STEP 1 Mathematics

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Hints and solutions

Introductory Remarks

This document should be read in conjunction with the corresponding mark scheme in order to gain full benefit from it. Since the complete solutions appear elsewhere, much of this Hints and Solutions document will concentrate more on the “whys and wherefores” of the solution approach to each question and less on the technical details.
The solutions that follow, presented either in outline or in full, are by no means the only ones, not even necessarily the ‘best’ ones. They are simply intended to be the ones that, on the evidence of the marking process, appear to be the ones which arose most frequently from the ideas produced by the candidates and that worked for those who could force them through to a conclusion. If you “see” things in a different way, I hope you can still both follow and appreciate what is given here.

Question 1

Despite the fact that this differs from the usual sort of routine A-level question of this kind – in that there are no "numbers" involved in the presentation of it – the underlying ideas are, essentially, unchanged. The opening part contains a line with a given (trigonometric) gradient, which leads to a (right-angled) triangle with sides of (non-numerical) lengths. The vertices and area of this triangle are thus 'write downs' and the obvious approach in part (i) of using calculus to determine the minimum value of A should be clear to all.

(Now that there is no Formula Book available, it is important that candidates have learnt the various relationships – or can find them out speedily by hand if not – between the trig. functions and their derivatives, including the possibilities for deploying various trig. identities along the way, when necessary, in order to tidy any answers up.)

In Part (ii), the length of the hypotenuse, XY, of this same triangle needs to be determined in some form and, in principle at least, this is just a GCSE-level use of *Pythagoras' theorem*. Finding the perimeter of the triangle and, again, using calculus to maximise it consists of a set of well-established routines; it is only the accompanying trig. work that requires a bit of skill, some background learning of the results, and a certain amount of care.

Question 2

Since the first curve, *C*, is given in parametric form, candidates should be guided along the path of using the *Chain rule* result for parametric differentiation, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, rather than reverting to finding its cartesian equation, hence avoiding issues with plus/minus signs that will clutter up the problem. $It\ is\ also\ important\ to\ think\ carefully\ about\ the\ difference\ between\ \ t$ and p: one is a general descriptive parameter for the curve while the other represents a particular value of it, even though this, too, is non-specific.

When finding the intersection, it is useful to simplify the expression using the difference of two cubes: $p^3 - q^3 \equiv (p - q)(p^2 + pq + q^2)$. This is the very useful cousin of the difference-of-two-squares factorisation. (Students might also like to look up the sum-of-two-cubes, the difference-of-two-fourth-powers, and similar results, for use in other problems … these are additional results often **not** flagged up automatically in STEPs but occurring sufficiently frequently to warrant learning in advance.) Once the intersections are in the required form, the constraint that the two tangents are perpendicular can now be used. Note the usefulness of the result that $p^2 + q^2 = (p + q)^2 - 2pq$ … another instance of a simple algebraic result that won't necessarily be sign-posted but which candidates *should* have in their 'toolkit' if they are to work through STEP questions fluently, rather than having to keep stopping to do odd bits of working *on one side* before being able to continue fluently with the solution to a problem.

There are various ways of finding the intersection of the two curves. One could turn everything into cartesian form, or just compare the parameterised x and y coordinates. It is tempting to try to use $u = p + q$ again, but that adds an additional constraint which will not necessarily hold at the point of intersection. We should end up with a cubic (or a disguised cubic) which needs to be factorised by first "spotting" a solution and then using the *factor theorem* and polynomial division. (Don't forget to check which of the solutions are valid … this is another routine STEP skill)

It should then be seen that one of the solutions is a double root – and this says something very significant about the intersection point: what? This should help with the final sketch. It is important to try to make sure that the sketch is consistent with all the information that has been given or found – for example, what is known about the gradient of *C* close to $x = 0$? What about the symmetry of C in the x axis?

Question 3

This question has a nice structure: firstly, an explicit "use this method" problem; followed by a "think about what's required for yourself, but it should be a similar kind of thing to the first bit" problem; with a "you're on your own now" finale.

