Targeted Advertising in Elections*

Maria Titova†

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LATEST VERSION

Abstract

Some elections seem unwinnable for challengers because pivotal voters prefer policies on the opposite sides of the status quo. In this paper, I argue that the challenger can win any such election with positive probability if he uses targeted advertising with verifiable messages. In private ads, the challenger makes each voter believe that his policy is a sufficient improvement over the status quo and wins the election when his policy is sufficiently moderate. Furthermore, his odds of winning increase when voters become more extreme, because more extreme voters are more dissatisfied by the status quo and can therefore be persuaded by a wider range of policies.

KEYWORDS: Persuasion, targeted advertising, elections

JEL CLASSIFICATION: D72, D82, D83

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†Department of Economics, Vanderbilt University. E-mail: maria.titova@vanderbilt.edu.

1. Introduction

Targeted advertising has played an essential role in many elections. In 2016, both Donald Trump's campaign in the U.S. presidential election and the Leave campaign in the U.K.'s Brexit referendum used the services of Cambridge Analytica to target voters via Facebook and Twitter. In the 2008 U.S. presidential election, Barack Obama's campaign pioneered the use of social networks to communicate with the electorate. Even before social media, in 2000, George W. Bush's presidential campaign targeted voters via direct mail. In all of these cases, the winning candidates had access to better technology or better voter data. [1] Would they have lost without targeted advertising? In other words, can targeted advertising swing elections?

To answer these questions, I propose a simple theoretical model that isolates the effects of targeted advertising from all other aspects of an electoral campaign. I assume that there are two candidates: the status-quo candidate and the challenger. To focus on information transmission, I abstract away from the policy selection stage and assume that both candidates are endowed with policies, which are elements of the policy space $[-1, 1]$. I then introduce two forms of asymmetry between the candidates. The first is informational: I assume that the status-quo candidate's policy is commonly known and fixed at the origin, while the challenger's policy is his private information, about which the voters have a common prior belief. This prior belief has full support over the policy space, reflecting total prior uncertainty. The second form of asymmetry is strategic: the status-quo candidate does not advertise, while the challenger can send messages to the voters to sway them. This setup allows me to reframe the motivating questions as follows: how much information should the challenger reveal to the voters? If the voters cannot be simultaneously convinced by a public advertising campaign, can the challenger convince them privately by revealing different amounts of information to each?

I assume that the challenger knows the preferences of the voters and communicates with them by providing verifiable information (in the spirit of Milgrom and Roberts, 1986). Each message sent by the challenger is a statement about his policy (formally, a subset of $[-1, 1]$) that contains a grain of truth (formally, it includes his actual policy x). Conceptually, this communication protocol allows lies of omission, but not commission: a message $[-1/2, 0]$ is informative because it restricts the support of the voter's belief about the challenger's policy to that interval, but it is only partially informative, because the challenger's policy could lie anywhere in that interval. The

[1] For comparisons of the candidates' advertising strategies, see Kim et al. (2018) and Wylie (2019) for the 2016 election, Harfoush (2009) and Katz, Barris, and Jain (2013) for the 2008 election, and Hillygus and Shields (2014) for the 2004 election.

protocol thus provides a reasonable middle ground between the possibilities identified by Persson and Tabellini (2002), who famously write (p. 483), "It is thus somewhat schizophrenic to study either extreme: where promises have no meaning or where they are all that matter." From a technical standpoint, the equilibria of games with verifiable information admit a simple characterization that roughly corresponds to how vague the challenger can be while still persuading the voters.

The voters in my model face a binary choice between approving the challenger's policy (voting "for" the challenger) and rejecting it (voting "against" the challenger, by either voting for the status-quo candidate or abstaining). I assume that voters have standard quadratic spatial preferences, meaning that each voter has an ideal policy and evaluates policies based on the distance from that bliss point. Each voter has an "approval set" consisting of the policies that are closer to her bliss point than the status quo; under complete information, she prefers to approve these policies and reject all others.

We start by analyzing an election with a single voter, whom the challenger wishes to convince. Define an equilibrium outcome of this election as a set of challenger's policies that the voter approves in equilibrium. I first ask, what is the challenger-preferred equilibrium outcome? Simply put, what is the largest set of policies that the voter is willing to approve in equilibrium?

Mathematically, our goal is to find the set of policies $W \subseteq [-1, 1]$ that maximizes the challenger's ex-ante odds of winning subject to the voter's obedience constraint. The obedience constraint states that if the voter approves all policies in W, then her expected utility from approval should exceed her expected utility from rejection. One way to implement this outcome is to have the challenger send the message W whenever his policy is $x \in W$ and the message $X \setminus W$ otherwise. On the equilibrium path, the voter interprets the first message as a recommendation to approve and the second message as a recommendation to reject. Off the equilibrium path, the voter is skeptical and assumes that the message comes from a challenger who most benefits from her approval, i.e. one whose policy is the furthest from her bliss point. By construction, if the challenger's policy is in W, he wins; in this case he has no profitable deviations because he is already collecting the highest payoff. If his policy is outside W, he loses; in this case he cannot deviate to the winning message because $x \notin W$. The problem of optimally convincing one voter is well studied in the communication literature (Alonso and Câmara, 2016; Titova, 2022). The solution features a cutoff policy: the voter approves all policies not too far from her bliss point. Geometrically, the challenger selects the largest interval of policies symmetric about the voter's bliss point that averages her utility from approval to her utility from rejection. Conceptually, the challenger tells the voter that his policy is not too far from her bliss point, but he does not elaborate further. The challenger-preferred equilibrium outcome is essentially the vaguest message that could conceivably persuade the voter, if every challenger with a policy in that set were sending it with probability one.

Now that we have established how much information the challenger should convey to one voter, let us study elections with multiple voters. We immediately run into the following impossibility result: if two voters have bliss points on opposite sides of the status quo, then under a common belief they will never both approve. This is due to the voters' risk aversion: if a voter's bliss point is left of the status quo (i.e., less than 0), then her approval set lies entirely to the left of the status quo, and therefore she approves only if she expects the challenger's policy to be on the left. Similarly, a voter whose bliss point is right of the status quo approves only if she expects the challenger's policy to be on the right. Obviously, the challenger's policy cannot (on average) be both left and right of the status quo. This impossibility result allows us to identify elections that I call *unwinnable* (under common belief) for the challenger. These elections are characterized by voter distributions and social choice rules such that, to win the election, the challenger must convince both left and right voters at the same time. If the voters hold a common belief (for example, if the challenger does not advertise, fully reveals his policy, or uses a public disclosure policy), then the challenger will lose an unwinnable election with probability one in every equilibrium, regardless of the communication protocol. Replacing verifiable messages with cheap talk or endowing the challenger with commitment power does not resolve the issue. The only way the challenger can win is by inducing different posterior beliefs among different voters.

Let us focus on the simplest unwinnable election, which requires the unanimous approval of two voters: a $left$ voter with bliss point $L < 0$ and a *right* voter with bliss point $R > 0$. We know that without targeted advertising, the challenger's odds of winning are zero. Can he win with positive probability if he can send a different private message to each voter? Provided the voters do not condition on the event of being pivotal, the answer is yes.

For this election, define an equilibrium outcome as a pair (W_L, W_R) of sets of policies approved by the two voters. Assume that each voter observes only her own message and does not condition on being pivotal. We can then use an equilibrium characterization similar to the one stated for the case of one voter: a voter will approve a set of policies only if her expected utility from approval over that set exceeds her expected utility from rejection. However, the challenger's objective now is to convince not just one voter, but both voters simultaneously. Consequently, to find the challenger-preferred equilibrium outcome (the one that maximizes the challenger's odds of winning), we must maximize the prior measure of the set $W_L \cap W_R$ of unanimously approved policies, subject to the two voters' obedience constraints. This means that, instead of sending each voter a message that lies close to her bliss point, the challenger should skew each message toward the other voter's bliss point. That is, his message to the *left* voter should include her approval set plus as many right policies as possible, and vice versa. Intuitively, the challenger's message to the *left* voter should not include policies that lie left of her approval set, because these policies are so far from the *right* voter's bliss point that they are too expensive (in terms of the *right* voter's obedience constraint) to include in the set of unanimously approved policies. In a sense, the unanimously approved policies must be "sufficiently moderate"—not too far from the status quo—because the *left* voter will approve only a limited set of right policies, and vice versa. However, each voter *is* persuadable by some policies outside her approval set, which means the set of unanimously approved policies is non-empty. In other words, by sending private messages, the challenger can win the unwinnable election with positive probability. The following motivating example illustrates the simple algebra behind the model; it also shows that the challenger has higher odds of swinging elections with more polarized electorates.

Motivating Example

Suppose that $L = -0.2$ and $R = 0.4$. For each $v \in \{L, R\}$, let voter v's utility be $-(x - v)^2$ if she approves the challenger's policy $x \in [-1, 1]$ and $-v^2$ if she rejects it. The challenger receives 1 if he convinces both voters and 0 otherwise. For simplicity, suppose that the voters' prior about the challenger's policy is uniform: ex ante, every x in $[-1, 1]$ is equally likely. Consider the following strategy of the challenger: to the *left* voter, he sends the message $[-0.4, 0.2]$ whenever his policy lies in $[-0.4, 0.2]$, and the message $[-1, 1]$ otherwise. In words, he tells the left voter that his policy is neither ultra-left nor moderately to ultra-right, whenever that is true, and says nothing otherwise. Similarly, to the *right* voter, the challenger sends the message $[-0.4, 0.8]$ whenever his policy lies in $[-0.4, 0.8]$, and $[-1, 1]$ otherwise. Figure 1 illustrates the challenger's strategy.

