Election Games with Verifiable Information[*]

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Abstract

This paper studies election games with verifiable information, in which the sender attempts to convince a group of receivers to approve his policy proposal over the status quo. The main result poses a version of the communication revelation principle that allows to restrict attention to not necessarily truthful, but direct equilibria, in which the sender recommends an action, and the receiver obediently follows the recommendation. The sender-worst equilibrium is truthful in the sense that full unraveling occurs and the receiver acts as if under complete information. In the sender-preferred equilibrium, his ex-ante utility is maximized subject to the receivers' obedience constraints, and the resulting set of approved states is the same as if the sender was solving a bayesian-persuasion model with ex-ante commitment power. I apply this model to study targeted advertising in a spatial model of voting. I show that all elections that are unwinnable under public disclosure become winnable when the challenging politician can advertise his position privately. While the targeting technology is valuable for swinging unwinnable elections, the value is limited in that only the challengers located sufficiently close to the status quo can benefit from it.

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1. INTRODUCTION

In a variety of real-life situations, a group of agents needs to make a collective decision. Oftentimes, they are uninformed and rely on a third party to provide some context.

There are many real-life examples of such situations. A politician challenges the incumbent, thus forcing the voting body to choose between his policy and the status quo; a company CEO attempts to convince the board of directors to decide on managerial compensation; a firm enters a market and launches an advertising campaign to get the consumers to adopt its product over an established competitor. In all these scenarios, the sender can prove any true claim, but he is not forced to say nothing but the truth. In his electoral campaign, the challenger may fail to mention his policy on some of the issues, but once elected, he must deliver on the promises he had made, or else he would bear extreme reputational costs. The CEO may omit some of the financial indicators in his report to the board, but those that he does present must be legitimate. The firm may exaggerate some of the product characteristics and fail to mention others, but it cannot advertise falsely. I focus attention on private disclosure, meaning that the challenging politician or the firm sends targeted ads to the voters or the consumers, and the CEO speaks to the board members in private.

I model such situations as a communication game between one sender and many receivers. The sender wishes to get his policy proposal approved by a collective vote, and sends private verifiable messages to the uninformed receivers in an attempt to convince them to vote in his favor. Each receiver independently compares the sender's proposal to the status quo and endorses the option that maximizes her expected utility. Whether the sender's proposal is approved is decided by a social choice function that aggregate receivers' votes.

I show that every equilibrium outcome can be implemented directly, but not truthfully. The direct implementation features a set of convincing messages, one for each receiver, that receivers interpret as a recommendation to approve the sender's proposal. While the sender cannot lie by commission, nothing prevents him from lying by omission and exaggerating the value of his proposal to each receiver.

I then focus attention on the scenario in which a politician challenges the incumbent in a spacial model of voting and ask the following question. If the challenger can make his policy look more appealing when making campaign promises in private, does targeted advertising change the outcomes of elections, relative to public disclosure? The answer is yes and the following motivating example walks through the intuition.

A privately informed politician (sender, challenger, he) sends verifiable messages to the only voter in an attempt to convince her to approve his policy proposal. The receiver (voter, she) compares the challenging proposal to the status quo ω° and votes for the policy that maximizes her expected utility. In this example, the receiver has spatial preferences on the $[0,1]$ interval, and her *approval set*, i.e. proposals that she approves under complete information, includes policies that are closer to her bliss point v than the status quo. This setup is illustrated in Figure 1.

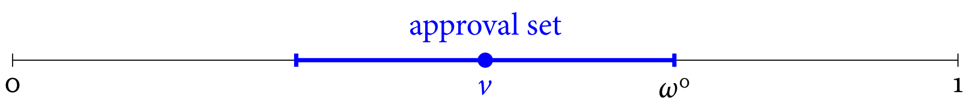

Figure 1: *voter v's approval set.*

What verifiable messages does the challenger send to convince the voter to approve his policy proposal? The first result of this paper summarized in Theorem 2.1, states that any of many equilibrium outcomes may be implemented directly. In a *direct implementation*, the policy space is partitioned into two subsets, the winning policies $\mathcal{W}$ and the losing policies $\mathcal{L} = \Omega \setminus \mathcal{W}$. In winning states, the challenger sends the convincing message $\mathcal{W}$, which is interpreted by the voter as a recommendation to approve his proposal. In losing states, he sends message $\mathcal{L}$, thus advising the receiver to maintain the status quo. Every direct equilibrium is characterized by one object – the set of winning states $\mathcal{W}$ that satisfies two constraints: *(i)* it contains the receiver's approval set because in those states the challenger can guarantee to get his proposal approved by deviating to truth-telling, and *(ii)* it satisfies the receiver's *obedience constraint*, which ensures that after hearing the convincing message the voter rationally expects to be weakly better off under the challenger's policy than the status quo. Using the terminology of Myerson (1982)'s communication revelation principle for mediated disclosure games, I conclude that when there is no external communication device but the information is verifiable, one can restrict attention to direct, but not truthful mechanisms. Full unraveling may not occur because the sender is not restricted by an external incentive-compatibility constraint, and while he cannot lie by commission, verifiability alone does not necessarily prevent him from lying by omission.

Equipped with this weaker version of the communication revelation principle, I characterize the full set of equilibrium outcomes of the election game. Unsurprisingly for games with verifiable information, the sender-worst equilibrium features full unraveling of information. Put differently, the set of winning states coincides with the approval set of the voter, and the challenging policy is approved if and only if the voter weakly prefers it to the status quo under complete information.

What is the best that the challenger can do? In a direct equilibrium, the sender's ex-ante utility is simply the measure of the set of states in which his proposal is approved, under the prior distribution. Hence, to find the sender-preferred equilibrium, one would expand the set of winning states beyond the approval set until the voter's obedience constraint binds. The resulting solution, described in Theorem 2.2, is the same as the solution when the sender has ex-ante commitment power, and is characterized by a set of cutoff states. Sender's proposal is approved if and only if his policy is sufficiently close (yet further than the status quo) to the voter, and whenever it is approved, the receiver is indifferent between the expected policy of the challenger and the status quo. The sender-preferred equilibrium of the verifiable information game, which coincides with the solution of the sender's bayesian-persuasion problem, is depicted in Figure 2.

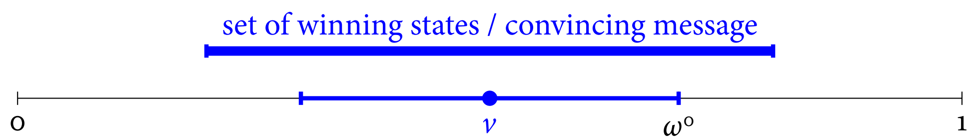

Figure 2: *sender-preferred set of winning states $\mathcal{W}$ that is symmetric around v; all challengers in $\mathcal{W}$ pool together and send convincing message $\mathcal{W}$.*

When there are many voters, the direct implementation result, summarized in Theorem 2.3, still holds as long as there are no information spillovers or strategic interactions among the voters. In other words, the voters must make their choice between the challenger's proposal and the status quo expressively and cannot share their private messages among themselves. With many voters, the direct implementation is characterized by a collection of convincing messages $\{\mathcal{W}_v\}$, one for each voter, rather than just one set of winning states. In each state, the challenger sends private convincing messages to the voters who he can feasibly convince (i.e. his electorate) and recommends others to vote for the status quo. Convincing messages may vary among the voters, but each voter only ever hears one of the two messages on the equilibrium path: one convinces her, one doesn't. Naturally, every voter must interpret the convincing message as a recommendation to vote for the challenger, which means that her obedience constraint (the one from the one-receiver case) must hold.

Equipped with the direct implementation result, let us study the setup of public disclosure, in which the challenger sends the same message to every voter. Observe that if two voters are located on the opposite sides of the status quo, then they are *incompatible* and do not agree on any policy under complete information, that is, their approval sets only intersect at the exact location of the status quo. As it turns out, these voters never both vote for the challenger under any common belief.

Consequently, if the social choice function is such that incompatible voters are jointly pivotal, then that election would be *unwinnable* for the challenger under public disclosure, even if he has ex-ante commitment power. Consequently, if the social choice function is such that the votes of incompatible voters are pivotal,

Can the challenger swing an unwinnable election by targeted advertising? The answer depends on the equilibrium selection criterion. In the sender-worst equilibrium, the answer is no because full unraveling occurs and brings the outcome back to complete information. In the sender-preferred equilibrium, however, the answer is always yes because targeting allows to induce heterogeneous posterior beliefs among the receivers. Convincing incompatible voters becomes possible because while their approval sets do not intersect almost surely, the challenger can always design private messages that do intersect. Consequently, under any social choice function, the challenger's set of approved states is no longer empty and an unwinnable election is won with positive probability.

Notice that in the one-dimensional case there are at most two incompatible voters, the left pivot $v_L < \omega^o$ (the representative voter of the coalition on the left) and the right pivot $v_R > \omega^o$ (the representative voter of the coalition on the right), convincing who is sufficient to win the election under any social choice rule. The reason is that all voters on the left are compatible, and convincing the utmost right voter on the left is necessary and sufficient to convince the whole left coalition. To convince both pivots at the same time, the challenger biases the private message toward policies preferred by the opposite-side voter, as illustrated in Figure 3.