As with Q1, there's a lot of different ways to go about these integrations, all governed by how readily one recognises what one has on the page at any stage of the process and on how easily it can be turned into something that can be integrated. A mixture of methods can be applied here, from direct integration, to use of the *Chain rule* in reverse (often referred to as "recognition" integration), through to the use of any one of a number of possible substitutions, or possibly even integration-by-parts.

In the first case, working on the integrand in the suggested way leads from

$$\frac{1}{1+\sin x} \text{ to } \frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \sec x \tan x$$

and both of these terms in the final expression are directly integrable (being "standard" derivatives of $\tan x$ and $\sec x$ respectively).

The obvious approach (using a similar idea to that used for the first integral) to the second integral is to multiply top and bottom by $(1 - \sec x)$, and this leads from

$$\frac{1}{1+\sec x} \text{ to } \frac{1-\sec x}{1-\sec^2 x} = \frac{1-\sec x}{\tan^2 x} = \cot^2 x - \operatorname{cosec} x \cot x$$

and this is definitely *similar* to the situation that arose in the first case, but now requires the extra step of replacing $\cot^2 x$ with $(\operatorname{cosec}^2 x - 1)$ in order to get all terms in a directly integrable form. Alternatively, one could start the ball rolling here by turning

$$\frac{1}{1+\sec x} \text{ into } \frac{\cos x}{\cos x + 1} \text{ or even } \frac{1+\cos x - 1}{\cos x + 1} = 1 - \frac{1}{\cos x + 1}$$

(some students favour turning all trig. functions into sine and cosine only, and this approach can work as well) and then multiplying top and bottom by $(1 - \cos x)$. In fact, this second approach yields working which ends up looking (essentially) *exactly* like the first case, only with the extra manipulation work having been done upfront.

For the third integral, the most helpful method is not immediately obvious and one may need to be prepared to explore various ideas before full progress can be made. In fact, the initial idea used in this question would suggest that the best approach is to multiply top and bottom by $(1 - \sin x)^2$, and this turns out to be the right thing to do, although one must then be prepared to split the result into (possibly) several distinct parts and work on them separately. (As an aside, it is surprising how helpful it is to do this … almost all candidates who try to work on "the whole thing" in such cases end up making a mess of things, even if it is only by muddling a negative sign or missing a factor outside a bracket somewhere along the line.) I would direct the reader to the Mark Scheme, which shows how this third integral ends up being split (using the method suggested) into a direct integration, a substitution integration, and an integration-by-parts.

Question 4

Part (i) is clearly a simple start for later reference – as it doesn't look as if it turns up very soon from what can immediately be seen of part (ii) – and you are helpfully directed to consider integers from the outset. Since it is clearly a case of dealing with small numbers, it really should be possible to spot immediately that the (positive) square-root of $3+2\sqrt{2}$ can only be $1+\sqrt{2}$. If this is not immediately clear, then there is always the direct approach:

expand $\left(m+n\sqrt{2}\right)^2$ to get $m^2+2n^2+2mn\sqrt{2}$; then compare the two parts with the known answer (hopefully avoiding a full algebraic method involving solving a quadratic-plus-linear pair of simultaneous equations, to find that $m=n=1$ works.

(But candidates are strongly advised to look out for every opportunity **not** to spend lots of time doing unnecessary algebra.)

Part (ii) begins with a little bit of an explanation, and this sort of thing must always be addressed as completely as possible – very many students tend to gloss over important details when it is important to be fully convincing.

Thereafter, multiplying out $(x^2+sx+p)(x^2-sx+q)$ and equating the coefficients with those of $\mathrm{f}(x)$ gives three equations for p,q and s. These can most helpfully be written in the following forms (suggested by the **given** form of the "Show that …" equation in the question):

$$(p+q)^2=(s^2-10)^2,$$
$$(p-q)^2=\frac{144}{s^2},$$
$$4pq=-8,$$

and it is clear that p and q are to be eliminated from this "system" of equations. Noting that the two squared terms on the left-hand-sides have a useful difference which leads to $(p+q)^2=(p-q)^2+4pq$ and rearranging then gives the required equation for s.

