STEP 1 2014 Hints and Solutions

Q1 This question is the traditional starter, involving ideas that should certainly be familiar to all candidates. That doesn't necessarily mean it is easy, just that everyone should be able to make a start with it, and make some good progress thereafter. In fact, parts (i), (ii) and (iii) are each especially amenable to all; and (i) is intended to help get you started along this particular road by having you first write out some numerical examples of differences of two squares, before asking you to move on to the algebra.

To begin with, (ii) requires only the observation that each odd number is the difference of consecutive squares, $2k - 1 \equiv k^2 - (k - 1)^2$; and then (iii) relies on noting that multiples of 4 arise from the difference of squares of numbers that are two apart, namely $4k \equiv (k + 1)^2 - (k - 1)^2$. Part (iv) develops these ideas further, although it is important now to deconstruct the problem into its building blocks; in this case, that means examining the four possible cases for $a^2 - b^2$ when a and b are either odd or even, none of which cases yield an even answer that is not automatically a multiple of 4. Alternatively, this is very easily addressed by examining squares modulo 4.

Part (v) draws on ideas of factorisation of (positive) integers into factor-pairs. Here, you are given the product pq where both p and q are odd. Using the *difference of two squares factorisation*, if $pq = a^2 - b^2 = (a - b)(a + b)$, then either $a - b = 1$ and $a + b = pq$ OR $a - b = p$ and $a + b = q$ (taking $p < q$ w.l.o.g.), giving the (exactly) two required factorisations. However, if $p = 2$, then pq is a multiple of 2, but not 4, and we have already shown this case to be impossible.

The final part of the question pulls all these ideas together in a numerical example, and we need only prime-factorise $675 = 3^3 \times 5^2$ and note that this yields $(3 + 1)(2 + 1) = 12$ factors and hence six factor-pairs.

Q2 The main ideas behind this question are relatively straightforward, but there are many difficulties in the execution of them. In (i), an integral of the form $\int \ln(x)\ dx$ would usually have to be approached by writing it as the product $\int \ln(x) \times 1\ dx$ and integrating by parts. On this occasion, however, with a given result, it is perfectly possible to verify by differentiation. Curve-sketching (rather than plotting) is a key skill and one that needs practice. The things to look for are crossing-points on the two coordinate axes (which arise here when $\ln(1)$ appears), asymptotes (from when we get ln(0)), symmetries and – if further detail is required – turning points. It should be obvious here, for instance, that the curve is an *even function*, so there is a turning point when $x = 0$. In (iii), you are given the area to be found so that you can check you are doing everything correctly before you go on to answer the rather tougher part (iv). You should first realise (from your sketch-graph) that it is required to integrate $\ln(4 - x^2)$ between $-\sqrt{3}$ and $\sqrt{3}$, though you should always be on the lookout for opportunities to make the working easier – here, you could integrate between 0 and $\sqrt{3}$ and then double the answer. Next, there is a very strong hint supplied by part (i) to split the log. term up as $\ln(2 - x) + \ln(2 + x)$, the first term of which has already been done and the second can be deduced from it with a bit of care.

Part (iv) actually asks nothing new. Curve B is just curve A with all portions drawn above the x-axis and it is only required that you deal with the extra bit(s) of area, and the awkwardness of finding an area up to the asymptote is covered for you by the footnote that follows (iv).

Q3 This question was actually devised to address what happens when students misunderstand or mis-apply a "rule" of mathematics and *it turns out to give the right answer*. Part (i) starts you off gently: integrating both terms, squaring the RHS and solving very quickly gives $b = \frac{4}{3}$. Part (ii) develops in much the same way, but with a non-zero lower limit to the integrals, and we immediately see that the algebra gets much more involved. Importantly, it should be very clear that whatever expression materialises must have (b – 1) as a factor (since setting $b = a$ would definitely give a zero area, thus trivially satisfying the given integral statement). This leads to the required cubic equation.

The final part of (ii) requires a mixture of different ideas (and can be done in a number of different ways). The most basic approach to demonstrating that a cubic curve has only one zero is to illustrate that both of its TPs lie on the same side of the x-axis (or to show there are no TPs). The popular *Change-of-Sign Rule* for continuous functions can be used to identify the position of this zero.

Having got you started with some simple lower limits, part (iii) develops matters more generally, and derives the (perhaps) surprising result that the exploration of this initial "stupid idea" requires b and a to be "not too far apart" to an extent that is easily identifiable.

