SI-2015 Hints and Solutions

(See the marking scheme for full details of the expected solution approach)

Q1 This question is intended to be a relatively straightforward entrée into the paper, and thus its demands are fairly routine in nature. That does not mean that it is easy, merely that the appropriate courses of action should be readily accessible to all candidates of a suitable standard. To begin with, the demand for a sketch (of any function) should lead you to consider things such as

- key points (such as where the curve meets either of the coordinate axes);
- asymptotes (note the information given in italics at the end of part (i) regarding what happens as $x \to -\infty$, which indicates that the negative x-axis is an asymptote in this case);
- turning points of the curve, which are clearly flagged as being of significance when considering what happens when $y = k$; i.e. when the curve meets a horizontal line;
- long-term behaviour (you already have sorted for you the "$x \to -\infty$" side of things, so there is only a quick decision to be made about what happens as $x \to +\infty$).

For the key points, first set $x = 0$ and then $y = 0$; the asymptote is effectively given; the TPs come from setting the first derivative to zero and solving for x again (noting, of course, that e^x is always positive); and the curve clearly grows exponentially as x increases positively. The rest of (i) then simply requires a bit of thought as to how many times a horizontal line will cut, or touch, the curve depending upon the value of k.

In part (ii), it is clear that the x in part (i) has now been replaced with an x^2, and this second curve must therefore have reflection symmetry in the y-axis, as all negative values of x are being squared to give the positive counterpart. Previously, when x was equal to zero, we now have$x^2 = 0$, and so each previous crossing-point on the positive x–axis leads to two, one on each side of the y-axis (and at the square-root of its former value). However, the previous y-intercept is unchanged, but must now appear at a TP of the curve (otherwise the symmetry of the gradient would be compromised). Also, the previous TP with positive x-value (the negative one has gone) occurs at the square-roots of the previous value, but again with unchanged y-coordinate.

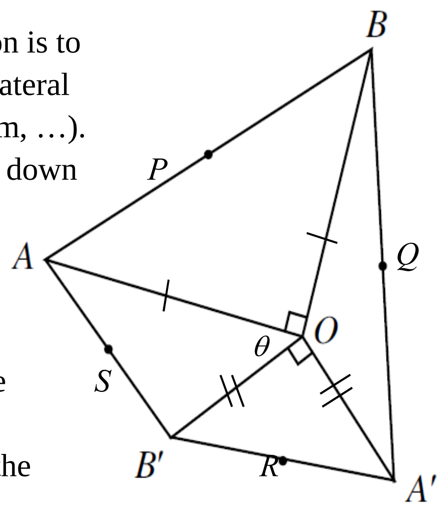
Q2 It is clear that (i) is an introductory part that requires the use of the $\cos(A - B)$ formula with suitably-chosen values of A and$B$. Using the $\sin(A - B)$ formula then leads to the second result, although there are alternative trig. identities that could be used in both cases, such as a double-angle formula. Repeated use of these, or the double-angle formulae (or *de Moivre's Theorem* for those from a further maths background) lead to the (relatively) well-known 'triple-angle formula' $\cos 3\alpha \equiv 4\cos^3\alpha - 3\cos\alpha$, which gives $x = \cos\alpha$ as a root of the equation $4x^3 - 3x - \cos 3\alpha = 0$. Although there are several possible methods here, a simple division/factorisation leads to $(x - c)\left\{4\left(x^2 + cx + c^2\right) - 3\right\}$ = 0, and the quadratic formula leads to the remaining two roots which, for their simplest form, requires the use of the most elementary of trig. identities, $s^2 = 1 - c^2$.

A bit of insight is needed in part (iii), where one should first realise that the constant term is intended be a cosine value (of 3 × some angle), and the most obvious candidate is $\frac{1}{2}\sqrt{2}$ = cos 45°, so that α is 15°. From here, it is now clear that "x" is $\frac{1}{2}y$, and that the three roots are those from part (ii) with the exact numerical forms of the sine and cosine of 15° from part (i) waiting to be deployed in order to find the surd forms requested.

