

Withholding Verifiable Information^{*}

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Abstract

We study a class of finite-action disclosure games where the sender’s preferences are state independent and the receiver’s optimal action depends only on the expected state. While receiver-preferred equilibria in these games involve full revelation, other equilibria are less well-understood. We show that any equilibrium payoff can be obtained with a disclosure strategy corresponding to a partition with a laminar structure that allows pooling nonadjacent states. In a sender-preferred equilibrium, such a structure balances inducing more sender-favorable actions and deterring deviations. Leveraging this insight, we identify conditions under which the sender does not benefit from commitment power and apply these results to study influencing voters and selling with quality disclosure.

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1 Introduction

When economists think of disclosure games, they usually think of unraveling (e.g., [Grossman, 1981](#); [Milgrom, 1981](#)). Specifically, if the sender can credibly prove that the state is highly favorable, he reveals it to induce a higher action from a receiver. Once the most favorable states are disclosed, slightly less favorable states must also be revealed to avoid being mistaken for worse ones, and this reasoning continues recursively until all states are revealed. This full revelation result, however, hinges on a crucial assumption: the receiver’s action space is sufficiently flexible, meaning that she can adjust her action continuously so that any marginal improvement in belief leads to a strictly higher action.

Nevertheless, this flexibility assumption often fails in settings where the receiver chooses among finitely many actions, such as a consumer deciding among a few products or product versions, an employer choosing among different position levels, or a policymaker deciding between a few policy alternatives. In such cases, the receiver cannot finely adjust her action in response to small changes in her belief. Previous studies (e.g., [Giovannoni and Seidmann, 2007](#); [Titova and Zhang, 2025](#)) show that this discreteness may prevent full unraveling and allow the sender to withhold some information, i.e., create scope for pooling. Yet little is known about exactly which states are pooled together in equilibrium, or about the full limits of what can be achieved through verifiable disclosure.

In this paper, we study a disclosure game in which the receiver’s preferred action is increasing in the expected state. The only essential difference from [Milgrom \(1981\)](#) is that the receiver’s action space is finite. We characterize the equilibrium payoff set, study the sender-preferred equilibrium payoff, and identify sufficient conditions under which the commitment payoff is achieved with verifiable disclosure. We illustrate our results with a motivating example.

Example. To illustrate our model and main results, consider a seller (he) promoting a product to a buyer (she) who chooses whether to buy nothing (action 1), buy the product (action 2), or buy the product bundled with an add-on (action 3). The players have a common prior that the product quality is uniformly distributed on $[0, 1]$. The buyer’s payoff depends on the posterior expectation of product quality ω and is such that she optimally buys the bundle if $\mathbb{E}[\omega] \in [\frac{3}{4}, 1]$, only the standalone product if $\mathbb{E}[\omega] \in [\frac{1}{2}, \frac{3}{4}]$, and nothing otherwise. The seller’s profit is 0 if he sells nothing, 1

if he sells the product with the add-on, and $p \in [0, 1]$ if he sells only the product. To persuade the buyer, the seller can disclose a piece of hard evidence about product quality after privately observing it. In particular, he can send a message corresponding to any nonempty closed subset of $[0, 1]$ containing the true quality ω . We will focus on partitional disclosure strategies associated with some (ordered) partition $\{B_1, B_2, B_3\}$ of $[0, 1]$ such that when the product quality is $\omega \in B_i$, the seller sends a message B_i . The interpretation is that the seller discloses which B_i the quality belongs to and recommends the buyer to take action i . Such a partition is an equilibrium partition if and only if it is *obedient* for the buyer so that she is willing to follow the recommendation, and *revelation-proof* for the seller so that he is almost never willing to deviate by revealing the true quality.

Before discussing the equilibria of the game, note that the seller's equilibrium payoff is bounded from above by his *commitment payoff* in this environment, that is, his maximal expected payoff in the case when he can commit to disclose information about ω using any experiment. The commitment problem can be seen as a relaxation of the problem of maximizing seller's payoff across equilibria because the former does not require the disclosure strategy to be revelation-proof. Therefore, if the commitment payoff is attainable in some equilibrium, this equilibrium must be a seller-preferred equilibrium of the game.

Suppose first that $p = 0$. Given the observation above, we start by identifying the commitment solution. Because $p = 0$, the action space is effectively binary and the seller is maximizing the probability of selling the bundle. Every commitment-optimal experiment corresponds to a partition given by $B_1 = [0, \frac{1}{2}]$, $B_2 = \emptyset$, $B_3 = [\frac{1}{2}, 1]$ so that the buyer is indifferent between buying and not buying the bundle following message B_3 . It is easy to check that this partition is revelation-proof, because almost no seller type in B_1 can induce any action higher than 1 by deviating. Therefore, there is a seller-preferred equilibrium that attains the commitment payoff.

Suppose next that $p = 0.5$. In this case, there are three non-trivial actions available to the buyer, and the seller is facing a trade-off between the likelihoods of selling the standalone product and selling the bundle. It turns out that the unique commitment-optimal partition is given by $B_1 = [0, \frac{1}{4}]$, $B_2 = [\frac{5}{16}, \frac{11}{16}]$, $B_3 = [\frac{1}{4}, \frac{5}{16}] \cup [\frac{11}{16}, 1]$. Note that in contrast to the case of $p = 0$, in this case the optimal partition is not monotone in the sense that not every element is an interval. Instead, it is associated with a class of *bi-pooling* distributions of posterior means which are known to be optimal in

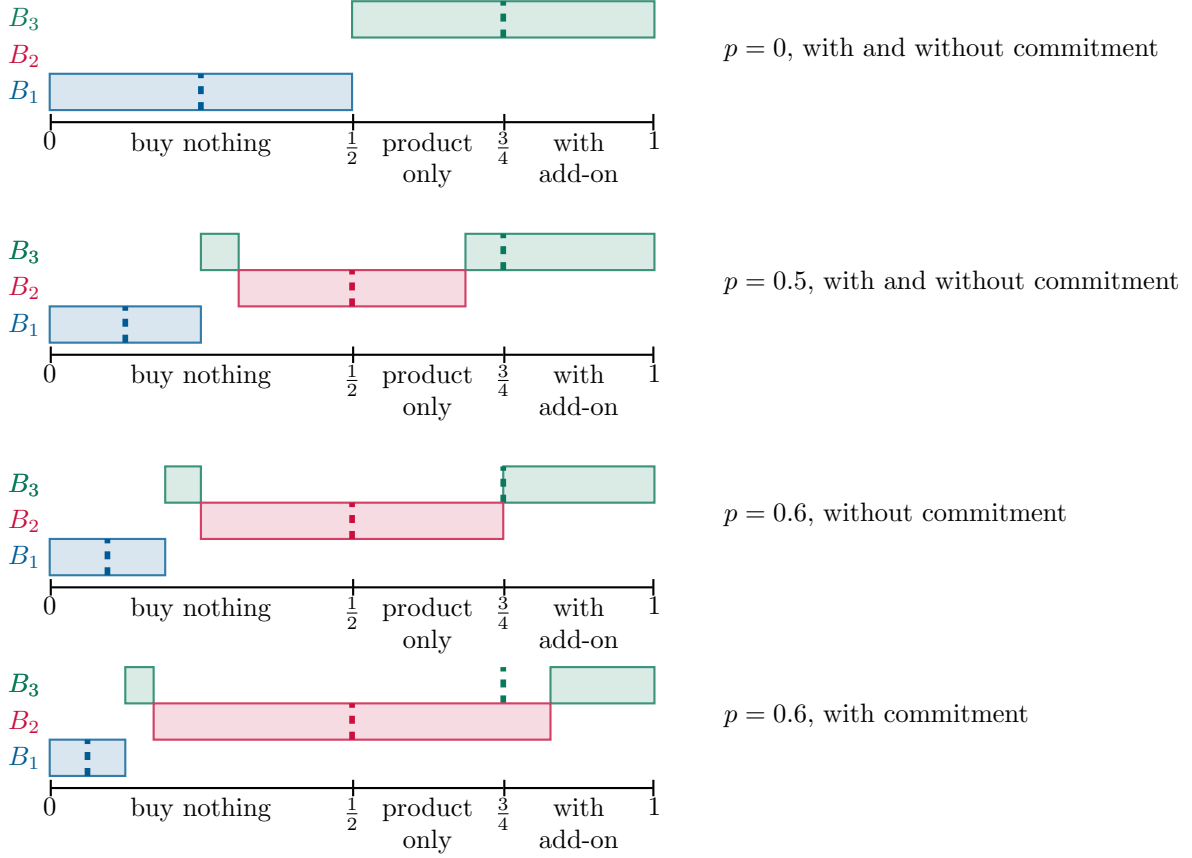


Figure 1. Commitment-optimal and sender-preferred equilibrium partitions in the example. The dashed lines are the (conditional) means of the partitional elements.

a general class of linear persuasion problems (Kleiner, Moldovanu, and Strack, 2021; Arieli, Babichenko, Smorodinsky, and Yamashita, 2023). Such bi-pooling partitions generalize monotone partitions by allowing pooling of non-adjacent types as follows. Each partitional element B_i can either be an interval, or consist of two intervals and “nest” a unique another interval B_j , in the sense that $\text{co}(B_i) \supseteq B_j$. To verify that this partition is revelation-proof, it is sufficient to check that almost no type in B_2 can deviate by revealing their type and convincing the buyer to buy the bundle. Indeed, since the highest type $\frac{11}{16}$ in B_2 is below the threshold $\frac{3}{4}$ of selling a bundle, this partition is an equilibrium partition.

Suppose next that $p = 0.6$. The unique commitment-optimal partition can be shown to be given by a bi-pooling partition $B_1 = [0, \frac{1}{8}]$, $B_2 = [\frac{11}{64}, \frac{53}{64}]$, $B_3 = [\frac{1}{8}, \frac{11}{64}] \cup [\frac{53}{64}, 1]$. Compared to the previous case, the profit from selling the standalone product increases and the set B_2 of types selling the standalone product dilates. In particu-

lar, there are now some types in B_2 which are above the bundle threshold of $\frac{3}{4}$ for whom revealing their type would be a profitable deviation. Therefore, the above partition is not revelation-proof and the maximal equilibrium seller profit in the disclosure game is strictly below the commitment benchmark. In this case, the seller-optimal equilibrium turns out to be associated with another bi-pooling partition given by $B_1 = [0, \frac{3-\sqrt{5}}{4}]$, $B_2 = [\frac{1}{4}, \frac{3}{4}]$, $B_3 = [\frac{3-\sqrt{5}}{4}, \frac{1}{4}] \cup [\frac{3}{4}, 1]$. To get some intuition why this is the best the seller can do in an equilibrium, note that this partition satisfies the following constraints. First, similarly to the commitment solution, the buyer's posterior means following messages B_2 and B_3 equal, respectively, $\mathbb{E}[\omega \mid \omega \in B_2] = \frac{1}{2}$ and $\mathbb{E}[\omega \mid \omega \in B_3] = \frac{3}{4}$, in other words the obedience constraint is binding and, as a result, the partition is *barely obedient*. Second, in contrast to the commitment solution, the upper bound of B_2 coincides with the threshold $\frac{3}{4}$, implying that the seller revelation-proofness constraint is binding.

Our model generalizes this example to any absolutely continuous prior distribution of the state, an ordered finite action space, and monotone preferences of the players.¹ Our first main result, [Theorem 1](#), shows that any equilibrium payoff of the sender can be obtained in a *laminar partitional equilibrium*, i.e., one that is associated with a laminar partition. The laminar property of a partition, introduced in [Candogan and Strack \(2023\)](#), generalizes the *bi-pooling* property known to characterize optima under sender's commitment. In a bi-pooling partition, each element's convex hull may nest at most one other lower-indexed element, while in a laminar partition, it may nest any number of lower-indexed elements. We prove this result by showing that laminar partitions are most revelation-proof among obedient partitions in the sense that any equilibrium partition can be transformed into a laminar equilibrium partition with the same sender payoff.

Our next result, [Theorem 2](#), establishes the properties of sender-preferred equilibrium partitions. First, sender-preferred (laminar) equilibrium partitions are barely obedient in the sense that they maximally exploit the receiver's obedience. This feature is common with the commitment solution: if the sender's payoff is maximized, then the obedience constraint must be binding. Second, we provide a necessary and sufficient condition for a barely obedient laminar partition to be revelation-proof. In particular, we show that instead of verifying revelation proofness for all states, one can consider a particular subset of on-path actions and then verify it for the highest state

¹The players have monotone preferences if the sender's payoff is strictly increasing in the receiver's action and the receiver's preferred action is increasing in the expected state.

in which each of those actions is recommended.

Our remaining set of results discusses the attainability of the commitment payoff in the disclosure game. First, every bi-pooling solution of the commitment problem pins down a barely obedient bi-pooling (and, therefore, laminar) partition that is most resistant to violations of revelation proofness ([Theorem 3](#), [Proposition 1](#)). Consequently, the revelation proofness of that partition determines whether the commitment payoff can be achieved in a disclosure game. With binary actions, [Titova and Zhang \(2025\)](#) show that every bi-pooling commitment solution is implementable. For three or more actions, our [Proposition 2](#) shows that every bi-pooling commitment solution is implementable if the sender’s utility is sufficiently convex in receiver’s action cutoffs. In the above example with three actions, the commitment payoff is attained when p is sufficiently low, which corresponds to the seller’s utility being sufficiently convex.

Implications. The results discussed above have several implications. In the existing literature, pooling nonadjacent states is typically attributed to the presence of the receiver’s private information.² The laminar structure provides an alternative rationale for pooling nonadjacent states, suggesting it could be a more general and robust feature that does not have to depend on the interaction of multiple information sources. Moreover, the coexistence of equilibria with distinct payoffs and simple messages may explain the diversity of disclosure policies observed in practice. This also suggests that focusing exclusively on the fully revealing outcomes in policy debates need not always be appropriate, especially when there is a small number of actions. Finally, this paper contributes to the ongoing debates over whether disclosure should be mandated. Our results identify communication environments where the sender can gain significantly by pooling a substantial portion of low states with high states. In such contexts, from the receiver’s perspective, the advantages of mandatory disclosure may outweigh the potential shortcomings of such a policy.

Applications. We apply these insights to study selling with quality disclosure and influencing voters. In the former setting, [Milgrom \(1981\)](#) shows that when the buyer can purchase any fraction of the product, unraveling takes place and every equilibrium

²This is documented in various contexts, such as signaling games (e.g., [Feltovich, Harbaugh, and To, 2002](#)), disclosure games (e.g., [Harbaugh and To, 2020](#)), information design (e.g., [Guo and Shmaya, 2019](#); [Candogan and Strack, 2023](#)), and in empirical studies (e.g., [Bederson, Jin, Leslie, Quinn, and Zou, 2018](#)).

features full revelation. If the buyer is restricted to buying integer units, however, we show that the seller may be able to achieve the same outcome as under commitment. In the second application, we consider an expert who discloses verifiable information to a group of voters in an amendment voting setting; the voters choose from three alternatives: the amended bill, the unamended bill, and the status quo. We demonstrate that the expert can be hurt even if, all else equal, all voters become more inclined toward the expert’s most preferred alternative.