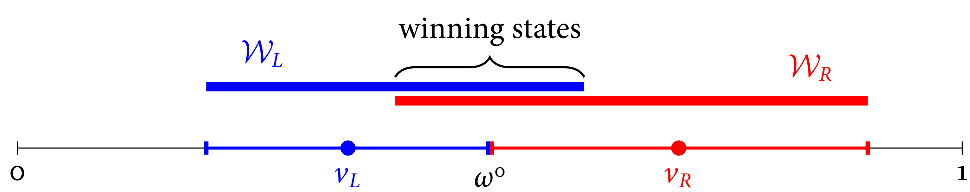

Figure 3: *sender-preferred equilibrium under targeted advertising: message $\mathcal{W}_i$ convinces voter v_i, in states $\mathcal{W}_L \cap \mathcal{W}_R$ challenger convinces both incompatible voters and wins the election.*

Recall the single point that the approval sets of the incompatible voters have in common is the status quo. Thus, to maximize the odds of convincing both voters, the challenger begins the construction of private message to voter v_R (v_L) with her approval set, and then adds policies to the left (right) of the status quo until her obedience constraint binds. The reason why this process is optimal is that adding policies that are immediately to the left of the status quo is "cheap" in terms of the right pivot's obedience constraint and "free" in terms of the left pivot's obedience constraint, because they lie in her approval set. Once both obedience constraints bind, the sender's set of winning states is pinned down by the left boundary of the right pivot's convincing message and the right boundary of the left pivot's convincing message. Consequently, the sender ends up convincing incompatible voters with an ex-ante positive probability, implying that every unwinnable election may be won by private messaging. Although the targeting technology is valuable for swinging unwinnable elections, its value is limited in that only challengers with policies sufficiently close to the status quo can benefit from it.

RELATED LITERATURE

This paper builds upon the seminal works of Milgrom (1981), Grossman (1981) and Milgrom and Roberts (1986) who introduced communication games with verifiable information. Their main result is that when the sender possesses hard evidence, all information unravels, since the good types have an incentive to separate themselves from the bad types because that yields a higher payoff, and the bad types do not possess the evidence to pass as good. In election games considered in this paper, the sender's utility is not strictly monotone and only takes two values (he can win or lose), meaning that the good types do not have a strict incentive to separate from the bad types. While the unraveling equilibrium still exists, there are other PBE in which the sender may do strictly better ex-ante. In particular, I show that in the sender-preferred equilibrium he can do as well as if he had commitment power, thus making a connection to the literature on bayesian persuasion started by Kamenica and Gentzkow (2011). The two bayesian persuasion papers closest to mine are Alonso and Câmara (2016) and Arieli and Babichenko (2019). In the former paper, the sender designs a public signal to reach his objective of persuading a group of voters. According to my results, every election that is unwinnable in their setting could be won by targeting, provided that the voters are sincere and do not share their private signals. Arieli and Babichenko (2019) consider a similar problem to mine, with the only differences being that the sender's utility may be more general, the sender has commitment power, and the state space is finite. I provide an example of how a finite-state problem can be transformed into a continuous setup, and that makes their solution applicable to the election games with verifiable information and no commitment power.

This paper also contributes to the literature on *discriminatory information disclosure*, in which the sender caters his messages to specific groups of receivers. Farrell and Gibbons (1989) and Goltsman and Pavlov (2011) consider a cheap talk setting and show that public disclosure may lead to more informative equilibria than private communication when receivers are biased in opposite directions because in private the sender has a significant incentive to overstate the truth, which prevents each receiver from believing him. In Koessler (2008)'s setting, the sender can send certifiable messages, but again, private communication forces him to reveal more information, and as a consequence, he is better off under public communication. Bar-Isaac and Deb (2014) confirm that public communication is better for the sender if his purpose is to build reputation. Overall, this literature concludes that if the voters are strategic (that is, if they either care explicitly about the outcome or take into account the probability of being pivotal), then they are extremely skeptical when receiving a private message. This, in turn, implies that the sender is always worse off under private disclosure, relative to public disclosure. In this paper, the receivers are not strategic and make their decisions in private. As a consequence, the sender is always weakly (and sometimes strictly) better off under private disclosure.

This paper contributes to the literature on *disclosure in electoral competition*. When candidates send verifiable messages about their policy, competition usually results in full unraveling of information. The reason is that the candidates play a zero-sum game, and it pushes them to voluntarily disclose all information, however unfavorable, as shown by Board (2009). Expanding the candidates' options beyond disclosing information about their own position does not help, either. Schipper and Woo (2019), like me, examine the effects of allowing candidates to microtarget voters (who may be unaware that some issues even exist), and, separately, allow candidates to "negatively campaign" by sending verifiable messages about each other. In both cases, they find that the voters always end up learning everything and making the same choices they would have made under complete information. Janssen and Teteryatnikova (2017) come to the same conclusion after studying a model of electoral competition in which the candidates are allowed to send (comparative) statements about their position relative to their opponent. While political candidates are symmetric is all these papers, in the application of my model, one candidate has a significant advantage over his opponent in that he is the only one who can send (targeted) messages to voters, perhaps because his budget is much higher, or because he has access to better technology. As a result, the challenger can improve (in some equilibria significantly) his ex-ante odds of winning, relative to the full unraveling benchmark.

Finally, I provide a theoretical explanation for the *empirical literature on voter persuasion*, surveyed by Prat and Strömberg (2013) and DellaVigna and Gentzkow (2010), which documents how catering campaign messages to specific groups of voters affects electoral outcomes. Other applicable examples include George and Waldfogel (2006), who argue that candidates who target educated voters who care more about global issues post their ads in the *New York Times*, while candidates with more local agenda post in local newspapers; DellaVigna and Kaplan (2007), who show that the introduction of Fox News in the local markets in the US increased the Republican vote in the 2000 presidential election; Enikolopov, Petrova, and Zhuravskaya (2011), who study the 1999 parliamentary election in Russia and show that introduction of a government-independent TV channel increased the vote for major opposition parties; Oberholzer-Gee and Waldfogel (2009), who document that an increase in the availability of Spanish-language news positively affected voter turnout in the US in 1994-2002, suggesting that the reason is that these news outlets are much more likely to report on issues that are of interest to Hispanics. The results of this paper suggest that targeted advertising may be so widespread in elections because it improves the candidate's odds of winning, and sometimes helps swing elections that are unwinnable otherwise.

The rest of the paper is organized as follows. Section 2 describes and analyses the model of election games with verifiable information. Section 3 focuses on the application of targeted advertising in a spatial model. Section 4 discusses assumptions and concludes.

2. Model

This section describes the election game between one sender and many receivers. Privately informed sender wishes to have his policy proposal approved by a collective vote and sends verifiable messages to the receivers to convince them to vote in his favor. Each receiver independently compares the sender's proposal to the status quo policy and votes for the option that maximizes her expected utility. Whether the sender's proposal is approved and his policy is implemented, or the status quo is maintained, is decided by the social choice function.

There is a continuous compact policy space $\Omega \subset \mathbb{R}^n$ and a finite space of receiver types $\mathbb{V}$. The sender privately observes his policy (*state of the world*) $\omega \in \Omega$, which is drawn from the common prior distribution $p > 0$ over Ω with measure $P(\cdot)$: $\forall W \subseteq \Omega$, $P(W) := \int_W p(\omega) d\omega$. The sender's preferences are state-independent, and his payoff is 1 if his proposal is approved and 0 otherwise. To convince the receivers to approve his proposal, the sender sends a private verifiable message $m_v \in \mathbb{M} := 2^\Omega$ such that $\omega \in m_v$, to every receiver type $v \in \mathbb{V}$.

There is a unit mass of receivers. Receiver of type $v \in \mathbb{V}$ chooses action $a_v \in \mathbb{A} = \{0, 1\}$, where $a_v = 0$ reflects the vote for the the known status quo policy $\omega^\circ \in \Omega$ and $a_v = 1$ corresponds to voting in favor of the sender's unknown policy $\omega \sim p$. Receivers' state-dependent preferences are characterized by a von Neumann-Morgenstern utility function $u : \mathbb{V} \times \Omega \to \mathbb{R}$. Receiver v's net payoff from approving sender's proposal under complete information is $\delta(v, \omega) := u(v, \omega) - u(v, \omega^\circ)$. I define for voter $v \in \mathbb{V}$:

- the approval set $\mathcal{A}_v := \{\omega \in \Omega \text{ s.t. } \delta(v, \omega) \geq 0\}$ to be the set of policies that are weakly preferred to the status quo under complete information;

- the set of approval beliefs $\mathcal{B}_v := \left\{q \in \Delta\Omega \text{ s.t. } \mathbb{E}_q[\delta(v,\omega)] \geq 0\right\}$ to be all the probability distributions over the policy space that make the perceived policy of the sender weakly more preferable than the status quo.

The distribution of receiver types is given by $g(\cdot) > 0$ over $\mathbb{V}$ with measure $G(\cdot)$: $\forall V \subseteq \mathbb{V}$, $G(V) = \sum_{v \in V} g(v)$. The outcome of the election, and hence the sender's utility, is decided by the social choice function $f : 2^{|\mathbb{V}|} \to \{0,1\}$, which for every set of voters who voted to approve sender's proposal $V \subseteq \mathbb{V}$ specifies whether his proposal is approved (1), or not (0). The only restriction on $f(\cdot)$ is that if the receivers unanimously agree on either alternative, then this alternative is implemented, i.e. $f(\varnothing) = 0$ and $f(\mathbb{V}) = 1$. Examples of decision rules that admit such representation are

- dictatorship: if v is a dictator, then $f(V) = 1 \iff v \in V$;
- simple majority: $f(V) = 1 \iff G(V) > \frac{1}{2}$;
- unanimity: $f(V) = 1 \iff V = \mathbb{V}$.

