STEP 2009 Paper I Principal Examiner's Report

There were significantly more candidates attempting this paper again this year (over 900 in total), and the scores were pleasing: fewer than 5% of candidates failed to get at least 20 marks, and the median mark was 48.

The pure questions were the most popular as usual; about two-thirds of candidates attempted each of the pure questions, with the exceptions of question 2 (attempted by about 90%) and question 5 (attempted by about one third). The mechanics questions were only marginally more popular than the probability and statistics questions this year; about one quarter of the candidates attempted each of the mechanics questions, while the statistics questions were attempted by about one fifth of the candidates.

A significant number of candidates ignored the advice on the front cover and attempted more than six questions. In general, those candidates who submitted answers to eight or more questions did fairly poorly; very few people who tackled nine or more questions gained more than 60 marks overall (as only the best six questions are taken for the final mark). This suggests that a skill lacking in many students attempting STEP is the ability to pick questions effectively. This is not required for A-levels, so must become an important part of STEP preparation.

Another "rubric"-type error was failing to follow the instructions in the question. In particular, when a question says "Hence", the candidate *must* make (significant) use of the preceding result(s) in their answer if they wish to gain any credit. In some questions (such as question 2), many candidates gained no marks for the final part (which was worth 10 marks) as they simply quoted an answer without using any of their earlier work.

There were a number of common errors which appeared across the whole paper. These included a noticeable weakness in algebraic manipulations, sometimes indicating a serious lack of understanding of the mathematics involved. As examples, one candidate tried to use the misremembered identity $\cos\beta = \sin\sqrt{1-\beta^2}$, while numerous candidates made deductions of the form "if $a^2+b^2=c^2$, then $a+b=c$" at some point in their work. Fraction manipulations are also notorious in the school classroom; the effects of this weakness were felt here, too.

Another common problem was a lack of direction; writing a whole page of algebraic manipulations with no sense of purpose was unlikely to either reach the requested answer or gain the candidate any marks. It is a good idea when faced with a STEP question to ask oneself, "What is the point of this (part of the) question?" or "Why has this (part of the) question been asked?" Thinking about this can be a helpful guide.

One aspect of this is evidenced by pages of formulæ and equations with no explanation. It is very good practice to explain *why* you are doing the calculation you are, and to write sentences in English to achieve this. It also forces one to focus on the purpose of the calculations, and may help avoid some dead ends.

Finally, there is a tendency among some candidates when short of time to write what they *would* do at this point, rather than using the limited time to actually try doing it. Such comments gain no credit; marks are only awarded for making progress in a question.

STEP questions do require a greater facility with mathematics and algebraic manipulation than the A-level examinations, as well as a depth of understanding which goes beyond that expected in a typical sixth-form classroom. It is wise to heed the sage advice on the

STEP Mathematics website, `http://www.admissionstests.cambridgeassessment.org.uk/adt/step`:

> From the point of view of admissions to a university mathematics course, STEP has three purposes. ... Thirdly, it tests motivation. It is important to prepare for STEP (by working through old papers, for example), which can require considerable dedication. Those who are not willing to make the effort are unlikely to thrive on a difficult mathematics course.

Question 1

The start of this question was attempted successfully by the majority of candidates, though a significant number failed to provide any justification of why $3^2 \times 5^3$ has *only* 10 factors. Disturbingly, many candidates expanded $3^2 \times 5^3 = 1125$ and then attempted to factorise the latter, rather than noticing that it is much easier to work from the already-factorised form. Few of these candidates made any further progress in the question.

Only about half of the candidates were able to make any significant progress on the second half of part (i); this required deducing a general formula for the number of factors. The idea was that by working through the specific example given at the beginning, it would be realised that every factor has the form $3^a \times 5^b$, and then a simple counting argument would do the job. There were numerous incorrect formulæ used at this point; candidates did not seem to understand that a formula cannot just be pulled from thin air, but requires some justification.

