Targeted Advertising in Elections*

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December 2022

LATEST VERSION

Abstract

Some elections are unwinnable for challengers because pivotal voters prefer policies on the opposite sides of the status quo. In this paper, I argue that the challenger can win any such election if he uses targeted advertising with verifiable messages. In his private ads, the challenger makes each voter believe that his policy is a sufficient improvement over the status quo and wins the election when his policy is sufficiently moderate. Targeted advertising makes voters regret their choices and minimizes voter welfare relative to the complete information and public advertising benchmarks. As a voter's favorite policy becomes more extreme, her dissatisfaction with the status quo grows, and she becomes persuadable by a wider range of policies. As a result, the challenger's odds of winning increase.

KEYWORDS: Persuasion, Targeted Advertising, Elections

JEL CLASSIFICATION: D72, D82, D83

*I thank Avidit Acharya, Arjada Bardhi, Renee Bowen, Steve Callander, Georgy Egorov, Aleksandr Levkun, Joel Sobel, and the audience members at Princeton University, Stanford GSB, Central European University, Linz, UC San Diego, SWET 2019, SITE 2021, and Virtual Formal Theory Workshop 2022. All errors are my own.

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1. INTRODUCTION

Targeted advertising played an important role in the recent US Presidential Elections. In 2016, the Trump campaign used voter data from Cambridge Analytica to target voters via Facebook and Twitter. In 2008, the Obama campaign pioneered the use of social networks to communicate with the electorate. Even before social media, in 2000, The Bush campaign targeted voters via direct mail. Given that the winning candidate had access to better technology or voter data in all these cases, one may wonder whether targeted advertising was why these candidates won.[1] Would they have lost without targeted advertising? In other words, can targeted advertising swing elections?

To answer this question, I consider the following baseline model of targeted advertising in elections. There is an underlying policy space $X = [-1, 1]$, and three players: the challenger and two voters, $left$ and $right$. The challenger is privately informed about his policy $x \in X$, which is drawn from a common prior distribution with full support. The status quo policy is known and fixed at 0. The $left$ voter has a bliss point $L < 0$ and the right voter has a bliss point $R > 0$. Voters have quadratic preferences and dislike policies that are far from their bliss point. Specifically, if voter $v \in \{L, R\}$ approves the challenger's policy $x \in X$, she receives a payoff of $-(x - v)^2$; if she rejects, she gets $-(v - 0)^2$. The challenger is office-motivated and receives a payoff of one of the two voters unanimously approve his policy and zero otherwise. The challenger communicates with the voters using verifiable messages. Each message m is a statement about his policy or $m \subseteq X$. Verifiability means that each message has to include the challenger's actual policy, or $x \in m$. In words, the challenger can lie by omission and send messages that include things other than his policy. At the same time, he cannot lie by commission and send messages that do not include his policy. The challenger knows the bliss points of the voters and uses targeted advertising to send different messages to the $left$ and the $right$ voter. Each voter only observes her own message and votes expressively for the candidate (the challenger or the status quo) whose policy she prefers the most.

Note that the baseline election is unwinnable for the challenger without targeted advertising. Specifically, his odds of winning are zero in every equilibrium, under every communication protocol that does not allow different messages to different voters. The left voter prefers policies to the left of the status quo, whereas the right voter prefers policies on the right. Since the challenger's policy cannot be left and right at the same time, at most one of the voters is willing to approve it under complete in-

[1] For comparison of advertising strategies between the candidates, see Kim et al. (2018) and Wylie (2019) for the 2016 election, Harfoush (2009) and Katz, Barris, and Jain (2013) for the 2008 election and Hillygus and Shields (2014) for the 2004 election.

formation. Similarly, the challenger's policy cannot be both left and right on average, meaning that at most one of the voters is willing to approve it under common belief. As a result, the challenger definitely loses if he does not advertise, if he fully discloses his policy, or if he advertises his policy publicly.

When the challenger has access to targeted advertising, he can tell different things to different voters. In his most preferred equilibrium, the challenger makes the left voter believe that his policy is, on average, to the left of the status quo. He induces that belief by pooling this voter's favorite policies on the left with as many right policies as possible. Similarly, in his private communication with the right voter, the challenger insists that his policy is, on average, on the right. The challenger wins the election whenever both voters approve, which when his policy is sufficiently close to the status quo: the further to the right (left) is his policy, the harder it becomes to convince the left (right) voter.

When a voter becomes more extreme, her dissatisfaction with the status quo grows, making her more persuadable. Consequently, as the electorate becomes more polarized, the challenger's odds of swinging an unwinnable election increase. As the right voter becomes more extreme, she becomes persuadable by a wider range of policies, including policies further to the left. As a result, the equilibrium set of unanimously approved policies shifts to the left.

MOTIVATING EXAMPLE

Suppose that $L = -0.4$ and $R = 0.2$. Also, suppose that the prior is uniform: every policy in $[-1, 1]$ is ex ante equally likely. Consider the following strategy of the challenger. When communicating with voter L, he sends the message $[-0.4, 0.2]$ whenever his policy is $x \in [-0.4, 0.2]$; otherwise, he sends the message $[-1, 1]$. In words, the challenger tells the left voter that his policy is not ultra left and not moderate- to ultra right, whenever that is true, and says nothing otherwise. Similarly, when talking to voter R, he sends the message $[-0.4, 0.8]$ whenever his policy is $x \in [-0.4, 0.8]$, and $[-1, 1]$ otherwise. Figure 1 illustrates the challenger's strategy.

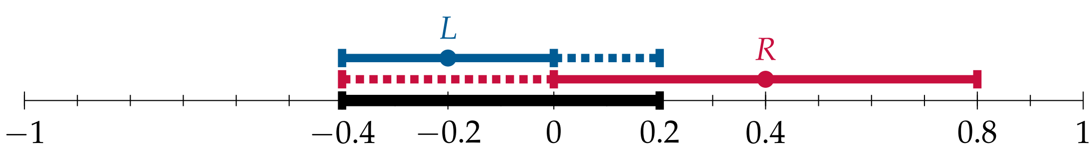

Figure 1. *Targeted messages that convince voter L (in blue) and R (in red). Each voter's convincing message includes policies that she prefers to the status quo (solid) and those that are dominated by the status quo (dashed). The challenger wins the election by unanimous approval whenever his policy is in the intersection of the convincing messages (in black).*

Now, suppose that voter L receives message $[-0.4, 0.2]$. The expected utility of approving the challenger's policy is $\int_{-0.4}^{0.2} -(x+0.2)^2 dx = -0.024$, which coincides with the expected utility of rejecting, which is $\int_{-0.4}^{0.2} -(0.2)^2 dx = -0.024$. In other words, if voter L hears the message $[-0.4, 0.2]$ and knows that every challenger with the policy $x \in [-0.4, 0.2]$ sends that message with probability one, then she is indifferent between approval and rejection. Suppose that she breaks the tie in favor of approval. Note that the message $[-0.4, 0.2]$ that convinces L comprises of the policies $[-0.4, 0]$ that she prefers to the status quo as well as the policies $(0, 0, 0.2]$ that are strictly dominated by the status quo. Although the voter is indifferent when approving, the average policy that voter L approves is $-0.1 < 0$, which reflects her risk aversion. Similarly, voter R approves after the message $[-0.4, 0.8]$. Whenever each voter hears the message $[-1, 1]$, she knows that it is sent by the challenger whose policy is further away than the status quo, so she rejects.

Now, these strategies of the challenger and voters lead to the following outcome. Voter L approves if and only if the challenger's policy belongs to the interval $[-0.4, 0, 2]$. At the same time, voter R approves the challenger's policy whenever it is in the interval $[-0.4, 0.8]$. We conclude that the challenger wins the election by unanimous approval whenever his policy is between -0.4 and 0.2. Given the uniform prior on $[-1, 1]$, the ex-ante odds that the challenger wins are 30%. It remains to show that the players' strategies constitute a perfect Bayesian equilibrium.

To complete the equilibrium characterization, we need to specify each voter's beliefs off the path. Let us impose that the voters are *skeptical*: they assume that each off-path message is sent by the challenger *(i)* who can send that message and *(ii)* whose policy is the farthest away from the voter's bliss point. Now, the challenger does not have profitable deviations: for any policy, he either wins the elections and secures the highest possible payoff or loses the election, but cannot verifiably send messages that convince both voters. The voters also do not have profitable deviations. Thus, the described strategies constitute an equilibrium. In fact, this is the challenger's preferred equilibrium, in which his odds of winning are the highest across all equilibria.

Next, suppose that the *right* voter becomes more extreme and her bliss point shifts from $R = 0.4$ to $R' = 0.5$. Following the same logic as above, the challenger can try to convince the $left$ voter with as many right policies, as possible, and vice versa. Since the $left$ voter's bliss point has not changed, her convincing message remains $[-0.4, 0.2]$. Re-doing the calculations for the new bliss point of the *right* voter, we find that her convincing message is now $[-0.5, 1]$. The challenger wins when his policy is between -0.4 and 0.2, which is the same interval as before.

Now, notice that the challenger can do even better. Specifically, notice that when his policy is between -0.5 and -0.4, he is convincing the *right* voter, but not the $left$ voter. At the same time, these policies are preferred by the $left$ voter to the policies she is currently approving, namely, the policies between 0.1 and 0.2, because they are closer to her bliss point. Hence, we can recalculate the $left$ voter's convincing message, forcing it to start at -0.5. The new message that convinces the $left$ voter and makes her indifferent between the challenger and the status quo is $[-0.5, 0.179]$. Figure 2 illustrates the new messages that convince the voters after the *right* voter becomes more extreme.

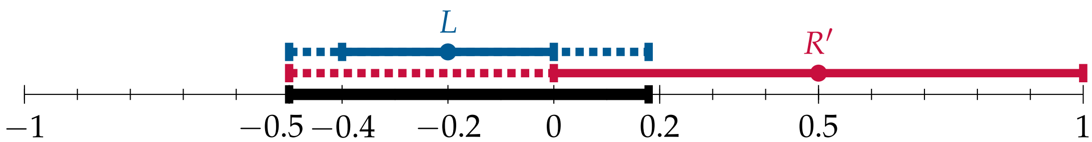

Figure 2. *Challenger-preferred targeted messages that convince voters L and R′. When the right voter becomes more extreme, she becomes persuadable by more left policies. The set of unanimously approved policies (in black) expands and shifts to the left.*

In the new equilibrium, the set of unanimously approved policies is $[-0.5, 0.179]$, and the odds of the challenger winning are 33.96%. An immediate observation is that more extreme voters increase the challenger's odds of swinging an unwinnable election. Essentially, when a voter becomes more extreme, her dissatisfaction with the status quo grows, making her more persuadable. A less obvious observation is that when the *right* voter's bliss point shifts to the right, the set of unanimously approved policies of the challenger shifts in the opposite direction, to the left. When a voter becomose more extreme, she becomes persuadable by a wider range of policies, in either direction, despite her risk aversion. As a result, the challenger can persuade the voter by more policies on the opposite side of the status quo from her bliss point.

