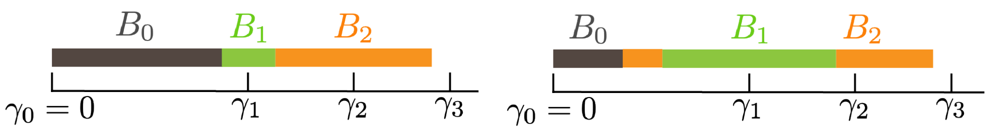

Figure 5: Pooling adjacent states versus pooling nonadjacent states. In both panels, the total probability of recommending actions a_0, a_1, and a_2 are the same.

of recommending a_1 without reducing the occurrence of a_2 too much (the right panel of Figure 5). This is particularly profitable when the gap in Sender's utility between actions a_2 and a_1 is significantly smaller than that between actions a_1 and a_0.

3.2.1 Proof Outline of Theorem 3.3

To establish Theorem 3.3, we first show that for every Sender's preferred equilibrium, we can construct an ORE in which Sender's payoff is the same. Thus, focusing on OREs is without loss.

To show that it suffices to focus on laminar partitions, we first observe that an ORE induces a distribution of posterior means G with support on at most n points, and this distribution must be a mean-preserving contraction (MPC) of the prior. Making use of a slight variant of a result in Candogan and Strack (2023), we show that such a distribution induces a laminar partition.[16] The argument is then completed by noting that laminar partitions exhibit the most resistance to violations of (IC). Furthermore, we find that if B_0 is nonempty, then it must be that $B_0 = [0, d]$ for some $d \leq \gamma_1$; thus, it is not contained in the convex hull of any other partitional elements. Then according to (2), for any $k \geq 1$, B_k must be the union of at most k intervals.

Now Lemma 3.1 implies that the search for Sender's preferred equilibrium can be reduced to finding a laminar partition that is both obedient and incentive compatible, and yields the highest Sender's payoff. To show that such a laminar partition exists, we first note that the space $\mathcal{C}_c$ of nonempty, closed, and convex subsets of $[0, 1]$ endowed with the Hausdorff distance is a compact metric space. Then since every component of a laminar partition is the union of at most $n-1$ closed intervals, it can be identified by an element of $\mathcal{C}_c^{n(n-1)}$;[17] $\mathcal{C}_c^{n(n-1)}$ is compact in the product topology. We show that (i) Sender's payoff is continuous on $\mathcal{C}_c^{n(n-1)}$, and (ii) the set of laminar partitions that satisfy obedience and

[16]This relies on the assumption that there is a continuum of states, and the prior distribution F has full support.

[17]$\mathcal{C}_c^{n(n-1)}$ denotes the $n(n-1)$-fold Cartesian product of $\mathcal{C}_c$ with itself.

incentive compatibility is a nonempty closed subset of $C_c^{n(n-1)}$. Then by the extreme value theorem, the desired laminar partition must exist; this begets the existence of Sender's preferred equilibrium.

The reason that Sender's payoff in her preferred equilibrium is strictly higher than that in a fully revealing equilibrium is already explained. To see that Sender must maximally exploit the obedience constraint, suppose $\mathbb{E}[\omega \mid \omega \in B_k] > \gamma_k$ for some $k > j = \min\{i : \operatorname{int}(B_i) \neq \emptyset\}$, then Sender can strictly increase her payoff by "taking out" a subset of B_j of strictly positive measure and "merge" it with B_k.

3.3 Other ORE and Equilibrium Payoff Set

Theorem 3.3 shows that Sender's highest payoff can be achieved in an equilibrium in which every on-path message acts as an action recommendation that Receiver finds optimal to follow, and the messages have a simple laminar structure. Two natural questions arise. First, do any other equilibria exhibit a similar laminar structure? Second, what is the set of Sender payoffs that are achievable in equilibria? These two questions are answered in Proposition 3.5.

Proposition 3.5. *Sender can attain a payoff in an equilibrium if and only if this payoff is less than Sender's payoff in a Sender's preferred equilibrium and is greater than that in a fully revealing equilibrium. Furthermore, any such payoff can be attained by an ORE associated with a laminar partition.*

Proposition 3.5 reveals that with a finite number of actions–no matter how large the number is–there is a continuum of OREs featuring distinct Sender payoffs. By definition, in all of these equilibria, both Sender and Receiver play pure strategies; in other words, these equilibria emerge without any need for mixed strategies. In particular, distinct equilibrium payoffs are supported by different sets of on-path messages, and every on-path message is relatively simple in that it is the union of at most $n-1$ disjoint intervals. Therefore, in settings where a rich message space is plausible, a large set of pooling messages can be sustained in equilibrium, even if attention is restricted to simpler ways of pooling. This may provide an explanation for the wide variety of disclosure strategies people observe.

The "only if" direction of Proposition 3.5 is straightforward. By definition, Sender's payoff in any equilibrium is lower than that in her preferred equilibria. Incentive compatibility implies that it must also be higher than her payoff in a fully revealing equilibrium, as otherwise Sender can always deviate to fully reveal the state. The proof of the other direction is more involved and constructive. Observe that the ORE defined by the laminar partition $\{A_i\}_{i=0}^{n-1}$ generates the same Sender payoff as in a fully revealing equilibrium, denoted as R_U. Let $\{B_i^S\}_{i=0}^{n-1}$ denote the laminar partition associated with a Sender's preferred equilibrium with Sender payoff R_S. For any $R \in [R_U, R_S]$, we identify a laminar partition "blending" $\{A_i\}_{i=0}^{n-1}$ and $\{B_i^S\}_{i=0}^{n-1}$ that achieves payoff R, and show that it is both obedient and incentive compatible. Lemma 3.1 then ensures that R can be induced by an ORE defined by this laminar partition.

4 When Are Commitment Outcomes Achievable?

The discussion on Sender's preferred equilibria in Section 3 does not address an important question: how well can Sender do in her preferred equilibria? Put differently, to what extent can Sender benefit from verifiable communication? Naturally, an upper bound of Sender's payoff is given by Sender's commitment payoff, that is, her payoff when she can *commit* on what messages to send in each of the states. In this section, we provide a partial answer to the aforementioned question by identifying conditions under which Sender can attain her commitment payoff in an equilibrium of the disclosure game.

This section presents two sets of results. The first set of results (Section 4.2) provides necessary and sufficient conditions for implementing a particular commitment outcome. The second set of results (Section 4.3) provides sufficient conditions on model primitives under which commitment has no value. To get there, we first characterize a salient class of solutions to the information design problem in Section 4.1.

4.1 Information Design Benchmark

We start by introducing a class of simple distributions of posterior means. A **bi-pooling distribution** partitions the state space to intervals such that in each of them,

(i) either all states are fully revealed,

(ii) or the same (deterministic) signal is sent in all states contained in this interval,

(iii) or two different, possibly random, signals are sent.

See Appendix A.2.1 for a formal definition.

Calling a bi-pooling distribution G_B that solves the information design problem (1) a **bi-pooling solution**, the following lemma is a direct consequence of results in Kleiner et al. (2021), Candogan (2022), and Arieli et al. (2023).

Lemma 4.1. *The information design problem admits a bi-pooling solution. Furthermore, if no three elements of the collection of points $\{(\gamma_i, v(a_i))\}_{i=0}^{n-1}$ are collinear, then a bi-pooling solution is the unique solution to the information design problem (1).*

Lemma 4.1 indicates that, despite the simplicity of bi-pooling distributions, finding a solution to the information design problem (1) merely requires searching within them. Moreover, under mild conditions, the *unique* solution to this problem is a bi-pooling distribution.[18]

Theorem 3.3 shows that a Sender's preferred equilibrium of the disclosure game can be identified by a laminar partition. If this laminar partition induces the same outcome as a bi-pooling solution, then Sender achieves the commitment payoff. It is, therefore, tempting to ask whether one can identify a bi-pooling solution by such a partition. Next, we define a special class of laminar partitions that are capable of identifying bi-pooling solutions.

Definition 5 (Nested intervals partition)**.** A partition is a **nested intervals partition** if it is laminar, and every partitional element is the union of at most two closed intervals.[19]

Recall that if $\{B_i\}_{i=0}^{n-1}$ is a laminar partition, for any two partitional elements B_j and B_k with $j < k$, either their respective convex hulls do not intersect except at the boundaries, or $\mathrm{co}(B_j) \subsetneq \mathrm{co}(B_k)$. If it is further a nested intervals partition, $\mathrm{co}(B_j) \subsetneq \mathrm{co}(B_k)$ implies that B_j is a closed interval, and B_k is the union of two closed intervals that "nests" B_j. Put differently, there exist $\underline{b}_k < \underline{b}_j < \overline{b}_j < \overline{b}_k$ such that $B_j = [\underline{b}_j, \overline{b}_j]$ and $B_k = [\underline{b}_k, \underline{b}_j] \cup [\underline{b}_k, \underline{b}_j]$. In this case, we say that $\{B_j, B_k\}$ is a pair of **nested intervals**. The properties of a nested intervals partition are illustrated in Figure 6.

Lemma 4.2, which is largely based on Lemma 4 in Arieli et al. (2023), shows that every bi-pooling solution can be identified by a *unique* nested intervals partition.

Lemma 4.2. *Let G_B be a bi-pooling solution. There exists a unique nested intervals partition $\{B_i\}_{i=0}^{n-1}$ such that G_B can be induced by an experiment such that in any $\omega \in B_i \setminus (\cup_{k<i} B_k)$, a single signal realizes with probability one. Moreover, for all but the lowest nonempty partitional element, $\mathbb{E}[\omega \mid \omega \in B_i] = \gamma_i$.*

[18]Bi-pooling solutions are also easy to identify: a bi-pooling solution can be found via solving a finite-dimensional convex program proposed by Candogan (2022), which admits a polynomial algorithm. That is, the number of computational steps in the algorithm can be expressed as a polynomial of n, the number of actions.

[19]The term "nested intervals" is borrowed from Guo and Shmaya (2019).

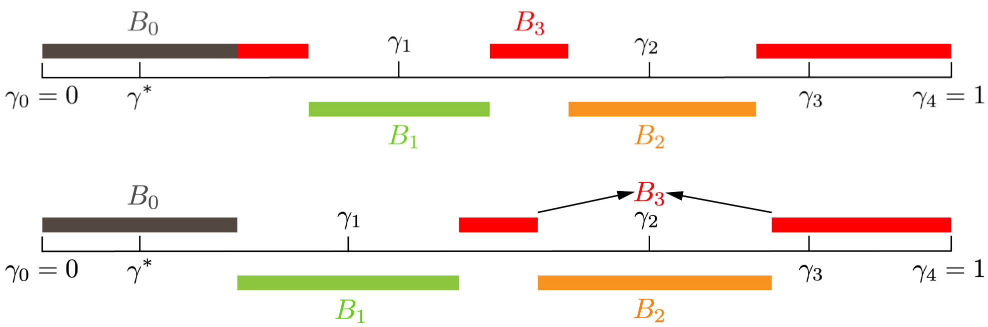

Figure 6: The laminar partition in the upper panel is not a nested intervals partition because B_3 is the union of *three* closed intervals; and the laminar partition in the lower panel is a nested intervals partition because for each $i = 0, 1, 2, 3$, B_i is either a closed interval or the union of two closed intervals. In particular, $\{B_2, B_3\}$ is a pair of nested intervals.

