**General Remarks**

There were significantly more candidates attempting this paper this year (an increase of nearly 25%), but many found it to be very difficult and only achieved low scores. The mean score was significantly lower than last year, although a similar number of candidates achieved very high marks. This may be, in part, due to the phenomenon of students in the Lower Sixth (Year 12) being entered for the examination before attempting papers II and III in the Upper Sixth. This is a questionable practice, as while students have enough technical knowledge to answer the STEP I questions at this stage, they often still lack the mathematical maturity to be able to apply their knowledge to these challenging problems.

Again, a key difficulty experienced by most candidates was a lack of the algebraic skill required by the questions. At this level, the fluent, confident and correct handling of mathematical symbols (and numbers) is necessary and is expected; many students were simply unable to progress on some questions because they did not know how to handle the algebra.

There were of course some excellent scripts, full of logical clarity and perceptive insight. It was also pleasing that one of the applied questions, question 13, attracted a very large number of attempts.

However, the examiners were again left with the overall feeling that some candidates had not prepared themselves well for the examination. The use of past papers and other available resources to ensure adequate preparation is strongly recommended. A student’s first exposure to STEP questions can be a daunting, demanding experience; it is a shame if that takes place during a public examination on which so much rides.

**Comments on individual questions**

**Q1** This question was primarily about logical thinking and structuring an argument. While it was a very popular question, the marks were disappointing: only 30% of candidates gained more than 6 marks.

Most candidates could describe vaguely what is meant by the term irrational, though only a handful gave a precise, accurate definition. The popular offering of ‘a number with an infinite decimal expansion’ was not acceptable.

It was pleasing to see, though, that the majority of candidates were capable of using proof by contradiction to prove statements A and B, and they then went on to provide a counterexample to statement C. A small number of very strong candidates justified their counterexamples by proving that the numbers they presented were in fact irrational, though any well-known irrational examples were given full marks without the need for justification.

It is important to stress the difference between proving a statement and disproving one; while a single (numerical) counterexample is adequate to disprove a statement, a proof of the truth of a statement requires a general argument. Too many candidates wrote things such as: ‘If $pq = \sqrt{3}$, then $p = \sqrt{3}$ and $q = 1$, so . . . .’ Also, it is unknown what an irrational number ‘looks like’, so the frequently occurring arguments such as ‘We know that e + $\pi$ must be irrational because the numbers are not of the same form' (when comparing this example to something like $(1 - \sqrt{2}) + \sqrt{2}$) are spurious.

Sadly, very few candidates made any significant progress on the main part of the question. Several attempted (unsuccessfully) to prove that all four of the given numbers are irrational. Others asserted that since $\pi$ and e are both irrational, $\pi$ + e must also be, despite having just disproved statement C. A number of candidates successfully showed that $\pi$ + e and $\pi$ - e cannot both be irrational by appealing to B, but then could not see how to continue. The best attempts proceeded by using A and B repeatedly to show that no pair of $\pi \pm e$ and $\pi^2 \pm e^2$ could simultaneously be rational (that is, they considered all six cases separately).

**Q2** This was by far the most popular question on the paper, with about six out of every seven candidates attempting it.

The very first part involving implicit differentiation was generally done very well with most candidates scoring full marks for this part.

A majority of candidates then went on to successfully see how to apply this result to the required integral, although a sizeable minority failed to understand that they were being asked to perform a substitution. Some candidates resorted to the formula book and quoted the standard integral $\int 1/\sqrt{x^2 + a^2}\,\mathrm{d}x$; however, this gained no credit as the question explicitly said 'hence'.

Having reached $\int 1/(t+b)\,\mathrm{d}t$, the vast majority of candidates became unstuck. Firstly, after integrating, some did not substitute back $t = x + \cdots$ to get an expression in terms of $x$. The fundamental problem, though, was that the candidates were mostly unaware of the need to use absolute values when integrating $1/x$: almost everyone gave the intermediate answer as $\ln(t + b) + c$ rather than $\ln |t + b| + c$. It turns out that in this case, $t + b$ is always positive so the absolute values may be replaced by parentheses, but this requires explicit justification (which no-one gave).