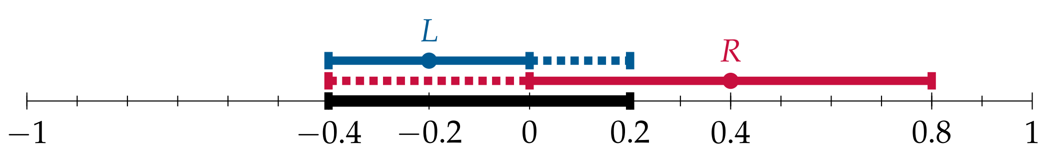

Figure 1. *Targeted messages that convince voter L (in blue) and voter R (in red). The challenger wins the election whenever his policy lies in the intersection of the convincing messages (in black).*

Now, suppose that the *left* voter receives the message $[-0.4, 0.2]$. Her expected utility from approving is $\int_{-0.4}^{0.2} -(x + 0.2)^2 dx = -0.024$, which coincides with her expected utility from rejecting, which is $\int\limits_{-0.4}^{0.2} -(0.2)^2 dx = -0.024$. In other words, if she receives the message $[-0.4, 0.2]$ and knows that every challenger whose policy lies in $[-0.4, 0.2]$ sends that message with probability one, then she is indifferent between approval and rejection. Assume that the voter breaks the tie in favor of approval. Note that the message that convinces her, $[-0.4, 0.2]$, comprises both policies that she prefers to the status quo (i.e., policies in $[-0.4, 0]$) and policies that are strictly dominated by the status quo (i.e., policies in $(0, 0.2]$). Her risk aversion is reflected in the fact that the average policy that she approves is -0.1 and is strictly to the left of the status quo. By similar reasoning, the *right* voter approves if she receives the message $[-0.4, 0.8]$. Finally, if either voter receives the message $[-1, 1]$, she knows that the challenger's policy is outside of her approval set, so she rejects.

The strategies described above lead to the following electoral outcome. The $left$ voter approves if and only if the challenger's policy is in the interval $[-0.4, 0, 2]$, while the *right* voter approves if and only if the challenger's policy is in the interval $[-0.4, 0.8]$. The challenger therefore wins the election whenever his policy is between -0.4 and 0.2. Given the uniform prior on $[-1, 1]$, his ex-ante odds of winning are 30%.

It remains to show that the players' strategies constitute a perfect Bayesian equilibrium. To complete the equilibrium characterization, we need to specify each voter's off-path beliefs. Let us impose that the voters are *skeptical*: each voter assumes that if the challenger sends an off-path message, then (i) the challenger *can* send that message, and (ii) the challenger's policy is outside of her approval set. Then the challenger has no profitable deviations: for any policy, either he wins the election (securing the highest possible payoff), or he loses the election (because he cannot verifiably send messages that convince both voters). The voters also do not have profitable deviations. Thus, the strategies we have described constitute an equilibrium. In fact, this is the challenger's preferred equilibrium—the one in which his odds of winning are the highest.

Now suppose the right voter's bliss point shifts from $R = 0.4$ to $R' = 0.5$; that is, she becomes more extreme. Following the same logic as above, the challenger can again send the $left$ voter a message that includes as many right policies as possible, and vice versa. Since the $left$ voter's bliss point has not changed, the message $[-0.4, 0.2]$ still convinces her. On the other hand, redoing the calculations for the bliss point $R' = 0.5$, we find that the message that convinces the *right* voter is now $[-0.5, 1]$. Using these messages, the challenger wins when his policy is between -0.4 and 0.2, exactly as before.

However, the challenger can do even better. Specifically, notice that when his policy is between -0.5 and -0.4, the strategy described above convinces the *right* voter, but not the $left$ voter. But the left voter actually prefers policies between -0.5 and

-0.4 over some of the policies she is currently approving—namely, those between 0.1 and 0.2—because the former are closer to her bliss point. Hence, we can recalculate the message that convinces the *left* voter (making her indifferent between the challenger and the status-quo), forcing it to start at -0.5. The resulting message is $[-0.5, 0.179]$. Figure 2 illustrates the messages with which the challenger can convince the voters after the *right* voter becomes more extreme.

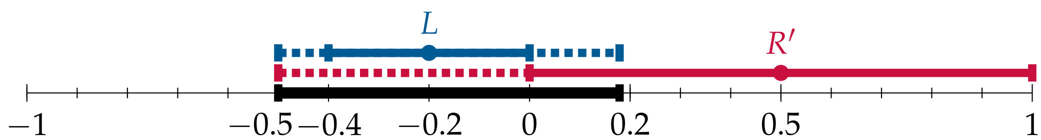

Figure 2. *Challenger-preferred targeted messages that convince voters L and R′. A more extreme right voter is now persuadable by policies farther to the left. As a result, the set of unanimously approved policies (in black) is larger and shifts to the left.*

In the new equilibrium, the set of unanimously approved policies is $[-0.5, 0.179]$, and the odds of the challenger winning are 33.96%. An immediate observation is that the challenger has better odds of swinging an unwinnable election when one of the voters is more extreme. Essentially, when a voter becomes more extreme, her dissatisfaction with the status quo grows, making her more persuadable. A less obvious observation is that when, for example, the *right* voter's bliss point shifts farther right, the set of unanimously approved policies (the policies with which the challenger can win) shifts in the opposite direction, to the left. When a voter becomes more extreme, she becomes persuadable by a wider range of policies, in either direction, despite her risk aversion. In particular, when the *right* voter becomes more extreme, the challenger can persuade her using more left policies, and vice versa.

Note that the results described above rely on both expressive voting and the equilibrium selection. On the one hand, while the voters are fully Bayesian, they do not condition on the event of being pivotal; the only information they take into account is their own signal. For that reason, the proposed model is more suitable for describing large elections.[2] On the other hand, even when the voters are expressive, the game under consideration has other equilibria than the one described. For example, the challenger could simply reveal his policy to every voter (that is, for each realization $x \in X$ of his policy, he could send the message $\{x\}$ to every voter); in this case he would lose with probability one. However, empirical evidence suggests that this equilibrium is

[2]The theory of expressive voting was pioneered by Brennan and Lomasky (1993) and Brennan and Hamlin (1998); Hamlin and Jennings (2011) provides a review. There is a large body of evidence that voters' behavior is consistent with expressive voting, e.g., in U.S. national elections (Kan and Yang, 2001; Degan and Merlo, 2007), Spanish general elections (Artabe and Gardeazabal, 2014), and Israeli general elections (Felsenthal and Brichta, 1985).

not necessarily realistic: politicians do, in fact, often make vague statements about the policies that they intend to pursue.[3] In this paper I select the equilibrium in which, in some sense, the challenger's messages to the voters are the most vague.

Related Literature

This paper contributes to the growing literature that compares public and private communication. In relation to games of verifiable information, the closest paper to mine is that of Schipper and Woo (2019), who study advertising competition with microtargeting. They show that even with targeted advertising, candidates tend to voluntarily disclose all of their private information. This unraveling result is fairly common in the verifiable information literature on voter persuasion (Board, 2009; Janssen and Teteryatnikova, 2017); it arises because the candidates play a zero-sum game. In contrast to these papers, I consider an asymmetric model in which only one candidate can communicate with the voters. Here, unraveling does not necessarily occur, and the challenger can improve his chances of winning beyond the full disclosure outcome.

Much progress has been made in comparing public and private communication in the cheap talk literature. This literature shows that senders often prefer to communicate in public, rather than in private (Farrell and Gibbons, 1989; Koessler, 2008; Goltsman and Pavlov, 2011; Bar-Isaac and Deb, 2014), because public communication reduces the number of deviations available to the sender in each state of the world. However, this effect is absent when the sender's messages are verifiable, because his message space is then already restricted. In fact, my main result shows that with verifiable messages, the opposite effect occurs: the sender strictly benefits from private communication, to the point that he can win elections that are unwinnable with public advertising.

The sender also prefers private communication to public advertising in information design (Arieli and Babichenko, 2019; Bardhi and Guo, 2018; Chan et al., 2019; Heese and Lauermann, 2021). I confirm this finding, which is not surprising: while my sender does not possess any commitment power, the sender-preferred equilibrium outcome is also a commitment outcome (Titova, 2022). Beyond that, my contribution to this literature is twofold: on the one hand, I conclude that the sender does not need commitment power to benefit from private communication. Moreover, private communication allows him not only to increase his ex-ante utility, but to increase it from 0 in every equilibrium to a positive number in his preferred equilibrium.

This paper also contributes to the literature on political ambiguity. Current expla-

[3] For empirical evidence of candidate ambiguity in the lab and in the field, see, e.g., Tomz and Houweling (2009), Somer-Topcu (2015), Bräuninger and Giger (2018), and Tolvanen, Tremewan, and Wagner (2022).

nations for why politicians advertise ranges of policies include voters' risk-seeking behavior (Shepsle, 1972), candidates' policy preferences (Alesina and Cukierman, 1990), candidates' preference for ambiguity (Aragonès and Neeman, 2000), future elections (Meirowitz, 2005; Alesina and Holden, 2008), and resolution of uncertainty after the election (Kartik, Van Weelden, and Wolton, 2017). This paper's equilibrium features ambiguity because sending vague messages allows the challenger to convince multiple voters at once without lying (by commission) to any of them. Notably, two previous papers have found that ambiguity may enable politicians to persuade voters on opposite sides of the status quo. In Callander and Wilson (2008), voters have context-dependent preferences, and in Tolvanen (2021), the voters' preferences are correlated with the state of the world. I reach a similar conclusion in a remarkably simple setup, with expressive voters who have standard quadratic preferences.

2. Baseline Election: Model

I study a game between a politician who challenges the status quo (the challenger, he/him) and the voters (each she/her). There is an underlying policy space $X := [-1, 1]$, with policy positions ranging from far left (-1) to far right (1). The status-quo candidate's policy is fixed, known, and normalized to 0. The game begins with the challenger privately observing his policy $x \in X$, which is drawn from a common prior distribution $\mu_0 \in \Delta X$ with full support.[4] The challenger is *office-motivated*: his goal is to win the election. I normalize his payoff from winning to 1 and his payoff from losing to 0. In the baseline election, there are two voters, $left$ and *right*, and the challenger needs their unanimous approval to win the election. Section 4 generalizes the model to large elections with many voters and arbitrary social choice functions.

The challenger advertises his policy to voters through private verifiable messages. Specifically, the challenger sends each voter a message $m \subseteq X$, and this message must contain a grain of truth: $x \in m$. That is, the challenger *can lie by omission* and send messages that contain policies other than x, but he *cannot lie by commission* and send messages that do not include x. The verifiability of the messages allows the voters to draw inferences about the challenger's policy. For example, a voter receiving the message $[-1/2, 0]$ infers that the challenger's policy is moderately left. She knows that the policy is not far-left or anywhere on the right, but she does not know its exact location between $-1/2$ and 0.

[4]For a compact metrizable space Y, I let ΔY denote the set of all Borel probability measures over Y, endowed with the weak* topology. For $\gamma \in \Delta Y$, I let supp γ denote the support of γ. I say that $\gamma \in \Delta Y$ is degenerate if $\gamma(y) = 1$ for some $y \in Y$, and non-degenerate otherwise.