Note that the results rely on both the assumption that the voters are expressive and the equilibrium selection. On the one hand, while the voters are fully Bayesian, they do not condition on the event of being pivotal. In other words, the only information they take into account is their own signal. In that sense, the proposed model is more suitable to describe large elections.[2] On the other hand, even when the voters are expressive, there exist other equilibria of the game under consideration. For example, the challenger could just reveal his policy (that is, send the message $\{x\}$ for each realization $x \in X$ of his policy) to each voter and lose with probability one. However,

[2]The theory of expressive voting was pioneered by Brennan and Lomasky (1993), Brennan and Hamlin (1998) and reviewed by Hamlin and Jennings (2011). There is a large body of evidence that the behavior of voters is consistent with expressive voting, e.g., in US national elections (Kan and Yang, 2001; Degan and Merlo, 2007), Spanish general elections (Artabe and Gardeazabal, 2014), Israeli general elections (Felsenthal and Brichta, 1985).

empirical evidence suggests that politicians do, in fact, often make vague statements about the policies that they intend to pursue.[3] I select the equilibrium in which, in some sense, the challenger's messages to the voters are the most vague.

Related Literature

This paper contributes to the growing literature that compares public and private communication. In the verifiable information literature (pioneered by Milgrom, 1981 and Grossman, 1981), the closest paper to mine is Schipper and Woo (2019), who study advertising competition with micro-targeting. They show that even with targeted advertising, candidates tend to voluntarily disclose all of their private information. This unraveling result is fairly common in the verifiable information literature on voter persuasion (Board, 2009; Janssen and Teteryatnikova, 2017), and arises because the candidates play a zero-sum game. In contrast to these papers, I consider an asymmetric model in which only one candidate can communicate with the voters. Unraveling does not necessarily occur, and the challenger can improve his chances of winning over full disclosure.

Much progress has been made in comparing public and private communication in the cheap talk literature. The sender often prefers to communicate in public, rather than in private (Farrell and Gibbons, 1989, Koessler, 2008, Goltsman and Pavlov, 2011, Bar-Isaac and Deb, 2014), because public communication reduces the number of deviations available to the sender in each state of the world. When his messages are verifiable, the sender's message space is already restricted, and there is no such effect. Thus, my main result is the opposite: the sender strictly benefits from private advertising when his messages are verifiable to the point that he can win elections that are unwinnable otherwise.

In information design, the sender prefers private communication to public advertising (Arieli and Babichenko, 2019), even if the receivers are strategic and condition on the event of being pivotal (Bardhi and Guo, 2018, Chan et al., 2019, Heese and Lauermann, 2021). I confirm this finding: while my sender does not possess any commitment power, the sender-preferred equilibrium outcome is also a commitment outcome (Titova, 2022). Beyond that, my contribution is twofold: on the one hand, I conclude that the sender does not need commitment power to benefit from targeted advertising. On the other hand, he does not only improve his ex ante utility by communicating in private; he improves it from 0 in every equilibrium to a positive number in his most preferred equilibrium.

[3] For empirical evidence of candidate ambiguity in the lab and in the field, see, e.g., Tomz and Houweling (2009), Somer-Topcu (2015), Bräuninger and Giger (2018), Tolvanen, Tremewan, and Wagner (2022).

This paper also contributes to the literature on political ambiguity. Current explanations for why politicians prefer to advertise policies imprecisely include voters' risk-seeking behavior (Shepsle, 1972); candidates' own policy preference (Alesina and Cukierman, 1990); candidates' preference for ambiguity (Aragonès and Neeman, 2000); future elections (Meirowitz, 2005, Alesina and Holden, 2008); policy-relevant information coming after the election (Kartik, Van Weelden, and Wolton, 2017). This paper's equilibrium features ambiguity because being vague allows the challenger to convince multiple voters at once without lying (by commission) to any of them. Notably, two papers in this strand of literature have previously found that politicians may be able to persuade voters on the opposite sides of the status quo. In Callander and Wilson (2008), voters have context-dependent preferences, and in Tolvanen (2021), the voters' preferences are correlated with the state of the world. I get to a similar conclusion in a remarkably simple setup with expressive voters who have standard risk averse preferences.

I also contribute to the growing literature on polarization and targeted political advertising through the media. As the number of media outlets increases, they become more specialized and target voters with more extreme preferences, which leads to social disagreement (Perego and Yuksel, 2022). If the electorate is polarized to begin with, so are the candidates' chosen policy platforms (Hu, Li, and Segal, 2019; Prummer, 2020). I find that as the electorate becomes more polarized, more challengers can swing elections that are unwinnable otherwise.

2. Baseline Election: Model

I study a game between a politician who challenges the status quo (the challenger, he/him) and the voters (each she/her). There is an underlying policy space $X := [-1, 1]$ with policy positions ranging from far left (-1) to far right (1). The status quo policy is fixed, known, and normalized to 0. The game begins with the challenger privately observing his policy $x \in X$, which is drawn from a common prior distribution $\mu_0 \in \Delta X$ with full support.[4] The challenger is *office-motivated* and his goal is to win the election. In the baseline election, there are two voters, $left$ and $right$, and the challenger needs unanimous approval of both voters to win the election. I normalize his payoff from winning to 1 and losing to 0.

The challenger advertises his policy to voters through private verifiable messages.

[4] For a compact metrizable space Y, I let ΔY denote the set of all Borel probability measures over Y, endowed with the weak* topology. For $\gamma \in \Delta Y$, I let $\operatorname{supp} \gamma$ denote the support of y. I say that $\gamma \in \Delta Y$ is degenerate if $\gamma(y) = 1$ for some $y \in Y$, and non-degenerate otherwise.

Specifically, each message m that the challenger may send *(i)* is a statement about his policy, $m \subseteq X$, and *(ii)* contains a grain of truth, $x \in m$. That is, the challenger *can lie by omission* and send messages that contain policies other than x. At the same time, he *cannot lie by commission* and send messages that do not include x. The verifiability of the messages allows the voters to draw inferences about the challenger's policy. For example, suppose that a voter hears the message $[-1/2, 0]$, or "my policy is moderately left". She concludes that the challenger's policy is not far-left or anywhere on the right. At the same time, she does not know the exact location of the challenger's policy between $-1/2$ and 0.

The voters have spatial preferences and vote expressively for the candidate they prefer the most. The voter with bliss point $v \in X$, to whom I will refer as "voter v" when there is no confusion, prefers to approve the challenger's policy $x \in X$ if she considers it a sufficient improvement over the status quo. Otherwise, she prefers to reject it. Mathematically, when the challenger's policy is $x \in X$, the payoff of the voter with bliss point $v \in X$ is

$$u_v(\text{approve}, x) = -(v-x)^2, \quad u_v(\text{reject}, x) = -(v-0)^2.$$

To simplify the analysis, let $\alpha_v(x) := u_v(\text{approve}, x) - u_v(\text{reject}, x) = -x^2 + 2vx$ be v's net payoff from approval. Note that v's net payoff from approval is a downward-sloping parabola that peaks at v, which reflects the voter's risk aversion. Now, this voter's best response is to approve the challenger's policy $x \in X$ whenever her net payoff from approval $\alpha_v(x)$ is non-negative. Next, let $A_v := \{x \in X \mid \alpha_v(x) \geq 0\}$ be her (complete information) approval set that includes all the policies of the challenger that she prefers to approve under complete information. Let $\lceil A_v \rceil := \max A_v$ be the largest and $\lfloor A_v \rfloor := \min A_v$ be the smallest elements of v's approval set.

In the baseline election, the $left$ voter has bliss point $L \in [-1, 0)$ and the *right* voter has bliss point $R \in (0, 1]$. Figure 3 illustrates the preferences of the $left$ voter.

I focus on the challenger-preferred Perfect Bayesian equilibrium of this game. Knowing his policy x, the challenger chooses verifiable messages $m_L \subseteq X$ and $m_R \subseteq X$ for voters L and R, respectively. Verifiability requires that $x \in m_v$ for all $v \in \{L, R\}$. Having observed message m_v, voter $v \in \{L, R\}$ forms a posterior belief over X. She then approves or rejects. Both voters are expressive and do not condition on the event of being pivotal.

In equilibrium, *(i)* the challenger sends messages that maximize his payoff, *(ii)* each voter approves the challenger's policy whenever her expected net payoff from approval is non-negative under her posterior belief, *(iii)* voters' posteriors on the equilibrium path are Bayes-rational. The challenger-preferred equilibrium is one in which

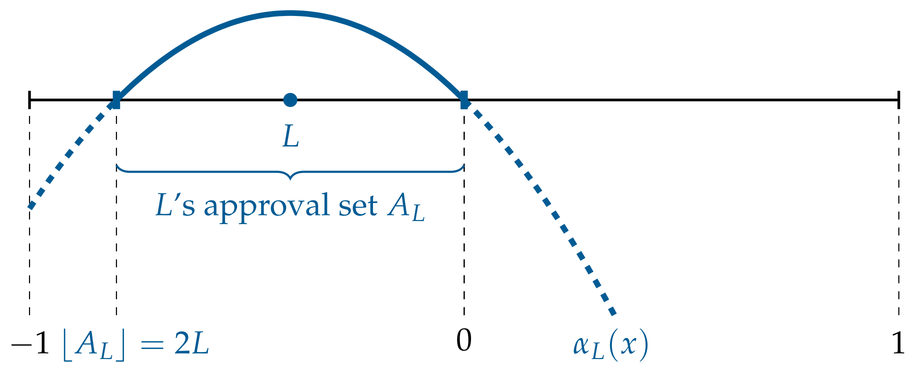

Figure 3. *The policy space $X = [-1, 1]$, the $left$ voter with bliss point $L < 0$, her net payoff from approval $\alpha_L(x)$, and her approval set A_L.*

his odds of unanimous approval are the highest across all equilibria.