Therefore, every signal realization can be interpreted as a recommended action: for every $i = 0, \ldots, n-1$, $B_i \backslash (\cup_{j>i} B_j)$ is the set of states in which action a_i is recommended with probability one. Every recommendation is obedient in the sense that Receiver's optimal action coincides with the action recommendation.

4.2 Characterizing Implementability

Recall that an information design solution is implementable if it can be induced by an equilibrium of the disclosure game. The following result characterizes the implementability of a bi-pooling solution.

Proposition 4.3. *A bi-pooling solution is implementable if and only if the associated nested interval partition $\{B_i\}_{i=0}^{n-1}$ is incentive compatible.*

[**MT:** check here, too] [**KZ:** Yes the following paragraph needs to be revised; we now need to compare it with TZ's Theorem 2. We can probably use "weaker" statements to comment on the differences between this and TZ Theorem 2. My rough attempt below.]

Proposition 4.3 says that to check whether a bi-pooling solution is implementable, it suffices to find the associated nested interval partition (identified in Lemma 4.2) and see if incentive compatibility holds. This result is closely related to Theorem 2 in Titova and Zhang (2025), which shows (adapted to our setting with more structures) that an *experiment* is implementable if and only if it is identified by an incentive compatible partition. In our setting, a bi-pooling solution can be induced by multiple experiments. In contrast, Proposition 4.3 shows that a bi-pooling solution, which is the unique solution under mild conditions, is implementable *if and only if* the associated nested interval partition is incentive compatible.

While the "if" direction is analogous, the "only if" direction is new. This direction builds on two important observations. First, if a bi-pooling solution is implementable, Theorem 3.3 implies that it can be induced by an ORE, and hence focusing on partitions suffices. Second, among all partitions that identify a bi-pooling solution, the nested intervals partition exhibits the greatest resilience against deviations to full revelation. This is somewhat expected because nested interval partitions are special cases of laminar partitions, and laminar partitions enjoy a similar property.

The next result, which is a corollary of Proposition 4.3 and Proposition 3.4, goes one step further: it reveals the exact features of the nested intervals partition that fail incentive compatibility and hence prevent the bi-pooling solution from being implementable.

Corollary 4.4. *A bi-pooling solution is implementable if and only if the associated nested intervals partition is such that*

(i) for $i \geq 1$, if a_i is skipped, the action a_j immediately below a_i is such that $\overline{b}_j \leq \gamma_i$, and

(ii) if $\{B_k, B_\ell\}$ is a pair of nested intervals, $\overline{b}_k \leq \gamma_\ell$.

Compared to Condition (IC), the two conditions in Corollary 4.4 are easier to verify when determining whether a bi-pooling solution is implementable. They also allow us to identify sufficient conditions under which commitment has no value in Section 4.3.

In the following example, taken from Gentzkow and Kamenica (2016), the two conditions in Corollary 4.4 are satisfied, and the commitment outcome is hence implementable.

Example 1 (Gentzkow and Kamenica, 2016)**.** Suppose $\mathcal{A} = \{a_0, a_1, a_2\}$, and $\gamma_1 = 1/3$, $\gamma_2 = 2/3$; that is, $A_0 = [0, 1/3]$, $A_1 = [1/3, 2/3]$, and $A_2 = [2/3, 1]$. The prior F is uniform. Assume also that $v(a_1) = 1$ and $v(a_2) = 3$. Gentzkow and Kamenica (2016) show that

$$B_0 = [0, 8/48]\,,\ B_1 = [11/48, 21/48]\,,\ B_2 = [8/48, 11/48] \cup [21/48, 1]$$

is the nested intervals partition of a bi-pooling solution to the information design problem.[20] Because no action is skipped, and the nested intervals partition $\{B_1, B_2\}$ satisfies $\overline{b}_1 \leq \gamma_2$, by Corollary 4.4, this bi-pooling solution can be implemented. Consequently, Sender's commitment payoff can be attained in an equilibrium of the disclosure game.

[20] Claim C.4 in the Supplementary Appendix shows that when there are three actions, all bi-pooling solutions have the same nested intervals partition.

Proposition 4.3 and Corollary 4.4 may have created a feeling that implementing a bi-pooling solution is relatively simple: one only needs to ensure that no action is recommended more often than incentive compatibility allows. As pointed out by Titova and Zhang (2025), incentive compatibility imposes no restrictions when there are only two actions.

Observation 4.5. *If there are two actions, the unique bi-pooling solution can be implemented.*

When there are three or more actions; however, the restriction imposed by incentive compatibility need not be weak in general. Example 2 and Example 3 illustrate that even a "slight perturbation" of binary actions environments could prohibit bi-pooling solutions from being implemented.

Example 2. Suppose $\mathcal{A} = \{a_0, a_1, a_2\}$, and $\gamma_1 = 0.5$, $\gamma_2 = 0.9$; that is, $A_0 = [0, 0.5]$, $A_1 = [0.5, 0.9]$, and $A_2 = [0.9, 1]$. The prior F is uniform. Assume that $v(a_1) = 1$ and $v(a_2) = 1.1$. The unique solution to the information design problem has nested interval partition $\{B_0, B_1, B_2\}$ with $B_1 = [0, 1]$ and $B_0 = B_2 = \emptyset$. Action a_2 is skipped, and $0.9 = \gamma_2 < \bar{b}_1 = 1$, which violates Condition (i) in Corollary 4.4. Therefore, this bi-pooling solution cannot be implemented.

When there are two actions, the Sender's sole objective is to maximize the probability that the "high action" a_1 is played. This, in turn, suggests that incentive compatibility is never an issue: in any state in which a_1 is played under complete information, there is no reason to recommend a_0. However, with three actions, as Gentzkow and Kamenica (2016) note, in the information design problem Sender needs to trade-off between how frequently a_1 and a_2 are taken. In Example 2, the gap between $v(a_1)$ and $v(a_2)$ is significantly smaller than that between $v(a_0)$ and $v(a_1)$, and hence it is more profitable to induce a_1 more often: to guarantee obedience, recommending a_2 more often must come with a_0 being played more frequently. Consequently, the unique information design solution recommends a_1 with probability one.[21] Then in states strictly higher than γ_2, Sender is strictly better off by fully revealing the state, rendering the information design solution not implementable.

Example 3. Suppose $\mathcal{A} = \{a_0, a_1, a_2\}$, and $\gamma_1 = 0.6$, $\gamma_2 = 0.7$; that is, $A_0 = [0, 0.6]$, $A_1 = [0.6, 0.7]$, and $A_2 = [0.7, 1]$. The prior F is uniform. Assume $v(a_1) = 1$ and $v(a_2) = 1.3$. The

[21] The same logic applies to the example in Section 2.1.

unique information design solution has the nested interval partition

$$B_0 = [0, 0.267],\ B_1 = [0.356, 0.845],\ \text{and } B_2 = [0.267, 0.356] \cup [0.845, 1].$$

Although no action is skipped, since $\overline{b}_1 = 0.845 > 0.7 = \gamma_2$, Condition (ii) in Corollary 4.4 fails to hold, and thus this bi-pooling solution cannot be implemented.

4.3 Sufficient Conditions for Implementability

In this section we identify conditions on model primitives under which an implementable commitment outcome must exist. To simplify notation, we write v_i instead of $v(a_i)$. Let $h(\gamma_i; \gamma_{i+1})$ denote the unique solution of $\mathbb{E}[\omega \mid \omega \in [h(\gamma_i; \gamma_{i+1}), \gamma_{i+1}]] = \gamma_i$ if the latter has a solution. If not, set $h(\gamma_i; \gamma_{i+1}) = 0$. Intuitively, $h(\gamma_i; \gamma_{i+1})$ is the lowest state such that the conditional mean of the states between this state and γ_{i+1} is no less than γ_i.

Proposition 4.6. *Suppose there are three or more actions. If*

$$\frac{v_{i+1} - v_i}{\gamma_{i+1} - \gamma_i} > \frac{v_i - v_{i-1}}{\min\{\gamma_i - \gamma_{i-1}, \gamma_i - h(\gamma_i; \gamma_{i+1})\}}, \tag{3}$$

for all $i = 1, \ldots, n-2$, then every bi-pooling solution is implementable. Consequently, the commitment payoff is attained in an equilibrium of the disclosure game.

In Condition (3), $v_{i+1} - v_i$ is Sender's marginal benefit of inducing a higher action evaluated at action a_i, and $\gamma_{i+1} - \gamma_i$ is the difference in cutoffs for inducing actions a_{i+1} and a_i under complete information, respectively. Condition (3) is satisfied whenever (i) the linear interpolation of the jump points in Sender's value function (illustrated in Figure 1) is convex, and (ii) the prior distribution does not accumulate too much mass around any point below the highest cutoff γ_{n-1}.

Proposition 4.6 suggests that in a communication environment, if inducing a marginally higher action is either sufficiently more profitable or sufficiently more difficult (requiring a sufficiently larger expected state), or both, then Sender does not benefit from commitment power. Put differently, Sender does not value commitment when her value function increases sufficiently fast in the expected state.

To get a closer look into how Condition (3) works, consider the following example. Let $\{B_i\}_{i=0}^{n-1}$ be a nested intervals partition of a bi-pooling solution, and suppose B_3 is a closed

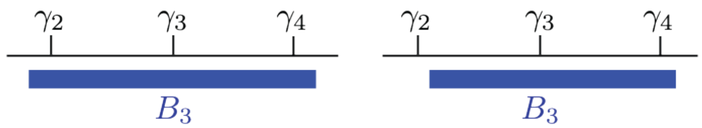

Figure 7: The two cases when B_3 is an interval with $\overline{b}_3 > \gamma_4$. In the left panel there exists $\gamma_k \in B_3$ with $k < 3$, and in the right panel $\gamma_2 \notin B_3$.

interval with $\overline{b}_3 > \gamma_4$.[22] By Corollary 4.4, this bi-pooling solution cannot be implemented; we show that Sender can strictly improve her payoff, which leads to a contradiction.

There are two cases: there exists $\gamma_k \in B_3$ with $k < 3$, or there is not; these two cases are illustrated in Figure 7. In the first case, without loss of generality, assume $\gamma_2 \in B_3$; and in the second case, it must be that $\gamma_2 < \underline{b}_3$. Observe that Condition (3) is equivalent to that both of the inequalities below must hold:

$$\frac{v_{i+1} - v_i}{\gamma_{i+1} - \gamma_i} > \frac{v_i - v_{i-1}}{\gamma_i - \gamma_{i-1}}, \tag{4}$$

$$\frac{v_{i+1} - v_i}{\gamma_{i+1} - \gamma_i} > \frac{v_i - v_{i-1}}{\gamma_i - h(\gamma_i; \gamma_{i+1})}. \tag{5}$$

Interpolating (γ_0, v_0), (γ_1, v_1), ..., and (γ_{n-1}, v_{n-1}), a piecewise linear function on $[0, \gamma_{n-1}]$ is obtained; illustrated in the left panel of Figure 8, inequality (4) says that the slope of any "piece" is strictly larger than that of the piece to its left. Then in the first case, as shown in the right panel of Figure 8, (4) implies–essentially because of the convexity–reducing the probability of recommending action a_3 and increasing the probability of recommending both a_2 and a_4 is strictly profitable. In the proof, we show that such a profitable deviation is feasible because the resulting distribution of posterior means is still feasible for the information design problem (1).

