Collective Search for a Public Good*

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November 2019

Abstract

This paper studies a model of costly sequential search among risky alternatives performed by a group of agents. The learning process stops and the best uncovered option is implemented when the agents unanimously agree to stop, or when all the projects have been researched. Both the implemented project and all the information gathered during the search process are public goods. I show that the equilibrium path implements the same project based on the same information, gathered in the same order as the social planner. At the same time, due to free-riding, search in teams does lead to a delay at each stage of the learning process, the size of which grows with search costs. Consequently, the team manager prefers to delegate search to an individual agent, while every agent prefers searching with a partner, since in the latter case she collects the same reward, but only pays the search cost half the time.

*I am particularly grateful to my friend and colleague Alex Weiss for productive conversations, support, and encouragement throughout the course of writing this paper. I thank my advisors Renee Bowen and Joel Sobel for useful discussions and detailed feedback. I am also grateful to the seminar participants at UC San Diego for their comments and feedback. All errors are my own.

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1. INTRODUCTION

In many situations, a group of people is presented with a problem that requires a collective solution. Oftentimes, the solutions are not readily available, and the team members must engage in costly search to learn what the options are. In the end, not only the resulting project but also all the information learned along the way is a public good.

There are many real-life examples of such situations. For instance, *cooperative consumer search*, with members of a household searching for a new house to move to; *search for investment opportunities*, with a team of hedge-fund managers deciding which asset to invest into; *adoption of new technologies*, with a team of employees searching to upgrade company-wide software; *research and development*, with a team of scientists deciding which research idea to pursue. In general, any situation that involves sequential social learning and that results in the final project benefiting everyone can be studied using this model. Broadly speaking, I study inefficiencies that arise as an artifact of sequential searching *in teams*, rather than individually. My most important result is that, compared to the socially optimal search process of just one agent searching individually, the team will use the exact same *order* and *stopping* rule while learning. In other words, the family will move to the same house, the funds will be invested in the same asset, the same software will be adopted company-wide, the same research idea will eventually be undertaken as if those decisions have been made by a social planner. Furthermore, the final decision in all these examples is made based on the exact same information gathered during the learning phase. Team search may be very inefficient, however, due to the free-riding effect, because agents procrastinate at each stage of the search process, and hope that their colleagues exert the search effort instead.

I model the sequential search process after the seminal model of Weitzman (1979). Each public good (project) is represented by a box with an uncertain reward that is revealed upon opening this box, which comes at a cost. At each stage of the game, one agent is randomly *chosen* and presented with three alternatives: opening a box of her choice, doing nothing, and proposing to terminate the game. In the latter case, her *opponent* can accept this offer, or reject it. In the event that the game was not terminated, a new stage begins, and if some box has been opened, then its contents are revealed. The game ends if the players agree to terminate the search, or if all boxes have been opened. At the end of the game, the best observed reward is collected (the best researched project is implemented).[1] I am studying *(i)* the optimal order of search among alternatives, *(ii)* incentives to shirk and free-ride on your colleague's search efforts, *(iii)* how the search in teams compares to the individual search.

I find that in the symmetric equilibrium, the chosen player acts very similarly to the way she would act had she been searching alone. More precisely, if she decides to open a box, she opens the "best" box according to Weitzman (1979) i.e. the box with the highest reservation value. She also wants to stop and proposes to terminate the game, at the same threshold – when no boxes are "good enough" to be opened, i.e. the highest reservation value among the unopened boxes is lower than the current best prize. In that case, her opponent always accepts the termination offer, so the search *order* and *termination* protocol on the equilibrium path are essentially indistinguishable from that of an individual searcher. The only difference is that agents free-ride when search costs are sufficiently large: the chosen player only decides to open some box *sometimes*, shirking the rest of the time, in hopes that her opponent gets to exert the search effort first.

I find that, compared to searching by herself, each agent is doing better on average when she is searching as a part of the team. A simple intuition behind this argument is that the search process is the same, but each team member only pays the search cost about half the time. In terms of social welfare, however, team search is inefficient due to *every* box being opened with a delay. The dynamics of the delay over time are inconclusive in my model. On the one hand, the players are more eager to open better boxes at the beginning of the search process in the hopes of terminating it early. On the other hand, the best revealed reward grows with time, and that speeds up the process towards the end.

[1] Weitzman's setup was generalized by Olszewski and Weber (2015) to account for cases when the searcher's payoff depends on all discovered prizes, rather than just the best one, and by Doval (2018) to allow the searcher to stop and take any unopened box, foregoing the inspection costs. I focus on Weitzman's setup because of the simple form of the solution that allows for a straightforward comparison of the team vs. social planner results.

Literature Review

First and foremost, this paper contributes to the literature on collective experimentation. In their seminar papers, Bolton and Harris (1999) extend the classic two-armed bandit problem to a many-agent setting. Information becomes a public good and that causes inefficiencies because agents have an incentive to wait and let their colleagues experiment instead, which is known as the *free-rider effect*. At the same time, the prospect of others experimenting forces every agent to experiment more, and that is known as the *encouragement effect*. The strength of each of these effects depends on the problem. For example, if the bandit is exponential, as in Keller, Rady, and Cripps (2005), then only the free-riding effect is present. With Poisson bandits, Keller and Rady (2010) show that the encouragement effect dominates. All these papers focus on a two-armed bandit that has one safe, one risky arm. In this paper, I consider a multi-armed bandit, such that the outcome of an arm is revealed after just one experiment. This allows me to study the *order* as well as the *stopping rule* of the search process for the public good. Furthermore, the agents in my model have to agree to stop. In the setting with heterogeneous preferences of agents, Bonatti and Rantakari (2016) show that forcing agents to agree on when to stop experimentation leads to polarization: it is optimal for one agent to develop her favorite project as quickly as possible, and for her opponent to agree to projects developed early in the game, since that way they will not have to exert effort themselves. Since I am studying public goods, there is no polarization, and I show that agents agree to stop the search process when it is individually (and socially) optimal to do so. They do shirk, however, in hopes of the opponent exerting the search effort, and that may lead to significant delays.

