Question 1

Given that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a(x - y + 2)^2 + b(cx + y)^2 + d,$$

find the values of the constants a, b, c and d.

We expand the right hand side, and then equate coefficients:

$$\begin{aligned}
5x^2 + 2y^2 - 6xy + 4x - 4y& \\
&\equiv a(x - y + 2)^2 + b(cx + y)^2 + d \\
&\equiv a(x^2 - 2xy + y^2 + 4x - 4y + 4) + b(c^2x^2 + 2cxy + y^2) + d \\
&\equiv (a + bc^2)x^2 + (2bc - 2a)xy + (a + b)y^2 + 4ax - 4ay + 4a + d,
\end{aligned}$$

so we require

$$\begin{aligned}
a + bc^2 &= 5 \\
2bc - 2a &= -6 \\
a + b &= 2 \\
4a &= 4 \\
-4a &= -4 \\
4a + d &= 0.
\end{aligned}$$

The fourth and fifth equations both give $a = 1$ immediately, giving $b = 1$ from the third equation. Then the second equation gives $c = -2$ and the final equation gives $d = -4$.

We must also check that this solution is consistent with the first equation. We have $a + bc^2 = 1 + 1 \times (-2)^2 = 5$, as required. (Why is this necessary? Well, if the second equation had begun with $7x^2 + \cdots$, then our method would still have given us $a = 1$, etc., but the coefficients for the x^2 term would not have matched, so we would not have been able to write the second equation in the same way as the first.)

We thus deduce that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv (x - y + 2)^2 + (-2x + y)^2 - 4.$$

Solve the simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9, \tag{1}$$
$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14. \tag{2}$$

Spurred on by our success in the first part, we will rewrite the first equation in the suggested form:

$$(x - y + 2)^2 + (y - 2x)^2 - 4 = 9. \tag{3}$$

We are led to wonder whether the same trick will work for the second equation, so let's try writing:

$$6x^2 + 3y^2 - 8xy + 8x - 8y \equiv a(x - y + 2)^2 + b(cx + y)^2 + d.$$

As before, we get equations:

$$\begin{aligned} a + bc^2 &= 6 \\ 2bc - 2a &= -8 \\ a + b &= 3 \\ 4a &= 8 \\ -4a &= -8 \\ 4a + d &= 0. \end{aligned}$$

(We can write these down as the right hand side is the same as before.)

This time, $a = 2$ from both the fourth and fifth equations, so we get $b = 1$ from the third equation. The second equation gives us $c = -2$. Finally, the sixth equation gives us $d = -8$.

We *must* now check that our solution is consistent with the first equation, which we have not yet used. The left hand side is $a + bc^2 = 2 + 1 \times (-2)^2 = 6$, which works, so we can write the second equation as

$$2(x - y + 2)^2 + (y - 2x)^2 - 8 = 14.$$

(If we had not checked for consistency, we might have wrongly concluded that $183x^2 + 3y^2 - 8xy + 8x - 8y$ can also be written in the same way.)

These two equations now look remarkably similar! In fact, let's move the constants to the right hand side and write them together:

$$\begin{aligned} (x - y + 2)^2 + (y - 2x)^2 &= 13 \\ 2(x - y + 2)^2 + (y - 2x)^2 &= 22. \end{aligned}$$

We now have two simultaneous equations which look almost linear. In fact, if we write $u = (x - y + 2)^2$ and $v = (y - 2x)^2$, we get

$$\begin{aligned} u + v &= 13 \\ 2u + v &= 22 \end{aligned}$$

which we can easily solve to get $u = 9$ and $v = 4$.

Therefore, we now have to solve the two equations

$$(x - y + 2)^2 = 9 \tag{4}$$
$$(y - 2x)^2 = 4. \tag{5}$$

We can take square roots, so that (4) gives $x - y + 2 = \pm 3$ and (5) gives $y - 2x = \pm 2$.

Thus we now have four possibilities (two from equation (4), and for each of these, two from equation (5)), and we solve each one, checking our results back in the original equations.

$2x-y$	$x-y+2$	x	y	LHS of (1)	LHS of (2)
2	3	1	0	9	14
2	−3	7	12	9	14
−2	3	−3	−4	9	14
−2	−3	3	8	9	14

Therefore we see that the four solutions are $(x, y) = (1, 0)$, $(7, 12)$, $(-3, -4)$ and $(3, 8)$.

An alternative is to observe that equation (2) looks almost double equation (1), so we consider $2 \times (1) - (2)$:

$$4x^2 + y^2 - 4xy = 4.$$

But the left hand side is simply $(2x - y)^2$, so we get $2x - y = \pm 2$.

Substituting this into equation (3) gives us

$$(x - y + 2)^2 + 4 - 4 = 9,$$

so that $x - y + 2 = \pm 3$.

Thus we have the four possibilities we found in the first approach, and we continue as above.

Yet another alternative approach is to subtract $(2) - (1)$ to get

$$x^2 + y^2 - 2xy + 4x - 4y = 5,$$

so that

$$(x - y)^2 + 4(x - y) = 5.$$

Writing $z = x - y$, we get the quadratic $z^2 + 4z - 5 = 0$, which we can then factorise to give $(z + 5)(z - 1) = 0$, so either $z = 1$ or $z = -5$, which gives $x - y = 1$ or $x - y = -5$.

Substituting $x - y = 1$ into (3) now gives

$$(1 + 2)^2 + (y - 2x)^2 - 4 = 9,$$

so that $(y - 2x)^2 = 4$; substituting $x - y = -5$, on the other hand, would lead us to

$$(-5 + 2)^2 + (y - 2x)^2 - 4 = 9,$$

and again we deduce $(y - 2x)^2 = 4$.

We have again reached the same deductions as in the first approach, so we continue from there.

Question 2

> *The curve* $y = \left(\dfrac{x-a}{x-b}\right)\mathrm{e}^x$, *where* a *and* b *are constants, has two stationary points. Show that*
> $$a-b<0 \qquad \text{or} \qquad a-b>4.$$

We begin by differentiating using first the product rule and then the quotient rule:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-a}{x-b}\right)\mathrm{e}^x + \frac{(x-a)}{(x-b)}\mathrm{e}^x \\
&= \frac{(x-b).1-(x-a).1}{(x-b)^2}\mathrm{e}^x + \frac{(x-a)}{(x-b)}\mathrm{e}^x \\
&= \frac{(a-b)}{(x-b)^2}\mathrm{e}^x + \frac{(x-a)(x-b)}{(x-b)^2}\mathrm{e}^x \\
&= \frac{x^2-(a+b)x+(ab+a-b)}{(x-b)^2}\mathrm{e}^x.
\end{aligned}$$

Now solving $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ gives $x^2-(a+b)x+(ab+a-b)=0$. Since the curve has two stationary points, this quadratic must have two distinct real roots. Therefore the discriminant must be positive, that is

$$(a+b)^2-4(ab+a-b)>0,$$

and expanding gives $a^2-2ab+b^2-4a+4b>0$, so $(a-b)^2-4(a-b)>0$. Factorising this last expression gives

$$(a-b)(a-b-4)>0,$$

so (sketching a graph to help, possibly also replacing $a-b$ with a variable like x), we see that we must either have $a-b<0$ or $a-b>4$.

> (i) *Show that, in the case* $a=0$ *and* $b=\frac{1}{2}$, *there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.*

We are studying the curve $y=\left(\dfrac{x}{x-\frac{1}{2}}\right)\mathrm{e}^x$.

We have $a-b=-\frac{1}{2}<0$, so the curve has two stationary points by the first part of the question. The x-coordinates of the stationary points are found by solving the quadratic

$$x^2-(a+b)x+(ab+a-b)=0,$$

as above.

Substituting in our values for a and b, we get $x^2-\frac{1}{2}x-\frac{1}{2}=0$, so $2x^2-x-1=0$, which factorises to $(x-1)(2x+1)=0$. Thus there are stationary points at $(1, 2\mathrm{e})$ and $(-\frac{1}{2}, \frac{1}{2}\mathrm{e}^{-1/2})$.

The vertical asymptote is at $x=b$, that is at $x=\frac{1}{2}$.

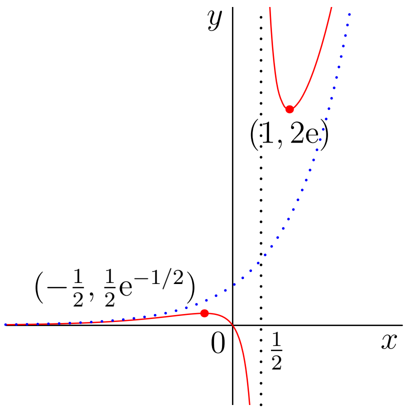
Therefore, since the two stationary points are at $x = 1$ and $x = -\frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote.

We note that the only time the curve crosses the x-axis is when $x = a$, so this is when $x = 0$, and this is also the y-intercept in this case.

As $x \to \pm\infty$, $y \sim e^x$ (meaning y is approximately equal to e^x; formally, we say that y is *asymptotically equal* to e^x), as the fraction $(x - a)/(x - b)$ tends to 1.

We can also note where the curve is positive and negative: since e^x is always positive, $y > 0$ whenever both $x - a > 0$ and $x - b > 0$, or when both $x - a < 0$ and $x - b < 0$, so $y < 0$ when x lies between a and b and is positive or zero otherwise.

Using all of this, we can now sketch the graph of the function. The nature of the stationary points will become clear from the graphs. In the graph, the dotted lines are the asymptotes ($x = \frac{1}{2}$ and $y = e^x$) and the red line is the graph we want, with the stationary points indicated.



(ii) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

This time, we are studying the curve $y = \left(\dfrac{x - \frac{9}{2}}{x}\right)e^x$.

Proceeding as in (i), we have $a - b = \frac{9}{2} > 4$, so again, the curve has two stationary points. The x-coordinates of the stationary points are given by solving the quadratic

$$x^2 - (a + b)x + (ab + a - b) = 0,$$

as above.

Substituting our values, we get $x^2 - \frac{9}{2}x + \frac{9}{2}$, so $2x^2 - 9x + 9 = 0$. Again, this factorises nicely to $(x - 3)(2x - 3) = 0$, giving stationary points at $(\frac{3}{2}, -2e^{3/2})$ and $(3, -\frac{1}{2}e^3)$.

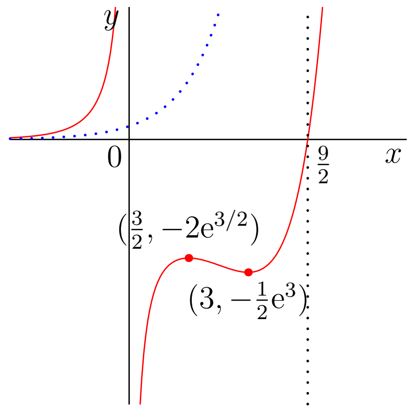
The vertical asymptote is at $x = b$, that is at $x = 0$. This time, therefore, the stationary points are both to the right of the vertical asymptote.

The x-intercept is at $x = a$, that is, at $(\frac{9}{2}, 0)$. There is no y-intercept as $x = 0$ is an asymptote.

Again, as $x \to \pm\infty$, $y \sim e^x$.

As in (i), $y < 0$ when x lies between a and b and is positive or zero otherwise.

Using all of this, we can now sketch the graph of this function. Note that the asymptote $y = e^x$ is much greater than y until x is greater than 20 or so, as even then $(x-a)/(x-b) \approx 15/20$, and only slowly approaches 1. We don't even attempt to sketch the function for such large values of x!



