STEP II 2014

Question 1:

Drawing a diagram and considering the horizontal and vertical distances will establish the relationships for $x\cos\theta$ and $x\sin\theta$ easily. The quadratic equation will then follow from use of the identity $\cos^2\theta + \sin^2\theta \equiv 1$. The same reasoning applied to a diagram showing the case where P and Q lie on AC produced and BC produced will show that the same equation is satisfied.

(*) will be linear if the coefficient of x^2 is 0, so therefore $\cos(\alpha+\beta)$ will need to equal $-\frac{1}{2}$, which gives a relationship between α and β. For (*) to have distinct roots the discriminant must be positive. Using some trigonometric identities it can be shown that the discriminant is equal to $4(1-(\sin\alpha - \sin\beta)^2)$ and it should be easy to explain why this must be greater than 0.

The first case in part (iii) leads to $x = \sqrt{2} \pm 1$ and so there are two diagrams to be drawn. In each case the line joining P to Q will be horizontal.

The second case in part (iii) is an example where (*) is linear. This leads to $x = \frac{\sqrt{3}}{3}$. Therefore Q is at the same point as C and so the point P is the midpoint of AC.

Question 2:

By rewriting in terms of $\cos 2nx$ it can be shown that $\int_0^\pi \sin^2 nx\, dx = \frac{\pi}{2}$ and $\int_0^\pi n^2\cos^2 nx\, dx = \frac{n^2\pi}{2}$. Therefore (*) must be satisfied as n is a positive integer. The function $f(x) = x$ does not satisfy (*) and $f(0) = 0$ but $f(\pi) \neq 0$. The function $g(x) = f(\pi - x)$ will therefore provide a counterexample where $g(\pi) = 0$, but $g(0) \neq 0$.

In part (ii), $f(x) = x^2 - \pi x$ will need to be selected to be able to use the assumption that (*) is satisfied. The two sides of (*) can then be evaluated:

$$\int_0^\pi x^4 - 2\pi x^3 + \pi^2 x^2\, dx = \frac{\pi^5}{30}$$

$$\int_0^\pi 4x^2 - 4\pi x + \pi^2\, dx = \frac{\pi^3}{3}$$

Substitution into (*) then leads to the inequality $\pi^2 \leq 10$.

To satisfy the conditions on $f(x)$ for the second type of function, the values of p, q and r must satisfy $q + r = 0$ and $p + r = 0$. Evaluating the integrals then leads to $\pi \leq \frac{22}{7}$.

Since $\left(\frac{22}{7}\right)^2 < 10$, $\pi \leq \frac{22}{7}$ leads to a better estimate for π^2.

Question 3:

By drawing a diagram and marking the shortest distance a pair of similar triangles can be used to show that $\frac{c/_m}{c\sqrt{m^2+1}/_m} = \frac{d}{c}$, which simplifies to $d = {}^{c}\!/\!\sqrt{m^2+1}$.

For the second part, the tangent to the curve at the general point (x, y) will have a gradient of y' and so the y-intercept will be at the point $(0, y - xy')$. Therefore the result from part (i) can be applied using $m = y'$ and $c = y - xy'$ to give $a = {}^{(y-xy')}\!\Big/\!\sqrt{(y')^2+1}$, which rearranges to give the required result.

Differentiating the equation then gives $y''\big(a^2y' + x(y - xy')\big) = 0$ and so either $y'' = 0$ or

$a^2y' + x(y - xy') = 0$.

If $y'' = 0$ then the equation will be of a straight line and the y-intercept can be deduced in terms of m.

If $a^2y' + x(y - xy') = 0$, then the differential equation can be solved to give the equation of a circle.

Part (iii) then requires combining the two possible cases from part (ii) to construct a curve which satisfies the conditions given. This must be an arc of a circle with no vertical tangents, with straight lines at either end of the arc in the direction of the tangents to the circle at that point.

Question 4:

In part (i), if the required integral is called I then the given substitution leads to an integral which can be shown to be equal to $-I$. This means that $2I = 0$ and so $I = 0$.

In part (ii), once the substitution has been completed, the integral will simplify to $\int_{1/b}^{b} \frac{\arctan\frac{1}{u}}{u}\,du$. Since $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ the integral can be shown to be equal to $\frac{1}{2}\int_{1/b}^{b} \frac{\pi}{2x}\,dx$, which then simplifies to the required result.

In part (iii), making with the substitution in terms of k and simplifying will show that the integral is equivalent to

$$\int_0^\infty \frac{ku^2}{(a^2u^2 + k^2)^2}\,du$$

Therefore choosing $k = a^2$, the integral can be simplified further to

$$\frac{1}{a^2}\int_0^\infty \frac{u^2}{(a^2 + u^2)^2}\,du = \frac{1}{a^2}\int_0^\infty \frac{1}{a^2 + u^2}\,du - \frac{1}{a^2}\int_0^\infty \frac{a^2}{(a^2 + u^2)^2}\,du$$

The result then follows by using the given value for $\int_0^\infty \frac{1}{a^2+x^2}\,dx$.