The literature on collective experimentation is closely related to the literature on delegation and approval of experimentation. Instead of two (or more) agents experimenting with bandits as a team, Manso (2011) studies a setting in which one agent (a principal) delegates the experimentation to another agent, and shows that the optimal mechanism must exhibit tolerance for early failure. Lewis (2012) examines a delegated sequential search problem and also concludes that the early efforts of the searcher must be rewarded more generously, even if unsuccessful. Halac et al. (2016) extend the multi-agent experimentation setup of Keller, Rady, and Cripps (2005) to have a principal who does not observe ability (adverse selection) or choice (moral hazard) of the agents. They find that asymmetric information leads to less experimentation, lower success rate, and more variance in success rates. On the bright side, all these inefficiencies are resolved if only moral hazard, or only adverse selection, is present. More recent papers by Guo (2016) and McClellan (2019) consider a different kind of asymmetric information: in their setup, the agent is privately informed about the state of nature, and the principal has to come up with an approval protocol for experimentation. This becomes a dynamic mechanism design problem, and the authors characterize optimal mechanisms of eliciting agent's information that feature cutoff approval rules. I abstract away from the principal's incentive-compatibility problem and study individual and team incentives instead. This allows me to conclude that from the outside (i.e. the manager's) perspective, search by an individual is more efficient because it happens without any delay, whereas each team member would rather search as part of a team.

This paper contributes to the literature on the search by committees. In the setup of Strulovici (2010), a group of heterogeneous voters is searching for policies and then votes for approval. He shows that under-experimentation occurs due to two phenomena: loser trap (the pivotal voters have overly optimistic beliefs and they have enough power to reform) and winner frustration (the pivotal voters have too pessimistic beliefs so the change never occurs), the sizes of which vary across voting rules. Compte and Jehiel (2010) focus on the implications of using the majority rule to decide when to stop searching among randomly arriving prospects. They show that many members do not have any real voting power (e.g. if prospects are single-dimensional, then only the median voter has real voting power), and the outcome is determined as the Nash bargaining solution for the key players who do. Albrecht et al. (2010) emphasize the ineffectiveness of committees relative to individual sequential search in that each member does worse, the acceptance threshold is lower, the search durations are shorter, and more conservative options are preferred. In my setting, the agents are collectively searching for the public good. As a result, their preferences are aligned so the voting rule does not have any effect on the approval process. Furthermore, the search and stopping rules have the same threshold as in the individual search case, and the only inefficiency that occurs is the delay due to free-riding.

As such, this paper also contributes to the literature on collaboration in teams. Bonatti and Hörner (2011) consider a setup in which the probability of a breakthrough positively depends on the agents' aggregate effort. They find that agents both free-ride on effort provision and procrastinate, which leads to delays in the project completion. Imposing deadlines helps solve both inefficiencies, while observability, as well as better monitoring, does not. Campbell et al. (2014) emphasize that inefficiencies in teams may also arise due to the lack of communication. When agents have private information about their progress, imposing a deadline achieves the social optimum, because not hearing about the colleague's success motivates the agent to work harder as the deadline approaches. Georgiadis (2015) considers a similar setup and studies the optimal team size problem. He finds that agents work harder in the early stages of the project, and when they are in larger teams. As a result, a team manager finds it optimal to dynamically decrease the size of the team as the project nears completion. My model is different in that the agents' goal is to choose the order in which to search for alternatives and decide when to stop. I show that delay occurs at every stage of the sequential search, although the search order and stopping rules are the same as in the individual case. I also show that, compared to individual search, my agents value being part of a team. However, since the team search results in free-riding, it is not socially optimal.

Finally, this paper contributes to the broad literature on the dynamic provision of public goods in which the public benefit is realized upon the completion of a project. Efficiency is usually not achieved because socially optimal public projects are not completed, like in Fershtman and Nitzan (1991) or Admati and Perry (1991), or they are completed, but with a delay, as in Marx and Matthews (2000). Compte and Jehiel (2004) further examine the delay in public contributions and find that it occurs because agents prefer to start low and increase contributions over time to discourage others from free-riding at the early stages of development of the project. Kessing (2007) shows that for this reason, some socially optimal projects may never be completed. Bowen et al. (2019) consider a case of heterogeneous agents and find another source of inefficiency: the public project may be implemented at a lower scale than socially optimal because that is the preference of the more "efficient" agents who have more bargaining power. I show that when searching the for public good, team members effectively use the individually-optimal search protocol, albeit with some free-riding when search costs are large enough. Consequently, all socially optimal projects are searched through, and the only source of inefficiency is the delay.

The rest of the paper is organized as follows. Section 2 introduces the dynamic model of sequential search among risky alternatives. Section 3 describes the results obtained for the case of one alternative, discusses comparative statics and welfare implications. Section 4 generalizes the model to the case of finitely many alternatives. Section 5 discusses future steps and concludes.

2. The Model

Two agents, $\mathcal{I} \equiv \{1, 2\}$, are sequentially searching for a public good. At each stage, one agent is chosen randomly with probability $1/2$. *The chosen player* has an option to *(i)* open exactly one box of her choice, *(ii)* do nothing, or *(iii)* propose to terminate the game. In the latter case, her *opponent* has a choice to reject the termination proposal or to accept it.

Each public good project is represented by a box that contains a stochastic prize. There are N unopened boxes. Box $b_k = \big(c_k, F_k(\cdot)\big)$ contains an uncertain reward $x_k \sim F_k(\cdot)$ distributed independently of all other rewards. If the chosen player decides to open this box, she has to pay the *search cost* c_k and wait one period to learn its contents. Once the contents are revealed, a new stage starts immediately and a new player is chosen.

The game ends either if both players agree to end it early (a termination proposal is offered, and this offer is accepted), or if there are no more boxes left to open. In either case, each player collects the maximum reward they have thus far uncovered. The initial fallback reward $z_0 = 0$ is available: players collect z_0 if no search takes place, or if all uncovered rewards happen to be negative.

Both players are risk-neutral and wish to maximize the expected present value of the best uncovered reward. Note that the search costs are sunk because they are paid during the search process, while the reward is only realized upon the end of the game. The players discount the time at the exponential rate $\delta = e^{-r\Delta t}$, where Δt is the length of the time interval between the stages. The goal of this paper is to find a dynamic rule that tells the chosen player when to search, the *order* in which to search, and when to (propose to) stop, and the opponent – when, and when not to, accept a termination proposal.

2.1. The Dynamic Problem

Let $\mathcal{B}^c$ denote the set of unopened boxes and let z be the best prize observed so far. Then, a pair $s \equiv (\mathcal{B}^c, z)$ is the state of the problem. Denote by $\Phi_i^{ch}(s)$ the value function of player i when she is chosen, and by $\Phi_i^{op}(s)$ when she is the opponent. Also, let

$$\bar{\Phi}_i(s) \equiv \frac{1}{2}\Phi_i^{ch}(s) + \frac{1}{2}\Phi_i^{op}(s)$$

be the *average* continuation value of player i (since she could be the chosen player or the opponent with equal probability each period).