I focus on perfect bayesian equilibria of the election game. The sender transmits a collection of messages $\boldsymbol{m} := \{m_v\}_{v \in \mathbb{V}} \in \mathbb{M}^{\mathbb{V}}$ to the receivers, where message m_v is intended for receiver of type $v \in \mathbb{V}$. The sender's behavior strategy assigns a probability distribution $\sigma(\cdot \mid \omega)$ over $\Delta\mathbb{M}^{\mathbb{V}}$, so that $\sigma(\boldsymbol{m} \mid \omega)$ is the probability that he sends collection $\boldsymbol{m}$ in state ω. For every distribution of message collections σ, one can calculate the marginal distribution of messages heard by receiver v as

$$\sigma_v(m_v \mid \omega) = \int_{\mathbb{M}^{|\mathbb{V}|-1}} \sigma\big(\{m_v, \boldsymbol{m}_{-v}\} \mid \omega\big) d\boldsymbol{m}_{-v},$$

where $\boldsymbol{m}_{-v} \in \mathbb{M}^{\mathbb{V}-1}$ is the collection of messages intended for every receiver, except for v.

Receiver $v \in \mathbb{V}$ observes message m_v, forms posterior belief $q_v(\cdot \mid m_v)$, and votes to approve sender's proposal if and only if $q_v(\cdot \mid m_v) \in \mathcal{B}_v$, thus breaking the ties in his favor.[1] When a receiver votes in favor of the sender, I say she is convinced.

Definition 2.1. *The equilibrium $(\sigma, \boldsymbol{\alpha}, \boldsymbol{q})$ of the election game $\big(\Omega, p, \omega^\circ, \mathbb{V}, g, f(\cdot)\big)$ with the policy space Ω, the common prior belief $p \in \Delta\Omega$, the status quo policy $\omega^\circ \in \Omega$, the set of receiver types $\mathbb{V}$, the distribution of receivers $g \in \Delta\mathbb{V}$, and the social choice function $f(\cdot)$ consists of the messaging strategy of the sender σ and the profiles of voting rules $\boldsymbol{\alpha} := \{\alpha_v\}_{v \in \mathbb{V}}$ and posterior beliefs $\boldsymbol{q} := \{q_v\}_{v \in \mathbb{V}}$ of the receivers, such that*

[1] This tie-breaker is without loss of generality and is assumed for convenience.

(i) $\forall \omega \in \Omega$, $\sigma(\boldsymbol{m} \mid \omega) > 0$ *only if*

$$\boldsymbol{m} \in \arg\max_{\boldsymbol{m}' \in \mathbb{M}^{\mathbb{V}}} f\left(\{v \in \mathbb{V} \text{ s.t. } a_v(m'_v) = 1\}\right) \text{ subject to } \omega \in m'_v \text{ for all } v \in \mathbb{V}.$$

The following conditions must hold for every receiver $v \in \mathbb{V}$:

(ii) $\forall m_v \in \mathbb{M}$, $\alpha_v(m_v) = \mathbb{1}\{q_v(\cdot \mid m_v) \in \mathcal{B}_v\}$, *where* $\mathbb{1}\{\cdot\}$ *is the indicator function;*

(iii) $\forall m_v \in \mathbb{M}$ *such that* $\int_\Omega \sigma_v(m_v \mid \omega) d\omega > 0$,

$$q_v(\omega \mid m_v) = \frac{\sigma_v(m_v \mid \omega) \cdot p(\omega)}{\int_\Omega \sigma_v(m_v \mid \omega') \cdot p(\omega') d\omega'}.$$

In words, *(i)* says that the sender only sends collections of messages that maximize his chances of getting his policy approved under to the social choice function $f(\cdot)$ that aggregates receivers' optimal voting rules; *(ii)* states that each receiver is sequentially rational and votes for the challenger if and only if her expected net payoff of approving the challenger's proposal is non-negative under her posterior belief; *(iii)* asserts that receivers' posterior beliefs are Bayes-rational on the equilibrium path, i.e. following any message that is heard in equilibrium with positive probability. Note that the receivers' posterior beliefs must be concentrated on the states in which the message was available to the sender, i.e. $\forall m_v \in \mathbb{M}$, $\operatorname{supp} q_v(\cdot \mid m_v) \subseteq m_v$.

One Receiver

Let us first focus on the model with one receiver, i.e. $\mathbb{V} = \{v\}$. The challenger gets a payoff of one if and only if he convinces voter v, i.e. $f(V) = 1 \iff V = \{v\}$. For ease of exposition, I drop all receiver-relevant subscripts.

direct implementation

Consider the following direct implementation. The state space is partitioned into the set of winning states $\mathcal{W} \subseteq \Omega$ and the set of losing states $\mathcal{L} := \Omega \setminus \mathcal{W}$. In winning states, the sender sends message $\mathcal{W}$, which is interpreted by the receiver as a recommendation to vote in favor of the sender's proposal. In losing states, the message is $\mathcal{L}$, and the receiver is advised to maintain the status quo. Since the receiver only hears two messages, calculating the posterior beliefs boils down to conditioning the prior distribution on sets $\mathcal{W}$ and $\mathcal{L}$.[2] Direct implementation characterized by set of winning states $\mathcal{W}$ is summarized in Table 1.

state	sender's message	receiver's belief	receiver's action
$\omega \in \mathcal{W}$	$\mathcal{W}$	$p(\cdot \mid \mathcal{W})$	approve sender
$\omega \in \mathcal{L}$	$\mathcal{L}$	$p(\cdot \mid \mathcal{L})$	status quo

Table 1: *direct implementation with set of winning states $\mathcal{W}$.*

For the direct implementation to constitute a direct equilibrium, one needs to make sure that the receiver obediently follows the sender's recommendations. First, observe that the approval set $\mathcal{A}$ must always be included in the set of winning states $\mathcal{W}$ or else the sender has a profitable deviation of fully disclosing his type $\omega \in \mathcal{A}$. Next, since $\mathcal{L} \cap \mathcal{A} = \varnothing$, the receiver is always strictly better off voting in favor of the status quo after message $\mathcal{L}$. Finally, if the set of winning states includes policies outside of the approval set, to be convinced by message $\mathcal{W} \supset \mathcal{A}$, the receiver must be expecting a higher payoff from approving the sender's proposal than that of the status quo. Equivalently, the receiver's posterior belief must lie within her set of approval beliefs:

$$p(\cdot \mid \mathcal{W}) \in \mathcal{B}. \qquad \text{(obedience)}$$

Theorem 2.1. *Direct implementation characterized by set of winning states $\mathcal{W} \subseteq \Omega$ constitutes a direct equilibrium $(\sigma^D, \alpha^D, q^D)$, where*

$$\begin{aligned} &\forall \omega \in \mathcal{W}, \;\; \sigma^D(\mathcal{W} \mid \omega) = 1; \qquad && q^D(\cdot \mid \mathcal{W}) = p(\cdot \mid \mathcal{W}); \qquad && \alpha^D(\mathcal{W}) = 1; \\ &\forall \omega \in \mathcal{L}, \;\; \sigma^D(\mathcal{L} \mid \omega) = 1; && q^D(\cdot \mid \mathcal{L}) = p(\cdot \mid \mathcal{L}); && \alpha^D(\mathcal{L}) = 0, \end{aligned}$$

if and only if $\mathcal{A} \subseteq \mathcal{W}$ and (obedience) constraint holds.

Furthermore, every equilibrium of the election game is outcome-equivalent to a direct equilibrium.

The proof of the last part of Theorem 2.1 boils down to showing that if the sender convinces the receiver in a non-direct equilibrium by sending various (possibly mixed) convincing messages, then he may as well just send one convincing message that includes all the winning states. The receiver's

[2] $\forall W \subseteq \Omega$, conditional prior distribution is defined as $p(\omega \mid W) := \frac{p(\omega)}{\int_W p(\omega') d\omega'}$.

(obedience) constraint in the direct equilibrium follows from the facts that *(i)* whenever the receiver is convinced in the non-direct equilibrium, she must be convinced with probability one, which implies *(ii)* the sender, in the non-direct equilibrium, must be sending convincing messages, and convincing messages only. The full proof, along with all proofs omitted in the main text, can be found in the appendix.

Theorem 2.1 is the main result of this paper and is an important contribution for several reasons. Firstly, it states that when studying election games with verifiable information, we may restrict attention to *pure strategies* of the sender.[3] Secondly, this theorem suggests that the *classic revelation principle* fails in settings where an uninformed principal (this model's receiver) wishes to retrieve information from a privately-informed agent (the sender) who possesses hard evidence.[4] This failure occurs because even though the sender cannot lie by commission, nothing is preventing him from lying by omission (which he does in every direct equilibrium with $\mathcal{W} \supset \mathcal{A}$). Finally, this result provides an insight on what happens to Myerson (1982)'s *communication revelation principle*, which states that in communication games the mediator can restrict attention to incentive-compatible direct mechanisms, in unmediated games.[5] When there is no external communication device but information is verifiable, mechanisms are still direct, and the sender himself sends a recommendation that the receiver obediently follows, but they are no longer truthful. Without a mediator, the sender is not bound by an external incentive-compatibility constraint, and verifiability alone does not prevent him from "exaggerating" his type to appear more appealing to the receiver.

EQUILIBRIUM RANGE

Theorem 2.1 permits restricting attention to direct equilibria, which significantly simplifies the task of describing the full equilibrium set. Recall that every direct equilibrium is characterized by the set of states $\mathcal{W} \supseteq \mathcal{A}$ in which the sender's proposal is approved. I will rank equilibria in terms of the sender's ex-ante utility, which is the same as his odds of getting his proposal approved, and equals $P(\mathcal{W})$.