Related literature. This paper belongs to a growing literature that characterizes equilibrium payoff set in disclosure games.³ The most closely related paper is [Titova and Zhang \(2025\)](#), which studies a more general disclosure game with a finite number of receiver actions. We specialize their model by assuming the state space is the unit interval and the receiver has monotonic preferences that depend only on the expected state. These assumptions allow us to provide a characterization of a sender-preferred equilibrium, identify the limits of verifiable communication, and establish sufficient conditions on model primitives for the sender to achieve the commitment payoff. While we focus on environments in which the receiver’s action set is more limited than in [Grossman \(1981\)](#) and [Milgrom \(1981\)](#)—our receiver chooses from a finite set of actions—[Ali, Kleiner, and Zhang \(2024\)](#) study settings with greater flexibility, where full revelation prompts an action that makes the sender no better off than inducing any other beliefs. Consequently, revelation proofness is not a concern. They provide conditions under which the sets of equilibrium payoff profiles are virtually the same as the set of achievable payoff profiles under commitment. [Gieczewski and Titova \(2025\)](#) consider disclosure games with a general message mapping, propose an equilibrium selection criterion related to neologism-proofness, and characterize the sender’s ex-ante payoff under this criterion when there is access to sufficiently rich stochastic evidence.⁴

Beyond verifiable disclosure, our work also contributes to the growing recent literature on the possibility of attaining commitment payoff without full commitment in other communication environments, for example, under cheap talk ([Lipnowski and Ravid, 2020](#); [Lipnowski, 2020](#)), repeated cheap talk ([Best and Quigley, 2023](#); [Kuvalekar, Lipnowski, and Ramos, 2022](#); [Mathevet, Pearce, and Stacchetti, 2022](#); [Pei, 2023](#)), infor-

³Disclosure games were introduced in [Grossman and Hart \(1980\)](#), [Grossman \(1981\)](#) and [Milgrom \(1981\)](#). For surveys of this literature, see [Milgrom \(2008\)](#), [Dranove and Jin \(2010\)](#), and [Ben-Porath, Dekel, and Lipman \(2025\)](#).

⁴“Stochastic evidence” means that the sender’s available messages depend on both the state and chance. In our game, however, the set of available messages is determined solely by the state.

mation design with privately (and fully) informed sender (Perez-Richet, 2014; Koessler and Skreta, 2023; Zapechelnyuk, 2023),⁵ a possibility to covertly revise a message from an experiment (Min, 2021; Lipnowski, Ravid, and Shishkin, 2022), an ability to covertly revise an experiment without affecting the marginal distribution over messages (Lin and Liu, 2024), costly misreporting (Guo and Shmaya, 2021; Nguyen and Tan, 2021), and sender-worst equilibrium selection (Lipnowski, Ravid, and Shishkin, 2024). In contrast, we study a one-shot communication game with verifiable information and absent any commitment.

Our paper is also closely related to Candogan and Strack (2023) which studies linear persuasion with one or more privately informed receivers. In that paper, laminar partitions arise as a solution to a standard linear maximization problem with a mean-preserving contraction constraint (Kleiner et al., 2021; Arieli et al., 2023), in which receivers’ incentive constraints are written as additional moment conditions. In contrast, we show that laminar partitions are ones that are most likely to be revelation-proof; yet, the revelation proofness constraint cannot be written as a moment condition.

2 The Model

We consider the following *disclosure game* between the sender (he) and the receiver (she).⁶ The state space is $\Omega = [0, 1]$, and the common prior F admits a strictly positive density f ; we let μ_F denote the probability measure induced by F . First, the sender learns the state. Then, the sender communicates with the receiver using verifiable messages. Specifically, the sender’s message space in state $\omega \in \Omega$ is $M(\omega) := \{m \in \mathcal{C} : \omega \in m\}$, where \mathcal{C} is the collection of all nonempty closed subsets of $[0, 1]$. Finally, the receiver observes the message, forms a posterior belief, and takes an action.

The receiver’s action space is $N = \{1, \dots, n\}$, where $n > 1$. The receiver’s optimal action depends only on her posterior mean, denoted by $x \in [0, 1]$. We assume that the receiver’s preferences are monotone in the sense that action i is optimal if and only if $x \in [\gamma_i, \gamma_{i+1}] =: A_i$ for some cutoffs $0 = \gamma_1 < \gamma_2 < \dots < \gamma_{n+1} = 1$.⁷ We also assume that the sender’s state-independent payoff u_i from receiver taking action i is strictly

⁵One can also interpret these models as disclosure games where the sender has access to stochastic evidence.

⁶We model (verifiable) disclosure the same way as in the seminal papers of Grossman and Hart (1980), Grossman (1981), and Milgrom (1981).

⁷One interpretation is the the receiver wants to match the state but is constrained by the number of available actions.

increasing in i .⁸ Without loss, we normalize $u_1 = 0$. Finally, let

$$v(x) = \begin{cases} u_i & \text{if } x \in [\gamma_i, \gamma_{i+1}) \text{ for some } i \in N \setminus \{n\}, \\ u_n & \text{if } x \in [\gamma_n, \gamma_{n+1}] \end{cases}$$

denote the sender's value function, which maps the receiver's posterior mean to the highest attainable sender payoff. By construction, $v(x)$ is upper semicontinuous.

We focus on perfect Bayesian equilibria of the disclosure game. An assessment is a triple (σ, τ, p) , where $\sigma : \Omega \rightarrow \Delta \mathcal{C}$ is the sender's strategy, $\tau : \mathcal{C} \rightarrow \Delta N$ is the receiver's strategy and $p : \mathcal{C} \rightarrow \Delta(\Omega)$ is the receiver's belief system.⁹ An assessment (σ, τ, p) is an (perfect Bayesian) equilibrium if:

1. for every $\omega \in \Omega$, $\sigma(\omega)$ is supported on $\arg \max_{m \in M(\omega)} \sum_{i \in N} \tau(i | m) u_i$;
2. for each $m \in \mathcal{C}$, $\tau(i | m) > 0 \implies \int_{\Omega} \omega \, dp(\omega | m) \in A_i$;
3. p is obtained from F given σ^* using Bayes rule;
4. for every $m \in \mathcal{C}$, $\text{supp}(p(m)) \subseteq m$.

In words, the first condition requires that the sender chooses verifiable messages that maximize his expected payoff. Second, the receiver chooses an action that is optimal given her posterior mean. Third, the receiver uses Bayes rule to calculate the posterior from the prior given the sender's strategy. Finally, the receiver's belief system is consistent with disclosure: she deems impossible any state in which the observed (on- or off-path) message is not verifiable.

3 Equilibrium Analysis

Partitions

We begin analysis by introducing the notion of a partition of the state space and its key properties. A sequence $\mathcal{B} := \{B_i\}_{i \in N} \subseteq \mathcal{C}$ of closed subsets of $[0, 1]$ is a (ordered) partition if $\bigcup_{i \in N} B_i = [0, 1]$ and $\mu_F(B_i \cap B_j) = 0$ for all $i, j \in N$. Note first that our partition is an ordered sequence, rather than a collection of sets. This is convenient because the receiver's action set is ordered. Second, the partition elements are closed

⁸We use “increasing,” “smaller,” and “greater” in the weak sense: “strictly” will be added whenever needed.

⁹For a compact metric space Y , let ΔY denote the set of all probability measures on the Borel subsets of Y . Endowing \mathcal{C} with the Hausdorff distance, it is a compact metric space.

sets that may have a non-empty intersection which, however, must have zero measure.

Partitions are useful in describing certain types of on-path behavior. We say that an assessment (σ, τ, p) and a partition \mathcal{B} are associated if for each $i \in N$:

- (a) $\sigma(B_i \mid \omega) = \mathbb{1}(\omega \in B_i \text{ and } \omega \notin B_{i+1} \cup \dots \cup B_n)$;
- (b) $\tau(i \mid B_i) = 1$;
- (c) $p(\cdot \mid B_i) = \mu_F(\cdot \mid B_i)$.

In an assessment associated with a partition \mathcal{B} , the sender's strategy is to reveal which element of the partition the state belongs to by sending message B_i when $\omega \in B_i$.¹⁰ Such a message can be interpreted as a recommendation for the receiver to choose action i . Then, the receiver's strategy is to always follow the recommendation. Finally, the receiver's posteriors on the path are calculated using Bayes rule. Note that a partition uniquely defines the on-path behavior of the players: if a partition is associated with multiple assessments, all of these assessments differ in receiver's off-path beliefs and actions only. If an assessment (σ, τ, p) associated with a partition \mathcal{B} is an equilibrium, then we call (σ, τ, p) a partitional equilibrium (PE) and \mathcal{B} an equilibrium partition. The following two properties are necessary and sufficient for \mathcal{B} to be an equilibrium partition.¹¹

Definition 1. A partition \mathcal{B} is

- obedient if $\mathbb{E}[\omega \mid \omega \in B_i] \in A_i$ for each $i \in N$.
- revelation-proof if $\omega \in B_i$ implies $\omega \in A_1 \cup \dots \cup A_i = [0, \gamma_{i+1}]$ for each $i \in N$.

In words, obedience requires that the receiver indeed prefers to take action i when she learns that $\omega \in B_i$, i.e., after message B_i . Revelation proofness, roughly speaking, ensures that the sender does not have profitable deviations toward fully revealing the state. Indeed, in the disclosure game, the sender has an option to fully reveal the state by sending message $\{\omega\}$ with probability one, thus convincing the receiver to take the action that she would take knowing ω . Thus, if $\omega \in B_i$, i.e., the partition prescribes that the receiver takes action i when the realized state is ω , then the receiver prefers to take *at most* action i when fully informed.

Next, we define the following structural property of partitions. Let $\text{cl}(\cdot)$ and $\text{co}(\cdot)$ denote the closure and the convex hull, respectively.

¹⁰We assume that if ω is in multiple partition elements, then the sender sends the highest one.

¹¹This result was first shown in [Titova and Zhang \(2025\)](#) for a disclosure game with a slightly different message space. We formulate and prove this result for our disclosure game in [Lemma 1](#) in the appendix.

Definition 2. A partition \mathcal{B} is laminar if for each $i \in N$

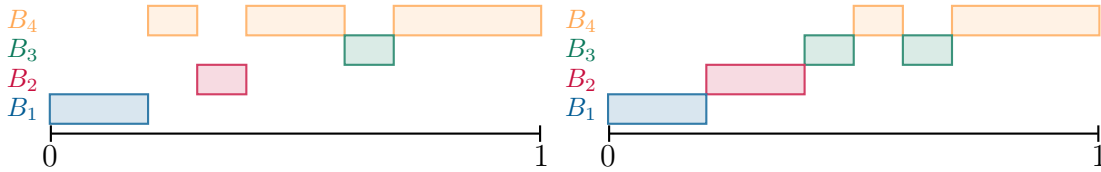
$$B_i = \text{cl} \left(\text{co}(B_i) \setminus \bigcup_{j < i} \text{co}(B_j) \right). \quad (1)$$

If an equilibrium is associated with a laminar partition \mathcal{B} , we call this equilibrium a laminar PE.

The intuitive interpretation of the laminar property is that each element B_i either has no “gaps” and is therefore an interval, or it has “gaps” that belong to lower-indexed partition elements B_1, \dots, B_{i-1} only. This implies that for each pair B_i and B_j with $i > j$, the convex hull $\text{co}(B_i)$ either nests or has a null intersection with $\text{co}(B_j)$.¹² When $\text{co}(B_i) \supset \text{co}(B_j)$, we say that action i *nests* action j . We illustrate this intuition in Figure 2.

The following observation is a direct consequence of the definition of a laminar partition.

Observation 1. *If \mathcal{B} is a laminar partition, then each B_i is either an empty set or a union of at most i intervals.*



(a) Only B_4 has gaps but they belong to the lower-indexed B_2 and B_3 .

(b) B_3 has a gap that belongs to the higher-indexed B_4 .

Figure 2. A laminar partition in panel (a) and a non-laminar partition in panel (b).

One prominent example of an obedient and revelation-proof laminar partition is $\mathcal{A} := \{A_i\}_{i \in N} = \{[\gamma_i, \gamma_{i+1}]\}_{i \in N}$, which we will call the fully informative partition. Note that while \mathcal{A} is not fully revealing, it provides the minimal information necessary for the receiver to take her complete-information-optimal action in each state.

¹²In combinatorics, a laminar set family has each pair of elements either nested or disjoint. We borrow the term from Candogan and Strack (2023) where the definition of a laminar partition is similar to ours but allows for any nesting partial order. This is essentially equivalent for finite partitions, in the sense that our partition comes with an exogenous linear order that must complete the nesting partial order.

Equilibrium Payoff Set

Our first result characterizes equilibrium ex-ante payoffs of the sender in terms of obedient and revelation-proof laminar partitions. We say that an equilibrium is sender-preferred if it yields the highest ex-ante payoff for the sender across all equilibria. Analogously, we refer to the ex-ante payoff-minimizing equilibrium as sender-worst.

Theorem 1. (a) *There exists a sender-worst equilibrium that is associated with the fully informative partition \mathcal{A} and yields $\underline{V} := \sum_{i \in N} \mu_F(A_i)u_i$.*
 (b) *There exists a sender-preferred equilibrium that is associated with an obedient and revelation-proof laminar partition and yields $\bar{V} > \underline{V}$.*
 (c) *For any $V \in [\underline{V}, \bar{V}]$, there exists an equilibrium associated with an obedient and revelation-proof laminar partition.*

We provide the intuition for [Theorem 1](#) below. Part (a) is straightforward: as in most disclosure games, there exists an equilibrium in which the receiver acts as if she is fully informed. In our setting, that equilibrium is associated with the fully informative partition \mathcal{A} . The sender's ex-ante payoff cannot fall below \underline{V} , or else the sender would have a profitable deviation toward fully revealing the state in a positive measure of states.

The key step to prove part (b) is showing that for any equilibrium partition there exists an equilibrium *laminar* partition that induces the same posterior mean distribution (PMD), and hence yields the same ex-ante payoff to the sender. We first observe that any partition induces a PMD with support on at most n points (posterior means); using the techniques from [Candogan and Strack \(2023\)](#), we show that any such PMD can be induced by a *laminar* partition. Then, we further show that if a PMD is induced by an obedient and revelation-proof partition, then the laminar partition that induces the same PMD is guaranteed to be obedient and revelation-proof. In that sense, laminar partitions are the most revelation-proof partitions.

We illustrate the intuition behind laminar partitions being the most revelation-proof ones in [Figure 3](#) using the simplest case when $n = 2$. Consider two partitions, \mathcal{B} (laminar) and \mathcal{W} (non-laminar) that induce the same posterior mean distribution and whose corresponding elements have the same prior mass; that is, $\mathbb{E}[\omega \mid \omega \in B_i] = \mathbb{E}[\omega \mid \omega \in W_i]$ and $\mu_0(B_i) = \mu_0(W_i)$ for all $i \in N$. Since \mathcal{B} is laminar, its lowest-indexed element B_1 is an interval. If W_1 is not interval, the only way to match B_1 's prior mass and expectation is to have $\max B_1 \leq \max W_1$. Consequently, if \mathcal{W} is revelation-proof

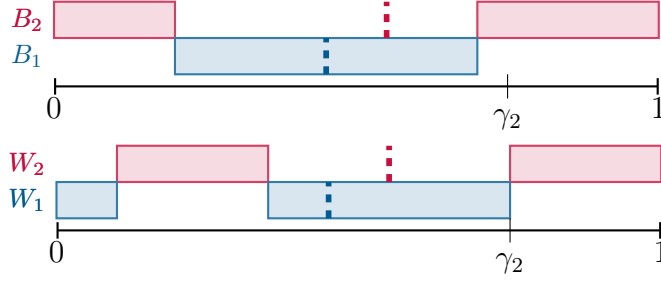


Figure 3. Two partitions, \mathcal{B} (laminar) and \mathcal{W} (non-laminar) that induce the same posterior mean distribution and whose corresponding elements have the same prior mass. Whenever \mathcal{W} is revelation-proof, so is \mathcal{B} .