When chosen, player i observes state s and decides whether to propose to terminate the game ($a_i = T$), do nothing ($a_i = \varnothing$), or open some box $b_k \in \mathcal{B}^c$ ($a_j = b_k$). In other words, her action space in state s is $A^{ch}(s) = \{T, \varnothing\} \cup \mathcal{B}^c$. When she is not chosen, player i may receive an offer to terminate the game, in which case she will have a choice between accepting ($r_i = 1$) and rejecting ($r_i = 0$), so her action space is $A^{op}(s) = \{0, 1\}$.

Given the strategy profile $(a_j(s), r_j(s)) \in A^{ch}(s) \times A^{op}(s)$ of player j, player i's value functions can be written in the form of Bellman equations. The first Bellman equation holds for the value function $\Phi_i^{ch}(s)$, when player i is chosen:

$$\begin{aligned} \Phi_i^{ch}(s) = \max_{a_i \in \{T,\varnothing\} \cup \mathcal{B}^c} & \left\{ \mathbb{1}_{\{a_i = T\}} \cdot r_j(s) \cdot z, \delta \bar{\Phi}_i(s), \max_{b_k \in \mathcal{B}^c} \left\{ -c_k + \delta \bar{\Phi}_i\left(s^{-b_k}\right) \right\} \right\}, \\ \text{s.t. } s^{-b_k} = & \left(\mathcal{B}^c \setminus \{b_k\}, \mathbb{E}\left[\max\{z, x_k\}\right] \right), \end{aligned} \tag{$\Phi_i^{ch}(s)$}$$

where the first term is the termination payoff that both players get if they agree to end the game; the second term is the continuation value of doing nothing; the third term reflects the value of opening box $b_k \in \mathcal{B}^c$ and drawing the prize $x_k \sim F_k(\cdot)$. In the latter case, the new state s^{-b_k} (the notation reflects box b_k having been opened) has a smaller set of unopened boxes $\mathcal{B}^c \setminus \{b_k\}$ and a (potentially larger) highest observed reward $\mathbb{E}\left[\max\{z, x_k\}\right]$. Recall that the discount factor δ reflects the time spent between opening the box and learning its contents, after which the next round begins immediately.

When player i is the opponent, her value function is

$$\Phi_i^{op}(s) = \max_{r_i \in \{0,1\}} \left\{ \mathbb{1}_{\{a_j(s)=T\}} \cdot r_i \cdot z,\ \delta \bar{\Phi}_i(s') \right\},$$
$$\text{s.t. } s' = \begin{cases} s & \text{if } a_j(s) = T, r_i = 0 \text{ or } a_j(s) = \varnothing, \\ s^{-b_k} & \text{if } a_j(s) = b_k. \end{cases} \tag{$\Phi_i^{op}(s)$}$$

Similarly to equation ($\Phi_i^{ch}(s)$) above, the first term of the maximization problem reflects the payoff from both players agreeing to terminate the game, while the second term is the continuation value. The state does not change if player j proposed to terminate the game and his proposal was rejected, or if he decided to skip his turn and do nothing.

When all boxes are open, the game ends, and players collect the safe option, i.e.

$$\Phi_i^{ch}(\varnothing, z) = \Phi_i^{op}(\varnothing, z) = z. \tag{$\Phi_i(\varnothing,z)$}$$

DEFINITION 1. *A profile of strategies* $\left(a_i(s), r_i(s)\right)_{i\in\{1,2\}} \in \left(A^{ch}(s), A^{op}(s)\right)^2$ *is a (pure-strategy) Markov-Perfect Nash* <u>*equilibrium*</u> *if for each* $i \in \mathcal{I}$ *and* $j \in \mathcal{I} \setminus \{i\}$*:*

- *the value functions* $\Phi_i^{ch}(s)$ *and* $\Phi_i^{op}(s)$ *satisfy the Bellman equations* ($\Phi_i^{ch}(s)$) *and* ($\Phi_i^{op}(s)$), *and the boundary condition* ($\Phi_i(\varnothing, z)$), *given* $\left(a_j(s), r_j(s)\right)$*;*
- $a_i(s)$ *is a policy function of equation* ($\Phi_i^{ch}(s)$), *and* $r_i(s)$ *is a policy function of equation* ($\Phi_i^{op}(s)$).

I will be looking for symmetric (possibly mixed strategy) equilibria.

3. ONE BOX

Suppose that there is only one unopened box, $\mathcal{B}^c = \{b_k\}$. Also, let $S(z, F_k) \equiv \mathbb{E}\big[\max\{z, x_k\}\big]$.

Recall that, according to Weitzman (1979), an individual searcher would open the box b_k if and only if the expected benefit minus the search cost exceeds the safe option:

$$-c_k + \delta S(z, F_k) \geq z. \tag{SR}$$

It is straightforward to show that the *solo rationality* condition (SR) holds if and only if

$z \le \bar{z}_k$, where the *reservation value* $\bar{z}_k$ solves $-c_k + \delta S(\bar{z}_k, F_k) = \bar{z}_k$. It is also easy to see that if (SR) holds, then opening box k for **the chosen player** weakly dominates proposing termination.

How does a player respond to a termination offer when she is **the opponent**? The best she can do by refusing to terminate the game is wait until she is chosen, which is discounted and happens with probability $\frac{1}{2}\delta + \left(\frac{1}{2}\delta\right)^2 + \cdots = \frac{\delta}{2-\delta}$ and then open the box herself. She prefers that over agreeing to end the game today and receiving z if the following *individual rationality* condition holds:

$$\frac{\delta}{2-\delta} \cdot \left[-c_k + \delta S(z, F_k) \right] \ge z. \tag{IR}$$

Similarly to the (SR) condition, (IR) condition holds if and only if $z \le z_k^{IR}$, where z_k^{IR} solves $\frac{\delta}{2-\delta}\left[-c_k + S(z_k^{IR}, F_k) \right] = z_k^{IR}$.[2] Furthermore, $z_i^{IR} \le \bar{z}_k$. In other words, (IR) implies (SR), but not vice versa. The intuition is that if player i is willing to wait until she is chosen to open box b_k, then she would definitely want to open it had she been chosen today. The only exception is when both players are perfectly patient, i.e. $\delta = 1$, in which case $\bar{z}_k = z_k^{IR}$.