Question 3

Show that

$$\sin(x+y) - \sin(x-y) = 2\cos x \sin y$$

and deduce that

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

We use the compound angle formulæ (also called the addition formulæ) to expand the left hand side, getting:

$$\begin{aligned}\sin(x+y) - \sin(x-y) &= (\sin x\cos y + \cos x\sin y) - (\sin x\cos y - \cos x\sin y)\\ &= 2\cos x\sin y,\end{aligned}$$

as required.

For the deduction, we want $A = x+y$ and $B = x-y$, so $x = \frac{1}{2}(A+B)$ and $y = \frac{1}{2}(A-B)$, solving these two equations simultaneously to find x and y. Then we simply substitute these values of x and y into our previous identity, and we reach the desired conclusion:

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

(This identity is known as one of the *factor formulæ*.)

Show also that

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

Likewise, we have

$$\begin{aligned}\cos(x+y) - \cos(x-y) &= (\cos x\cos y - \sin x\sin y) - (\cos x\cos y + \sin x\sin y)\\ &= -2\sin x\sin y,\end{aligned}$$

so again substituting $x = \frac{1}{2}(A+B)$ and $y = \frac{1}{2}(A-B)$ gives

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

The points P, Q, R and S have coordinates $(a\cos p, b\sin p)$, $(a\cos q, b\sin q)$, $(a\cos r, b\sin r)$ and $(a\cos s, b\sin s)$ respectively, where $0 \leqslant p < q < r < s < 2\pi$, and a and b are positive.

Given that neither of the lines PQ and SR is vertical, show that these lines are parallel if and only if

$$r + s - p - q = 2\pi.$$

Remark: The points P, Q, R and S all lie on an ellipse, *which can be thought of as a stretched circle, as their coordinates all have $\frac{x}{a} = \cos\theta$ and $\frac{y}{b} = \sin\theta$, so they satisfy the equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.*

The lines PQ and SR are parallel if and only if their gradients are equal (and neither are vertical, so their gradients are well-defined), thus

$$\begin{aligned}
PQ \parallel RS \iff{}& \frac{b\sin q - b\sin p}{a\cos q - a\cos p} = \frac{b\sin s - b\sin r}{a\cos s - a\cos r} \\
\iff{}& \frac{\sin q - \sin p}{\cos q - \cos p} = \frac{\sin s - \sin r}{\cos s - \cos r} \\
\iff{}& \frac{2\cos\frac{1}{2}(q+p)\sin\frac{1}{2}(q-p)}{-2\sin\frac{1}{2}(q+p)\sin\frac{1}{2}(q-p)} = \frac{2\cos\frac{1}{2}(s+r)\sin\frac{1}{2}(s-r)}{-2\sin\frac{1}{2}(s+r)\sin\frac{1}{2}(s-r)} \\
\iff{}& \frac{\cos\frac{1}{2}(q+p)}{-\sin\frac{1}{2}(q+p)} = \frac{\cos\frac{1}{2}(s+r)}{-\sin\frac{1}{2}(s+r)} \\
\iff{}& \cot\tfrac{1}{2}(q+p) = \cot\tfrac{1}{2}(s+r) \\
\iff{}& \tfrac{1}{2}(q+p) = \tfrac{1}{2}(s+r) + k\pi \qquad \text{for some } k \in \mathbb{Z} \\
\iff{}& q + p = s + r + 2k\pi \qquad \text{for some } k \in \mathbb{Z} \\
\iff{}& r + s - p - q = 2n\pi \qquad \text{for some } n \in \mathbb{Z}.
\end{aligned}$$

The last four lines could have also been replaced by the following:

$$\begin{aligned}
PQ \parallel RS \iff{}& \cdots \\
\iff{}& \frac{\cos\frac{1}{2}(q+p)}{-\sin\frac{1}{2}(q+p)} = \frac{\cos\frac{1}{2}(s+r)}{-\sin\frac{1}{2}(s+r)} \\
\iff{}& \cos\tfrac{1}{2}(q+p)\sin\tfrac{1}{2}(s+r) = \cos\tfrac{1}{2}(s+r)\sin\tfrac{1}{2}(q+p) \\
\iff{}& \sin\tfrac{1}{2}(s+r)\cos\tfrac{1}{2}(q+p) - \cos\tfrac{1}{2}(s+r)\sin\tfrac{1}{2}(q+p) = 0 \\
\iff{}& \sin\tfrac{1}{2}((q+p) - (s+r)) = 0 \\
\iff{}& \tfrac{1}{2}(q + p - s - r) = \pi \qquad \text{for some } k \in \mathbb{Z} \\
\iff{}& r + s - p - q = 2n\pi \qquad \text{for some } n \in \mathbb{Z}.
\end{aligned}$$

We are almost there; we now only need to show $n = 1$ in the final line. We know that $0 \leqslant p < q < r < s < 2\pi$, so $r+s < 4\pi$ and $0 < p+q < r+s$, so that $0 < r+s-p-q < 4\pi$, which means that n must equal 1 if PQ and RS are parallel.

Thus PQ and RS are parallel if and only if $r + s - p - q = 2\pi$.

Question 4

Use the substitution $x = \dfrac{1}{t^2-1}$*, where* $t > 1$*, to show that, for* $x > 0$*,*

$$\int \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x = 2\ln\left(\sqrt{x} + \sqrt{x+1}\right) + c.$$

*[**Note:** You may use without proof the result* $\displaystyle\int \frac{1}{t^2-a^2}\,\mathrm{d}t = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| +$ *constant.]*

Using the given substitution, we first use the chain rule to calculate

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -(t^2-1)^{-2}\cdot 2t = -\frac{2t}{(t^2-1)^2}.$$

(We could alternatively have used the quotient rule to reach the same conclusion.)

We can now perform the requested substitution, simplifying the algebra as we go:

$$\begin{aligned}
\int \frac{1}{\sqrt{x(x+1)}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{\frac{1}{t^2-1}\cdot\frac{t^2}{t^2-1}}}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int \frac{1}{\left(\frac{t}{t^2-1}\right)}\cdot\frac{-2t}{(t^2-1)^2}\,\mathrm{d}t \\
&= \int \frac{-2}{t^2-1}\,\mathrm{d}t \\
&= -2\times\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c \qquad \text{using the given result} \\
&= \ln\left|\frac{t+1}{t-1}\right| + c.
\end{aligned}$$

At this point, we wish to substitute t for x, so we rearrange the original substitution to get

$$t = \sqrt{\frac{1}{x}+1} = \sqrt{\frac{x+1}{x}}.$$

This now yields:

$$\ln\left|\frac{t+1}{t-1}\right| + c = \ln\left|\frac{\sqrt{\frac{x+1}{x}}+1}{\sqrt{\frac{x+1}{x}}-1}\right| + c.$$

We note immediately that we can drop the absolute value signs, since both the numerator and denominator of the fraction are positive (the denominator is positive as $t > 1$ or $\sqrt{(x+1)/x} > 1$). So we get, on multiplying the numerator and denominator of the fraction by $\sqrt{x}$ to clear the fractions,

$$\begin{aligned}
\ln\left|\frac{t+1}{t-1}\right| + c &= \ln\left(\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right) + c \\
&= \ln\left(\frac{\bigl(\sqrt{x+1}+\sqrt{x}\bigr)\bigl(\sqrt{x+1}+\sqrt{x}\bigr)}{\bigl(\sqrt{x+1}-\sqrt{x}\bigr)\bigl(\sqrt{x+1}+\sqrt{x}\bigr)}\right) + c \\
&= \ln\left(\frac{\bigl(\sqrt{x+1}+\sqrt{x}\bigr)^2}{(x+1)-x}\right) + c \\
&= \ln\bigl(\sqrt{x+1}+\sqrt{x}\bigr)^2 + c \\
&= 2\ln\bigl(\sqrt{x+1}+\sqrt{x}\bigr) + c,
\end{aligned}$$

which is what we were after.

> *The section of the curve*
> $$y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$$
> *between* $x = \frac{1}{8}$ *and* $x = \frac{9}{16}$ *is rotated through* 360° *about the* x*-axis. Show that the volume enclosed is* $2\pi\ln\frac{5}{4}$.

To find the volume of revolution, we need to calculate the definite integral $\int_{1/8}^{9/16} \pi y^2\,\mathrm{d}x$:

$$\begin{aligned}
&\int_{1/8}^{9/16} \pi y^2\,\mathrm{d}x \\
&\quad= \pi\int_{1/8}^{9/16} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}\right)^2 \mathrm{d}x \\
&\quad= \pi\int_{1/8}^{9/16} \frac{1}{x} - 2\frac{1}{\sqrt{x(x+1)}} + \frac{1}{x+1}\,\mathrm{d}x \\
&\quad= \pi\Bigl[\ln x - 4\ln\bigl(\sqrt{x}+\sqrt{x+1}\bigr) + \ln(x+1)\Bigr]_{1/8}^{9/16} \qquad \text{using the above result} \\
&\quad= \pi\Bigl(\ln\tfrac{9}{16} - 4\ln\bigl(\sqrt{\tfrac{9}{16}}+\sqrt{\tfrac{25}{16}}\bigr) + \ln\tfrac{25}{16}\Bigr) - \pi\Bigl(\ln\tfrac{1}{8} - 4\ln\bigl(\sqrt{\tfrac{1}{8}}+\sqrt{\tfrac{9}{8}}\bigr) + \ln\tfrac{9}{8}\Bigr) \\
&\quad= \pi\Bigl(2\ln\tfrac{3}{4} - 4\ln\bigl(\tfrac{3}{4}+\tfrac{5}{4}\bigr) + 2\ln\tfrac{5}{4}\Bigr) - \pi\Bigl(2\ln\tfrac{1}{2\sqrt{2}} - 4\ln\bigl(\tfrac{1}{2\sqrt{2}}+\tfrac{3}{2\sqrt{2}}\bigr) + 2\ln\tfrac{3}{2\sqrt{2}}\Bigr) \\
&\quad= \pi\bigl((2\ln 3 - 2\ln 4) - 4\ln 2 + (2\ln 5 - 2\ln 4)\bigr) - \\
&\qquad\quad \pi\bigl(-2\ln(2\sqrt{2}) - 4\ln\sqrt{2} + (2\ln 3 - 2\ln(2\sqrt{2}))\bigr) \\
&\quad= \pi(2\ln 3 - 4\ln 2 - 4\ln 2 + 2\ln 5 - 4\ln 2) - \\
&\qquad\quad \pi(-3\ln 2 - 2\ln 2 + 2\ln 3 - 3\ln 2) \\
&\quad= \pi(-4\ln 2 + 2\ln 5) \\
&\quad= 2\pi(-2\ln 2 + \ln 5) \\
&\quad= 2\pi\ln\tfrac{5}{4}.
\end{aligned}$$

Question 5

> *By considering the expansion of $(1+x)^n$ where n is a positive integer, or otherwise, show that:*
>
> *(i)* $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$;

We take the advice and begin by writing out the expansion of $(1+x)^n$:

$$(1+x)^n=\binom{n}{0}x^0+\binom{n}{1}x^1+\binom{n}{2}x^2+\cdots+\binom{n}{n}x^n, \qquad (*)$$

where we have pedantically written in x^0 and x^1 in the first two terms, as this may well help us to understand what we are looking at.

Now comparing this expansion to the expression we are interested in, we see that the only difference is the presence of the xs. If we substitute $x=1$, we will get exactly what we want:

$$(1+1)^n=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n},$$

as all powers of 1 are just 1.

> *(ii)* $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}=n2^{n-1}$;

For the rest of the question, there are two very distinct approaches, one via calculus and one via properties of binomial coefficients.