The voters have spatial preferences and vote expressively for the candidate they prefer. A voter with bliss point $v \in X$, to whom I will refer as "voter v" when there is no possibility of confusion, prefers to approve (i.e., to vote for the challenger) if she considers the challenger's policy $x \in X$ a sufficient improvement over the status quo. Otherwise, she prefers to reject it. Mathematically, when the challenger's policy is $x \in X$,

$$u_v(\text{approve}, x) = -(v-x)^2, \quad u_v(\text{reject}, x) = -(v-0)^2.$$

To simplify the analysis, define voter v's *net payoff from approval* as $\alpha_v(x) := u_v(\text{approve}, x) - u_v(\text{reject}, x) = -x^2 + 2vx$. Note that $\alpha_v(x)$ is a downward-sloping parabola that peaks at v, which reflects the voter's risk aversion. Voter v's best response is to approve the challenger's policy $x \in X$ whenever $\alpha_v(x) \geq 0$. Next, define voter v's *approval set* as $A_v := \{x \in X \mid \alpha_v(x) \geq 0\}$; this is the set of all policies that she prefers to approve under complete information. Let $\lfloor A_v \rfloor := \min A_v$ and $\lceil A_v \rceil := \max A_v$ respectively denote the smallest and the largest elements of the approval set.

In the baseline election, the $left$ voter has bliss point $L \in [-1, 0)$ and the *right* voter has bliss point $R \in (0, 1]$. Figure 3 illustrates the preferences of the $left$ voter.

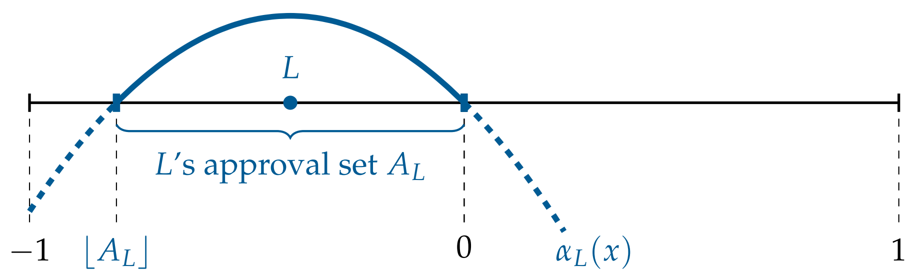

Figure 3. *The policy space $X = [-1, 1]$, the $left$ voter's bliss point $L < 0$, her net payoff from approval $\alpha_L(x)$, and her approval set A_L.*

I focus on the challenger-preferred perfect Bayesian equilibrium of this game. Knowing his policy x, the challenger chooses verifiable messages $m_L \subseteq X$ and $m_R \subseteq X$ for voters L and R, respectively. Verifiability requires that $x \in m_v$ for all $v \in \{L, R\}$. Having observed the message m_v, voter $v \in \{L, R\}$ forms a posterior belief over X. She then approves or rejects.

In equilibrium, (i) the challenger sends messages that maximize his payoff, (ii) each voter approves the challenger's policy whenever her expected net payoff from approval is non-negative under her posterior belief, and (iii) each voter calculates her posterior using Bayes' rule, without conditioning on being pivotal. The challenger-preferred equilibrium is the one in which his odds of unanimous approval are the highest across all equilibria.

3. Baseline Election: Analysis

Unwinnable Elections

We begin by establishing that the baseline election is unwinnable for the challenger under complete information. First, we show that a voter prefers to approve only if she believes the challenger's policy is on the same side of the status quo as her bliss point.

Lemma 1. *If voter with bliss point $v \in X$ approves under a non-degenerate belief $\mu \in \Delta X$, then $\mathbb{E}_\mu[x] \in \big(\lfloor A_v \rfloor, \lceil A_v \rceil\big)$.*

Proof. By Jensen's inequality for the strictly concave function $\alpha_v(x)$ and the non-degenerate belief μ, $\alpha_v\big(\mathbb{E}_\mu[x]\big) > \mathbb{E}_\mu\big[\alpha_v(x)\big]$. Since voter v approves under μ, the right-hand side, which is her expected net payoff from approval, is non-negative. Now, $\alpha_v(x) = -x^2 + 2vx = -x(x-2v) > 0$ if and only if $x \in \big(\lfloor A_v \rfloor, \lceil A_v \rceil\big)$. Consequently, if v approves under μ, then $\mathbb{E}_\mu[x] \in \big(\lfloor A_v \rfloor, \lceil A_v \rceil\big)$. □

Second, we observe that if the voters hold a common belief, they will not both approve, because they cannot both expect the challenger's policy to be on their side of the status quo.

Lemma 2. *For any non-degenerate common belief $\mu \in \Delta X$, at most one of the voters prefers to approve.*

Proof. From Lemma 1, if the $left$ voter approves under belief μ, then $\mathbb{E}_\mu[x] \in (2L, 0)$. Similarly, if the *right* voter approves under μ, then $\mathbb{E}_\mu[x] \in (0, 2R)$. Thus, at least one voter prefers to reject under μ. □

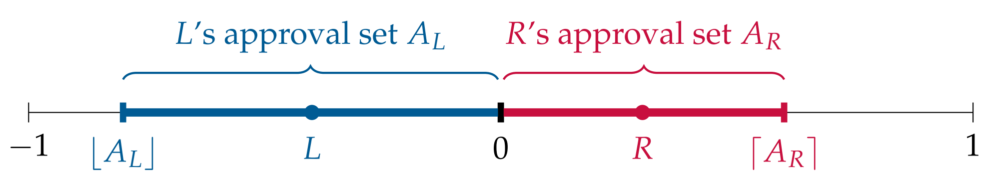

Figure 4. *Approval sets of the $left$ and right voters: under complete information, L approves left policies, and R approves right policies.*

Figure 4 illustrates each voter's approval set. Simply put, the $left$ voter prefers to approve only left (blue) policies, while the *right* voter prefers to approve only right (red) policies. Since the challenger's policy cannot be both left and right at the same time, at least one voter prefers to reject it. The same argument applies when the voters have a common belief. That is, since both voters cannot expect the challenger's policy to be on their side of the status quo, at least one of them prefers to reject.

Lemma 2 implies that without targeted advertising, the baseline election is *unwinnable* for the challenger. If he does not advertise at all, the voters hold a common prior, and at most one of them approves. If he advertises publicly, the voters' common prior is updated to a common posterior, but again, at most one voter is convinced to approve. The only event in which the challenger wins is when his policy coincides with the status quo. However, this event has a zero prior measure.

EQUILIBRIUM OUTCOMES UNDER TARGETED ADVERTISING

Let us now characterize the (challenger-preferred) equilibrium payoff of the baseline election game with targeted advertising. According to Titova (2022), every equilibrium is payoff equivalent to a direct equilibrium with *approved messages* $W_L \subseteq X$ and $W_R \subseteq X$ that satisfy certain constraints. In the direct equilibrium, the challenger sends the message W_v to voter v if $x \in W_v$, and its complement $W_v^c := X \smallsetminus W_v$ otherwise.[5] When voter v hears W_v, she approves; otherwise, she rejects. By construction, a voter's approved message is also the set of policies that she approves in the (direct) equilibrium, while its complement is the set of policies she rejects. Off the (direct) equilibrium path, each voter is "skeptical": she assumes that if the challenger sends an unexpected message, his policy must lie outside of her approval set, whenever possible.[6]

To be implementable in equilibrium, voter v's approved message W_v must satisfy two conditions. The first is the sender's incentive-compatibility constraint, $A_v \subseteq W_v$, which guarantees that the challenger does not want to deviate towards a fully informative strategy. This constraint is automatically satisfied in the challenger-preferred equilibrium, because there the challenger sends each voter as vague a message as possible (i.e., each of his messages includes as many policies as possible). The second condition is the receiver's obedience constraint, which ensures that voter v only approves when her average net payoff from approval is non-negative:

$$\int_{W_v} \alpha_v(x) d\mu_0(x) \geq 0.$$

The challenger wins the baseline election (and gets a payoff of 1) whenever the

[5]More specifically, the challenger's strategy is to send (with probability one in all cases) W_L to L and W_R to R if $x \in W_L \cap W_R$; W_L to L and W_R^c to R if $x \in W_L \cap W_R^c$; W_L^c to L and W_R to R if $x \in W_L^c \cap W_R$; and W_L^c to L and W_R^c to R if $x \in W_L^c \cap W_R^c$.

[6]Let $m \subseteq X$ such that $m \notin \{W_v, W_v^c\}$ be an off-path message, and let $\mu(\cdot \mid m) \in \Delta X$ be voter v's posterior belief after hearing m. If $m \nsubseteq A_v$, let supp $\mu(m) \subseteq m \smallsetminus A_v$ so that $\int \alpha_v(x) d\mu(x \mid m) < 0$ and v prefers to reject. If $m \subseteq A_v$, then supp $\mu(\cdot \mid m) \subseteq A_v$ since the messages are verifiable, and v prefers to approve. If $A_v \subseteq W_v$, then a challenger with a policy $x \in A_v$ does not wish to deviate, as he is already convincing voter v on the equilibrium path.

$left$ and the *right* voters approve, that is, when his policy is in the *set of unanimously approved policies* $W_L \cap W_R$. Thus, the approved messages that maximize his odds of winning across all equilibria solve

$$\max_{W_L, W_R \subseteq X} \int_{W_L \cap W_R} d\mu_0(x) \text{ subject to } \int_{W_v} \alpha_v(x) d\mu_0(x) \geq 0, \; \forall v \in \{L, R\}. \tag{1}$$

I refer to the pair $(\overline{W}_L, \overline{W}_R)$ that solves Problem (1) as a (challenger-preferred) equilibrium outcome (under targeted advertising). The main result of this paper states that the challenger can always win an unwinnable election by advertising privately.