In the mixed-strategy equilibrium, let π_k be the equilibrium probability of opening the box b_k. The chosen player i must be indifferent between *(i)* opening the box today, and *(ii) someone* opening the box in the future, which translates into

$$-c_k + \delta S(z, F_k) = \frac{\pi_k \delta}{1-(1-\pi_k)\delta} \cdot \left[-\frac{c_k}{2} + \delta S(z, F_k) \right]. \tag{IC}$$

In words, her expected payoff from not opening the box today is the surplus from the box being opened eventually $\delta S(z, F_k)$, less her having to pay the search cost c_k *half of the time* on average, infinitely discounted according to the time discount factor δ, and the probability that the box is not opened in the current period, $1 - \pi_k$.

By solving the *indifference condition* (IC), provided that there is a solution, we obtain the equilibrium probability of opening the box π_k as a function of the safe option z. The full search and termination protocol for the case of only one unopened box is summarized

[2] The formal proof of this statement and all other results can be found in the appendix.

in the following proposition.

PROPOSITION 1. *Let z be the safe option and let $\mathcal{B}^c = \{b_k\}$. In equilibrium,*

the chosen player:

- *if $z \leq \bar{z}_k$, opens the box b_k with probability*

$$\pi_k(z) = \begin{cases} \frac{2(1-\delta)}{\delta c_k} \cdot \left[-c_k + \delta S(z, F_k) \right] < 1 & \text{if } c_k > S(z, F_k) \cdot \frac{2\delta(1-\delta)}{2-\delta}, \\ 1 & \text{otherwise}, \end{cases}$$

  and does nothing otherwise;
- *proposes to terminate the game if $z > \bar{z}_k$.*

the opponent:

- *accepts the termination proposal if $z > z_k^{IR}$;*
- *rejects the proposal if $z \leq z_k^{IR}$.*

Comparing this to the optimal search and stopping protocol of a single decision-maker, we can see that on the equilibrium path, the box will eventually be opened as long as $z \leq \bar{z}_k$, the same cutoff as in Weitzman (1979). Put differently, this means that that the project is eventually searched through if and only if it is individually rational (and hence socially optimal) to do so. If search costs are large enough, however, the box is opened with a probability less than one each period. Hence, it may take several periods to open the same box that an individual searcher would have opened right away. Hence, collaborative search results in *delay* as a consequence of the free-rider problem. I measure the size of the delay and discuss comparative statics in the section that follows.

In the case of one unopened box, the value functions take an especially simple form.

PROPOSITION 2. *In equilibrium, the value functions are*

$$\begin{aligned} \Phi^{ch}(s) &= \max\left\{z, -c_k + \delta S(z, F_k)\right\}, \\ \Phi^{op}(s) &= \max\left\{z, \frac{2-\delta}{\delta} \cdot \left[-c_k + \delta S(z, F_k) \right]\right\}, \\ \delta\bar{\Phi}(s) &= \Phi^{ch}(s). \end{aligned}$$

Notice that when the box is worth opening, i.e. when $z \leq -c_k + \delta S(z, F_k)$, we have $\Phi^{ch}(s) = \delta\bar{\Phi}(s) = \frac{\delta}{2-\delta} \cdot \Phi^{op}(s)$. In words, when a player is chosen, she has an incentive to free-ride because both her average value $\bar{\Phi}(s)$ and especially her value of being the opponent $\Phi^{op}(s)$ are strictly larger than $\Phi^{ch}(s)$ since $\delta < 1$. It is also worth mentioning that $\Phi^{ch}(s)$ coincides exactly with Weitzman's value function for the individual searcher. Hence, *on average*, each searcher is doing strictly better than under an individual search protocol (since $\bar{\Phi}(s) = \frac{1}{\delta}\Phi^{ch}(s) > \Phi^{ch}(s)$), but overall the team search is less efficient due to free-riding.

LEMMA 1. *In equilibrium, the value functions $\Phi^{ch}(s)$ and $\bar{\Phi}(s)$ increase as*

- *the search cost c_k decreases,*
- *players' patience increases, i.e. as δ increases.*

The value function $\Phi^{op}(s)$ exhibits similar behavior, except $\frac{\partial \Phi^{op}(z)}{\partial \delta}$ could be positive, or negative.

When the search cost decreases, two effects take place. First of all, the expected net benefit of opening the box $-c_k + \delta S(z, F_k)$ increases. Secondly, the reservation value $\bar{z}_k$ of the box also increases, i.e. this box becomes ex-ante more attractive. As a result, the box is now opened for values of z for which it was not opened before, which drives the value functions even higher. The same argument applies to studying the dynamics of the average value function and the value function of the chosen player with respect to δ.

The dynamics of the opponent's value function with respect to δ are inconclusive: on the one hand, more patience leads to higher value because the box becomes ex-ante more attractive. On the other hand, as players' patience increases, so does the incentive to free-ride, and that drives the equilibrium probability of opening the box down. As a consequence, the opponent's value function drops, since it is determined by the likelihood of the chosen player opening the box. Either effect may prevail, depending on other parameters.

As I discuss at the end of this paper, the results that *(i)* the chosen player uses the same cutoff as an individual searcher to make box-opening and game-termination decisions, *(ii)* the selected box is opened with probability less than one, *(iii)* this is inefficient, *(iv)* comparative statics, *(v)* each searcher on average does better than she would have done alone, hold for any finite number of boxes.

Recall that it is efficient to open the box immediately whenever an individual searcher would do so. Hence, whenever the chosen player is opening the box with probability less than one for $z \le \bar{z}_k$, she is free-riding, which results in delay, and there is welfare loss. If π_k is the probability that box b_k is opened each period and Δt is the time between periods (Δt solves $\delta = e^{-r\Delta t}$), then the expected *delay* before the box is opened is

$$D(\pi_k, \Delta t) = 0 \cdot \pi_k + \Delta t \cdot \pi_k \cdot (1 - \pi_k) + 2\Delta t \cdot (1 - \pi_k)^2 \cdot \pi_k + \cdots = \Delta t \cdot \frac{1 - \pi_k}{\pi_k}.$$

Obviously, the delay is increasing as the agents shirk more, i.e. if $\pi_k(z)$ is decreasing. To understand when the delay occurs and how severe it is, I analyze the properties of $\pi_k(z)$. Recall that each player only mixes between opening the box b_k and doing nothing when she is indifferent between these two options, as described by equation (IC). When the search costs are small, i.e. when $c_k < S(z, F_k) \cdot \frac{2\delta(1-\delta)}{2-\delta}$, it is strictly dominant to open the box right away, i.e. $\pi_k(z) = 1$. Since $S(z, F_k)$ is strictly increasing and convex,[3]

