STEP 3 2015 Hints and solutions

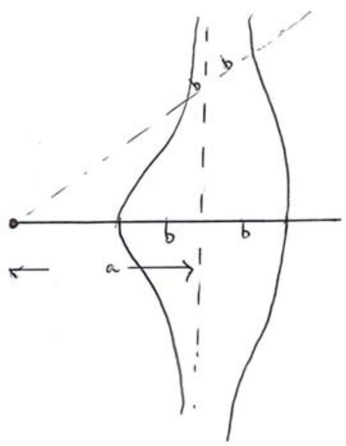
1. The first result can be obtained by simplifying the LHS and then writing it as $\int_0^\infty u \frac{u}{(1+u^2)^{n+1}} du$ and integrating this by parts. To obtain the evaluation of I_{n+1}, the first result can be re-arranged to make I_{n+1} the subject, and then iterating the result to express it in terms of I_1 which is a standard integral. The expression can be tidied by multiplying numerator and denominator by $(2n)(2n-2)\dots(2)$. The first result for (ii) is obtained by means of the substitution $u = x^{-1}$, the second by adding the two versions of J, and the third by the substitution $u = x - x^{-1}$, being careful with limits of integration and employing symmetry. Part (iii) is solved by expressing the integrand as $\frac{x^{-2}}{((x-x^{-1})^2+1)^n}$ and then employing first part (ii) then part (i) to obtain I_n, which is $\frac{(2n-2)!\pi}{2^{2n-1}\left((n-1)!\right)^2}$.

2. Part (ii) is the only false statement, and a simple counter-example is $s_n = 1$ and $t_n = 2$ for n odd, and $s_n = 2$ and $t_n = 1$ for n even. Part (i) $m = 1000$ is a suitable value, then $1000 \leq n$ and as n is positive, the inequality can be multiplied by it giving the required result. Part (iii) requires the use of the definition twice with values m_1 and m_2 say, and then using $m = \max(m_1, m_2)$. For part (iv), we can choose $m = 4$, and an inductive argument such as

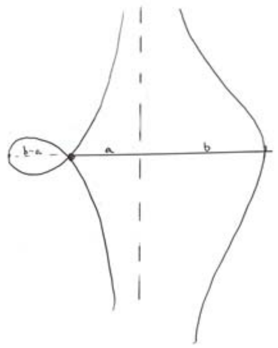
$(k+1)^2 = \left(1+\frac{1}{k}\right)^2 k^2 \leq \left(1+\frac{1}{4}\right)^2 k^2 < 2k^2 \leq 2 \times 2^k = 2^{k+1}$ works.

3. The part (i) inequality for $\sec\theta$ can be obtained by making r the subject of the formula as $r = a\sec\theta \pm b$ and invoking $a > b$ remembering that $r < 0$ is not permitted.

Then the points lie on a conchoid of Nicomedes with *A* being the pole (origin), *d* being *b*, and L being the line $r = a\sec\theta$ ("x" $= a$). A sketch is



In part (ii), the extra feature is the loop as specified with end-points at the pole corresponding to $\sec\theta = \frac{-b}{a}$. A sketch is



So in the given case, the area is given by $2\times\frac{1}{2}\int_{\frac{2\pi}{3}}^{\pi}(\sec\theta+2)^2 d\theta$ which is $\frac{4\pi}{3}+\sqrt{3}-4\ln\left|2+\sqrt{3}\right|$.

4. Part (i) is imply shown by considering the image of the function $f(z)=z^3+az^2+bz+c$ as $z\to\pm\infty$ and then observing that the function is continuous and exhibits a sign-change. Part (ii) can be approached by writing $z^3+az^2+bz+c=(z-z_1)(z-z_2)(z-z_3)$ giving $a=-S_1$, $=\frac{{S_1}^2-S_2}{2}$, which can be obtained by considering $(z_1+z_2+z_3)^2$ and the required result for $6c$ which can be neatly obtained by considering $f(z_1)+f(z_2)+f(z_3)=0$.

Writing $z_k=r_k(\cos\theta_k+i\sin\theta_k)$ for $k=1,2,3$, employing de Moivre's theorem, the three sums imply the reality of S_1 , S_2 , and S_3 , and hence a , b , and c which by virtue of the result of part (i) yields the reality of z_1 , z_2 , or z_3 and hence the required result. The final result can be considered as two cases, the trivial one of all three roots being real, and the one where the other two are complex. The latter can be shown to give the required result by considering the real and imaginary parts of the roots of a real quadratic.