3. Baseline Election: Analysis

Incompatible Voters and Unwinnable Elections

First, observe that the challenger faces an electorate of voters who prefer diametrically opposing policies. As a result, the baseline election is unwinnable for him without targeted advertising. Firstly, observe that each voter expects that the challenger's policy is on the same side of the status quo as her bliss point when she prefers to approve.

Lemma 1. *If voter with bliss point $v \in X$ approves under a non-degenerate belief $\mu \in \Delta X$, then $\mathbb{E}_\mu[x]$ is strictly between 0 and $2v$.*

Proof. By Jensen's inequality for the strictly concave function $\alpha_v(x)$ and the non-degenerate belief μ, $\alpha_v\big(\mathbb{E}_\mu[x]\big) > \mathbb{E}\big[\alpha_v(x)\big]$. Since voter v approves under belief μ, the right-hand side, which is her expected net payoff from approval, is non-negative. Now, $\alpha_v(x) = -x^2 + 2vx = -x(x - 2v)$ is positive whenever x is between 0 and $2v$. Consequently, if v approves under the non-degenerate belief μ, then $\mathbb{E}_\mu[x]$ is strictly between 0 and $2v$.

□

Second, if the voters hold a common belief, they cannot both expect the challenger's policy to be on the left and on the right of the status quo at the same time.

Lemma 2. *For any common non-degenerate belief $\mu \in \Delta X$, at most one of the voters prefers to approve.*

Proof. Using Lemma 1, if the $left$ voter approves under belief μ, then $\mathbb{E}_\mu[x] \in (2L, 0)$. Similarly, if the *right* voter approves under μ, then $\mathbb{E}_\mu[x] \in (0, 2R)$. Clearly, both are impossible at the same time, which means that at least one of the voters prefers to reject under μ.

□

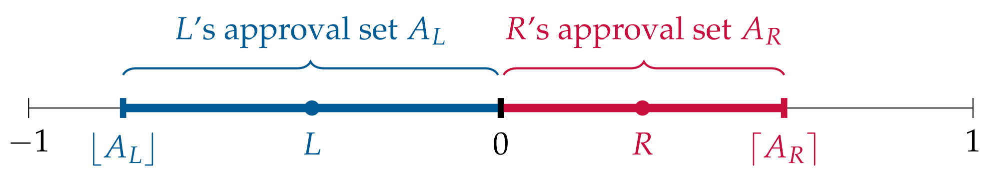

Figure 4. *Voters L and R are incompatible: L prefers to approve left policies and R prefers to approve right policies.*

Figure 4 illustrates the approval sets of the voters. Simply put, the $left$ voter prefers the left (blue) policies, while the *right* voter prefers the right (red) policies. Since the challenger's policy cannot be both left and right at the same time, at least one of the voters prefers to reject it. The same argument applies when the voters have a common belief.

Lemma 2 implies that the baseline election is *unwinnable* for the challenger *without targeted advertising*. If he does not advertise at all, the voters hold a common prior, and at most one of them votes to approve. If he advertises publicly, the voters' common prior is updated to a common posterior, but again, at most one voter is convinced to approve. The only event in which the challenger wins is when his policy coincides with the status quo. However, this event has a zero prior measure.

Equilibrium Outcomes under Targeted Advertising

Let us now characterize the (challenger-preferred) equilibrium payoff of the baseline election game with targeted advertising. According to Titova (2022), every equilibrium is payoff equivalent to a direct equilibrium with *sets of approved policies* $W_L \subseteq X$ and $W_R \subseteq X$ that satisfy certain constraints. In the direct equilibrium, the challenger sends the message W_v to voter $v \in \{L, R\}$ if $x \in W_v$, and its complement $W_v^c := X \setminus W_v$ otherwise.[5] When voter v hears W_v, she approves; otherwise, she rejects the challenger's policy. Therefore, we can interpret the message W_v as the challenger's *recommendation to approve* and the message W_v^c as the *recommendation to reject*. Off the (direct) equi-

[5]More specifically, the challenger's strategy is to send (with probability 1 in all cases) W_L to L and W_R to R if $x \in W_L \cap W_R$; W_L to L and W_R^c to R if $x \in W_L \cap W_R^c$; W_L^c to L and W_R to R if $x \in W_L^c \cap W_R$; W_L^c to L and W_R^c to R if $x \in W_L^c \cap W_R^c$.

librium path, each voter is "skeptical" and assumes that every unexpected message comes from the challenger whose policy is outside of her approval set, whenever possible.[6]

To be implementable in equilibrium, voter v's set of approved policies W_v must satisfy two conditions. On the one hand, there is the sender's incentive-compatibility constraint, $A_v \subseteq W_v$, which guarantees that the challenger does not want to deviate towards a fully informative strategy. This constraint is automatically satisfied in the challenger-preferred equilibrium because the challenger attempts to convince the voters with as many policies as possible. On the other hand, there is the receiver's obedience constraint that ensures that voter v only approves when her average net payoff from approval is non-negative:

$$\int\limits_{W_v} \alpha_v(x) d\mu_0(x) \geq 0. \tag{obedience}$$

Note that the described direct equilibrium implementation relies on the assumption that the voters are expressive. Specifically, when voter L hears message W_L, she updates her prior μ_0 to a posterior $\mu_0(\cdot \mid W_L)$ using the Bayes rule.[7] Then, she calculates her expected net payoff from approval $\int \alpha_L(x) d\mu_0(x \mid W_L) = \frac{1}{\mu_0(W_L)} \int\limits_{W_L} \alpha_L(x) d\mu_0(x)$ and approves because it is non-negative by (obedience). If L conditioned on the event of being pivotal, then her posterior, given the same strategy of the challenger, would be $\mu_0(\cdot \mid W_L \cap W_R)$, because R approves if and only if $x \in W_R$. Consequently, the described equilibrium characterization does not apply if the voters are not expressive.

The challenger wins the election whenever both voters approve, or when $x \in W_L \cap W_R$, and his odds of winning are $\mu_0(W_L \cap W_R)$. Thus, the (challenger-preferred) equilibrium sets of approved policies $(\overline{W}_L, \overline{W}_R)$ solve

$$\max_{W_L, W_R \subseteq X} \mu_0(W_L \cap W_R) \quad \text{subject to} \quad \int\limits_{W_v} \alpha_v(x) d\mu_0(x) \geq 0 \text{ for each } v \in \{L, R\} \tag{1}$$

[6] Let $m \subseteq X$ such that $m \notin \{W_v, W_v^c\}$ be an off-path message and $\mu(\cdot \mid m) \in \Delta X$ be v's posterior belief after m. If $m \nsubseteq A_v$, let supp $\mu(m) \subseteq m \setminus A_v$ so that $\int \alpha_v(x) d\mu(x \mid m) < 0$ and v prefers to reject. If $m \subseteq A_v$, then supp $\mu(\cdot \mid m) \subseteq A_v$ since the messages are verifiable, and v prefers to approve. If $x \in W_v$ then the challengers with policies $x \in A_v$ do not wish to deviate as they are already convincing voter v on the equilibrium path.

[7] Here, $\mu_0(\cdot \mid W)$ denotes the prior belief truncated to set $W \subseteq X$ such that $\mu_0(W) > 0$. By the Bayes rule, $\mu_0(x \mid W) = \frac{\mu_0(x) \cdot \mathbb{1}(x \in W)}{\mu_0(W)}$.

I refer to the pair $(\overline{W}_L, \overline{W}_R)$ that solves Problem (1) as the (challenger-preferred) equilibrium outcome (under targeted advertising). The main result of this paper establishes that the challenger can always win an unwinnable election by advertising privately.

PROPOSITION 1. *In equilibrium of the baseline election game, the challenger's ex-ante odds of winning are positive.*

The proof of this result is straightforward. First, observe that each voter's approval set is guaranteed to convince this voter, i.e., $\int_{A_v} \alpha_v(x) d\mu_0(x) > 0$ for each $v \in \{L, R\}$. Next, for each voter $v \in \{L, R\}$, select a subset $B_v \subseteq A_{-v}$ of the other voter's approval set that satisfies $\int_{A_v} \alpha_v(x) d\mu_0(x) + \int_{B_v} \alpha_v(x) d\mu_0(x) \geq 0$ and $\mu_0(B_v) > 0$. Let $W_v := A_v \cup B_v$ be voter v's set of approved policies. Although W_L and W_R may not be equilibrium sets of approved policies, they do, by construction, satisfy the constraints of Problem (1). At the same time, $\mu_0(W_L \cap W_R) = \mu_0(B_R \cup B_L) > 0$, implying that the challenger's ex ante odds of winning in equilibrium must be positive.

Before characterizing the equilibrium sets of approved policies, let us focus on the problem of maximizing the odds of convincing just one voter.

ONE VOTER'S INTERVALS OF APPROVED POLICIES

Consider a voter with bliss point $v \in X \setminus \{0\}$. Let us focus on the following auxiliary problem of finding this voter's largest (in terms of prior measure) set of approved policies constrained by $l \in \big[-1, \lfloor A_v \rfloor\big]$ from the left and $r \in \big[\lceil A_v \rceil, 1\big]$ from the right.

$$I_v(l, r) := \max_{W \subseteq [l,r]} \mu_0(W) \quad \text{subject to} \quad \int_W \alpha_v(x) d\mu_0(x) \geq 0. \tag{AUX}$$

The solution to the auxiliary is characterized by a cutoff value for the voter's net payoff from approval (see, for example, Alonso and Câmara, 2016 and Titova, 2022). In other words, voter v approves every policy with a not too negative payoff from approval (those $x \in X$ for which $\alpha_v(x) \geq -c_v^*$). The cutoff value $c_v^* > 0$ is obtained from the binding obedience constraint. The set $\{x \in [l, r] \mid \alpha_v(x) \geq -c_v^*\}$ is an interval: it is the upper contour set of the concave function $\alpha_v(x)$. Corollary 1 characterizes the solution of the auxiliary problem.

COROLLARY 1. *Consider a voter with bliss point $v \in X \setminus \{0\}$. The solution $I_v(l, r) \subseteq [l, r]$ to Problem (AUX) with $l \in \big[-1, \lfloor A_v \rfloor\big]$ and $r \in \big[\lceil A_v \rceil, 1\big]$ is almost surely an interval such that*

- *if $\int_l^r \alpha_v(x) d\mu_0(x) \geq 0$, then $I_v(l, r) = [l, r]$;*

- *otherwise,* $I_v(l,r) = \{x \in [l,r] \mid \alpha_v(x) \geq -c_v^*(l,r)\}$*, where* $c_v^*(l,r) > 0$ *is obtained from the binding constraint* $\int_{I_v(l,r)} \alpha_v(x) d\mu_0(x) = 0$.