This lack of appreciation of absolute values prevented all but the strongest candidates from making a decent attempt at the last part of the question, the consideration of the case $c = b^2$. Some candidates successfully substituted this in to the earlier result as instructed, but many claimed that $\sqrt{2x^2 + 2x + b^2} = x + b$. However, the correct expression is $|x + b|$, which is $x + b$ when this is positive, but $-(x + b)$ when $x + b < 0$. Only the tiny handful of candidates who appreciated this subtlety managed to correctly explain the distinction between these two cases.

**Q3** This was another popular question, although the scores were again fairly poor.

The proof of (*) was often done quite well. The main difficulties here arose because of a lack of clarity in the logic; it is important to make clear where the starting point is and what steps are being taken to move forward from there. A significant number of candidates attempted to work backwards, and then divided by $d - b$ or the like without realising that this might be zero. Also, inequalities were multiplied without any regard to the sign of the numbers under consideration; for example, while $0 > -1$ and $1 > -2$, it is not true that $0 \times 1 > (-1) \times (-2)$.

The beginning of part (i) was completed correctly by a majority of candidates. It is important to stress again that if a question specifies "use (*)", then this must be done to gain any credit; no marks were given for the numerous answers which began with "as $(x - y)^2 \geq 0$ for all $x$ and $y$, we have $x^2 - 2xy + y^2 \geq 0$" or similar.

The last part of (i) was often poorly tackled. It was sometimes interpreted to mean "when $x < 0$" or other spurious cases, without understanding that the inequality had so far only been shown in the case $x \geq y \geq z$. (Indeed, the intermediate result $z^2 + xy \geq xz + yz$ does not hold in the case $x > z > y$.) Many other candidates ignored their preceding work and went on to prove the result from scratch using the inequality $(x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0$. Very few candidates explained the symmetry of the situation.

Part (ii) was problematic because of the wording of the question. It turned out that there is a very straightforward way to answer this part by making use of the results proved in part (i). While this was not what was actually asked ("Show similarly . . . "), it was felt unfair to penalise candidates too harshly for taking this route. Thus they were awarded partial credit and **all** such candidates were referred to the Chief Examiner for individual consideration. Nonetheless, the attempts at this part, by whichever method, were generally either close to perfect or non-starters.

**Q4** The initial graph-sketching part of this question was designed to help candidates solve the quadratic equation which was to come up later in the question. Whilst almost all of the candidates successfully sketched $y = \sin x$, the attempts at $y = \frac{2}{3}\cos^2 x$ were significantly poorer. Many candidates sketched curves with cusps at the $x$-axis, presumably confusing $\cos^2 x$ with $|\cos x|$; others had curves which fell below the $x$-axis in places. Perhaps few candidates had seen graphs of $y = \cos^2 x$ before or considered that $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$, making $\cos^2 x$ sinusoidal itself. Also, a large number of candidates appeared to have spent a significant amount of time drawing beautiful and accurate graphs on graph paper; it is important to appreciate the nature of a sketch as a rough drawing which captures the essential features of a situation. In general, STEP questions will not require accurate graphs, only accurate sketches.

The first derivative of $f(x)$ was generally computed correctly, though a sizable proportion of candidates failed to correctly apply the product rule to determine the second derivative. Those candidates who obtained $f''(x)$ correctly generally realised that they needed to solve the inequality $\frac{2}{3}\cos^2 x \geq \sin x$. Some appear to have guessed a value of $x$ which makes this an equality - this method is perfectly acceptable as long as some justification of the claimed result is given (such as by explicitly substituting $x = \pi / 6$ into the two sides). Most candidates who got this far correctly understood the connection with the graph sketch and went on to give the correct intervals.

In part (ii), there was a lot of difficulty performing the differentiation. A number of candidates made their life more difficult by substituting $k = \sin 2\alpha$ before differentiating $g(x)$; this just made the expressions appear more complex and increased the likelihood of error. Some candidates, for example, tried differentiating with respect to both $x$ and $\alpha$ simultaneously.