(which in the case of two actions reduces to $\max W_1 \leq \gamma_2$), then so is \mathcal{B} .

In a sender-preferred equilibrium, the receiver breaks the ties in favor of the sender, so a single action is taken with probability one in each state. Therefore, every sender-preferred equilibrium is payoff equivalent to a partitional equilibrium, i 'th element of the partition contains states in which action the receiver takes action i . Consequently, to find a sender-preferred equilibrium, it suffices to focus on laminar partitional equilibria. A sender-preferred laminar PE is associated with a partition that solves

$$\begin{aligned} \max_{\mathcal{B}} \quad & \sum_{i \in N} \mu_F(B_i) u_i \\ \text{subject to} \quad & \mathcal{B} \text{ is an obedient and revelation-proof laminar partition.} \end{aligned} \tag{2}$$

We show that Problem (2) admits a solution,¹³ which implies that a sender-preferred equilibrium exists. To show that $\bar{V} > \underline{V}$, we construct an obedient and revelation-proof (and hence equilibrium) laminar partition that yields a strictly higher payoff than \underline{V} . For example, we could take the fully revealing partition $\{[\gamma_i, \gamma_{i+1}]\}_{i \in N}$ and “move” the interval $[0, \varepsilon]$ from the first element of the partition to the n -th one. If $\varepsilon > 0$ is sufficiently small, the resulting partition would remain obedient, revelation-proof and laminar, but the sender’s ex-ante utility in the associated equilibrium would be strictly higher than \underline{V} . Therefore, sender’s ex-ante payoff in his most preferred equilibrium must also exceed \underline{V} .

To prove Part (c), we show that for any payoff between \underline{V} and \bar{V} there exists an laminar equilibrium partition that blends the sender-worst (fully informative and

¹³The existence of a solution follows from the extreme value theorem. To apply it, we endow the space of laminar partitions with a topology that makes the sender’s payoff continuous and the constraint set compact.

laminar) partition \mathcal{A} and the sender-preferred laminar equilibrium partition \mathcal{B} that yields that payoff.

Sender-preferred Equilibrium

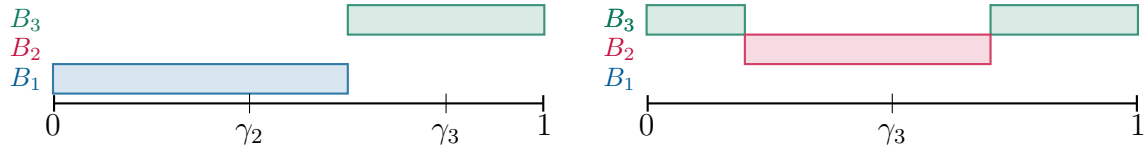
Next, we characterize the sender-preferred laminar partitional equilibrium. We say that action i is skipped in partition \mathcal{B} if B_i is a null set and unskipped otherwise. We refer to a partition \mathcal{B} as barely obedient if $\mathbb{E}[\omega \mid \omega \in B_i] = \gamma_i$ for all unskipped $i \in N$, except the lowest.

Theorem 2. *There exists a sender-preferred laminar equilibrium partition \mathcal{B} such that (i) B_i is the union of at most $\max\{i - 1, 1\}$ closed intervals for all $i \in N$, and (ii) \mathcal{B} is barely obedient. Furthermore, a barely obedient laminar partition is an equilibrium partition if and only if $\max B_i \leq \gamma_{i+1}$ for any unskipped $i > 1$ such that $i + 1$ either nests i or is skipped.*

[Theorem 2](#) characterizes the solution to Problem (2), which is an obedient and revelation-proof laminar partition that maximizes the sender's ex-ante payoff. Condition (i) follows from laminarity of \mathcal{B} . First, we find that $B_j = [0, d]$ for some $d \leq \gamma_{j+1}$ for the lowest unskipped action j . That is, the lowest unskipped action is an interval that is not in the convex hull of any other partitional elements. Then, each partitional element with index $k \geq 2$ must be a union of at most $k - 1$ intervals by the definition of a laminar partition. Next, \mathcal{B} must be barely obedient; otherwise, one could pool additional low states with high states without violating obedience or revelation-proofness, thereby obtaining an equilibrium with a strictly higher ex-ante sender payoff. Specifically, if j is the lowest unskipped action, one can reassign a subset of B_j of strictly positive measure to B_k for some $k > j$.

Next, revelation proofness is equivalent to $\max B_i \leq \gamma_{i+1}$ for each action i . The second part of [Theorem 2](#) shows that for a barely obedient laminar partition, this constraint is only binding for an action in two specific scenarios, illustrated in [Figure 4](#): either the next-highest action is skipped, or the action's corresponding partitional element is nested within the convex hull of the partitional element for the next-highest action. In all other instances, revelation proofness is automatically satisfied. This finding further emphasizes that the laminar structure makes revelation proofness easier to fulfill.

As in [Candogan and Strack \(2023\)](#), the laminar structure emerges in a sender-preferred equilibrium of our game due to incentive constraints. In [Candogan and](#)



(a) If action 2 is skipped and $\gamma_2 < \max B_1$, then full revelation induces the higher skipped action 2 for all $\omega \in [\gamma_1, \max B_1) \subseteq B_1$.
(b) If $\gamma_3 < \max B_2$ and $\text{co}(B_2) \subseteq \text{co}(B_3)$, then full revelation induces the higher action 3 for all $\omega \in [\gamma_2, \max B_2) \subseteq B_2$.

Figure 4. Violations of revelation proofness ruled out in [Theorem 2](#).

[Strack \(2023\)](#), where the receiver is privately informed, it serves to prevent the receiver from misreporting her private information. In our setting, it instead optimally balances the trade-off between deterring the sender's deviations and inducing the most desirable action distribution.

As discussed after [Theorem 1](#), laminar partitions are the most revelation-proof among all obedient partitions. However, interval partitions—a special case of laminar partitions—often fail to be sender-preferred equilibrium partitions. This underscores the importance of pooling nonadjacent states in inducing the most desirable distribution over actions subject to obedience and revelation proofness. To illustrate, recall our introductory example with $p = 0.6$. The interval partition that generates the highest sender's ex-ante payoff is given by $B_1^I = [0, \frac{1}{4}]$, $B_2^I = [\frac{1}{4}, \frac{3}{4}]$, $B_3^I = [\frac{3}{4}, 1]$. Since $\mathbb{E}[\omega \mid \omega \in B_3^I] > \gamma_3 = \frac{3}{4}$, we can pool the states from the top of B_1^I with B_3^I , until the partition becomes barely obedient. This process results in yields a sender-preferred equilibrium partition $B_1 = [0, \frac{3-\sqrt{5}}{4}]$, $B_2 = [\frac{1}{4}, \frac{3}{4}]$, $B_3 = [\frac{3-\sqrt{5}}{4}, \frac{1}{4}] \cup [\frac{3}{4}, 1]$.

4 When Is Commitment Payoff Achievable?

While the extent to which the sender benefits from verifiable communication depends on the specific parameters, an upper bound on the sender's payoff is given by his commitment payoff; i.e., his payoff when he can *commit* to what messages to send in each state. In this section, we identify conditions under which the sender can attain his commitment payoff in an equilibrium of the disclosure game.

4.1 Commitment Benchmark

We start by introducing the commitment problem, or information-design problem, as a benchmark. In this problem, the sender can commit to any experiment that reveals

information about the state. An experiment is a mapping $\chi: [0, 1] \rightarrow \Delta(S)$, where S is a sufficiently rich signal space. For each state $\omega \in [0, 1]$, a signal $s \in S$ realizes according to $\chi(\omega)$. Because the receiver's optimal action only depends on the expected state, it is without loss to restrict attention on the class of experiments where $S = [0, 1]$, and each $s \in S$ is calibrated to equal the induced posterior mean: $s = \mathbb{E}[\omega \mid s]$. Such a calibrated experiment χ induces the posterior mean distribution with a CDF $G(x) = \int_0^1 \int_0^x d\chi(s|\omega) dF(\omega)$.

It is well known that a posterior mean distribution G is induced by some experiment if and only if G is a mean-preserving contraction of the prior CDF F .¹⁴ Consequently, the commitment problem can be stated as a maximization of the expected value v with respect to the PMD:

$$\max_{G \in \text{MPC}(F)} \int_0^1 v(x) dG(x), \quad (3)$$

where $\text{MPC}(F)$ is the set of all mean-preserving contractions of F . We call any solution to problem (3) a commitment solution, and call the value of problem (3) the commitment payoff. Clearly, the commitment payoff is an upper bound of the sender's equilibrium payoff in the disclosure game.

Finally, we say that a commitment solution G is implementable with verifiable messages, or just implementable, for short, if there is an equilibrium in which the sender's strategy induces a distribution of the receiver's posterior means G .

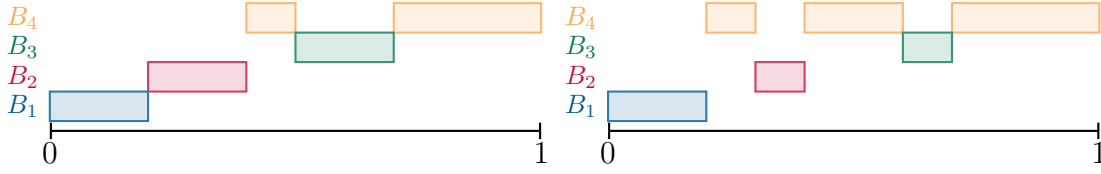
We say that an experiment χ is associated with a partition \mathcal{B} if it maps almost each $\omega \in B_i$ into the degenerate distribution centered on $\mathbb{E}[\omega \mid \omega \in B_i]$. In other words, such an experiment discloses only which element of the partition the state belongs to. In this case, we also say that the induced posterior mean distribution G is associated with this partition.

Next, we define a refinement of the laminar property which will be key for optimality under commitment.

Definition 3. A laminar partition is a bi-pooling partition if every partitional element B_i is either an interval, or there exists a unique $j \in \{0, \dots, n-1\}$ such that $j < i$ and $B_j \subseteq \text{co}(B_i)$.

While laminar partitions allow any B_j with $j < i$ to be nested within $\text{co}(B_i)$,

¹⁴See, for example, [Gentzkow and Kamenica \(2016\)](#) and [Kolotilin \(2018\)](#). A distribution $G \in \Delta([0, 1])$ is a mean-preserving contraction of F if $\int_0^x G(s) ds \leq \int_0^x F(s) ds$ for all $x \in [0, 1]$, where the inequality binds at $x = 1$.



(a) B_1 , B_2 , and B_3 are intervals, and $\text{co}(B_4)$ only contains B_3 .

(b) B_4 is not an interval, and $B_2, B_3 \subseteq \text{co}(B_4)$.

Figure 5. A bi-pooling partition in panel (a) and a laminar partition that is not a bi-pooling partition in panel (b), which is the same as that in Figure 2a.

bi-pooling partitions allow only for a single such B_j to be nested within $\text{co}(B_i)$ (see Figure 5).

The following result regarding bi-pooling partitions is a direct consequence of results in Kleiner et al. (2021), Candogan (2022), and Arieli et al. (2023). Say that the communication environment is generic if no three elements of the collection of points $\{(\gamma_i, u_i)\}_{i \in N}$ are collinear.

Theorem 3.

- (i) *There exists a commitment solution associated with a unique barely obedient bi-pooling partition.*
- (ii) *For generic communication environments, the commitment solution is unique.*

Theorem 3 indicates that, despite the simplicity of bi-pooling partitions, there always exist a commitment solution associated with a bi-pooling partition. Moreover, under mild conditions, the *unique* commitment solution is associated with a bi-pooling partition.

4.2 Characterizing Implementability

The following result characterizes the implementability of a commitment solution associated with a bi-pooling partition.

Proposition 1. *Let G be a commitment solution associated with a bi-pooling partition \mathcal{B} . Then, G is implementable if and only if \mathcal{B} is revelation-proof.*

It follows directly from Theorem 3 and Proposition 1 that in a generic communication environment, the unique commitment solution is implementable if and only if the associated bi-pooling partition is revelation-proof.

The “if” part of [Proposition 1](#) is a direct consequence of Theorem 2 in [Titova and Zhang \(2025\)](#), which states that for a partition associated with a commitment solution, implementability is equivalent to revelation proofness. Our primary contribution in [Proposition 1](#) is to show that among all sender strategies that induce a commitment solution, the one associated with a bi-pooling partition has the best shot at being an equilibrium strategy. Roughly, this stems from the fact that bi-pooling partitions are a special case of laminar partitions, which exhibit a similar property.

[Proposition 1](#) is useful in that it suggests a “guess and verify” approach for finding the sender-preferred equilibrium. First, one finds the commitment solution using standard information design methods. Second, one identifies the associated bi-pooling partition. If this partition proves to be revelation-proof, then the sender-preferred equilibrium has been successfully identified.

The next result, which is a corollary of [Proposition 1](#) and [Theorem 2](#), goes one step further: it reveals the exact features of bi-pooling partitions that fail revelation proofness and hence prevent the commitment solutions from being implementable.

Corollary 1. *Let G be a commitment solution associated with a bi-pooling partition \mathcal{B} . Then, G is implementable if and only if \mathcal{B} is such that $\max B_i \leq \gamma_{i+1}$ for all unskipped i such that $i + 1$ either nests i or is skipped.*

Compared to the definition of revelation proofness, the two conditions in [Corollary 1](#) are easier to verify when determining whether a commitment solution is implementable. They also allow us to identify sufficient conditions under which commitment has no value in [Section 4.3](#).

To illustrate this result, recall our introductory example with $A = \{1, 2, 3\}$, $u_1 = 0$, $u_2 = p$, $u_3 = 1$, $\gamma_2 = \frac{1}{2}$, $\gamma_3 = \frac{3}{4}$, and uniform F . When $p = 0$, there is a unique commitment solution associated with a bi-pooling partition given by $B_1 = [0, \frac{1}{2}]$, $B_2 = \emptyset$, $B_3 = [\frac{1}{2}, 1]$. Because this partition is monotone, no action is nested. As for condition (i), note that action 2 is skipped and action 1 is the highest unskipped action below. Therefore, (i) is satisfied because $\max B_1 = \frac{1}{2} \leq \frac{1}{2} = \gamma_2$. In other words, the commitment-optimal partition is revelation-proof because the sender cannot induce an action higher than 1 by revealing themselves in any state in B_1 .

When $p = 0.5$, there is a unique commitment solution associated with a bi-pooling partition given by $B_1 = [0, \frac{1}{4}]$, $B_2 = [\frac{5}{16}, \frac{11}{16}]$, $B_3 = [\frac{1}{4}, \frac{5}{16}] \cup [\frac{11}{16}, 1]$. Because no action is skipped, (i) is trivially satisfied. As for condition (ii), note that B_2 is nested by $\text{co}(B_3)$. Therefore, (ii) is satisfied because $\max B_2 = \frac{11}{16} \leq \frac{3}{4} = \gamma_3$.