Of those candidates who reached a correct formula, several became stuck attempting to solve the equation $(m + 1)(n + 1) = 12$; expanding the left hand side is likely to lead nowhere (especially in part (ii) of the question). The form $mn + m + n = 11$ used by some candidates was particularly unhelpful, as it makes it very hard to justify that all solutions have been found. Furthermore, to answer this part successfully, some systematic approach is required, as we want the *number* of solutions to this equation. Several candidates slipped up at this point by failing to realise that 0 is, in fact, an integer, giving $m = 0$, $n = 11$ and vice versa as solutions.

Those candidates who understood part (i) generally made good progress on part (ii). However, a sizeable number of candidates failed to write down the most general form for N, often writing things likes $N = p^m \times q^n$ or $N = 3^m \times 5^n$. Also, many failed to consider all possible factorisations of 428. Furthermore, they needed to give some justification for the choice of primes and exponents corresponding to each factorisation of 428, and this was often lacking.

Nevertheless, there were some very good answers to this question, and those who understood the principles involved generally made very good progress.

Question 2

This was the most popular question on the paper by far, being attempted by about 90% of the candidates. It was also found to be one of the most straightforward questions; almost 60% of the attempts successfully completed the first two parts of the question, gaining 10 of the available 20 marks in the process, and about 20% of the attempts gained 15 marks or above.

In the first part, most candidates understood the procedures required to find the tangent to a curve given implicitly, although some went to the effort of taking the cube root of the original equation. Disturbingly, a few of them could not do this correctly, and wrote $y = x + a + b$. A much more common and equally concerning error was to differentiate the equation and correctly deduce that $\mathrm{d}y/\mathrm{d}x = x^2/y^2$, and then use *this* as the gradient without substituting in the values of x and y at the point under consideration. The resulting purported equation for the tangent,

$$y - b = \frac{x^2}{y^2}(x + a),$$

is clearly not a linear equation. The candidates then rearranged it and figured out that some substitutions were necessary to reach the given answer, and fudged their way to it. It is always recommended that students go back through their answer to find the source of the error, rather than trying to fudge, as the latter is almost certainly going to get no credit.

For the second part of the question, most candidates realised that they needed to solve the two equations simultaneously (although there were some substitution errors encountered at this point). The process of substituting one equation into another and rearranging to reach the given cubic was done well, although there was a reasonable amount of fudging present here, too. Also, some candidates attempted to perform the substitutions while keeping the as and bs present in the equations; it is much simpler to substitute at the start, so that one only needs to work with two variables (x and y).

A sizeable number of candidates gave up at this point. This is unfortunate, as with the question saying “Hence”, the obvious thing to do is to solve the cubic. Even without any idea of how to go beyond this, actually solving the cubic might have given a clue. Those who attempted this step were reasonably successful, primarily once they realised that $x = -1$ is a root of the cubic. Again, though, a number of candidates were let down by poor algebraic manipulation at this point.

Finally, having reached the two solutions of the cubic, many candidates floundered, assuming that $x = \frac{17}{7}$ could not possibly give them a solution. Unfortunately, $x = -1$ does not help, either. Those who did press ahead with $x = \frac{17}{7}$ often struggled to determine the corresponding value of y, frequently leading to some very long pieces of arithmetic as they attempted to calculate $x^3 + a^3 + b^3$ and then take the cube root, rather than using the tangent equation $4y - x = 9$. Often, the long arithmetical approaches failed to reach the correct answer because of a simple slip, which was a shame. A significant number of candidates did, however, manage to correctly reach the final conclusion.

As mentioned in the introduction, many candidates simply quoted a solution to this final equation (namely $6^3 = 5^3 + 4^3 + 3^3$), and this was given no credit, as the question had specifically said “Hence”.

Question 3

This was found to be one of the hardest questions on the whole paper, with a median mark of 2 and a mean mark of 3.9. A relatively small number of candidates appreciated the subtleties involved and rest consequently produced fairly nonsensical answers.