- if $c_k > \bar{c}_k \equiv S(\bar{z}_k, F_k) \cdot \frac{2\delta(1-\delta)}{(2-\delta)}$, the search costs are large enough, and there is always an interior solution $\pi_k \in (0, 1)$ on the whole segment $z \in [0, \bar{z}_k]$;
- if $c_k < \underline{c}_k \equiv S(0, F_k) \cdot \frac{2\delta(1-\delta)}{(2-\delta)}$, the search costs are so small that opening the box b_k right away is strictly dominant for all $z \in [0, \bar{z}_k]$;
- if $c \in \left[\underline{c}_k, \bar{c}_k\right]$, there is an interior solution $\pi_k(z) \in (0, 1)$ for low values of z, but once z becomes larger than $\tilde{z}_k$ that solves $c_k = S(\tilde{z}_k, F_k) \cdot \frac{2\delta(1-\delta)}{(2-\delta)}$, the chosen player prefers to open the box b_k right away.

The properties of $\pi_k(z)$ are summarized in Proposition 3 and illustrated in Figure 1.

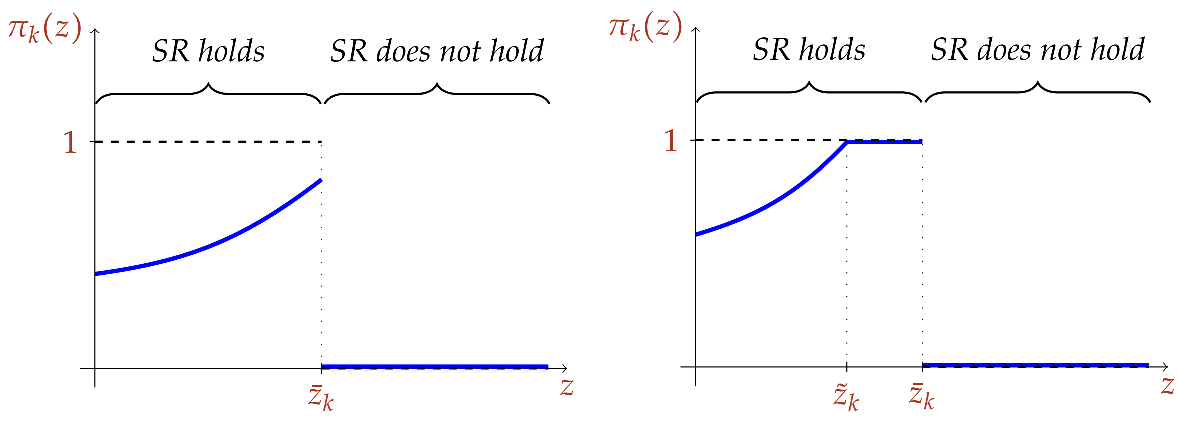
PROPOSITION 3 (PROPERTIES OF $\pi_k(z)$).
When individual rationality holds, i.e. for $z \in [0, \bar{z}_k]$,

(1) if $c_k \ge \bar{c}_k$, then $\pi_k(z) \in (0, 1)$, and is strictly increasing and strictly convex;

(2) if $c_k \le \underline{c}_k$, then $\pi_k(z) = 1$;

[3] Since $S(z, F_k) = \mathbb{E}\left[\max\{z, x_k\}\right] = zF_k(z) + \int_z^{+\infty} x dF_k(x)$, it follows directly that $\frac{\partial S(z,F_k)}{\partial z} = F_k(z) > 0$ and $\frac{\partial^2 S(z,F_k)}{\partial z^2} = f_k(z) > 0$.

(3) *if* $c_k \in (\underline{c}_k, \bar{c}_k)$, *then* $\pi_k(z)$ *exhibits the same properties as in case (1) for* $z \in [0, \tilde{z}_k]$, *and as in case (2) for* $z \in [\tilde{z}_k, \bar{z}_k]$.



(a) $c_k \geq \bar{c}_k$.

(b) $c_k \in (\underline{c}_k, \bar{c}_k)$.

Figure 1: *The equilibrium probability of opening the box* $\pi_k(z)$. *The dashed black lines represent the efficient level of* $\pi_k(z)$.

Strikingly, the probability of the chosen player opening the box is *increasing* in the value of the safe option, at an increasing rate, strictly for search costs that are large enough.[4] This result is somewhat counter-intuitive considering that z is the safe option, and this finding says that raising the safe option will raise the probability of opening the box (as opposed to collecting the safe option today) and reduce the delay. The key to understanding this result is recalling that the safe option z is not the outside option for the chosen player, because she cannot unilaterally deviate to collect it (her opponent must accept the termination offer). The effective outside option is to open the box today, and increasing z makes it more appealing to do so: once the (last) box is opened, the game ends, and *at least* z is collected. To remain indifferent between opening the box and not, the chosen player must rationally expect that the box is more likely to be opened in the future, and that drives $\pi_k(z)$ up and the delay down.

Notice that the least amount of free-riding occurs at $z = \bar{z}_k$ when the (SR) constraint binds and the chosen player is indifferent between *(i)* opening the box, *(ii)* not opening the box, and *(iii)* collecting the safe reward $\bar{z}_k$ (if that was an option). At that point, the

[4]This result is proved easily by observing that $\pi_k(z) = a \cdot S(z, F_k)$ with $a \in (0,1)$ whenever there is an interior solution.

expected benefit from opening the box $-c_k + \delta S(z, F_k)$ is the highest among all $z \in [0, \bar{z}_k]$ which, by the logic described above, makes opening the box sooner more desirable.[5] The most inefficiency occurs when the safe option is zero (e.g. at the beginning of the search process, since $z_0 = 0$), when there seems to be the most to gain from opening the box. However, the expected reward from opening the box is actually the lowest as compared to higher safe options, and the indifference condition dictates that the box is the least likely to be opened.

Next, I discuss the comparative statics of the expected delay with respect to various model parameters. In general, as the value of opening the box today increases, so does the continuation value of not opening the box today. To remain indifferent, the chosen player must rationally expect a higher probability of the box being opened in the future, as prescribed by equation (IC). Hence, *less delay* is associated with *higher reservation values* as a consequence of

- lower search cost c_k: the root of free-riding lies in the unwillingness to pay the search cost. Reducing the search cost reduces the incentive to shirk;
- "better" rewards: changing $F_k(\cdot)$ to $G_k(\cdot)$ such that $G_k(x) \leq F_k(x)\ \forall x$ leads to a higher expected reward $S(z, G_k)$ and higher π_k. Performing a mean-preserving spread on $F_k(\cdot)$ (making the box riskier) has the same effect.