Q3 In this question, it is important to draw suitable diagrams in order to visualise what is going on, and these are not difficult to manage, with the guard either at a corner of the yard (C) or at its middle (M). However, this second case has two possible sub-cases to consider, depending upon whether the 'far' corners of the yard are visible to him/her (which, in fact, turns out to be the $b = 3a$ case which separates the two cases that the question invites you to consider), and it is the extra length of the opposite wall that is visible that makes for different working. These lengths are $\frac{4b^2}{b+a}$ (from C), $\frac{b^2}{a}$ (from M, with the 'far' corners not visible) and $\frac{2b(2b-3a)}{b-a}$ (from M, with the 'far' corners visible). Once obtained, these should be compared in order to find that the guard should stand at C for $b < 3a$ and at M for $b > 3a$ (and at either when $b = 3a$).

Q4 A quick read-through of the question should make it clear that it is the lower end of the rod that is being referred to (as do the subtractions within the given coordinates x and y). The fact that the rod is tangent to the given parabola means that its direction is $\tan\theta = \tfrac{1}{2}x$, which gives the coordinates of the rod's midpoint as $(\tan\theta, \tan^2\theta)$; a simple right-angled triangle and some accompanying basic trig. then leads to the given answer. The second part of the question is equally straightforward once one realises that when $x_A = 0$, $2\tan\alpha = b\cos\alpha \Rightarrow b = \frac{2\tan\alpha}{\cos\alpha} \Rightarrow y_A = -\tan^2\alpha$. There are several ways to attack the area between two curves – e.g. as $\int (y_1 - y_2)\,\mathrm{d}x$, or by translating the bit below the x-axis up by $\tan^2\alpha$ and calculating the difference between area under the "new" curve and a triangle; the key is to eliminate the "b" and then the given answer follows.

Q5 Once one realises that the x within the integral is not the variable, then both integrations are relatively straightforward. In (i), we get $\mathrm{f}(x) = \frac{2^x}{x}$, while in (ii) $\mathrm{g}(x) = \frac{1}{x}\left(|1+x| - |1-x|\right)$ … remember to use the modulus function when taking square-roots (although one could, alternatively, work out a piece-wise definition for g; that is, in bits). The sketch of f should prompt the solver to differentiate in order to identify the turning point at $\left(\frac{1}{\ln 2},\ \mathrm{e}\ln 2\right)$. Noting that $y \to +\infty$ as $x \to 0$ and that $y \to +\infty$ as $x \to +\infty$ gives all else that is needful to draw the graph in (i). In (ii), the piece-wise definition of g is certainly more useful now since its graph is made up of two 'reciprocal' curve bits joined by a horizontal straight-line in the middle.



Q6 The best way to start any geometrically-inclined question is to have a good diagram that doesn't make the shape of the quadrilateral look too specialised in any way (square, rectangle, parallelogram, …). Next, label the midpoints sensibly (see diagram) and then write down their position vectors in terms of **a** and **b**.

It is relatively easy to prove that the opposite sides of this quadrilateral are equal and opposite, but you must then also show that adjacent sides are equal in length and that they are perpendicular. This last outcome is going to follow from the use of the scalar product.

For the final part, you should label one of the angles at the

centre θ (say) and note that the fourth angle at O is thus $180^\circ - \theta$.
Having already calculated the squares of the lengths of the square's sides in the form

$$\tfrac{1}{4}\left(a^2 + (a')^2 - 2aa'\cos\left[90^\circ + \theta\right]\right)$$

the required result follows from noting that this is maximal when $\cos\left[90^\circ + \theta\right] = -1$; i.e. when $\theta = 90^\circ$.

Q7 The crucial observation here is that a (continuous) function takes its maximum value on a finite interval either at a maximum turning-point *or* at an endpoint. Differentiating (a 'negative' cubic – so we know what its shape is) gives a MIN. TP at (0, 0) and a MAX. TP at $(\tfrac{1}{3}a, \tfrac{1}{9}a^3)$, and evaluating at the endpoints gives $f(-\tfrac{1}{3}) = \tfrac{1}{9}(3a + 2)$ and $f(1) = 3a - 6$.

Now, a comparison of these possible values for f then yields that $\tfrac{1}{9}(3a + 2) \geq \tfrac{1}{9}a^3 \Leftrightarrow a \geq 0, a \leq 2$; and that $\tfrac{1}{9}a^3 \geq 3a - 6$ holds for all $a \geq 0$; and also that $\tfrac{1}{9}(3a + 2) \geq 3a - 6 \Leftrightarrow a \leq \tfrac{7}{3}$ (which, actually, affects nothing, but the working should be done anyhow). Thus $M(a) = \begin{cases} \tfrac{1}{9}(3a+2) & 0 \leq a \leq 2 \\ \tfrac{1}{9}a^3 & 2 \leq a \leq 3 \\ 3a - 6 & a \geq 3 \end{cases}$.