5. (i) Step 3 is straightforward on the basis of steps 1 and 2, noting that no lowest terms restriction need be made in part 1. Step 5 requires that the given expression is a positive integer as well as well as being integer when multiplied by root two. Step 6 requires justification that $\sqrt{2}-1<1$.

(ii) The rationality of $2^{2/3}$ on the basis of $2^{1/3}$ being rational is simply obtained by squaring the latter, and the opposite implication can be made by squaring the former or dividing 2 by the former. To construct the similar argument, let the set T be the set of positive integers with the following property: n is in T if and only if $n2^{1/3}$ and $n2^{2/3}$ are integers, and taking t to be the smallest positive integer in that set, consider $t\left(2^{2/3}-1\right)$ to produce the argument.

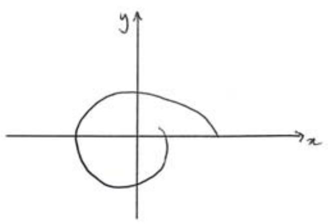
6. Treating the equations for u and v as simultaneous equations for w and z , one finds that $w=\frac{u\pm\sqrt{2v-u^2}}{2}$ and $z=\frac{u\mp\sqrt{2v-u^2}}{2}$ which demonstrates that if $u\in\mathbb{R}$ and $u^2\leq 2v$, i.e. $v\in\mathbb{R}$, w and z are real. If w and z are real, then u and v are (trivially) and $2v-u^2=(w-z)^2\geq 0$.

In (ii), the first two equations yield $3wz=1$, making it possible to write the third equation as $u(u^2-1)=\lambda(u-1)$ which has an obvious factor of $(u-1)$ leading to $u=1$ or $u=\frac{-1\pm\sqrt{1+4\lambda}}{2}$ from the quadratic equation. If one of the solutions of the quadratic equation gives the same root $u=1$, then there are not three possible values, i.e. if $\lambda=2$. From the first part of the question, for w and z to be real, we would want u to be real, $u^2-\frac{2}{3}$ to be real, and $u^2\leq 2\left(u^2-\frac{2}{3}\right)$, in other words $u^2 \geq \frac{4}{3}$. So a counter-example could be $u = 1$ giving $2w^2 - 2w + \frac{2}{3} = 0$ which has a negative discriminant.

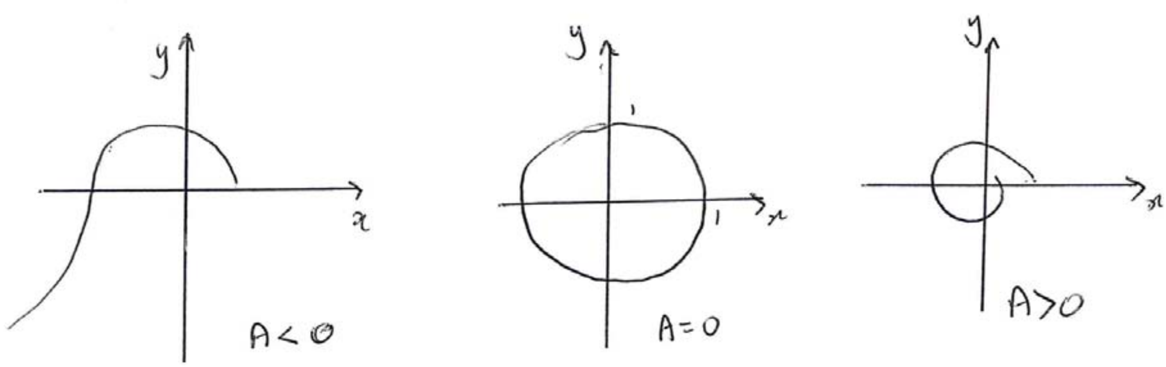
7. The opening result is simply achieved by following the given explanation for $D^2 f(x)$ with $f(x) = x^a$. Parts (i) and (ii) can both be shown usual the principle of mathematical induction with initial statements

"Suppose $D^k P(x)$ is a polynomial of degree r i.e. $D^k P(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_0$ for some integer k ." and "Suppose $D^k (1-x)^m$ is divisible by $(1-x)^{m-k}$ i.e. $D^k (1-x)^m = f(x)(1-x)^{m-k}$ for some integer k , with $k < m - 1$." Part (iii) is obtained by expressing $(1-x)^m$ in sigma notation (by the binomial theorem), then carrying out $D^n (1-x)^m$ using the idea in the stem, and finally invoking the result of part (ii) and then substituting $x = 1$.