In the second case where $\gamma_2 \notin B_3$, $h(\gamma_3; \gamma_4)$, which is the state that the conditional mean of the states between this state and γ_4 is exactly γ_3, must be in B_3 since $\gamma_4 \in B_3$ and $\mathbb{E}[\omega \,|\, \omega \in B_3] = \gamma_3$ by Lemma 4.2. Replacing γ_2 by $h(\gamma_3; \gamma_4)$, by inequality (5), the previous argument goes through *mutatis mutandis*.

Imposing an assumption on the prior, Condition (3) can be substantially simplified.

Corollary 4.7. *Suppose there are three or more actions. If for all* $i = 1, \dots, n-1$, $v_{i+1} - v_i \geq v_i - v_{i-1}$ *and* $\gamma_{i+1} - \gamma_i \leq \gamma_i - \gamma_{i-1}$ *with one of the inequalities being strict, and* f *is increasing, then every bi-pooling solution is implementable. Consequently, the commitment payoff is attained in an equilibrium.*

[22] If B_3 is the union of two closed intervals, a bit more work is needed, but the underlying idea is similar.

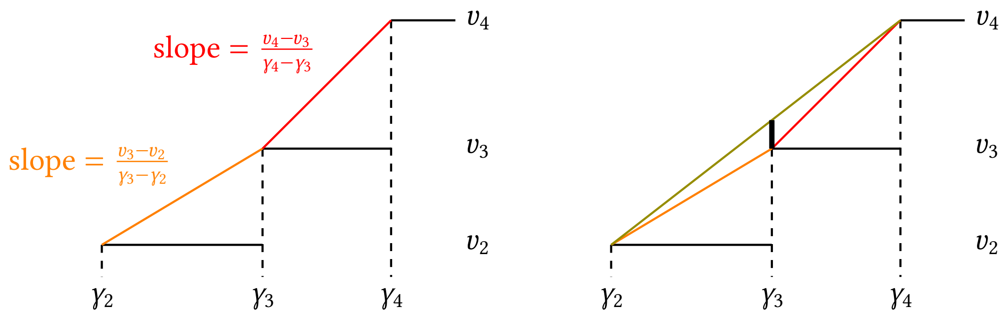

Figure 8: The left panel illustrates the geometric interpretation of inequality (4). The right panel shows that, when inequality (4) holds and $\gamma_2 \in B_3$, induce actions a_2 and a_4 more often, and action a_3 less often is strictly profitable to Sender.

4.4 The Special Case of Ternary Actions

It is instructive to take a deeper dive into the case in which Receiver has three actions. In this case, a partition can be written as $\{B_0, B_1, B_2\}$. As noted at the end of the second paragraph in Section 3.2.1, in a laminar partition corresponding to a Sender's preferred equilibrium, B_0 must be an interval if not empty; and B_1 and B_2 form a pair of nested intervals.[23] Consequently, Proposition 3.4 implies that Condition (IC) boils down to that $\sup B_1 \leq \gamma_2$.

Armed with these observations, a Sender's preferred equilibrium can be explicitly solved.

Claim 4.8. *The Sender's preferred equilibrium outcome either coincides with the commitment outcome, or it can be induced by an ORE defined by* $\{B_0, B_1, B_2\}$ *such that* $B_2 = [y, h] \cup [\gamma_2, 1]$, $B_1 = [h, \gamma_2]$, *and* $B_0 = [0, y]$, *where* $h > 0$, $y \geq 0$ *solve the system of equations*

$$\mathbb{E}[\omega \mid \omega \in [h, \gamma_2]] = \gamma_1;$$
$$\mathbb{E}[\omega \mid \omega \in [y, h] \cup [\gamma_2, 1]] = \gamma_2.$$

When there are only three actions, the only reason that a commitment outcome is not implementable is that the "middle action" a_1 is recommended too often. Therefore, if no commitment outcome is implementable, in the nested intervals partition corresponding to a Sender's preferred equilibrium, a_1 is recommended as frequently as incentive compatibility allows: that is, the upper bound of B_1 must coincide with γ_2. The proofs of all results in this subsection are relegated to the Supplementary Appendix.

[23] Both B_0 and B_1 may be empty, but B_2 cannot be: otherwise, (IC) must be violated.

When there are three actions, the sufficient conditions can be further simplified.

Corollary 4.9. *If $|\mathcal{A}| = 3$, f is increasing, and $v_2 > 2v_1$, then all commitment outcomes are implementable.*

Example 1 revisited. With the help of Corollary 4.9, to assert that Sender does not benefit from commitment power, one does not need to solve the information design problem. Because the prior F is uniform, its density $f = 1$ is constant, and hence increasing. Then since $v_2 = 3v_1 > 2v_1$, by Corollary 4.9, the commitment payoff must be attained in an equilibrium of the disclosure game.

5 Applications

5.1 Selling with Quality Disclosure

This section studies a variant of the model of a sales encounter studied in Section 5 of Milgrom (1981). The state of the world, ω, is interpreted as the quality of the seller's product. Let $p > 0$ be the unit price, and there is no quantity discount. Denote the seller's constant unit cost by c, where $0 \leq c < p$. The product is indivisible: the buyer can only buy integer units of the product. The buyer's utility from purchasing q units is $\omega U(q) - pq$, where $U : \mathbb{R}_+ \to \mathbb{R}$ is a bounded, strictly increasing, strictly concave threetimes differentiable function with $U(0) = 0$. We further assume that $U(q) - pq$ is maximized at $n > 1$. As a consequence, the buyer buys at most n units of the product, and she buys nothing if ω is close enough to 0.

The only significant difference between this model and Milgrom's is that he considers a perfectly divisible product, and hence the seller's value function is strictly increasing. In this model; however, indivisibility makes the seller's value function a step function with n jumps. For a perfectly divisible product, Milgrom shows that every equilibrium of the game features full revelation: the seller sends $m = \{\omega\}$ for each $\omega \in [0, 1]$, resulting in the buyer's preferred outcome. With indivisibility; however, the seller may be able to gain considerably from verifiable communication, attaining her commitment payoff.

To state the result, let

$$A(x) = -\frac{U''(x)}{U'(x)}, \quad \text{and} \quad P(x) = -\frac{U'''(x)}{U''(x)}$$

denote the coefficients of absolute risk aversion and absolute prudence, respectively.

Claim 5.1. *If f is increasing and $P > 2A$, there exists an equilibrium of this game in which the seller is as well off as having commitment power.*

The proofs of results in this section are left to the Supplementary Appendix. The assumption of increasing prior density can be interpreted as it is common knowledge that the consumer is relatively confident about the quality of the product. $P > 2A$ is satisfied by, for example, CRRA utility function with parameter $0 < \sigma < 1$.

5.2 Influencing Voters

Consider an amendment voting setting where a voting rule satisfying the Condorcet winner criterion is employed to determine which one of the three alternatives, the (unamended) bill (b), the amended bill (ab), and no bill (maintaining the status quo; $\varnothing$), will prevail.[24] The state of the world is $\omega \in [0, 1]$, and there are $N > 1$ voters; for simplicity, assume that N is an odd number. Voters have linear preferences: for $j = 1, \ldots, N$, voter j's utilities are given by $u_k^j(\omega) = \alpha_k^j + \beta_k^j \omega$, where $k \in \{b, ab, \varnothing\}$ and where $\omega \in [0, 1]$ is the state of the world. Moreover, $\beta_b^j > \beta_{ab}^j > \beta_\varnothing^j = 0$ and $0 = \alpha_\varnothing^j > \alpha_{ab}^j > \alpha_b^j$ for all $j = 1, \ldots, N$. Let γ_1^j and γ_2^j denote the cutoff states that voter j is indifferent between $\varnothing$ and ab, and ab and b, respectively.[25] We impose further assumptions so that $\gamma_1^j < \gamma_2^j$ for all $j = 1, \ldots, N$.

In this model, voter j's preferences over alternative k are characterized by two parameters: one is α_k^j, we call it voter j's *reference point* for alternative k as it is the voter's cardinal utility when the state is zero; another is β_k^j, we call it voter j's *state sensitivity* for alternative k because it measures how fast the voter's cardinal utility increases in the state. Furthermore, all voters agree that when the state is low (intermediate, high), no bill (the amended bill, the bill, respectively) is optimal, but for the amended bill or the bill, different voters may have different reference points and different state sensitivity levels. Figure 9 illustrates a voter's utilities and the resulting cutoffs.

There is an expert who observes the state and can communicate to the voters. We follow Jackson and Tan (2013) to assume that the expert discloses verifiable information.

[24] See, for example, Enelow and Koehler (1980) and Enelow (1981) for examples of amendment voting.

[25] For every $j = 1, \ldots, N$,

$$\gamma_1^j = -\frac{\alpha_{ab}^j}{\beta_{ab}^j} \quad \text{and} \quad \gamma_2^j = \frac{\alpha_{ab}^j - \alpha_b^j}{\beta_b^j - \beta_{ab}^j}.$$

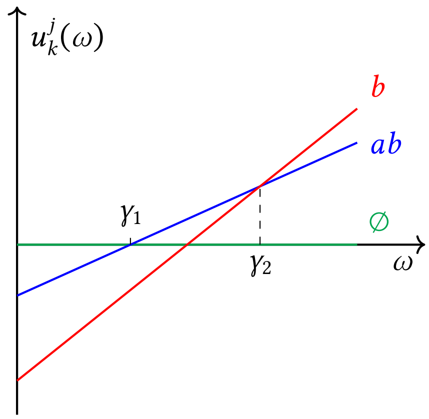

Figure 9: Voter j's utilities.

Unlike their work with two states; however, we consider a continuum of states, and the expert's messages are closed subsets of the state space that contain the true state. Assume that the expert's preferences satisfy $v(b) > v(ab) > v(\varnothing) = 0$; that is, the expert strictly prefers the bill to the amended bill, and the amended bill is strictly preferred to no bill.

Because the voting rule satisfies the Condorcet winner criterion, and the preferences are single-peaked, the Condorcet winner is the median voter's most preferred alternative.[26] Therefore, it suffices to consider the median voter; further assumptions are imposed to make sure that the median voter is the same voter, say voter m, for all states. Consequently, the expert's problem is equivalent to communicating to voter m.[27] Thus, the expert's preferred equilibrium outcome is characterized by Claim 4.8.

Perhaps surprisingly, Claim 5.2 shows that the expert can be hurt if all voters are "more inclined toward" the bill, in the sense that all else equal, either the reference point or the state sensitivity for the bill increases (or both) for all voters.

Claim 5.2. *If no commitment outcome is implementable, and at least one of the following happens:*

(i) voter j's state sensitivity for the bill, β_b^j, increases for all $j = 1, \ldots, N$,

[26]One voting rule used for amendment voting that satisfies the Condorcet winner criterion when voters' preferences are single-peaked is the pairwise majority rule, in which alternatives will be considered sequentially and two at a time using the majority rule.