Proposition 1 and Corollary 1 may have created a feeling that implementing a commitment solution associated with a bi-pooling partition is relatively simple: one only needs to ensure that no action is recommended more often than revelation proofness allows. Indeed, when the receiver is choosing between two actions, Titova and Zhang (2025) showed that revelation proofness is automatically satisfied. When there are three or more actions, however, the restriction imposed by revelation proofness can be substantial. We illustrated this in our introductory example with $p = 0.6$. In this case, the commitment-optimal bi-pooling partition is given by $B_1 = [0, \frac{1}{8}]$, $B_2 = [\frac{11}{64}, \frac{53}{64}]$, $B_3 = [\frac{1}{8}, \frac{11}{64}] \cup [\frac{53}{64}, 1]$. Note that in this case (ii) is violated because B_2 is nested by $\text{co}(B_3)$, but $\max B_2 = \frac{53}{64} > \frac{3}{4} = \gamma_3$.

When there are two actions, the sender's sole objective is to maximize the probability that the "high action" 2 is played. This, in turn, suggests that revelation proofness is never an issue: in any state in which action 2 is played under complete information, there is no reason to recommend action 1. However, with three actions, as Gentzkow and Kamenica (2016) note, in the commitment problem the sender faces a trade-off between inducing actions 2 and 3. When p is high, the gap between u_2 and u_3 is significantly smaller than that between u_1 and u_2 , and hence it is more profitable to induce action 2 more often: to guarantee obedience, recommending action 3 more often must come with action 1 being played more frequently. Consequently, the unique commitment-optimal partition recommends action 2 so frequently that $\max B_2 > \gamma_3$. Then in states strictly higher than γ_3 , the sender is strictly better off by fully revealing the state, rendering the commitment solution not implementable.

4.3 Sufficient Conditions for Implementability

In what follows we identify conditions on model primitives that guarantee an implementable commitment solution exists. Under these conditions, the sender would not benefit from commitment relative to the sender-preferred equilibrium. The equilibrium payoff set and the sender-preferred equilibrium partition can thus be found by solving the corresponding commitment problem. Conversely, the commitment assumption is unnecessary for any information-design problem that satisfies these conditions.

To state the result, let $h(\gamma_i; \gamma_{i+1})$ denote the unique solution of $\mathbb{E}[\omega \mid \omega \in [h(\gamma_i; \gamma_{i+1}), \gamma_{i+1}]] = \gamma_i$ if it exists, and set it to 0 otherwise. Intuitively, $h(\gamma_i; \gamma_{i+1})$ is the lowest state such that the conditional mean of the states between this state and γ_{i+1} is no less than γ_i .

Proposition 2. *Suppose there are three or more actions. If*

$$\frac{u_{i+1} - u_i}{\gamma_{i+1} - \gamma_i} > \frac{u_i - u_{i-1}}{\gamma_i - \max\{\gamma_{i-1}, h(\gamma_i; \gamma_{i+1})\}}, \quad (4)$$

for all $i = 2, \dots, n-1$, then every commitment solution associated with a bi-pooling partition is implementable. Consequently, the commitment payoff is attained in an equilibrium of the disclosure game.

In Condition (4), $u_{i+1} - u_i$ is the sender's marginal benefit of inducing a higher action evaluated at action i , and $\gamma_{i+1} - \gamma_i$ is the difference in cutoffs for inducing actions $i+1$ and i under complete information, respectively. Proposition 2 suggests that in a communication environment, if inducing a marginally higher action is either sufficiently more profitable or sufficiently more difficult (requiring a sufficiently larger expected state), or both, then the sender does not benefit from commitment power. Put differently, the sender does not value commitment when his value function increases sufficiently fast in the expected state.

The rough intuition behind Proposition 2 is as follows. To establish the sufficiency of (4), we will argue that any partition that is not revelation-proof must also fail optimality in the commitment problem. Take any barely obedient bi-pooling partition \mathcal{B} that is not revelation-proof. Then, some action i is recommended in some states in which the sender would prefer to fully reveal the state to induce a higher action instead; that is, $\max B_i > \gamma_{i+1}$. To illustrate how \mathcal{B} can then be strictly improved, suppose B_i is an interval. Then, $\max B_i > \gamma_{i+1}$ and obedience implies $\gamma_{i+1} \in B_i$. Next, modify the partition by shrinking B_i and shifting the probability of recommending action i to actions $i-1$ and $i+1$. Recommending action $i+1$ can be still made barely obedient by inducing belief γ_{i+1} . At the same time, action $i-1$ can now be induced at γ_{i-1} if γ_{i-1} is not too low, and otherwise at $h(\gamma_i; \gamma_{i+1})$. Either way, condition (4) ensures that such a local mean-preserving spread is profitable by requiring the sender's utility to be "convex enough" with respect to the cutoffs $\{\gamma_i\}_{i \in N}$. Consequently, any barely obedient partition associated with a commitment solution must be revelation-proof, and thus every such commitment solution is implementable.

Imposing a further assumption on the prior, Condition (4) can be simplified.

Corollary 2. *Suppose there are three or more actions. If for all $i = 1, \dots, n-1$, $u_{i+1} - u_i \geq u_i - u_{i-1}$ and $\gamma_{i+1} - \gamma_i \leq \gamma_i - \gamma_{i-1}$ with one of the inequalities being strict, and f is increasing, then every commitment solution associated with a bi-pooling partition*

is implementable. Consequently, the commitment payoff is attained in an equilibrium.

An increasing prior density is equivalent to a convex prior CDF. This condition is satisfied by the uniform distribution, and more generally the family of power distributions on $[0, 1]$ with CDF given by $F(x) = x^\alpha$, $\alpha \geq 1$.

4.4 The Special Case of Ternary Actions

It is instructive to take a deeper dive into the case in which the receiver has three actions. In this case, a partition can be written as $\mathcal{B} = \{B_1, B_2, B_3\}$. As implied by [Theorem 2](#) (i), in a sender-preferred laminar equilibrium partition, B_1 must be an interval if not null. Moreover, B_2 and B_3 are either both intervals or are such that $B_2 \subseteq \text{co}(B_3)$, meaning that \mathcal{B} is a bi-pooling partition.¹⁵ Consequently, [Theorem 2](#) implies that revelation proofness boils down to $\max B_1 \leq \gamma_2$.

Armed with these observations, a sender-preferred equilibrium can be explicitly solved.

Claim 1. *Suppose that $|N| = 3$.¹⁶ A sender-preferred equilibrium is associated with a barely obedient bi-pooling partition \mathcal{B} that either is also associated with the commitment solution, or is such that $B_1 = [0, y]$, $B_2 = [h, \gamma_3]$, and $B_3 = [y, h] \cup [\gamma_3, 1]$, where $h > 0$, $y \geq 0$ solve the system of equations*

$$\begin{aligned}\mathbb{E}[\omega \mid \omega \in [h, \gamma_3]] &= \gamma_2; \\ \mathbb{E}[\omega \mid \omega \in [y, h] \cup [\gamma_3, 1]] &= \gamma_3.\end{aligned}$$

When there are only three actions, the only reason that a commitment solution is not implementable is that the “middle action” 2 is recommended too often. Therefore, if no commitment solution is implementable, in the bi-pooling partition associated with a sender-preferred equilibrium, action 2 is recommended as frequently as revelation proofness allows: that is, the upper bound of B_2 must coincide with γ_3 . The proofs of all results in this subsection are relegated to a supplementary appendix.

The sufficient conditions can be further simplified when $|N| = 3$.

Corollary 3. *If $|N| = 3$, f is increasing, and $u_3 > 2u_2$, then all commitment solutions are implementable.*

¹⁵Both B_1 and B_2 may be empty, but B_3 cannot be: otherwise, revelation proofness must be violated.

¹⁶For a finite set K , let $|K|$ denote the cardinality of K .

Applying [Corollary 3](#) to our introductory example, since $u_2 = p$ and $u_3 = 1$, as long as $p < 0.5$, the seller does not benefit from commitment power. In other words, even prior to solving the commitment problem, we know that a sender-preferred equilibrium partition can be identified from the commitment solution.

5 Applications

5.1 Selling with Quality Disclosure

We first consider a variant of the model of a sales encounter studied in [Section 5 of Milgrom \(1981\)](#). The state of the world, ω , is interpreted as the quality of the seller's product. Let $p > 0$ be the unit price, and for simplicity, assume that there is no quantity discount. Denote the seller's constant unit cost by c , where $0 \leq c < p$. The product is indivisible: the buyer can only buy integer units of the product. The buyer's utility from purchasing q units is $\omega U(q) - pq$, where $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a bounded, strictly increasing, strictly concave threetimes differentiable function with $U(0) = 0$. We further assume that $U(q) - pq$ is maximized at $n > 1$. As a consequence, the buyer buys at most n units of the product, and she buys nothing if ω is close enough to 0.

The only significant difference between this model and that of [Milgrom \(1981\)](#) is that he considers a perfectly divisible product, and hence the seller's value function is strictly increasing. In this model, however, indivisibility makes the seller's value function a step function with n jumps. For a perfectly divisible product, [Milgrom](#) shows that every equilibrium of the game features full revelation: the seller sends $m = \{\omega\}$ for each $\omega \in [0, 1]$, resulting in the buyer-preferred outcome. With indivisibility, however, the seller may be able to gain considerably from verifiable communication, attaining his commitment payoff.

To state the result, let

$$A(x) = -\frac{U''(x)}{U'(x)}, \quad \text{and} \quad P(x) = -\frac{U'''(x)}{U''(x)}$$

denote the coefficients of absolute risk aversion and absolute prudence, respectively.

Claim 2. *If f is increasing and $P > 2A$, there exists an equilibrium of this game in which the seller is as well off as having commitment power.*

The proofs of results in this section are left to a supplementary appendix. The

assumption of increasing prior density can be interpreted as it is common knowledge that the consumer is relatively confident about the quality of the product. $P > 2A$ is satisfied by, for example, CRRA utility function with parameter $0 < \sigma < 1$.

5.2 Influencing Voters

Consider an amendment voting setting where a voting rule satisfying the Condorcet winner criterion is employed to determine which one of the three alternatives, the (unamended) bill (b), the amended bill (ab), and no bill (maintaining the status quo; \emptyset), will prevail.¹⁷ The state of the world is $\omega \in [0, 1]$, and denote the set of voters by J ; for simplicity, assume that $|J| > 1$ is an odd number. Voters have linear preferences: for $j \in J$, voter j 's utilities are given by $u_k^j(\omega) = \alpha_k^j + \beta_k^j \omega$, where $k \in \{b, ab, \emptyset\}$ and where $\omega \in [0, 1]$ is the state of the world. Moreover, $\beta_b^j > \beta_{ab}^j > \beta_{\emptyset}^j = 0$ and $0 = \alpha_{\emptyset}^j > \alpha_{ab}^j > \alpha_b^j$ for all $j \in J$. Let γ_2^j and γ_3^j denote the cutoff states that voter j is indifferent between \emptyset and ab , and ab and b , respectively.¹⁸ We impose further assumptions so that $\gamma_2^j < \gamma_3^j$ for all $j \in J$.

In this model, voter j 's preferences over alternative k are characterized by two parameters: one is α_k^j , we call it voter j 's *reference point* for alternative k as it is the voter's cardinal utility when the state is zero; another is β_k^j , we call it voter j 's *state sensitivity* for alternative k because it measures how fast the voter's cardinal utility increases in the state. Furthermore, all voters agree that when the state is low (intermediate, high), no bill (the amended bill, the bill, respectively) is optimal, but for the amended bill or the bill, different voters may have different reference points and different state sensitivity levels. Figure 6 illustrates a voter's utilities and the resulting cutoffs.

There is an expert who observes the state and can communicate to the voters. We follow Jackson and Tan (2013) to assume that the expert discloses verifiable information. Unlike their work with two states, however, we consider a continuum of states, and the expert's messages are closed subsets of the state space that contain the true state. Assume that the expert's preferences satisfy $u(b) > u(ab) > u(\emptyset) = 0$; that is,

¹⁷See, for example, Enelow and Koehler (1980) and Enelow (1981) for examples of amendment voting.

¹⁸For every $j \in J$,

$$\gamma_2^j = -\frac{\alpha_{ab}^j}{\beta_{ab}^j} \quad \text{and} \quad \gamma_3^j = \frac{\alpha_{ab}^j - \alpha_b^j}{\beta_b^j - \beta_{ab}^j}.$$

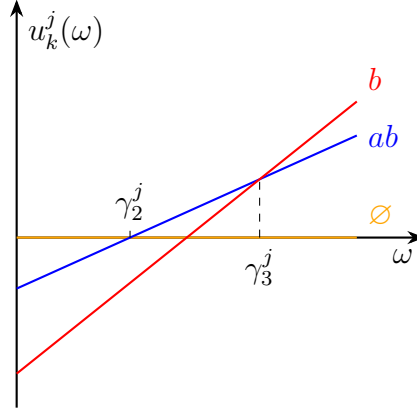


Figure 6. Voter j 's utilities.

the expert strictly prefers the bill to the amended bill, and the amended bill is strictly preferred to no bill.

Because the voting rule satisfies the Condorcet winner criterion, and the preferences are single-peaked, the Condorcet winner is the median voter's most preferred alternative.¹⁹ Therefore, it suffices to consider the median voter; further assumptions are imposed to make sure that the median voter is the same voter, say voter m , for all states.²⁰ Consequently, the expert's problem is equivalent to communicating to voter m alone. Thus, the expert-preferred equilibrium is characterized by [Claim 1](#).

[Claim 3](#) shows that the expert can be hurt if all voters become “more inclined toward” the bill, in the sense that all else equal, either the reference point or the state sensitivity for the bill increases (or both) for all voters.

Claim 3. *If no commitment solution is implementable, and at least one of the following happens:*

- (i) *voter j 's state sensitivity for the bill, β_b^j , increases for all $j \in J$,*
- (ii) *voter j 's reference point for the bill, α_b^j , increases for all $j \in J$,*

then the expert's payoff in his preferred equilibria may decrease.

¹⁹One voting rule used for amendment voting that satisfies the Condorcet winner criterion when voters' preferences are single-peaked is the pairwise majority rule, in which alternatives will be considered sequentially and two at a time using the majority rule.

²⁰That is, we assume that for any two voters where one is indexed higher than the other ($i > j$), $\gamma_2^i \geq \gamma_2^j$ and $\gamma_3^i > \gamma_3^j$ (see [Footnote 18](#) for the definition of the cutoffs). A sufficient condition for this assumption is that (i) all voters share the same reference points for both b and ab : $\alpha_b^j = \alpha_b$ and $\alpha_{ab}^j = \alpha_{ab}$ for all $j \in J$; (ii) among voters, those with a higher index not only exhibit higher state sensitivity for both b and ab but also show a strictly greater *difference* in state sensitivity between these alternatives: $i > j$ implies $\beta_{ab}^i \geq \beta_{ab}^j$, $\beta_b^i \geq \beta_b^j$, and $\beta_b^i - \beta_{ab}^i > \beta_b^j - \beta_{ab}^j$.

When either (i) or (ii) occurs (or both), γ_3^j decreases for every voter j , and hence γ_3^m must also decrease. This implies that the expected state required to pass the bill is lowered. Recall from the discussion after [Claim 1](#) that in scenarios where no commitment solution is implementable, the expert is too tempted to recommend the amended bill at the ex-ante stage. This may occur when the utility gap between the bill and the amended bill is smaller than that between the amended bill and the status quo (i.e., $u(b) - u(ab) < u(ab) - u(\emptyset)$), which is plausible in many voting scenarios. As γ_3^m decreases, the expert can more frequently induce the passage of the original bill. However, this also introduces an adverse indirect effect: the revelation proofness constraint tightens, limiting the chance of inducing the amended bill. Therefore, the expert is harmed if the indirect effect dominates.