8. Transforming the differential equation in part (i) is made by substituting for x and y as given, for $\frac{dy}{dx}$ using $\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta$ and a similar result for $\frac{dx}{d\theta}$, and then simplifying the algebra by multiplying out and collecting like terms bearing in mind that a factor r can be cancelled as $r \neq 0$. The transformed equation can be solved by separating variables or using an integrating factor, to give $r = ke^{-\theta}$, the sketch of which is



The same techniques for part (ii) yields a differential equation $r - r^3 + \frac{dr}{d\theta} = 0$ which is solved by separating the variables and then employing partial fractions giving a variety of possible solution sketches



($A \neq -1$ but it is possible to consider $A < -1$ in which case $\theta < 0$)

9. Whilst the first part can be obtained otherwise, the simplest approach is by conserving energy, when $\frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 + \frac{\lambda}{2a}\left(\sqrt{a^2 + x^2} - a\right)^2$ leads to the required answer simply. x_0 is found by setting $\dot{x} = 0$ leading to $x_0 = \sqrt{\frac{v^2}{k^2} + \frac{2av}{k}}$. The acceleration can be found by applying

Newton's 2nd Law or by differentiating the equation found in the first part, and substituting leads to the result $-\frac{kv\sqrt{v^2+2akv}}{v+ak}$ for the acceleration when $x = x_0$. Treating the equation found in the first part as a differential equation for x in terms of , the expression for the period is

$\tau = 4\int_0^{x_0} \frac{1}{\left[v^2-k^2\left(\sqrt{a^2+x^2}-a\right)^2\right]^{\frac{1}{2}}} dx$. Making the substitution $u^2 = \frac{k\left(\sqrt{a^2+x^2}-a\right)}{v}$, leads to

$x = \sqrt{2kav}\frac{u}{k}\left(1+\frac{v}{2ka}u^2\right)^{\frac{1}{2}}$, which making a binomial expansion and using the given condition to approximate $x \approx \sqrt{2kav}\frac{u}{k}$ results in the final given expression.

10. The position vector of the upper particle is $\begin{pmatrix} x + a\sin\theta \\ y + a\cos\theta \end{pmatrix}$ so differentiating with respect to time yields the correct velocity and acceleration which gives the second result when used in Newton's second law resolving horizontally and vertically. The corresponding equations are $m\begin{pmatrix} \ddot{x} - a\,\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta \\ \ddot{y} + a\,\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta \end{pmatrix} = T\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} - mg\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the other particle by merely swapping the direction of the tension and the displacement from the midpoint. The deductions are obtained by adding the two equations of motion, and in the case of $\ddot{\theta}$, subtracting the two equations and then eliminating T between the equations for each component. Using $\begin{pmatrix} \dot{x} + a\,\dot{\theta}\cos\theta \\ \dot{y} - a\,\dot{\theta}\sin\theta \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ and a similar equation for the lower particle, initial values of $\dot{y}$ and $\dot{\theta}$ can be found and then the time for the rod to rotate by $\frac{1}{2}\pi$ can be obtained and substituted in the displacement equation under uniform acceleration to obtain the final result.

11. (i) The first result is obtained, as the question prompts, by considering a component of force on the rod due to P, and taking moments about the hinge to find that that component is zero with the consequence that any force exerted on the rod by P must be parallel to the rod. Bearing in mind the horizontal acceleration of P towards the centre of the circle it describes, resolving perpendicular to the rod and writing the equation of motion for P leads directly to the given equation with the stated substitution being made. The force exerted by the hinge on the rod is along the rod towards P and resolving vertically for forces on P and rearranging gives $F = mg\sec\alpha$.

(ii) Taking moments for the whole system about the hinge gives

$$m_1 g\, d_1 \sin\beta + m_2 d_2 g\sin\beta = m_1 d_1 (r - d_1\sin\beta)\omega^2\cos\beta + m_2 d_2 (r - d_2\sin\beta)\omega^2\cos\beta$$

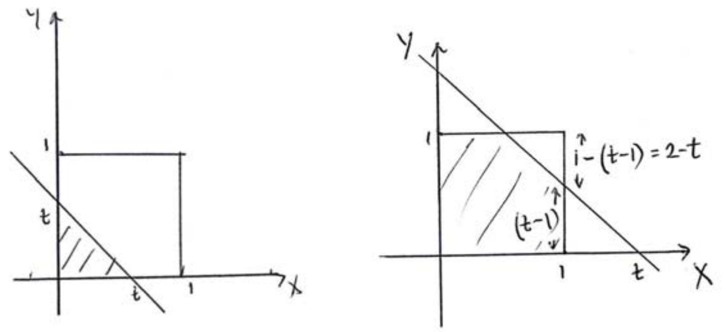
which can be rearranged into the required form with $b = \frac{m_1 {d_1}^2 + m_2 {d_2}^2}{m_1 d_1 + m_2 d_2}$.