[27]That is, we assume that for any two voters where one is indexed higher than the other ($i > j$), $\gamma_1^i \geq \gamma_1^j$ and $\gamma_2^i > \gamma_2^j$ (see Footnote 25 for the definition of the cutoffs). A sufficient condition for this assumption is that (i) all voters share the same reference points for both b and ab: $\alpha_b^j = \alpha_b$ and $\alpha_{ab}^j = \alpha_{ab}$ for all $j = 1, \ldots, N$; (ii) among voters, those with a higher index not only exhibit higher state sensitivity for both b and ab but also show a strictly greater *difference* in state sensitivity between these alternatives: $i > j$ implies $\beta_{ab}^i \geq \beta_{ab}^j$, $\beta_b^i \geq \beta_b^j$, and $\beta_b^i - \beta_{ab}^i > \beta_b^j - \beta_{ab}^j$.

(ii) voter j's reference point for the bill, α_b^j, increases for all $j = 1, \ldots, N$,

then the expert's payoff in her preferred equilibria may decrease.

When either (i) or (ii) happens–or both–γ_2^j decreases for all j, and hence γ_2^m must decrease. This means that the expected state required for passing the bill is lowered. Recall from the discussion after Claim 4.8 that, in scenarios where no commitment outcome is implementable, it must be that the expert is too tempted to recommend the amended bill at the ex ante stage. This is because the utility gap between the bill and the amended bill is not as large as that between the amended bill and the status quo (i.e., $v(b) - v(ab) < v(ab) - v(\emptyset)$), which is plausible in many voting scenarios. As γ_2^m decreases, even though the expert may more frequently induce the bill, this also introduces an indirect effect whereby the incentive compatibility constraint tightens, limiting the chance of inducing the amendment bill. Therefore, the expert is hurt if the indirect effect dominates.

6 Discussion

6.1 Assumption on the Message Space

Although our assumption on the message space is standard in the literature, it is useful to discuss the meaning of this assumption as well as how our results rely on it. Importantly, Sender does not need to literally use a closed subset of the state space as her message. What we require is that in each state ω, Sender's message space, or the set of evidence, $\mathcal{M}(\omega)$ is *sufficiently rich* in the sense that for every closed subset C of the state space $[0, 1]$, there is a message m such that in every state contained in C this message is feasible, and for every state not in C this message is infeasible. Formally, for every $\omega \in C$, $m \in \mathcal{M}(\omega)$; and for every $\omega \notin C$, $m \notin \mathcal{M}(\omega)$. In other words, message m provides hard evidence that the state is in C.

Two implications of this assumption are crucial to our results. First, Sender can always fully reveal the state; that is, the fully revealing message $m = \{\omega\}$ is feasible in every $\omega \in [0, 1]$. This implies that in every state, Sender's equilibrium payoff must be no lower than her payoff from fully revealing the state, which gives rise to Condition (IC). It turns out that full revelation is the only kind of deviation that needs to be accounted for, which largely simplifies the analysis. Second, Sender can use messages that are the unions of a finite number of closed intervals; the importance of this feature is already explained in the previous sections.[28]

6.2 Robustness

It is natural to ask whether the equilibria of the disclosure game considered in this paper are credible in that they survive certain equilibrium refinements. We consider the following two equilibrium refinements:

- The Never-a-Weak-Best-Response (NWBR) Criterion, proposed by Cho and Kreps (1987), is a strengthening of a few equilibrium refinements that are extensively used in the literature, which includes the Intuitive Criterion, D1, and D2.[29]

- The Grossman-Perry-Farrell equilibrium, proposed by Bertomeu and Cianciaruso (2018), is based on the perfect sequential equilibrium of Grossman and Perry (1986) and neologism-proofness of Farrell (1993).

It can be shown that every obedient recommendation equilibrium (ORE) of the disclosure game we study is a Grossman-Perry-Farrell equilibrium. Furthermore, after a slight adjustment in the belief system, one can show that every ORE also survives the NWBR Criterion. Interested readers are referred to the Supplementary Appendix for formal details.

6.3 Further Cheap Talk Opportunities

One may wonder what Sender would be able to achieve if she were also allowed to send cheap talk messages. Indeed, Wu (2022) and Dasgupta (2023) show that this benefits Sender in their respective settings. However, this is not the case in our setting.

Define Sender's **value function in beliefs**, denoted by $w : \Delta([0,1]) \to \mathbb{R}$, by $w(G) = u(\mathbb{E}G)$ for any $G \in \Delta([0,1])$, where u is Sender's value function. Theorem 2 in Lipnowski and Ravid (2020) asserts that Sender's optimal value is given by the quasiconcave hull of

[28] One might believe that the payoffs of a laminar partition can be replicated using only closed interval messages, by identifying each message with its convex hull. However, this approach fails. Consider a pair of nested intervals $\{B_\ell, B_h\}$ with $\ell < h$. If one wants $\mathrm{co}(B_h)$ to be sent in every $\omega \in B_h$, this message is also available to every $\omega \in B_\ell$ since $B_\ell = \mathrm{co}(B_\ell) \subseteq \mathrm{co}(B_h)$. Consequently, in almost every $\omega \in B_\ell \cup B_h = \mathrm{co}(B_h)$, the message $\mathrm{co}(B_h)$ is sent, which renders replicating the laminar partition impossible.

[29] Although closely related, this is not precisely the same as the NWBR property proposed by Kohlberg and Mertens (1986). For this reason, Fudenberg and Tirole (1991) call it "NWBR in signaling games."

her value function in beliefs evaluated at the prior F.[30] Observe that w is the composite function of u and the expectation operator $\mathbb{E}$. Because u is increasing, and an increasing transformation of an affine function is quasiconcave, w coincides with its quasiconcave hull. The aforementioned theorem therefore suggests that Sender never benefits from further cheap-talk opportunities.

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[30]The quasiconcave hull of a function $w : \Delta([0,1]) \to \mathbb{R}$ is the pointwise lowest quasiconcave and upper semicontinuous function that majorizes w.

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A Omitted Proofs and Details

A.1 Proofs for Section 3

A.1.1 Proof of Theorem 3.3

We start by presenting a roadmap for the proof:

1. Lemma A.1 discovers that the search for Sender's preferred equilibrium can be confined to the set of OREs.
2. Lemma A.2 establishes Part (c) of the theorem.
3. Lemma A.3, Lemma A.4, and Lemma A.5 together indicate that one can further restrict attention to OREs defined by laminar partitions.
4. Lemma A.6 reveals that every element of a laminar partition defining a Sender's preferred ORE is the union of at most $n-1$ disjoint intervals.
5. Lemma A.7 shows that a Sender's preferred equilibrium exists.
6. Lemma A.8 argues that Sender's payoff in her preferred equilibrium is strictly higher than that in any fully revealing equilibrium, which is Part (b).

Lemma A.1. *If there exists a Sender's preferred equilibrium (σ, τ, p), then there exists an ORE that generates the same payoff to Sender.*

Proof of Lemma A.1. Because (σ, τ, p) is a Sender's preferred equilibrium, Receiver must break ties in favor of Sender; consequently, $\tau(a_i \,|\, m) = 1$ if and only if $\mathbb{E}[\omega \,|\, \omega \in m] \in [\gamma_i, \gamma_{i+1})$. Suppose in some $\omega \in [0,1]$, $|\text{supp }\sigma(\cdot \,|\, \omega)| > 1$. Because (σ, τ, p) is an equilibrium, for every $m, m' \in \text{supp }\sigma(\cdot \,|\, \omega)$, there must exist $i \in \{0, 1, \dots, n-1\}$ such that $\tau(a_i \,|\, m) = \tau(a_i \,|\, m') = 1$. As a consequence, for every $\omega \in [0,1]$, there is an action a_i played with probability 1. For $i = 0, 1, \dots, n-1$, let

$$B_i = \text{cl}\left\{\omega \in [0,1] \,:\, \tau(a_i \,|\, m) = 1 \text{ for all } m \in \text{supp }\sigma(\cdot \,|\, \omega)\right\},$$

and it is straightforward that $\{B_i\}_{i=0}^{n-1}$ is a partition. We claim that $\{B_i\}_{i=0}^{n-1}$ is both obedient and incentive compatible. To see obedience, note that for all $\omega \in B_i$, for all $m \in \operatorname{supp} \sigma(\cdot \mid \omega)$, $\mathbb{E}[\omega \mid \omega \in m] \in [\gamma_i, \gamma_{i+1}]$, and also $\omega \in m$; therefore, $\mathbb{E}[\omega \mid \omega \in \cup_{\omega \in B_i, m \in \operatorname{supp} \sigma(\cdot \mid \omega)} m] \in [\gamma_i, \gamma_{i+1}]$, and $B_i \subseteq \cup_{\omega \in B_i, m \in \operatorname{supp} \sigma(\cdot \mid \omega)} m$. Thus, it must be that $\mathbb{E}[\omega \mid \omega \in B_i] \in [\gamma_i, \gamma_{i+1}]$. Moreover, incentive compatibility follows because (σ, τ, p) is an equilibrium, and hence in any $\omega \in B_i$ using $m = \{\omega\}$ cannot increase Sender's payoff. Then by Lemma 3.1, there exists an ORE defined by the partition $\{B_i\}_{i=0}^{n-1}$. By construction, this ORE generates the same payoff to Sender. ∎

Consequently, it suffices to focus on Sender's preferred ORE henceforth. Lemma A.2 provides a useful characterization of such OREs.

Lemma A.2. *Let a Sender's preferred ORE be defined by partition $\{B_i\}_{i=0}^{n-1}$, then it must be that $\mathbb{E}[\omega \mid \omega \in B_j] \in [\gamma_j, \gamma_{j+1})$ and $\mathbb{E}[\omega \mid \omega \in B_k] = \gamma_k$ for all k with $k > j$ such that a_k is not skipped.*

Proof of Lemma A.2. Suppose to the contrary that $\{B_i\}_{i=0}^{n-1}$ has $\mathbb{E}[\omega \mid \omega \in B_k] > \gamma_k$ for some $k > j$ where a_k is not skipped. Because $\{B_i\}_{i=0}^{n-1}$ defines an ORE, it must be both obedient and incentive compatible. Let $\underline{b}_j = \inf\left(\operatorname{int}(B_j)\right)$; because $f > 0$, there exists $x \in B_j$ with $x > \underline{b}_j$ such that $\mathbb{E}[\omega \mid \omega \in [\underline{b}_j, x] \cup B_k] \geq \gamma_k$, and clearly $\mathbb{E}[\omega \mid \omega \in B_j \backslash [\underline{b}_j, x]] > \gamma_j$. Now consider the partition $\{\widetilde{B}_i\}_{i=0}^{n-1}$ where $\widetilde{B}_j = B_j \backslash [\underline{b}_j, x]$, $\widetilde{B}_k = [\underline{b}_j, x] \cup B_k$, and $\widetilde{B}_i = B_i$ for $i \neq j, k$. It can be seen that the new partition strictly improves Sender's payoff, and it is still obedient and incentive compatible. A contradiction. ∎

Because an ORE can be identified by the partition associated with it, a Sender's preferred ORE induces a distribution over posterior mean G_S with support on at most n points, which must be a MPC of the prior F.[31] In particular, by Lemma A.2, it must be that $\operatorname{supp}(G_S) \subseteq \{\gamma_i\}_{i=1}^{n-1} \cup \{\gamma_S\}$, where $\gamma_S \in [0, 1] \backslash \{\gamma_i\}_{i=1}^{n-1}$.