Approach 1: Use calculus

This one looks a little more challenging, and we must observe carefully that there is no $\binom{n}{0}$ term. Comparing to the binomial expansion, we see that the term $\binom{n}{r}x^r$ has turned into $\binom{n}{r}.r$. Now, setting $x=1$ will again remove the x, but where are we to get the r from? Calculus gives us the answer: if we differentiate with respect to x, then x^r becomes rx^{r-1}, and then setting $x=1$ will complete the job. Now differentiating $(*)$ gives

$$n(1+x)^{n-1}=\binom{n}{1}.1x^0+\binom{n}{2}.2x^1+\binom{n}{3}.3x^2+\cdots+\binom{n}{n}.nx^{n-1},$$

so by setting $x=1$, we get the desired result.

Approach 2: Use properties of binomial coefficients

We know that

$$\binom{n}{r}=\frac{n!}{(n-r)!r!}$$

so we can manipulate this formula to pull out an r, using $r! = r.(r-1)!$ and similar expressions. We get

$$\begin{aligned}\binom{n}{r} &= \frac{n!}{(n-r)!r!} \\ &= \frac{1}{r}.\frac{n!}{(n-r)!(r-1)!} \\ &= \frac{n}{r}.\frac{(n-1)!}{(n-r)!(r-1)!} \\ &= \frac{n}{r}\binom{n-1}{r-1}\end{aligned}$$

so that $r\binom{n}{r} = n\binom{n-1}{r-1}$. This is true as long as $r \geqslant 1$ and $n \geqslant 1$, so we get

$$\begin{aligned}\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} &= n\binom{n-1}{0} + n\binom{n-1}{1} + \cdots + n\binom{n-1}{n-1} \\ &= n.2^{n-1}\end{aligned}$$

where we have used the result from part (i) with $n-1$ in place of n to do the last step.

(iii) $\displaystyle\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}(2^{n+1}-1)$;

Approach 1: Use calculus

Spurred on by our previous success, we see that now the $\binom{n}{2}x^2$ term gives us $\binom{n}{2}.\frac{1}{3}$, so we think of integration instead. Integrating $(*)$ gives

$$\frac{1}{n+1}(1+x)^{n+1} = \binom{n}{0}.x^1 + \binom{n}{1}.\frac{1}{2}x^2 + \binom{n}{2}.\frac{1}{3}x^3 + \cdots + \binom{n}{n}.\frac{1}{n+1}x^{n+1} + c\,.$$

We do need to determine the constant of integration, so we put $x = 0$ to do this; this gives

$$\frac{1}{n+1}1^{n+1} = \binom{n}{0}.0 + \binom{n}{1}.\frac{1}{2}.0 + + \cdots + \binom{n}{n}.\frac{1}{n+1}.0 + c,$$

so $c = \frac{1}{n+1}$. Now substituting $x = 1$ gives

$$\frac{1}{n+1}(1+1)^{n+1} = \binom{n}{0} + \frac{1}{2}\binom{n}{1} + + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} + \frac{1}{n+1}.$$

Finally, subtracting $\frac{1}{n+1}$ from both sides gives us our required result.

(An alternative way to think about this is to integrate both sides from $x = 0$ to $x = 1$.)

Approach 2: Use properties of binomial coefficients

We can try rewriting our identity so that the $\frac{1}{r}$ stays with the $r-1$ term; this gives us

$$\frac{1}{n}\binom{n}{r} = \frac{1}{r}\binom{n-1}{r-1}.$$

Unfortunately, though, our expressions involve $\binom{n}{r-1}$ terms rather than $\binom{n-1}{r-1}$ terms, but we can fix this by replacing n by $n+1$ to get

$$\frac{1}{n+1}\binom{n+1}{r} = \frac{1}{r}\binom{n}{r-1}.$$

We substitute this in to get

$$\begin{aligned}\binom{n}{0} + \frac{1}{2}\binom{n}{1} + &\dots + \frac{1}{n+1}\binom{n}{n} \\ &= \frac{1}{n+1}\binom{n+1}{1} + \frac{1}{n+1}\binom{n+1}{2} + \dots + \frac{1}{n+1}\binom{n+1}{n+1} \\ &= \frac{1}{n+1}(2^{n+1} - 1),\end{aligned}$$

where we have again used the result of part (i), this time with $n+1$ replacing n.

(iv) $\displaystyle \binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \dots + n^2\binom{n}{n} = n(n+1)2^{n-2}.$

Approach 1: Use calculus

This looks similar to (ii), in that we have increasing multiples. So we try differentiating $(*)$ twice, giving us:

$$n(n-1)(1+x)^{n-2} = \binom{n}{1}.1.0 + \binom{n}{2}.2.1x^0 + \binom{n}{3}.3.2x^1 + \dots + \binom{n}{n}.n(n-1)x^{n-2}.$$

Unfortunately, though, the coefficient of $\binom{n}{r}$ is $r(r-1)$ rather than the r^2 we actually want. But no matter: we can just add r and we will be done, as $r^2 = r(r-1) + r$, and we know from (ii) what terms like $\binom{n}{2}.2$ sum to give us. So we have, putting $x = 1$ in our above expression:

$$n(n-1)(1+1)^{n-2} = \binom{n}{1}.1.0 + \binom{n}{2}.2.1 + \binom{n}{3}.3.2 + \dots + \binom{n}{n}.n(n-1).$$

Now adding the result of (ii) gives

$$\begin{aligned}n(n-1)2^{n-2} + n2^{n-1} = \binom{n}{1}.(1.0+1) + \binom{n}{2}.(2.1+2) + \binom{n}{3}.(3.2+3) + \dots + \\ \binom{n}{n}.(n(n-1)+n)\end{aligned}$$

so

$$(n(n-1)+2n)2^{n-2} = \binom{n}{1} + \binom{n}{2}.2^2 + \binom{n}{3}.3^2 + \dots + \binom{n}{n}.n^2.$$

The left side simplifies to $n(n+1)2^{n-2}$, and thus we are done.

An alternative (calculus-based) method is as follows. The first derivative of ($*$), as we have seen, is

$$n(1+x)^{n-1} = \binom{n}{1}.1x^0 + \binom{n}{2}.2x^1 + \binom{n}{3}.3x^2 + \cdots + \binom{n}{n}.nx^{n-1}.$$

Now were we to differentiate again, we would end up with terms like $n(n-1)x^{n-2}$, rather than the desired n^2x^k (for some k). We can remedy this problem by multiplying the whole identity by x before we differentiate, so that we are differentiating

$$nx(1+x)^{n-1} = \binom{n}{1}.1x^1 + \binom{n}{2}.2x^2 + \binom{n}{3}.3x^3 + \cdots + \binom{n}{n}.nx^n.$$

Differentiating this now gives (using the product rule for the left hand side):

$$n(1+x)^{n-1}+n(n-1)x(1+x)^{n-2} = \binom{n}{1}.1^2x^0+\binom{n}{2}.2^2x^1+\binom{n}{3}.3^2x^2+\cdots+\binom{n}{n}.n^2x^{n-1}.$$

Substituting $x=1$ into this gives our desired conclusion (after a small amount of algebra on the left hand side).

Approach 2: Use properties of binomial coefficients

As this looks similar to the result of part (ii), we can start with what we worked out there, namely $r\binom{n}{r} = n\binom{n-1}{r-1}$, giving us

$$\binom{n}{1}+2^2\binom{n}{2}+\cdots+n^2\binom{n}{n} = n\binom{n-1}{0}+n.2\binom{n-1}{1}+n.3\binom{n-1}{2}+\cdots+n.n\binom{n-1}{n-1}$$

Taking out the factor of n leaves us having to work out

$$\binom{n-1}{0}+2\binom{n-1}{1}+3\binom{n-1}{2}+\cdots+n\binom{n-1}{n-1}$$

This looks very similar to the problem of part (ii) with n replaced by $n-1$, but now the multiplier of $\binom{n}{r}$ is $r+1$ rather than r, and there is also an $\binom{n}{0}$ term. We can get over the first problem by splitting up $(r+1)\binom{n}{r}$ as $r\binom{n}{r}+\binom{n}{r}$, so this expression becomes

$$\begin{aligned} &\binom{n-1}{1}+2\binom{n-1}{2}+\cdots+(n-1)\binom{n-1}{n-1}+ \\ \binom{n-1}{0}+&\binom{n-1}{1}+\binom{n-1}{2}+\cdots+\binom{n-1}{n-1} \end{aligned}$$

The first line is just part (ii) with n replaced by $n-1$, so that it sums to $(n-1).2^{n-2}$, and the second line is just $2^{n-1} = 2.2^{n-2}$ by part (i). So the answer to the original question (remembering the factor of n we took out earlier) is

$$n\big((n-1).2^{n-2}+2.2^{n-2}\big) = n(n-1+2).2^{n-2} = n(n+1).2^{n-2}.$$

Question 6

Show that, if $y = \mathrm{e}^x$, *then*

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0. \qquad (*)$$

If $y = \mathrm{e}^x$, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{e}^x$. Substituting these into the left hand side of $(*)$ gives

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = (x-1)\mathrm{e}^x - x\mathrm{e}^x + \mathrm{e}^x = 0,$$

so $y = \mathrm{e}^x$ satisfies $(*)$.

In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, *where* u *is a function of* x. *By substituting this into* $(*)$, *show that*

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0. \qquad (**)$$

We have $y = u\mathrm{e}^x$, so we apply the product rule to get:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x + u\mathrm{e}^x \\
&= \left(\frac{\mathrm{d}u}{\mathrm{d}x} + u\right)\mathrm{e}^x \\
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{e}^x + \frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x\right) + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x + u\mathrm{e}^x\right) \\
&= \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\mathrm{e}^x + 2\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^x + u\mathrm{e}^x \\
&= \left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + u\right)\mathrm{e}^x.
\end{aligned}$$

(If you know Leibniz's Theorem, then you could write down $\mathrm{d}^2u/\mathrm{d}x^2$ directly.)

We now substitute these into $(*)$ to get

$$(x-1)\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + u\right)\mathrm{e}^x - x\left(\frac{\mathrm{d}u}{\mathrm{d}x} + u\right)\mathrm{e}^x + u\mathrm{e}^x = 0.$$

Dividing by $\mathrm{e}^x \neq 0$ and collecting the derivatives of u then gives

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (2x-2-x)\frac{\mathrm{d}u}{\mathrm{d}x} + (x-1-x+1)u = 0,$$

which gives $(**)$ on simplifying the brackets.

By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$ *in* $(**)$ *and solving the resulting first order differential equation for* v*, find* u *in terms of* x*. Hence show that* $y = Ax + B\mathrm{e}^x$ *satisfies* $(*)$*, where* A *and* B *are any constants.*

As instructed, we set $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$, so that $\dfrac{\mathrm{d}^2u}{\mathrm{d}x^2} = \dfrac{\mathrm{d}v}{\mathrm{d}x}$, which gives us

$$(x-1)\frac{\mathrm{d}v}{\mathrm{d}x} + (x-2)v = 0.$$

This is a standard separable first-order linear differential equation, so we separate the variables to get

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{x-2}{x-1}$$

and then integrate with respect to x to get

$$\int \frac{1}{v}\,\mathrm{d}v = \int -\frac{x-2}{x-1}\,\mathrm{d}x.$$

Performing the integrations now gives us

$$\begin{aligned}\ln|v| &= \int -\Big(1 - \frac{1}{x-1}\Big)\,\mathrm{d}x \\ &= -x + \ln|x-1| + c,\end{aligned}$$

which we exponentiate to get

$$|v| = |k|\mathrm{e}^{-x}|x-1|,$$

where k is some constant, so we finally arrive at

$$v = k\mathrm{e}^{-x}(x-1).$$

We now recall that $v = \mathrm{d}u/\mathrm{d}x$, so we need to integrate this last expression once more to find u. We use integration by parts to do this, integrating the e^{-x} part and differentiating $(x-1)$, to give us

$$\begin{aligned}u &= \int k\mathrm{e}^{-x}(x-1)\,\mathrm{d}x \\ &= k(-\mathrm{e}^{-x})(x-1) - \int k(-\mathrm{e}^{-x}).1\,\mathrm{d}x \\ &= k(-\mathrm{e}^{-x})(x-1) - k\mathrm{e}^{-x} + c \\ &= -kx\mathrm{e}^{-x} + c,\end{aligned}$$

which is the solution to $(**)$.