Q4 This is quite a sweet little question, and deals with the movements of the hands of a clock. Using the *Cosine Rule* and differentiating gives $\frac{dx}{d\theta} = \frac{ab\sin\theta}{\sqrt{b^2 + a^2 - 2ab\cos\theta}}$. However, it is important to note that we can only work with $\frac{dx}{d\theta}$ and $\frac{d^2x}{d\theta^2}$, rather than $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$, since $\frac{d\theta}{dt}$ is constant (so that extra terms in the use of the *Chain Rule* simply cancel). Then, differentiating again with respect to θ, equating to zero and solving leads to the quadratic equation in $C = \cos\theta$:

$$abC^2 - \left(b^2 + a^2\right)C + ab = 0$$

which can be factorised (or solved otherwise). Only one of these solutions gives $|\ C\ | < 1$, and substituting $\cos\theta = \frac{a}{b}$ gives the required result. [In the normal course of events, one should justify that this is a maximum value rather than a minimum – but it is rather clear that it must be so in this case.] Finally, $\frac{d\theta}{dt} = \frac{11}{6}\pi$ is the constant rate of change of θ, the angle between the two hands, and solving $\cos\left(\frac{11}{6}\pi\right) = \frac{1}{2}$ gives a time of $\frac{2}{11}$ hours ≈ 11 minutes, as required.

Q5 This is another question that asks you to deploy your graph-sketching skills – in association with the supporting algebra, of course – and finding the coordinates of the TPs of the family of cubics given here, at $(a,\ 0)$ and $(-5a,\ 108a^3)$ makes it clear that the requested result holds. However, the case $a = 0$ (despite its relative triviality) still needs to be addressed separately.

Using (i)'s result with $a = y$ then immediately gives $27xy^2 \le (x + 2y)^3 \le 3^3$ so that $xy^2 \le 1$. Equality holds (from part (i)) when $x = a = y$ and from when $x + 2y = 3 \Rightarrow x = y = 1$.

Part (iii) requires a little more care and perseverance, but setting $x = p$ and $2a = q + r$ gets you off to a good start. There is now a little bit of "working to one side" needed if you are to convince yourself that $\left(\dfrac{q+r}{2}\right)^2 \ge qr$. Those students who have previously encountered the *Arithmetic Mean – Geometric Mean Inequality* will have spotted this straightaway; otherwise, this result becomes obvious upon rearrangement, as it is true $\Leftrightarrow (q - r)^2 \ge 0$, which is clearly true. Having to write it out in this way does have the advantage of highlighting that equality holds if and only if $q = r$ here (leading to $p = q = r$ for the main result).

Q6 Applying the given recurrence relation leads to

$$u_1 = 4\sin^2\theta\,(1 - \sin^2\theta) = 4\sin^2\theta\cos^2\theta = \sin^2(2\theta)\,,$$

though the question becomes very difficult indeed without the use of the two very basic trigonometric identities involved to this point. A moment's thought will help you avoid unnecessary further working at this stage, as the determination of u_2 from $u_1 = \sin^2$(some angle) should obviously give us the result $u_2 = \sin^2$(twice that angle) in exactly the same way. Thus $u_2 = \sin^2(4\theta)$ and, in general, $u_n = \sin^2(2^n\theta)$. Establishing the result inductively requires no more working than that which was given two sentences ago, with the $n = 1$ "baseline" case having been given to you.

In part (ii), we approach a more awkward looking recurrence relation by transforming it using the given substitution, and then comparing it with the required form. This gives the following relationships:

$$p\alpha = 4,\quad q - 2p\beta = 4 \quad\text{and}\quad (q-1)\,\beta + r = p\beta^2$$

from which the required result follows. Upon checking, the final r.r. satisfies the appropriate conditions, and since $u_0 = \sin^2\left(\tfrac{1}{4}\pi\right)$, we have $u_n = \sin^2\left(2^n.\tfrac{1}{4}\pi\right)$ and so $v_n = 4\sin^2\left(2^n.\tfrac{1}{4}\pi\right) - 1$. However, it should be noted that $\sin\left(2^n.\tfrac{1}{4}\pi\right) = 0$ for all $n \ge 2$, so that $\{v_n\} = \{1, 3, -1, -1, -1\ \ldots\}$.

Q7 When it boils down to it, this vectors question actually involves little more than the use of a standard ratio result, vector equations of lines, and finding intersections of lines. Note first that, if a point T divides a line segment XY in the ratio $p : q = \frac{p}{p+q} : \frac{q}{p+q}$, then the respective position vectors of these points are related thus: $\mathbf{t} = \left(\frac{q}{p+q}\right)\mathbf{x} + \left(\frac{p}{p+q}\right)\mathbf{y}$. Indeed, it is precisely this result that leads to the vector equation of a line in the form $\mathbf{r} = (1 - \alpha)\mathbf{a} + \alpha\mathbf{d}$ (for the line AD here). Writing BE's equation similarly and substituting for $\mathbf{d}$ and $\mathbf{e}$ in terms of r, s, $\mathbf{a}$ and $\mathbf{b}$ then gives the point of intersection of AD and BE by equating the two and comparing terms in $\mathbf{a}$ and $\mathbf{b}$, deriving the given form for $\mathbf{g}$.