Substituting $t=s^2$ produces a cubic equation and the *factor theorem* provides us with a method for finding the three possible values of s^2 and we can soon find one solution, $t=2$. Now take out the corresponding factor, and solve a quadratic equation for the other two possible values.

If $s=\sqrt{2}$ (using the question's directing hint towards the use of the smallest value of s^2) the usefulness of the opening result should be revealed, and we now have the simultaneous equations for p and q: $p+q=-8$, $q-p=6\sqrt{2}$ … and now the two quadratic equations $x^2+sx+p=0$ and $x^2-sx+q=0$ can be solved separately. Each of the solutions will involve a term of the form $\sqrt{18\pm12\sqrt{2}}$, and the fact that $\sqrt{3+2\sqrt{2}}=1+\sqrt{2}$ from (i), together with a similar expression for $\sqrt{3-2\sqrt{2}}$, can be used to simplify the roots.

Question 5

Once again, there are two opening specific questions which require little more than a knowledge of the properties of quadrilaterals and the meaning that accompanies the statement that two vectors are equal. As with Q4, this is for later reference.

The start of part (i) can be approached in so many different ways, depending upon the depth of one's knowledge of further vector methods (where the equations of planes, independence of vectors, the vector product, the scalar triple product, etc. *could* all be considered). However, for knowledge of vectors at single maths level only, the obvious tactic is to show that the two diagonals intersect and this can be done using the standard sorts of method for finding the vector equations of the lines *PR* and *QS* and showing that they do indeed intersect provided the given condition holds.

Part (a) may seem unfamiliar territory, but the question gives all the necessaries, including the definition of the centroid of a quadrilateral. Note that an "if an only if" proof consists of two parts: the *if* ($\Rightarrow$) direction of the argument **and** the *only if* ($\Leftarrow$) part. Alternatively, in simple cases, it may be clear that any steps taken in the reasoning are entirely reversible ($\Leftrightarrow$) but this still needs to be stated clearly and not just left to be invisibly implied.

In (b), once one has now realised that *PQRS* is a rhombus, one can proceed by working with the magnitudes of the relevant vectors (i.e. the lengths of the sides of the quadrilateral), and by using the immediately preceding result, in order to replace q, r and s with p. Then, to wrap things up (with the scalar product not now required for Paper 1) the *Cosine rule* can be used in triangle *PQR* to obtain the displayed result … thereafter, moving from rhombus to square by setting $\cos PQR = 0$. At the very end, there is a simple bit of inequality work to demonstrate the final given answer.

Question 6

From its appearance, one's initial impression of Q6 is that there's rather a lot to it … and, in fact, this does turn out to be the case. Nonetheless, the two parts of (i) are quite straightforward in terms of clearly stated demands and can be approached by everyone. The process of completing the square is relatively routine, even though one must be prepared to think of $\cos\theta$ as a numerical part of the coefficient of x (etc.) It is possible to factor out the 9 from the leading terms before commencing the process, but not really necessary, since the first two terms (to all intents and purposes the constant term, 4, at the end can be dealt with separately) will not require the use of fractions:

$$9x^2 - 12x\cos\theta \equiv (3x - 2\cos\theta)^2 - \text{an adjusting constant.}$$

The final, overall adjusted, constant turns out to be $4 - 4\cos^2\theta$ and it is helpful to replace this immediately by $4\sin^2\theta$. The purpose of the completed-square form for the original expression is that the minimum value, and the value of x which gives it, can then be written down without further ado.

The second of these given expressions is a quadratic in (x^2) and one could complete the square (effectively hinted at from the first request) or – since there is now no clear direction as to which method should be used – find the required maximum (also $4\sin^2\theta$) using calculus. The summary result of part (i) now follows by separating off the two bits previously re-formatted and realising that the minimum of one side can only hit the maximum of the other – in this case – at the one instant, $4\sin^2\theta$.

Setting the x's equal and then turning the resulting equation into a quadratic in ($\sin x$) leads to the single solution $\sin x = \frac{1}{2}$. Now, this is one of the "standard" exact trig. results which all candidates should recognise, though it does lead to two values of x in the given interval (right back at the very start of the question) of $0 < x < \pi$. It is then important both to consider the positive *and* negative square-roots of the values for x AND to remember that, having squared along the way, there are likely to be "extraneous solutions" that shouldn't be there and hence need to be checked for validity at the end.