Nevertheless, most candidates who were able to correctly compute g''(x) went on to solve the resulting trigonometric equation, finding the solution $x = \alpha$, but many failed to determine the second interval. A sketch of some sort would very likely have been useful.

**Q5** This was the least popular of the Pure Mathematics questions. There was a fair amount of confusion as to the meaning of the summation, with the majority of attempts at the $n$ =1 case in part (i) failing to understand that the polynomial would be $p(x)= x+a_0$ rather than just $p(x)= x$. A small number thought that the summation indicated a geometric series, and proceeded to claim that $p(x) = (1 - x^n)/(1 - x)$ or other such things.

Nonetheless, there were many good answers to the rest of part (i), with candidates showing that they understood the statement of Chebyshev's theorem. A small number of strong candidates had a mature enough understanding of mathematics to use the alternative method given in the sample solutions; most were content with finding the maxima and minima. Some forgot to check the value of p($x$) at the ends of the interval, which was not penalised as long as they did not incorrectly assert that they had found the value of $M$.

One recurrent incorrect assertion was that from the inequality $p(x) \geq \frac{1}{2}$, it necessarily follows that $M = \frac{1}{2}$, without showing that equality is obtained for *some* value of $x$.

There were few serious attempts at part (ii), but most of those achieved full marks or very close to it. Several candidates had difficulty in explaining their reasoning: a sketch would certainly have helped clarify why a maximum value of $|p(x)|$ occurring in the interval $-1 < x < 1$ necessarily forces this point to be a turning point.

Of the other attempts, many could not see the relevance of Chebyshev's theorem to this situation, or even if they did, then failed to divide the given polynomial by 64. Arguments which did not invoke Chebyshev's theorem were not given any credit (the main alternative being to differentiate, then to find points where the derivative was positive and negative, and use the intermediate value theorem to assert that there is a point where the derivative must be zero).

**Q6** This was another popular question which was gained a pleasing number of good marks.

The sketch was generally done well. A significant number of candidates did not realised that f(0) = 0 and f(1) = 1, so either had non-intersecting graphs or graphs which were tangent to each other at the origin. A number of candidates sketched the graph of f($x$) for all real $x$, in spite of the question stating $x \geq 0$; they were not penalised for this. Most understood how the graphs of f($x$) and g($x$) related.

The determination of g($x$) algebraically was performed correctly by a majority of candidates. However, a disturbing number of candidates introduced absolute value signs, writing $g(x)= \ln |(e-1)x+1|$. Whilst technically correct in this range (and therefore not penalised here), it is indicative of confusion about when absolute values are used

with logs: when integrating $1/x$ (as in Question 2 above) they are required; when inverting exponentiation they are not. A smaller number made very significant errors in their handling of the logarithm function, writing such things as
$\ln(\mathrm{e}x - x + 1) = \ln \mathrm{e}x - \ln x + \ln 1$.

The majority of candidates correctly integrated $f(x)$. A small minority bizarrely asserted that $\int_0^{1/2} f(x)\,dx = f\left(\frac{1}{2}\right) - f(0)$which was somewhat disturbing.

The integration of $g(x)$ proved much more troublesome. Despite $\int \ln x \, dx$ being a standard integral and explicitly mentioned in both the STEP Specification and A2 Mathematics specifications, the introduction of the linear function of $x$ flummoxed most candidates. Some differentiated instead of integrating, others just gave up. A small number either attempted to use parts or to substitute, and a good proportion of such attempts were successful. Some candidates confused differentiation with integration during this process and tried to use a mixture of parts and the product rule.

Finally, of those who managed to reach this point, a decent number gave a very convincing explanation of why $\int f + \int g = \frac{1}{2}k$.

**Q7** This was a reasonably popular question, tackled by about half of the candidates. Most confidently showed that $y = \frac{1}{2}(y - \sqrt{3}\,x)$ and went on to deduce the result for the clockwise rotation. A small number of candidates lost marks here because their presentation either failed to make clear which answer corresponded to which direction of rotation, or the directions were reversed. Several candidates would have been helped by including a sketch in their solution.