Before proceeding to further characterize the structure of Sender's preferred OREs, we introduce the following auxiliary result first.

Lemma A.3. *There exist $0 = d_0 \leq d_1 \leq \ldots \leq d_{m-1} \leq d_m = 1$ with $m \leq n$ such that $\int_0^x G_S(q)\,\mathrm{d}q \leq \int_0^x F(q)\,\mathrm{d}q$ on $[d_i, d_{i+1}]$ for all $i = 0, 1, \ldots, m-1$, and the inequality holds with equality only at d_i and d_{i+1}.*

[31] See Footnote 7 for the definition of a mean-preserving contraction (MPC).

Proof of Lemma A.3. Without loss of generality, let $\operatorname{supp}(G_S) = \{\gamma_S, \gamma_{(1)}, \dots, \gamma_{(k-1)}\}$, where $\gamma_S \notin \{\gamma_i\}_{i=1}^{n-1}$, $k \leq n$, and $\gamma_S < \gamma_{(1)} < \gamma_{(2)} < \dots < \gamma_{(k-1)}$. Because $F(0) = G(0) = 0$, F is strictly increasing (since $f > 0$), and G_S is a step function, there exists $\varepsilon > 0$ such that $F(x) > G(x)$ for all $x \in [0, \varepsilon]$. By definition, $\int_0^1 G_S(q)\,\mathrm{d}q = \int_0^1 F(q)\,\mathrm{d}q$, and hence the set

$$D_1 = \left\{ x \in [\varepsilon, 1] \,:\, \int_0^x G_S(q)\,\mathrm{d}q = \int_0^x F(q)\,\mathrm{d}q \right\}$$

is nonempty. Let $d_1 = \inf D_1$. If $d_1 = 1$, set $m = 1$ and the proof is complete; suppose $d_1 < 1$.

Observe that $\gamma_S < d_1 < \gamma_{(k-1)}$: if $d_1 \leq \gamma_S$ then $F(x) > G(x) = 0$ for all $x \in [0, d_1)$, contradicting the definition of d_1; and if $d_1 \geq \gamma_{(k-1)}$, $F(x) < G_S(x) = 1$ for all $x \in [d_1, 1]$, and hence it cannot be that $\int_0^1 G_S(q)\,\mathrm{d}q = \int_0^1 F(q)\,\mathrm{d}q$, again a contradiction. Next we claim that $F(d_1) \geq G_S(d_1)$. Suppose to the contrary that $F(d_1) < G_S(d_1)$, then because G_S is a CDF and hence right-continuous, there must exist $\delta > 0$ such that $F(x) < G_S(x) = G_S(d_1)$ for all $x \in [d_1, d_1 + \delta]$, where the equality follows from the fact that G_S is a step function. Then

$$\begin{aligned} \int_0^{d_1+\delta} F(q)\,\mathrm{d}q &= \int_0^{d_1} F(q)\,\mathrm{d}q + \int_{d_2}^{d_1+\delta} F(q)\,\mathrm{d}q = \int_0^{d_1} G_S(q)\,\mathrm{d}q + \int_{d_2}^{d_1+\delta} F(q)\,\mathrm{d}q \\ &< \int_0^{d_1} G_S(q)\,\mathrm{d}q + \int_{d_1}^{d_1+\delta} G_S(q)\,\mathrm{d}q, \end{aligned}$$

which contradicts the assumption that G_S is a MPC of F.

Given these, let $j = \min\{i \,:\, \gamma_{(i)} > d_1\}$. Because $d_1 < \gamma_{(j)}$, $F(d_1) \geq G_S(d_1)$, F is strictly increasing, and G_S is a step function, there exists $\eta > 0$ such that $F(x) > G_S(x)$ for all $x \in [d_1, d_1 + \eta]$. Now $\int_0^1 G_S(q)\,\mathrm{d}q = \int_0^1 F(q)\,\mathrm{d}q$ implies that the set

$$D_2 = \left\{ x \in [d_1 + \eta, 1] \,:\, \int_0^x G_S(q)\,\mathrm{d}q = \int_0^x F(q)\,\mathrm{d}q \right\}$$

is nonempty. Let $d_2 = \inf D_2$. If $d_2 = 1$, set $m = 2$ and the proof is complete; suppose $d_2 < 1$. Similar to the previous paragraph, one can show that $d_2 > \gamma_{(j)}$ and $F(d_2) \geq G_S(d_2)$. Proceed inductively, eventually one can find d_t with $t \leq k \leq n$ such that $d_t > \gamma_{(k-1)}$. It must be that $d_t = 1$: suppose not, then because F is strictly increasing, $F(x) < 1$ on $(d_k, 1)$; but $G_S(x) = 1$ on the same interval, which implies that $\int_0^1 G_S(q)\,\mathrm{d}q > \int_0^1 F(q)\,\mathrm{d}q$, a contradiction. Now set $m = t$, the proof is complete. ∎

The structure of G_S discovered in Lemma A.3 allows us to show that G_S induces a laminar partition: by Lemma A.3, on each of the intervals such that the MPC constraint does not bind, the mass is redistributed to at most n points, and there can be at most n such intervals. We show that every such interval admits a laminar partition; the definition readily implies that the resulting partition is still laminar by taking the union. Consequently, a laminar partition indeed exists.

Lemma A.4. *If G_S is the posterior mean distribution that is induced by a Sender's preferred ORE, there exists a laminar partition that is induced by it.*

As it mostly follows from the results in Candogan and Strack (2023), the proof of Lemma A.4 is relegated to the Supplementary Appendix.

Lemma A.5 shows that in searching for Sender's preferred equilibrium, one can further restrict attention to OREs defined by laminar partitions.

Lemma A.5. *For every partition that is associated with a Sender's preferred ORE, there exists a laminar partition that does the same.*

Proof of Lemma A.5. By Lemma 3.1, it suffices to show that there exists a laminar partition that is both obedient and incentive compatible, and generates the same payoff to Sender. Fix a partition $\{B_i\}_{i=0}^{n-1}$ associated with a Sender's preferred ORE, and denote the posterior mean distribution that corresponds to it by G^*. By Lemma A.2, $\mathrm{supp}(G^*) \subseteq \{\gamma_i\}_{i=1}^{n-1} \cup \{\gamma_S\}$; and by Lemma A.4, G^* induces a laminar partition $\{B_i^L\}_{i=0}^{n-1}$. Because the two partitions are induced by the same posterior mean distribution, it must be that that (i) $\mathrm{int}(B_i) = \varnothing$ if and only if $\mathrm{int}(B_i^L) = \varnothing$, and (ii) for every k such that $\mathrm{int}(B_k) \neq \varnothing$, $\mathbb{E}[\omega \,|\, \omega \in B_k] = \mathbb{E}[\omega \,|\, \omega \in B_k^L] \in \{\gamma_k, \gamma_S\}$ and $\mu_F(B_k) = \mu_F(B_k^L) \in \{g^*(\gamma_k), g^*(\gamma_S)\}$, where g^* is the probability mass function of G^*. Thus, the two partitions generate the same payoff to Sender, and $\{B_i^L\}_{i=0}^{n-1}$ must be obedient because $\{B_i\}_{i=0}^{n-1}$ is. Consequently, it only suffices to show that $\{B_i^L\}_{i=0}^{n-1}$ satisfies (IC).

To this end, consider any $m = 1, \dots, n-1$; we need to show that $A_m \subseteq \cup_{i\geq m} B_i$. Suppose first that a_m is not skipped. By Lemma A.3, there must exist an interval I on which the MPC constraint only binds at the endpoints, and $\gamma_m \in I$. Moreover, for every $\gamma_k \in (\mathrm{supp}\,(G^*) \cap I)$, it must be that $B_k \subseteq I$ and $B_k^L \subseteq I$.

If $\gamma_m = \min(\mathrm{supp}\,(G^*) \cap I)$, it must be that $\cup_{i\geq m} B_i^L = \cup_{i\geq m} B_i$, and thus $A_m \subseteq \cup_{i\geq m} B_i$ implies $A_m \subseteq \cup_{i\geq m} B_i^L$. Consider next that $\gamma_m > \min(\mathrm{supp}\,(G^*) \cap I)$. Clearly,

$$\gamma_m \geq \sup(\mathrm{int}\,(B_k)) \text{ for all } k < m \text{ such that } \mathrm{int}\,(B_k) \neq \varnothing, \tag{6}$$

as otherwise (IC) is violated.[32] Because $\{B_i^L\}_{i=0}^{n-1}$ is laminar, either $\text{co}\,(B_{m-1}^L) \cap \text{co}\,(B_m^L) = \emptyset$ or $\text{co}\,(B_{m-1}^L) \subseteq \text{co}\,(B_m^L)$.[33] In the former case, B_m^L must be an interval; consequently, $\gamma_m \in B_m^L$. Furthermore, the laminar structure of $\{B_i^L\}_{i=0}^{n-1}$ implies that for any $\omega > \sup B_m^L$, $\omega \in B_q^L$ for some $q > m$, and hence $A_m \subseteq \cup_{i \geq m} B_i^L$. In the latter case, to show that $A_m \subseteq \cup_{i \geq m} B_i^L$, it suffices to show that $\gamma_m \geq \sup(\text{co}\,(B_{m-1}^L))$.[34] Let

$$\mathcal{N} = \left\{ i \leq m-1 \,:\, \gamma_i \in \text{supp}\,(G^*) \cap I, \text{ and } \text{co}\,(B_i^L) \subseteq \text{co}\left(B_{m-1}^L\right) \right\},$$

and define $J = \cup_{i \in \mathcal{N}} B_i$. By (6), $\sup(\text{int}\,(J)) \leq \gamma_m$; thus, to show that $\gamma_m \geq \sup(\text{co}\,(B_{m-1}^L))$, we only need to prove that $\sup(\text{int}\,(J)) \geq \sup(\text{co}\,(B_{m-1}^L))$. Suppose not, so $\sup(\text{int}\,(J)) < \sup(\text{co}\,(B_{m-1}^L))$. Because $f > 0$, $\mu_F\,(\text{int}\,(J)) = \mu_F\left(\text{co}\,(B_{m-1}^L)\right)$, and that $\text{co}\,(B_{m-1}^L)$ is an interval, it must be that $\mathbb{E}[\omega \,|\, \omega \in \text{int}(J)] < \mathbb{E}[\omega \,|\, \omega \in \text{co}\,(B_{m-1}^L)]$. However, $\mathbb{E}[\omega \,|\, \omega \in \text{int}\,(J)] = \mathbb{E}[\omega \,|\, \omega \in \text{co}\,(B_{m-1}^L)]$ by construction; a contradiction. Therefore, $\gamma_m \geq \sup(\text{co}\,(B_{m-1}^L))$, and hence $A_m \subseteq \cup_{i \geq m} B_i^L$.