Now recalling that $y = u\mathrm{e}^x$ gives us $y = -kx + c\,\mathrm{e}^x$ as our general solution to $(*)$. In particular, letting $k = -A$ and $c = B$, where A and B are any constants, shows that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, as required.

Question 7

Relative to a fixed origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b},$$

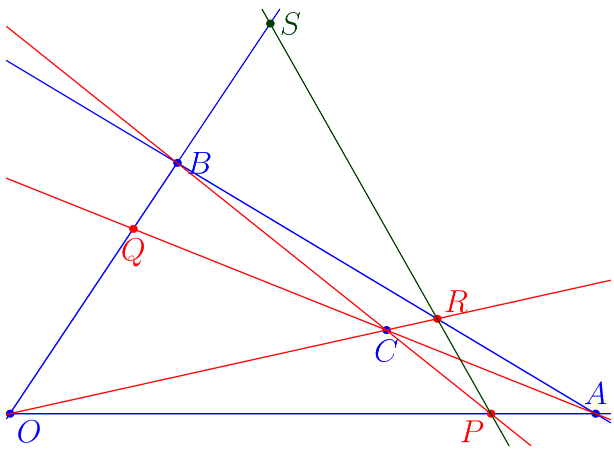
where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\mathbf{p}$, and the lines OB and AC meet at the point Q with position vector $\mathbf{q}$. Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta},$$

and write down $\mathbf{q}$ in terms of α, β and $\mathbf{b}$.

The condition $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b}$ with $\alpha + \beta < 1$ and α and β both positive constants means that C lies strictly inside the triangle OAB. Can you see why?

We start by sketching the setup so that we have something visual to help us with our thinking.



The line OA has points with position vectors given by $\mathbf{r}_1 = \lambda\mathbf{a}$, and the line BC has points with position vectors given by

$$\mathbf{r}_2 = \overrightarrow{OB} + \mu\overrightarrow{BC} = \mathbf{b} + \mu(\mathbf{c} - \mathbf{b}) = (1-\mu)\mathbf{b} + \mu\mathbf{c}.$$

The point P is where these two lines meet, so we must have

$$\begin{aligned}\mathbf{p} = \lambda\mathbf{a} &= (1-\mu)\mathbf{b} + \mu\mathbf{c} \\ &= (1-\mu)\mathbf{b} + \mu(\alpha\mathbf{a} + \beta\mathbf{b}) \\ &= (1-\mu+\beta\mu)\mathbf{b} + \alpha\mu\mathbf{a}.\end{aligned}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, we must have $1 - \mu + \beta\mu = 0$ and $\alpha\mu = \lambda$. The first equation gives $(1 - \beta)\mu = 1$, so $\mu = 1/(1 - \beta)$. This gives $\lambda = \alpha/(1 - \beta)$, so that

$$\mathbf{p} = \frac{\alpha \mathbf{a}}{1 - \beta}.$$

Now swapping the roles of $\mathbf{a}$ and $\mathbf{b}$ (and hence also of α and β) will give us the position vector of Q:

$$\mathbf{q} = \frac{\beta \mathbf{b}}{1 - \alpha}.$$

Show further that the point R with position vector $\mathbf{r}$ given by

$$\mathbf{r} = \frac{\alpha \mathbf{a} + \beta \mathbf{b}}{\alpha + \beta}$$

lies on the lines OC and AB.

We could approach this question in two ways, either by finding the point of intersection of OC and AB or by showing that the given point lies on both given lines. We give both approaches.

Approach 1: Finding the point of intersection

We require $\mathbf{r}$ to lie on OC, so $\mathbf{r} = \lambda \mathbf{c}$, and $\mathbf{r}$ to lie on AB, so $\mathbf{r} = (1 - \mu)\mathbf{a} + \mu \mathbf{b}$, as before. Substituting for $\mathbf{c}$ and equating coefficients gives

$$\alpha\lambda \mathbf{a} + \beta\lambda \mathbf{b} = (1 - \mu)\mathbf{a} + \mu \mathbf{b},$$

so that

$$\begin{aligned} \alpha\lambda &= 1 - \mu \\ \beta\lambda &= \mu. \end{aligned}$$

Adding the two equations gives $(\alpha + \beta)\lambda = 1$, so $\lambda = 1/(\alpha + \beta)$ and hence

$$\mathbf{r} = \frac{\mathbf{c}}{\alpha + \beta} = \frac{\alpha \mathbf{a} + \beta \mathbf{b}}{\alpha + \beta}.$$

Approach 2: Showing that the given point lies on both lines

The equation of line OC is $\mathbf{r}_1 = \lambda \mathbf{c}$, and

$$\mathbf{r} = \frac{\alpha \mathbf{a} + \beta \mathbf{b}}{\alpha + \beta} = \frac{1}{\alpha + \beta}\mathbf{c}$$

is of the required form, so R lies on OC.

The equation of the line AB can be written as $\mathbf{r}_2 - \mathbf{a} = \mu\overrightarrow{AB}$, so we want $\mathbf{r} - \mathbf{a} = \mu(\mathbf{b} - \mathbf{a})$. Now we have

$$\begin{aligned}
\mathbf{r} - \mathbf{a} &= \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta} - \frac{\alpha + \beta}{\alpha + \beta}\mathbf{a} \\
&= \frac{-\beta\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta} \\
&= \frac{\beta}{\alpha + \beta}(\mathbf{b} - \mathbf{a}),
\end{aligned}$$

which is of the form $\mu(\mathbf{b} - \mathbf{a})$, so R lies on both lines.

The lines OB and PR intersect at the point S. Prove that $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

S lies on both OB and PR, so we need to find its position vector, $\mathbf{s}$. Once again, we require $\mathbf{s} = \lambda\mathbf{b} = (1 - \mu)\mathbf{p} + \mu\mathbf{r}$, so we substitute for $\mathbf{p}$ and $\mathbf{r}$ and compare coefficients:

$$\begin{aligned}
\mathbf{s} = \lambda\mathbf{b} &= (1 - \mu)\mathbf{p} + \mu\mathbf{r} \\
&= (1 - \mu)\frac{\alpha\mathbf{a}}{1 - \beta} + \mu\frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta} \\
&= \frac{(1 - \mu)\alpha(\alpha + \beta)\mathbf{a} + \mu(1 - \beta)(\alpha\mathbf{a} + \beta\mathbf{b})}{(1 - \beta)(\alpha + \beta)} \\
&= \frac{((1 - \mu)\alpha(\alpha + \beta) + \mu\alpha(1 - \beta))\mathbf{a} + \mu(1 - \beta)\beta\mathbf{b}}{(1 - \beta)(\alpha + \beta)} \\
&= \frac{\alpha(\alpha + \beta - \mu\alpha - 2\mu\beta + \mu)\mathbf{a} + \mu(1 - \beta)\beta\mathbf{b}}{(1 - \beta)(\alpha + \beta)}
\end{aligned}$$

Since the coefficient of $\mathbf{a}$ in this expression must be zero, we deduce that

$$\mu = \frac{\alpha + \beta}{\alpha + 2\beta - 1},$$

so that

$$\begin{aligned}
\mathbf{s} &= \frac{\mu(1 - \beta)\beta}{(1 - \beta)(\alpha + \beta)}\mathbf{b} \\
&= \frac{\alpha + \beta}{\alpha + 2\beta - 1}\frac{\beta}{(\alpha + \beta)}\mathbf{b} \\
&= \frac{\beta}{\alpha + 2\beta - 1}\mathbf{b}
\end{aligned}$$

Now, since $\overrightarrow{OQ}$ and $\overrightarrow{BQ}$ are both multiples of the vector q, we can compare the lengths OQ and BQ in terms of their multiples of $\mathbf{q}$. This might come out to be negative, depending on the relative directions, but at the end, we can just consider the magnitudes.

We thus have, since $\mathbf{q} = \dfrac{\beta \mathbf{b}}{1-\alpha}$,

$$\begin{aligned}
\frac{OQ}{BQ} &= \Big(\frac{\beta}{1-\alpha}\Big) \Big/ \Big(\frac{\beta}{1-\alpha} - 1\Big) \\
&= \Big(\frac{\beta}{1-\alpha}\Big) \Big/ \Big(\frac{\beta - 1 + \alpha}{1-\alpha}\Big) \\
&= \frac{\beta}{\beta - 1 + \alpha},
\end{aligned}$$

while

$$\begin{aligned}
\frac{OS}{BS} &= \Big(\frac{\beta}{\alpha + 2\beta - 1}\Big) \Big/ \Big(\frac{\beta}{\alpha + 2\beta - 1} - 1\Big) \\
&= \Big(\frac{\beta}{\alpha + 2\beta - 1}\Big) \Big/ \Big(\frac{-\alpha - \beta + 1}{\alpha + 2\beta - 1}\Big) \\
&= \frac{\beta}{-\alpha - \beta + 1}.
\end{aligned}$$

Thus these two ratios of lengths are equal, as the magnitude of both of these is

$$\frac{\beta}{1 - (\alpha + \beta)}.$$

Question 8

> (i) *Suppose that a, b and c are integers that satisfy the equation*
>
> $$a^3 + 3b^3 = 9c^3.$$
>
> *Explain why a must be divisible by 3, and show further that both b and c must also be divisible by 3. Hence show that the only integer solution is $a = b = c = 0$.*

We have $a^3 = 9c^3 - 3b^3 = 3(3c^3 - b^3)$, so a^3 is a multiple of 3. But as 3 is prime, a itself must be divisible by 3. (Why is this? If 3 divides a product rs, then 3 must divide either r or s, as 3 is prime. Therefore since $a^3 = a.a.a$, and 3 divides a^3, it follows that 3 must divide one of the factors, that is, 3 must divide a.)

Now we can write $a = 3d$, where d is an integer. Therefore we have

$$(3d)^3 + 3b^3 = 9c^3,$$

which, on dividing by 3, gives

$$9d^3 + b^3 = 3c^3.$$

By the same argument, as $b^3 = 3(c^3 - 3d^3)$, it follows that b^3, and hence also b, is divisible by 3.

We repeat the same trick, writing $b = 3e$, where e is an integer, so that

$$9d^3 + (3e)^3 = 3c^3.$$

We again divide by 3 to get

$$3d^3 + 9e^3 = c^3,$$

so that c^3, and hence also c, is divisible by 3.

We then write $c = 3f$, where f is an integer, giving

$$3d^3 + 9e^3 = (3f)^3.$$

Finally, we divide this equation by 3 to get

$$d^3 + 3e^3 = 9f^3.$$

Note that this is the same equation that we started with, so if a, b, c are integers which satisfy the equation, then so are $d = a/3$, $e = b/3$ and $f = c/3$. We can repeat this process indefinitely, so that $a/3^n$, $b/3^n$ and $c/3^n$ are also integers which satisfy the equation. But if $a/3^n$ is an integer for all $n \geqslant 0$, we must have $a = 0$, and similarly for b and c.

Therefore the only integer solution is $a = b = c = 0$.