Two special cases of the auxiliary problem will be useful in further analysis. Firstly, there is the unconstrained (by anything other than the voter's obedience constraint) version with $l = -1$ and $r = 1$. Figure 5 illustrates the largest unconstrained interval of approved policies of the $left$ voter.

DEFINITION 1. *Consider a voter with bliss point* $v \in X$*. Then, this voter's* largest unconstrained interval of approved policies *is* $I_v^{UC} = [a_v^{UC}, b_v^{UC}] := I_v(-1,1)$.

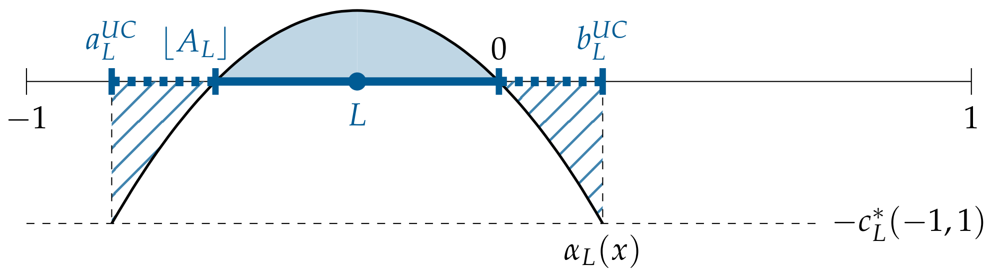

Figure 5. *$[a_L^{UC}, b_L^{UC}]$ is the $left$ voter's largest unconstrained interval of approved policies. Under uniform prior, c_L^* is obtained from equating the solid area (expected value of $\alpha_L(x)$ over A_L) to the dashed area (expected value of $\alpha_L(x)$ outside of A_L).*

The second relevant case is the largest asymmetric interval of approved policies that includes the most policies on the opposite side of the status quo from the voter's approval set. Figure 6 illustrates the $left$ voter's largest asymmetric interval of approved policies.

DEFINITION 2.

- *The* $left$ voter's largest asymmetric interval of approved policies *is*

$$I_L^{AS} = [\lfloor A_L \rfloor, b_L^{AS}] := I_L(\lfloor A_L \rfloor, 1).$$

- *The right* voter's largest asymmetric interval of approved policies *is*

$$I_R^{AS} = [a_R^{AS}, \lceil A_R \rceil] := I_R(-1, \lceil A_R \rceil).$$

Note that since $L < 0$ and $R > 0$, we get $b_L^{AS} > 0$ and $a_R^{AS} < 0$. In words, since each voter's approval set has a positive measure, she is persuadable by a positive measure of policies outside of her approval set. Also, recall from Lemma 1 that each voter expects

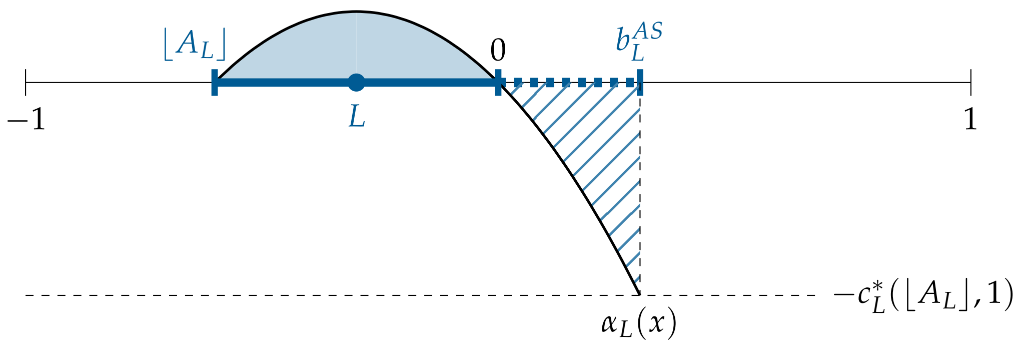

Figure 6. *$[\lfloor A_L \rfloor, b_L^{AS}]$ is the $left$ voter's largest asymmetric interval of approved policies. Under uniform prior, c_L^* is obtained from equating the solid area (expected value of $\alpha_L(x)$ over A_L) to the dashed area (expected value of $\alpha_L(x)$ outside of A_L).*

the challenger's policy to be within her approval set when she prefers to approve. This immediately implies that $\int_{\lfloor A_L \rfloor}^{b_L^{AS}} x d\mu_0(x) \in (\lfloor A_L \rfloor, 0)$ and $\int_{a_R^{AS}}^{\lceil A_R \rceil} x d\mu_0(x) \in (0, \lceil A_R \rceil)$.

Convincing Two Voters at the Same Time

Let us now solve Problem (1), or, put simply, attempt to convince both voters at the same time as often as possible. One thing the challenger can do is convince the $left$ (*right*) voter with as many policies to the right (left) of her approval set as possible. That is, he can let each voter's set of approved policies be her largest asymmetric interval of approved policies. I illustrate the outcome (I_L^{AS}, I_R^{AS}) in Figure 7. As it turns out, (I_L^{AS}, I_R^{AS}) is often an equilibrium outcome.

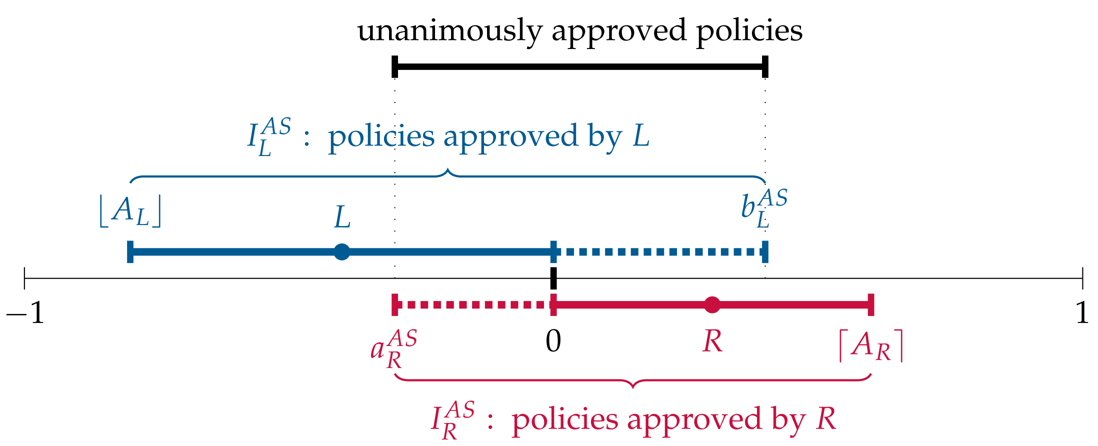

Figure 7. *The largest asymmetric intervals of approved policies of voter L (in blue) and R (in red). To convince voter L (R), the challenger pools policies that she prefers (solid) together with policies preferred by the other voter (dashed). The winning policies of the challenger (in black) are those approved by both voters.*

One case when $[a_R^{AS}, b_L^{AS}]$ may not be an equilibrium set of approved policies is if $\lfloor A_L \rfloor > -1$ and $\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x) > 0$, which jointly imply that $a_R^{AS} < \lfloor A_L \rfloor$.[8] Intuitively, in this case, the *right* voter is so persuadable that her largest asymmetric interval of approved policies includes the *left* voter's entire approval set, and then some. Note that $\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_v(x) d\mu_0(x) > 0$ cannot hold for both $v = L$ and $v = R$ at the same time.[9] To simplify notation, I let $\rho_v(L, R) := \int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_v(x) d\mu_0(x)$. For ease of exposition, I do not characterize the case when $\rho_L(L, R) > 0$ as it is symmetric to when $\rho_R(L, R) > 0$.

PROPOSITION 2. *Suppose that the $left$ voter is not significantly more persuadable than the right voter, that is, $\rho_L(L, R) \leq 0$. Almost surely, the right voter's equilibrium set of approved policies is her largest asymmetric interval of approved policies, or $\overline{W}_R = I_R^{AS}$. Furthermore, almost surely,*

(1) if $\rho_R(L, R) \leq 0$, then
- *the equilibrium set of approved policies of the $left$ voter is her largest asymmetric interval of approved policies, or $\overline{W}_L = I_L^{AS}$;*
- *the equilibrium set of unanimously approved policies is $\overline{W} = [a_R^{AS}, b_L^{AS}]$;*

(2) if $\rho_R(L, R) > 0$, then
- *the equilibrium set of approved policies of the $left$ voter is the largest interval of approved policies constrained from the left by a_R^{AS}, or $\overline{W}_L = I_L(a_R^{AS}, 1)$;*
- *the equilibrium set of unanimously approved policies is $\overline{W} = \overline{W}_L$.*

The formal proof of Proposition 2 is in the Appendix, but I outline it below. Since the *right* voter is the more persuadable one, let us add as many left policies to her message as possible. That is, let $\overline{W}_R = I_R^{AS} = \left[a_R^{AS}, \lceil A_R \rceil\right]$. Now, Proposition 2 states that the equilibrium depends on how a_R^{AS} is related to $\lfloor A_L \rfloor$. Specifically, if we are in *Case (2)* of Proposition 2, then $a_R^{AS} < \lfloor A_L \rfloor$. Otherwise, we are in *Case (1)*. Let us first consider the lower values of a_R^{AS}. Note that since $\rho_L(L, R) \leq 0$, the *left* voter can never

[8]Observe that $\phi(t) := \int_{t}^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x)$ is strictly increasing in $t < 0$ since $\frac{\partial \phi(t)}{\partial t} = -\alpha_R(t)\mu_0(t) > 0$. Consequently, $\phi(\lfloor A_L \rfloor) > 0 = \phi(a_L^{AS})$ is possible if and only if $\lfloor A_L \rfloor > a_L^{AS}$.

[9]Use Lemma 2 for the prior belief truncated to $\left[\lfloor A_L \rfloor, \lceil A_R \rceil\right]$ to conclude that $\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_v d\mu_0(x) \geq 0$ implies that $\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_{-v} d\mu_0(x) < 0$, for each $v \in \{L, R\}$ and $-v \in \{L, R\} \setminus \{v\}$.

be persuaded by policies to the right of the *right* voter's approval set, i.e. $b_L^{AS} \leq \lceil A_R \rceil$.