PROPOSITION 1. *In the equilibrium of the baseline election game, the challenger's ex-ante odds of winning are positive.*

Proof. First, observe that each voter is guaranteed to be convinced by her approval set, i.e., $\int_{A_v} \alpha_v(x) d\mu_0(x) > 0$ for each $v \in \{L, R\}$. Next, for each voter $v \in \{L, R\}$, select a subset $B_v \subseteq A_{v'}$ of the other voter's approval set that satisfies $\int_{A_v} \alpha_v(x) d\mu_0(x) + \int_{B_v} \alpha_v(x) d\mu_0(x) \geq 0$ and $\mu_0(B_v) > 0$. Let $W_v := A_v \cup B_v$ be voter v's approved message. Although (W_L, W_R) may not be an equilibrium pair of approved messages, they do, by construction, satisfy the constraints of Problem (1). At the same time, $\mu_0(W_L \cap W_R) = \mu_0(B_R \cup B_L) > 0$, implying that the challenger's ex-ante odds of winning in equilibrium must be positive. Before characterizing the equilibrium approved messages, let us focus on the problem of maximizing the challenger's odds of convincing just one voter. □

ONE VOTER'S INTERVALS OF APPROVED POLICIES

Consider a voter with bliss point $v \in X \setminus \{0\}$. Let us focus on the auxiliary problem of finding this voter's largest (in terms of prior measure) set of approved policies, constrained by $l \in \big[-1, \lfloor A_v \rfloor\big]$ from the left and $r \in \big[\lceil A_v \rceil, 1\big]$ from the right:

$$I_v(l, r) := \max_{W \subseteq [l, r]} \mu_0(W) \quad \text{subject to} \quad \int_W \alpha_v(x) d\mu_0(x) \geq 0. \tag{AUX}$$

The solution to the auxiliary problem is characterized by a cutoff value for the voter's net payoff from approval (for proofs, see Alonso and Câmara, 2016, and Titova, 2022). In words, voter v approves every policy with a not too negative payoff from approval (i.e., every $x \in X$ for which $\alpha_v(x) \geq -c_v^*$). The cutoff value $c_v^* > 0$ is obtained from the binding constraint. The set $\{x \in [l, r] \mid \alpha_v(x) \geq -c_v^*\}$ is an interval: it is the upper contour set of the concave function $\alpha_v(x)$. Lemma 3 characterizes the solution of the auxiliary problem.

LEMMA 3. *Consider a voter with bliss point $v \in X \setminus \{0\}$. The solution $I_v(l,r) \subseteq [l,r]$ to Problem (AUX) with $l \in \big[-1, \lfloor A_v \rfloor\big]$ and $r \in \big[\lceil A_v \rceil, 1\big]$ is almost surely an interval such that*

- *if $\int_l^r \alpha_v(x) d\mu_0(x) \geq 0$, then $I_v(l,r) = [l,r]$;*
- *otherwise, $I_v(l,r) = \big\{x \in [l,r] \mid \alpha_v(x) \geq -c_v^*(l,r)\big\}$, where $c_v^*(l,r) > 0$ is obtained from the binding constraint $\int_{I_v(l,r)} \alpha_v(x) d\mu_0(x) = 0$.*

Two special cases of the auxiliary problem will be useful in further analysis. Firstly, there is the version with $l = -1$ and $r = 1$, which is unconstrained except by the voter's obedience constraint. Figure 5 illustrates the largest unconstrained interval of approved policies of the $left$ voter.

DEFINITION 1. *Voter v's* largest unconstrained interval of approved policies *is $I_v^{UC} = [a_v^{UC}, b_v^{UC}] := I_v(-1,1)$.*

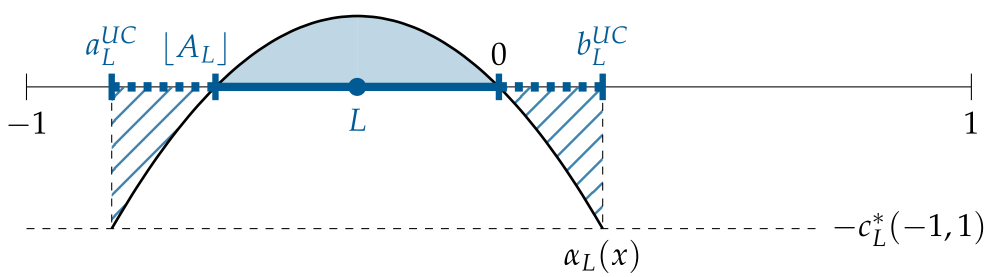

Figure 5. *The $left$ voter's largest unconstrained interval of approved policies. Under uniform prior, c_L^* is obtained from equating the solid area to the dashed area.*

The second relevant case is the voter's largest asymmetric interval of approved policies, which includes as many policies as possible on the opposite side of the status quo from the voter's approval set. Figure 6 illustrates this interval for the $left$ voter.

DEFINITION 2. *The $left$* voter's largest asymmetric interval of approved policies *is $I_L^{AS} = \big[\lfloor A_L \rfloor, b_L^{AS}\big] := I_L(\lfloor A_L \rfloor, 1)$. The right* voter's largest asymmetric interval of approved policies *is $I_R^{AS} = \big[a_R^{AS}, \lceil A_R \rceil\big] := I_R(-1, \lceil A_R \rceil)$.*

CONVINCING TWO VOTERS AT THE SAME TIME

We now consider how the challenger can solve Problem (1), that is, convince both voters at the same time as often as possible. One option is for him to send the $left$ (*right*)

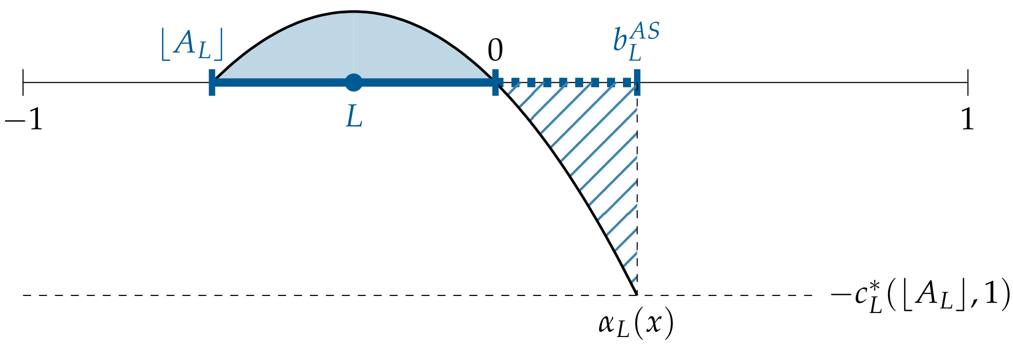

Figure 6. *The $left$ voter's largest asymmetric interval of approved policies. Under uniform prior, c_L^* is obtained from equating the solid area to the dashed area.*

voter a message containing as many policies to the right (left) of her approval set as possible. That is, he can let each voter's approved message be her largest asymmetric interval of approved policies. Figure 7 illustrates the outcome (I_L^{AS}, I_R^{AS}). As it turns out, (I_L^{AS}, I_R^{AS}) is often an equilibrium outcome.

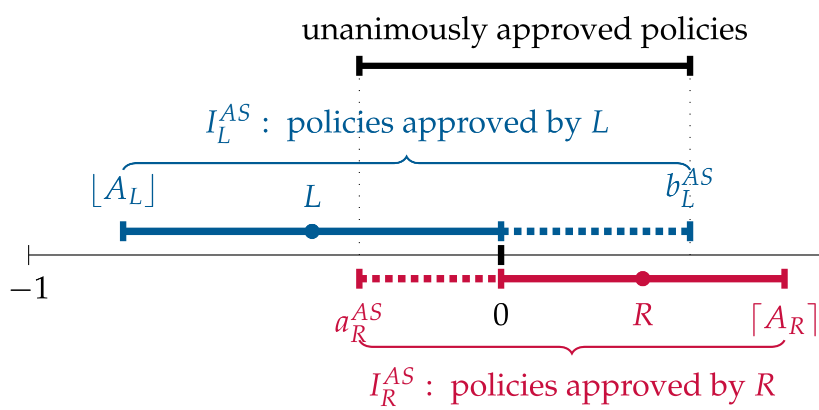

Figure 7. *The challenger convinces each voter using her largest asymmetric interval of approved policies. Unanimously approved policies of the challenger are indicated in black.*

One case when (I_L^{AS}, I_R^{AS}) may not be an equilibrium outcome is if $\lfloor A_L \rfloor > -1$ and $\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x) > 0$, which jointly imply that $a_R^{AS} < \lfloor A_L \rfloor$.[7] Intuitively, in this case, the *right* voter is so persuadable that her largest asymmetric interval of approved policies includes the $left$ voter's entire approval set, and then some. Note that

[7]Observe that $\phi(t) := \int_{t}^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x)$ is strictly increasing in $t < 0$, since $\frac{\partial \phi(t)}{\partial t} = -\alpha_R(t)\mu_0(t) > 0$. Thus, $\phi(\lfloor A_L \rfloor) > 0 = \phi(a_L^{AS})$ is possible if and only if $\lfloor A_L \rfloor > a_L^{AS}$.

$\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_v(x) d\mu_0(x) > 0$ cannot hold for both $v = L$ and $v = R$ at the same time.[8] To simplify notation, I let $\rho_v(L, R) := \int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_v(x) d\mu_0(x)$. For ease of exposition, I characterize only the case when $\rho_R(L, R) > 0$; the case when $\rho_L(L, R) > 0$ is symmetric.

PROPOSITION 2. *Suppose that the $left$ voter is not significantly more persuadable than the right voter, that is, $\rho_L(L, R) \leq 0$. Then, almost surely, the right voter's equilibrium approved message is her largest asymmetric interval of approved policies; that is, $\overline{W}_R = I_R^{AS}$. Furthermore, almost surely, the following hold:*

(1) If $\rho_R(L, R) \leq 0$, then
- *the left voter's equilibrium approved message is her largest asymmetric interval of approved policies; that is, $\overline{W}_L = I_L^{AS}$;*
- *the equilibrium set of unanimously approved policies is $\overline{W} = [a_R^{AS}, b_L^{AS}]$.*

(2) If $\rho_R(L, R) > 0$, then
- *the left voter's equilibrium approved message is the largest interval of approved policies constrained from the left by a_R^{AS}; that is, $\overline{W}_L = I_L(a_R^{AS}, 1)$;*
- *the equilibrium set of unanimously approved policies is $\overline{W} = \overline{W}_L$.*

For this and the following results, I outline the proof in the main text and provide the formal arguments in the appendix.

The idea of the proof of Proposition 2 is as follows. Since the *right* voter is relatively more persuadable, let us add as many left policies to her message as possible. That is, let $\overline{W}_R = I_R^{AS} = [a_R^{AS}, \lceil A_R \rceil]$. Now, Proposition 2 states that the equilibrium depends on how a_R^{AS} is related to $\lfloor A_L \rfloor$. Specifically, if $a_R^{AS} < \lfloor A_L \rfloor$, then we are in Case 2 of the proposition; otherwise, we are in Case 1.