[In fact, we can say even more. If a, b and c are all *rational*, say $a = d/r$, $b = e/s$, $c = f/t$ (where d, e, f are integers and r, s, t are non-zero integers), then we have

$$\left(\frac{d}{r}\right)^3 + 3\left(\frac{e}{s}\right)^3 = 9\left(\frac{f}{t}\right)^3.$$

Now multiplying both sides by $(rst)^3$ gives

$$(dst)^3 + 3(ert)^3 = 9(frs)^3,$$

with dst, ert and frs all integers, and so they must all be zero, and hence $d = e = f = 0$. Therefore, the only *rational* solution is also $a = b = c = 0$.]

> *(ii) Suppose that p, q and r are integers that satisfy the equation*
>
> $$p^4 + 2q^4 = 5r^4.$$
>
> *By considering the possible final digit of each term, or otherwise, show that p and q are divisible by 5. Hence show that the only integer solution is $p = q = r = 0$.*

We consider the final digit of fourth powers:

a	a^4	$2a^4$
0	0	0
1	1	2
2	6	2
3	1	2
4	6	2
5	5	0
6	6	2
7	1	2
8	6	2
9	1	2

So the last digits of fourth powers are all either 0, 5, 1 or 6, and of twice fourth powers are all either 0 or 2.

Also, $5r^4$ is a multiple of 5, so it must end in a 0 or a 5.

Therefore if $2q^4$ ends in 0 (that is, when q is a multiple of 5), the possibilities for the final digit of $p^4 + 2q^4$ are

$$(0 \text{ or } 1 \text{ or } 5 \text{ or } 6) + 0 = 0 \text{ or } 1 \text{ or } 5 \text{ or } 6,$$

so it can equal $5r^4$ (which ends in 0 or 5) only if p^4 ends in 0 or 5, which is exactly when p is a multiple of 5.

Similarly, if $2q^4$ ends in 2 (so q is not a multiple of 5), the possibilities for the final digit of $p^4 + 2q^4$ are

$$(0 \text{ or } 1 \text{ or } 5 \text{ or } 6) + 2 = 2 \text{ or } 3 \text{ or } 7 \text{ or } 8,$$

so it can not be equal to $5r^4$ (which ends in 0 or 5).

Therefore, if $p^4 + 2q^4 = 5r^4$, we must have p and q both being multiples of 5.

Now as in part (i), we write $p = 5a$ and $q = 5b$, where a and b are both integers, to get

$$(5a)^4 + 2(5b)^4 = 5r^4.$$

Dividing both sides by 5 gives

$$5^3a^4 + 2.5^3b^4 = r^4,$$

where we are using dot to mean multiplication, so as before, r^4 must be a multiple of 5 as the left hand side is $5(5^2a^4 + 2.5^2b^4)$. Thus, since 5 is prime, r itself must be divisible by 5. Then writing $r = 5c$ gives

$$5^3a^4 + 2.5^3b^4 = (5c)^4,$$

which yields

$$a^4 + 2b^4 = 5c^4$$

on dividing by 5^3.

So once again, if p, q, r give an integer solution to the equation, so do $a = p/5$, $b = q/5$ and $c = r/5$. Repeating this, so are $p/5^n$, $q/5^n$, $r/5^n$, and as before, this shows that the only integer solution is $p = q = r = 0$.

[Again, the same argument as before shows that this is also the only rational solution.]

This is an example of the use of Fermat's Method of Descent, which he used to prove one special case of his famous Last Theorem: he showed that $x^4 + y^4 = z^4$ has no positive integer solutions. In fact, he proved an even stronger result, namely that $x^4 + y^4 = z^2$ has no positive integer solutions.

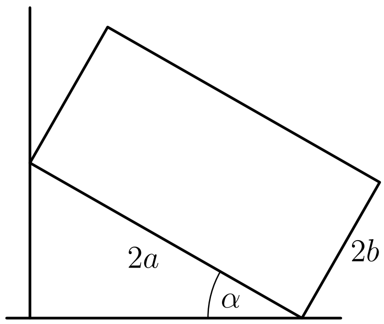
Another approach to solving the first step of part (ii) of this problem is to use modular arithmetic, *where we only consider remainders when dividing by a certain fixed number. In this case, we would consider arithmetic modulo 5, so the only numbers to consider are 0, 1, 2, 3 and 4, and we want to solve $p^4 + 2q^4 \equiv 0 \pmod 5$, where $\equiv$ means "leaves the same remainder". Now a quick calculation shows that $p^4 \equiv 1$ unless $p \equiv 0$, while $2q^4 \equiv 2$ unless $q \equiv 0$, so that*

$$p^4 + 2q^4 \equiv (0 \text{ or } 1) + (0 \text{ or } 2) \equiv 0,\ 1,\ 2 \text{ or } 3 \pmod 5$$

with $p^4 + 2q^4 \equiv 0$ if and only if $p \equiv q \equiv 0$.

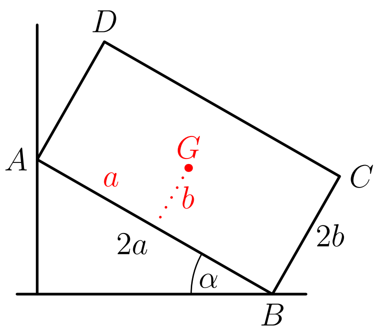
Incidentally, Fermat has another theorem relevant to this problem, which turns out to be relatively easy to prove (Fermat himself claimed to have done so), and is known as Fermat's Little Theorem. *This states that, if p is prime, then $a^{p-1} \equiv 1 \pmod p$ unless $a \equiv 0 \pmod p$. In our case, $p = 5$ gives $a^4 \equiv 1 \pmod 5$ unless $a \equiv 0 \pmod 5$, as we wanted.*

Question 9



The diagram shows a uniform rectangular lamina with sides of lengths $2a$ and $2b$ leaning against a rough vertical wall, with one corner resting on a rough horizontal plane. The plane of the lamina is vertical and perpendicular to the wall, and one edge makes an angle of α with the horizontal plane. Show that the centre of mass of the lamina is a distance $a\cos\alpha + b\sin\alpha$ from the wall.

We start by redrawing the sketch, labelling the corners and indicating the centre of mass as G, as well as showing various useful lengths.



It is now clear that the distance of G from the wall is $a\cos\alpha$ (horizontal distance from wall to midpoint of AB) plus $b\sin\alpha$ (horizontal distance from midpoint of AB to G), so a total of $a\cos\alpha + b\sin\alpha$.

Also, in case it is useful later, we note that the vertical distance above the horizontal plane is, by a similar argument from the same sketch, $a\sin\alpha + b\cos\alpha$.

The coefficients of friction at the two points of contact are each μ and the friction is limiting at both contacts. Show that

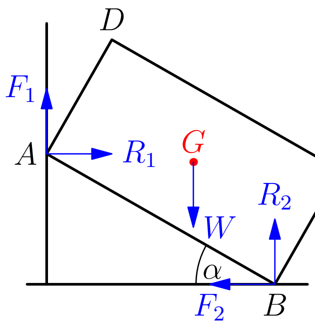
$$a\cos(2\lambda + \alpha) = b\sin\alpha,$$

where $\tan\lambda = \mu$.

There are two approaches to this. One is to indicate the reaction and friction forces separately, while the other is to use the Three Forces Theorem. We show both of these.

Approach 1: All forces separately

We start by sketching the lamina again, this time showing the forces on the lamina, separating the normal reactions from the frictional forces.



We now resolve and take moments:

$$\begin{array}{ll} \mathscr{R}(\uparrow) & F_1 + R_2 - W = 0 \\ \mathscr{R}(\rightarrow) & R_1 - F_2 = 0 \\ \mathscr{M}(\widehat{A}) & W(a\cos\alpha + b\sin\alpha) - R_2.2a\cos\alpha + F_2.2a\sin\alpha = 0 \end{array}$$

Since friction is limiting at both points of contact, we have $F_1 = \mu R_1$ and $F_2 = \mu R_2$. Substituting these gives:

$$\begin{array}{lll} \mathscr{R}(\uparrow) & \mu R_1 + R_2 - W = 0 & (1) \\ \mathscr{R}(\rightarrow) & R_1 - \mu R_2 = 0 & (2) \\ \mathscr{M}(\widehat{A}) & W(a\cos\alpha + b\sin\alpha) - 2aR_2\cos\alpha + 2a\mu R_2\sin\alpha = 0 & (3) \end{array}$$

Equation (2) gives $R_1 = \mu R_2$, so we can substitute this into (1) to get $W = (1+\mu^2)R_2$. Substituting this into (3) now leads to

$$(1+\mu^2)R_2(a\cos\alpha + b\sin\alpha) = 2aR_2(\cos\alpha - \mu\sin\alpha).$$

We can clearly divide both sides by R_2, and we are given that $\tan\lambda = \mu$, so we substitute this in as well, to get

$$(1+\tan^2\lambda)(a\cos\alpha + b\sin\alpha) = 2a(\cos\alpha - \tan\lambda\sin\alpha).$$

We spot $1+\tan^2\lambda = \sec^2\lambda$, and so multiply the whole equation through by $\cos^2\lambda$, as the form we are looking for does not involve $\sec\lambda$:

$$a\cos\alpha + b\sin\alpha = 2a(\cos^2\lambda\cos\alpha - \sin\lambda\cos\lambda\sin\alpha).$$

Since the form we are going for is $b\sin\alpha = a\cos(2\lambda+\alpha)$, we make use of double angle formulæ, after rearranging:

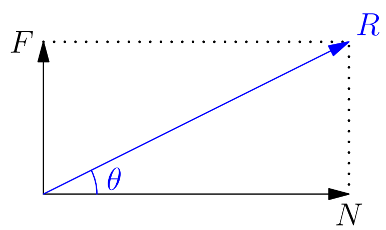
$$\begin{aligned} b\sin\alpha &= a(2\cos^2\lambda\cos\alpha - 2\sin\lambda\cos\lambda\sin\alpha) - a\cos\alpha \\ &= a((2\cos^2\lambda - 1)\cos\alpha - 2\sin\lambda\cos\lambda\sin\alpha) \\ &= a(\cos 2\lambda\cos\alpha - \sin 2\lambda\sin\alpha) \\ &= a\cos(2\lambda+\alpha), \end{aligned}$$

and we are done with this part.

Approach 2: Three Forces Theorem

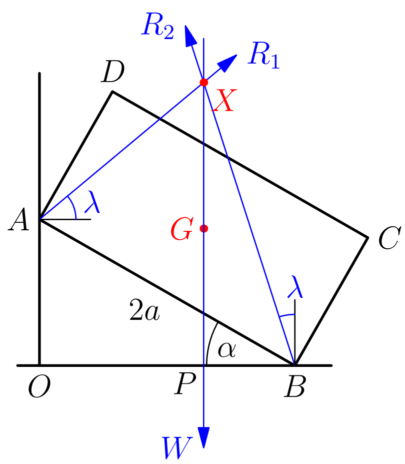
The 'Three Forces Theorem' states that if three (non-zero) forces act on a large body in equilibrium, and they are not all parallel, then they must pass through a single point. (Why is this true? Let's say two of the forces pass through point X. Taking moments about X, the total moment must be zero, so the moment of the third force about X must be zero. Therefore, the force itself is either zero or it passes through X. Since the forces are non-zero, the third force must pass through X.)

In our case, we have a normal reaction and a friction force at each point of contact. We can combine these into a single reaction force as shown in the sketch. Here we have written N for the normal force, F for the friction and R for the resultant, which is at an angle of θ to the normal.



We see from this sketch that $\tan\theta = F/N = \mu N/N = \mu$. In our case, since $\tan\lambda = \mu$ we must have $\theta = \lambda$.

We can now redraw our original diagram with the three (combined) forces shown:



We can now use the Three Forces Theorem is as follows. Looking at the diagram, we know that the distance $OB = 2a\cos\alpha = OP + PB$. Now we know $OP = a\cos\alpha + b\sin\alpha$, so we need only calculate PB.