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[3] This is a purification argument – when the state space is continuous and the information structure is atomless the existence of pure strategy equilibria is often guaranteed: Radner and Rosenthal (1982), Aumann et al. (1983), Milgrom and Weber (1985).

[4] This is in line with past work of Green and Laffont (1986), Bull and Watson (2007), Deneckere and Severinov (2008) and Ben-Porath, Dekel, and Lipman (2019), all of whom show that the revelation principle in mechanism design with verifiable information often does not hold.

[5] The communication revelation principle in mediated disclosure games was introduced by Myerson (1982), extended to multistage games by Myerson (1986) and Forges (1986), verifiable disclosure by Forges and Koessler (2005) and cheap talk games with a sender-biased mediator by Salamanca (2019).

In the sender-worst equilibrium, the set of winning states is the smallest in terms of ex-ante utility, and is $\mathcal{W} = \mathcal{A}$. In this equilibrium, the receiver effectively learns all the information that is relevant for her decision (whether the sender's policy is better than the status quo, or not), thus making her decision as if under complete information. Notice that this equilibrium is outcome-equivalent to *full unraveling*, which is persistent in the verifiable information literature.

In the sender-preferred equilibrium, the set of winning states maximizes the sender's ex-ante utility subject to the (obedience) constraint. Since the policy space is continuous, in the optimum this constraint will bind, and the solution will feature a set of cutoff states.

Theorem 2.2. *The problem of finding sender-preferred equilibrium*

$$\max_{\mathcal{W}} P(\mathcal{W}) \text{ subject to } p(\cdot \mid \mathcal{W}) \in \mathcal{B}$$

has the same solution as the sender's problem in the bayesian persuasion game.[6] *The sender-optimal set of winning states* $\mathcal{W}^*$ *is such that*

- $\{\omega \in \Omega \text{ s.t. } \delta(v, \omega) = c^*\} \subset \mathcal{W}^*$ *is the set of cutoff states;*
- *sender's proposal is rejected if* $\delta(v, \omega) < c^* < 0$ *and approved if* $\delta(v, \omega) > c^*$*;*
- *whenever sender's proposal is approved, the receiver is indifferent, i.e.* $\mathbb{E}_p\big[\delta(v, \omega) \mid \mathcal{W}^*\big] = 0$.

The solution is more clear-cut when receiver's preferences are strictly monotone in the sense that the measure if the indifference set $\{\omega \in \Omega \text{ s.t. } \delta(v, \omega) = c\}$ is zero for every constant level $c \in \mathbb{R}$.[7] Then, the sender-preferred set of winning states becomes $\mathcal{D}(v, c^*) := \{\omega \in \Omega \text{ s.t. } \delta(v, \omega) \geq c^* > 0\}$, and the sender's proposal is approved if and only if $\omega \in \mathcal{D}(v, c^*)$. I refrain from assuming that the receiver has a strict ranking of states because it does not hold in some important applications.[8]

[6] In the bayesian persuasion game, the sender chooses and commits to an experiment π that consists of a realization space S and a family of likelihood functions $\pi(\cdot \mid \omega)$, $\omega \in \Omega$, over S. Upon the realization of the state, he transmits to the receiver the signal $\pi(\cdot \mid \omega)$ that he had committed to, along with its realization $s \in S$. The solution is due to Alonso and Câmara (2016).

[7] If Ω is one-dimensional, this condition translates into a strict ranking of states.

[8] For example, I show in Section 4 that receiver's utility in the continuous version of a finite state space model becomes a step function, with steps corresponding to the utility levels at elements of the finite state space.

Many Receivers

direct implementation

With many receivers, the direct implementation is characterized by a collection of convincing messages $\{\mathcal{W}_v\}_{v\in\mathbb{V}} \subseteq \Omega^{\mathbb{V}}$, rather that just one set of winning states. In state $\omega \in \Omega$, the sender partitions the set of receivers $\mathbb{V}$ into two groups:

- his electorate $\mathcal{E}(\omega) := \{v \in \mathbb{V} \text{ s.t. } \omega \in \mathcal{W}_v\}$;
- receivers who vote in favor of the status quo, $\mathbb{V} \setminus \mathcal{E}(\omega)$.

The sender's pure strategy in state ω is to send private convincing message $\mathcal{W}_v$ to every receiver in his electorate $v \in \mathcal{E}(\omega)$, and the non-convincing message $\mathcal{L}_v := \Omega \setminus \mathcal{W}_v$ to everyone else, i.e. $v \notin \mathcal{E}(\omega)$. Receiver $v \in \mathbb{V}$ observes one of two messages, $\mathcal{W}_v$ or $\mathcal{L}_v$.[9] She interprets the former as a recommendation to vote in favor of the sender's proposal as long as her obedience constraint $p(\cdot \mid \mathcal{W}_v) \in \mathcal{B}_v$ holds. The outcome of the election in state ω is decided by the social choice function $f(\mathcal{E}(\omega))$. The set of winning states becomes $\mathcal{W} := \{\omega \in \Omega \text{ s.t. } f(\mathcal{E}(\omega)) = 1\}$.[10] Sender's ex-ante utility is $P(\mathcal{W})$.

Theorem 2.3. *Direct implementation characterized by collection of convincing messages $\{\mathcal{W}_v\}_{v\in\mathbb{V}} \in \Omega^{\mathbb{V}}$ constitutes a direct equilibrium $(\sigma^D, \boldsymbol{a}^D, \boldsymbol{q}^D)$, where*

- *the sender's pure strategy entails a binary recommendation*

$$\forall \omega \in \Omega, \quad \sigma^D\Big(\{\mathcal{W}_v\}_{v\in\mathcal{E}(\omega)} \text{ and } \{\mathcal{L}_v\}_{v\notin\mathcal{E}(\omega)} \;\Big|\; \omega\Big) = 1;$$

- *for every receiver $v \in \mathbb{V}$ and message $m_v \in \{\mathcal{W}_v, \mathcal{L}_v\}$*
  - *her posterior belief is $q_v^D(\cdot \mid m_v) = p(\cdot \mid m_v)$,*
  - *her action is $\alpha_v^D(m_v) = \mathbb{1}\{m_v = \mathcal{W}_v\}$,*

  if and only if the (obedience) constraint holds for every $v \in \mathbb{V}$.

Furthermore, every equilibrium of the election game is outcome-equivalent to a direct equilibrium.

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[9] Either one of $\mathcal{W}_v$, $\mathcal{L}_v$ may be empty, in which case this message is not sent in any $\omega \in \Omega$, and receiver v only hears its complement that becomes Ω.

[10] Note that this definition is consistent with the one-receiver case presented above. When $\mathbb{V} = \{v\}$, if $\mathcal{W}_v$ is the convincing message, then the sender's electorate is $\mathcal{E}(\omega) = \{v\}$ if $\omega \in \mathcal{W}_v$ and $\varnothing$ if $\omega \notin \mathcal{W}_v$. Consequently, the set of winning states is $\mathcal{W} = \{\omega \in \Omega \text{ s.t. } \omega \in \mathcal{W}_v\}$ and coincides with the convincing message $\mathcal{W}_v$.

The last part of Theorem 2.3 that states that every equilibrium in outcome-equivalent to some direct equilibrium is established by *(i)* figuring out which receivers vote to approve the sender's proposal with probability one in each state, *(ii)* for every receiver, pulling states in which she votes for the sender with probability one into one convincing message, *(iii)* showing that this convincing message satisfies this receiver's obedience constraint. One caveat to outcome-equivalence is that some receivers may vote to approve the sender's proposal with a probability between zero and one in some states, and that fact is not accounted for in the direct implementation, meaning that those states will not be part of this receiver's convincing message. Thus, technically, for these voters in these states the outcome in the direct implementation is not the same, and their probability of approval is set to zero. However, a receiver never votes in favor of the sender's proposal with a positive probability less than one if her vote is pivotal in that state, since once a receiver is "convincible", the sender can convince her with certainty. Thus, eliminating these receivers from the sender's electorate in appropriate states is inconsequential for the outcome of the election.

EQUILIBRIUM RANGE

As with one receiver, the worst that the sender can do in any equilibrium is truthfully announce his policy in every state. Thus, in the sender-worst equilibrium, the set of convincing messages becomes $\{\mathcal{A}_v\}$, and the outcome is the same as under complete information.

In the sender-preferred equilibrium, the collection of winning messages is selected to maximize the prior measure of the set of winning states (which is pinned down by the social choice function) subject to each receiver's (obedience) constraint.

$$\max_{\{\mathcal{W}_v\}} P(\mathcal{W}) \quad \text{subject to} \quad \mathcal{W} = \Big\{\omega \in \Omega \text{ s.t. } f\Big(\underbrace{\{v \in \mathbb{V} \text{ s.t. } \omega \in \mathcal{W}_v\}}_{\mathcal{E}(\omega)}\Big) = 1\Big\},$$

$$\forall v \in \mathbb{V}, \quad p(\cdot \mid \mathcal{W}_v) \in \mathcal{B}_v.$$

Once again, the problem of finding the sender-preferred equilibrium in the verifiable information election game has the same solution as the sender's problem in the bayesian persuasion game. In the bayesian persuasion game, the sender commits to a signaling policy (that is known to receivers) $F : \Omega \to \Delta\{\Theta_v\}_{v \in \mathbb{V}}$, where Θ_v is the private signal set of receiver v. Once state $\omega \in \Omega$ is realized, a profile of signals $\{\theta_v\}_{v \in \mathbb{V}}$ is generated according to F, and receiver v observes her private signal realization θ_v. Since there is no strategic interaction among the receivers, one can restrict attention to *straightforward* signaling policies, in which the sender recommends action, i.e. $\Theta_v = \{0, 1\}$ and each receiver's best response is to follow this recommendation, i.e. $\alpha(\theta_v) = \theta_v$.[11] In the end, whether the sender has commitment power or not, his problem boils down to convincing individual receivers in as many states, as their obedience constraints permit. As has been shown in Theorem 2.2, continuity of the policy space is a sufficient condition for the set of convincing states to coincide in the two problems.