6 Discussion

6.1 Assumption on the Message Space

Although our assumption on the message space is standard in the literature, it is useful to discuss the meaning of this assumption as well as how our results rely on it. Importantly, the sender does not need to literally use a closed subset of the state space as his message; what we require is *complete provability*: the sender can prove any true fact. More specifically, in each state ω , the sender's message space, or the set of evidence, $M(\omega)$, is *sufficiently rich* in the sense that for every closed subset C of the state space $[0, 1]$, there is a message m that provides hard evidence that the state is contained in C . Formally, $m \in M(\omega)$ if and only if $\omega \in C$.

Two implications of this assumption are crucial to our results. First, the sender can always fully reveal the state; that is, the fully revealing message $m = \{\omega\}$ is feasible in every $\omega \in [0, 1]$. This implies that in every state, the sender's equilibrium payoff must be no lower than his payoff from fully revealing the state, which gives rise to revelation proofness. It turns out that full revelation is the only kind of deviation that needs to be accounted for, which largely simplifies the analysis. Second, the sender can use messages that are the unions of a finite number of closed intervals; the importance of this feature is already explained in the previous sections.²¹

²¹One might believe that the payoffs of a laminar partition can be replicated using only closed interval messages, by identifying each message with its convex hull. However, this approach fails. Consider two partitional elements B_ℓ, B_h with $B_\ell \subseteq \text{co}(B_h)$. If one wants $\text{co}(B_h)$ to be sent in every

6.2 Robustness

It is natural to ask whether the equilibria of the disclosure game considered in this paper are credible in that they survive certain equilibrium refinements. We consider the following two equilibrium refinements:

- The Never-a-Weak-Best-Response (NWBR) Criterion, proposed by [Cho and Kreps \(1987\)](#), is a strengthening of a few equilibrium refinements that are extensively used in the literature, which includes the Intuitive Criterion, D1, and D2.²²
- The Grossman-Perry-Farrell equilibrium, proposed by [Bertomeu and Cianciaruso \(2018\)](#), is based on the perfect sequential equilibrium of [Grossman and Perry \(1986\)](#) and neologism-proofness of [Farrell \(1993\)](#).

It can be shown that every PE of the disclosure game we study is a Grossman-Perry-Farrell equilibrium. Furthermore, every PE outcome survives the NWBR Criterion. Interested readers are referred to the supplementary appendix for formal details.

6.3 Further Cheap Talk Opportunities

One may wonder what the sender would be able to achieve if he were also allowed to send cheap talk messages. Indeed, [Wu \(2022\)](#) and [Dasgupta \(2023\)](#) show that this benefits the sender in their respective settings. However, this is not the case in our setting.

Define the sender's value function in beliefs, denoted by $w : \Delta(\Omega) \rightarrow \mathbb{R}$, by $w(G) := v(\mathbb{E}G)$ for any $G \in \Delta(\Omega)$, where v is the sender's value function. Theorem 2 in [Lipnowski and Ravid \(2020\)](#) asserts that the sender's optimal value is given by the quasiconcave hull of his value function in beliefs evaluated at the prior F .²³ Observe that w is the composite function of v and the expectation operator \mathbb{E} . Because v is increasing, and an increasing transformation of an affine function is quasiconcave, w coincides with its quasiconcave hull. The aforementioned theorem therefore suggests that the sender never benefits from further cheap-talk opportunities.

$\omega \in B_h$, this message is also available to every $\omega \in B_\ell$ since $B_\ell = \text{co}(B_\ell) \subseteq \text{co}(B_h)$. Consequently, in almost every $\omega \in B_\ell$, the message $\text{co}(B_h)$ is sent, which renders replicating the laminar partition impossible.

²²Although closely related, this is not precisely the same as the NWBR property proposed by [Kohlberg and Mertens \(1986\)](#). For this reason, [Fudenberg and Tirole \(1991\)](#) call it “NWBR in signaling games.”

²³The quasiconcave hull of a function $w : \Delta(\Omega) \rightarrow \mathbb{R}$ is the pointwise lowest quasiconcave and upper semicontinuous function that majorizes w .

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A Omitted Proofs and Details

A.1 Proofs for Section 3

In what follows, we employ the following result, which is essentially Theorem 1 in [Titova and Zhang \(2025\)](#) adapted to our setting.²⁴ We provide a proof below for completeness.

Lemma 1. *\mathcal{B} is an equilibrium partition if and only if it is obedient and revelation-proof.*

Proof. Sufficiency is straightforward: if \mathcal{B} is not revelation-proof or obedient, then in every associated assessment, the sender has a profitable deviation to full revelation or the receiver is not best responding to an on-path message. For necessity, suppose that \mathcal{B} is an obedient and revelation-proof partition. Let (σ, τ, p) be an associated partition such that the receiver (1) has *maximally skeptical* off-path beliefs and (2) best responds according to her posterior mean and breaks ties in the sender-adversarial manner when indifferent. Specifically, for each $m \notin \mathcal{B}$, let $p(\min m \mid m) = 1$, and $\tau(j \mid m) = 1$ if $\min m \in [\gamma_{j-1}, \gamma_j)$ and $j \in N \setminus \{n\}$, or $\min m \in [\gamma_{n-1}, \gamma_n]$ and $j = n$. Then, (σ, τ, p) is an equilibrium, and thus \mathcal{B} is an equilibrium partition. Indeed, equilibrium conditions 2, 3, and 4 are satisfied by construction. Equilibrium condition 1 (sender has no profitable deviations for each ω) is satisfied because if $\omega \in B_i$ and $\omega \notin B_{i+1} \cup \dots \cup B_n$, then the sender’s interim payoff is u_i ; deviations to on-path messages with a higher index are not feasible, deviations to on-path messages with a lower index or to off-path messages yield an interim payoff of at most u_i . ■

To prove [Theorem 1](#), we will need three auxiliary results.

Claim 4. *Let H be a mean-preserving contraction of F with $|\text{supp}(H)| \leq n$. Then there exist cutoffs $0 =: d_0 \leq d_1 \leq \dots \leq d_{m-1} \leq d_m := 1$ with $m \leq |\text{supp}(H)|$ such that*

²⁴The messages in this paper are closed subsets of the state space, while in [Titova and Zhang \(2025\)](#) they are Borel subsets of the state space.

$\int_0^x H(q) dq \leq \int_0^x F(q) dq$ on $[d_{i-1}, d_i]$ for all $i = 1, \dots, m$ and the inequality binds only at d_{i-1} and d_i .

Proof of Claim 4. First, let $\text{supp}(G_S) = \{\tilde{\gamma}_i\}_{i=1}^k$, where $k \leq n$ since $|\text{supp}(H)| \leq n$ and $\tilde{\gamma}_i < \tilde{\gamma}_j$ if $i < j$. Since $F(0) = H(0) = 0$, F is strictly increasing (since $f > 0$), and H_S is a step function, there exists $\varepsilon > 0$ such that $F(x) > H(x)$ for all $x \in [0, \varepsilon]$. Since $H \in \text{MPC}(F)$, we have $\int_0^1 H(q) dq = \int_0^1 F(q) dq$, and hence the set

$$D_1 = \left\{ x \in [\varepsilon, 1] : \int_0^x H(q) dq = \int_0^x F(q) dq \right\}$$

is nonempty. Let $d_1 = \inf D_1$. If $d_1 = 1$, then set $m = 1$ and the proof is complete. For the rest of the proof, suppose that $d_1 < 1$.

Observe that $\tilde{\gamma}_1 < d_1 < \tilde{\gamma}_k$: if $d_1 \leq \tilde{\gamma}_1$, then $F(x) > H(x) = 0$ for all $x \in [0, d_1]$, which contradicts the definition of d_1 ; if $d_1 \geq \tilde{\gamma}_k$, then $F(x) < H(x) = 1$ for all $x \in [d_1, 1]$, so that $\int_0^1 H(q) dq \neq \int_0^1 F(q) dq$, a contradiction.

Next, we argue that $F(d_1) \geq H(d_1)$. Suppose to the contrary that $F(d_1) < H(d_1)$. Then, since H is a CDF and hence right-continuous, there must exist $\delta > 0$ such that $F(x) < H(x) = H(d_1)$ for all $x \in [d_1, d_1 + \delta]$, where the equality follows from the fact that H is a step function. Then,

$$\begin{aligned} \int_0^{d_1+\delta} F(q) dq &= \int_0^{d_1} F(q) dq + \int_{d_1}^{d_1+\delta} F(q) dq = \int_0^{d_1} H(q) dq + \int_{d_1}^{d_1+\delta} F(q) dq \\ &< \int_0^{d_1} H(q) dq + \int_{d_1}^{d_1+\delta} H(q) dq, \end{aligned}$$

which contradicts the assumption that H is a MPC of F .

Now, let $j = \min\{i : \tilde{\gamma}_i > d_1\}$. Given that $d_1 < \tilde{\gamma}_j$, $F(d_1) \geq H(d_1)$, F is strictly increasing, and H is a step function, there exists $\eta > 0$ such that $F(x) > H(x)$ for all $x \in [d_1, d_1 + \eta]$. Consequently, $\int_0^1 H(q) dq = \int_0^1 F(q) dq$ implies that the set

$$D_2 = \left\{ x \in [d_1 + \eta, 1] : \int_0^x H(q) dq = \int_0^x F(q) dq \right\}$$

is nonempty. Let $d_2 = \inf D_2$. If $d_2 = 1$, then set $m = 2$ and the proof is complete. For the remainder of the proof, suppose that $d_2 < 1$. Using the same steps as above, one can show that $d_2 > \tilde{\gamma}_j$ and $F(d_2) \geq H(d_2)$. Proceeding inductively, one can find d_t with $t \leq k \leq n$ such that $d_t > \tilde{\gamma}_k$. It must be that $d_t = 1$: suppose not, then because

F is strictly increasing, $F(x) < 1$ on $(d_k, 1)$; but $H(x) = 1$ on the same interval, which implies that $\int_0^1 H(q) dq > \int_0^1 F(q) dq$, a contradiction. Now set $m = t$, the proof is complete. \blacksquare

Claim 5. *If H is a mean-preserving contraction of F with $|\text{supp}(H)| \leq n$, then it induces a laminar partition.*

Proof. By Claim 4, on each of the m intervals such that the MPC constraint only binds at the endpoints, the mass is redistributed to at most n points, and there can be at most m such intervals. We show that every such interval admits a laminar partition; the definition of a laminar partition then implies that the resulting partition is still laminar by taking the union.

Denote an arbitrary interval on which the MPC constraint only binds at the endpoints by $I := [a, b]$; that is, $\int_0^x H(q) dq \leq \int_0^x F(q) dq$ for all $q \in I$, and the inequality binds only at a and b .

The remainder of the proof is very similar to the proof of Lemma 11 in Candogan and Strack (2023), and hence we only provide an outline here; readers interested in details are directed to that paper. Let $K = |\text{supp}(H) \cap I| \leq n$; the proof proceeds by induction on K . If $K = 1$, let $\text{supp}(H) \cap I = \{\tilde{\gamma}_\ell\}$; then clearly $\{B_\ell\}$ where $B_\ell = I$ is a laminar partition of I . If $K = 2$, let $\text{supp}(H) \cap I = \{\tilde{\gamma}_\ell, \tilde{\gamma}_m\}$ where $\ell < m$; then by Lemma 4 in Arieli et al. (2023), $\{B_\ell, B_m\}$ can be chosen such that $B_\ell = [c, d]$ and $B_m = [a, c] \cup [b, d]$, which is laminar.

Taking $K = 2$ as the base case, consider $K > 2$; the induction hypothesis holds for $K - 1$. One can find a closed interval B_ℓ such that (i) $\mu_F(B_\ell) = h(\tilde{\gamma}_\ell)$, where h is the probability mass function (pmf) of H , and $\tilde{\gamma}_\ell = \min(\text{supp}(H) \cap I)$,²⁵ and (ii) $\mathbb{E}[\omega \mid \omega \in B_\ell] = \tilde{\gamma}_\ell$. Consequently, conditional on $\omega \notin B_\ell$, H only has $K - 1$ mass points, and Lemma 12 in Candogan and Strack (2023) shows that it is a MPC of F . Invoking the inductive hypothesis, a laminar partition of I , $\{\hat{B}_i\}_{i \in \mathcal{T}}$, is obtained, where $\mathcal{T} := \{k \neq \ell : \gamma_k \in \text{supp}(H) \cap I\}$. For every $i \in \mathcal{T}$, let $B_i := \text{cl}(\hat{B}_i \setminus B_\ell)$. Since $\{\hat{B}_i\}_{i \in \mathcal{T}}$ is laminar, and $\ell < i$ for all $i \in \mathcal{T}$, $\{B_i\}_{i \in \mathcal{T} \cup \{\ell\}}$ is also laminar. \blacksquare

Claim 6. *Suppose that $W \subseteq [0, 1]$ and $B = [x, y] \subseteq [0, 1]$ such that $\mu_F(W) = \mu_F(B)$ and $\mathbb{E}[\omega \mid \omega \in W] = \mathbb{E}[\omega \mid \omega \in B]$. Then, $\inf W \leq x$ and $y \leq \sup W$.*

²⁵The only difference between our proof and Candogan and Strack (2023)'s is that they choose the mass point of H that has the largest index number on I , and for our purpose we work with the smallest. Their proof, however, goes through despite this difference.

Proof. We prove that $y \leq \sup W$; the proof of $\inf W \leq x$ is analogous. Since $\mu_F(W) = \mu_F(B)$ and $\mathbb{E}[\omega \mid \omega \in W] = \mathbb{E}[\omega \mid \omega \in B]$, we have $\int_{W \setminus B} \omega dF(\omega) = \int_{B \setminus W} \omega dF(\omega)$ and $\mu_F(W \setminus B) = \mu_F(B \setminus W)$.

Suppose, by contradiction, that $\sup W < y$. Then, $W \setminus B \subseteq [0, x]$, $B \setminus W \subseteq [x, y]$ and $\mu_F(W \setminus B) = \mu_F(B \setminus W) > 0$. Furthermore,

$$\int_{W \setminus B} \omega dF(\omega) < x\mu_F(W \setminus B) = x\mu_F(B \setminus W) < \int_{B \setminus W} \omega dF(\omega),$$

a contradiction. Therefore, $y \leq \sup W$. ■

A.1.1 Proof of Theorem 1

Proof of Part (a). First, observe that the fully revealing partition \mathcal{A} is obedient and revelation-proof since $A_i = [\gamma_{i-1}, \gamma_i]$ for each $i \in N$. By Lemma 1, it is an equilibrium partition.

Next, we show that the sender's ex-ante payoff cannot be lower than \underline{V} in any other equilibrium. Let $\underline{w}(\omega) := \min\{u_i : i \in N, \omega \in A_i\}$ be the sender's payoff in state ω when the receiver knows the state and breaks ties in the sender-adversarial manner. By definition, $\int_{\Omega} \underline{w}(\omega) dF(\omega) = \sum_{i \in N} \mu_F(A_i) u_i = \underline{V}$. If the sender's ex-ante payoff is strictly below \underline{V} in an equilibrium, then his interim payoff is strictly below $\underline{w}(\omega)$ in a positive measure of states. Then, in each of those states, the sender has a profitable deviation toward sending message $\{\omega\}$ and receiving $\underline{w}(\omega)$. Therefore, the sender's ex-ante payoff in a sender-worst equilibrium is exactly \underline{V} . This completes the proof of Part (a).