Let us start by considering the lower values of a_R^{AS} (Case 2). Note that since $\rho_L(L, R) \leq 0$, the $left$ voter can never be persuaded by policies to the right of the *right* voter's approval set, i.e. $b_L^{AS} \leq \lceil A_R \rceil$. Suppose first that the *right* voter is so persuadable that she is willing to approve *all* left policies, i.e. $a_R^{AS} = -1$. In this case, (I_L^{UC}, I_R^{AS}) solves Problem (1), and the set of unanimously approved policies, which is essentially

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[8] To see this, use Lemma 2 for the prior belief truncated to $[\lfloor A_L \rfloor, \lceil A_R \rceil]$ to conclude that, for each $v \in \{L, R\}$, $\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_v d\mu_0(x) \geq 0$ implies $\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_{v'} d\mu_0(x) < 0$ for $v' \in \{L, R\} \setminus \{v\}$.

determined by the $left$ voter, is I_L^{UC}.[9] By construction, there is no way to increase the objective (the challenger's odds of winning) beyond $\mu_0(I_L^{UC})$ while still satisfying the $left$ voter's constraint. The same argument applies for every $a_R^{AS} \in [-1, a_L^{UC}]$. This case is illustrated in the left panel of Figure 8.

Next, suppose that $a_L^{UC} < a_R^{AS} < \lfloor A_L \rfloor$ (Case 2, continued). Now, (I_L^{UC}, I_R^{AS}) is no longer optimal: the challenger does not persuade the *right* voter with policies in $[a_L^{UC}, a_R^{AS})$, yet "wastes" the constraint of the $left$ voter on them. Instead, the challenger should select the $left$ voter's message out of $[a_R^{AS}, 1]$, since the *right* voter rejects the policies outside of that interval anyway. The proposed solution is therefore $\big(I_L(a_R^{AS}, 1), I_R^{AS}\big)$, and the set of unanimously approved policies is $I_L(a_R^{AS}, 1)$.[10] The challenger cannot increase his objective beyond $\mu_0\big(I_L(a_R^{AS}, 1)\big)$: this would require unanimous approval of policies to the left of a_R^{AS}, which are strictly more expensive in terms of the *right* voter's constraint than those that she already approves. Therefore, the proposed solution is optimal. This case is illustrated in the right panel of Figure 8.

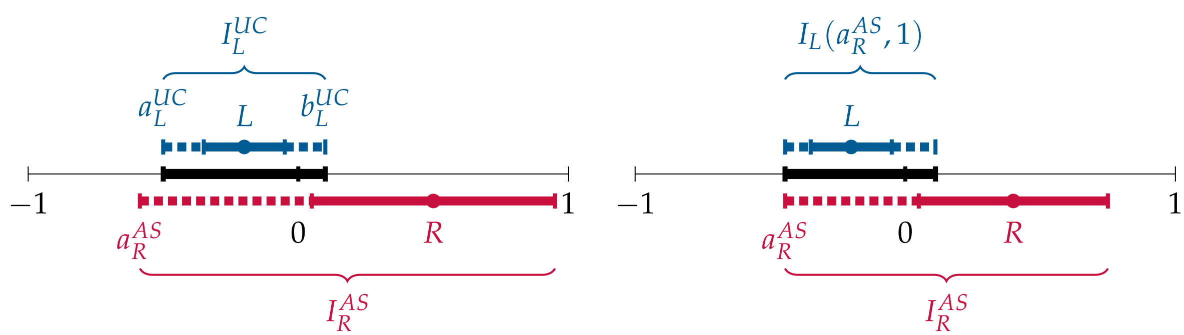

Figure 8. *Equilibrium approved messages when the right voter is significantly more persuadable than the $left$ voter.*

The last case we need to consider is when $\lfloor A_L \rfloor \leq a_R^{AS} < 0$ (Case 1 of the proposition). It remains to show that the proposed solution $(\overline{W}_L, \overline{W}_R) = (I_L^{AS}, I_R^{AS})$ with the set of unanimously approved policies $\overline{W} = [a_R^{AS}, b_L^{AS}]$ maximizes the objective of Problem (1). Indeed, any alternative set of unanimously approved policies would include policies to the left of a_R^{AS} or to the right of b_L^{AS}, which are more expensive in terms of the *right* or the $left$ voter's constraint, respectively.

Note that the obedience constraint of the relatively less persuadable ($left$) voter

[9]Since there is no constraint from the *right* voter, we can apply Lemma 3 to conclude that $\overline{W}_L = I_L^{UC}$. The set of unanimously approved policies is then $\overline{W}_L \cap \overline{W}_R = \overline{W}_L$ since $\lfloor \overline{W}_L \rfloor = a_R^{AS} \leq a_L^{UC} = \lfloor \overline{W}_R \rfloor$ and $\lceil \overline{W}_L \rceil = b_L^{UC} \leq b_L^{AS} \leq \lceil A_R \rceil = \lceil \overline{W}_R \rceil$.

[10]This is because $\overline{W}_L \cap \overline{W}_R = \overline{W}_L$, since $\lfloor \overline{W}_L \rfloor = a_R^{AS} = \lfloor \overline{W}_R \rfloor$ and $\lceil \overline{W}_L \rceil \leq b_L^{AS} \leq \lceil A_R \rceil = \lceil \overline{W}_R \rceil$.

always binds. The *right* voter's constraint also binds unless $a_R^{AS} < a_L^{UC}$. Finding the equilibrium approved messages is not computationally difficult; it requires solving at most two auxiliary optimization problems. In the following example, I calculate the equilibrium illustrated in Figure 7.

EXAMPLE 1 (UNIFORM PRIOR, $L = -0.4$, $R = 0.3$.). In this example, $A_L = [-0.8, 0]$ and $A_R = [0, 0.6]$. First, we check the relative persuadability of each voter by calculating $\int_{\lfloor A_L \rfloor}^{\lceil A_R \rceil} \alpha_v(x)dx$ for each $v \in \{L, R\}$. Both of these values are negative, so it remains to calculate b_L^{AS} and a_R^{AS}.

To find b_L^{AS}, solve $\int_{\lfloor A_L \rfloor}^{b_L^{AS}} \alpha_L(x)dx = 0$. Plugging in $\alpha_L(x) = -x^2 + 2Lx$, we arrive at the following equation: $(b_L^{AS})^3 - 3L \cdot (b_L^{AS})^2 = -4L^3$. It is not hard to check that the unique solution is $b_L^{AS} = -L = 0.4$, so that $\overline{W}_L = [-0.8, 0.4]$.[11] Similarly, we find that $a_R^{AS} = -R$ and $\overline{W}_R = [-0.3, 0.6]$.

Recall that one way to implement the equilibrium outcome is by pooling all policies in $\overline{W}_v$ into one message $\overline{W}_v$ that convinces voter $v \in \{L, R\}$. In this example, $\overline{W}_L = [-0.8, 0.4]$, meaning that the challenger tells the *left* voter that his policy is not ultra-left and not moderately to ultra-right, but does not clarify any further. Furthermore, the message $[-0.8, 0.4]$ averages to -0.2, which is to the left of the status quo; thus, the *left* voter expects that the challenger's policy is aligned with her preferences.

Both voters approve and the challenger wins if $x \in \overline{W}_L \cap \overline{W}_R = [-0.3, 0.4]$, that is, if his policy is sufficiently moderate. His odds of winning, calculated as the length of the interval of winning policies (0.7) relative to the length of the policy space (2), equal 0.35. We conclude that targeted advertising allows the challenger to improve his odds of winning from 0% to 35%.

COMPARATIVE STATICS

Next, let us analyze what happens when the voters become more extreme.

DEFINITION 3.

- *The left voter* becomes more extreme *if L strictly decreases; the right voter* becomes more extreme *if R strictly increases.*
- *The baseline electorate* $\{L, R\}$ becomes more polarized *if R increases and L decreases, with one of the changes being strict.*

[11] Note that under the uniform prior, the solution $b_L^{AS} = -L$ is applicable for any $L \in [0.5, 0)$ and any quadratic net payoff from approval $\alpha_v(x) = -d(v) \cdot (x^2 - 2vx)$ with $d(v) > 0$.

Note that a larger distance between the voters does not necessarily imply greater polarization. Polarization increases only if one voter becomes more extreme while the other either stays fixed or also becomes more extreme (in the opposite direction).

Observe that when a voter becomes more extreme, she becomes less satisfied with the status quo, and this makes her more persuadable. While there are multiple ways to formalize this observation, for now we will focus on how becoming more extreme affects a voter's largest asymmetric interval of approved policies.[12] Lemma 4 states that a more extreme voter is persuadable by a wider range of policies, and Figure 9 illustrates the argument for the *right* voter.

LEMMA 4. *As a voter becomes more extreme, her largest asymmetric interval of approved policies expands. Specifically,*

- *if* $L' < L$, *then* $\left[\lfloor A_{L'} \rfloor, b_{L'}^{AS}\right] \supseteq \left[\lfloor A_L \rfloor, b_L^{AS}\right]$, *with* $\lfloor A_{L'} \rfloor \leq \lfloor A_L \rfloor$ *and* $b_{L'}^{AS} \geq b_L^{AS}$; *the latter inequality is strict unless* $b_L^{AS} = 1$;
- *if* $R' > R$, *then* $\left[a_{R'}^{AS}, \lceil A_{R'} \rceil\right] \supseteq \left[a_R^{AS}, \lceil A_R \rceil\right]$, *with* $a_{R'}^{AS} \leq a_R^{AS}$ *and* $\lceil A_{R'} \rceil \geq \lceil A_R \rceil$; *the former inequality is strict unless* $a_R^{AS} = -1$.