But $PB = PX\tan\lambda$ (using the triangle PBX), and

$$\begin{aligned} PX &= OA + \text{height of } X \text{ above } A \\ &= 2a\sin\alpha + (a\cos\alpha + b\sin\alpha)\tan\lambda. \end{aligned}$$

Putting these together gives

$$
\begin{aligned}
OB = 2a\cos\alpha = OP + PB& \\
&= a\cos\alpha + b\sin\alpha + (2a\sin\alpha + (a\cos\alpha + b\sin\alpha)\tan\lambda)\tan\lambda \\
&= a\cos\alpha(1+\tan^2\lambda) + b\sin\alpha(1+\tan^2\lambda) + 2a\sin\alpha\tan\lambda \\
&= a\cos\alpha\sec^2\lambda + b\sin\alpha\sec^2\lambda + 2a\sin\alpha\tan\lambda.
\end{aligned}
$$

We can now rearrange to get

$$b\sin\alpha\sec^2\lambda = 2a\cos\alpha - a\cos\alpha\sec^2\lambda - 2a\sin\alpha\tan\lambda.$$

Since we want an expression for $b\sin\alpha$, we now multiply by $\cos^2\lambda$ to get

$$
\begin{aligned}
b\sin\alpha &= 2a\cos\alpha\cos^2\lambda - a\cos\alpha - 2a\sin\alpha\sin\lambda\cos\lambda \\
&= a((2\cos^2\lambda - 1)\cos\alpha - (2\sin\lambda\cos\lambda)\sin\alpha) \\
&= a(\cos 2\lambda\cos\alpha - \sin 2\lambda\sin\alpha) \\
&= a\cos(2\lambda + \alpha).
\end{aligned}
$$

An alternative argument using the Three Forces Theorem proceeds by considering the distance XP. Using the left half of the diagram, we have

$$
\begin{aligned}
XP &= OA + OP\tan\lambda \\
&= 2a\sin\alpha + (a\cos\alpha + b\sin\alpha)\tan\lambda.
\end{aligned}
$$

From the right half of the diagram, we also have $XP = PB/\tan\lambda$, and $PB = 2a\cos\alpha - OP = a\cos\alpha - b\sin\alpha$, so that

$$2a\sin\alpha + (a\cos\alpha + b\sin\alpha)\tan\lambda = (a\cos\alpha - b\sin\alpha)/\tan\lambda.$$

Now multiplying by $\tan\lambda$ and collecting like terms gives

$$b\sin\alpha(1+\tan^2\lambda) = a\cos\alpha(1-\tan^2\lambda) - 2a\sin\alpha\tan\lambda.$$

Then using $1+\tan^2\lambda = \sec^2\lambda$ and then multiplying by $\cos^2\lambda$ gives us

$$
\begin{aligned}
b\sin\alpha &= a\cos\alpha(\cos^2\lambda - \sin^2\lambda) - 2a\sin\alpha\sin\lambda\cos\lambda \\
&= a\cos\alpha\cos 2\lambda - a\sin\alpha\sin 2\lambda \\
&= a\cos(\alpha + 2\lambda),
\end{aligned}
$$

as we wanted.

Show also that if the lamina is square, then $\lambda = \frac{\pi}{4} - \alpha$.

We have $a = b$ as the lamina is square, so that our previous equation becomes

$$a\sin\alpha = a\cos(2\lambda + \alpha).$$

Dividing by a gives

$$\sin\alpha = \cos(2\lambda + \alpha).$$

Now we can use the identity $\sin\alpha = \cos(\frac{\pi}{2} - \alpha)$, so that

$$\frac{\pi}{2} - \alpha = 2\lambda + \alpha.$$

(Being very careful, we should check that we can take inverse cosines of both sides to deduce this equality. This will be the case if both $0 < \frac{\pi}{2} - \alpha < \frac{\pi}{2}$ and $0 < 2\lambda + \alpha < \frac{\pi}{2}$. But $0 < \alpha < \frac{\pi}{2}$ so the first inequality is clearly true. For the second inequality, we have $0 < \lambda < \frac{\pi}{2}$ so that $0 < 2\lambda + \alpha < \frac{3\pi}{2}$. But since the cosine of this is positive (being $\sin\alpha$), it must lie in the range $0 < 2\lambda + \alpha < \frac{\pi}{2}$ as required.)

Subtracting α and dividing by 2 now gives our desired result:

$$\frac{\pi}{4} - \alpha = \lambda.$$

Question 10

> *A particle P moves so that, at time t, its displacement $\mathbf{r}$ from a fixed origin is given by*
>
> $$\mathbf{r} = \big(\mathrm{e}^t \cos t\big)\mathbf{i} + \big(\mathrm{e}^t \sin t\big)\mathbf{j}.$$
>
> *Show that the velocity of the particle always makes an angle of $\frac{\pi}{4}$ with the particle's displacement, and that the acceleration of the particle is always perpendicular to its displacement.*

To find the velocity, $\mathbf{v}$, and acceleration, $\mathbf{a}$, we differentiate with respect to t (using the product rule).

We have

$$\begin{aligned}
\mathbf{r} &= \big(\mathrm{e}^t \cos t\big)\mathbf{i} + \big(\mathrm{e}^t \sin t\big)\mathbf{j}\\
\mathbf{v} = \mathrm{d}\mathbf{r}/\mathrm{d}t &= \big(\mathrm{e}^t \cos t - \mathrm{e}^t \sin t\big)\mathbf{i} + \big(\mathrm{e}^t \sin t + \mathrm{e}^t \cos t\big)\mathbf{j}\\
\mathbf{a} = \mathrm{d}\mathbf{v}/\mathrm{d}t &= \big((\mathrm{e}^t \cos t - \mathrm{e}^t \sin t\big) - (\mathrm{e}^t \sin t + \mathrm{e}^t \cos t))\mathbf{i}+\\
&\qquad \big((\mathrm{e}^t \sin t + \mathrm{e}^t \cos t) + (\mathrm{e}^t \cos t - \mathrm{e}^t \sin t\big)\mathbf{j}\\
&= (-2\mathrm{e}^t \sin t)\mathbf{i} + (2\mathrm{e}^t \cos t)\mathbf{j}.
\end{aligned}$$

We can rewrite these, if we wish, by taking out the common factors:

$$\begin{aligned}
\mathbf{r} &= \mathrm{e}^t\big((\cos t)\mathbf{i} + (\sin t)\mathbf{j}\big)\\
\mathbf{v} &= \mathrm{e}^t\big((\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j}\big)\\
\mathbf{a} &= 2\mathrm{e}^t\big((-\sin t)\mathbf{i} + (\cos t)\mathbf{j})\big).
\end{aligned}$$

From these, we can easily find the magnitudes of the displacement, velocity and acceleration:

$$\begin{aligned}
|\mathbf{r}| &= \mathrm{e}^t\sqrt{(\cos t)^2 + (\sin t)^2} = \mathrm{e}^t\\
|\mathbf{v}| &= \mathrm{e}^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}\\
&= \mathrm{e}^t\sqrt{2\cos^2 t + 2\sin^2 t}\\
&= \mathrm{e}^t\sqrt{2}\\
|\mathbf{a}| &= 2\mathrm{e}^t\sqrt{(-\sin t)^2 + (\cos t)^2} = 2\mathrm{e}^t.
\end{aligned}$$

We can now find the angles between these using $\mathbf{a}.\mathbf{b} = 2|\mathbf{a}||\mathbf{b}|\cos\theta$; firstly, for displacement and velocity we have

$$\begin{aligned}
\mathbf{r}.\mathbf{v} &= \mathrm{e}^{2t}\big(\cos t(\cos t - \sin t) + \sin t(\sin t + \cos t)\big)\\
&= \mathrm{e}^{2t}(\cos^2 t + \sin^2 t)\\
&= \mathrm{e}^{2t}, \qquad \text{while}\\
\mathbf{r}.\mathbf{v} &= \mathrm{e}^t.\mathrm{e}^t\sqrt{2}\cos\theta,
\end{aligned}$$

so that $\cos\theta = 1/\sqrt{2}$, so that $\theta = \frac{\pi}{4}$ as required.

Next, for displacement and acceleration we have

$$\mathbf{r}.\mathbf{a} = 2\mathrm{e}^{2t}\big(\cos t(-\sin t) + \sin t\cos t\big) = 0,$$

so they are perpendicular.

Geometric-trigonometric approach

There is another way to find the angles involved which does not use the scalar (dot) product.

Recall that the velocity is $\mathbf{v} = \mathrm{e}^t\big((\cos t - \sin t)\mathbf{i} + (\sin t + \cos t)\mathbf{j}\big)$. We can use the "$R\cos(\theta+\alpha)$" technique, thinking of $\cos t - \sin t$ as $1\cos t - 1\sin t$, so that

$$\begin{aligned}
\cos t - \sin t &= \sqrt{2}\big(\tfrac{1}{\sqrt{2}}\cos t - \tfrac{1}{\sqrt{2}}\sin t\big)\\
&= \sqrt{2}(\cos t\cos\tfrac{\pi}{4} - \sin t\sin\tfrac{\pi}{4})\\
&= \sqrt{2}\cos(t+\tfrac{\pi}{4})\\
\sin t + \cos t &= \sqrt{2}\big(\tfrac{1}{\sqrt{2}}\sin t + \tfrac{1}{\sqrt{2}}\cos t\big)\\
&= \sqrt{2}(\sin t\cos\tfrac{\pi}{4} + \cos t\sin\tfrac{\pi}{4})\\
&= \sqrt{2}\sin(t+\tfrac{\pi}{4})
\end{aligned}$$

Thus $\mathbf{v} = \sqrt{2}\mathrm{e}^t\big(\cos(t+\frac{\pi}{4})\mathbf{i} + \sin(t+\frac{\pi}{4})\mathbf{j}\big)$, so $\mathbf{v}$ is at an angle of $\frac{\pi}{4}$ with $\mathbf{r}$.

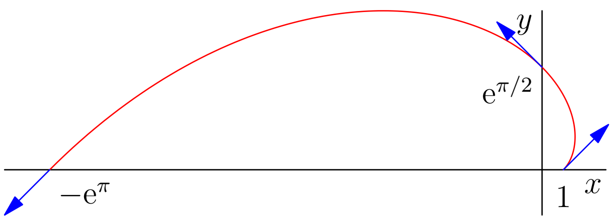
Likewise, $\mathbf{a}$ makes an angle of $\frac{\pi}{4}$ with $\mathbf{v}$, and so an angle of $\frac{\pi}{2}$ with $\mathbf{r}$.

Sketch the path of the particle for $0 \leqslant t \leqslant \pi$.

One way of thinking about the path of the particle is that its displacement at time t is given by $\mathbf{r} = \mathrm{e}^t\big((\cos t)\mathbf{i} + (\sin t)\mathbf{j}\big)$, so that it is at distance e^t from the origin and at an angle of t (in radians) to the x-axis (as $\big((\cos t)\mathbf{i}+(\sin t)\mathbf{j}\big)$ is a unit vector in this direction). Thus its distance at time $t=0$ is $\mathrm{e}^0 = 1$, and when it has gone a half circle, its distance is e^π, which is approximately $\mathrm{e}^3 \approx 20$. So the particle moves away from the origin *very* quickly!

Another thing to bear in mind is that its velocity is always at an angle of $\frac{\pi}{4}$ to its displacement. Since it is moving away from the origin, its velocity is directed away from the origin, so initially it is moving at an angle of $\frac{\pi}{4}$ above the positive x-axis.

As we sketch the path, we also indicate the directions of the velocities at the times $t=0$, $t=\frac{\pi}{2}$ and $t=\pi$.