The dynamic of the equilibrium probability of opening the box π_k with respect to the players' patience parameter δ is inconclusive and could go either way. On the one hand, increasing patience increases the value of opening the box today, and by the argument discussed above, decreases the delay. On the other hand, more patient players are more willing to wait for their opponent to perform the search, and that drives the delay up.

4. Many Boxes

Now, let $\mathcal{B}^c$ contain an arbitrary number of boxes.

Suppose player i is **the opponent** and she has received a termination offer, i.e. $a_j(s) = T$. Then, her value function is $\Phi_i^{op}(s) = \max\{z, \delta\bar{\Phi}_i(s)\}$ and she rejects this offer if and

[5] For every $z < \bar{z}_k$ it is true that $-c_k + \delta S(z, F_k) < -c + \delta S(\bar{z}_k, F_k) = \bar{z}_k$.

only if $z \leq \delta\bar{\Phi}_i(s)$. Since player j proposes termination in state s, when player i gets chosen in the future (which will happen eventually), the state will still be s, and her value function will be

$$\Phi_i^{ch}(s) = \max_{b_l \in \mathcal{B}^c} \left\{ -c_l + \delta\bar{\Phi}_i\left(s^{-b_l}\right) \right\}.$$

In other words, when she is the opponent, player i rejects the termination offer if and only if her discounted average value $\delta\bar{\Phi}_i(s)$ is weakly greater than the termination payoff z, and if there exists *some* order of opening boxes that makes it worthwhile to wait until she gets chosen. Using the definition of the average value function $\bar{\Phi}_i(s)$, in the scenario that the offer is rejected we get

$$(2-\delta)\bar{\Phi}_i(s) = \max_{b_l \in \mathcal{B}^c} \left\{ -c_l + \delta\bar{\Phi}_i\left(s^{-b_l}\right) \right\}.$$

Upon close inspection, it becomes clear that this equation that describes the *average* value function of player i searching collaboratively with player j also describes a law of motion of the value function of player i *searching alone* with a discount factor of $\delta/(2-\delta)$, instead of δ. In other words, player i, upon receiving a termination offer, faces the same problem *as if she was searching alone*, discounting the time when she is not chosen, since her colleague does not assist with the search, and nothing happens in periods in which the colleague is chosen. According to Weitzman (1979), player i's optimal policy is to open the boxes in the order of decreasing reservation value. When the highest reservation valuer becomes less than the maximum observed reward, she proposes termination, and her opponent agrees. To summarize,

PROPOSITION 4. *Let b_k be the unopened box with the highest reservation value, i.e. $\bar{z}_k = \max_{b_l \in \mathcal{B}^c} \bar{z}_l$. Then, in equilibrium*

***the opponent**:*

- *accepts the termination proposal if $z > \frac{\delta}{2-\delta} \cdot \left[-c_k + \delta\bar{\Phi}_i\left(s^{-b_k}\right)\right]$,*
- *rejects the proposal otherwise.*

Here, for every state $\tilde{s} = \left(\tilde{\mathcal{B}}^c, \tilde{z}\right)$, $\bar{\Phi}_i(\tilde{s})$ is recursively defined by

$$\delta\bar{\Phi}_i(\tilde{s}) = \max\left\{ \tilde{z}, \frac{\delta}{2-\delta} \cdot \left[-c_l + \delta\bar{\Phi}_i\left(\tilde{s}^{-b_l}\right)\right]\right\},$$

where b_l is such that $\bar{z}_l = \max_{b_m \in \tilde{\mathcal{B}}^c} \bar{z}_m$, and $\bar{\Phi}_i(\varnothing, \tilde{z}) = \tilde{z}$.

Notice that there is no explicit solution for the threshold of accepting the termination offer in the sense that finding the value function requires iterating the search process forward (using the optimal individual search protocol) until no more boxes are left. However, two observations can be made using simpler arguments:

LEMMA 2.

- *$z \leq z_k^{IR}$ is sufficient to reject the termination offer,*
- *$z > \bar{z}_k$ is sufficient to accept the termination offer.*

Intuitively, if it is worth waiting to be chosen to open just one box b_k, then the termination offer should be rejected. Conversely, if player i does not want to open the best unopened box even when she is chosen, then she should definitely accept the termination offer when she is the opponent.

Next, I consider the problem of **the chosen player**. First observe that if $z > -c_k + \delta S(z, F_k)$ for the box b_k with the highest reservation value among the unopened boxes, the weakly dominant strategy is to offer to terminate the game. The offer will be accepted since $z > \bar{z}_k$ (by Lemma Lemma 2), and both players will collect z.

On the other hand, if $\bar{z}_k \geq z$, proposing termination is weakly dominated by opening some box, or doing nothing. Let player i play a mixed strategy of opening box b_l with probability $\tilde{\pi}_l$ and doing nothing with probability $1 - \sum_{b_l \in \mathcal{B}^c} \tilde{\pi}_l$. Since she must be indifferent between all these actions to mix, her value function satisfies

$$\Phi_i^{ch}(s) = \delta \bar{\Phi}_i(s) = -c_l + \delta \bar{\Phi}_i\left(s^{-b_l}\right) \text{ for all } b_l \in \mathcal{B}^c \text{ s.t. } \tilde{\pi}_l > 0.$$

If we rewrite the last part of this equation as $\bar{\Phi}_i(s) = -\frac{c_l}{\delta} + \bar{\Phi}_i\left(s^{-b_l}\right)$, we can again recognize the Bellman equation of an individual searcher who is perfectly patient and does not discount the future. From Weitzman (1979) we know that the optimal search policy prescribes to open the box *with the highest reservation value* b_k and that all other search protocols are valued strictly lower, i.e

$$-c_k + \delta \bar{\Phi}_i\left(s^{-b_k}\right) > -c_l + \delta \bar{\Phi}_i\left(s^{-b_l}\right) \text{ for all } b_l \in \mathcal{B}^c \setminus \{b_k\}.$$

Consequently, player i will not be opening *every* box with a positive probability, but only the box with the highest reservation value. Her mixed strategy becomes: open the box b_k with probability $\tilde{\pi}_k$ and do nothing with probability $1 - \tilde{\pi}_k$.