For the first half of part (i), relatively few candidates successfully attempted to relate the two given equations. It was encouraging that the majority of candidates were able to correctly quote the quadratic formula (especially as it is given in the formula book!), but there was considerable difficulty in evaluating it in this case. It seemed that candidates were also confused by the fact that the formula is given for the equation $ax^2 + bx + c = 0$, whereas here the constant term involves the variable a. Other candidates became confused about the distinction between negative roots and non-real (complex) roots; negative numbers are real!

Of those who related the two equations, very few appreciated that the solutions of an equation involving a square root may be different from the corresponding squared equation. This was compounded by many students asserting that $1 + 4a < 0$ if and only if $a < 0$ (where the "if and only if" may have been implied). Few understood that proving that there was no real solution when $a < 0$ was insufficient to show that there was one real solution when $a > 0$.

A small number of candidates proved this part of the question using an effective graphical method, which was very impressive.

Interestingly, the second half of part (i) proved to be more straightforward for a number of candidates, possibly because there were no square roots involved. However, some were confused and thought that the conditions on a in the first half of part (i) would be inherited in the second half.

Only a handful of students made any significant attempt at part (ii), and of those, some used algebraic methods following the ideas of part (i) while others used graph-sketching approaches, often quite successfully.

Question 4

This question was found to have a moderate level of difficulty. There were some very good attempts as well as some very poor ones.

The initial part of the question confused some candidates; the sketches produced showed that the statement: "The largest and smallest angles of the triangle are α and β, respectively" caused a surprising amount of difficulty. The use of "respectively", in particular, is standard mathematical language and should be recognised—a number of candidates swapped the two angles. (It might be that some genuinely thought that the angles were the other way round, but most simply probably misunderstood the question.) There were also a fair number of candidates who labelled the angle opposite p as either α or β, which prevented them from making much further progress in the question.

Most candidates were able to correctly use the cosine rule in the context of this question, which was very pleasing. Manipulating it to deduce the required equation was much harder, though: as usual, the lack of algebraic fluency was the sticking point. There were many careless errors, including sign errors, transcription errors and the like. Also, a number of candidates did not simplify the algebraic fractions when they could, thereby making their task even more challenging. Many demonstrated that they did not understand how to add fractions, and the monstrous messes which resulted were quite painful to see.

The straightforward method for approaching this question was to substitute expressions for $\cos\alpha$ and $\cos\beta$ into the left and right hand sides of the required equation and to show that they turned out to be equal. A significant number of candidates did not realise that this would be useful, and instead produced pages of algebra leading nowhere.

In the second part of the question, a number of candidates who had not succeeded in the first part nonetheless realised that they could use the given result to solve the second part, and some were successful at this.

There were some good attempts at the second part. The sticking points were once again mostly algebraic: some did not know the double angle formulæ or they used incorrect expressions for them; of those who did know them, many then erred in their expansion of the brackets. The next sticking point was solving the cubic; some did not attempt to do so, while others made algebraic slips. Those who did succeed in deducing $\cos\beta$ usually went on to correctly find the side lengths and hence the required ratio.

Question 5

This was the least popular of the pure questions, being attempted by barely one-third of the candidates. The marks followed suit; one third of the attempts scored 0 while another third scored 1: these were generally candidates who tried the question and then quickly gave up. Of those who got past the initial stages of the question, the significant majority gained full marks or very close to it.

The question essentially required only one idea, and was then a straightforward application of "connected rates of change" or the chain rule. The idea needed was that r, h and ℓ are related via Pythagoras, so that if A is fixed (i.e., it's a constant), then h can be expressed as a function of r.

Those students who realised that h is a function of r mostly got the whole question correct; those who regarded h as a constant got nowhere.

Question 6

This was found to be another hard question. While the integral in part (i) looks deceptively easy, many candidates had little idea how to approach it. Simplifying algebraic fractions via division is a standard A-level technique, and students should recognise that this is a useful approach when attempting to integrate rational functions. Others successfully used the substitution $u = x + 1$ to simplify the integral.

One very common approach to part (i) was to use integration by parts. While this is often a useful technique, here it proves fruitless, and many candidates spent much effort without appearing to realise that they were not getting anywhere. Some thought they were making progress, but had actually misapplied the parts formula.