12. (i) The required probability generating function is $G(x) = \frac{1}{6}(1 + x + x^2 + x^3 + x^4 + x^5)$ and it is simple to write down the probability distribution function of S_2 and hence of R_2 and arrive at the same pgf. As a consequence, it can be argued that the pgf for R_n is also $G(x)$ and so the required probability is $^1/_6$.

(ii) $G_1(x) = \frac{1}{6}(1 + 2x + x^2 + x^3 + x^4) = \frac{1}{6}(x + y)$. $G_2(x)$ would be $\left(G_1(x)\right)^2$ except that the powers must be multiplied congruent to modulus 5 , and it can be shown that $xy = y$ and $y^2 = 5y$ so obtaining the required result for $G_2(x)$. Obtaining $G_n(x) = \frac{1}{6^n}\left(x^{n-5p} + \frac{6^n - 1}{5}y\right)$

where p is an integer such that $0 \leq n - 5p \leq 4$, and the probability that S_n is divisible by 5 will be the coefficient of x^0 which in turn is the coefficient of y as required. If n is divisible by 5, the probability that S_n is divisible by 5 will be $\frac{1}{5}\left(1 + \frac{4}{6^n}\right)$ as $x^{n-5p} = x^0$.

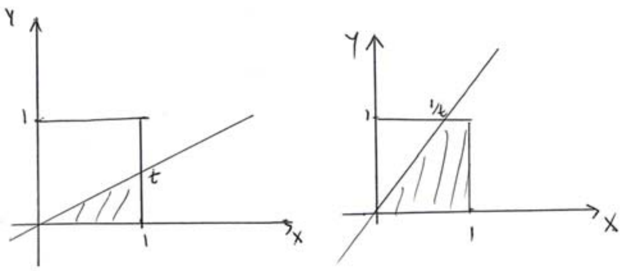
13. (i)



lead to $P(X+Y<t) = \frac{1}{2}t^2$ if $0 \leq t \leq 1$, $P(X+Y<t) = 1 - \frac{1}{2}(2-t)^2$ if $1 < t \leq 2$, $P(X+Y<t) = 0$ if $t < 0$ and $P(X+Y<t) = 1$ if $t > 2$.

From this, the cumulative distribution function of $(X+Y)^{-1}$ by means of the logic $P((X+Y)^{-1} < t) = P\left(X+Y > \frac{1}{t}\right) = 1 - P\left(X+Y < \frac{1}{t}\right)$ and the required probability density function can be found by differentiation. From that, by integration, $E\left(\frac{1}{X+Y}\right) = 2\ln 2$

(ii)



lead to

$$P\left(\frac{Y}{X} < t\right) = \begin{cases} \frac{1}{2}t \ for\ 0 \leq t \leq 1 \\ 1 - \frac{1}{2}t^{-1} \ for\ t > 1 \end{cases}$$

That $P\left(\frac{X}{X+Y} < t\right) = P\left(\frac{X+Y}{X} > \frac{1}{t}\right) = P\left(\frac{Y}{X} > \frac{1}{t} - 1\right) = 1 - P\left(\frac{Y}{X} < \frac{1}{t} - 1\right)$ and differentiation leads to

the probability density function $f(t) = \begin{cases} \frac{1}{2}t^{-2} \ for\ \frac{1}{2} \leq t \leq 1 \\ \frac{1}{2}(1-t)^{-2} \ for\ 0 \leq t < \frac{1}{2} \end{cases}$.

$E\left(\frac{X}{X+Y}\right) = \frac{1}{2}$ can be written down because by symmetry, $E\left(\frac{X}{X+Y}\right) = E\left(\frac{Y}{X+Y}\right)$

and $E\left(\frac{X}{X+Y}\right) + E\left(\frac{Y}{X+Y}\right) = E\left(\frac{X+Y}{X+Y}\right) = E(1) = 1$. This is simply verified by integration using the probability density function found.

[An earlier potential version of the question had X , Y , and Z independently uniformly distributed on $[0,1]$, considered the distribution of $\ln X$,

went on to find the pdf of U , where $U = -\ln(XY)$ and finished by showing that $(XY)^Z$ is also uniformly distributed on $[0,1]$.]

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