Finally, consider the case that a_m is skipped. If $m < \min\{i \,:\, \text{int}\,(B_j) \neq \emptyset\}$, we are done. Otherwise, define $a_k = \max\{a_i \in A \,:\, a_i \text{ is not skipped, and } i < m\}$, and $a_q = \min\{a_i \in A \,:\, a_i \text{ is not skipped, and } i > m\}$. Replace m and $m-1$ in the previous paragraph by q and k, respectively, it can be shown that $A_m \subseteq \cup_{i \geq q} B_i^L = \cup_{i \geq m} B_i^L$. This completes the proof. ∎

Lemma A.6 establish an important property of laminar partitions associated with Sender's preferred OREs.

Lemma A.6. *Let $\{B_i\}_{i=0}^{n-1}$ be a laminar partition associated with a Sender's preferred ORE. Every member of $\{B_i\}_{i=0}^{n-1}$ is the union of at most $n-1$ intervals.*

Proof of Lemma A.6. Recall that a laminar partition is defined by (2). Because $\text{co}\,(B_k)$ must be an interval for all k, $\cup_{k<i}\,\text{co}\,(B_k)$ is the union of at most i intervals. By Claim C.1 in the Supplementary Appendix, since $\{B_i\}_{i=0}^{n-1}$ defines a Sender's preferred ORE, $\text{int}\,(B_0) \cap B_k = \emptyset$ for all $k \geq 1$. Thus, by taking out $\cup_{k<i}\,\text{co}\,(B_k)$ from $\text{co}\,(B_i)$, at most $i-1$ intervals are removed, and hence the remainder, namely $\text{co}\,(B_i) \setminus \cup_{k<i}\,\text{co}\,(B_k)$, must be the union of at

[32] This is obvious for k's with $k < m$ and $\mathbb{E}[\omega \,|\, \omega \in B_k] \notin \text{supp}\,(G^*) \cap I$. If $\gamma_k \in \text{supp}\,(G^*) \cap I$, the only possibility that makes $\gamma_m \leq \sup(\text{int}\,(B_k))$ hold without violating $A_m \subseteq \cup_{i \geq m} B_i$ is that $A_m \subseteq \text{co}\,(\text{int}\,(B_k))$. Then there must exist $q > m$ such that $A_q \subseteq \cup_{i \geq q} B_i$, a contradiction.

[33] Here we are assuming that a_{m-1} is not skipped. This is just for simplifying notation: if instead a_{m-1} is skipped, replace it with the action immediately below a_m in the set of unskipped actions.

[34] This is because, again, the laminar structure guarantees that (1) $B_m \subseteq \text{co}\,(B_m) \setminus \left(\cup_{i \in I, i<m}\,\text{co}\,(B_i)\right)$, and (2) for any $\omega > \sup(\text{co}\,(B_m))$, $\omega \in B_q$ for some $q > m$, and hence $A_m \subseteq \cup_{i \geq m} B_i^L$.

most i intervals. By taking closure, B_i is also the union of at most i intervals. Because $i \leq n-1$, the lemma thus follows. ■

Lemma A.5 and Lemma A.6 together imply that, in a Sender's preferred ORE, the associated partition is laminar, and each of the elements of the laminar partition is the union of at most $n-1$ intervals. These properties of Sender's preferred ORE allow us to prove the existence of Sender's preferred equilibrium.

Lemma A.7. *A Sender's preferred equilibrium exists.*

Since the proof of Lemma A.7 is mostly technical, it is deferred to the Supplementary Appendix. Lastly,

Lemma A.8. *A Sender's preferred ORE must generate a strictly higher payoff to Sender than a fully revealing equilibrium.*

Proof of Lemma A.8. Observe that Sender's payoff in a fully revealing equilibrium is the same as the ORE defined by the partition $\{A_i\}_{i=0}^{n-1}$. Let a Sender's preferred ORE be defined by $\{B_i\}_{i=0}^{n-1}$; it must generate a weakly higher payoff to Sender than $\{A_0, A_1, \dots, A_{n-1}\}$ because incentive compatibility guarantees that in every $\omega \in B_i \setminus \cup_{j>i} B_j$ and all $i = 0, 1, \dots, n-1$, sending message B_i makes Sender no worse off than fully revealing the state. Moreover, for incentive compatibility to hold, action a_{n-1} cannot be skipped; consequently, $\mu_F(B_{n-1}) > 0$. Because $\mathbb{E}[\omega \,|\, \omega \in B_{n-1}] = \gamma_{n-1} = \inf A_{n-1}$, $f > 0$ implies that $\mu_F(B_{n-1} \setminus A_{n-1}) > 0$. Consequently, in a positive measure of states, Sender is strictly better off in Sender's preferred ORE than in a fully revealing equilibrium, which implies that her payoff must be strictly higher in the former. ■

A.1.2 Proof of Proposition 3.4

Let $\{B_i\}_{i=0}^{n-1}$ be a laminar partition satisfying Condition (c) in Theorem 3.3, and suppose Conditions (i) and (ii) in the statement of Proposition 3.4 hold. Claim A.9 is a direct consequence of the laminar structure.

Claim A.9. *For any $j \neq k$ such that $B_j, B_k \neq \emptyset$, if $\mathrm{co}(B_j) \subseteq \mathrm{co}(B_k)$, then $j < k$.*

Proof of Claim A.9. Suppose not, so $j > k$; but then the definition of laminar partition and $\mathrm{co}(B_j) \subseteq \mathrm{co}(B_k)$ together imply that $B_j = \emptyset$, a contradiction. ■

Claim A.10 is implied by Condition (c).

Claim A.10. *1) For any $j < q$ such that* $\operatorname{int}\left(\operatorname{co}(B_q)\right) \cap \operatorname{int}\left(\operatorname{co}(B_j)\right) = \emptyset$*, every* $\omega' \in B_j$ *and* $\omega'' \in B_q$ *satisfy* $\omega' \leq \omega''$.

2) For every $k > q$ such that $\operatorname{int}\left(\operatorname{co}(B_q)\right) \cap \operatorname{int}(\operatorname{co}(B_k)) = \emptyset$*, every* $\omega' \in B_q$ *and* $\omega'' \in B_k$ *satisfy* $\omega' \leq \omega''$.

Proof of Claim A.10. We only prove 1); the argument for 2) is completely analogous. Because $\operatorname{int}\left(\operatorname{co}(B_q)\right) \cap \operatorname{int}\left(\operatorname{co}(B_j)\right) = \emptyset$, either $\omega' \leq \omega''$ for every $\omega' \in B_j$ and $\omega'' \in B_q$, or $\omega' \geq \omega''$ for every $\omega' \in B_j$ and $\omega'' \in B_q$. The second case; however, cannot be true: $j < q$ implies $\underline{b}_j \leq \gamma_q$, but Condition (c) requires that $\mathbb{E}[\omega \,|\, \omega \in B_q] \geq \gamma_q$, and hence $\overline{b}_q > \gamma_q \geq \underline{b}_j$, a contradiction. Therefore, it must be that $\omega' \leq \omega''$ for every $\omega' \in B_j$ and $\omega'' \in B_q$. ∎

Finally, Claim A.11 is a consequence of Claim A.10.

Claim A.11. *For two unskipped actions a_j and a_k with $j < k$, $\overline{b}_j < \overline{b}_k$.*

Proof of Claim A.11. Because both a_j and a_k are not skipped, $B_j, B_k \neq \emptyset$. The laminar structure then implies that there are two cases: either $\operatorname{int}\left(\operatorname{co}(B_j)\right) \cap \operatorname{int}(\operatorname{co}(B_k)) = \emptyset$, or if $\operatorname{co}(B_j) \subsetneq \operatorname{co}(B_k)$. In the first case, Claim A.10 implies that $\overline{b}_j < \overline{b}_k$; in the second case, $\operatorname{co}(B_j) \subsetneq \operatorname{co}(B_k)$ readily implies that $\overline{b}_j < \overline{b}_k$. ∎

We are now ready to prove Proposition 3.4.

Proof of Proposition 3.4. Suppose Conditions (i) and (ii) hold, consider any $q \leq n-1$. If a_q is skipped, (ii) implies that $\overline{b}_k \leq \gamma_q$ for unskipped action a_k immediately below a_q. By Claim A.11, $\overline{b}_j < \overline{b}_k$ for all $j < k$. Thus, $A_q \subseteq \cup_{i \geq q+1} B_i = \cup_{i \geq q} B_i$, which is precisely (IC). If instead a_q is not skipped, Claim A.9 implies that there are two cases: (a) B_q is an interval such that $\operatorname{int}\left(\operatorname{co}(B_q)\right) \cap \operatorname{int}(\operatorname{co}(B_i)) = \emptyset$ for all $i \neq q$; (b) either $\operatorname{co}(B_q) \subsetneq \operatorname{co}(B_k)$ for some $k > q$, or $\operatorname{co}(B_j) \subsetneq \operatorname{co}(B_q)$ for some $j < q$.

In case (a), Theorem 3.3 (c) implies that $\underline{b}_q \leq \gamma_q$. Moreover, Claim A.10 indicates that $\omega > \overline{b}_q$ if and only if $\omega \in B_k$ for some $k > q$. Thus, $A_q = [\gamma_q, \gamma_{q+1}] \subseteq [\underline{b}_q, 1] \subseteq \cup_{i \geq q} B_i$. In case (b), by Claim A.11 it suffices to show that $\overline{b}_\ell \leq \gamma_q$ for unskipped action a_ℓ immediately below a_q. Because $\operatorname{co}(B_j) \subsetneq \operatorname{co}(B_q)$ for some $j < q$ and a_ℓ is immediately below a_q, it must be that $\operatorname{co}(B_\ell) \subsetneq \operatorname{co}(B_q)$. Then (ii) implies that $\overline{b}_\ell \leq \gamma_q$. Therefore, in both cases, (IC) must hold.

For the other direction, we prove the contrapositive: if any of the two conditions is violated, the laminar partition $\{B_i\}_{i=0}^{n-1}$ must violate (IC). If (i) is violated, then there exist a skipped action a_i and the unskipped action a_j immediately below it are such that $\gamma_i < \overline{b}_j$. Consequently, there exists $k \geq i$ such that $A_k \cap \text{int}(B_j) \neq \emptyset$, and hence (IC) is also violated. Now suppose (ii) is violated; that is, there exist B_k and B_ℓ are such that $\text{co}(B_k) \subsetneq \text{co}(B_\ell)$, and $\overline{b}_k > \gamma_\ell$. This implies that there exists $j \geq \ell > k$ such that $A_j \cap \text{int}(B_k) \neq \emptyset$, which violates (IC). Therefore, if a laminar partition $\{B_i\}_{i=0}^{n-1}$ satisfies (IC), (i) and (ii) must also be true. This completes the proof. ∎

A.1.3 Proof of Proposition 3.5

By Theorem 3.3, one can find a laminar partition that corresponds to Sender's preferred equilibrium, denote it by $\{B_i^S\}_{i=0}^{n-1}$; and let Sender's payoff in this equilibrium be R_S. Clearly, $\{A_i\}_{i=0}^{n-1}$ is a laminar partition that corresponds to an ORE and yields Sender the same payoff as in a fully revealing equilibrium; let Sender's payoff in this equilibrium be R_U. The "only if" direction is straightforward. By definition, in no equilibrium can Sender achieve a payoff strictly higher than R_S. Moreover, if in an equilibrium, Sender's payoff is strictly less than her fully revealing payoff, it must be that in a positive measure of states, Sender's interim payoff is strictly less than her interim payoff from fully revealing the state. Therefore, in these states, Sender is going to deviate to full revelation, a contradiction. Therefore, in any equilibrium, Sender's payoff must be no lower than the fully revealing payoff.