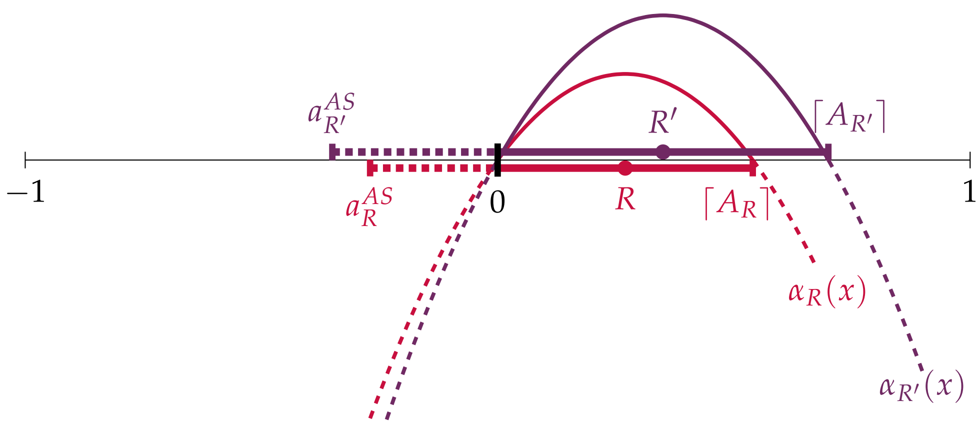

Figure 9. *As the right voter becomes more extreme (i.e., her bliss point increases from* R *to* R'*), she becomes persuadable by a wider range of policies: her largest asymmetric interval of approved policies* $\left[a_R^{AS}, \lceil A_R \rceil\right]$ *expands.*

The intuition behind the proof of Lemma 4 is as follows. When the *right* voter's bliss point increases from R to R', two effects occur. On the one hand, her approval set expands, and the expected value of her net payoff from approval over her approval set,

[12] I provide alternative but related measures of the "higher persuadability" of more extreme voters in Section 4, Lemma 6.

$\int_0^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x)$, strictly increases.[13] On the other hand, due to the voter's aversion to risk, her net payoff from approval to the left of her approval set strictly decreases. These two effects work in opposite directions, meaning that a_R^{AS} decreases if the former effect dominates and increases if the latter does. However, it turns out that the quadratic utility is not concave enough for the latter effect ever to dominate. Note that Lemma 4 is the first result of this paper that relies on the quadratic utility of the voters; all previous results require risk aversion only.

Now, consider a baseline electorate that satisfies the assumption of Proposition 2; that is, the $left$ voter is not significantly more persuadable than the *right* voter. Proposition 3 describes what happens to the equilibrium approved messages as the *right* voter becomes more extreme. Figure 10 illustrates the result.[14]

PROPOSITION 3. *Suppose that the $left$ voter is not significantly more persuadable than the right voter, that is, $\rho_L(L, R) \leq 0$. Then, as the right voter becomes more extreme,*

- *the challenger's odds of winning increase;*
- *the equilibrium set of unanimously approved policies shifts to the left.*

Proof. Proposition 3 compares the equilibrium outcomes of two baseline elections, with the $left$ voter's bliss point fixed at L and the *right* voter's bliss point increasing from R to R'. Assume that $a_R^{AS} > a_L^{UC}$, or else no changes will take place.[15] Let $(\overline{W}_L, \overline{W}_R)$ (respectively, $(\overline{W}'_L, \overline{W}'_R)$) be the equilibrium outcome when the *right* voter's bliss point is R (respectively, R'). Let $\overline{W} = \overline{W}_L \cap \overline{W}_L$ and $\overline{W}' = \overline{W}'_L \cap \overline{W}'_R$ be the equilibrium sets of unanimously approved policies before and after the change. Note that by Lemma 4, the *right* voter's constraint is looser after the change, which immediately implies that the change can only increase the value of the objective (the challenger's odds of winning). Furthermore, increasing R decreases the left endpoint a_R^{AS} of the *right* voter's largest interval of approved policies (strictly so, unless $a_R^{AS} = -1$). From Proposition 2, a_R^{AS} is also the left endpoint of the set of unanimously approved policies. It remains to prove

[13] If $2R \geq 1$, then the approval set itself remains the same, unlike in Figure 9. However, $\int_0^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x)$ strictly increases, because $\alpha_{R'}(x) - \alpha_R(x) = 2x(R' - R) > 0$ if $x > 0$.

[14] Figure 10 presents the numerical solution for the uniform prior, with bliss point $L = -0.15$ for the $left$ voter and successive bliss points $R_1 = 0.15$, $R_2 = 0.25$, $R_3 = 0.35$, and $R_4 = 0.50$ (top to bottom) for the *right* voter. The sets of unanimously approved policies (in black) are $[-0.15, 0.15]$, $[-0.25, 0.15]$, $[-0.35, 0.1436]$, and $[-0.4098, 0.1098]$, respectively.

[15] If $a_R^{AS} \leq a_L^{UC}$, then $\overline{W} = \overline{W}_L = I_L^{UC}$. In this case, loosening the *right* voter's constraint does not change the equilibrium set of unanimously approved policies, because the objective cannot be improved from $\mu_0(I_L^{UC})$ without violating the $left$ voter's constraint, which does not change.

that the right endpoint of $\overline{W}$ also decreases. The general idea is that this endpoint cannot shift to the right, as it is determined by the $left$ voter's constraint, which does not change when the *right* voter becomes more extreme. The remainder of this section describes the conditions under which the decrease is strict.

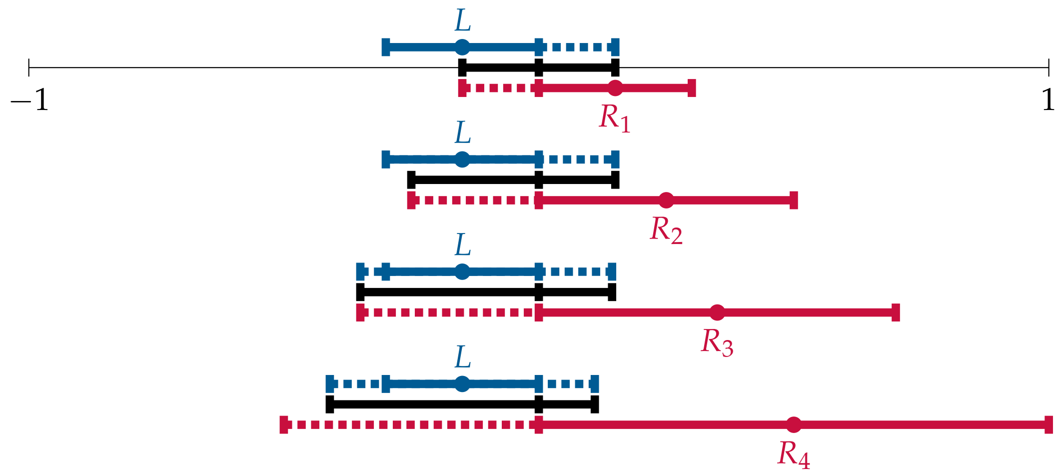

Figure 10. *Equilibrium approved messages as the right voter becomes more extreme (top to bottom). As the right voter becomes persuadable by a wider range of policies (in red), the set of unanimously approved policies (in black) shifts to the left.*

Recall that Proposition 2 is broken into two cases: one in which both voters are moderately persuadable and one in which the *right* voter is significantly more persuadable. Also, recall that increasing R increases $\lceil A_R \rceil$ and decreases $\rho_L(L, R)$. That is, when the *right* voter becomes more extreme, the $left$ voter remains moderately persuadable, while $\rho_R(L, R)$ increases, making the *right* voter more persuadable. We have three cases to consider.

Case (i): the *right* voter is moderately persuadable both before and after the change; that is, $\rho_R(L, R) < \rho_{R'}(L, R') \leq 0$. Then, applying Case 1 of Proposition 2, we get $\overline{W}_v = I_v^{AS}$ for each $v \in \{L, R\}$ and $\overline{W}'_v = I_v^{AS}$ for each $v \in \{L, R'\}$. In particular, the right boundary of the set of unanimously approved policies is fixed at b_L^{AS} both before and after the change. In Figure 10, Case (i) can be seen in the transition from the first to the second line (when the *right* voter's bliss point increases from R_1 to R_2).

Case (ii): the *right* voter is moderately persuadable before the change and significantly more persuadable than the $left$ voter after the change; that is, $\rho_R(L, R) \leq 0 < \rho_{R'}(L, R')$. Then before the change, applying Case 1 of Proposition 2, we get $\overline{W}_v = I_v^{AS}$ for each $v \in \{L, R\}$, with $\overline{W} = [a_R^{AS}, b_L^{AS}]$. After the change, applying Case 2 of Proposition 2, we get $\overline{W}'_R = I_{R'}^{AS}$, $\overline{W}'_L = I_L(a_{R'}^{AS}, 1) = \left[\max\{a_{R'}^{AS}, a_L^{UC}\}, b'_L\right]$, with $\overline{W}' = \overline{W}'_L$.

Now, by the $left$ voter's obedience constraint,

$$\int\limits_{I_L^{AS}} \alpha_L(x) d\mu_0(x) = \int\limits_{\lfloor A_L \rfloor}^{b_L^{AS}} \alpha_L(x) d\mu_0(x) = 0$$

$$\leq \int\limits_{I_L(a_{R'}^{AS},1)} \alpha_L(x) d\mu_0(x) = \int\limits_{\max\{a_{R'}^{AS}, a_L^{UC}\}}^{b'_L} \alpha_L(x) d\mu_0(x).$$

Since $\lfloor A_L \rfloor > \max\{a_{R'}^{AS}, a_L^{UC}\}$, we must have $b_L^{AS} > b'_L$, meaning that both endpoints of the set of unanimously approved policies strictly decrease. In Figure 10, Case (ii) can be seen in the transition from the second to the third line (when the right voter's bliss point increases from R_2 to R_3).

Case (iii): the $right$ voter is significantly more persuadable than the $left$ voter both before and after the change; that is, $0 < \rho_R(L, R) < \rho_{R'}(L, R')$. Then, applying Case 2 of Proposition 2 before and after the change, we conclude that $\overline{W} = \overline{W}_L = I_L(a_R^{AS}, 1) = [a_R^{AS}, b_L]$ and $\overline{W}' = \overline{W}'_L = I_L(a_{R'}^{AS}, 1) = [\max\{a_{R'}^{AS}, a_L^{UC}\}, b'_L]$. Once again, from the $left$ voter's obedience constraint, $a_R^{AS} > \max\{a_{R'}^{AS}, a_L^{UC}\} \implies b_L > b'_L$. In Figure 10, *Case (iii)* can be seen in the transition from the third to the fourth line (when the right voter's bliss point increases from R_3 to R_4). $\square$

Welfare

Consider an outcome in which a voter with bliss point $v \in X$ approves some set of policies $W_v \subseteq X$. When voter v approves, her payoff is $-(v-x)^2$, and when she rejects, it is $-v^2$. Hence, her ex-ante utility is $-\mathbb{E}_{\mu_0}\left[\mathbb{1}(x \in W_v) \cdot (v-x)^2 + \mathbb{1}(x \in W_v^c) \cdot v^2\right]$. If we add v^2 to that expression we get $\int\limits_{W_v} \alpha_v(x) d\mu_0(x)$. I use the latter object as a measure of voter v's welfare.