About two-thirds of the candidates were unable to progress beyond this point. Of those who continued, the majority succeeded in finding $h_1$, either by a direct argument or, more usually, by using the earlier result as intended by the question. Despite the hint of $h_1$ being given with absolute value signs, a large number of candidates then claimed that $h_2 = y$ rather than the correct $|y|$, suggesting that they do not understand what absolute values mean and when they should be used.

Very few candidates correctly determined $h_3$, the most common incorrect answer being $h_3 = \frac{1}{2}|y + \sqrt{3}\,x|$ to parallel the answer for $h_1$. Again, clear diagrams are essential if marks are to be gained for questions such as these. There was also evidence of confusion in the algebraic manipulation of absolute values, with some candidates confusing $|a-b|$ with $|a|-|b|$, thereby giving answers such as $h_3 = \frac{1}{2}|y + \sqrt{3}\,x| - \frac{1}{2}\sqrt{3}$

Only a handful of candidates made a significant attempt at the final part of the question, and of those who did, the main difficulty stemmed from not appreciating that to prove an “if and only if” statement, one has to prove the implication in both directions. The sample solutions use the triangle inequality; it could equally and straightforwardly be argued by considering all eight possible cases of where the point $P$ might lie with respect to each of the three sides of the triangle.

**Q8** This was another popular question, and many candidates achieved decent marks on this question.

Many candidates were correctly able to differentiate (*), although a significant number ran into difficulties with the $(y')^2$ term, where things such as $2y'\frac{\mathrm{d}}{\mathrm{d}x}(y') = 2y'.y''\frac{\mathrm{d}y}{\mathrm{d}x}$ were common errors. Although the rest of (*) was usually differentiated correctly by these candidates, since the rest of part (i) depended upon getting this first step correct, they floundered from then on. Many of these candidates nevertheless went on to gain additional marks by at least making a good start to part (ii).

Also, candidates must remember to read the question and to follow its guidance; the question instructed them to differentiate (*), and those who tried rearranging it instead got nowhere.

From $y'' = 0$, most candidates deduced that $y' = m$ and substituted this back into (*) to determine $y = mx - m^2$. However, when working like this, it is vital to check that the purported $y$, call it $\hat{y}$ say, satisfies $\mathrm{d}\hat{y}/\mathrm{d}x = y'$, since y and $y'$ must be related by both the given differential equation and also by $y' = \mathrm{d}y/\mathrm{d}x$. There may be other arguments which would allow one not to differentiate the obtained $y$, but these would have to be given explicitly. The alternative method of determining that $y = mx + c$ and then substituting this into (*) was noticeably less common, but avoided this subtlety.

For the $2y'' = x$ case, similar comments again apply, although here it was concerning how many candidates integrated to get $y = \frac{1}{4}x^2$ without including a constant of integration.

In part (ii), few candidates succeeded in correctly differentiating the differential equation. One of the most common errors was to claim that $\frac{\mathrm{d}}{\mathrm{d}x}(y^2) = 2y'$. The few candidates who correctly differentiated the equation mostly applied the techniques from part (i) to solve the equation successfully. Several fudged the solution of the resulting equation $(x^2 - 1)\,y' = xy$ by conveniently forgetting the absolute value signs when integrating (as shown in the sample solutions), but this error was not penalised on this occasion.

**Q9** This was an unpopular question and the marks were very disappointing; only half of the attempts gained over one mark out of twenty, and only a handful of candidates gained over six marks.

Nonetheless, of the candidates who made a reasonable start, many were capable of drawing a clear diagram of the position of the hoop after it had rolled, but few were able to show how the position after it had rolled related to its initial position. This allowed them to correctly determine the $y$-coordinate of $P$, but they became very unstuck when attempting to determine the $x$-coordinate.

The next difficulty encountered was in calculating the components of the velocity of $P$, as many candidates appeared unable to differentiate a function of $\theta$ with respect to $t$.

For the determination of the kinetic energy, several candidates $\dot{\theta}$ used the expected method. It was also very encouraging to see a number of candidates correctly using the formula total KE = linear KE + rotational KE, and then determining the rotational KE using either moments of inertia or the explicit formula $\frac{1}{2}mr^2\dot{\theta}^2$.