The "if" direction requires more work. By Lemma 3.1, it suffices to show that for any $R \in [R_U, R_S]$, one can find an obedient and incentive compatible laminar partition $\{B_i\}_{i=0}^{n-1}$ that generates R. To this end, for all $z \in [0, \gamma_1]$, let $\hat{B}_0^z = B_0^S \cup [0, z]$ and $\hat{B}_i^z = B_i^S \backslash [0, z]$ for all $i \geq 1$. Then for any $z \in [0, \gamma_1]$, because $\mathbb{E}\left[\omega \,\middle|\, \omega \in \hat{B}_0^z\right] \in A_0$, $\mathbb{E}\left[\omega \,\middle|\, \omega \in \hat{B}_0^z\right] \in A_0$ by construction; and for all $i \geq 1$, $\mathbb{E}\left[\omega \,\middle|\, \omega \in \hat{B}_i^z\right] \in A_i$ since $\mathbb{E}\left[\omega \,\middle|\, \omega \in \hat{B}_i^S\right] \in A_i$. Consequently, $\{\hat{B}_i^z\}_{i=0}^{n-1}$ is obedient. To see that it is also incentive compatible, observe that

$$\bigcup_{j \geq i}^{n-1} \hat{B}_j^z = \left(\bigcup_{j \geq i}^{n-1} B_j^S\right) \backslash [0, z] \text{ for all } i \geq 1,$$

and hence if $A_j \subseteq \cup_{j \geq i}^{n-1} B_j^S$ it must be that $A_j \subseteq \cup_{j \geq i}^{n-1} \hat{B}_j^z$. Then by Lemma 3.1, for every $z \in [0, \gamma_1]$ there exists an equilibrium that corresponds to $\{\hat{B}_i^z\}_{i=0}^{n-1}$.

Let $P(z)$ denote Sender's payoff in the equilibrium that corresponds to $\{\hat{B}_i^z\}_{i=0}^{n-1}$: $P(z) = \sum_{i=0}^{n-1} v(a_i)\, \mu_F\left(\hat{B}_0^z\right)$; because $f > 0$, $P(z)$ is continuous in z. Since $P(0) = R_S > R$, if $P(\gamma_1) := R_1 \leq R$, by the intermediate value theorem, there exists $z^* \in (0, \gamma_1]$ such that $P(z^*) = R$. If instead $R_1 > R$, we start from $\{\hat{B}_i^{\gamma_1}\}_{i=0}^{n-1}$, and for every $z \in [\gamma_1, \gamma_2]$, define

$\hat{B}_0^z = \hat{B}_0^{\gamma_1}$, $\hat{B}_1^z = \hat{B}_1^{\gamma_1} \cup [\gamma_1, z]$, and $\hat{B}_i^z = B_i^s \setminus [v_1, z]$ for all $i \geq 2$. It is routine to check that $\{\hat{B}_i^z\}_{i=0}^{n-1}$ is obedient and incentive compatible. Because $P(\gamma_1) = R_1 > R$, if $P(\gamma_2) := R_2 \leq R$, by the intermediate value theorem, there exists $z^* \in (\gamma_1, \gamma_2]$ such that $P(z^*) = R$. If instead $R_2 < R$, proceed in a kindred manner *mutatis mutandis*. Because $P(\gamma_{n-1}) = R_U < R$, eventually one must be able to find some $z^* \in [0, \gamma_{n-1}]$ such that $P(z^*) = R$. Consequently, by letting $B_i = \hat{B}_i^z$ for all i, $\{B_i\}_{i=0}^{n-1}$ identifies an equilibrium with Sender's payoff being R. Finally, it can be checked using (2) that this is a laminar partition, which completes the proof.

A.2 Proofs and Details for Section 4

A.2.1 Some Details for Section 4.1

Below is a formal definition of a bi-pooling solution.

Definition 6 (Bi-pooling distribution)**.** A distribution $G \in \mathrm{MPC}(F)$ is a **bi-pooling distribution** if there exists a collection of pairwise disjoint intervals $\left\{ \left(\underline{\omega}_i, \overline{\omega}_i\right) \right\}_{i \in \mathcal{I}}$ such that

- for all $i \in \mathcal{I}$, $G(\overline{\omega}_i) - G\left(\underline{\omega}_i\right) = F(\overline{\omega}_i) - F\left(\underline{\omega}_i\right)$ and $\left|\mathrm{supp}\left(G|_{\left(\underline{\omega}_i, \overline{\omega}_i\right)}\right)\right| \leq 2$;[35]
- $G|_{[0,1] \setminus \cup_{i \in \mathcal{I}} \left(\underline{\omega}_i, \overline{\omega}_i\right)} = F|_{[0,1] \setminus \cup_{i \in \mathcal{I}} \left(\underline{\omega}_i, \overline{\omega}_i\right)}$.

In particular, $\left(\underline{\omega}_i, \overline{\omega}_i\right)$ is called a **pooling interval** if $\left|\mathrm{supp}\left(G|_{\left(\underline{\omega}_i, \overline{\omega}_i\right)}\right)\right| = 1$, and it is called a **bi-pooling interval** if $\left|\mathrm{supp}\left(G|_{\left(\underline{\omega}_i, \overline{\omega}_i\right)}\right)\right| = 2$.

The following observation is proved useful.

Lemma A.12 (Candogan, 2019)**.** *Every bi-pooling solution to the information design problem satisfies* $\mathrm{supp}(G_B) \subseteq [0, \tilde{\gamma}] \cup \{\gamma_i\}_{i=1}^{n-1}$, *where* $\tilde{\gamma} \in [0, 1]$ *is such that* $\tilde{\gamma} \leq \gamma$ *for all* $\gamma \in \mathrm{supp}(G_B) \cap \{\gamma_i\}_{i=1}^{n-1}$.

An important consequence of Lemma A.12 is that every signal realization can be identified by the action it induces. In particular, every $x \in \mathrm{supp}(G_B) \cap [0, \tilde{\gamma}]$ induces the "lowest" action,[36] and γ_i induces action a_i for each $i = 1, 2, \ldots, n-1$.

[35]For any cumulative distribution function H, let $H|_{[c,d]}$ denote the restriction of G to $[c, d] \subseteq [0, 1]$. Moreover, for a finite set K, let $|K|$ denote the cardinality of K.

[36]For example, if a_0 is (almost) never induced, the "lowest" action is a_1.

A.2.2 Proof of Lemma 4.1

The first part of the statement is established by Kleiner et al. (2021) and Arieli et al. (2023). For any bi-pooling solution G_B, let $\mu_i := G_B(\gamma_i) - G_B(\gamma_{i-1})$ for each $i = 2, \dots, n-1$. If $\text{supp}\,(G_B) \setminus \{\gamma_i\}_{i=1}^{n-1} = \emptyset$, set $\mu_1 = G_B(\gamma_1)$ and $\tilde{\mu} = 0$; otherwise, let $\tilde{\mu} = G_B(\tilde{\gamma})$ and $\mu_1 = G_B(\gamma_1) - \tilde{\mu}$; Lemma A.12 indicates that every bi-pooling solution is identified by $\{\tilde{\mu}\} \cup \{\mu_i\}_{i=1}^{n-1}$. By Lemma D.1 in Candogan (2022), no three elements of the collection of points $\{(\gamma_i, v(a_i))\}_{i=0}^{n-1}$ are collinear implies that any two bi-pooling solutions of problem (1) induces the same collection of $\{\tilde{\mu}\} \cup \{\mu_i\}_{i=1}^{n-1}$. Then because every bi-pooling solution is a mean-preserving spread of F, any two such bi-pooling solutions must be identical. As noted in Kleiner et al. (2021) and Arieli et al. (2023), every extreme point of the set of solutions to problem (1) is a bi-pooling solution, and hence the solution to problem (1) must be a unique bi-pooling solution.

A.2.3 Proof of Lemma 4.2

By Lemma A.12, in any bi-pooling solution G_B, if $\text{supp}(G_B) \setminus \{\gamma_i\}_{i=1}^{n-1}$ is nonempty but not a singleton, there must exist an interval $[0, d]$ on which action a_0 is recommended, and $[d, 1]$ is comprised of pooling intervals and/or bi-pooling intervals; in this case, $[0, d]$ can be viewed as a pooling interval. Otherwise, $[0, 1]$ comprises pooling and/or bi-pooling intervals. The existence of a nested intervals partition now follows from Lemma 4 in Arieli et al. (2023): on every pooling interval, a single signal realizes, which induces a single action and hence ties to a single partitional element. On every bi-pooling interval $(\underline{\omega}, \overline{\omega})$, there exists $(\underline{q}, \overline{q}) \subseteq (\underline{\omega}, \overline{\omega})$ such that one (deterministic) signal realizes when $\omega \in (\underline{q}, \overline{q})$, and another signal realizes when $\omega \in (\underline{\omega}, \underline{q}) \cup (\overline{q}, \overline{\omega})$, with the latter signal recommending a higher action. Then $[\underline{q}, \overline{q}]$ and $[\underline{\omega}, \underline{q}] \cup [\overline{q}, \overline{\omega}]$ correspond to two partitional elements B_ℓ and B_h with $\ell < h$, and $\{B_\ell, B_h\}$ is a pair of nested intervals.

To see uniqueness, first note that every pooling interval necessarily ties to a unique partitional element; hence if a bi-pooling solution admits two distinct nested interval partitions, $\{B_i\}_{i=0}^{n-1}$ and $\{B_i\}_{i=0}^{n-1}$, they must differ on a bi-pooling interval $(\underline{\omega}, \overline{\omega})$. Then there must exist $\{B_\ell, B_h\}$ and $\{B'_\ell, B'_h\}$ such that $B_\ell = [\underline{b}_\ell, \overline{b}_\ell]$, $B_h = [\underline{\omega}, \underline{b}_\ell] \cup [\overline{b}_\ell, \overline{\omega}]$, $B'_\ell = [\underline{b}'_\ell, \overline{b}'_\ell]$, and $B'_h = [\underline{\omega}, \underline{b}'_\ell] \cup [\overline{b}'_\ell, \overline{\omega}]$, with either $\underline{b}'_\ell \neq \underline{b}_\ell$, or $\overline{b}'_\ell \neq \overline{b}_\ell$, or both. Since $f > 0$ and $\underline{\omega}$ and $\overline{\omega}$ are fixed, either $\mathbb{E}[\omega \,|\, \omega \in B_\ell] \neq \gamma_\ell$, or $\mathbb{E}[\omega \,|\, \omega \in B'_\ell] \neq \gamma_\ell$, or both. A contradiction.

Finally, the last statement can be established by an argument analogous to the proof of Lemma A.2.