In part (ii), it soon becomes clear that the given curve has two branches, a $\cup$-shaped bit and a $\cap$-shaped bit. A standard calculus approach (helped by a consideration of what happens as $x \to 0\pm$ and as $x \to \pm\infty$) shows that there is a minimum at $(2\theta, 4\theta)$ and a maximum at (0, 0), as required. Next, in the given rational expression, it can be quickly noted that the numerator is always less than or equal to 1 while the denominator is greater than or equal to 1, all of which gives us the desired result. Finally, re-arranging the second of the initial two established inequalities, and setting it alongside the next one gives

$$\frac{x^2}{4\theta(x-\theta)} \geq 1 \geq \frac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x}$$

and, with the same notion as appeared in part (i), the two can only be equal when top and bottom both take their maximum/minimum values respectively.

Question 7

This is very much a "reasoning" question, where the explanations form a major part of the deal and there is no point skimming through these and expecting to come out with high marks. Although it is not essential to have some grasp of the ideas behind modular arithmetic, the notations used are immensely brief compared to the written word … so, for instance, the statement

$$x \equiv 2 \pmod 3$$

is actually saying that the number (integer) x leaves a remainder of 2 upon division by 3; as such, this is a statement being made about an infinite set of numbers which have such a property, without worrying about the actual whole number part when the division is undertaken (a bit like the *Remainder theorem*). It is also worth noting that we can switch between positive and negative "remainders" without any need for additional explanation; and $x \equiv 2 \pmod 3$ is, in fact, exactly the same as saying that $x \equiv -1 \pmod 3$ since saying a number is 2 more than one multiple of 3 is equivalent to saying that it is 1 less than another (the next, in fact).

So, with this basic notation in mind, let's try to avoid using it as far as possible (though it appears within the published MS to some extent). To begin with part (i):

Step 1. If a is not a multiple of 3, dividing by 3 leaves a remainder of 1 or 2, so then $a = 3j + 1$ or $a = 3j - 1$. Squaring then gives $a^2 = 9k^2 \pm 6k + 1 = 3(3k^2 \pm 2k) + 1$, which is clearly **shown** to be 1 more than a multiple of 3.

Step 3. We have $\left(\frac{a}{b}\right)^2 = \left(\sqrt{2} + \sqrt{3}\right)^2 = 5 + 2\sqrt{6}$, and $\left(\frac{a}{b}\right)^4 = \left(\sqrt{2} + \sqrt{3}\right)^4 = 49 + 20\sqrt{6}$. So we then have $\left(\frac{a}{b}\right)^4 = 10\left(\frac{a}{b}\right)^2 - 1$, a relationship clearly suggested by the result to which we are working, and this can then be rearranged to the desired form.

Step 4. Writing $a = 3k$ and rearranging, $b^4 = 90k^2b^2 - 81k^4$, which is clearly a multiple of 3. So b must also be a multiple of 3.

Step 5. It follows that if a is a multiple of 3 then a and b have 3 as a common factor, contradicting the assumption of step 2. This is not yet enough to conclude that $\sqrt{2} + \sqrt{3}$ is irrational, since it is also necessary to deal with the case when a is not a multiple of 3. If b is a multiple of 3, then so is a, by symmetric reasoning, so the final case is when neither is a multiple of 3. In this case we use Step 1: a^4, b^4 and a^2b^2 are all squares of numbers which are not multiples of 3, so each of these is 1 more than a multiple of 3. It follows that $a^4 + b^4$ is 2 more than a multiple of 3, but $10a^2b^2$ is 1 more than a multiple of 3, a contradiction.

In part (ii), it should be clear that a similar type of working will be employed **but** there is going to be at least one significant difference (see the comment required at the very end) so one must be careful to notice things work slightly differently in this slightly new situation.

As expected, then, this works similarly to part (i), but is more complicated. Numbers which are not multiples of 5 can have the form $5k \pm 1$ or $5k \pm 2$. Squaring each of these separately, their squares all have the form $5m \pm 1$, and the fourth powers all have the form $5n + 1$.