Q8 The standard "bookwork" approach to this opening part is to write the sum (S) both forwards and backwards, add the terms in pairs (n pairs, each of value $n + 1$) and then to half this to get $S = \tfrac{1}{2}n(n + 1)$. As with any such invitation to establish a result, one should not simply seek to quote a result and thus merely "write down" the given answer. When looking at part (ii)'s question, the *binomial theorem* should really be screaming at you from the page, and all that is needed is to observe that the binomial expansion of $(N - m)^k$ consists of $k + 1$ terms, the first k of which contain a factor of (at least one) N. The final term, since k is odd must be $-m^k$ which then conveniently cancels with the $+ m^k$ term to leave something that is clearly divisible by N.

In the next part of the question, you are invited to explore the cases n odd and n even separately (indeed the results that follow are slightly different). To begin with,

$$S = 1^k + 2^k + \ldots + n^k \text{ (an odd no. of terms)} = 0^k + 1^k + 2^k + \ldots + n^k \text{ (an even no. of terms)}$$

So these terms can now be paired up:

$$n \text{ with } 0, \quad n - 1 \text{ with } 1, \quad \ldots, \quad (\tfrac{1}{2}n + \tfrac{1}{2}) \text{ with } (\tfrac{1}{2}n - \tfrac{1}{2}),$$

so that all pairs are of the form $(n - m)^k + m^k$, which was just established as being divisible by n. Next, in the case when

$$S = 1^k + 2^k + \ldots + n^k \text{ (an even no. of terms)} = 0^k + 1^k + 2^k + \ldots + n^k \text{ (an odd no. of terms)},$$

the pairs are now

$$n \text{ with } 0, \quad n - 1 \text{ with } 1, \quad \ldots, \quad (\tfrac{1}{2}n + 1) \text{ with } (\tfrac{1}{2}n - 1),$$

but with an odd term, $(\tfrac{1}{2}n)^k$, left over. This gives us (from the same previous result as before) a sum consisting of terms divisible by n and one that is divisible by $\tfrac{1}{2}n$, giving the second result.

Then, for n even, so that $(n + 1)$ is odd, $S + (n + 1)^k$ is divisible by $n + 1$ (by the previous result) $\Rightarrow S$ is divisible by $n + 1$; and for n odd, so that $(n + 1)$ is even, $S + (n + 1)^k$ is divisible by $\tfrac{1}{2}(n + 1)$. Thus, since $\text{hcf}(n, n + 1) = 1 \Rightarrow \text{hcf}(\tfrac{1}{2}n, n + 1) = 1$ for n even, and $\text{hcf}(n, \tfrac{1}{2}(n + 1)) = 1$ for n odd, it follows that S is divisible by $\tfrac{1}{2}n(n + 1)$ for all positive integers n.

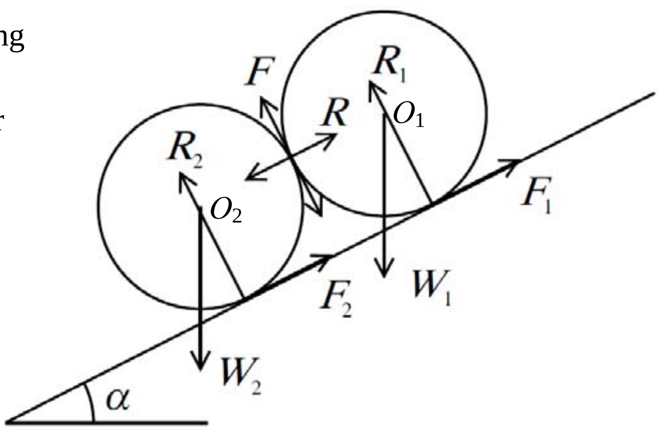
Q9 The standard time taken to land (at the level of the projection) of a projectile is $t = \dfrac{2u\sin\alpha}{g}$. Thus, a bullet fired at time t, $0 \le t \le \dfrac{\pi}{6\lambda}$, lands at time $T_L = t + \dfrac{2u}{g}\sin\left(\dfrac{\pi}{3} - \lambda t\right)$. Differentiating this w.r.t. t and setting it equal to zero, gives $k = \cos\left(\dfrac{\pi}{3} - \lambda t\right)$. The horizontal range is then given by $R = \dfrac{2u^2\sin\alpha\cos\alpha}{g}$ and this gives the required answer. Moreover, substituting the endpoints of the given time interval $0 \le t \le \dfrac{\pi}{6\lambda}$ into $k = \cos\left(\dfrac{\pi}{3} - \lambda t\right)$ gives $\dfrac{1}{2} \le k \le \dfrac{\sqrt{3}}{2}$. However, if $k < \frac{1}{2}$, then one sees that $\dfrac{\mathrm{d}T_L}{\mathrm{d}t} < 0$ throughout the gun's firing, so that T_L is a (strictly) decreasing function. Hence its maximum value occurs at $t = 0$, i.e. $\alpha = \dfrac{\pi}{3}$, whence $R = \dfrac{2u^2}{g} \times \dfrac{1}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{u^2\sqrt{3}}{2g}$.