Other candidates tried manipulating the integral in completely incorrect ways, for example by trying to use the formula for the derivative of a quotient, or the derivative of a logarithm, or stating things like $\int fg\,\mathrm{d}x = f\int g\,\mathrm{d}x$.

About one third of attempts scored zero on this question, and fewer than one third of the attempts progressed beyond part (i). The main sticking point was part (ii), where few of the candidates realised that the substitution was $u = 1/x$. There were two major hints in the question itself: firstly that the limits were $1/m$ to m, and these remained unchanged, and secondly that the $1/x^n$ changed into u^{n-1}, which is about the same as u^n. Many candidates failed to spot this, and spent time trying other substitutions unsuccessfully. Even among those who did realise what they needed to do, many seemed unsure how to handle the limits correctly when performing a substitution.

Some of part (iii) could be done even without successfully completing parts (i) and (ii). Very few candidates realised this, though, and did not even attempt part (iii) if they had got stuck earlier. Of those who did try part (iii), some used partial fractions instead of the earlier results (which was perfectly acceptable). Some of these were very successful, whereas others did not fully understand partial fractions. The most common error was writing

$$\frac{1}{x^3(1+x)} = \frac{A}{x^3} + \frac{B}{1+x},$$

which cannot possibly get the correct answer.

Other candidates applied the results of parts (i) and (ii) with varying degrees of success. A common error was being careless about the limits when trying to use the result of part (ii) in the integral in (iii)(b); the limits are not of the form $1/m$ to m, and so more care is needed.

Question 7

Candidates found this question very approachable, and was one of the highest scoring questions on the paper. The standard integral at the start was generally well done, and about three-quarters of candidates gained full marks for it. Common mistakes included incorrect notation for integration by parts, forgetting to use the $[\cdots]_0^{2\pi}$ notation when using parts with a definite integral, and of course making sign errors.

The trigonometrical identity in part (i) was done perfectly by most candidates who tried it. The following integral was usually performed successfully, and over half of all attempts got this far. The errors which occurred were similar to above, along with the standard inability to add and simplify fractions.

The integral in part (ii) caused significantly more problems. Most candidates wrote down the product formula for $\sin A \sin B$, but then many got stuck: they could not see how to proceed. It is very common in STEP questions for the later parts of a question to depend on the ideas developed earlier parts. In this case, we end up after expansion with two integrals like the one seen in part (i), but it seems as though this was not noticed.

The other common problem which occurred in this part was missing or losing the factor of $\frac{1}{2}$ in this part; this is another example of algebraic carelessness costing marks.

Question 8

The first part of this question was generally done very well, but the second half stumped most candidates. Most students confidently used the double angle formula for $\tan 2\alpha$, and most realised that the line $y = \frac{4}{3}x$ is tangent to the circle because it is at an angle of 2α to the x-axis and passes through the origin. There were some candidates who misremembered or misapplied the double angle formula, or were unable to explain why the circle touches the line. In questions such as these, a good sketch is very helpful.

The second part of the question was done relatively poorly in comparison; fewer than a quarter of the candidates gained more than one mark for it. Even after drawing a good sketch, many candidates showed little idea of how to proceed. They wrote pages of calculations with no direction, and so gained no credit. It is not worth performing calculations without a strategy; how will knowing the coordinates of the vertices, for example, help find the value of t? (This was a very common calculation, perhaps because it is something the candidates are very comfortable with.) About half of the attempts noted that the two given lines were perpendicular, but then did not do anything useful with this fact.

Many students attempted to substitute the equation of the line $4y+3x = 15$ into the equation of the circle to eliminate y, but then went on to eliminate t by asserting that $x = 2t$. This cannot succeed, as the point of tangency with $4y+3x = 15$ clearly does not lie on the vertical line through the centre of the circle.

Some candidates, though, successfully used the ideas of part (i) to find the equation of the angle bisector of the x-axis and the side AB, and then found the value of t which had $(2t, t)$ lying on this bisector.