Question 5:

Using the substitution $y = xu$, the differential equation can be simplified to

$$x\frac{du}{dx} = \frac{1 + 4u - u^2}{u - 2}$$

This can be solved by separating the variables after which making the substitution $u = \frac{y}{x}$ and substituting the point on the curve gives the required quadratic in x and y.

In part (ii), $\frac{dY}{dX}$ can be shown to be equal to $\frac{dy}{dx}$. The values of a and b need to be chosen so that the right hand side of the differential equation has no constant terms in the numerator or denominator. This leads to the simultaneous equations:

$$a - 2b - 4 = 0$$

$$2a + b - 3 = 0$$

Solving these and substituting the values into the differential equation gives $\frac{dY}{dX} = \frac{X-2Y}{2X+Y}$, and so

$$\frac{dX}{dY} = \frac{2X + Y}{X - 2Y}$$

This is the same differential equation as in part (i), with $x = Y$ and $y = X$. Most of the solution in part (i) can therefore be applied, but the point on the curve is different, so the constant in the final solution will need to be calculated for this case.

Question 6:

One of the standard trigonometric formulas can be used to show that

$$\sin\left(r + \tfrac{1}{2}\right)x - \sin\left(r - \tfrac{1}{2}\right)x = 2\cos rx \sin\tfrac{1}{2}x.$$

Summing these from $r = 1$ to $r = n$ will then give the required result.

In part (i), the definition can be rewritten as $S_2(x) = \sin x + \frac{1}{2}\sin 2x$. The stationary points can then be evaluated by differentiating the function. The sketch is then easy to complete.

For part (ii), differentiating the function gives $S_n'(x) = \cos x + \cos 2x + \cdots + \cos nx$. Applying the result from the start of the question, this can be written as

$$S_n'(x) = \frac{\sin\left(n + \frac{1}{2}x\right) - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

Since $\sin\frac{1}{2}x \neq 0$ in the given range, the stationary points are where $\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x = 0$. This can then be simplified to the required form by splitting $\sin\left(n+\frac{1}{2}\right)x$ into functions of nx and $\frac{1}{2}x$ and noting that $\sin\frac{1}{2}x \neq 0$ and $\cos\frac{1}{2}x \neq 0$ in the given range, so both can be divided by. By noting that the difference between $S_{n-1}(x)$ and $S_n(x)$ is $\frac{1}{n}\sin nx$ the result just shown can be used to show the final result of part (ii). Part (iii) then follows by induction.

Question 7:

By considering the regions $x \leq a$, $a < x < b$ and $x \geq b$, $f(x)$ can be written as

$$f(x) \quad = \quad \begin{matrix} a+b-2x & x \leq a \\ b-a & a < x < b \\ 2x-a-b & x \geq b \end{matrix}$$

Therefore the graph of $y = f(x)$ will be made up of two sloping sections (with gradients 2 and -2 and a horizontal section). The graph of $y = g(x)$ will have the same definition in the regions $x \leq a$ and $x \geq b$, with the sloping edges extending to a point of intersection on the x-axis. The quadrilateral with therefore have sides of equal length and right angles at each vertex, so it is a square.

In part (ii), sketches of the cases where $c = a$ and $c = b$ show that these cases give just one solution. If $a < c < b$ there will be no solutions and in the other regions there will be two solutions.

In part (iii) the graphs for the two sides of the equation can be related to graphs of the form of $g(x)$ (apart from the section which is replaced by a horizontal line) in the first part of the question. Since $d - c < b - a$, the horizontal sections of the two graphs must be at different heights so the number of solutions can be seen to be the same as the number of intersections of the graphs of the form of $g(x)$.

Question 8:

The coefficients from the binomial expansion should be easily written down. It can then be shown that

$$\frac{c_{r+1}}{c_r} = \frac{b(n-r)}{a(r+1)}$$

This will be greater than 1 (indicating that the value of c_r is increasing) while $b(n-r) > a(r+1)$, which simplifies to $r < \frac{nb-a}{a+b}$. Similarly, $\frac{c_{r+1}}{c_r} = 1$ if $r = \frac{nb-a}{a+b}$ and $\frac{c_{r+1}}{c_r} < 1$ if $r > \frac{nb-a}{a+b}$. Therefore the maximum value of c_r will be the first integer after $\frac{nb-a}{a+b}$ (and there will be two maximum values for c_r if $\frac{nb-a}{a+b}$ is an integer. The required inequality summarises this information.

In parts (i) and (ii) the values need to be substituted into the inequality. Where there are two possible values, it needs to be checked that they are equal before taking the higher if this has not been justified in the first case.

In part (iii) the greatest value will be achieved when the denominator takes the smallest possible value, therefore $a = 1$, and then in part (iv) the greatest value will be achieved by maximising the numerator. Since the maximum possible value of $G(n, a, b)$ is n, $b \geq n$ will achieve this maximum.