In the symmetric equilibrium, when player i becomes the opponent, she expects player j to play the same mixed strategy when he is chosen. Player i's Bellman equations when she is chosen and when she is the opponent become

$$\Phi_i^{ch}(s) = (1 - \tilde{\pi}_k) \cdot \delta \bar{\Phi}_i(s) + \tilde{\pi}_k \Big[- c_k + \delta \bar{\Phi}_i\big(s^{-b_k}\big)\Big],$$
$$\Phi_i^{op}(s) = (1 - \tilde{\pi}_k) \cdot \delta \bar{\Phi}_i(s) + \tilde{\pi}_k \cdot \delta \bar{\Phi}_i\big(s^{-b_k}\big).$$

Solving these equations, we obtain the equilibrium strategy of the chosen player.

Proposition 5. *Let b_k be the unopened box with the highest reservation value, i.e. $\bar{z}_k = \max_{b_l \in \mathcal{B}^c} \bar{z}_l$. Then, in equilibrium*

***the chosen player**:*

- *if $z \leq \bar{z}_k$, opens the box b_k with probability*

$$\tilde{\pi}_k(s) = \min\left\{\frac{2(1-\delta)}{\delta c_k}\Big[- c_k + \delta \bar{\Phi}_i\big(s^{-b_k}\big)\Big], 1\right\},$$

  and does nothing otherwise;

- *proposes to terminate the game if $z > \bar{z}_k$.*

Here, for every state $\tilde{s} = \big(\tilde{\mathcal{B}}^c, \tilde{z}\big)$, $\bar{\Phi}_i(\tilde{s})$ is recursively defined by

$$\delta \bar{\Phi}_i\big(\tilde{s}\big) = \max\left\{\tilde{z}, -c_l + \delta \bar{\Phi}_i\big(\tilde{s}^{-b_l}\big)\right\},$$

where b_l is such that $\bar{z}_l = \max_{b_m \in \tilde{\mathcal{B}}^c} \bar{z}_m$, and $\bar{\Phi}_i(\varnothing, \tilde{z}) = \tilde{z}$.

Recall that in Weitzman (1979), the individual searchers make decisions myopically in the sense that at each stage of the search they only compare the reservation value of the next best box to the best reward sampled so far. With two searchers, this is no longer true, and both the chosen player and the opponent take the future into account when making decisions: the chosen player needs the future value function to calculate $\tilde{\pi}_k$, while the opponent needs it to calculate the acceptance cutoff rule. At the same time, the *order* and the *stopping rule* on the equilibrium path are identical to that of Weitzman (1979): the box with the highest reservation value is opened next, and the game is terminated when the best sampled reward exceeds the reservation value of all unopened boxes. The only difference is that every box is opened with a delay when the search costs are high because each player is hoping that her opponent with end up opening the box and paying that cost.

Note that π_k, the equilibrium probability of opening the box b_k if that was the last unopened box, is the *lower bound* for $\tilde{\pi}_k$ (keeping the safe option z constant). This means, among other things, that if search costs are sufficiently low, then there is no inefficiency and the dual search on the equilibrium path is completely identical to the individual search protocol.

Recall from the section that deals with the single-box problem that boxes with higher reservation values are opened faster. In the multi-box setting, boxes with the higher reservation value are opened earlier rather than later, suggesting that the delay grows over the search process. On the other hand, the delay also decreases with the value of the outside option z, which grows as more rewards are uncovered. In the end, it is unclear whether the players shirk more or less as the search process goes on.

5. Conclusion

This paper examined a model of sequential search for a public good performed by a team of agents. I showed that group search results in the socially optimal search and stopping rule, but delay occurs at every stage of the learning process because agents free-ride. Overall, the team manager prefers to delegate research to individual agents, but each agent would rather search in a team.

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APPENDIX

LEMMA 3. *Let* $H(z, b_k) \equiv \frac{\delta}{2-\delta} \cdot [-c_k + S(z, F_k)]$ *and let* z_k^{IR} *solve* $H(z_i^{IR}, b_k) = z_k^{IR}$. *Then,* (IR) *holds if and only if* $z \le z_k^{IR}$.

Proof. Since $\frac{\partial S(z,F_k)}{\partial z} = F_k(z) \in [0,1]$, $H(z, b_k)$ is increasing in z at the rate less than one. Since $z_k^{IR} = H(z_k^{IR}, b_k)$, $z \le H(z, b_k)$ if and only if $z \le z_k^{IR}$. □

PROOF OF PROPOSITION 1 AND PROPOSITION 2

When player i is **the opponent**, her Bellman equation in the state $s = \big(\{b_k\}, z\big)$ is

$$\Phi_i^{op}(s) = \max_{r_i \in \{0,1\}} \big\{ r_i \cdot z, \delta \bar{\Phi}_i(s) \big\}.$$

Once she is chosen, if the state is still s, her Bellman equation becomes

$$\Phi_i^{ch}(s) = \max_{a_i \in \{T, \varnothing, b_k\}} \Big\{ \mathbb{1}_{\{a_i = T\}} \cdot r_j \cdot z, \delta \bar{\Phi}_i, -c_k + \delta \bar{\Phi}_i(s') \Big\},$$

where $s' = \big(\varnothing, S(z, F_k)\big)$, so $\bar{\Phi}_i(s') = S(z, F_k)$ by the boundary condition. First observe that if $-c_k + \delta S(z, F_k) < z$, then player i will immediately accept the termination offer since she does not expect to do better than z even if she is chosen in the future.

If $-c_k + \delta S(z, F_k) \ge z$, in other words, if it is solo rational to open the box b_k, the offer may or may not be rejected. Suppose that the offer is rejected, in which case $z \le \delta \bar{\Phi}_i(s)$. This is only possible if player i opens the box once chosen, i.e. $\Phi_i^{ch}(s) = -c_k + \delta S(z, F_k)$.

Then, using the facts that *(i)* it may take time for her to get chosen $\Phi_i^{op}(s) = \delta\bar{\Phi}_i(s)$ and *(ii)* her value is an average of her being chosen and being the opponent $\bar{\Phi}_i(s) = \frac{\Phi_i^{ch}(s)+\Phi_i^{op}(s)}{2}$, we get that the termination offer is rejected if and only if:

$$\frac{\delta}{2-\delta} \cdot \left[-c_k + S(z, F_k) \right] \geq z.$$

When i is **the chosen player**, she knows that her opponent will accept the termination offer if and only if $z > z_k^{IR}$. First observe that when $z > \bar{z}_k$ (which implies $z > z_k^{IR}$), proposing termination (this offer is accepted) weakly dominates doing anything else.