Another very common successful approach was to find the distance of the point $(2t, t)$ from the line $4y + 3x = 15$ using the formula given in the formula book for the distance of a point from a line.

It is encouraging to see a significant number of candidates offering creative (and correct) solutions to a problem such as this.

Question 9

The start of this question was found to be fairly approachable. Most candidates were able to draw a correct sketch of the situation. However, a significant number became unstuck at this very early stage, getting the directions incorrect (for example, having Q at an angle of β *above* the horizontal) or swapping P and Q or worse.

The next challenge was to correctly apply the "suvat" equations (motion with constant acceleration) to this situation. This was done successfully for P by most candidates, although some made sign errors; it is *vital* to *always* indicate which direction is positive in problems such as these. The motion of Q, however, caused many problems: what is s when the motion does not start from the ground? Some candidates determined the displacement from the starting position and then tried adding or subtracting d, others introduced d later on, while others seemed to ignore d entirely. The general principle is to be clear about the meaning of one's symbols; being explicit whether s is the vertical displacement upwards or downwards from the ground or from D would have averted many of the problems.

Most candidates were able to show that $\cos\alpha = k\cos\beta$, but the quadratic proved too much for the majority of them. There were many attempts, but because of the earlier errors in calculating the displacements of the particles or mistakes in algebraic manipulations, the quadratic did not appear. Many did not realise that it was necessary to use the earlier result about the cosines, while others asserted that $\sin\alpha = k\sin\beta$.

About one-fifth of the attempts at the question reached the final part of the question. Of those, most were able to determine a formula for T, but many became stuck at this point. Once again, there were frequent attempts to rearrange formulæ, but few were successful.

Only a handful of candidates managed to progress beyond the requirement that $\sin^2\alpha \leqslant 1$ to reach the desired conclusion.

Question 10

This was (marginally) the most popular of the mechanics questions, attempted by close to one third of the candidates. Those who attempted it generally gained reasonably good marks, and there were a few very pleasing answers.

In the first part, however, there was a clear lack of understanding among the candidates of the basic principles of mechanics. While they showed a good understanding of resolving forces, the once commonplace mantra of "Apply Newton's Second Law to each particle, then combine as needed" seemed to be all but forgotten: too few candidates indicated the tension in the string on their diagram. Many therefore treated the two particles as a single system which could be regarded as one particle with various forces, but offered no justification for this assertion. In this case, such an approach does work, but had the pulley not been smooth, it would not have done. Furthermore, applying Newton's Second Law at each particle is a much more straightforward, less error-prone approach, and may have saved a number of candidates from unfortunate sign errors.

Those who managed to write down a correct equation of motion were generally quite confident to show the required inequality. However, most fudged the argument, saying things like "the force on M is bigger than the force on m so $Mg\sin\alpha > mg\cos\alpha$," which does not show an appreciation of the effect of tension.

The trigonometric manipulations required at the start of part (ii) were done fairly well, and those who got this far were very comfortable differentiating the expression using the quotient rule. However, having done so, very few were then able to perform the requisite manipulations on the derivative to reach the result that $\tan^3\alpha = 2$, and hence deduce the relationship between M and m.

It was also disappointing that of those good candidates who had reached this point, only three realised that they needed to justify that this was a (local) maximum; it should be standard for students to ascertain the nature of any stationary point they have found if they require the maximum or minimum of a function.

Question 11

This was found to be another very hard question, with most attempts being fragmentary; almost half of students gained 2 or less, though the handful who made it beyond half way achieved either 19 or 20 marks.

The first part of the question required correctly stating the laws of conservation of momentum and restitution, together with the formula for kinetic enery. Almost all candidates realised this and attempted to write down the appropriate formulæ. It was particularly disappointing, though, that so many candidates struggled to do even this correctly. Most were comfortable with the conservation of momentum, but having learnt the law of restitution in the form:

$$e = \frac{\text{speed of separation}}{\text{speed of approach}},$$

they were at a loss when all of the velocities were in the same direction, and consequently introduced various sign errors. A number of candidates remembered the formula upside-down (as speed of approach over speed of separation), and this was invariably fatal.