Definition 4. *Consider a voter with bliss point $v \in X \setminus \{0\}$ and the approved message W_v. Then voter v's* welfare *is $\int\limits_{W_v} \alpha_v(x) d\mu_0(x)$.*

The table below compares voter welfare and the challenger's odds of winning across three communication protocols. Firstly, there is the first-best *full disclosure* outcome, (A_L, A_R), which delivers the complete information payoff for all players.[16] Secondly, there is the *public disclosure* outcome (W_L^{PD}, W_R^{PD}), which is the outcome of the baseline model with the additional restriction that the challenger must always send the same (verifiable) message to both voters.[17] Thirdly, there is the *targeted advertising* outcome $(\overline{W}_L, \overline{W}_R)$. In the latter, recall from the discussion after Proposition 2 that the obedience constraints $\int_{\overline{W}_v} \alpha_v(x) d\mu_0(x) \geq 0$ of both voters bind unless one voter is very extreme/persuadable, in which case her constraint may be loose.

	v's welfare	challenger's odds of winning
full disclosure	$\int_{A_v} \alpha_v(x) d\mu_0(x) > 0$	0
public disclosure	$\int_{W_v^{PD}} \alpha_v(x) d\mu_0(x) \geq 0$	0
targeted advertising	$\int_{\overline{W}_v} \alpha_v(x) d\mu_0(x) = 0$	$\mu_0(\overline{W}_L \cap \overline{W}_R) > 0$

Notice that targeted advertising maximizes the challenger's odds of winning, at the expense of minimizing voter welfare.

4. Large Elections

This section generalizes the baseline model to electorates with more than two voters and any social choice rule. A *large election* $(\mathcal{V}, scf)$ consists of

- a finite set $\mathcal{V} = \{v_1, \ldots, v_n\} \subseteq X$ of voters' bliss points;
- a social choice function $scf : 2^{\mathcal{V}} \to \{0, 1\}$ that indicates, for each subset of voters $V \subseteq \mathcal{V}$ whether the challenger wins (1) or loses (0) the election if every voter in V approves.

Let $\mathcal{D} := \{D \subseteq \mathcal{V} \mid scf(D) = 1\}$ denote the set of all *decisive coalitions*. That is, to win the election, the challenger must convince all voters in some decisive coalition $D \in$

[16]Under full disclosure, voter v's approved message is A_v. Each voter learns whether the challenger's policy is in her approval set, and thus acts as if under complete information. Note that full disclosure is the sender-worst equilibrium outcome of the baseline model.

[17]We do not cover the public disclosure outcome in detail in this paper; however, note that it must satisfy each voter v's obedience constraint, $\int_{W_v^{PD}} \alpha_v(x) d\mu_0(x) \geq 0$, or else voter v would not approve these policies. Furthermore, the challenger's odds of winning are $\mu_0(W_L^{PD} \cap W_R^{PD}) = 0$, because the election is unwinnable.

$\mathcal{D}$. The first result characterizes large elections that are unwinnable without targeted advertising.

LEMMA 5. *The following statements about a large election* $(\mathcal{V}, scf)$ *are equivalent:*

(1) *No decisive coalition consists of voters with only strictly negative or only strictly positive bliss points.*
(2) *Under complete information, the status-quo candidate is almost surely socially preferred to the challenger; that is, for any decisive coalition* $D \in \mathcal{D}$*, the prior measure of the set* $\bigcap_{v \in D} A_v$ *is zero.*
(3) *Under any non-degenerate belief* $\mu \in \Delta X$*, the status-quo candidate is socially preferred to the challenger. That is, for any decisive coalition* $D \in \mathcal{D}$*, at least one voter* $v \in D$ *prefers to reject under belief* μ*.*

Lemma 5 generalizes the idea that left and right voters never simultaneously approve under complete information (Condition 2) or under common belief (Condition 3). Condition 1 provides a simple characterization of an unwinnable election in terms of the voters' locations relative to the status quo. Under a simple majority rule, Condition 1 describes a version of the median voter theorem.[18]

COROLLARY 1. *Under a simple majority rule, a large election is unwinnable for the challenger without targeted advertising if and only if the status quo is the bliss point of the median voter.*

Now, recall from Lemma 4 that more extreme voters have larger asymmetric intervals of approved policies. Relatedly, more extreme voters also prefer to approve whenever their less extreme counterparts (on the same side of the status quo) do.

LEMMA 6. *If voter* $v \in X \setminus \{0\}$ *prefers to approve under belief* $\mu \in \Delta X$*, then any voter* v' *who is more extreme than voter* v *also prefers to approve under belief* μ*.*

Thus, if the challenger can convince a given voter v whose bliss point is, say, to the left of the status quo, then he can convince every voter to the left of v using exactly the same strategy.[19]

DEFINITION 5. *Consider a decisive coalition* $D \in \mathcal{D}$*. The* left pivot *of* D *is* $L(D) := \max_{v \in D, v<0} v$*, and the* right pivot *of* D *is* $R(D) := \min_{v \in D, v>0} v$*.*

[18] Black (1948) states the median voter theorem as follows: "If X is a single-dimensional issue and all voters have single-peaked preferences defined over X, then the median position could not lose under majority rule."

[19] By "using exactly the same strategy" we mean that for every $x \in X$, the challenger sends the same message, with equal probability, to voter v and to every voter v' who is more extreme than v.

If a given decisive coalition D does not include any voters whose bliss point is exactly 0, then for the challenger to convince all voters in D, it is necessary and sufficient that he convinces the pivotal voters $L(D)$ and $R(D)$. This reduces the problem of winning the large election to that of winning the baseline election (with only two voters)—and we have already shown that every unwinnable baseline election can be won with targeted advertising. However, it may be the case that *every* decisive coalition in a large election includes a voter whose bliss point is 0; in this case, the election cannot be won with targeted advertising.[20] Thus, many, but not all, unwinnable large elections can be won with targeted advertising. The result below summarizes this observation.

PROPOSITION 4. *Consider an unwinnable large election (V, scf). In equilibrium, the challenger's ex-ante odds of winning are positive if and only if there exists a decisive coalition $D \in \mathcal{D}$ such that $0 \notin D$.*

Proof. The proof of Proposition 4 is straightforward: if every decisive coalition includes a voter with bliss point 0, then the challenger's odds of winning are 0 because his odds of convincing that voter are 0. Otherwise, we can select a decisive coalition $D \in \mathcal{D}$ that does not include a voter with bliss point 0, and solve the baseline election game for $L(D)$ and $R(D)$ as described in Proposition 2. □

To maximize his odds of winning, the challenger must search over all decisive coalitions that do not include voter 0. Proposition 3 reduces the domain of this optimization problem: for two decisive coalitions $D, D' \in \mathcal{D}$, if the baseline electorate $\big(L(D), R(D)\big)$ is more polarized than $\big(L(D'), R(D')\big)$, then the challenger's odds of winning are higher in the former baseline election. Beyond that, the challenger must solve Problem (1) for the pair $\big(L(D), R(D)\big)$ of pivotal voters in every decisive coalition $D \in \mathcal{D}$ such that $0 \notin D$.

EXAMPLE 2. Let $V = \{-0.5, -0.2, 0, 0.2, 0.3\}$ and $scf(V) = \mathbb{1}(|V| \geq 3)$ (that is, voting is by simple majority: to win, the challenger must convince at least 3 voters). By Corollary 1, this large election is unwinnable for the challenger without targeted advertising, because the median voter is 0. The (minimal) decisive coalitions and their pivotal voters are given in the table below.

Since -0.5 is more extreme than -0.2 and 0.3 is more extreme than 0.2, the challenger's odds of winning are higher if he targets coalition D_2 or D_3 than if he targets D_1 or D_4. Whether his odds are maximized by targeting D_2 or D_3 depends on the

[20] If $v = 0$, then $\alpha_v = -x^2$ and $A_v = \{0\}$. If a belief $\mu \in \Delta X$ convinces this voter to approve (i.e., if $\int \alpha_v(x) d\mu(x) \geq 0$), then $\mu(0) = 1$. Thus, the ex-ante probability of convincing such a voter is 0.

coalition	left pivot	right pivot
$D_1 = \{-0.5, -0.2, 0.2\}$	-0.2	0.2
$D_2 = \{-0.5, -0.2, 0.3\}$	-0.2	0.3
$D_3 = \{-0.5, 0.2, 0.3\}$	-0.5	0.2
$D_4 = \{-0.2, 0.2, 0.3\}$	-0.2	0.2

shape of the prior distribution. For example, if the prior distribution is heavily skewed to the right of 0, then targeting D_2 may maximize his odds, whereas if the prior distribution is symmetric about 0, then targeting D_3 will maximize his odds.[21] In particular, under the uniform prior, if we solve the baseline election for $L = -0.5$ and $R = 0.2$ (the pivotal voters for D_3), we get $\overline{W}_L = [-1, 0.5]$, $\overline{W}_R = [-0.179, 0.5]$, and $\overline{W} = [-0.179, 0.5]$. Thus, in his preferred equilibrium, the challenger sends the message $\overline{W}_L = [-1, 0.5]$ to all voters left of the status quo whenever $x \in \overline{W}_L$, and he sends the message $\overline{W}_R = [-0.179, 0.5]$ to all voters right of the status quo whenever $x \in \overline{W}_R$; otherwise, he says nothing. On the equilibrium path, the challenger convinces voters -0.5, 0.2, and 0.3 whenever his policy is $x \in [-0.179, 0.5]$. His ex-ante odds of winning are therefore 33.96%.

5. Conclusion

In an election where pivotal voters are on opposite sides of the status quo, a challenger cannot win without targeted advertising, because no public message will simultaneously convince voters on both sides of the status quo to approve the challenger's policy. In this paper I show that targeted advertising makes it possible for the challenger to win any such election. He does so by sending each voter a private message to convince her that his policy is a sufficient improvement over the status quo. He wins the election if his policy is sufficiently moderate. Furthermore, his odds of winning increase when voters become more extreme.