The problem of finding the sender-preferred equilibrium in a multiple-receiver setting is computationally hard (and goes beyond the scope of this paper), because every receiver needs to be convinced as much as possible (subject to her obedience constraint), but different receivers may be pivotal in different states. For some social choice functions, ready solutions to the bayesian persuasion problem are available. Most notably, Arieli and Babichenko (2019) provide a solution for the case of supermodular utility of the sender, and Babichenko and Barman (2016) provide an approximation for when it is submodular.

3. Application: Targeted Advertising in a Spatial Model

In this application, the sender is a politician (the challenger) who challenges the status quo, and the receivers are sincere voters. Assume $\Omega = [0, 1]$, and voters have spatial preferences. Specifically, $\mathbb{V} \subseteq \Omega$, and voter type $v \in \mathbb{V}$ is her ideal policy. All other policies are evaluated based on how far they are from the bliss point, i.e. receiver utility is $u(v, \omega) = -|v - \omega|$. Voter v's approval set comprises of policies that are weakly closer to v than the status quo: $\mathcal{A}_v = \left\{\omega \in \Omega \text{ s.t. } |v - \omega| \leq |v - \omega^\circ|\right\}$. I will use the model to answer the following question: does targeted advertising improve the odds of getting elected over public disclosure?

public disclosure

Under public disclosure, the challenger is restricted to sending the same public message m to every voter. Since voters have a common prior belief regarding the challenger's policy position, hearing the same message induces a common posterior belief. It is the identical posterior beliefs of the voters that make the case of public disclosure significantly simpler to analyze. First, notice that some voters never vote for the challenger at the same time when they possess the same information.

[11] The fact that straightforward implementation applies in a setup with multiple receivers who do not strategically interact was established by Arieli and Babichenko (2019).

Definition 3.1.

- *Voters* $v_L, v_R \in \mathbb{V}$ *are incompatible if their ideal policies are located on the opposite sides of the status quo, i.e.*

$$|v_L - \omega^o| + |v_R - \omega^o| = |v_L - v_R|.$$

- *Set of voters* $\mathcal{CV} \subseteq \mathbb{V}$ *is a coalition of compatible voters if* $v < \omega^o$ *or* $\omega^o < v$ *for every* $v \in \mathbb{V}$.

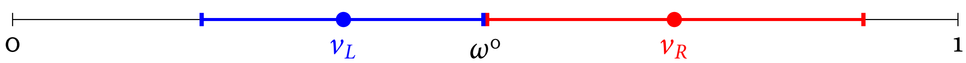


Figure 4: *voters* v_1 *and* v_2 *are incompatible and their approval sets do not intersect a.s.*

It is easy to see in Figure 4 that the approval sets of incompatible voters do not intersect a.s., meaning that these voters do not prefer any challenging policy to the status quo. In fact, these voters do not prefer the same *perceived* policy of the challenger to the status quo, either.

Lemma 3.1. *Sets of approval beliefs of incompatible voters do not intersect a.s.*

The challenger cannot convince incompatible voters by public disclosure, meaning that his electorate must be a coalition of compatible voters. Consequently, some elections are unwinnable for the challenger under public disclosure and his odds of winning these elections is zero in every equilibrium.

Lemma 3.2. *(Unwinnable Elections). Challenger loses with probability one under public disclosure if and only if no coalition of compatible voters is pivotal, i.e.* $\nexists \mathcal{CV}$ *such that* $f(\mathcal{CV}) = 1$.

The proof of this lemma is straightforward. On the one hand, if no $\mathcal{CV}$ is pivotal, then there does not exist a common belief (including those that may be induced by public disclosure) that convinces enough voters to vote for the challenger. The sufficiency becomes clear by contraposition: if some $\mathcal{CV}$ is pivotal, then $\bigcap_{v \in \mathcal{CV}} \mathcal{A}_v$ is non-empty a.s.[12] Consequently, even in the worst equilibrium for the challenger, in which he fully discloses his type, his odds of winning are positive.

In this single-dimensional spatial model, there are two "largest" coalitions of compatible voters – the types to the left, and the right of the status quo. Whether these coalitions are pivotal depends on the social choice function. Under majority rule, neither of them is pivotal if the mass of voters to

[12] More precisely, if $v < \omega^o$ ($\omega^o < v$) for all $v \in \mathcal{CV}$, then $\bigcap_{v \in \mathcal{CV}} \mathcal{A}_v = \mathcal{A}(v^r)$, where $v^r = \max_{v \in \mathcal{CV}} v$ ($v^r = \min_{v \in \mathcal{CV}} v$).

the left (and to the right) of the status quo is less than one half. Consequently, we arrive at a version of the median voter theorem for games with public disclosure.[13]

Corollary 3.1. *Under majority rule, challenger cannot win by means of public disclosure if and only if ω° is the ideal policy of the median voter.*

TARGETED ADVERTISING

Consider an election that is unwinnable under public disclosure. By Lemma 3.2, no coalition of compatible voters is pivotal. Let $\mathcal{CV}_L := \{v \in \mathbb{V} \text{ s.t. } v < \omega^{\circ}\}$ and $\mathcal{CV}_R := \{v \in \mathbb{V} \text{ s.t. } \omega^{\circ} < v\}$ be the sets of (compatible) voters who agree to the left and to the right, respectively. Let $v_L = \max_{v \in \mathcal{CV}_L} v$ and $v_R = \min_{v \in \mathcal{CV}} v$ be the representative voters of coalitions $\mathcal{CV}_L$ and $\mathcal{CV}_R$, respectively. Observe that $\mathcal{A}(v_i) = \bigcap_{v \in \mathcal{CV}_i} \mathcal{A}_v$ for $i \in \{L, R\}$, meaning that the whole coalition $\mathcal{CV}_i$ is convinced if and only if its representative voter is convinced. Recall that the social choice function exhibits the property of unanimous agreement, which implies that convincing every voter in $\mathcal{CV}_L$ and $\mathcal{CV}_R$ is sufficient for the challenger to win the election.[14] Representative voters v_L and v_R thus become the left and the right pivots, respectively. The arguments presented in this paragraph are illustrated in Figure 5 and formalized in Corollary 3.2.

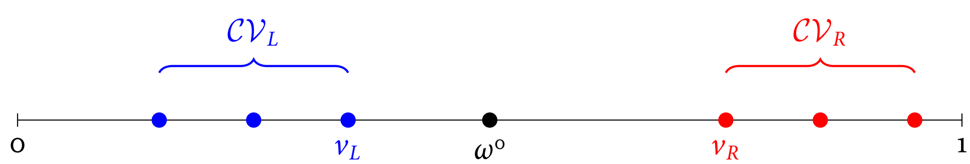


Figure 5: *the left pivot v_L and the right pivot v_R represent the preferences of coalitions of compatible voters $\mathcal{CV}_L$ and $\mathcal{CV}_R$.*

Corollary 3.2. *Every unwinnable election admits two incompatible voters, the left pivot $v_L < \omega^{\circ}$ and the right pivot $\omega^{\circ} < v_R$, convincing who is sufficient for the challenger to win under targeted advertising.*

Convincing incompatible voters v_L and v_R is impossible under public disclosure (which makes this election unwinnable), but here is how the challenger can win by targeting in his most preferred equilibrium. In the sender-preferred equilibrium, the set of states in which the challenger convinces

[13] Black (1948) states the median voter theorem as "If Ω is a single-dimensional issue and all voters have single-peaked preferences defined over Ω, then ω°, the median position, could not lose under majority rule."

[14] The only exception is when voter $v = \omega^{\circ}$ is a dictator but I ignore this case.

both voters is maximized, and each voter obediently follows her private recommendation:

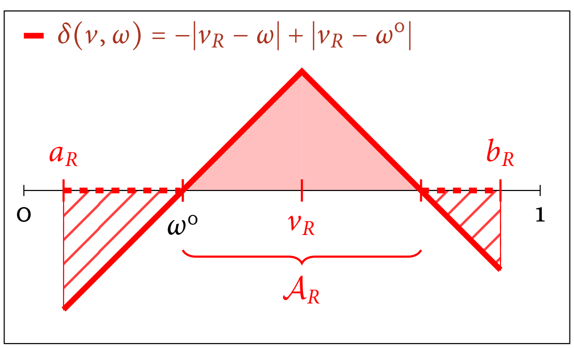
$$\max_{\mathcal{W}_L,\mathcal{W}_R} P(\mathcal{W}_L \cap \mathcal{W}_R) \text{ subject to } p(\cdot \mid \mathcal{W}_i) \in \mathcal{B}_i \text{ for } i \in \{L, R\}.$$

The solution to this problem is characterized in the following theorem.