The next stumbling block was the formula for loss of kinetic energy. A sizeable minority of students wrote that the loss of KE in the first particle is $\frac{1}{2}m(u-p)^2$ instead of $\frac{1}{2}mu^2-\frac{1}{2}mp^2 = \frac{1}{2}m(u^2-p^2)$ and similarly for the second particle; this is a disturbing error, although it is not clear whether it arose because of a belief that $a^2 - b^2 = (a-b)^2$ or a lack of thought about what loss of KE means.

A number of students compounded their problems by confusing M and m, often due to careless handwriting. It is a cardinal principle of written mathematics that distinct symbols should look obviously distinct!

Those students who made one of these errors generally failed to make any significant further progress with the question.

The key step in part (i) was to manipulate the formula for loss of KE to eliminate M and q, making use of the other two equations to achieve this. Most candidates who had reached this point succeeded in doing this and reaching the required formula. Some made their lives more difficult by expanding every bracket fully; it is a valuable skill to recognise when this would be helpful and when not. For the final deduction in part (i), the instruction to "deduce" the result $u \geqslant p$ meant that no credit was given to those students who simply stated "it's obvious"; some justification on the basis of the formula for loss of KE was required.

The majority of candidates did not even attempt part (ii), which is a shame, as the ideas were essentially the same as in part (i), albeit with a little more sophistication: the task was to eliminate some of the variables. The first half required writing down the word statement in symbols and eliminating either M or M, while the second half required elimination of p and q followed by some rearrangement. Any attempt to perform these eliminations would have led to the correct results.

Question 12

This was the more popular of the probability questions, and was attempted by almost a quarter of the candidates. It gained the highest scores on the whole paper: the median mark was 14.

The introductory part caused some problems. This is a standard result which should have been recognised as such. Some candidates began with $(x+y)^2$ and became stuck. Others attempted to prove the result by induction, which will be challenging as x and y are any real numbers.

Part (i) was generally done extremely well, with many candidates drawing an appropriate tree diagram. However, some candidates failed to add the probabilities (more commonly when they had not drawn a tree diagram), asserting things such as P(same colour) $= \frac{ab}{(a+b)^2}$ instead of $\frac{2ab}{(a+b)^2}$.

Part (ii) caused more problems. Many candidates decided that it would be useful to expand $(a+b+c)^3$ and wasted a lot of time doing so. Another common issue was not adequately justifying the probability stated in the question; the arguments proposed for counting the number of possibilities were often weak or confused. Students would have done well to either draw a tree diagram (possibly abbreviated) or to explicitly list all of the possibilities. While taking time, it would have ensured that they reached the correct answer.

Most of the candidates were able to calculate the probabilty of all three being the same colour correctly, although some left out the factor of 3.

The last part of the question caused the most difficulty. Some suggested that since there were 18 ways of having exactly two the same colour and 3 ways of having them all the same colour, the former was 6 times more likely than the latter. This totally ignored the probabilities of these events and the work done earlier, as well as the "deduce" in the question. Others assumed $a \leqslant b \leqslant c$ and then wrote $a + x = b$, $a + y = c$ to try to relate it to the initial inequality, but struggled to get further. Nevertheless, there were a good number of students who correctly related the earlier inequality to the new context and succeeded in making the required deduction.

Question 13

This was a very unpopular question and most attempts were fragmentary, while a small number were essentially perfect. Most attempts were very weak at conveying their combinatorial ideas clearly, leading to confused and incorrect arguments. A few candidates completely misunderstood what was being asked and tried to answer something different entirely.

For part (i), very few candidates realised that the best way to begin part (i) was to regard the girls as a block dividing the boys into two sections. Even those who did spot this still struggled to deduce the required probability. Most of those who counted the total number of arrangements as $(n + 3)!$ and the number of arrangements of the girls as $3!$ failed to see that the girls could be positioned as a block in only $(n + 1)$ positions.

For part (ii), most offerings were vain attempts to reach the given answer, but demonstrated no understanding of the problem.