Expanding powers of $\sqrt{6} + \sqrt{7}$ and proceeding as for Step 3 of part (i) gives the relationship $a^4 + b^4 = 26a^2b^2$. If either a or b is a multiple of 5, a similar argument to part (i) shows that a and b have a common factor of 5, contradicting the assumption on a and b. If not, we can use what we know about squares and fourth powers to argue that the left-hand side is 2 more than a multiple of 5 but the right-hand side is either 1 more than, or 1 less than, a multiple of 5, giving a contradiction.

As a final thought, it is easy to demonstrate that a contradiction purely by considering multiples of 3 is not possible. This is because 26 is 2 more than a multiple of 3, so if a^2 and b^2 are each 1 more than a multiple of 3, both sides of the equation will be 2 more than a multiple of 3.

Question 8

This question deals with the process of substitution integration itself and how it can be used to show how things are related *functionally*.

In order to be entirely comfortable with this, one must first realise the roles being played by the various letters. At this level, it should be clear that

$$\int_1^2 \mathrm{f}(x)\,\mathrm{d}x = \int_1^2 \mathrm{f}(y)\,\mathrm{d}y = \int_1^2 \mathrm{f}(t)\,\mathrm{d}t \text{ etc.}$$

since the letter being used within the integrand is irrelevant to whatever is going on … it is called a *dummy variable* and is only there as an indicator. (Of course, these letters may contain within them a wider, geometric or graphical, significance to the whole integral, such as indicating the area between the curve and the x- or y-axes in the first two cases above; but the result itself – a number – is unchanged by the choice of letter. So, when one sees the given definition of f(x) in integral form, the upper limit of x is what now guarantees that the answer is indeed a function of x rather than of t and it has a rather different role.

For (i), the $\mathrm{f}(\frac{1}{2}x)$ in the given answer gives the big hint: setting $t = \frac{1}{2}u$ in the original integral should (and does) do the trick … but it is still very important to **show** that everything works out properly, such as demonstrating how the limits change from those for t to those for u.

In (ii), one can either take up where (i) finished … setting $v = u - 2$ (say) … or go back to the original integral and set $v = 2t - 2$, again being careful to show how everything works out.

In part (iii), the first real challenge is presented. What must be done *can* be seen intuitively, but there is also the perfectly sensible tactic – also requiring insight – that a linear substitution must be required, but it is not immediately clear what it is. So, try setting $u = at + b$ and forcing it through, then comparing the outcomes with what is needed. Doing this reveals that $a = 3$ and $b = 2$ do the trick admirably.

Having decided that a linear substitution worked in (iii), the jump now for part (iv) would seem to require a quadratic substitution. Even something simple like $y = u^2$ would give $\mathrm{d}y = 2u\,\mathrm{d}u$ in the substitution working and, at first glance, though it looks as if the numerator of the final integrand is going to cause trouble, we can write $\int \frac{u^2}{\sqrt{u^2+4}}\,\mathrm{d}u$ as $\int \sqrt{\frac{u^2}{u^2+4}}\,u\,\mathrm{d}u$ and it is now seen that the integrand is now starting to look remarkably similar to the form of (ii)'s integral, namely $\int \sqrt{\frac{X}{X+4}}\,\mathrm{d}X$, which has already been addressed. It is only the remaining details that must be dealt with, along with the fact that the final answer is not just a single f(--) thing.

Question 9

As with many mechanics questions, a sensibly large diagram is important; if not actually essential, then at least a considerable advantage. In this case, it is best to consider a 2-d vertical "cross-section" through ladder and box. There are really three bits of physical insight required when setting up this diagram. The first is that the reaction force between the box and the ladder acts at right angles to the ladder. The second is that, if the box topples, then it will rotate around the edge diagonally opposite the contact point between the ladder and the box. The third is that at the point of toppling the normal force from the ground on the box will act through this tipping point.

Once the diagram has been set up suitably, the question boils down to choosing which points to take moments about or which direction(s) to resolve in.