A.2.4 Proof of Proposition 4.3

Suppose first that a bi-pooling solution is implementable. Because the commitment payoff is an upper bound of equilibrium payoffs, it must be achieved in a Sender's preferred equilibrium. Theorem 3.3 shows that to look for a Sender's preferred equilibrium, it suffices to search within the class of obedient recommendation equilibria, which, by Lemma 3.1, can be simplified to partitions that are obedient and incentive compatible. Thus, an obedient and incentive-compatible partition that identifies the bi-pooling solution must exist. The proof of this direction is completed by noting that if there exists an IC partition of a bi-pooling solution, the (unique) nested interval partition of the same bi-pooling solution must also be IC. Because nested interval partitions are special cases of laminar partitions, the proof of this fact is relegated to the Supplementary Appendix.

For the other direction, suppose the nested interval partition $\{B_i\}_{i=0}^{n-1}$ of a bi-pooling solution is incentive compatible. By Lemma 4.2, this nested interval partition is also obedient. Then by Lemma 3.1, this partition is associated with an ORE, which in turn implies implementability.

A.2.5 Proof of Proposition 4.6

We use the following result due to Arieli et al. (2023) to prove Proposition 4.6. Say that $\{\gamma_\ell, \gamma_h\}$ **is a feasible bi-pooling support for the interval** $(\underline{\omega}, \overline{\omega})$ (or just **feasible** for simplicity) if there exists a mean preserving contraction of $F|_{[\underline{\omega},\overline{\omega}]}$ whose support is $\{\gamma_\ell, \gamma_h\}$.

Lemma A.13 (Arieli et al. 2023)**.** *Fix an interval $(\underline{\omega}, \overline{\omega})$, and let $y(\gamma_\ell)$ satisfy $\mathbb{E}[\omega \mid \omega \in [\underline{\omega}, y(\gamma_\ell))] = \gamma_\ell$. Then $\{\gamma_\ell, \gamma_h\}$ is feasible for the interval $(\underline{\omega}, \overline{\omega})$ if and only if*

(i) $\underline{\omega} \leq \gamma_\ell \leq \mathbb{E}[\omega \mid \omega \in [\underline{\omega}, \overline{\omega}]] \leq \gamma_h \leq \overline{\omega}$, *and*

(ii) $\mathbb{E}[\omega \mid \omega \in [y(\gamma_\ell), \overline{\omega}]] \geq \gamma_h$.

Proof of Proposition 4.6. Suppose to the contrary that there exists a bi-pooling solution G_B that is not implementable. Then by Corollary 4.4, the nested intervals partition of G_B, $\{B_i\}_{i=0}^{n-1}$, violates at least one of the two conditions in Corollary 4.4. First suppose condition (i) is violated; that is, there exists a skipped action a_i, such that $\gamma_i < \overline{b}_j$ for action a_j immediately below a_i. There are two cases.

Case 1. B_j is an interval. There are two subcases:

(I) There exists $\gamma_k \in B_j$ with $k < j$.

For $z \in (\gamma_i, \overline{b}_j)$, let $h(z)$ solve $\mathbb{E}[\omega \mid \omega \in [\underline{b}_j, h(z)] \cup [z, \overline{b}_j]] = \gamma_i$. By Lemma A.13, for z close enough to $\overline{b}_j$, $\{\gamma_k, \gamma_j\}$ is feasible for $[h(z), z]$. Consequently, Sender's payoff on $[\underline{b}_j, \overline{b}_j]$, as a function of z, is

$$P(z) = v_i[F(\overline{b}_j) - F(z) + F(h(z)) - F(\underline{b}_j)] + \left[\frac{m(z) - \gamma_k}{\gamma_j - \gamma_k} v_j + \frac{\gamma_j - m(z)}{\gamma_j - \gamma_k} v_k\right] (F(z) - F(h(z)))$$

where $m(z) := \mathbb{E}[\omega \mid \omega \in [z, h(z)]]$. To show that this is a profitable deviation, it suffices to show that $P'(\overline{b}_j) := \lim_{z \nearrow \overline{b}_j} P'(z) < 0$. To this end, we first calculate

$$P'(z) = \frac{f(z)}{\gamma_j - \gamma_k} \frac{z - h(z)}{\gamma_i - h(z)} \left[(v_j - v_k)\gamma_i + (v_i - v_j)\gamma_k - (v_i - v_k)\gamma_j\right].$$

Letting $z \nearrow \overline{b}_j$, then $h(z) \searrow \underline{b}_j$, and

$$P'(\overline{b}_j) = \frac{f(\overline{b}_j)}{\gamma_j - \gamma_k} \frac{\overline{b}_j - \underline{b}_j}{\gamma_i - \underline{b}_j} \left[(v_j - v_k)\gamma_i + (v_i - v_j)\gamma_k - (v_i - v_k)\gamma_j\right];$$

clearly $P'(\overline{b}_j) < 0$ if and only if $(v_i - v_k)\gamma_j - (v_j - v_k)\gamma_i - (v_i - v_j)\gamma_k > 0$, and this is in turn equivalent to

$$\frac{v_i - v_j}{\gamma_i - \gamma_j} > \frac{v_j - v_k}{\gamma_j - \gamma_k},$$

which is implied by (3).

(II) There does not exist $\gamma_k \in B_j$ with $k < j$. In particular, it must be that $\gamma_{j-1} < \underline{b}_j$. WLOG, let $i = j + 1$.[37] Define $B_j^\varepsilon = [\underline{b}_j + \varepsilon, \overline{b}_j]$; for small enough $\varepsilon > 0$, $\{\gamma_j, \gamma_{j+1}\}$ is feasible for B_j^ε. Let $m(\varepsilon)$ denote the mean of B_j^ε; note that $m(0) = \gamma_j$. Then Sender's payoff as a function of ε on B_j is

$$P(\varepsilon) = \left[\frac{m(\varepsilon) - \gamma_j}{\gamma_{j+1} - \gamma_j} v_{j+1} + \frac{\gamma_{j+1} - m(\varepsilon)}{\gamma_{j+1} - \gamma_j} v_j\right] [F(\overline{b}_j) - F(\underline{b}_j + \varepsilon)] + v_{j-1}[F(\underline{b}_j + \varepsilon) - F(\underline{b}_j)].$$

To show that this is a profitable deviation, it suffices to show that $P'(0) := \lim_{\varepsilon \searrow 0} P'(\varepsilon) > 0$. Algebra reveals that

$$P'(\varepsilon) = f(\underline{b}_j + \varepsilon) \left[v_{j-1} - \frac{\gamma_{j+1} v_j - \gamma_j v_{j+1}}{\gamma_{j+1} - \gamma_j} - (\underline{b}_j - \varepsilon) \frac{v_{j+1} - v_j}{\gamma_{j+1} - \gamma_j}\right];$$

[37] If $i > j + 1$, it must be that $\overline{b}_j > \gamma_i > \gamma_{j+1}$, and hence one can work with a_{j+1} instead.

consequently, as $\varepsilon \searrow 0$,

$$P'(0) = \frac{f(\underline{b}_j)}{\gamma_{j+1} - \gamma_j} \left[(\gamma_j - \underline{b}_j)(v_{j+1} - v_j) - (\gamma_{j+1} - \gamma_j)(v_j - v_{j-1}) \right]. \tag{7}$$

Because $f(\underline{b}_j) > 0$ and $\gamma_{j+1} - \gamma_j > 0$ by assumption, the sign of $P'(0)$ is the same as the sign of term in the square brackets in the right-hand side of (7). Thus, $P'(0) > 0$ if and only if

$$\frac{v_{j+1} - v_j}{\gamma_{j+1} - \gamma_j} > \frac{v_j - v_{j-1}}{\gamma_j - \underline{b}_j}.$$

Because $h(\gamma_j; \gamma_{j+1}) > \underline{b}_j$ by definition, (3) implies the inequality above.

Case 2. There is a partitional element B_k such that $\{B_j, B_k\}$ is a pair of nested intervals. There are three cases: (a) $\ell < j < i$, (b) $j < \ell < i$, and (c) $j < i < \ell$. In Cases (b) and (c), B_j must be the middle interval, and hence they are completely analogous to Case 1; therefore, it remains to consider Case (a). In this case, $B_\ell = [\underline{b}_\ell, \overline{b}_\ell]$, and $B_j = [\underline{b}_j, \underline{b}_\ell] \cup [\overline{b}_\ell, \overline{b}_j]$. WLOG, assume that $\underline{b}_j < \underline{b}_\ell$; by Lemma A.13, $\{\gamma_\ell, \gamma_j\}$ is feasible for $[\underline{b}_j, \overline{b}_j]$, and

$$\mathbb{E}[\omega \mid \omega \in [y(\gamma_\ell), \overline{b}_j]] > \gamma_j. \tag{8}$$

Furthermore, it must be that $\gamma_i \in [\overline{b}_\ell, \overline{b}_j]$: if instead $\gamma_i \in [\underline{b}_j, \underline{b}_\ell]$, it must be that $\gamma_i < \gamma_\ell$, which violates the assumption that $\ell < i$.

For $z \in (\gamma_i, \overline{b}_j)$, let $h(z)$ solve $\mathbb{E}[\omega \mid \omega \in [\underline{b}_j, h(z)] \cup [z, \overline{b}_j]] = \gamma_i$. By (8) and Lemma A.13, for z close enough to $\overline{b}_j$, $\{\gamma_\ell, \gamma_j\}$ is feasible for $[h(z), z]$. Consequently, one can find a profitable deviation similar to in Case 1 (I) *mutatis mutandis*.

Now suppose condition (ii) is violated; that is, for any pair of nested intervals $\{B_\ell, B_h\}$, $\gamma_h < \overline{b}_\ell$. Since $B_\ell = [\underline{b}_\ell, \overline{b}_\ell]$, this case is isomorphic to Case 1 when condition (i) is violated. As a consequence, (3) implies that (ii) must also hold for all bi-pooling solutions.

Therefore, every bi-pooling solution must satisfy both (i) and (ii), and hence implementable. This completes the proof. ∎

A.2.6 Proof of Corollary 4.7

By definition of $h(\gamma_i; \gamma_{i+1})$, when f is increasing, $\gamma_{i+1} - \gamma_i \leq \gamma_i - h(\gamma_i; \gamma_{i+1})$. Then since $\gamma_{i+1} - \gamma_i \leq \gamma_i - \gamma_{i-1}$, it must be that $\gamma_{i+1} - \gamma_i \leq \min\{\gamma_i - \gamma_{i-1}, \gamma_i - h(\gamma_i; \gamma_{i-1})\}$. Now there are two cases. If $v_{i+1} - v_i > v_i - v_{i-1}$, (3) must hold; then by Proposition 4.6, every bi-pooling solution can be implemented. If instead $\gamma_{i+1} - \gamma_i > \gamma_i - \gamma_{i-1}$, it must be that $\gamma_{i+1} - \gamma_i < \min\{\gamma_i - \gamma_{i-1}, \gamma_i - h(\gamma_i; \gamma_{i-1})\}$; this inequality and $v_{i+1} - v_i \geq v_i - v_{i-1}$ together imply (3). Again by Proposition 4.6, every bi-pooling solution can be implemented.