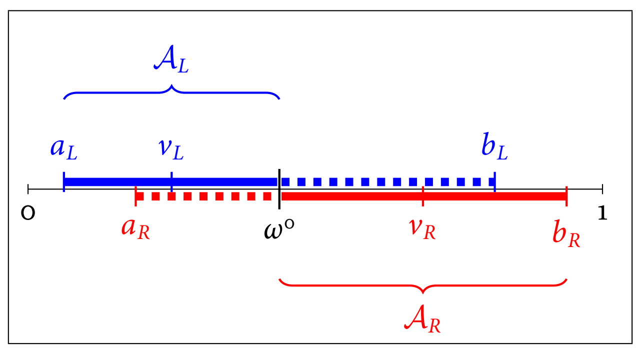
Theorem 3.1. *In the sender-preferred equilibrium of an unwinnable election with left pivot $v_l < \omega^o$ and right pivot $\omega^o < v_R$,*

- *convincing message $\mathcal{W}_i$ is an interval $[a_i, b_i] \supset \mathcal{A}_i$, $\forall i \in \{L, R\}$;*
- *at least one of the obedience constraints $\int_{a_i}^{b_i} \left(-|v_i - \omega| + |v_i - \omega^o| \right) \cdot p(\omega) d\omega \geq 0$, $i \in \{L, R\}$ is binding;*
- *the set of winning states is $[a_R, b_L]$ with $a_R < \omega^o < b_L$.*

The intuition behind this result can be observed in Figure 6. On the left, there is a picture of the right pivot's net payoff from voting for the challenger, i.e. $\delta(v, \omega) = -|v_R - \omega| + |v_R - \omega^o|$. For this voter to be convinced by message $[a_R, b_R]$, her expectation of this function over $[a_R, b_R]$ must be non-negative. Under uniform distribution, for example, this obedience constraint states that the red solid area (which integrates $\delta(v, \omega)$ over the approval set $\mathcal{A}_R$ and is positive) must exceed the sum of the red dashed areas (which integrate $\delta(v, \omega)$ over $[a_R, b_R] \setminus \mathcal{A}_R$ and are negative).



***(a)** Obedience constraint for convincing message $[a_R, b_R]$ of the right pivot v_R: $\int_{a_R}^{b_R} \delta(v, \omega) p(\omega) d\omega \geq 0$. Under uniform prior, the solid area exceeds the dashed area.*



***(b)** Convincing private messages $[a_L, b_L]$ and $[a_R, b_R]$ in the sender-preferred equilibrium under TA. Each includes approval set and is skewed towards the other pivot.*

***Figure 6:** right pivot's net payoff from voting for challenger and sender-preferred convincing messages that swing an unwinnable election.*

When there are two voters, as in Figure 6 on the right, the optimal way to increase the set of winning states is to skew the convincing messages in the direction of the other pivot's position. Intuitively, the challenger can start with $\mathcal{W}_i = \mathcal{A}_i$, $\forall i \in \{L, R\}$. Since neither voter's obedience constraint binds, the challenger can increase his odds of winning by decreasing a_R (which is only costly in terms of the right pivot's constraint) and increasing b_L (which is only costly in terms of the left pivot's constraint). Once both constraints bind, the challenger has maximized his ex-ante utility. Notice that the set of winning states $[a_R, b_L]$ has positive measure, and the status quo belongs to its interior. In other words, only the challengers that are sufficiently close to the status quo benefit from targeting and can swing unwinnable elections.

If at least one of the obedience constraints is not satisfied when $a_R = 2v_L - \omega^\circ$, which is the left boundary of the left pivot's approval set, or when $b_L = 2v_R - \omega$, which is the right boundary of the left pivot's approval set, then the solution may look different. Figure 7 illustrates the possible variations of the relative position of convincing messages for a fixed v_L and increasing v_R and Theorem 3.2 summarizes the comparative statics.

Theorem 3.2. *Consider an unwinnable election with left pivot $v_l < \omega^\circ$, right pivot $\omega^\circ < v_R$ and a uniform prior. Then,*

- *as v_R increases, the challenger's ex-ante utility in the sender-preferred equilibrium increases;*
- *for high enough v_R, as v_R increases, a_R and b_L, the left and the right bounds of the set of winning states, shift to the left.*

First notice that under a uniform prior, increasing v_R increases this voter's approval set thus loosening her obedience constraint. This can be seen in Figure 6 on the left: as v_R increases, the solid area increases with respect to set inclusion, providing more room for increasing the dashed areas. Consequently, the sender's maximized ex-ante utility may only go up as one of the pivots is moved away from the status quo.

The intuition for the second part of the result can be observed in Figure 7. For every left pivot v_L there exists a right pivot v_R such that both obedience constraints bind when $a_R \in \mathcal{A}_L$ and $b_L \in \mathcal{A}_R$ (middle case of Figure 7). In this mutually-binding equilibrium, the bounds of the left pivot's convincing message are chosen as far to the right as her obedience constraint permits. As v_R increases (second case from the top of Figure 7), the right pivot is moved away from the status quo, which increases her approval set and loosens her obedience constraint. At this point, the right pivot is far enough from the status quo that she is willing to be convinced by messages that include policies that are to the left of the left pivot's approval set. Going back to v_L's obedience constraint, the policies that are immediately to the left of her approval set are "cheaper" than those that are far to the right. Thus, shifting the bounds of v_L's convincing message increases the challenger's ex-ante utility and results

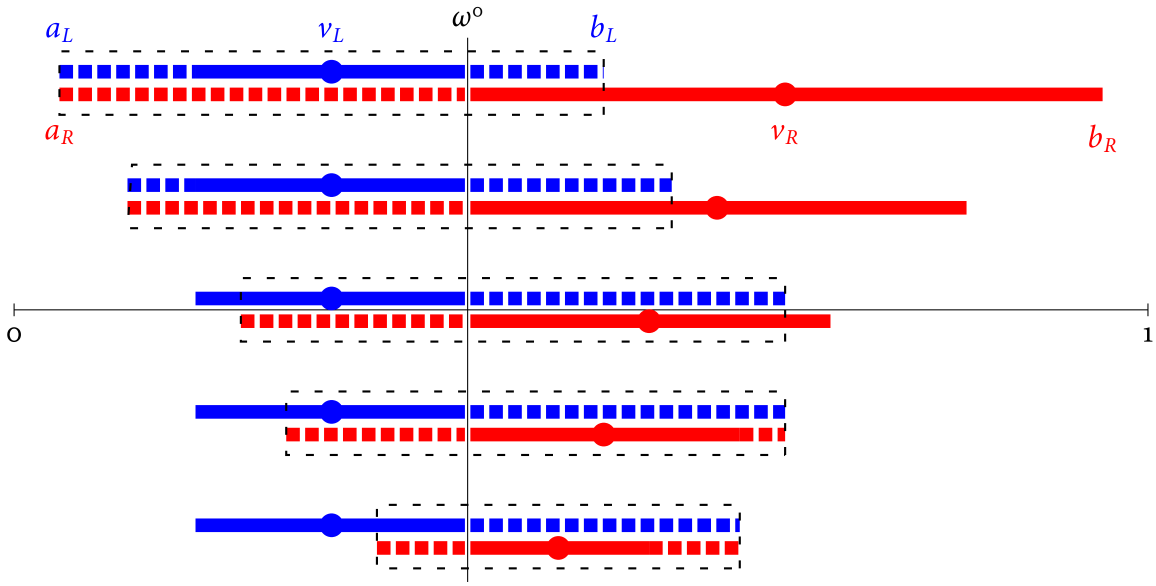


Figure 7: *comparative statics of approval sets (solid lines), convincing messages (solid plus dotted lines) and winning states (black dashed area) for fixed v_L (in blue) and varying v_R (in red), uniform prior.*

in a new set of winning states that is shifted to the left, as well. In the extreme case, v_R is located so far to the right that her obedience constraint does not bind whenever v_L's constraint binds (the top case of Figure 7). Now, the left pivot is essentially the only receiver, and Theorem 2.2 states that the one-receiver solution (v_L's convincing message becomes the set of winning states) is an interval that is centered around v_L.

Finally, notice how comparative statics of the set of winning states are not monotone with respect to v_R. The reason behind this is that once v_R is small enough (fourth case from the top, Figure 7), it is the left pivot who is located far enough from the status quo to be convinced by policies that are to the right of the right pivot's approval set. In the extreme case, if v_R is very small, the left pivot's obedience constraint no longer binds, and the right pivot becomes effectively the only receiver.

4. Discussion

Robustness of results

The continuity of the state space Ω is not a restrictive assumption. Suppose the state space is $\Omega = \{0, 1\}$, the prior belief is $\{p_0, p_1\}$, receiver utility is $u(v, \omega) = \omega$ for $\omega \in \Omega \cup \{\omega^o\}$, and $\omega^o > p_1$, which ensures that the receiver votes for the status quo under the prior belief. When the sender has commitment power, his optimal signal features messaging "$\omega = 1$" when the state is indeed $\omega = 1$, and messaging "$\omega = 1$" with probability $\alpha = \frac{p_1}{p_0}\frac{1-\omega^\circ}{\omega^\circ} \in (0,1)$ when the state is $\omega = 0$. The frequency α of misreporting the low state is obtained from the bayes-plausibility constraint and receiver indifference: the posterior belief is $q(\omega = 1 \mid \text{“}\omega = 1\text{”}) = \frac{p_1}{p_1+\alpha p_0}$, and receiver is indifferent between her two actions when $q(\omega = 1 \mid \text{“}\omega = 1\text{”}) = \omega^\circ$. At the end, the sender's proposal is approved with probability α in state $\omega = 0$ and with certainty in state $\omega = 1$, making his ex-ante utility total $\alpha p_0 + p_1$.