The equation of the line OG is then $\mathbf{r} = \lambda\mathbf{g}$, which meets AB at F, which is known to cut AB in the ratio $t : 1$, which gives us two forms for the position vector of F. Comparing terms again yields the required answer.

Incidentally, those students and teachers more familiar with standard results in Euclidean Geometry will spot that this problem is actually an application of, first, *Menelaus' Theorem* and then *Ceva's Theorem*.

Q8 Part (i) directs you to finding the equation of L_a, which is $y = \left(1 - \frac{1}{a}\right)(x - a)$, and it follows immediately that L_b has equation $y = \left(1 - \frac{1}{b}\right)(x - b)$ similarly. Solving simultaneously yields their point of intersection at $\left(ab, (1-a)(1-b)\right)$. As $b \to a$, this point of intersection $\to \left(a^2, (1-a)^2\right)$, and it is clear that $a = \sqrt{x}$, so that $0 < \sqrt{x} < 1$ and $0 < x < 1$, and that $y = \left(1 - \sqrt{x}\right)^2$.

Differentiating this gives $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{1}{\sqrt{x}}$ which, at the point $C\left(\sqrt{c}, (1-\sqrt{c})^2\right)$, gives a tangent with equation $y - \left(1 - \sqrt{c}\right)^2 = \left(1 - \frac{1}{\sqrt{c}}\right)\left(x - \sqrt{c}\right)$. This rearranges into the form $y = \left(1 - \frac{1}{\sqrt{c}}\right)\left(x - \sqrt{c}\right)$, which is simply $L_{\sqrt{c}}$ where $0 < \sqrt{c} < 1$, as required.

Q9 This question provides rather a nice twist to the standard sort of projectiles question. Firstly, there is a non-zero horizontal component of the acceleration. However, since this doesn't affect the vertical motion, it is still the case that $T_H = \dfrac{U\sin\theta}{g}$ and $T_L = \dfrac{2U\sin\theta}{g}$. Next, the time $T = \dfrac{U\cos\theta}{kg}$ is introduced, which is quickly identified as the time when the horizontal component of the velocity is zero. Writing it in the form $T = \dfrac{U\sin\theta}{g} \times \dfrac{1}{k\tan\theta}$ at an early stage is really helpful, partly because it pulls out the common factor of $\dfrac{U\sin\theta}{g}$, but mostly as it identifies the factor of $k\tan\theta$ which is the major determining feature of the question. Having done this, you should realise that the three cases do little more than ask you to sketch what happens depending upon when the wind (for surely that is what is supplying this opposing force on the projectile) causes the particle to "trun round" horizontally, relative to when the maximum height and greatest range are achieved. In the first case, we get a "shortened" parabola compared to the usual shape of a non-wind-resisted projectile (so that T isn't reached before landing); in the second, the particle turns round horizontally before landing; and in the third, the wind is so strong that the particle begins to be blown backwards before reaching its greatest height. In the case of the "afterthought", the particle is constrained to move up and down in a straight line, returning to the point of projection.

Q10 In this question, we examine the dropping of two balls together, smaller on top of larger, which leads to the surprising outcome where the smaller ball bounces so much higher than one would expect it to. To begin with, we examine the bounce of the larger ball when it falls on its own. Since it falls a distance H, it hits the ground with speed u given by $u^2 = 2gH$, either by energy considerations or by using the ("suvat") constant acceleration formulae. It then leaves the ground with speed $v = eu$ according to *Newton's (Experimental) Law of Restitution*, attaining a height H_1 given by $v^2 = 2g(H_1 - R)$. Substituting for v and then u gives the printed answer.

In the extended situation, we again use $u = \sqrt{2gH}$ (for both balls) and $v = eu$ (for the larger ball after it bounces on the ground), though these do not need to be substituted immediately into the "collision" statements that are gained when using the principle of *Conservation of Linear Momentum* and *N*(*E*)*LR* for the subsequent collision between the balls. Solving simultaneously, the upwards speed of the smaller ball is found to be $x = \dfrac{\left(Me^2 + 2Me - m\right)}{M + m}u$. Using $x^2 = 2gd$, where d is the distance travelled by the small ball's centre of mass, and $u^2 = 2gH$, we deduce the result that $d = \left(\dfrac{Me^2 + 2Me - m}{M + m}\right)^2 H$; and then $h = 2R + r + d$.