Proof of Part (b). We proceed in three steps. First, we show that if there exists a sender-preferred equilibrium, then there exists a sender-preferred laminar PE (Lemma 2). Second, we show that a sender-preferred laminar PE exists (Lemma 3). Third, we show that $\bar{V} > \underline{V}$ (Lemma 4).

Lemma 2. *For any equilibrium in which the receiver plays pure strategy, there exists a laminar equilibrium partition that induces the same posterior mean distribution.*

Furthermore, if there exists a sender-preferred equilibrium, then there exists a sender-preferred laminar partitional equilibrium.

Proof of Lemma 2. Let (σ, τ, p) be an equilibrium in which the receiver plays pure strategies. For any state $\omega \in \Omega$ in which the sender mixes, i.e., $|\text{supp } \sigma(\cdot \mid \omega)| > 1$,

since (σ, τ, p) is an equilibrium, there exists $i \in N$ such that $\tau(i \mid m) = \tau(i \mid m') = 1$ for every $m, m' \in \text{supp } \sigma(\cdot \mid \omega)$. Thus, in every $\omega \in \Omega$, there is an action i played with probability 1 in this equilibrium. For every $i \in N$, let

$$W_i := \{\omega \in \Omega: \tau(i \mid m) = 1 \text{ for all } m \in \text{supp } \sigma(\cdot \mid \omega)\}.$$

By construction, $\cup_{i \in N} W_i = \Omega$, and $W_i \cap W_j = \emptyset$ for any $i \neq j$. By Theorem 1(a) of [Titova and Zhang \(2025\)](#), for every $i \in N$, $\tilde{\gamma}_i := \mathbb{E}[\omega \mid \omega \in W_i] \in [\gamma_{i-1}, \gamma_i]$, and $W_i \subseteq [0, \gamma_i]$.

Consider the PMD \tilde{G} with $\text{supp } \tilde{G} \subseteq \{\tilde{\gamma}_i\}_{i=1}^n$ whose probability mass function is given by $\tilde{g}(\tilde{\gamma}_i) = \mu_F(W_i)$. By construction, \tilde{G} is a MPC of the PMD induced by the equilibrium (σ, τ, p) ; since G is induced by an equilibrium, it is a MPC of F . Thus, \tilde{G} is a mean-preserving contraction of F with $|\text{supp}(\tilde{G})| \leq n$. Then by [Claim 5](#), \tilde{G} is also induced by a laminar partition \mathcal{B} with generic element B_i . By construction, for every $i \in N$, $\mathbb{E}[\omega \mid \omega \in B_i] = \tilde{\gamma}_i$, which implies obedience. Also by construction, $\mu_F(B_i) = \mu_F(W_i)$ for each $i \in N$.

We show next that \mathcal{B} is also revelation-proof, i.e., $B_i \subseteq [0, \gamma_{i+1}]$ for all $i \in N$. Fix $i \in N$. Because \mathcal{B} is laminar, there exists $k \leq i$ such that for every action $j \in \{k, \dots, i\}$, $\text{co}(B_j) \subseteq \text{co}(B_i)$; and for every action $j \in \{1, \dots, k-1, i+1, \dots, n\}$, $\mu_F(B_j \cap \text{co}(B_i)) = 0$. This implies that $\widehat{B}_i := B_k \cup \dots \cup B_i = \text{co}(B_i)$ is an interval. Let $\widehat{W}_i := W_k \cup \dots \cup W_i$. Since for each $j = k, \dots, i$, $\mu_F(B_j) = \mu_F(W_j)$ and $\mathbb{E}[\omega \mid \omega \in B_j] = \mathbb{E}[\omega \mid \omega \in W_j]$, and since $W_j \cap W_\ell = \emptyset$ and $\mu_F(B_j \cap B_\ell) = 0$ for any $j \neq \ell$, we have $\mu_F(\widehat{B}_i) = \mu_F(\widehat{W}_i)$ and $\mathbb{E}[\omega \mid \omega \in \widehat{B}_i] = \mathbb{E}[\omega \mid \omega \in \widehat{W}_i]$. By [Claim 6](#), we have $\widehat{B}_i \subseteq \widehat{W}_i \subseteq [0, \gamma_{i+1}]$, where the last inclusion follows since $W_i \subseteq [0, \gamma_{i+1}]$ for all $i \in N$.

To prove the second statement, suppose that there exists a sender-preferred equilibrium. Without loss, we focus on a sender-preferred equilibrium (σ, τ, p) in which the receiver breaks ties in favor of the sender.²⁶ Then by the first statement, there must exist a sender-preferred laminar equilibrium partition. ■

Lemma 3. *Among all obedient and revelation-proof laminar partitions, there is one that maximizes the sender's ex-ante payoff.*

Proof. The problem of finding a laminar partition equilibrium that maximizes the

²⁶If the receiver plays a mixed strategy in some sender-preferred equilibrium (σ', τ', p') , then the assessment (σ', τ'', p') , where τ'' is obtained from τ' by breaking the receiver's ties in favor of the highest action in the support of τ' is also an equilibrium.

sender's ex-ante payoff can be written as

$$\begin{aligned}
& \max_{\mathcal{B} \text{ laminar}} \sum_{i \in N} u_i \mu_F(B_i) \\
& \text{s.t.} \quad \bigcup_{i \in N} B_i = [0, 1] \\
& \quad \mu_F(B_i \cap B_j) = \emptyset \text{ for all } i \neq j \\
& \quad \gamma_i \leq \mathbb{E}[\omega \mid \omega \in B_i] \leq \gamma_{i+1} \text{ for all } i \in N \\
& \quad B_i \subseteq [0, \gamma_{i+1}] \text{ for all } i \in N
\end{aligned}$$

To prove the lemma, it suffices to show that this problem has a solution.

Although an action j may be never recommended, one can still assume that B_j is nonempty by setting $\mu_F(B_j) = 0$. Because \mathcal{B} is laminar, it is without loss of generality to assume that for each $i \in N$, B_i is the union of at most n intervals (cf. [Observation 1](#)). Consequently, adding singletons if necessary, one can always set B_i as the union of exactly n convex sets $\{B_{i,s}\}_{s=1}^n$ such that $\mu_F(B_{i,s'} \cap B_{i,s''}) = 0$ for all $s' \neq s''$.

Let $\mathcal{C}_c([0, 1])$ denote the set of closed, nonempty, and convex subsets of $[0, 1]$ endowed with the Hausdorff distance; to simplify notation, we write \mathcal{C}_c henceforth. By Proposition 1 in [Ely \(2022\)](#), \mathcal{C}_c is compact; by Tychonoff's theorem, $\mathcal{C}_c^{n^2}$ is compact in the product topology. The problem above can be transformed to

$$\begin{aligned}
& \max_{\{B_{i,s}\} \in \mathcal{C}_c^{n^2}} \sum_{i \in N} u_i \sum_{s=1}^n \mu_F(B_{i,s}) \tag{5} \\
& \text{s.t.} \quad \bigcup_{i,s} B_{i,s} = [0, 1] \\
& \quad \mu_F(B_{i',s'} \cap B_{i'',s''}) = 0 \text{ for all } (i', s') \neq (i'', s'') \\
& \quad \mu_F(\bigcup_{s=1}^n B_{i,s}) \gamma_i \leq \int_{\bigcup_{s=1}^n B_{i,s}} \omega \, d\mu_F(\omega) \leq \mu_F(\bigcup_{s=1}^n B_{i+1,s}) \gamma_{i+1} \\
& \quad \bigcup_{s=1}^n B_{i,s} \subseteq [0, \gamma_{i+1}]
\end{aligned}$$

where the third constraint is equivalent to the conditional mean condition.

Define

$$\mathcal{D} = \left\{ \{B_{i,s}\} \in \mathcal{C}_c^{n^2} : \bigcup_{i,s} B_{i,s} = [0, 1], \text{ and } \mu_F(B_{i',s'} \cap B_{i'',s''}) = 0 \text{ for all } (i', s') \neq (i'', s'') \right\};$$

We claim that \mathcal{D} is compact. To show this, it is enough to show that \mathcal{D} is a closed

subset of $\mathcal{C}_c^{n^2}$. Take any $\{B_{i,s}^m\}$ that converges to $\{B_{i,s}\}$ in the product topology, then $B_{i,s}^m \rightarrow B_{i,s}$ for each i and s . Consequently, because the limit of convergence in Hausdorff distance is preserved under finite unions,²⁷ $\cup_{i,s} B_{i,s}^m \rightarrow \cup_{i,s} B_{i,s}$. Therefore, if $\cup_{i,s} B_{i,s}^m = [0, 1]$, it must be that $\cup_{i,s} B_{i,s} = [0, 1]$. Furthermore, if $\mu_F(B_{i',s'}^m \cap B_{i'',s''}^m) = 0$ for all m and $(i', s') \neq (i'', s'')$, the same argument as the second paragraph in the proof of Lemma 2 in Ely (2022) shows that $\mu_F(B_{i',s'} \cap B_{i'',s''}) = 0$ for all $(i', s') \neq (i'', s'')$. Therefore, if $\{B_{i,s}^m\} \in \mathcal{D}$ for each m and $\{B_{i,s}^m\} \rightarrow \{B_{i,s}\}$, it must be that $\{B_{i,s}\} \in \mathcal{D}$. Thus, \mathcal{D} is a closed subset of $\mathcal{C}_c^{n^2}$.

Problem (5) is equivalent to

$$\max_{\{B_{i,s}\} \in \mathcal{D}} \sum_{i \in N} u_i \sum_{s=1}^n \mu_F(B_{i,s}) \quad (6)$$

$$\text{s.t.} \quad \left(\sum_{s=1}^n \mu_F(B_{i,s}) \right) \gamma_i \leq \sum_{s=1}^n \int_{B_{i,s}} \omega \, d\mu_F(\omega) \leq \left(\sum_{s=1}^n \mu_F(B_{i,s}) \right) \gamma_{i+1} \quad (7)$$

$$\cup_{s=1}^n B_{i,s} \subseteq [0, \gamma_{i+1}] \quad (8)$$

where constraint (7) supersedes the the third constraint in problem (5) because for any $\{B_{i,s}\} \in \mathcal{D}$, $\mu_F(B_{i',s'} \cap B_{i'',s''}) = 0$ for all $(i', s') \neq (i'', s'')$.

By the extreme value theorem, to show that a solution to problem (6) exists, it suffices to show that (i) the objective function is continuous, and (ii) the constraint set is nonempty and compact. Clearly, the constraint set is nonempty: for each $i \in N$, consider

$$B_{i,1} = A_i, \text{ and } B_{i,s} = \{\gamma_{i+1}\} \text{ for all } s = 2, \dots, n,$$

then $\{B_{i,s}\}$ is feasible for this problem. Furthermore, by Proposition 1 in Ely (2022), μ_F is continuous on \mathcal{C}_c , and hence the objective is continuous.

Since \mathcal{D} is compact, to show that the constraint set is compact, it suffices to show that each of the constraints, (7) and (8), defines a closed subset of \mathcal{D} . Observe that if $C^m \rightarrow C$ and $D^m \rightarrow D$ with $C^m \subseteq D^m$ for all m , then $C \subseteq D$.²⁸ Then because the limit of convergence in Hausdorff distance is preserved under finite unions, if $\cup_{s=1}^n B_{i,s}^m \subseteq [0, \gamma_{i+1}]$ for each i , it must be that $\cup_{s=1}^n B_{i,s} \subseteq [0, \gamma_{i+1}]$ for each i . Hence, (8) defines a closed subset of \mathcal{D} .

Next, we show that (7) does the same, which is equivalent to showing that if

²⁷See, for example, Lemma 4.5 in McLennan (2018).

²⁸See, for example, page 15 in Ely (2022).

$\{B_{i,s}^m\} \rightarrow \{B_{i,s}\}$ where $\{B_{i,s}^m\} \in \mathcal{D}$ for each m , then

$$\left(\sum_{s=1}^n \mu_F(B_{i,s}^m) \right) \gamma_i \leq \sum_{s=1}^n \int_{B_{i,s}^m} \omega \, d\mu_F(\omega) \leq \left(\sum_{s=1}^n \mu_F(B_{i,s}^m) \right) \gamma_{i+1}$$

for all m and i implies that

$$\left(\sum_{s=1}^n \mu_F(B_{i,s}) \right) \gamma_i \leq \sum_{s=1}^n \int_{B_{i,s}} \omega \, d\mu_F(\omega) \leq \left(\sum_{s=1}^n \mu_F(B_{i,s}) \right) \gamma_{i+1}$$

for all i . Because μ_F is continuous on \mathcal{C}_c , $\sum_{s=1}^n \mu_F(B_{i,s}^m) \rightarrow \sum_{s=1}^n \mu_F(B_{i,s})$ for all i . Consequently, $(\sum_{s=1}^n \mu_F(B_{i,s}^m)) \gamma_i \rightarrow (\sum_{s=1}^n \mu_F(B_{i,s})) \gamma_i$, and $(\sum_{s=1}^n \mu_F(B_{i,s}^m)) \gamma_{i+1} \rightarrow (\sum_{s=1}^n \mu_F(B_{i,s})) \gamma_{i+1}$ for each i . Therefore, it only remains to show that

$$\sum_{s=1}^n \int_{B_{i,s}^m} \omega \, d\mu_F(\omega) \rightarrow \sum_{s=1}^n \int_{B_{i,s}} \omega \, d\mu_F(\omega),$$

which is a consequence of $E \mapsto \int_E \omega \, d\mu_F(\omega)$ being continuous on \mathcal{C}_c .

To prove this claim, we show that $E \mapsto \int_E \omega \, d\mu_F(\omega)$ is both upper- and lower-semicontinuous. To see that it is upper-semicontinuous, pick any $\varepsilon > 0$, and let $E \in \mathcal{C}_c$; we show that there exists $\delta > 0$ such that for every $E' \in N(E, \delta)$, where $N(E, \delta)$ is the δ -neighborhood of E , $\int_{E'} \omega \, d\mu_F(\omega) < \int_E \omega \, d\mu_F(\omega) + \varepsilon$. Because $E \in \mathcal{C}_c$, there exist $a, b \in [0, 1]$ such that $E = [a, b]$. The key observation here is that for any $E' \in N(E, \delta)$, it must be that $E' \subseteq [a - \delta, b + \delta]$. Then

$$\begin{aligned} \int_{a-\delta}^{b+\delta} \omega \, d\mu_F(\omega) &= \int_{a-\delta}^a \omega \, d\mu_F(\omega) + \int_a^b \omega \, d\mu_F(\omega) + \int_b^{b+\delta} \omega \, d\mu_F(\omega) \\ &\leq \mu_F([a - \delta, a]) + \int_E \omega \, d\mu_F(\omega) + \mu_F([b, b + \delta]) \end{aligned}$$

where the inequality holds because $\omega \in [0, 1]$. For δ small enough, since μ_F is absolutely continuous with respect to the Lebesgue measure,

$$\int_{E'} \omega \, d\mu_F(\omega) \leq \int_{a-\delta}^{b+\delta} \omega \, d\mu_F(\omega) < \int_E \omega \, d\mu_F(\omega) + \varepsilon.$$

Next, we show that $\int_E \omega \, d\mu_F(\omega)$ is lower semicontinuous, that is, there exists $\delta > 0$ such that for every $E' \in N(E, \delta)$, $\int_{E'} \omega \, d\mu_F(\omega) > \int_E \omega \, d\mu_F(\omega) - \varepsilon$. Without loss of

generality, assume $\delta < (b - a)/2$. Consequently, $[a + \delta, b - \delta]$ is an interval of positive measure, and for any $E' \in N(E, \delta)$, $[a + \delta, b - \delta] \subseteq E'$. Then

$$\begin{aligned} \int_{a+\delta}^{b-\delta} \omega \, d\mu_F(\omega) &= \int_a^b \omega \, d\mu_F(\omega) - \int_a^{a+\delta} \omega \, d\mu_F(\omega) - \int_{b-\delta}^b \omega \, d\mu_F(\omega) \\ &\geq \int_E \omega \, d\mu_F(\omega) - a\mu_F([a - \delta, a]) - a\mu_F([b, b + \delta]) \end{aligned}$$

where the inequality follows from the fact that $\omega \geq a$ on $[a, b]$. Consequently, $\int_E \omega \, d\mu_F(\omega)$ is both upper- and lower-semicontinuous, and hence continuous. This completes the proof. \blacksquare

Let \mathcal{B} denote a laminar partition associated with a sender-preferred equilibrium; by the previous two lemmas, such a partition exists. Denote the sender's ex-ante payoff from \mathcal{B} by \bar{V} ,

Lemma 4. $\bar{V} > \underline{V}$.