Without commitment, such outcome would be impossible because if there is a winning message in state $\omega = 0$, then the sender would want to send it with probability one, while his probability of approval in the low state must be $\alpha \in (0,1)$. Instead, extend the policy space to a continuous $\tilde{\Omega} = [0,1]$, and make the following adjustments to the prior distribution and receiver utility:

$$\tilde{p}(\omega) = \begin{cases} \dfrac{p_0}{\omega^\circ}, & \omega \in [0,\omega^\circ), \\ \dfrac{p_1}{1-\omega^\circ}, & \omega \in [\omega^\circ, 1], \end{cases} \quad \text{and} \quad \tilde{u}(v,\omega) = \begin{cases} 0, & \omega \in [0,\omega^\circ), \\ \omega^\circ, & \omega = \omega^\circ, \\ 1, & \omega \in (\omega^\circ, 1]. \end{cases}$$

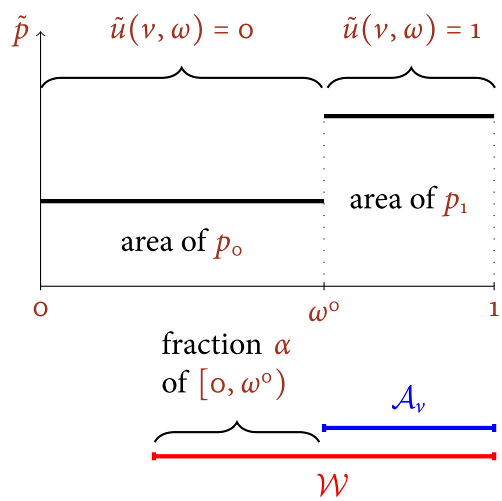


Figure 8: *prior distribution $\tilde{p}$, receiver utility $\tilde{u}(v,\omega)$, and the set of winning states $\mathcal{W}$ in the continuous version of model with binary state space $\{0,1\}$.*

The receiver's approval set is $\mathcal{A}_v = [\omega^\circ, 1]$. In the sender-preferred equilibrium, the winning message $\mathcal{W}$ comprises of the approval set $[\omega^\circ, 1]$ and fraction α of the low states $[0,\omega^\circ)$, where α is obtained from the binding (obedience) constraint $\tilde{p}(\cdot \mid \mathcal{W}) \in \mathcal{B}_v$ that becomes $\frac{p_1}{p_1+\alpha p_0} = \omega^\circ$. This is, of course, the same constraint as in the case with a binary state space featuring sender commitment. Thus, the tools presented in this paper are applicable to setups with finite state spaces that are considered in much of the literature on bayesian persuasion and information design.

Binary action of the receiver is not a restrictive assumption, either. Suppose there is one receiver and let $\mathbb{A} = \{a_1, \ldots, a_K\}$ be her finite set of actions. Next, partition the action space into $\mathbb{A}^W \subseteq \mathbb{A}$, which are the actions that lead to the sender's proposal being approved, and $\mathbb{A}^L = \mathbb{A} \setminus \mathbb{A}^W$, which are the actions that maintain the status quo. Theorem 2.1 still holds with the only exception being that the sender's winning message $\mathcal{W}$ is interpreted not as a recommendation to vote in favor of his proposal, but rather a statement "these are the states in which my proposal will be approved".

Conclusion

This paper analyzes election games with verifiable information, in which the sender attempts to convince a group of receivers to undertake an action.

I show that a weaker version of the communication revelation principle holds in this setup, provided that the sender's preferences are state-independent (he simply wants to get his proposal approved), receivers do not interact strategically, and there are no information spillovers and private messages stay private. Under these assumptions, any equilibrium may be implemented directly, meaning that the sender provides an action recommendation to each receiver, which the receivers obediently follow. While direct, the implementation is not truthful, and while the sender cannot lie by commission, nothing prevents him from lying by omission, but not too much, or else the receivers would prefer the status quo.

Focusing on direct equilibria allows us to characterize the full equilibrium set in terms of the sender's ex-ante utility. On the lower bound, information unravels and the implementation is direct and truthful. On the upper bound, the sender can do as well as if he had ex-ante commitment power, which is a novel result that provides a link to the literature on bayesian persuasion.

When it comes to applications, the present model is useful for comparing outcomes under public and private disclosure. In particular, assuming that the sender-preferred equilibrium is selected, I show that in a spatial model of voting the challenger can win elections that are unwinnable (in any equilibrium) under public disclosure. Since the challenger should not be elected under complete information, there is welfare loss, suggesting that targeted advertising creates inefficiencies.

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Appendix

Proof of Theorem 2.1

Part I: if $(\sigma^D, \alpha^D, q^D)$ is a direct equilibrium with set of winning states $\mathcal{W}$, then $\mathcal{A} \subseteq \mathcal{W}$ and $p(\cdot \mid \mathcal{W}) \in \mathcal{B}$, or else the sender in state $\omega \in \mathcal{A} \setminus \mathcal{W}$ or receiver after hearing message $\mathcal{W}$ has a profitable deviation.

Part II: for the direct implementation with set of winning states $\mathcal{W} \supseteq \mathcal{A}$ that satisfies (obedience) to constitute a direct equilibrium, one needs to specify off-path beliefs of the receiver. There are two restrictions on these beliefs: *(i)* the density must be supported on the set of states in which the message was available to the sender: $\forall m \in \mathbb{M}$, supp $q^D(\cdot \mid m) \subseteq m$, *(ii)* the sender may not have profitable deviations in the losing states $\omega \in \mathcal{L}$. One way to ensure that does not happen is to impose "skeptical beliefs"

$$\forall m \subseteq \mathcal{A},\ \ \text{supp } q^D(\cdot \mid m) \subseteq m, \text{so that } q^D(\cdot \mid m) \in \mathcal{B},$$

$$\forall m \notin \mathcal{A}, m \neq \mathcal{W}\ \ \text{supp } q^D(\cdot \mid m) \subseteq m \setminus \mathcal{A}, \text{so that } q^D(\cdot \mid m) \notin \mathcal{B}$$

that assign positive probability to states within the approval set if and only if the message comprises of these states only.

Part III: consider equilibrium (σ, α, q). Let

- $\mathcal{M} := \left\{m \in \mathbb{M} \text{ s.t. } \int_\Omega \sigma(m \mid \omega) d\omega > 0 \text{ and } q(\cdot \mid m) \in \mathcal{B}\right\}$ be the set of convincing messages. These are the messages that are *(i)* heard on equilibrium path and *(ii)* convince the receiver;
- $\mathcal{W} := \left\{\omega \in \Omega \text{ s.t. } \exists m \in \mathcal{M} \text{ s.t. } \omega \in m\right\}$ be the set of winning states, in which at least one winning message is available. Notice that if $\omega \in \mathcal{W}$, then once ω is realized, the sender's proposal must be getting approved with probability one, or else he could deviate to the convincing message and win the election with certainty. Put differently, in every winning state, the sender must be sending convincing messages, and convincing messages only, or $\forall \omega \in \mathcal{W}$, $\int_\mathcal{M} \sigma(m \mid \omega) dm = 1$.

Next consider the direct implementation with the set of winning states $\mathcal{W}$. Since (σ, α, q) is an equilibrium, it must be the case that $\mathcal{A} \subseteq \mathcal{W}$. All that remains to show is that the (obedience) constraint holds. Recall that convincing messages in equilibrium (σ, α, q) convince the receiver to vote in favor of the challenger's proposal, so

$$\forall m \in \mathcal{M},\ \ q(\cdot \mid m) \in \mathcal{B} \iff \int\limits_{\text{supp } q(\cdot \mid m) \subseteq m \subseteq \mathcal{W}} \delta(v, \omega) \cdot q(\omega \mid m) d\omega \geq 0$$

$$\iff \int\limits_\mathcal{W} \delta(v, \omega) \cdot \frac{\sigma(m \mid \omega) \cdot p(\omega)}{\int_\mathcal{W} \sigma(m \mid \omega') \cdot p(\omega') d\omega'} d\omega \geq 0 \iff \int\limits_\mathcal{W} \delta(v, \omega) \cdot \sigma(m \mid \omega) \cdot p(\omega) d\omega \geq 0.$$

Next, integrate the above expression over all convincing messages $m \in \mathcal{M}$:

$$\int\limits_\mathcal{M} \int\limits_\mathcal{W} \delta(v, \omega) \cdot \sigma(m \mid \omega) \cdot p(\omega) d\omega dm = \int\limits_\mathcal{W} \delta(v, \omega) \cdot p(\omega) \underbrace{\int\limits_\mathcal{M} \sigma(m \mid \omega) dm}_{=1,\ \forall \omega \in \mathcal{W}} d\omega$$

$$= \int_{\mathcal{W}} \delta(v,\omega)\cdot p(\omega)d\omega \geq 0 \iff \int_{\mathcal{W}} \delta(v,\omega)\cdot \frac{p(\omega)}{\int_{\mathcal{W}} p(\omega')d\omega'} d\omega \geq 0,$$

which is equivalent to (obedience)

$$p(\cdot \mid \mathcal{W}) \in \mathcal{B}.$$

PROOF OF THEOREM 2.2

In this proof, I borrow the notation and the results of Alonso and Câmara (2016). In the bayesian persuasion game, the optimal signal π is supported on $\{s^-, s^+\}$, where s^+ induces posterior $q^+ \in \mathcal{B}$ and recommends to aprove the sender's proposal and s^- induces posterior $q^- \notin \mathcal{B}$ and recommends voting for the status quo.