Suppose first that the *right* voter is so persuadable that she is willing to approve *all* left policies, i.e. $a_R^{AS} = -1$. In this case, (I_L^{UC}, I_R^{AS}) solves Problem (1), and the set of unanimously approved policies, which is essentially determined by the *left* voter, is I_L^{UC}.[10] By construction, there is no way to increase the objective beyond $\mu_0(I_L^{UC})$ while still satisfying the *left* voter's constraint. The same argument applies for every $a_R^{AS} \in [-1, a_L^{UC}]$. This case is illustrated in Figure 8 on the left.

Next, suppose that $a_L^{UC} < a_R^{AS} < \lfloor A_L \rfloor$. Now, (I_L^{UC}, I_R^{AS}) is no longer optimal: the challenger does not persuade the *right* voter with policies in $[a_L^{UC}, a_R^{AS})$, yet "wastes" the constraint of the *left* voter on them. Instead, select the *left* voter's message out of $[a_R^{AS}, 1]$, since the *right* voter rejects the policies outside of that interval anyway. Now, the proposed solution is $\big(I_L(a_R^{AS}, 1), I_R^{AS}\big)$, with the set of unanimously approved policies $I_L(a_R^{AS}, 1)$.[11] The challenger cannot increase his objective beyond $\mu_0\big(I_L(a_R^{AS}, 1)\big)$: it would require unanimous approval of policies to the left of a_R^{AS}, which are strictly more expensive in terms of the *right* voter's constraint than those that she already approves. Therefore, the proposed solution is optimal. This case is illustrated in Figure 8 on the right.

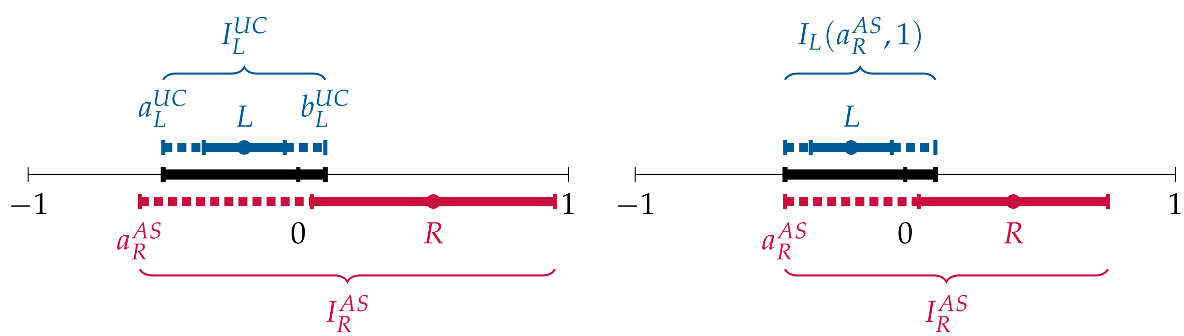

Figure 8. *Equilibrium sets of approved policies when the right voter is significantly more persuadable than the left voter.*

The last case we need to consider is when $\lfloor A_L \rfloor \leq a_R^{AS} < 0$. It remains to show that the proposed solution $(\overline{W}_L, \overline{W}_R) = (I_L^{AS}, I_R^{AS})$ with the set of unanimously approved policies $\overline{W} = [a_R^{AS}, b_L^{AS}]$ maximizes the objective of Problem (1). Indeed, any alternative set of unanimously approved policies would include policies to the left of a_R^{AS} or to the right of b_L^{AS}, which are more expensive in terms of the *right* or the *left* voter's

[10]Without the *right* voter's constraint, apply Corollary 1 to conclude that $\overline{W}_L = I_L^{UC}$. The set of unanimously approved policies is then $\overline{W}_L \cap \overline{W}_R = \overline{W}_L$ since $\lfloor \overline{W}_L \rfloor = a_R^{AS} \leq a_L^{UC} = \lfloor \overline{W}_R \rfloor$ and $\lceil \overline{W}_L \rceil = b_L^{UC} \leq b_L^{AS} \leq \lceil A_R \rceil = \lceil \overline{W}_R \rceil$.

[11]$\overline{W}_L \cap \overline{W}_R = \overline{W}_L$ since $\lfloor \overline{W}_L \rfloor = a_R^{AS} = \lfloor \overline{W}_R \rfloor$ and $\lceil \overline{W}_L \rceil \leq b_L^{AS} \leq \lceil A_R \rceil = \lceil \overline{W}_R \rceil$.

constraint, respectively.

Note that the obedience constraint of the relatively less persuadable (*left*) voter always binds. The *right* voter's constraint also binds unless $a_R^{AS} < a_L^{UC}$. Finding the equilibrium sets of approved policies is not computationally difficult and requires solving at most two auxiliary optimization problems. Algorithm 1 describes the steps.

Algorithm 1 Calculating the equilibrium sets of approved policies $(\overline{W}_L, \overline{W}_R)$

calculate $\rho_v(\lfloor A_L \rfloor, \lceil A_R \rceil) = \int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_v(x) d\mu_0(x)$ for each $v \in \{L, R\}$

if $\rho_v(\lfloor A_L \rfloor, \lceil A_R \rceil) \leq 0$ for each $v \in \{L, R\}$ **then**

  find a_R^{AS} and b_L^{AS} ▷ two (AUX) problems

  $\overline{W}_L = I_L^{AS}, \overline{W}_R = I_R^{AS}, \overline{W} = [a_R^{AS}, b_L^{AS}]$

else if $\rho_R(\lfloor A_L \rfloor, \lceil A_R \rceil) > 0$ **then**

  find a_R^{AS} ▷ one (AUX) problem

  find $I_L(a_R^{AS}, 1)$ ▷ one (AUX) problem

  $\overline{W}_R = I_R^{AS}, \overline{W}_L = I_L(a_R^{AS}, 1), \overline{W} = \overline{W}_L$

else

  find b_L^{AS} ▷ one (AUX) problem

  find $I_R(-1, b_L^{AS})$ ▷ one (AUX) problem

  $\overline{W}_L = I_L^{AS}, \overline{W}_R = I_R(-1, b_L^{AS}), \overline{W} = \overline{W}_R$

The following example calculates the equilibrium illustrated in Figure 7.

EXAMPLE 1 (UNIFORM PRIOR, $L = -0.4$, $R = 0.3$.). In this example, $A_L = [-0.8, 0]$ and $A_R = [0, 0.6]$. First, we check the relative persuadability of each voter by calculating $\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_v(x) dx$ for each $v \in \{L, R\}$. Both of these values are negative, so it remains to calculate b_L^{AS} and a_R^{AS}.

To find b_L^{AS}, solve $\int_{\lfloor A_L \rfloor}^{b_L^{AS}} \alpha_L(x) dx = 0$. Plugging in $\alpha_L(x) = -x^2 + 2Lx$, we arrive at the following equation: $(b_L^{AS})^3 - 3L \cdot (b_L^{AS})^2 = -4L^3$. It is not hard to check that the unique solution is $b_L^{AS} = -L = 0.4$, so that $\overline{W}_L = [-0.8, 0.4]$.[12] Similarly, we find that $a_R^{AS} = -R$ and $\overline{W}_R = [-0.3, 0.6]$.

Recall that one way to implement the equilibrium outcome is by pooling all policies in $\overline{W}_v$ into one message $\overline{W}_v$ that convinces voter $v \in \{L, R\}$. In this example, $\overline{W}_L = [-0.8, 0.4]$, meaning that the challenger says that his policy is not ultra left and not moderate to ultra right, but does not clarify any further. Furthermore, that message averages to -0.2, which is to the left of the status quo, making the *left* voter think that the challenger's policy is aligned with her preferences.

Both voters approve and the challenger wins if $x \in \overline{W}_L \cap \overline{W}_R = [-0.3, 0.4]$, that is, if his policy is sufficiently moderate. His odds of winning, calculated as the length of the interval of winning policies (0.7) relative to the length of the policy space (2), equal 0.35. We conclude that targeted advertising allows the challenger to improve his odds of winning from 0% to 35%!

Comparative Statics

Next, let us analyze what happens when the electorate becomes more polarized.

Definition 3.

- *The left voter* becomes more extreme *if L decreases; the right voter* becomes more extreme *if R increases.*
- *The baseline electorate* becomes more polarized *if the left and/or the right voter becomes more extreme.*

Note that the larger distance between the voters does not necessarily imply higher polarization. To increase polarization, one voter has to become more extreme, while the other voter has to stay fixed, or also become more extreme (in the opposite direction).

Observe that when a voter becomes more extreme, she becomes more persuadable.In particular, the voter becomes persuadable by a wider range of the left policies as well. The technical proof of the following result is in the Appendix, but I illustrate the argument for the *right* voter in Figure 9.

Lemma 3. *As a voter becomes more extreme, her largest asymmetric interval of approved policies expands, i.e.* $I_{v'}^{AS} \supseteq I_v^{AS}$. *Specifically,*

- *if* $L' < L$, *then* $\left[\lfloor A_{L'} \rfloor, b_{L'}^{AS}\right] \supseteq \left[\lfloor A_L \rfloor, b_L^{AS}\right]$, *with* $\lfloor A_{L'} \rfloor \leq \lfloor A_L \rfloor$ *and* $b_{L'}^{AS} \geq b_L^{AS}$; *the latter inequality is strict unless* $b_L^{AS} = 1$;

[12] Note that under the uniform prior, the solution $b_L^{AS} = -L$ is applicable for any $L \in [0.5, 0)$ and any quadratic net payoff from approval $\alpha_v(x) = -d(v) \cdot (x^2 - 2vx)$ with $d(v) > 0$.