Proof. To prove the statement, we construct an equilibrium partition $\mathcal{B} = \{B_i\}_{i \in N}$ that yields $V(\mathcal{B}) > \underline{V}$. For each $i \in N \setminus \{1, n\}$, let $B_i = A_i$. Also, for some $\varepsilon \in (0, \gamma_2)$, let $B_1 = [\varepsilon, \gamma_2]$ and $B_n = [0, \varepsilon] \cup A_n$. It is easy to see that $\{B_i\}_{i \in N}$ is a revelation-proof partition. Furthermore, $\mathbb{E}[\omega \mid \omega \in B_i] \in A_i$ for all $i \in N \setminus \{n\}$ by construction. Finally, the function $\psi(\varepsilon') := \mathbb{E}[\omega \mid \omega \in B_n] = \int_0^{\varepsilon'} \omega dF(\omega) + \int_{A_n} \omega dF(\omega)$ is strictly decreasing, continuous and $\psi(0) > \gamma_n$. Consequently, there exists $\varepsilon > 0$ such that $\psi(\varepsilon) \in [\gamma_n, 1]$, which makes \mathcal{B} an obedient partition. By Lemma 1, \mathcal{B} is an equilibrium partition; the sender's ex-ante payoff in that equilibrium is

$$V(\mathcal{B}) = \underline{V} + \mu_F([0, \varepsilon])(u_n - u_1) > \underline{V},$$

which completes the proof. \blacksquare

Proof of Part (c). For any $V \in [\underline{V}, \bar{V}]$, we construct an obedient and revelation-proof laminar partition that yields V by “blending” the fully informative partition \mathcal{A} and a sender-preferred laminar equilibrium partition \mathcal{B} (which exists by part (b)). For all $z \in [0, \gamma_n]$,

- for every $i \in N$ such that $\gamma_{i+1} \leq z$, let $\hat{B}_i^z := A_i$;
- if i is such that $\gamma_i \leq z \leq \gamma_{i+1}$, $\hat{B}_i^z := \text{cl}(B_i \setminus [0, \gamma_i]) \cup [\gamma_i, z]$;

- for every $i \in N$ such that $z < \gamma_i$, let $\hat{B}_i^z := \text{cl}(B_i \setminus [0, z])$.

For any $z \in [0, \gamma_n]$, and any $i \in N$, $\mathbb{E}[\omega \mid \omega \in B_i] \in A_i$ implies $\mathbb{E}[\omega \mid \omega \in \hat{B}_i^z] \in A_i$. Consequently, $\hat{\mathcal{B}}(z) := \{\hat{B}_i^z\}_{i=1}^n$ is obedient. It is also revelation-proof because both \mathcal{A} and \mathcal{B} are. Finally, because both \mathcal{A} and \mathcal{B} are laminar, it can be checked using (1) that $\hat{\mathcal{B}}(z)$ is a laminar partition.

Let $V(z)$ denote the sender's ex-ante payoff from $\hat{\mathcal{B}}(z)$: $V(z) := \sum_{i \in N} u_i \mu_F(\hat{B}_i^z)$; because $f > 0$, $V(z)$ is continuous in z . Since $V(0) = \bar{V} \geq V \geq \underline{V} = V(\gamma_n)$, by the intermediate value theorem, there exists $z^V \in [0, \gamma_n]$ such that $V(z^V) = V$. Thus, the partition $\hat{\mathcal{B}}(z^V)$ is an obedient and revelation-proof laminar partition that yields V .

A.1.2 Proof of Theorem 2

Proof of (i). We first prove the following preliminary result.

Claim 7. *Let \mathcal{B} be a laminar partition associated with a sender-preferred equilibrium, and let $j = \min\{i : \mu_F(B_i) > 0\}$. If $\mathbb{E}[\omega \mid \omega \in B_j] > \gamma_j$, then $B_j = [0, d]$ for some $d \leq \gamma_{j+1}$.*

Proof. Because \mathcal{B} is a laminar partition, B_j must be an interval, and hence one can write $B_j = [\underline{b}_j, \bar{b}_j]$. Furthermore, revelation proofness implies that $[\underline{b}_j, \bar{b}_j] \subseteq [0, \gamma_{j+1}]$.

To show that $\underline{b}_j = 0$, it suffices to show that $\text{int}(B_j) \cap \text{co}(B_k) = \emptyset$ for all $k > j$.²⁹ Suppose not, so $B_j \subseteq \text{co}(B_k)$ for some $k > j$. The laminar structure implies that B_k is the union of at most k closed intervals; let $[\alpha, \beta]$ and $[\zeta, \eta]$ be an interval such that $\beta \leq \underline{b}_j$ and $[\alpha, \beta] \subseteq B_k$; this interval is well-defined because \mathcal{B} is laminar and $B_j \subseteq \text{co}(B_k)$. Because $\mathbb{E}[\omega \mid \omega \in B_k] = \gamma_k$ and $\mathbb{E}[\omega \mid \omega \in B_j] := \gamma_S > \gamma_j$, for small enough $\varepsilon > 0$, one can find $h(\varepsilon)$ such that

$$\mathbb{E}[\omega \mid \omega \in (B_k \cup [\bar{b}_j - h(\varepsilon), \bar{b}_j]) \setminus (\beta - \varepsilon, \beta)] = \gamma_k,$$

and

$$\mathbb{E}[\omega \mid \omega \in [\beta - \varepsilon, \beta] \cup [\bar{b}_j - h(\varepsilon), \bar{b}_j]] \geq \gamma_j.$$

Now define a new partition $\tilde{\mathcal{B}}$ with generic element \tilde{B}_i by $\tilde{B}_i = B_i$ if $i \neq j, k$, $\tilde{B}_j = [\beta - \varepsilon, \beta] \cup [\bar{b}_j - h(\varepsilon), \bar{b}_j]$, and $\tilde{B}_k = (B_k \cup [\bar{b}_j - h(\varepsilon), \bar{b}_j]) \setminus (\beta - \varepsilon, \beta)$. $\tilde{\mathcal{B}}$ is obedient by construction, and it is also revelation-proof because \mathcal{B} is; then by Lemma 1, $\tilde{\mathcal{B}}$ is an equilibrium partition. Furthermore, it must be that $\mu_F(\tilde{B}_k) > \mu_F(B_k)$: this is because

²⁹For a subset S of $[0, 1]$, $\text{int}(S)$ denotes the interior of S .

by construction, $\mathbb{E}[\omega \mid \omega \in B_k] = \mathbb{E}[\omega \mid \omega \in \tilde{B}_k] = \gamma_k$ and $\int_{\tilde{B}_k} \omega dF(\omega) > \int_{B_k} \omega dF(\omega)$. Consequently, the sender's payoff is strictly higher in this new equilibrium since $j < k$, which contradicts the assumption that \mathcal{B} is a partition associated with a sender-preferred equilibrium. \blacksquare

We are now ready to prove (i). Recall that a laminar partition is defined by (1). Because $\text{co}(B_k)$ must be an interval for all k , $\cup_{k < i} \text{co}(B_k)$ is the union of at most $i - 1$ intervals. By Claim 7, since \mathcal{B} is associated with a sender-preferred equilibrium, $\text{int}(B_0) \cap B_k = \emptyset$ for all $k \geq 2$. Thus, for any $i \geq 1$, by taking out $\cup_{k < i} \text{co}(B_k)$ from $\text{co}(B_i)$, at most $\max\{i - 2, 0\}$ intervals are removed, and hence the remainder, namely $\text{co}(B_i) \setminus \cup_{k < i} \text{co}(B_k)$, must be the union of at most $\max\{i - 1, 1\}$ intervals. By taking closure, B_i is also the union of at most $\max\{i - 1, 1\}$ intervals.

Proof of (ii). Suppose to the contrary that \mathcal{B} , a laminar partition associated with a sender-preferred equilibrium, has $\mathbb{E}[\omega \mid \omega \in B_k] > \gamma_k$ for some non-null $k > j$. By Lemma 1, \mathcal{B} is both obedient and revelation-proof. Let $\underline{b}_j := \min B_j$; because $f > 0$, there exists $x \in B_j$ with $x > \underline{b}_j$ such that $\mathbb{E}[\omega \mid \omega \in [\underline{b}_j, x] \cup B_k] \geq \gamma_k$, and $\mathbb{E}[\omega \mid \omega \in B_j \setminus [\underline{b}_j, x]] > \gamma_j$. Now consider the partition $\tilde{\mathcal{B}}$ where $\tilde{B}_j = B_j \setminus [\underline{b}_j, x]$, $\tilde{B}_k = [\underline{b}_j, x] \cup B_k$, and $\tilde{B}_i = B_i$ for $i \neq j, k$. $\tilde{\mathcal{B}}$ is obedient and revelation-proof because \mathcal{B} is, and the sender's ex-ante payoff from $\tilde{\mathcal{B}}$, $V(\tilde{\mathcal{B}})$, satisfies

$$V(\tilde{\mathcal{B}}) - \bar{V} = [F(x) - F(\underline{b}_j)] (u_k - u_j) > 0,$$

a contradiction.

Proof of the “Furthermore” statement. Let \mathcal{B} be a barely obedient laminar partition. For convenience, for any $B_i \in \mathcal{B}$, denote $\underline{b}_i := \min B_i$ and $\bar{b}_i := \max B_i$.³⁰ We first prove the following auxiliary result.

Claim 8. *Let \mathcal{B} be a barely obedient laminar partition, and let i, j be two unskipped actions with $i < j$. Then if $\mu_F(\text{co}(B_i) \cap \text{co}(B_j)) = 0$, then every $\omega' \in B_j$ and $\omega'' \in B_i$ satisfy $\omega' \leq \omega''$.*

Proof of Claim 8. Because $\mu_F(\text{co}(B_i) \cap \text{co}(B_j)) = 0$, either $\omega' \leq \omega''$ for every $\omega' \in B_j$ and $\omega'' \in B_i$, or $\omega' \geq \omega''$ for every $\omega' \in B_j$ and $\omega'' \in B_i$. The second possibility, however,

³⁰If $B_i = \emptyset$, we let $\underline{b}_i = \bar{b}_i = 0$.

cannot be true: by obedience, $j < i$ implies that $\underline{b}_j \leq \gamma_i$, and $\mathbb{E}[\omega \mid \omega \in B_i] \geq \gamma_i$. Consequently, $\bar{b}_i > \gamma_i \geq \underline{b}_j$, a contradiction. Therefore, it must be that $\omega' \leq \omega''$ for every $\omega' \in B_j$ and $\omega'' \in B_i$. \blacksquare

We are now ready to prove the equivalence between revelation proofness and the condition that $\max B_i \leq \gamma_{i+1}$ for all unskipped i such that $i + 1$ either nests i or is skipped.

Proof of the equivalence. Suppose first that $\max B_i \leq \gamma_{i+1}$ for all unskipped i such that $i + 1$ either nests i or is skipped, and we show that for any action $i \in N$, $B_i \subseteq [0, \gamma_{i+1}]$. Since \mathcal{B} is laminar, it is without loss of generality to assume that if action i is skipped, $B_i \subseteq [0, \gamma_{i+1}]$.

Now suppose i is not skipped. There are two possibilities: either action $i + 1$ is skipped or not. If $i + 1$ is skipped, by assumption $\bar{b}_i \leq \gamma_{i+1}$, implying $B_i \subseteq [0, \gamma_{i+1}]$. If instead $i + 1$ is not skipped, there are two cases: either $B_i \subseteq \text{co}(B_{i+1})$ or not. If $B_i \subseteq \text{co}(B_{i+1})$, then by assumption $\bar{b}_i \leq \gamma_{i+1}$, implying that $B_i \subseteq [0, \gamma_{i+1}]$.

Suppose instead that B_i is not a subset of $\text{co}(B_{i+1})$, and suppose to the contrary that $\bar{b}_i > \gamma_{i+1}$. Because \mathcal{B} is laminar, since B_i is not a subset of $\text{co}(B_{i+1})$ (which means that $\text{co}(B_{i+1})$ does not nest $\text{co}(B_i)$), it must be that $\mu_F(\text{co}(B_i) \cap \text{co}(B_{i+1})) = \emptyset$. By [Claim 8](#), for every $\omega' \in B_i$ and every $\omega'' \in B_{i+1}$, $\omega' \leq \omega''$. This implies that $\omega \geq \bar{b}_i > \gamma_{i+1}$ for every $\omega \in B_{i+1}$, violating bare obedience, a contradiction. Thus, $\bar{b}_i \leq \gamma_{i+1}$, which implies that $B_i \subseteq [0, \gamma_{i+1}]$.

For the other direction, it suffices to prove the contrapositive. Note that if the condition is violated; that is, $\max B_i > \gamma_{i+1}$ for some unskipped i such that $i + 1$ either nests i or is skipped, implying that B_i is not a subset of $[0, \gamma_{i+1}]$. Thus, the laminar partition \mathcal{B} must violate revelation proofness. \blacksquare

A.2 Proofs and Details for Section 4

We first introduce the notion of a bi-pooling distribution.

Definition 4 (Bi-pooling distribution). A distribution $G \in \text{MPC}(F)$ is a bi-pooling distribution if there exists a collection of pairwise disjoint intervals $\{(\underline{\omega}_i, \bar{\omega}_i)\}_{i \in \mathcal{I}}$ such that

- for all $i \in \mathcal{I}$, $G(\bar{\omega}_i) - G(\underline{\omega}_i) = F(\bar{\omega}_i) - F(\underline{\omega}_i)$ and $|\text{supp}(G|_{(\underline{\omega}_i, \bar{\omega}_i)})| \leq 2$,³¹

³¹For any cumulative distribution function H , let $H|_{[c,d]}$ denote the restriction of G to $[c, d] \subseteq [0, 1]$.

- $G|_{[0,1] \setminus \cup_{i \in \mathcal{I}} (\underline{\omega}_i, \bar{\omega}_i)} = F|_{[0,1] \setminus \cup_{i \in \mathcal{I}} (\underline{\omega}_i, \bar{\omega}_i)}$.

In particular, $(\underline{\omega}_i, \bar{\omega}_i)$ is called a pooling interval if $|\text{supp}(G|_{(\underline{\omega}_i, \bar{\omega}_i)})| = 1$, and it is called a bi-pooling interval if $|\text{supp}(G|_{(\underline{\omega}_i, \bar{\omega}_i)})| = 2$.

We call a bi-pooling distribution G_B that solves the commitment problem (3) a bi-pooling solution.