If $z \leq \bar{z}_k$, then offering termination is weakly dominated by opening the box b_k right away, since $-c_k + \delta S(z, F_k) \geq z$. Hence, player i is effectively choosing between opening b_k and doing nothing. Her Bellman equation when she is chosen is

$$\Phi_i^{ch}(s) = \max_{a_i \in \{\varnothing, b_k\}} \left\{ \delta\bar{\Phi}_i(s), -c_k + \delta\bar{\Phi}_i(s') \right\},$$

where $s' = \big(\varnothing, S(z, F_k)\big)$, so $\bar{\Phi}_i(s') = S(z, F_k)$ by the boundary condition ($\Phi_i(\varnothing, z)$). Player i's Bellman equation when she is the opponent is

$$\Phi_i^{op}(s) = \begin{cases} \delta S(z, F_k) & \text{if } a_j(s) = b_k, \\ \delta\bar{\Phi}_i(s) & \text{if } a_j(s) \in \{\varnothing, T\}. \end{cases}$$

Let both players use a mixed strategy of opening the box b_k with probability π_k and doing nothing with probability $(1 - \pi_k)$. When chosen, player i is indifferent between these two actions:

$$\Phi_i^{ch}(s) = \delta\bar{\Phi}_i(s) = -c_k + \delta S(z, F_k) = (1 - \pi_k) \cdot \delta\Phi_i^{ch}(s) + \pi_k \cdot \left[-c_k + \delta S(z, F_k) \right].$$

When player i is the opponent, she expects player j to stick to the same mixed strategy. As a result, the second Bellman equation simplifies to

$$\Phi_i^{op}(s) = (1 - \pi_k) \cdot \delta\bar{\Phi}_i(s) + \pi_k \cdot \delta S(z, F_k).$$

Solving these equations, we get

$$\pi_k = \frac{2(1-\delta)}{\delta c_k} \cdot \big[-c_k + \delta S(z, F_k)\big], \quad \bar{\Phi}_i(s) = \frac{1}{\delta} \cdot \big[-c_k + \delta S(z, F_k)\big],$$

$$\Phi_i^{ch}(s) = c_k + \delta S(z, F_k), \quad \Phi^{op}(s) = \frac{2-\delta}{\delta} \cdot \big[-c_k + \delta S(z, F_k)\big].$$

Note that there is no interior solution $\pi_k \in (0,1)$ when opening the box strictly dominates doing nothing, in which case $\pi_k = 1$ (the box is opened immediately). This happens iff

$$-c_k + \delta S(z, F_k) > \delta \bar{\Phi}_i(s), \quad \bar{\Phi}_i(s) = \frac{\Phi_i^{ch}(s) + \Phi_i^{op}(s)}{2}, \quad \Phi_i^{op} = \delta S(z, F_k)$$

$$\iff c_k < S(z, F_k) \cdot \frac{2\delta(1-\delta)}{2-\delta}.$$

PROOF OF LEMMA 1

Firstly, let us find how the reservation value $\bar{z}_k$ changes with c_k and δ. Recall that $\bar{z}_k$ is implicitly defined by

$$\bar{z}_k = -c_k + \delta S(z, F_k).$$

Differentiating this equation with respect to the variables of interest, we get

$$\frac{\partial \bar{z}_k}{\partial c_k} = -1 + \delta F_k(\bar{z}_k) \cdot \frac{\partial \bar{z}_k}{\partial c_k} \Rightarrow \frac{\partial \bar{z}_k}{\partial c_k} = \frac{-1}{1 - \delta F_k(\bar{z}_k)} < 0,$$

$$\frac{\partial \bar{z}_k}{\partial \delta} = S(z, F_k) + \delta F_k(\bar{z}_k) \cdot \frac{\partial \bar{z}_k}{\partial \delta} \Rightarrow \frac{\partial \bar{z}_k}{\partial \delta} = \frac{S(z, F_k)}{1 - \delta F_k(\bar{z}_k)} > 0,$$

using the fact that $\frac{\partial S(z)}{\partial z} = F_k(z)$.

Since the reservation value increases as c_k decreases and δ increases, the condition for opening the box $z \le \bar{z}_k$ is met for a wider range of z. Since $\frac{\partial \Phi^{ch}(s)}{\partial \delta} = S(z, F_k) > 0$ and $\frac{\partial \bar{\Phi}(s)}{\partial \delta} = \frac{c_k}{\delta^2} > 0$ when $z \le \bar{z}_k$, it follows directly that $\Phi^{ch}(s)$ and $\bar{\Phi}(s)$ increase. The same argument can be applied for dynamics of $\Phi^{op}(s)$ with respect to c_k. The sign of $\frac{\partial \Phi^{op}(s)}{\partial \delta}$ is inconclusive.

PROOF OF PROPOSITION 3

When there is an interior solution,

$$\pi_k(z) = \frac{2(1-\delta)}{\delta c_k} \cdot \big[-c_k + \delta S(z, F_k) \big].$$

Then,

$$\frac{\partial \pi_k(z)}{\partial z} = \frac{2(1-\delta)}{c_k} \cdot F_k(z) > 0, \text{ and } \frac{\partial^2 \pi_k(z)}{\partial z^2} = \frac{2(1-\delta)}{c_k} \cdot f_k(z) > 0.$$

PROOF OF LEMMA 2

- Since $\bar{\Phi}_i(s)$ is the value function of an individual searcher, it must hold that $\bar{\Phi}_i(s) \geq S(z, F_k)$ for any state s, since $S(z, F_k)$ is the (possibly non-optimal) value of opening only the box b_k. Hence,

$$z \leq z_k^{IR} \iff z \leq \frac{\delta}{2-\delta} \cdot \big[-c_k + \delta S(z, F_k) \big] \leq \frac{\delta}{2-\delta} \cdot \Big[-c_k + \delta \bar{\Phi}_i\big(s^{-b_k}\big) \Big].$$

- If the highest reservation value among the unopened boxes is less than the safe option z, then no search is optimal today or in the future, so it is strictly dominant to accept the termination offer:

$$z > \bar{z}_k \iff z > -c_k + \delta S(z, F_k) = -c_k + \delta \bar{\Phi}_i\big(s^{-b_k}\big) > \frac{\delta}{2-\delta} \cdot \Big[-c_k + \delta \bar{\Phi}_i\big(s^{-b_k}\big) \Big].$$