- *if $R' > R$, then $[a_{R'}^{AS}, \lceil A_{R'} \rceil] \supseteq [a_R^{AS}, \lceil A_R \rceil]$, with $a_{R'}^{AS} \leq a_R^{AS}$ and $\lceil A_{R'} \rceil \geq \lceil A_R \rceil$; the former inequality is strict unless $a_R^{AS} = -1$.*

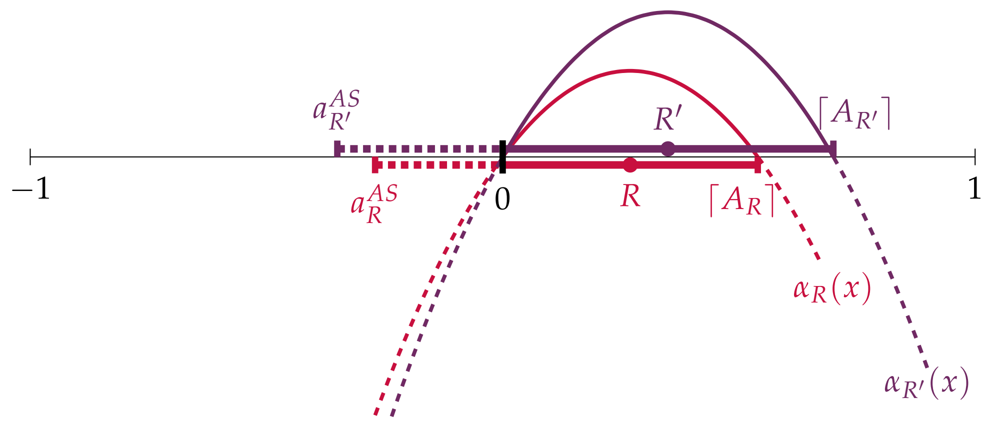

Figure 9. *The right voter becomes more persuadable by a wider range of policies as she becomes more extreme (her bliss point increases from R to R'): her approval set $[0, \lceil A_R \rceil]$ and her largest asymmetric interval of approved policies $[a_R^{AS}, \lceil A_R \rceil]$ expand.*

When her bliss point increases from R to R', two effects occur. On the one hand, her approval set expands, and the expected value of her net payoff from approval over her approval set, $\int_0^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x)$, strictly increases.[13] On the other hand, the voter becomes more risk averse as she becomes more extreme, and her net payoff from approval to the left of her approval set strictly decreases. The two changes work in opposite directions, meaning that a_R^{AS} decreases if the former effect dominates and increases if the latter effect is stronger. As it turns out, the quadratic utility is not concave enough for the latter effect to ever be stronger.

Now, consider the baseline electorate that satisfies the assumption of Proposition 2. That is, the *left* voter is not significantly more persuadable than the *right* voter. Proposition 3 describes what happens to the equilibrium sets of approved policies as the *right* voter becomes more extreme. Figure 10 illustrates.[14]

PROPOSITION 3. *Suppose that the left voter is not significantly more persuadable than the right voter, that is, $\rho_L(L, R) \leq 0$. Then, as the right voter becomes more extreme,*

[13] If $2R > 1$, then the approval set itself remains the same, unlike in Figure 9. However, $\int_0^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x)$ strictly increases because $\alpha_{R'}(x) > \alpha_R(x)$ for $x \in [0, \lceil A_R \rceil]$.

[14] Figure 10 presents the numerical solution for the uniform prior, $L = -0.15$, and $R_1 = 0.15$, $R_2 = 0.25$, $R_3 = 0.35$, $R_4 = 0.50$ (top to bottom). The sets of unanimously approved policies (in black) are $[-0.15, 0.15]$, $[-0.25, 0.15]$, $[-0.35, 0.1436]$, $[-0.4098, 0.1098]$, respectively.

- *the challenger's odds of winning increase;*
- *the equilibrium set of unanimously approved policies shifts to the left.*

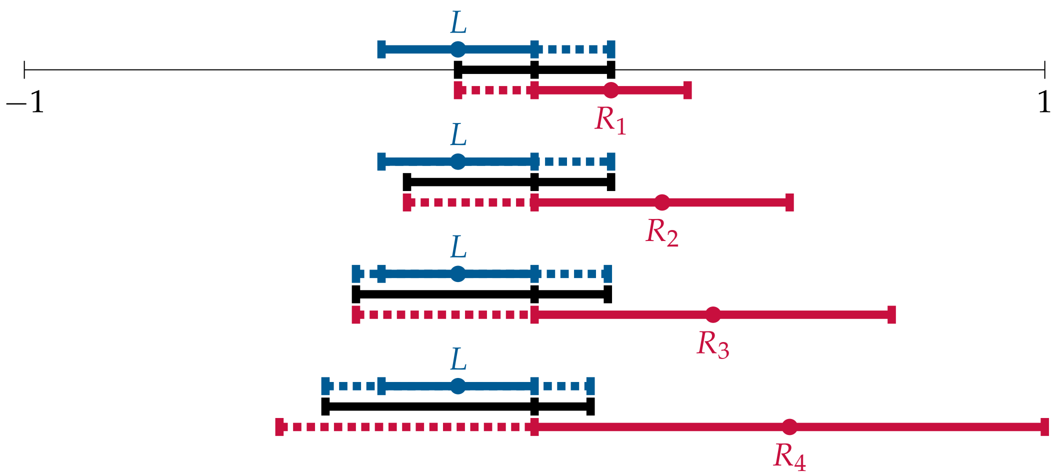

Figure 10. *Equilibrium set of approved policies as the right voter becomes more extreme (top to bottom). She becomes persuadable by a wider range of policies (in red), and the set of unanimously approved policies (in black) shifts to the left.*

Proposition 3 compares the equilibrium outcomes of two baseline elections, fixing the *left* voter's bliss point at L and increasing the *right* voter's bliss point from R to R'. Assume that $a_R^{AS} > a_L^{UC}$, or else no changes will take place.[15] Let $(\overline{W}_L, \overline{W}_R)$ and $(\overline{W}'_L, \overline{W}'_R)$ be the equilibrium outcome when the *right* voter's bliss point is R and R', respectively. Also, let $\overline{W} = \overline{W}_L \cap \overline{W}_L$ and $\overline{W}' = \overline{W}'_L \cap \overline{W}'_R$ be the equilibrium sets of unanimously approved policies before and after the change. Note that by Lemma 3, the *right* voter's constraint is looser after the change, immediately implying that the value of the objective (the challenger's odds of winning) can only go up. Furthermore, increasing R decreases the left boundary a_R^{AS} of the *right* voter's largest interval of approved policies (strictly so, unless $a_R^{AS} = -1$). From Proposition 2, a_R^{AS} is also the left boundary of the set of unanimously approved policies. It remains to prove that the right boundary of $\overline{W}$ also decreases. The general idea is that this boundary cannot shift to the right, as it is determined by the *left* voter's constraint, which does not change. The remainder of this section describes the conditions under which the decrease is strict.

Recall that Proposition 2 had two cases: one in which both voters are moderately persuadable and one in which the *right* voter is significantly more persuadable. Also,

[15] If $a_R^{AS} \leq a_L^{UC}$, then $\overline{W} = \overline{W}_L = I_L^{UC}$. Loosening the *right* voter's constraint does not change the equilibrium set of unanimously approved policies because the objective cannot be improved upon $\mu_0(I_L^{UC})$ while still satisfying the *left* voter's constraint, which does not change.

recall that increasing R increases $\lceil A_R \rceil$ and decreases $\rho_L(L,R)$. That is, the $left$ voter remains moderately persuadable after the change. At the same time, $\rho_R(L,R)$ increases, making the *right* voter more persuadable. We have three cases to consider.

Case (i): the *right* voter is moderately persuadable before and after the change, or $\rho_R(L,R) < \rho_{R'}(L,R') \leq 0$. Applying Part *(1)* of Proposition 2, we get $\overline{W}_v = I_v^{AS}$ for each $v \in \{L,R\}$ and $\overline{W}'_v = I_v^{AS}$ for each $v \in \{L,R'\}$. In particular, the right boundary of the set of unanimously approved policies is fixed at b_L^{AS} before and after the change. In Figure 10, *Case (i)* can be seen in the transition from the first to the second exhibit (when R_1 increases to R_2).

Case (ii): the *right* voter is moderately persuadable before and significantly more persuadable than the $left$ voter after the change, or $\rho_R(L,R) \leq 0 < \rho_{R'}(L,R')$. Apply Part *(1)* of Proposition 2 before the change to get $\overline{W}_v = I_v^{AS}$ for each $v \in \{L,R\}$, with $\overline{W} = [a_R^{AS}, b_L^{AS}]$. After the change, apply Part *(2)* of Proposition 2 to get $\overline{W}'_R = I_{R'}^{AS}$, $\overline{W}'_L = I_L(a_{R'}^{AS},1) = \left[\max\{a_{R'}^{AS}, a_L^{UC}\}, b'_L\right]$, with $\overline{W}' = \overline{W}'_L$. Now, by the $left$ voter's obedience constraint,

$$\int_{I_L^{AS}} \alpha_L(x)d\mu_0(x) = \int_{\lfloor A_L \rfloor}^{b_L^{AS}} \alpha_L(x)d\mu_0(x) = 0 \leq$$

$$\int_{I_L(a_{R'}^{AS},1)} \alpha_L(x)d\mu_0(x) = \int_{\max\{a_{R'}^{AS},a_L^{UC}\}}^{b'_L} \alpha_L(x)d\mu_0(x).$$

Since $\lfloor A_L \rfloor > \max\{a_{R'}^{AS}, a_L^{UC}\}$, we must have $b_L^{AS} > b'_L$, meaning that both ends of the set of unanimously approved policies strictly decrease. In Figure 10, *Case (ii)* can be seen in the transition from the second to the third exhibit (when R_2 increases to R_3).

Case (iii): the *right* voter is significantly more persuadable than the $left$ voter before and after the change, or $0 < \rho_R(L,R) < \rho_{R'}(L,R')$. Applying Part 2 of Proposition 2 before and after the change, we conclude that $\overline{W} = \overline{W}_L = I_L(a_R^{AS},1) = \left[a_R^{AS}, b_L\right]$ and $\overline{W}' = \overline{W}'_L = I_L(a_{R'}^{AS},1) = \left[\max\{a_{R'}^{AS}, a_L^{UC}\}, b'_L\right]$. Once again, from the $left$ voter's obedience constraint, $a_R^{AS} > \max\{a_{R'}^{AS}, a_L^{UC}\} \implies b_L > b'_L$. In Figure 10, *Case (iii)* can be seen in the transition from the third to the fourth exhibit (when R_3 increases to R_4).

Welfare

Consider an outcome in which a voter with a bliss point $v \in X$ approves some set of policies $W_v \subseteq X$. When v approves, her payoff is $-(v-x)^2$, and when she rejects, it is $-v^2$. Hence, her ex-ante utility is $-\mathbb{E}_{\mu_0}\left[\mathbb{1}(x \in W_v) \cdot (v-x)^2 + \mathbb{1}(x \in W_v^c) \cdot v^2\right]$.