The following observation is proved useful.

Lemma 5 (Candogan, 2019). *Every bi-pooling solution to the commitment problem satisfies $\text{supp}(G_B) \subseteq [0, \tilde{\gamma}] \cup \{\gamma_i\}_{i=2}^n$, where $\tilde{\gamma} \in [0, 1]$ is such that $\tilde{\gamma} \leq \gamma$ for all $\gamma \in \text{supp}(G_B) \cap \{\gamma_i\}_{i=2}^n$.*

An important consequence of Lemma 5 is that every signal realization can be identified by the action it induces. In particular, every $x \in \text{supp}(G_B) \cap [0, \tilde{\gamma}]$ induces the lowest unskipped action, and γ_i induces action i for each $i = 2, \dots, n$. Moreover, if a bi-pooling solution is associated with a bi-pooling partition, then Lemma 5 implies that the partition must be barely obedient.

A.2.1 Proof of Theorem 3

Part (i). By results in Kleiner et al. (2021) and Arieli et al. (2023), the commitment problem (3) admits a bi-pooling solution.

By Lemma 5, in any bi-pooling solution G_B , if $\text{supp}(G_B) \setminus \{\gamma_i\}_{i=2}^n$ is nonempty but not a singleton, there must exist an interval $[0, d]$ on which action 1 is recommended, and $[d, 1]$ is comprised of pooling intervals and/or bi-pooling intervals; in this case, $[0, d]$ can be viewed as a pooling interval. Otherwise, $[0, 1]$ comprises pooling and/or bi-pooling intervals. The fact that every bi-pooling solution G_B is associated with a bi-pooling partition therefore follows from Lemma 4 in Arieli et al. (2023): on every pooling interval, a single signal realizes, which induces a single action and hence ties to a single partitional element. On every bi-pooling interval $(\underline{\omega}, \bar{\omega})$, there exists $(\underline{q}, \bar{q}) \subseteq (\underline{\omega}, \bar{\omega})$ such that one (deterministic) signal realizes when $\omega \in (\underline{q}, \bar{q})$, and another signal realizes when $\omega \in (\underline{\omega}, \underline{q}) \cup (\bar{q}, \bar{\omega})$, with the latter signal recommending a higher action. Then $[\underline{q}, \bar{q}]$ and $[\underline{\omega}, \underline{q}] \cup [\bar{q}, \bar{\omega}]$ correspond to two partitional elements B_ℓ and B_h with $\ell < h$, respectively. This observation and Lemma 5 together imply that the bi-pooling partition is barely obedient.

To see uniqueness of the barely obedient bi-pooling partition, first note that every pooling interval necessarily ties to a unique partitional element; hence if a bi-pooling

solution admits two distinct bi-pooling partitions, \mathcal{B}^1 and \mathcal{B}^2 , they must differ on a bi-pooling interval $(\underline{\omega}, \bar{\omega})$. Then there must exist $\{B_\ell, B_h\}$ and $\{B'_\ell, B'_h\}$ such that $B_\ell = [\underline{b}_\ell, \bar{b}_\ell]$, $B_h = [\underline{\omega}, \underline{b}_\ell] \cup [\bar{b}_\ell, \bar{\omega}]$, $B'_\ell = [\underline{b}'_\ell, \bar{b}'_\ell]$, and $B'_h = [\underline{\omega}, \underline{b}'_\ell] \cup [\bar{b}'_\ell, \bar{\omega}]$, with either $\underline{b}'_\ell \neq \underline{b}_\ell$, or $\bar{b}'_\ell \neq \bar{b}_\ell$, or both. Since $f > 0$ and $\underline{\omega}$ and $\bar{\omega}$ are fixed, either $\mathbb{E}[\omega \mid \omega \in B_\ell] \neq \gamma_\ell$, or $\mathbb{E}[\omega \mid \omega \in B'_\ell] \neq \gamma_\ell$, or both. A contradiction.

Part (ii). Based on part (i), it suffices to show that the solution to the commitment problem (3) is unique, and the unique solution is a unique bi-pooling distribution. For any bi-pooling solution G_B , let $\mu_i := G_B(\gamma_{i+1}) - G_B(\gamma_i)$ for each $i = 2, \dots, n$. If $\text{supp}(G_B) \setminus \{\gamma_i\}_{i=1}^n = \emptyset$, set $\mu_1 = G_B(\gamma_2)$ and $\tilde{\mu} = 0$; otherwise, let $\tilde{\mu} = G_B(\tilde{\gamma})$ and $\mu_1 = G_B(\gamma_2) - \tilde{\mu}$; Lemma 5 then indicates that every bi-pooling solution is identified by $\{\tilde{\mu}\} \cup \{\mu_i\}_{i=1}^n$. By Lemma D.1 in Candogan (2022), no three elements of the collection of points $\{(\gamma_i, u_i)\}_{i=1}^n$ are collinear implies that any two bi-pooling solutions of problem (3) induces the same collection of $\{\tilde{\mu}\} \cup \{\mu_i\}_{i=2}^n$. Then because every bi-pooling solution is a mean-preserving spread of F , any two such bi-pooling solutions must be identical. As noted in Kleiner et al. (2021) and Arieli et al. (2023), every extreme point of the set of solutions to problem (3) is a bi-pooling solution, and hence the solution to problem (3) must be a unique bi-pooling solution.

A.2.2 Proof of Proposition 1

Since the bi-pooling partition is barely obedient, the “if” direction follows directly from Lemma 1. For the other direction, suppose first that a bi-pooling solution G_B is implementable. Because the commitment payoff is an upper bound of equilibrium payoffs, G_B is the PMD induced by a sender-preferred equilibrium (σ, τ, p) . By Theorem 3, G_B is associated with a unique barely obedient bi-pooling partition \mathcal{B} ; an argument analogous to the one in the proof of Lemma 2 shows that \mathcal{B} is revelation-proof.

A.2.3 Proof of Proposition 2

We use the following result due to Arieli et al. (2023) to prove Proposition 2. Say that $\{\gamma_\ell, \gamma_h\}$ is a feasible bi-pooling support for the interval $(\underline{\omega}, \bar{\omega})$ (or just feasible for simplicity) if there exists a mean preserving contraction of $F|_{[\underline{\omega}, \bar{\omega}]}$ whose support is $\{\gamma_\ell, \gamma_h\}$.

Lemma 6 (Arieli et al., 2023). *Fix an interval $(\underline{\omega}, \bar{\omega})$, and let $y(\gamma_\ell)$ satisfy $\mathbb{E}[\omega \mid \omega \in [\underline{\omega}, y(\gamma_\ell)]] = \gamma_\ell$. Then $\{\gamma_\ell, \gamma_h\}$ is feasible for the interval $(\underline{\omega}, \bar{\omega})$ if and only if*

- (i) $\underline{\omega} \leq \gamma_\ell \leq \mathbb{E}[\omega \mid \omega \in [\underline{\omega}, \bar{\omega}]] \leq \gamma_h \leq \bar{\omega}$, and
(ii) $\mathbb{E}[\omega \mid \omega \in [y(\gamma_\ell), \bar{\omega}]] \geq \gamma_h$.

Proof of Proposition 2. Suppose to the contrary that there exists a bi-pooling solution G_B that is not implementable. Then by Corollary 1, the bi-pooling partition of G_B , \mathcal{B} , must violate the condition therein. First suppose that there exists an unskipped action i such that $\bar{b}_i > \gamma_{i+1}$. There are two cases.

Case 1. B_i is an interval; i.e., $B_i = [\underline{b}_i, \bar{b}_i]$. There are two subcases:

(I) $\gamma_{i-1} \in \text{int}(B_i)$.

For $z \in (\gamma_{i+1}, \bar{b}_i)$, let $h(z)$ solve $\mathbb{E}[\omega \mid \omega \in [\underline{b}_i, h(z)] \cup [z, \bar{b}_i]] = \gamma_{i+1}$. By Lemma 6, for z close enough to \bar{b}_i , $\{\gamma_{i-1}, \gamma_i\}$ is feasible for $[h(z), z]$. Consequently, the sender's payoff on B_i , as a function of z , is

$$P(z) := u_{i+1}[F(\bar{b}_i) - F(z) + F(h(z)) - F(\underline{b}_i)] + \left[\frac{m(z) - \gamma_{i-1}}{\gamma_i - \gamma_{i-1}} u_i + \frac{\gamma_i - m(z)}{\gamma_i - \gamma_{i-1}} u_{i-1} \right] (F(z) - F(h(z)))$$

where $m(z) := \mathbb{E}[\omega \mid \omega \in [z, h(z)]]$. To show that this is a profitable deviation, it suffices to show that $P'(\bar{b}_i) := \lim_{z \nearrow \bar{b}_i} P'(z) < 0$. To this end, we first calculate

$$P'(z) = \frac{f(z)}{\gamma_j - \gamma_k} \frac{z - h(z)}{\gamma_{i+1} - h(z)} [(u_i - u_{i-1})\gamma_{i+1} + (u_{i+1} - u_i)\gamma_{i-1} - (u_{i+1} - u_{i-1})\gamma_i].$$

Letting $z \nearrow \bar{b}_i$, then $h(z) \searrow \underline{b}_i$, and

$$P'(\bar{b}_i) = \frac{f(\bar{b}_i)}{\gamma_i - \gamma_{i-1}} \frac{\bar{b}_i - \underline{b}_i}{\gamma_{i+1} - \underline{b}_i} [(u_i - u_{i-1})\gamma_{i+1} + (u_{i+1} - u_i)\gamma_{i-1} - (u_{i+1} - u_{i-1})\gamma_i];$$

$P'(\bar{b}_i) < 0$ if and only if $(u_{i+1} - u_{i-1})\gamma_i - (u_i - u_{i-1})\gamma_{i+1} - (u_{i+1} - u_i)\gamma_{i-1} > 0$, and this is equivalent to

$$\frac{u_{i+1} - u_i}{\gamma_{i+1} - \gamma_i} > \frac{u_i - u_{i-1}}{\gamma_i - \gamma_{i-1}},$$

which is implied by (4).

(II) $\gamma_{i-1} \notin \text{int}(B_i)$ (and therefore $\gamma_{i-1} \leq \underline{b}_i$).

Define $B_i^\varepsilon = [\underline{b}_i + \varepsilon, \bar{b}_i]$; for small enough $\varepsilon > 0$, $\{\gamma_i, \gamma_{i+1}\}$ is feasible for B_i^ε . Let $m(\varepsilon)$ denote the mean of B_i^ε ; note that $m(0) = \gamma_i$. Then the sender's payoff as a

function of ε on B_i is

$$P(\varepsilon) := \left[\frac{m(\varepsilon) - \gamma_i}{\gamma_{i+1} - \gamma_i} u_{i+1} + \frac{\gamma_{i+1} - m(\varepsilon)}{\gamma_{i+1} - \gamma_i} u_i \right] [F(\bar{b}_i) - F(\underline{b}_i + \varepsilon)] + u_{i-1} [F(\underline{b}_i + \varepsilon) - F(\underline{b}_i)].$$

To show that this is a profitable deviation, it suffices to show that $P'(0) := \lim_{\varepsilon \searrow 0} P'(\varepsilon) > 0$. Algebra reveals that

$$P'(\varepsilon) = f(\underline{b}_i + \varepsilon) \left[u_{i-1} - \frac{\gamma_{i+1} u_i - \gamma_i u_{i+1}}{\gamma_{i+1} - \gamma_i} - (\underline{b}_i - \varepsilon) \frac{u_{i+1} - u_i}{\gamma_{i+1} - \gamma_i} \right];$$

consequently, as $\varepsilon \searrow 0$,

$$P'(0) = \frac{f(\underline{b}_i)}{\gamma_{i+1} - \gamma_i} [(\gamma_i - \underline{b}_i)(u_{i+1} - u_i) - (\gamma_{i+1} - \gamma_i)(u_i - u_{i-1})]. \quad (9)$$

Because $f(\underline{b}_i) > 0$ and $\gamma_{i+1} - \gamma_i > 0$ by assumption, the sign of $P'(0)$ is the same as the sign of term in the square brackets in the right-hand side of (9). Thus, $P'(0) > 0$ if and only if

$$\frac{u_{i+1} - u_i}{\gamma_{i+1} - \gamma_i} > \frac{u_i - u_{i-1}}{\gamma_i - \underline{b}_i}.$$

Because $\gamma_{i+1} < \bar{b}_i$, $h(\gamma_i; \gamma_{i+1}) > \underline{b}_i$. Thus, (4) implies the inequality above.

Case 2. There is a partitional element B_j with $j < i$ such that $B_j \subseteq \text{co}(B_i)$. In this case, $B_j = [\underline{b}_j, \bar{b}_j]$, and $B_i = [\underline{b}_i, \underline{b}_j] \cup [\bar{b}_j, \bar{b}_i]$. WLOG, assume that $\underline{b}_i < \underline{b}_j$; by Lemma 6, $\{\gamma_j, \gamma_i\}$ is feasible for $[\underline{b}_i, \bar{b}_i]$, and

$$\mathbb{E}[\omega \mid \omega \in [y(\gamma_j), \bar{b}_i]] > \gamma_i. \quad (10)$$

Furthermore, it must be that $\gamma_i \in [\bar{b}_j, \bar{b}_i]$: if instead $\gamma_i \in [\underline{b}_i, \underline{b}_j]$, it must be that $\gamma_i < \gamma_j$, which violates the assumption that $j < i$.

For $z \in (\gamma_i, \bar{b}_i)$, let $h(z)$ solve $\mathbb{E}[\omega \mid \omega \in [\underline{b}_i, h(z)] \cup [z, \bar{b}_i]] = \gamma_i$. By (10) and Lemma 6, for z close enough to \bar{b}_i , $\{\gamma_j, \gamma_i\}$ is feasible for $[h(z), z]$. Consequently, one can find a profitable deviation similar to in Case 1 (I) *mutatis mutandis*.

Now suppose instead that there exists a pair of partitional elements B_i and B_{i+1} with $B_i \subseteq \text{co}(B_{i+1})$, and $\gamma_{i+1} < \bar{b}_i$. Since $B_i = [\underline{b}_i, \bar{b}_i]$, this case is isomorphic to Case 1 when action $i + 1$ is skipped. As a consequence, we can similarly find a profitable deviation.

Therefore, every bi-pooling solution must satisfy the condition in [Corollary 1](#), and hence implementable. This completes the proof. ■

A.2.4 Proof of [Corollary 2](#)

By definition of $h(\gamma_i; \gamma_{i+1})$, when f is increasing, $\gamma_{i+1} - \gamma_i \leq \gamma_i - h(\gamma_i; \gamma_{i+1})$. Then since $\gamma_{i+1} - \gamma_i \leq \gamma_i - \gamma_{i-1}$, it must be that $\gamma_{i+1} - \gamma_i \leq \min\{\gamma_i - \gamma_{i-1}, \gamma_i - h(\gamma_i; \gamma_{i+1})\}$. Now there are two cases. If $u_{i+1} - u_i > u_i - u_{i-1}$, (4) must hold; then by [Proposition 2](#), every bi-pooling solution can be implemented. If instead $\gamma_{i+1} - \gamma_i < \gamma_i - \gamma_{i-1}$, it must be that $\gamma_{i+1} - \gamma_i < \min\{\gamma_i - \gamma_{i-1}, \gamma_i - h(\gamma_i; \gamma_{i+1})\}$; this inequality and $u_{i+1} - u_i \geq u_i - u_{i-1}$ together imply (4). Again by [Proposition 2](#), every bi-pooling solution can be implemented.