Let $\alpha(\omega) = \text{Prob}(s^+ \mid \omega)$ so that $\int_{\Omega} \alpha(\omega)p(\omega)d\omega$ is the probability of approval, i.e. the sender's objective function. The voter is convinced after observing s^+ if and only if $\mathbb{E}_{q^+}\left[\delta(v,\omega)\right] \geq 0$. This constraint can be written as

$$\int_{\Omega} \delta(v,\omega)\cdot q^+(\omega)d\omega = \int_{\Omega} \delta(v,\omega)\cdot \frac{\alpha(\omega)\cdot p(\omega)}{\int_{\Omega}\alpha(\omega')\cdot p(\omega')d\omega'} d\omega \geq 0$$

$$\iff \int_{\Omega} \delta(v,\omega)\cdot\alpha(\omega)\cdot p(\omega)d\omega \geq 0.$$

Putting these two together, the sender's problem when he has commitment power becomes

$$\max_{\alpha(\cdot)} \int_{\Omega} \alpha(\omega)\cdot p(\omega)d\omega, \quad \text{subject to} \quad \begin{array}{c} \int_{\Omega} \alpha(\omega)\cdot\delta(v,\omega)\cdot p(\omega)d\omega \geq 0, \\ 0 \leq \alpha(\omega) \leq 1 \quad \forall \omega \in \Omega. \end{array} \tag{BP}$$

Observe that $\forall \omega \in \mathcal{A}$, $\alpha(\omega) = 1$, or else increasing $\alpha(\omega)$ would relax the constraint (because $\delta(v,\omega) \geq 0$) and increase the objective function. Also, unless $\alpha(\omega) = 1$ for every $\omega \in \Omega$, the constraint is binding. Assume for the rest of this proof that the measure of states that are not approved is positive, which happens when $p \notin \mathcal{B}$.

Next, if $\alpha(\omega) \in (0,1)$ for some $\omega \in \Omega$, then $\alpha(\omega') = 1$ for all $\omega' \in \Omega$ such that $\delta(v,\omega) < \delta(v,\omega') < 0$, since these types are "cheaper" in terms of the constraint and hence should be added the set of approved states first. Next, let c^* be the cutoff value of the receiver's net payoff of approving the sender's proposal, i.e. $c^* = \delta(v,\omega)$ for all ω such that $\alpha(\omega) \in (0,1)$. Also let

- $\bar{\mathcal{D}}(v,c^*) := \{\omega \in \Omega \text{ s.t. } \delta(v,\omega) \leq c^*\}$;

- $\mathcal{D}(v, c^*) := \{\omega \in \Omega \text{ s.t. } \delta(v, \omega) < c^*\}$;
- $\partial\mathcal{D}(v, c^*) := \bar{\mathcal{D}}(v, c^*) \setminus \mathcal{D}(v, c^*) = \{\omega \in \Omega \text{ s.t. } \delta(v, \omega) = c^*\}$.[15]

We know that the solution $\alpha(\cdot)$ takes form

$$\alpha(\omega) = \begin{cases} 1, & \omega \in \mathcal{D}(v, c^*); \\ (0, 1), & \omega \in \partial\mathcal{D}(v, c^*); \\ 0, & \omega \in \Omega \setminus \bar{\mathcal{D}}(v, c^*). \end{cases}$$

The (binding) obedience constraint is thus

$$\int\limits_{\mathcal{D}(v,c^*)} \delta(v, \omega) \cdot p(\omega) d\omega + \int\limits_{\partial\mathcal{D}(v,c^*)} \alpha(\omega) \cdot \delta(v, \omega) \cdot p(\omega) d\omega = 0.$$

If measure of set $\partial\mathcal{D}(v, c^*)$ is not zero, partition it into two sets, $X \subseteq \partial\mathcal{D}(v, c^*)$ and $Y = \partial\mathcal{D}(v, c^*) \setminus X$. Let $\alpha(\omega) = 1$ for $\omega \in X$ and $\alpha(\omega) = 0$ for $\omega \in Y$. Set X must satisfy the obedience constraint

$$\int\limits_{\partial\mathcal{D}(v,c^*)} \alpha(\omega) \cdot \delta(v, \omega) \cdot p(\omega) d\omega = \int\limits_{X} \delta(v, \omega) \cdot p(\omega) d\omega = -\int\limits_{\mathcal{D}(v,c^*)} \delta(v, \omega) \cdot p(\omega) d\omega.$$

From equation above it follows immediately that if measure of set $\partial\mathcal{D}(v, c^*)$ is positive, so is measure of set X.

To summarize, the solution may be represented as

$$\alpha(\omega) = \begin{cases} 1, & \omega \in \mathcal{D}(v, c^*) \cup X; \\ 0, & \omega \in \left(\Omega \setminus \bar{\mathcal{D}}(v, c^*)\right) \cup Y, \end{cases}$$

or simply $\alpha(\omega) = \mathbb{1}\{\omega \in W\}$, where $W \supseteq \mathcal{A}$ also solves

$$\max_{W \subseteq \Omega} \int\limits_{\Omega} \mathbb{1}\{\omega \in W\} \cdot p(\omega) d\omega \;\; \text{subject to} \int\limits_{\Omega} \delta(v, \omega) \cdot \mathbb{1}\{\omega \in W\} \cdot p(\omega) d\omega,$$

which is the same problem as finding the sender-preferred equilibrium of the election game with

[15] Note that $\partial\mathcal{D}(v, c^*)$ may not technically be the boundary of set $\bar{\mathcal{D}}(v, c^*)$ and have positive measure.

verifiable information, specified in Theorem 2.2. By the argument above, the solution is characterized by the cutoff value of receiver's net payoff of approving the sender's proposal c^*.

Proof of Theorem 2.3

When applicable, I will refer to the proof of Theorem 2.1 that deals with the one-receiver case.

Part I: if $(\sigma^D, \alpha^D, q^D)$ is a direct equilibrium with collection of winning messages $\{\mathcal{W}_v\}$, then $p(\cdot \mid \mathcal{W}_v) \in \mathcal{B}_v$ for every $v \in \mathbb{V}$, or else the receiver has a profitable deviation.

Part II: direct implementation becomes direct equilibrium for "skeptical" off-path beliefs of the receiver as in the one-receiver case.

Part III: consider a non -direct equilibrium $(\sigma, \boldsymbol{\alpha}, \boldsymbol{q})$. Let

- $\pi_v(\omega) := \int_{\mathbb{M}} \sigma_v(m \mid \omega) \cdot \mathbb{1}\{q_v(\cdot \mid m) \in \mathcal{B}_v\} dm$ be the probability with which receiver $v \in \mathbb{V}$ is convinced in state $\omega \in \Omega$;
- $\mathcal{M}_v := \{m \in \mathbb{M} \text{ s.t. } \sigma_v(m \mid \omega) > 0 \text{ and } q_v(\cdot \mid m) \in \mathcal{B}_v\}$ be the set messages that are heard on the equilibrium path and convince receiver $v \in \mathbb{V}$;
- $\mathcal{E}(\omega) := \{v \in \mathbb{V} \text{ s.t. } \pi_v(\omega) = 1\}$ be the sender's electorate in state ω.[16]

Observe that receivers with $\pi_v(\omega) \in (0, 1)$ are not included in state ω's electorate despite being "convincible", because they are not pivotal in ω. If they were, then the sender would deviate to the convincing message, convince this receiver, and swing the election outcome, thus making a profitable deviation.

For every receiver $v \in \mathbb{V}$, let $\mathcal{W}_v := \{\omega \in \Omega \text{ s.t. } \pi_v(\omega) = 1\}$ be the set of states in which she is convinced with probability one. Note that $\mathcal{W}_v$ may be empty, e.g. when $v \notin \mathcal{E}(\omega)$, $\forall \omega \in \Omega$, which happens when this receiver is not pivotal in any state.

Next consider the direct implementation with collection of convincing messages $\{\mathcal{W}_v\}$. Each receiver's (obedience) constraint follows (as it did in the one-receiver case) from the fact that in the non-direct equilibrium she was convinced with probability one in each of those states, i.e. $\forall \omega \in \mathcal{W}_v$, $\int_{\mathcal{M}_v} \sigma_v(m \mid \omega) = 1$.

[16] Sender's electorate in a direct equilibrium with collection of convincing messages $\{\mathcal{W}_v\}$ was previously defined as $\mathcal{E}(v) := \{v \in \mathbb{V} \text{ s.t. } \omega \in \mathcal{W}_v\}$. While defining it again for non-direct equilibria is an abuse of notation, the sender's electorate in the non-direct equilibrium will be the same as its direct counterpart.

Proof of Lemma 3.1

By the definition of the set of approval beliefs, for every $v_i \in \mathbb{V}$

$$q \in \mathcal{B}_i \iff \int_{\mathbb{R}} |v_i - \omega| \cdot q(\omega) d\omega \leq |v_i - \omega^{\circ}|.$$

Adding up the right hand sides for $i \in \{1, 2\}$ and employing the definition of voter incompatibility,

$$\pi \in \mathcal{B}_1 \cap \mathcal{B}_2 \Longrightarrow \int_{\mathbb{R}} \left[|v_1 - \omega| + |v_2 - \omega|\right] \cdot q(\omega) d\omega \leq |v_1 - v_2|.$$

However, the right hand side almost surely violates the triangle inequality, which states that $|v_1 - \omega| + |\omega - v_2| \geq |v_1 - v_2|$ for every $\omega \in \Omega$.