In the final part of the question, substituting in the given numbers yields the answer $\dfrac{M}{m} = 9$. [Of course, this part of the question has really been asked back-to-front, as the real issue is to explore the speed of the smaller ball after the collision; however, the numerical working would have been much less favourable that way round.]

Q11 Part (i) of this question is a very straightforward single-pulley scenario, with the tension in the string equal throughout its length, which is constant, and the accelerations of the particles equal (relative to the string). Using *Newton's Second Law* for each particle gives the acceleration (as shown) and the tension in the string given by $T = \dfrac{4Mmg}{M+m}$.

In the second part of the question, we apply exactly the same assumptions and principles in exactly the same way, but there is a complication imposed now by the relative accelerations of the two particles on the P_1 pulley system. If we assume that the P_1 system has the "same" acceleration as the particle on the LHS of P, then (assuming that the m_1 particle accelerates downwards within this sub-system) the two particles on the RHS have accelerations $b - a_2$ and $b + a_2$ respectively. Also, since P_1 is taken to have zero mass, the tension in the main string is twice the tension in the sub-system. Without this set-up, the following working is most unlikely to be meaningful in any mark-scoring capacity. *N2L* applied several times, for the different particles, then leads to a set of equations that gives the second printed answer.

With two given answers, the very final part of the question can be done as a "stand alone" piece of work. Notice, however, that this is an *if and only if* proof and thus requires either two separate arguments or one in which every step is reversible. In point of fact, it transpires that $a_1 = a_2$ if and only if $(m_1 - m_2)^2 = 0$, which is equivalent to $m_1 = m_2$.

Q12 As with all such questions, one can make this much easier to deal with by being systematic, and by presenting one's working in a clear and coherent manner. To begin with, for instance, set out a table of the six possible outcomes, along with their associated probabilities. Of course, the reason why you are then asked for $\mathrm{E}(X^2)$ rather than $\mathrm{E}(X)$ is clearly because squaring negates any concerns about the sign of whatever is inside a pair of "modulus" lines, so that they can simply be replaced by ordinary brackets. You are then told that the (given) answer is a positive integer, which restricts $(k - 1)$ to being a multiple of 6 … namely, 1 or 7. Checking each of these in the possible $\mathrm{E}(X)$'s that arise enables you to eliminate $k = 1$ and confirm that $k = 7$.

Now that this information is known, it is possible to draw up the probability distribution for X (again, a table works very well), work out the probabilities $\mathrm{P}(X > 25) = \frac{21}{144}$ and $\mathrm{P}(X = 25) = \frac{4}{144}$, and then evaluate the expected payout

$$\mathrm{E}(W) = \sum\left(w \times \mathrm{P}(w)\right) = \left(w \times \tfrac{21}{144}\right) + \left(1 \times \tfrac{4}{144}\right) + 0$$

for the gambler (where W represents "winnings"). For this to be in the casino's favour, this expression must be negative, giving $w < 7$ and so the largest integer value of w is 6.

Q13 Although you are not asked for a sketch, a quick diagram might well help prevent stupid mistakes. Since the area under the triangle's two sloping sides is equal to 1, it follows that the height of the triangle is $h = \dfrac{2}{b-a}$, and so the gradient of the first line is this divided by ($c - a$) and the equation – that is, g(x) – follows immediately. Without further working, h(x) can be written down immediately by substituting b for a appropriately (i.e. *not* in the bit of the expression for h; and remembering that $c - b$ is negative).

In (i), it is definitely not enough to think "Aha! I recognise that expression" and simply throw the word "centroid" at the problem and hope to get all the points. You won't. The question makes it clear that it is intended for you to do the work to show that this expression is the mean. (For a start, the values a, b and c lie on a line and aren't "masses" placed at the vertices of the triangle.) The value of E(X) must be found by integrating over the two regions, and then sorted out algebraically.

For the final part of the question, it is important first to spend a moment making sure that you can identify the different possible cases that arise. These depend upon whether c is < , = , or > than $\frac{1}{2}(a+b)$. The equality case is the easiest, and the easiest to overlook, since then $m = c$ (by symmetry). In the first case, the median lies under the first of the sloping line segments, and will be gained by setting the area of a triangle equal to $\frac{1}{2}$. In a similar way to earlier, the third case can be dealt with by "working down from b" instead of "working up from a). These give, respectively, $m = a + \sqrt{\frac{1}{2}(b-a)(c-a)}$ and $m = b - \sqrt{\frac{1}{2}(b-a)(b-c)}$.