Finally, a small number of candidates correctly considered forces or energy and deduced that the hoop rolls at constant speed.

**Q10** This was the most popular mechanics question, and the question which gained the best marks across the entire paper.

The sketch of the particle's trajectories in the two different scenarios was generally well done, with almost all candidates successfully completing the sketch. It was a little disappointing, though, that very few attempted to justify their assumption that the particle does, in fact, reach height *h*.

The next stage, using the "suvat" equations to deduce *d*, was generally either done very well or very poorly. Of those who had difficulty, some were stuck trying to figure out how to go about the question, others were unsure of which of the "suvat" equations to use (despite all of the individual components of this question being very standard A-level problems), while some derived a quadratic equation (having used $s = ut + \frac{1}{2}at^2$) but were incapable of then solving it.

Nonetheless, this question did require a sustained chain of logical steps, and it was pleasing to see over a quarter of the candidates who attempted this question gaining close to full marks on it.

**Q11** This was the least popular of the mechanics questions, and of the candidates who attempted it, only a handful made any progress beyond drawing a usually incorrect sketch and writing down some equations.

The majority were aware that $F = \mu R$ as the equilibrium is limiting. Unfortunately, though, they often either missed forces from their diagram or drew at least one of the frictional forces in the wrong direction. Another frequent problem was that they labelled both normal reaction forces with the same variable *R*, thereby implicitly implying that they are equal, whereas this is not the case. A small annoyance was the number of candidates who used the notation *Fr* for friction; an unhealthy practice as it can so easily be confused with $F \times r$. Also, several failed to mark the angle $\alpha$ correctly on their diagram.

After this, a small number of candidates correctly resolved in two directions and took moments. Those who understood how to then manipulate the resulting equations to eliminate most of the variables went on to produce essentially perfect solutions, whereas everyone else became stuck at this point and found themselves unable to progress any further.

No attempts using the theorem regarding three forces on a large body were seen, which is a shame, as it made the problem significantly easier.

**Q12** This was by far the least popular question on the paper, as is often the case with Probability and Statistics questions.

Of the candidates who attempted it, most successfully answered part (i), and a significant number were also confident in the manipulation of sums required for part (ii). Success was clearly dependent upon taking great care to ascertain the meaning of the event $X = r$ in terms of $X_1$ and $X_2$.

Part (iii) proved much more problematic, as almost no-one made use of *both* defining inequalities for the median; one inequality on its own may appear to give the correct answer, but is insufficient to gain the marks.

The handful of candidates who attempted part (iv) were generally successful in their attempts.

**Q13** This combinatorics question was attempted by close to half of all candidates, a very encouraging statistic. About two-thirds of the attempts did not progress very far, gaining five marks or fewer, but of those who did get further, the marks were fairly evenly distributed.

For part (i), most attempts reached the stated answer, although a significant number used very creative, if inaccurate or meaningless, ways of doing so. The majority of candidates used counting methods, and many of these were successful to a greater or lesser extent. The other method used by many candidates was to consider the probability of the first wife sitting next to her husband ($\frac{2}{5}$) and the conditional probability of the spouse of the other person sitting next to the first husband sitting next to them (this is $\frac{1}{3}$), and then multiplying these.

It is crucial at this point to reinforce that candidates must explain their reasoning in their answers, especially when they are working towards a given answer. Simply writing $\frac{2}{5} \times 13 = \frac{2}{15}$ is woefully inadequate to gain all of the available marks; there must be a justification of the reasoning behind it.

Parts (ii) and (iii) were found to be a lot more challenging. A number of candidates attempted to construct probabilistic arguments, which are very challenging in this case. The successful attempts all used pure counting arguments. The examiners often found it challenging to decipher their thinking, though, as the explanations were often somewhat incoherent. Those who used counting arguments usually made good progress on both parts.

The favoured method for part (iii) was to use P(no pairs) = 1 − P(≥ 1 pair). It would have certainly been worth checking the answer obtained using a direct method, as this would have caught a number of errors.

The main errors encountered in good attempts at the later parts of the question were a failure to consider all possible cases or a miscounting of the number of ways each possible case could occur.

Overall, this question was answered well by a significant number of candidates.