Q10 The difficulty in this question lies in ignoring unnecessary information (not given). Firstly, then, note that the speed of the rain relative to the bus is $v\cos\theta - u$ (or $u - v\cos\theta$ if negative), and when $u = 0$, the area of the bus getting wet, A, is such that $A \propto hv\cos\theta + av\sin\theta$. Now the given result follows from observing that when $v\cos\theta - u > 0$, the rain hitting the top of the bus is the same, while the rain hits the back of the bus as before, but with speed $v\cos\theta - u$ instead of $v\cos\theta$; and when $v\cos\theta - u < 0$, the rain hitting the top of the bus is the same, while the rain hits the front of the bus as before, but with $u - v\cos\theta$ instead of $v\cos\theta$.

Next, as the journey time $\propto \dfrac{1}{u}$, we need to minimise $J = \dfrac{av\sin\theta}{u} + \dfrac{h\,|\,v\cos\theta - u\,|}{u}$. For $v\cos\theta - u > 0$ and $w \le v\cos\theta$, we minimise $J = \dfrac{av\sin\theta}{u} + \dfrac{hv\cos\theta}{u} - h$, and this decreases as u increases, and this is done by choosing u as large as possible; i.e. $u = w$. For $u - v\cos\theta > 0$, we minimise $J = \dfrac{av\sin\theta}{u} - \dfrac{hv\cos\theta}{u} + h$, and this decreases as u increases if $a\sin\theta > h\cos\theta$, so we again choose u as large as possible; i.e. $u = w$. Next, if $a\sin\theta < h\cos\theta$, then J increases with u when u exceeds $v\cos\theta$, so we choose $u = v\cos\theta$ in this case. Finally, if $a\sin\theta = h\cos\theta$ then J is independent of u, so we may as well take $u = w$.

For the return journey, simply replace θ by $180^{\circ} - \theta$ to give $J = \dfrac{av\sin\theta}{u} + \dfrac{hv\cos\theta}{u} + h$, which always decreases as u increases, so take $u = w$ again.

Q11 As with all statics problems, the key to getting a good start, and to making life as easy as possible for the working that follows, is to have a good, clear diagram with all relevant forces, in appropriate directions, marked on it (see alongside).



To begin with, take moments about the respective cylinders' axes yields $F = F_1 = F_2$, as required. Next, write down the four equations that arise from resolving for each cylinder in the directions parallel and perpendicular to the plane.

These are:-

$F_1 + R = W_1 \sin\alpha$ ①; $R_1 + F = W_1 \cos\alpha$ ②; $F_2 - R = W_2 \sin\alpha$ ③ and $R_2 - F = W_2 \cos\alpha$ ④.

(Note that one could replace some of these with equivalent equations gained from resolving for the whole system.) Replacing F_1 and F_2 by F, equating for $\sin\alpha$, re-arranging for F in terms of R and using the Friction Law, $F \le \mu R$, appropriately leads to the first given answer in (ii). A bit more determination is needed to gain the second given answer, however. Firstly, $\tan\alpha$ can be gained by division in at least two ways, and both F and R must be eliminated from any equations being used. Thereafter, it is simply a matter of forcing the working through correctly and, hopefully, concisely.