Next, add v^2 from to that expression, to get $\int_{W_v} \alpha_v(x) d\mu_0(x)$. I use the latter object as a measure of v's welfare. I also define voter v's amount of regret as the difference between her welfare in the outcome under consideration (when she approves W_v) and under complete information (when she approves A_v).

DEFINITION 4. *Consider a voter with bliss point $v \in X \setminus \{0\}$ and her set of approved policies W_v. Then, v's*

- welfare *is* $\int_{W_v} \alpha_v(x) d\mu_0(x)$*;*
- amount of regret *is* $\int_{A_v} \alpha_v(x) d\mu_0(x) - \int_{W_v} \alpha_v(x) d\mu_0(x)$.

The table below compares voter welfare and challenger's odds of winning across three communication protocols. Firstly, there is the first-best *full disclosure* outcome (A_L, A_R) that delivers the complete information payoff for all players.[16] Secondly, there is the *public disclosure* outcome (W_L^{PD}, W_R^{PD}) of the baseline model with an additional restriction that the challenger must always send the same (verifiable) message to both voters.[17] Thirdly, there is the *targeted advertising* outcome $(\overline{W}_L, \overline{W}_R)$. Recall from the discussion after Proposition 2 that the obedience constraints $\int_{\overline{W}_v} \alpha_v(x) d\mu_0(x) \geq 0$ of both voters bind unless one of them is very extreme/persuadable, in which case her constraint may be loose.

	v's welfare	v's regret	challenger's odds of winning
full disclosure	$\int_{A_v} \alpha_v(x) d\mu_0(x) > 0$	0	0
public disclosure	$\int_{W_v^{PD}} \alpha_v(x) d\mu_0(x) \geq 0$	≥ 0	0
targeted advertising	$\int_{\overline{W}_v} \alpha_v(x) d\mu_0(x) = 0$	> 0	$\mu_0(\overline{W}_L \cap \overline{W}_R) > 0$

[16]Under full disclosure, the set of approved policies of voter $v \in \{L, R\}$ is A_v. Each voter learns whether the challenger's policy is in her approval set, and thus acts as if under complete information. Note that full disclosure is the sender-worst equilibrium outcome of the baseline model.

[17]While not covered in this paper, the public disclosure outcome has to satisfy voter v's obedience constraint, $\int_{W_v^{PD}} \alpha_v(x) d\mu_0(x) \geq 0$, or else v would not approve these policies. Furthermore, the challenger's odds of winning are $\mu_0(W_L^{PD} \cap W_R^{PD}) = 0$ by the definition of an unwinnable election.

Notice that targeted advertising maximizes the challenger's odds of winning at the expense of minimizing voter welfare and maximizing voter regret.

4. LARGE ELECTIONS

This section generalizes the baseline model to electorates with more than two voters and any social choice rule. A *large election* $(\mathcal{V}, scf)$ consists of

- a finite set $\mathcal{V} = \{v_1, \ldots, v_n\} \subseteq X$ of voters' bliss points;
- a social choice function $scf : 2^{\mathcal{V}} \to \{0, 1\}$ that describes, for every subset of voters $V \subseteq \mathcal{V}$ whether the challenger wins (1) or loses (0) the election if every voter in V approves his proposal.

Now, let $\mathcal{D} := \{D \subseteq \mathcal{V} \mid scf(D) = 1\}$ be the set of all *decisive coalitions*. Then, to win the election, the challenger must convince all voters of some decisive coalition $D \in \mathcal{D}$. The first result characterizes large elections that are unwinnable without targeted advertising.

LEMMA 4. *The following statement about the large election $(\mathcal{V}, scf)$ are equivalent*

(1) no decisive coalition consists of voters with only strictly negative or only strictly positive bliss points;

(2) the status quo policy is almost surely socially preferred to the challenger's policy under complete information, that is, for any decisive coalition $D \in \mathcal{D}$, the prior measure of the set $\bigcap_{v \in D} A_v$ is zero;

(3) the status quo policy is socially preferred to the challenger's policy under any non-degenerate belief $\mu \in \Delta X$, that is, for any decisive coalition $D \in \mathcal{D}$, at least one of the voters $v \in D$ prefers to reject under belief μ, or $\int \alpha_v(x) d\mu(x) < 0$.

Lemma 4 generalizes the idea that left and right voters never prefer to approve the challenger's policy at the same time under complete information (condition *2*) and under common belief (condition *3*). Condition *1* provides a simple characterization of an unwinnable election in terms of the locations of the voters relative to the status quo. Under simple majority, the first condition describes a version of the median voter proposition.[18]

COROLLARY 2. *Under simple majority, a large election is unwinnable for the challenger without targeted advertising if and only if the status quo is the bliss point of the median voter.*

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[18]Black (1948) states the median voter proposition as "If X is a single-dimensional issue and all voters have single-peaked preferences defined over X, then the median position could not lose under majority rule."

Now, recall from Lemma 3 that more extreme voters have larger asymmetric intervals of approved policies. Relatedly, more extreme voters also prefer to approve whenever their less extreme counterparts do.

LEMMA 5. *If voter $v \in X \setminus \{0\}$ prefers to approve under belief $\mu \in \Delta X$, then v', who is more extreme than v, also prefers to approve under belief μ.*

At this point, we can conclude that if the challenger convinces voter v whose bliss point is, say, to the left of the status quo, then he can convince every voter to the left of v using the exact same strategy.[19]

DEFINITION 5. *Consider a decisive coalition $D \in \mathcal{D}$. The* left pivot *of D is $L(D) := \max_{v \in D, v<0} v$ and the* right pivot *of D is $R(D) := \min_{v \in D, v>0} v$.*

Now, if D does not include any voters whose bliss point is exactly 0, then convincing the pivotal voters $L(D)$ and $R(D)$ is necessary and sufficient to convince all voters in D. While every unwinnable baseline election can be won with targeted advertising, it may be the case that *every* decisive coalition of a large election includes voters whose bliss point is 0.[20] As a result, many, but not all, unwinnable large elections can be won with targeted advertising.

PROPOSITION 4. *Consider an unwinnable large election (V, scf). In equilibrium, the challenger's ex-ante odds of winning are positive if and only if there exists a decisive coalition $D \in \mathcal{D}$ such that $0 \notin D$.*

The proof of Proposition 4 is straightforward: if every decisive coalition includes a voter with a bliss point of 0, then the election is unwinnable by Lemma 4. Otherwise, select a winning coalition $D \in \mathcal{D}$ that does not have a voter with a bliss point of 0, and solve the baseline election game for $L(D)$ and $R(D)$. Proposition 1 states that the challenger's odds of winning are positive, and Proposition 2 describes the solution.

Maximizing the challenger's odds of winning involves the search over all decisive coalitions that do not include voter 0. Proposition 3 reduces the domain of this optimization problem: for two decisive coalitions $D, D' \in \mathcal{D}$, if the baseline electorate $\big(L(D), R(D)\big)$ is more polarized than $\big(L(D'), R(D')\big)$, then the challenger's odds of winning are higher if he targets the former pivotal voters. Beyond that, the challenger must solve Problem (1) for the pair $\big(L(D), R(D)\big)$ of pivotal voters of all decisive coalitions $D \in \mathcal{D}$ such that $0 \notin D$.

[19] The same strategy means that for every $x \in X$ the challenger sends the same messages with equal probabilities to voter v and every voter v' who is more extreme than v.

[20] If $v = 0$, then $\alpha_v = -x^2$ and $A_v = \{0\}$. Every belief $\mu \in \Delta X$ that convinces this voter to approve, or $\int \alpha_v(x) d\mu(x) \geq 0$ is such that $\mu(0) = 1$. The ex-ante probability of convincing such voter is 0.

EXAMPLE 2. Let $V = \{-0.5, -0.2, 0, 0.2, 0.3\}$ and $scf(V) = \mathbb{1}(|V| \geq 3)$ (simple majority). That is, to win, the challenger must convince at least 3 voters.

Now, clearly, every decisive coalition includes at least three voters. At the same time, there are only 2 voters on each side of the status quo, meaning that condition *(1)* of Lemma 4 is satisfied. Next, the approval sets of the voters are $A_1 = [-1, 0]$, $A_2 = [-0.4, 0]$, $A_3 = \{0\}$, $A_4 = [0, 0.4]$, and $A_5 = [0, 0.6]$. Thus, an intersection of any combination of three or more approval sets is $\{0\}$, which has prior measure zero, satisfying condition *(2)* of Lemma 4. Finally, consider a non-degenerate belief $\mu \in \Delta X$. Voter v_3 prefers to reject, since $\alpha_3 = -x^2$ is negative for every $x \in X \setminus \{0\}$. By Lemma 2, neither one of the left voters -0.5 and -0.2 prefers to approve if the right voters 0.2 and 0.3 prefer to approve. Consequently, condition *(3)* of Lemma 4 is also satisfied. We conclude that this election is unwinnable without targeted advertising. A shorter way to establish this result is by using Corollary 2: the median voter is 0.

Now, the (minimal) decisive coalitions are

coalition	left pivot	right pivot
$D_1 = \{-0.5, -0.2, 0.2\}$	-0.2	0.2
$D_2 = \{-0.5, -0.2, 0.3\}$	-0.2	0.3
$D_3 = \{-0.5, 0.2, 0.3\}$	-0.5	0.2
$D_4 = \{-0.5, 0.2, 0.3\}$	-0.2	0.2

Since -0.5 is more extreme than -0.2 and 0.3 is more extreme than 0.2, targeting coalitions D_2 and D_3 results in higher odds of winning than targeting D_1 and D_4. Now, without any assumption on the shape of the prior distribution, targeting either coalition may maximize the challenger's odds of winning. For example, if the prior distribution is heavily skewed to the right of 0, then D_2 may result in the highest odds of winning. At the same time, for any prior distribution that is symmetric about 0, targeting D_3 maximizes the challenger's objective. Under the uniform prior, if we solve the baseline election for $L = -0.5$ and $R = 0.2$, we get $\overline{W}_L = [-1, 0.5]$, $\overline{W}_R = [-0.179, 0.5]$, and $\overline{W} = [-0.179, 0.5]$.