A second particle Q moves on the same path, passing through each point on the path a fixed time T after P does. Show that the distance between P and Q is proportional to e^t.

We write $\mathbf{r}_P = \mathbf{r}$ for the position vector of P and $\mathbf{r}_Q$ for the position vector of Q. We therefore have

$$\begin{aligned}\mathbf{r}_P &= \big(\mathrm{e}^t \cos t\big)\mathbf{i} + \big(\mathrm{e}^t \sin t\big)\mathbf{j}\\ \mathbf{r}_Q &= \big(\mathrm{e}^{t-T} \cos(t-T)\big)\mathbf{i} + \big(\mathrm{e}^{t-T} \sin(t-T)\big)\mathbf{j},\end{aligned}$$

and so we can calculate $|\mathbf{r}_P - \mathbf{r}_Q|^2$:

$$\begin{aligned}|\mathbf{r}_P - \mathbf{r}_Q|^2 &= \big(\mathrm{e}^t \cos t - \mathrm{e}^{t-T}\cos(t-T)\big)^2 + \big(\mathrm{e}^t \sin t - \mathrm{e}^{t-T}\sin(t-T)\big)^2\\ &= \big(\mathrm{e}^{2t}\cos^2 t - 2\mathrm{e}^t\mathrm{e}^{t-T}\cos t\cos(t-T) + 2\mathrm{e}^{2(t-T)}\cos^2(t-T)\big)+\\ &\qquad \big(\mathrm{e}^{2t}\sin^2 t - 2\mathrm{e}^t\mathrm{e}^{t-T}\sin t\sin(t-T) + \mathrm{e}^{2(t-T)}\sin^2(t-T)\big)\\ &= \mathrm{e}^{2t} - 2\mathrm{e}^{2t-T}\big((\cos t\cos(t-T) + \sin t\sin(t-T)\big) + \mathrm{e}^{2(t-T)}\\ &= \mathrm{e}^{2t} - 2\mathrm{e}^{2t-T}\cos(t-(t-T)) + \mathrm{e}^{2t-2T}\\ &= \mathrm{e}^{2t}\big(1 - 2\mathrm{e}^{-T}\cos T + \mathrm{e}^{-2T}\big),\end{aligned}$$

so that

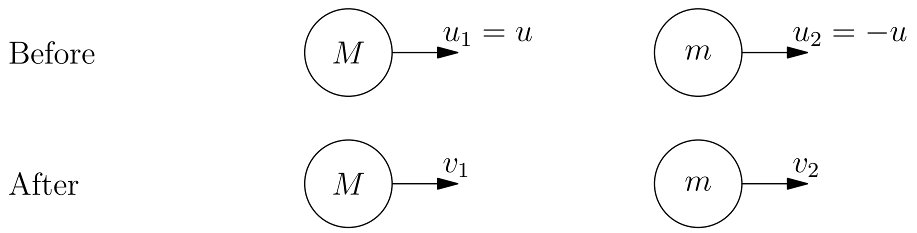
$$|r_P - r_Q| = \mathrm{e}^t\sqrt{1 - 2\mathrm{e}^{-T}\cos T + \mathrm{e}^{-2T}},$$

which is clearly proportional to e^t, as required, since T is a constant.

Question 11

Two particles of masses m and M, with $M > m$, lie in a smooth circular groove on a horizontal plane. The coefficient of restitution between the particles is e. The particles are initially projected round the groove with the same speed u but in opposite directions. Find the speeds of the particles after they collide for the first time and show that they will both change direction if $2em > M - m$.

This begins as a standard collision of particles question. ***ALWAYS*** *draw a diagram for collisions questions; you will do yourself (and the examiner) no favours if you try to keep all of the directions in your head, and you are very likely to make a mistake. My recommendation is to always have all of the velocity arrows pointing in the same direction. In this way, there is no possibility of messing up the Law of Restitution; it* **always** *reads* $v_1 - v_2 = e(u_2 - u_1)$ *or* $\dfrac{v_1 - v_2}{u_2 - u_1} = e$*, and you only have to be careful with the signs of the given velocities; the algebra will then keep track of the directions of the unknown velocities for you.*



Then Conservation of Momentum gives

$$Mu_1 + mu_2 = Mv_1 + mv_2$$

and Newton's Law of Restitution gives

$$v_2 - v_1 = e(u_1 - u_2).$$

Substituting $u_1 = u$ and $u_2 = -u$ gives

$$Mv_1 + mv_2 = (M - m)u \qquad (1)$$
$$v_2 - v_1 = 2eu. \qquad (2)$$

Then solving these equations (by (1) $- m \times$ (2) and (1) $+ M \times$ (2)) gives

$$v_1 = \frac{(M - m - 2em)u}{M + m} \qquad (3)$$
$$v_2 = \frac{(M - m + 2eM)u}{M + m}. \qquad (4)$$

The speeds are then (technically) the absolute values of these, but we will stick with these formulæ as they are what are needed later.

Now, the particles both change directions if v_1 and v_2 have the opposite signs from u_1 and u_2, respectively, so $v_1 < 0$ and $v_2 > 0$. Thus we need

$$M - m - 2em < 0 \qquad \text{and} \tag{5}$$
$$M - m + 2eM > 0. \tag{6}$$

But (6) is always true, as $M > m$, so we only need $M - m < 2em$ from (5).

> *After a further $2n$ collisions, the speed of the particle of mass m is v and the speed of the particle of mass M is V. Given that at each collision both particles change their directions of motion, explain why*
>
> $$mv - MV = u(M - m),$$
>
> *and find v and V in terms of m, M, e, u and n.*

The fact that the particles both change their directions of motion at each collision means that if they have velocities v_1 and v_2 after some collision, they will have velocities $-v_1$ and $-v_2$ before the next collision. This is because they are moving around a circular track, and therefore next meet on the opposite site, and hence are each moving in the opposite direction from the one they were moving in. (We do not concern ourselves with precisely where on the track they meet, and we are thinking of our velocities as one-dimensional directed speeds.)

Therefore, $mv_m + Mv_M$ is constant in value after each collision, where v_m is the velocity of the particle of mass m, and v_M that of the particle of mass M, but it reverses in sign before the next collision. So after the first collision, it it $Mu - mu$ to the right (in our above sketch), and hence after an even number of further collisions, it will still be $Mv_M + mv_m = Mu - mu$ to the right. But after an even number of further collisions, the particle of mass M is moving to the left, so $v_M = -V$, $v_m = v$. Thus

$$mv - MV = (M - m)u.$$

Also, since there are a total of $2n+1$ collisions, we have, by $2n+1$ applications of Newton's Law of Restitution,

$$V + v = e^{2n+1}(u + u).$$

Solving these two equations simultaneously as before then yields

$$V = \frac{(2me^{2n+1} - M + m)u}{M + m}$$
$$v = \frac{(2Me^{2n+1} + M - m)u}{M + m}.$$

Question 12

A discrete random variable X takes only positive integer values. Define $\mathrm{E}(X)$ *for this case, and show that*

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} P(X \geqslant n).$$

For the definition of $\mathrm{E}(X)$, we simply plug the allowable values of X into the definition of $\mathrm{E}(X)$ for discrete random variables, to get

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} n\,\mathrm{P}(X = n).$$

Now, we can think of $n, \mathrm{P}(X = n)$ as the sum of n copies of $\mathrm{P}(X = n)$, so that we get

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{n=1}^{\infty} n\,\mathrm{P}(X = n) \\
&= 1.\mathrm{P}(X = 1) + 2.\mathrm{P}(X = 2) + 3.\mathrm{P}(X = 3) + 4.\mathrm{P}(X = 4) + \cdots \\
&= \mathrm{P}(X = 1) + \\
&\quad\ \mathrm{P}(X = 2) + \mathrm{P}(X = 2) + \\
&\quad\ \mathrm{P}(X = 3) + \mathrm{P}(X = 3) + \mathrm{P}(X = 3) + \\
&\quad\ \mathrm{P}(X = 4) + \mathrm{P}(X = 4) + \mathrm{P}(X = 4) + \mathrm{P}(X = 4) + \cdots
\end{aligned}$$

Adding each column now gives us something interesting: the first column is $\mathrm{P}(X = 1) + \mathrm{P}(X = 2) + \mathrm{P}(X = 3) + \cdots = \mathrm{P}(X \geqslant 1)$, the second column is $\mathrm{P}(X = 2) + \mathrm{P}(X = 3) + \cdots = \mathrm{P}(X \geqslant 2)$, the third column is $\mathrm{P}(X = 3) + \mathrm{P}(X = 4) + \cdots = \mathrm{P}(X \geqslant 3)$, and so on. So we get

$$\begin{aligned}
\mathrm{E}(X) &= \mathrm{P}(X \geqslant 1) + \mathrm{P}(X \geqslant 2) + \mathrm{P}(X \geqslant 3) + \mathrm{P}(X \geqslant 4) + \cdots \\
&= \sum_{n=1}^{\infty} \mathrm{P}(X \geqslant n),
\end{aligned}$$

as we wanted.

An alternative, more formal, way of writing this proof is as follows, using what is sometimes called "summation algebra":

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{n=1}^{\infty} nP(X=n) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{n} \mathrm{P}(X=n) && \text{summing } n \text{ copies of a constant} \\
&= \sum_{1\leqslant m\leqslant n<\infty} \mathrm{P}(X=n) && \text{writing it as one big sum} \\
&= \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} \mathrm{P}(X=n) && \text{see below} \\
&= \sum_{m=1}^{\infty} \mathrm{P}(X\geqslant m),
\end{aligned}
$$

which is the sum we wanted. For the penultimate step, note the we are originally summing all pairs of values (m,n) where n is any positive integer and m lies between 1 and n, so we have $1 \leqslant m \leqslant n < \infty$, as written on the third line. This can also be thought of as summing over all pairs of values (m,n) where m is any positive integer (i.e., $1 \leqslant m < \infty$), and n is chosen so that $m \leqslant n < \infty$, that is, we are summing on n from m to ∞.

[One final technical note: we are allowed to reorder the terms of this *infinite* sum because all of the summands (the things we are adding) are non-negative. If some were positive and others were negative, we might get all sorts of weird things happening if we reordered the terms. An undergraduate course in Analysis will usually explore such questions.]

I am collecting toy penguins from cereal boxes. Each box contains either one daddy penguin or one mummy penguin. The probability that a given box contains a daddy penguin is p and the probability that a given box contains a mummy penguin is q, where $p \neq 0$, $q \neq 0$ and $p+q=1$.

Let X be the number of boxes that I need to open to get at least one of each kind of penguin. Show that $\mathrm{P}(X \geqslant 4) = p^3 + q^3$, and that

$$\mathrm{E}(X) = \frac{1}{pq} - 1.$$

We ask ourselves: what needs to happen to have $X \geqslant 4$? This means that we need to open at least 4 boxes to get both a daddy and a mummy penguin. In other words, we can't have had both a daddy and a mummy among the first three boxes, so they must have all had daddies or all had mummies. Therefore $\mathrm{P}(X \geqslant 4) = p^3 + q^3$.

This immediately generalises to give $\mathrm{P}(X \geqslant n) = p^{n-1} + q^{n-1}$, at least for $n \geqslant 3$. For $n = 1$, $\mathrm{P}(X \geqslant 1) = 1$, and for $n = 2$, $\mathrm{P}(X \geqslant 2) = 1 = p^1 + q^1$, as we argued above.