For part (i) we want to link the painter's mass and the reaction force, so taking moments about the point of contact of the ladder and the ground seems most sensible. Note also, that there is no interest in what happens to the ladder … taking moments about any other point would require a consideration of what is going on at the foot of the ladder.

This has not really brought in anything to do with the box toppling, so if we are looking for another crucial equation we should take moments about one of the base corners of the box. This is what is required in part (ii). The geometry of this can be quite tricky to deal with – the reaction force is not pointing in a convenient direction to deal with directly. However, some angle chasing will allow you to resolve it into vertical and horizontal components.

The final part brings in friction, so we will make use of the fact that the box topples before it slides so the friction force must not have gone past its maximum possible value. This means that $F \leq \mu N$. Resolving vertically and horizontally gives enough information about F and N to make progress with this expression, but then it is necessary to use some of the previous parts (as so often in STEP!) and some trigonometric identities to form the required expression.

Question 10

Although not essential in this question, it is still a good idea to begin with a clear diagram; especially since there is a slight twist to the traditional set-up here as the angle given is with the vertical (not the horizontal), so standard results need to be modified carefully to fit.

Given this slight variation, it is best in part (i) to derive the usual *trajectory equation* rather than "quote it", though there is no harm in the latter approach: remember that t plays no part in this equation so we find a simple equation involving t, rearrange it and eliminate t in the other equations. Then, substituting in $y = h$ when $x = h \tan\beta$, and making use of some trig. identities, leads to the required answer.

In part (a), it should be clear that one can use the results for the sum of the roots of a quadratic. This might also suggest the idea that the product of the roots may be useful later on. We now have lots of information involving cots so you might think that, in order to show that $\alpha_1 + \alpha_2 = \beta$, one needs only to track back to $\cot(\alpha_1 + \alpha_2) = \cot\beta$, and this indeed turns out to be the case. Although the compound angle formula for cot is not commonly used, it is easily derived from the compound angle formula for tan. There is one technical issue here, however, which is this: just because $\tan A = \tan B$ it does not follow that $A = B$. For example, $\tan 0 = \tan 180$. You will have to come up with an argument about why we can equate the arguments of the cot function in this case.

Up to this point we've not really made use of the fact that there are two solutions. Looking back, it is seen that the equation (*) is a quadratic in c so there is a common method for determining how many solutions it has.

In part (ii), an expression for the greatest height is needed, but we must be careful that u and α are fixed in the question, so we are looking for the greatest height over the time of flight. This will occur when the vertical velocity is zero. We can then get the required inequality by comparing this value to the known achieved height h.

Question 11

In the situation described in part (i), in each round, a decision is made if the coins are the same way up (with probability $p^2 + q^2$) and the process continues if they are different (with probability $pq + qp = 2pq$). So the probability that they continue $(n - 1)$ times and then make a decision on the nth round is $(2pq)^{n-1}(p^2 + q^2)$.

The probability that they don't make a decision on the first n rounds is $(2pq)^n$, and so the probability they do is $1 - (2pq)^n$. Now $2pq = 2p - 2p^2$, and completing the square will show that this is at least $\frac{1}{2}$, giving the required bound.

In part (ii), a decision is made in the first round if all coins are the same; i.e. with probability $p^3 + q^3$. In order to make a decision on the second round, there are two possibilities. One possibility is that the first round has two heads and a tail, the tail is turned over to become a head, and the other two coins are tossed, both resulting in heads. This has probability $3p^4q$. The other possibility is the same with heads and tails exchanged, so has probability $3pq^4$.

So we wish to find the minimum value of $p^3 + q^3 + 3p^4q + 3pq^4$. One way to do this is to write it as a function of p only, using the fact that $q = 1 - p$, and to differentiate. If done correctly, this will give three turning points at $p = 0$, $p = \frac{1}{2}$ and $p = 1$. By considering the second derivative we find that $p = 0$ and $p = 1$ are local maxima, but $p = \frac{1}{2}$ is a local minimum; so it follows that the minimum value for $0 \leq p \leq 1$ is at $p = \frac{1}{2}$, and evaluating the function at this point gives $\frac{7}{16}$.