Q12 Here, you are given the relevant Poisson result at the outset, and this is intended to guide your thinking later on in the question. To begin with, though, part (i) is actually a Binomial situation … requiring just a single general term. In part (ii), you were asked to prove *algebraically* a result that you might usually be required to quote and use. This requires a good understanding of the use of the sigma-notation and a clear grasp as to which of the various terms are constant relative to the summation, and then combining the remaining terms together appropriately to give the requested Poisson answer. Most important of all, of course, it is essential to have the first line of working correct; this is

$$\mathrm{P}(S = r) = \sum_{n=r}^{\infty} \frac{\mathrm{e}^{-8} 8^n}{n!} \times \frac{n!}{r!(n-r)!} \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{n-r}$$

and one follows this through to the point where the result $\dfrac{\mathrm{e}^{-8} \times 2^r}{r!} \displaystyle\sum_{n=r}^{\infty} \frac{6^{n-r}}{(n-r)!}$ is obtained. At this stage another simple trick is required – effectively a re-labelling of the starting-point, using $m = n - r$ to re-write this as $\dfrac{\mathrm{e}^{-8} \times 2^r}{r!} \displaystyle\sum_{m=0}^{\infty} \frac{6^m}{m!}$. The required result follows immediately since the infinite sum is just e^6.

Having established this, the final part of the question is relatively straightforward, requiring only the use of the conditional probability formula applied to $\mathrm{P}(M = 8 \mid M + T = 12)$.

Q13 The first three parts of this question are very easy indeed, if looked at in the right way. In part (i) it is not necessary at all that you recognise the Geometric Distribution (indeed, some of you may not have encountered it at all), but the result asked for is simply "(n – 1) failures followed by 1 success", and one can write down immediately, and without explanation, the answer $\mathrm{P}(A) = \left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right)$. In (ii), you have a situation in which one can apply the principle of symmetry: either a 5 arises before a 6, or *vice versa*, so the required probability is just $\mathrm{P}(B) = \frac{1}{2}$. Part (iii) can be approached similarly, in that the first 4s, 5s, 6s can arise in the orders **<u>456</u>**, 465, **<u>546</u>**, 564, 645, 654 $\Rightarrow$ $\mathrm{P}(B \cap C) = \frac{1}{3}$ (i.e. also "by symmetry", but with three pairings to consider).

Parts (iv) and (v), however, each turn out to require the use of the result given at the end of the question, as the outcomes (theoretically) stretch off to infinity. For (iv), it is best to consider only on which throw the first 6 occurs (since we stop at that point). It cannot occur on the first throw, so we have the sum of the situations:

a 5 occurs on the first throw, followed by a 6 on the second;

one 5 and a 1-4 occur, in either order, followed by the 6 on the third;

one 5 and two 1-4s occur, in any of three possible orders, followed by a 6 on the fourth;

etc.

Thus P(D) = $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)+\binom{2}{1}\left(\frac{1}{6}\right)\left(\frac{4}{6}\right)\left(\frac{1}{6}\right)+\binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{4}{6}\right)^2\left(\frac{1}{6}\right)+\ldots$, and this factorises as $\left(\frac{1}{36}\right)\left\{1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2+\ldots\right\}$, and the big bracket is just the given result with x = $\frac{2}{3}$ and n = 2.

Before getting too deeply into part (v), a couple of simple results should be noted. Firstly, we use the fact that P(E) = P(D) = $\frac{1}{4}$, the answer to (iv); and then that we will need to use the basic probability result P($D\cup E$) = P(D) + P(E) – P($D\cap E$) = $\frac{1}{2}$ – P($D\cap E$). Turning this around, since it is far easier to calculate the probability, P($D\cap E$), that *both* one 4 and one 5 occur before the first 6. Again, looking at this from the viewpoint of finishing after the first 6 is thrown, we see that

$$P(D\cap E) = \left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)+\binom{3}{1}\left(\tfrac{3}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)+\binom{4}{2}\left(\tfrac{3}{6}\right)^2\left(\tfrac{2}{6}\right)\left(\tfrac{1}{6}\right)\left(\tfrac{1}{6}\right)+\ldots = \left(\tfrac{1}{108}\right)\left\{1+3\left(\tfrac{1}{2}\right)+6\left(\tfrac{1}{2}\right)^2+\ldots\right\}$$

and the big bracket is the given result with x = $\frac{1}{2}$ and n = 3 , leading to the answer P($D\cup E$) = $\frac{23}{54}$.