Question 9:

Once a diagram has been drawn the usual steps will lead to the required result:

Resolving vertically:

$$F + T\cos\theta = mg$$

Resolving horizontally:

$$T\sin\theta = R$$

Taking moments about A:

$$mg(a\cos\varphi + b\sin\varphi) = Td\sin(\theta+\varphi)$$

Limiting equilibrium, so $F = \mu R$:

$$\mu T\sin\theta + T\cos\theta = mg$$

Therefore:

$$Td\sin(\theta+\varphi) = T(\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

And so:

$$d\sin(\theta+\varphi) = (\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

If the frictional force were acting in the opposite direction, then the only change to the original equations would be the sign of F in the first equation. Therefore the final relationship will change to

$$d\sin(\theta+\varphi) = (-\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

For the final part, the first and third of the equations above can be used to show that

$$F = \frac{Td\sin(\theta+\varphi)}{a\cos\varphi + b\sin\varphi} - T\cos\theta$$

Since $F > 0$ if the frictional force is upwards, this then leads to the condition $d > \frac{a+b\tan\varphi}{\tan\theta+\tan\varphi}$. Since the string must be attached to the side AB, d cannot be bigger than $2b$, which leads to the final result of the question.

Question 10:

Consideration of the motion horizontally and vertically and eliminating the time variable leads to a Cartesian equation for the trajectory:

$$y = \lambda x - \frac{gx^2}{2u^2}(1+\lambda^2)$$

The maximum value can be found either by differentiation or by completing the square. Completing the square gives:

$$y = -\frac{gx^2}{2u^2}\left(\lambda - \frac{u^2}{gx}\right) + \frac{u^2}{2g} - \frac{gx^2}{2u^2}$$

Which shows that $Y = \frac{u^2}{2g} - \frac{gx^2}{2u^2}$. If this graph is sketched then the region bounded by the graph and the axes will represent all the points that can be reached.

The maximum achievable distance must lie on the curve and the distance, d, of a point on the curve can be shown to satisty $d^2 = \left(\frac{u^2}{2g} + \frac{gx^2}{2u^2}\right)^2$, which must be maximised when x takes the maximum value possible.

Question 11:

A diagram shows that the coordinates of P are $(x + (L - x)\sin\alpha\,, -(L - x)\cos\alpha)$

Therefore, by differentiating the y-coordinate of P shows that the vertical acceleration of P is $\ddot{x}\cos\alpha$ and applying Newton's Second Law gives

$$T\cos\alpha - kmg = km\ddot{x}\cos\alpha$$

A similar method for the horizontal motion of P and R gives the two equations

$$T\sin\alpha = -km(1 - \sin\alpha)\ddot{x}$$

$$T - T\sin\alpha = -m\ddot{x}$$

For part (ii), eliminating T from the last two equations gives the required relationship. A sketch of the graph of $y = \frac{x}{(1-x)^2}$ will then show that for any value of k there is a possible value between 0 and 1 for $\sin\alpha$.

In part (iii), elimination of T from the two equations formed by considering the motion of P gives the required result.

Question 12:

The required probability in the first part is given by

$$\frac{P(t < T < t + \delta t)}{P(T > t)} = \frac{F(t + \delta t) - F(t)}{1 - F(t)}$$

In the case of small values of δt, $F(t + \delta t) - F(t) \approx f(t)\delta t$, which leads to the correct probability.

In part (ii), differentiation gives $f(t) = \frac{1}{a}$, and substituting into the definition of the hazard function gives $h(t) = \frac{1}{a-t}$. Both graphs are simple to sketch.

In part (iii), using the definition of the hazard function gives $\frac{F'(t)}{1-F(t)} = \frac{1}{t}$. Integrating gives $-\ln|1 - F(t)| = \ln|kt|$, and so the probability density function can be found by rearranging to find $F(t)$ and then differentiating.

A similar method in part (iv) shows that if $h(t)$ is of the form stated then $f(t)$ will be of the given form. Similarly, if $f(t)$ has the given form then $h(t)$ can be shown to have the form stated.

In part (v), a differential equation can again be written using the definition of the hazard function and this can again be solved by integrating both sides with respect to t.

Question 13:

Considering the sequence of events for $X = 4$, the 1st, 2nd and 3rd numbers must all be different and then the 4th must be the same as one of the first three. The probability is therefore

$$P(X = 4) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n}$$

The same reasoning applied to $X = r$ gives

$$P(X = r) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{r-2}{n}\right)\frac{r-1}{n}$$

The result of part (i) is then found by observing that the probabilities of all possible outcomes add up to 1.

Substituting the probabilities into the formula for $E(X)$ gives

$$E(X) = \frac{2}{n} + 3\left(1 - \frac{1}{n}\right)\frac{2}{n} + 4\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots + (n+1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$

For part (iii) observe that any case where $X \geq k$ will have the first $k - 1$ numbers all different from each other. Therefore

$$P(X \geq k) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-2}{n}\right)$$

The first formula in part (iv) can be shown by considering $kP(Y = k)$ to be equal to the sum of k copies of $P(Y = k)$ and then regrouping the sum for $E(Y)$. Finally this gives two different expressions for $E(Y)$, which must be equal to each other:

$$\begin{aligned}\frac{2}{n} + 3\left(1 - \frac{1}{n}\right)\frac{2}{n} + 4\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots + (n+1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\ = 1 + 1 + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)\end{aligned}$$

Rearranging and using the result from part (i) then gives the required result.