Therefore, in his most preferred equilibrium, the challenger sends message $\overline{W}_L = [-1, 0.5]$, whenever $x \in \overline{W}_L$, to all voters to the left of the status quo, and message $\overline{W}_R = [-0.179, 0.5]$, whenever $x \in \overline{W}_R$, to all voters to the right of the status quo. Otherwise, he says nothing. On the equilibrium path, the challenger convinces voters -0.5, 0.2, 0.3 whenever his policy is $x \in [-0.179, 0.5]$. His ex-ante odds of winning are 33.96%.

5. Conclusion

An election is unwinnable for a challenger without targeted advertising if pivotal voters are on the opposite sides of the status quo because no public message convinces them to approve the challenger's policy. I show that the challenger can win any such election with targeted advertising. In private, the challenger makes each voter believe that his policy is a sufficient improvement over the status quo and wins the election if his policy is sufficiently moderate. When voters become more extreme, the challenger's odds of winning increase.

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APPENDIX: OMITTED PROOFS

PROOF OF PROPOSITION 2

The case where $a_R^{AS} \leq a_L^{UC}$ is proved in the main text.

Suppose that $a_L^{UC} < a_R^{AS} < \lfloor A_L \rfloor$. Let $\overline{W}_L = I_L(a_R^{AS}, 1)$ and $\overline{W}_R = I_R^{AS}$. It remains to show that the challenger's odds of winning cannot be higher than $\mu_0(\overline{W}_L)$ for any other pair (W_L, W_R) that satisfies both voters' constraints. Indeed, any W_L such that $\mu_0(W_L) > \mu_0(\overline{W}_L)$ satisfies the constraint of the *left* voter has to include a positive-measure set $Y \subseteq [-1, a_R^{AS}]$. However, every policy $y \in Y$ is more expensive in terms of R's constraint than any policy $x \in [a_R^{AS}, 0]$ (because $\alpha_R(y) < \alpha_R(x)$). Consequently, including Y in the set of unanimously approved policies increases the objective by $\mu_0(Y)$ but decreases it by more than $\mu_0(Y)$. Hence, $(I_L(a_R^{AS}, 1), I_R^{AS})$ solves Problem (1) if $a_L^{UC} < a_R^{AS} < \lfloor A_L \rfloor$, or more generally, whenever $a_R^{AS} < \lfloor A_L \rfloor$, since if $a_R^{AS} \leq a_L^{UC}$ then $I_L(a_R^{AS}, 1) = I_L^{UC}$.

The last case is where $\lfloor A_L \rfloor \leq a_R^{AS} \leq 0$. Here I show that the proposed solution $(\overline{W}_L, \overline{W}_R) = (I_L^{AS}, I_R^{AS})$ with the set of unanimously approved policies $\overline{W} = [a_R^{AS}, b_L^{AS}]$ maximizes the objective of Problem (1). Consider another solution (W_L, W_R) with the set of unanimously approved policies $W = W_L \cap W_R$. First, observe that W cannot be left of $\overline{W}$, i.e. the set $W \cap [b_L^{AS}, 1]$ must have a positive prior measure. If not, then the *right* voter's constraint has to be spent on policies further than a_R^{AS}, which decreases the objective. Specifically, from R's constraint, $\int_{\overline{W}_R \cap W_R^c} \alpha_R(x) d\mu_0(x) \leq \int_{W_R \cap [-1, a_R^{AS}]} \alpha_R(x) d\mu_0(x)$. Also, $\alpha_R(\overline{x}) > \alpha_R(x)$ for all $\overline{x} \in \overline{W}_R$ and $x \in [-1, a_R^{AS}]$, which implies $\mu_0(\overline{W}_R \cap W_R^c) > \mu_0(W_R \cap [-1, a_R^{AS}])$. Finally, we have $\overline{W}_R \cap W_R^c = \overline{W} \setminus W$ and $W_R \cap [-1, a_R^{AS}] \supseteq W \setminus \overline{W}$. It follows that $\mu_0(\overline{W}) > \mu_0(W)$. By a symmetric argument for the *left* voter, W cannot be to the right of $\overline{W}$, either, and the set $W \cap [-1, a_L^{AS}]$ has to have a positive prior measure.

Next, observe that sets $W \cap \big[-1, 0\big]$ and $W \cap \big[0, 1\big]$ must be intervals that end at 0 and start at 0, respectively. Otherwise, W can be improved upon. For example, if $W \cap \big[-1, 0\big] \neq [a, 0]$ for some $a \geq -1$, then there exist two sets $Y = [y_1, y_2] \subseteq W$ and $Z = [z_1, z_2] \subseteq W^c$ such that $-1 \leq y_1 < y_2 \leq z_1 < z_2 \leq 0$ and $\mu_0(Y) = \mu_0(Z)$. Then, for every $y \in Y$ and $z \in Z$, $\alpha_R(y) < \alpha_R(z) < 0$ and either $\alpha_L(y) < \alpha_L(z)$ or $\alpha_L(y) > \alpha_L(z) \geq 0$. Let $\widehat{W}_L = \big(W_L \setminus (Y \cap A_L^c)\big) \cup Z$ and $\widehat{W}_R = \big(W_R \setminus Y\big) \cup Z$. By construction, $(\widehat{W}_L, \widehat{W}_R)$ satisfies both constraints and maintains the objective at $\mu_0(W)$. However, since R's constraint is now loose, we can further increase the objective, a contradiction.

PROOF OF LEMMA 3

I prove this lemma for the *right* voter whose bliss point increases from R to R'. The case of the *left* voter is symmetric. To simplify the notation, I let $a := a_R^{AS}$ and $a' := a_{R'}^{AS}$.

First, note that $R' > R$ implies $\lceil A_{R'} \rceil \geq \lceil A_R \rceil$ since $\lceil A_{R'} \rceil = \min\{1, 2R'\} \geq \min\{1, 2R\} = \lceil A_R \rceil$. Thus, $A_{R'} \supseteq A_R$.

Next, observe that unless $a = -1$, R's constraint binds and $\int\limits_a^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x) = 0$. If the constraint does not bind for R', then $a' = -1 < a$. If it does bind, then

$$\int\limits_{a'}^{\lceil A_{R'} \rceil} \alpha_{R'}(x) d\mu_0(x) = 0 \iff$$

$$\int\limits_{a'}^{a} \underbrace{\alpha_{R'}(x)}_{<0 \text{ since } x<0} d\mu_0(x) + \int\limits_{a}^{\lceil A_R \rceil} \big(\alpha_{R'}(x) - \alpha_R(x)\big) d\mu_0(x) + \int\limits_{\lceil A_R \rceil}^{\lceil A_{R'} \rceil} \underbrace{\alpha_{R'}(x)}_{\geq 0 \text{ since } x \in A_{R'}} d\mu_0(x) = 0,$$

where I subtracted $\int\limits_a^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x) = 0$ from the second term.

Now, if $\int\limits_a^{\lceil A_R \rceil} \big(\alpha_{R'}(x) - \alpha_R(x)\big) d\mu_0(x) \geq 0$, then $a' < a$. In what follows, I show that the former inequality holds for the quadratic $\alpha_R(x)$. Indeed, $\alpha_{R'}(x) - \alpha_R(x) = 2(R' - R)x$. Let $\overline{x}_R := \int\limits_a^{\lceil A_R \rceil} x d\mu_0(x)$. By Jensen's inequality for the concave $\alpha_R(x)$,

$$\alpha_R(\overline{x}_R) \geq \int\limits_a^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x) = 0,$$

and $\alpha_R(\overline{x}_R) \geq 0$ if and only if $\overline{x}_R \in \big[0, \lceil A_R \rceil\big]$. Therefore, $\int\limits_a^{\lceil A_R \rceil} 2(R' - R) x d\mu_0(x) \geq 0$ and $a' < a$.

PROOF OF LEMMA 4

(1) $\Longrightarrow$ *(2),(3)*: there are two cases to consider, Case 1 is when all decisive coalitions include 0, and Case 2 is when every decisive coalition consists of voters with strictly negative and strictly positive bliss points.

In Case 1, $\bigcap\limits_{v \in D} A_v = \{0\}$ for all $D \in \mathcal{D}$ because $A_0 = \{0\}$. At the same time, 0 only approves under belief $\mu \in \Delta X$ if $\mu(0) = 1$, which is a degenerate belief. She prefers to reject for any non-degenerate belief.

In Case 2, $\bigcap\limits_{v \in D} A_v = \{0\}$ for all $D \in \mathcal{D}$ because $A_L \cap A_R = \{0\}$ for each $L \in D$ and $R \in D$. Similarly, by Lemma 2, at most one of the voters L and R prefers to approve under a common noon-degenerate belief, for all $L \in D$ and $R \in D$.

(2),(3) $\Longrightarrow$ *(1)* (by contrapositive). If *(1)* is not true, then there exists a decisive coalition $D \in \mathcal{D}$

of voters with strictly negative or strictly positive bliss points. For that coalition (say, all bliss points are strictly positive), let $\hat{v} := \min_{v \in D} v$ be the voter closest to the status quo. Then, $\bigcap_{v \in D} A_v = A_{\hat{v}}$ because $A_v = \left[0, \min\{1, 2v\}\right]$ for all $v \in D$. Since $\hat{v} > 0$, the set $A_{\hat{v}}$ has a positive prior measure. Now, let $\mu \in \Delta X$ be the prior belief μ_0 truncated to $\hat{v}$'s approval set $A_{\hat{v}}$. Note that μ is a non-degenerate belief because μ_0 has no atoms and $A_{\hat{v}}$ has a positive prior measure. Every voter's expected net payoff from approval is strictly positive under μ because $\alpha_v(x) \geq 0$ for all $x \in \operatorname{supp} \mu = A_{\hat{v}}$ (strictly so in the interior of $A_{\hat{v}}$) and $v \in D$. Consequently, every voter in D prefers to approve.

Proof of Lemma 5

I prove this result for $R' > R > 0$. The case of $L' < L < 0$ is symmetric.

Since voter R approves, we have

$$\begin{aligned}
\int \alpha_R(x) d\mu(x) &= \int (-x^2 + 2Rx) d\mu(x) \\
&= \int (-x^2 + 2R'x) d\mu(x) + 2(R - R') \int x d\mu(x) \\
&= \int \alpha_{R'}(x) d\mu(x) + 2(R - R')\mathbb{E}_\mu[x] \geq 0.
\end{aligned}$$

Since by Lemma 1 $\mathbb{E}_\mu[x] > 0$ whenever R approves, the following inequality implies that $\int \alpha_{R'}(x) d\mu(x) > 0$ if $R' > R$.