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[21] If the prior is symmetric about 0, then left and right voters are symmetric. Consequently, the challenger's odds of winning with the baseline electorate $(-0.2, 0.3)$ (which corresponds to D_2) are the same as his odds with the baseline electorate $(-0.3, 0.2)$. But the latter is less polarized than the baseline electorate $(-0.5, 0.2)$, which corresponds to D_3. Therefore, the odds of winning are higher with D_3 than with D_2.

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APPENDIX: OMITTED PROOFS

PROOF OF PROPOSITION 2

The case where $a_R^{AS} \leq a_L^{UC}$ is proved in the main text.

Suppose that $a_L^{UC} < a_R^{AS} < \lfloor A_L \rfloor$. Let $\overline{W}_L = I_L(a_R^{AS}, 1)$ and $\overline{W}_R = I_R^{AS}$. It remains to show that the challenger's odds of winning cannot be higher than $\mu_0(\overline{W}_L)$ for any other pair (W_L, W_R) that satisfies both voters' constraints. Indeed, any W_L such that $\mu_0(W_L) > \mu_0(\overline{W}_L)$ satisfies the constraint of the $left$ voter has to include a positive-measure set $Y \subseteq [-1, a_R^{AS}]$. However, every policy $y \in Y$ is more expensive in terms of the *right* voter's constraint than any policy $x \in [a_R^{AS}, 0]$ (because $\alpha_R(y) < \alpha_R(x)$). Consequently, including Y in the set of unanimously approved policies increases the challenger's odds of winning by $\mu_0(Y)$ but decreases it by more than $\mu_0(Y)$. Hence, $(I_L(a_R^{AS}, 1), I_R^{AS})$ solves Problem (1) if $a_L^{UC} < a_R^{AS} < \lfloor A_L \rfloor$, or, more generally, whenever $a_R^{AS} < \lfloor A_L \rfloor$, since if $a_R^{AS} \leq a_L^{UC}$ then $I_L(a_R^{AS}, 1) = I_L^{UC}$.

The last case is where $\lfloor A_L \rfloor \leq a_R^{AS} \leq 0$. Here I show that the proposed solution $(\overline{W}_L, \overline{W}_R) = (I_L^{AS}, I_R^{AS})$ with the set of unanimously approved policies $\overline{W} = [a_R^{AS}, b_L^{AS}]$ maximizes the objective of Problem (1). Consider another solution (W_L, W_R) with the set of unanimously approved policies $W = W_L \cap W_R$. First, observe that W cannot lie entirely to the left of $\overline{W}$, i.e. the set $W \cap [b_L^{AS}, 1]$ must have positive prior measure. Otherwise, the *right* voter's constraint would have to be spent on policies farther left than a_R^{AS}, which decreases the objective. Specifically, from the *right* voter's constraint, $\int_{\overline{W}_R \cap W_R^c} \alpha_R(x) d\mu_0(x) \leq \int_{W_R \cap [-1, a_R^{AS}]} \alpha_R(x) d\mu_0(x)$. Also, $\alpha_R(\overline{x}) > \alpha_R(x)$ for all $\overline{x} \in \overline{W}_R$ and $x \in [-1, a_R^{AS}]$, which implies $\mu_0(\overline{W}_R \cap W_R^c) > \mu_0(W_R \cap [-1, a_R^{AS}])$. Finally, we have $\overline{W}_R \cap W_R^c = \overline{W} \setminus W$ and $W_R \cap [-1, a_R^{AS}] \supseteq W \setminus \overline{W}$. It follows that $\mu_0(\overline{W}) > \mu_0(W)$. By a symmetric argument for the $left$ voter, W cannot lie entirely to the right of $\overline{W}$, either; that is, the set $W \cap [-1, a_L^{AS}]$ must have positive prior measure.

Next, observe that sets $W \cap \big[-1, 0\big]$ and $W \cap \big[0, 1\big]$ must be intervals that end at 0 and start at 0, respectively. Otherwise, W could be improved upon. For example, if

$W \cap \big[-1,0\big] \neq [a,0]$ for some $a \geq -1$, then there exist two sets $Y = [y_1, y_2] \subseteq W$ and $Z = [z_1, z_2] \subseteq W^c$ such that $-1 \leq y_1 < y_2 \leq z_1 < z_2 \leq 0$ and $\mu_0(Y) = \mu_0(Z)$. Then, for every $y \in Y$ and $z \in Z$, we have $\alpha_R(y) < \alpha_R(z) < 0$ and either $\alpha_L(y) < \alpha_L(z)$ or $\alpha_L(y) > \alpha_L(z) \geq 0$. Let $\widehat{W}_L = \big(W_L \setminus (Y \cap A_L^c)\big) \cup Z$ and $\widehat{W}_R = \big(W_R \setminus Y\big) \cup Z$. By construction, $(\widehat{W}_L, \widehat{W}_R)$ satisfies both constraints and maintains the objective at $\mu_0(W)$. However, since the *right* voter's constraint is now loose, we can further increase the objective, a contradiction.

PROOF OF LEMMA 4

I prove this lemma for the case in which the *right* voter's bliss point increases from R to R'. The case of the *left* voter is symmetric. To simplify the notation, I let $a := a_R^{AS}$ and $a' := a_{R'}^{AS}$.

First, note that $R' > R$ implies $\lceil A_{R'} \rceil \geq \lceil A_R \rceil$, since $\lceil A_{R'} \rceil = \min\{1, 2R'\} \geq \min\{1, 2R\} = \lceil A_R \rceil$. Thus, $A_{R'} \supseteq A_R$.

Next, observe that if $a > -1$, then the constraint for voter R binds, and $\int\limits_a^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x) = 0$. If the constraint for voter R' does not bind, then $a' = -1 < a$. If it does bind, then

$$\int\limits_{a'}^{\lceil A_{R'} \rceil} \alpha_{R'}(x) d\mu_0(x) = 0 \iff \int\limits_{a'}^{a} \underbrace{\alpha_{R'}(x)}_{<0 \text{ since } x<0} d\mu_0(x) + \int\limits_{a}^{\lceil A_R \rceil} \big(\alpha_{R'}(x) - \alpha_R(x)\big) d\mu_0(x) + \int\limits_{\lceil A_R \rceil}^{\lceil A_{R'} \rceil} \underbrace{\alpha_{R'}(x)}_{\geq 0 \text{ since } x \in A_{R'}} d\mu_0(x) = 0,$$

where I have subtracted $\int\limits_a^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x) = 0$ from the first term in the second line.

Now, if $\int\limits_a^{\lceil A_R \rceil} \big(\alpha_{R'}(x) - \alpha_R(x)\big) d\mu_0(x) \geq 0$, then $a' < a$. In what follows, I show that the former inequality holds for the quadratic $\alpha_R(x)$. Indeed, $\alpha_{R'}(x) - \alpha_R(x) = 2(R' - R)x$. Let $\overline{x}_R := \int\limits_a^{\lceil A_R \rceil} x d\mu_0(x)$. By Jensen's inequality for the concave $\alpha_R(x)$, $\alpha_R(\overline{x}_R) \geq \int\limits_a^{\lceil A_R \rceil} \alpha_R(x) d\mu_0(x) = 0$, and $\alpha_R(\overline{x}_R) \geq 0$ if and only if $\overline{x}_R \in \big[0, \lceil A_R \rceil\big]$. Therefore, $\int\limits_a^{\lceil A_R \rceil} 2(R' - R)x d\mu_0(x) \geq 0$ and $a' < a$.

Proof of Lemma 5

(1) $\Longrightarrow$ *(2)*,*(3)*: There are two cases to consider: Case 1, in which all decisive coalitions include 0, and Case 2, in which every decisive coalition includes both voters with strictly negative bliss points and voters with strictly positive bliss points.

In Case 1, $\bigcap_{v \in D} A_v = \{0\}$ for all $D \in \mathcal{D}$ because $A_0 = \{0\}$. Voter 0 approves under the belief $\mu \in \Delta X$ only if $\mu(0) = 1$, which means μ is a degenerate belief. Given any non-degenerate belief, she prefers to reject.

In Case 2, $\bigcap_{v \in D} A_v = \{0\}$ for all $D \in \mathcal{D}$ because each decisive coalition D contains a voter L with a strictly negative bliss point, as well as a voter R with a strictly positive bliss point, and for any such pair of voters we have $A_L \cap A_R = \{0\}$. Similarly, by Lemma 2, under a non-degenerate common belief, at most one of the voters L and R prefers to approve.

(2),*(3)* $\Longrightarrow$ *(1)*: We prove the contrapositive. If *(1)* is not true, then there exists a decisive coalition $D \in \mathcal{D}$ in which either all voters have strictly negative bliss points, or all voters have strictly positive bliss points. For that coalition (in which, say, all bliss points are strictly positive), let $\hat{v} := \min_{v \in D} v$ be the voter closest to the status quo. Then $\bigcap_{v \in D} A_v = A_{\hat{v}}$, because $A_v = \big[0, \min\{1, 2v\}\big]$ for all $v \in D$. Since $\hat{v} > 0$, the set $A_{\hat{v}}$ has positive prior measure. Now, let $\mu \in \Delta X$ be the prior belief μ_0 truncated to voter $\hat{v}$'s approval set $A_{\hat{v}}$. Note that μ is a non-degenerate belief, because μ_0 has no atoms and $A_{\hat{v}}$ has positive prior measure. Every voter's expected net payoff from approval is strictly positive under μ, because $\alpha_v(x) \geq 0$ for all $x \in \operatorname{supp} \mu = A_{\hat{v}}$ (the inequality is strict for all x in the interior of $A_{\hat{v}}$) and $v \in D$. Consequently, every voter in D prefers to approve.

Proof of Lemma 6

I prove this result for $R' > R > 0$. The case of $L' < L < 0$ is symmetric. Since voter R approves, we have

$$\begin{aligned}
\int \alpha_R(x) d\mu(x) &= \int (-x^2 + 2Rx) d\mu(x) \\
&= \int (-x^2 + 2R'x) d\mu(x) + 2(R - R') \int x d\mu(x) \\
&= \int \alpha_{R'}(x) d\mu(x) + 2(R - R') \mathbb{E}_\mu[x] \geq 0.
\end{aligned}$$

Since by Lemma 1 $\mathbb{E}_\mu[x] > 0$ whenever voter R approves, the last inequality implies that $\int \alpha_{R'}(x) d\mu(x) > 0$.