Therefore, we have

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{n=1}^{\infty} \mathrm{P}(X \geqslant n) \\
&= 1 + (p^1 + q^1) + (p^2 + q^2) + (p^3 + q^3) + \cdots \\
&= (1 + p + p^2 + p^3 + \cdots) + (1 + q + q^2 + q^3 + \cdots) - 1 \\
&= \frac{1}{1-p} + \frac{1}{1-q} - 1 \qquad \text{adding the geometric series} \\
&= \frac{1}{q} + \frac{1}{p} - 1 \qquad \text{as } p + q = 1 \\
&= \frac{p+q}{qp} - 1 \\
&= \frac{1}{pq} - 1 \qquad \text{again using } p + q = 1.
\end{aligned}$$

Hence show that $\mathrm{E}(X) \geqslant 3$.

To show that $\mathrm{E}(X) \geqslant 3$, we simply need to show that $\frac{1}{pq} \geqslant 4$. But this is the same as showing that $pq \leqslant \frac{1}{4}$, by taking reciprocals.

Now, recall that $q = 1 - p$, so we need to show that $p(1-p) \leqslant \frac{1}{4}$. To do this, we rewrite the quadratic in p by completing the square:

$$p(1-p) = p - p^2 = \tfrac{1}{4} - (p - \tfrac{1}{2})^2.$$

Since $(p - \frac{1}{2})^2 \geqslant 0$ for all p (even outside the range $0 < p < 1$), we have $p(1-p) \leqslant \frac{1}{4}$, as required, with equality only when $p = q = \frac{1}{2}$.

This can also be proved using calculus, or using the AM–GM inequality, or by writing $1/pq = (p+q)^2/pq$ and then rearranging to get $(p-q)^2 \geqslant 0$.

Question 13

The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$pe^{2\lambda} - e^{\lambda} + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

Let X be the number of texts George receives in the first hour of the morning and Y be the number he receives in the second hour.

Then $X \sim \text{Po}(\lambda)$ and $Y \sim \text{Po}(\lambda)$, with X and Y independent random variables.

We thus have

$$\begin{aligned}
\text{P(George waits between 1 and 2 hours for first text)} &= \text{P}(X = 0 \text{ and } Y > 0) \\
&= \text{P}(X = 0).\text{P}(Y > 0) \\
&= e^{-\lambda}.(1 - e^{-\lambda}) \\
&= p,
\end{aligned}$$

so that $e^{-\lambda} - e^{-2\lambda} = p$.

Multiplying this last equation by $e^{2\lambda}$ gives $e^{\lambda} - 1 = pe^{2\lambda}$; a straightforward rearrangement yields our desired equation.

(This equation can also be deduced by considering the *waiting time* until the first text; this is generally not studied until university, though.)

The solutions of the quadratic equation in e^{λ} are given by

$$e^{\lambda} = \frac{1 \pm \sqrt{1 - 4p}}{2p}.$$

But we are given that $4p < 1$, so that $1 - 4p > 0$ and there are real solutions. We need to show, though, that the two values of e^{λ} that we get are both greater than 1, so that the resulting values of λ itself are both greater than 0.

We have

$$\begin{aligned}
\frac{1 \pm \sqrt{1 - 4p}}{2p} > 1 &\iff 1 \pm \sqrt{1 - 4p} > 2p \\
&\iff \pm\sqrt{1 - 4p} > 2p - 1.
\end{aligned}$$

Now, since $4p < 1$, we have $2p < \frac{1}{2}$, so $2p - 1 < 0$, from which it follows that for the positive sign in the inequality, $\sqrt{1 - 4p} > 0 > 2p - 1$. It therefore only remains to show that $-\sqrt{1 - 4p} > 2p - 1$. But

$$\begin{aligned}
-\sqrt{1 - 4p} > 2p - 1 &\iff \sqrt{1 - 4p} < -(2p - 1) \\
&\iff 1 - 4p < (2p - 1)^2 \\
&\iff 1 - 4p < 4p^2 - 4p + 1,
\end{aligned}$$

which is clearly true as $4p^2 > 0$. (We were allowed to square between the first and second lines as both sides are positive.)

Thus the two solutions to our quadratic in e^λ are both greater than 1, so there are two positive values of λ which satisfy the equation.

> *The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables but with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.*

Each phone behaves in the same way as George's phone above, so the two possible values of λ are those found above. That is, the values of e^{λ_1} and e^{λ_2} are the two roots of $pe^{2\lambda} - e^\lambda + 1 = 0$.

We know that the product of the roots of the equation $ax^2 + bx + c = 0$ is c/a,[1] so in our case, $e^{\lambda_1}e^{\lambda_2} = 1/p$, so that $e^{\lambda_1+\lambda_2} = 1/p$, giving

$$\lambda_1 + \lambda_2 = \ln(1/p) = -\ln p.$$

> *Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.*

Let X_1 be the number of texts she receives on the first phone during the first hour and Y_1 be the number of texts that she receives on the first phone during the second hour. Then X_1 and Y_1 are both distributed as $\mathrm{Po}(\lambda_1)$, so

$$\begin{aligned} \mathrm{P}(X_1 = 0) &= e^{-\lambda_1} \\ \mathrm{P}(Y_1 = 0) &= e^{-\lambda_1}. \end{aligned}$$

Now let X_2 and Y_2 be the corresponding random variables for the second phone, so we have

$$\begin{aligned} \mathrm{P}(X_2 = 0) &= e^{-\lambda_2} \\ \mathrm{P}(Y_2 = 0) &= e^{-\lambda_2}. \end{aligned}$$

We must now consider the possible situations in which she receives her first text between 1 and 2 hours in the morning. She must receive no texts on either phone in the first hour, and at least one text on one of the phones in the second hour. We use the above result that $\lambda_1 + \lambda_2 = -\ln p$, so that $e^{-\lambda_1-\lambda_2} = p$.

[1] Why is this? If the roots of $ax^2 + bx + c = 0$ are α and β, then we can write the quadratic as $a(x-\alpha)(x-\beta) = a(x^2 - (\alpha+\beta)x + \alpha\beta)$, so that $c = a\alpha\beta$, or $\alpha\beta = c/a$. Likewise, $b = -a(\alpha+\beta)$ so that $\alpha + \beta = -b/a$.

Thus

$$
\begin{aligned}
&\mathrm{P}(\text{first text between 1 and 2 hours})\\
&\qquad = \mathrm{P}(X_1 = 0 \text{ and } X_2 = 0 \text{ and } Y_1 > 0 \text{ or } Y_2 > 0 \text{ or both})\\
&\qquad = \mathrm{P}(X_1 = 0).\mathrm{P}(X_2 = 0).\big(1 - \mathrm{P}(Y_1 = 0 \text{ and } Y_2 = 0)\big)\\
&\qquad = \mathrm{P}(X_1 = 0).\mathrm{P}(X_2 = 0).\big(1 - \mathrm{P}(Y_1 = 0).\mathrm{P}(Y_2 = 0)\big)\\
&\qquad = \mathrm{e}^{-\lambda_1}.\mathrm{e}^{-\lambda_2}.\big(1 - \mathrm{e}^{-\lambda_1}.\mathrm{e}^{-\lambda_2}\big)\\
&\qquad = \mathrm{e}^{-\lambda_1-\lambda_2}.(1 - \mathrm{e}^{-\lambda_1-\lambda_2})\\
&\qquad = p(1-p),
\end{aligned}
$$

and we are done.

Alternative approach

This approach uses a result which you may not have come across yet: the sum $X + Y$ of two independent Poisson random variables $X \sim \mathrm{Po}(\lambda)$ and $Y \sim \mathrm{Po}(\mu)$ is itself a Poisson variable with $X + Y \sim \mathrm{Po}(\lambda + \mu)$.

Since the number of texts received on the two phones together is the sum of the number of texts received on each one, the total can be modelled by a Poisson random variable with mean $\Lambda = \lambda_1 + \lambda_2$ texts per hour.

Then the probability of waiting between 1 and 2 hours in the morning for the first text is given by q, where

$$q\mathrm{e}^{2\Lambda} - \mathrm{e}^{\Lambda} + 1 = 0,$$

using the result from the very beginning of the question, replacing p with q and λ with Λ. Since $\Lambda = \lambda_1 + \lambda_2 = \ln(1/p)$ from above, $\mathrm{e}^{\Lambda} = 1/p$.

Therefore

$$
\begin{aligned}
q &= \frac{\mathrm{e}^{\Lambda} - 1}{\mathrm{e}^{2\Lambda}}\\
&= \frac{1/p - 1}{(1/p)^2}\\
&= p^2(1/p - 1)\\
&= p(1-p).
\end{aligned}
$$

Hints & Solutions for STEP II 2010

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1 When two curves meet they share common coordinates; when they "touch" they also share a common gradient. In the case of the *osculating circle*, they also have a common curvature at the point of contact. Since curvature (a further maths topic) is a function of both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, the question merely states that C and its osculating circle at P have equal rates of change of gradient. It makes sense then to differentiate twice both the equation for C and that for a circle, with equation of the form $(x - a)^2 + (y - b)^2 = r^2$, and then equate them when $x = \frac{1}{4}\pi$. The three resulting equations in the three unknowns a, b and r then simply need to be solved simultaneously.

For $y = 1 - x + \tan x$, $\frac{dy}{dx} = -1 + \sec^2 x$ and $\frac{d^2y}{dx^2} = 2\sec^2 x \tan x$.

For $(x - a)^2 + (y - b)^2 = r^2$, $2(x - a) + 2(y - b)\frac{dy}{dx} = 0$ and $2 + 2(y - b)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$.

When $x = \frac{1}{4}\pi$, $y = 2 - \frac{1}{4}\pi$ and so $\left(\frac{1}{4}\pi - a\right)^2 + \left(2 - \frac{1}{4}\pi - b\right)^2 = r^2$;

$$\frac{dy}{dx} = -\frac{(x-a)}{(y-b)} = 1 \text{ then gives a relationship between } a \text{ and } b;$$

$$\text{and } \frac{d^2y}{dx^2} = 4 = -\frac{4}{2(y-b)} \text{ gives the value of } b.$$

Working back then gives a and r.

Answers: The osculating circle to C at P has centre $\left(\frac{1}{4}\pi - \frac{1}{2},\ \frac{5}{2} - \frac{1}{4}\pi\right)$ and radius $\frac{1}{\sqrt{2}}$.

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2 The single-maths approach to the very first part is to use the standard trig. "Addition" formulae for sine and cosine, and then to use these results, twice, in (i); firstly, to rewrite $\sin^3 x$ in terms of $\sin 3x$ so that direct integration can be undertaken; then to express $\cos 3x$ in terms of $\cos^3 x$ in order to get the required "polynomial" in $\cos x$. Using the given "misunderstanding" in (ii) then leads to a second such polynomial which, when equated to the first, gives an equation for which a couple of roots have already been flagged. Unfortunately, the several versions of the question that were tried, in order to help candidates, ultimately led to the inadvertent disappearance of the interval 0 to π in which answers had originally been intended. This meant that there was a little bit more work to be done at the end than was initially planned.

$\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x = (2c^2 - 1)c - 2sc.s = (2c^2 - 1)c - 2c(1 - c^2)$
$= 4c^3 - 3c$.

$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x = 2sc.c + (1 - 2s^2)s = 2s(1 - s^2) + s(1 - 2s^2)$
$= 3s - 4s^3$

(i) $I(\alpha) = \int_0^\alpha \left(7\sin x - 8\sin^3 x\right) dx = \int_0^\alpha \left(\sin x + 2\sin 3x\right) dx = \left[-\cos x - \frac{2}{3}\cos 3x\right]_0^\alpha$

$= -\cos\alpha - \frac{2}{3}(4\cos^3\alpha - 3\cos\alpha) + 1 + \frac{2}{3} = -\frac{8}{3}c^3 + c + \frac{5}{3}$

and $I(\alpha) = 0$ when $c = 1$ $(\alpha = 0)$