OXFORD AND CAMBRIDGE AND RSA EXAMINATIONS

STEP MATHEMATICS PAPER 2: 9470: JULY 2004 : HINTS AND ANSWERS

Q1(i) Put the terms with radicals on one side and the terms without on the other and square. Repeat this strategy (S) and the equation $x^4 - 6x^3 + 9x^2 - 4x = 0(*)$ will be obtained. The roots of (*) are $x = 0,\ 1,\ 4$.

Squaring may introduce spurious roots, so these numbers must be checked to see that they are roots of the original equation. In fact, they are.

(ii) Application of S again leads to (*). Checking shows that $x = 0$, $x = 1$ are roots of the second equation but that $x = 4$ is not.

(iii)Again application of S leads to (*). Checking shows that $x = 1$, $x = 4$ are roots of the third equation but that $x = 0$ is not.

Q2 Write $Q \equiv x^2 - \alpha|x| + 2 = [|x| - \alpha/2]^2 + 2 - \alpha^2/4$.

Thus $\alpha < 2\sqrt{2} \Rightarrow 2 - \alpha^2/4 > 0 \Rightarrow Q > 0$ for all x.

It is therefore unnecessary to consider $x > 0$ and $x < 0$ separately and even more unnecessary to use calculus methods.

• if $\alpha = 3$ then $Q \equiv (|x| - 1)(|x| - 2)$, in which case the solution set of $Q < 0$ is $\{x : -2 < x < -1\} \cup \{x : 1 < x < 2\}$.

• The solutions in x of the equation $Q = 0$ are of the form $-x_2,\ -x_1,\ x_1,\ x_2,$ where $0 < x_1 < x_2$, so that $S = 2(x_2 - x_1)$. Use of the identity $x_2 - x_1 = \sqrt{(x_2 + x_1)^2 - 4x_1x_2}$ will lead immediately to $S = 2\sqrt{\alpha^2 - 8}$. Thus $S < 2\sqrt{\alpha^2} = 2\alpha$.

• The graph of S as a function of α is that part of the hyperbola $4\alpha^2 - S^2 = 32$ which is in the first quadrant. A sketch of this graph should, therefore, leave the other quadrants empty. It should also show the curve starting at the point $(2\sqrt{2},\ 0)$ and asymptotically approaching the line $S = 2\alpha$.

Q3 The obtaining of dy/dx in the form required is a routine exercise in differentiation followed by some algebra.

Setting $dy/dx = 0$ shows that there are stationary points where $x = -2/3,\ 1/2,\ 2$. Moreover $d^2y/dx^2 = (x - 2)^3(12x + 1)$ + a term which is necessarily zero when $x = -2/3,\ 1/2,\ 2$. Thus d^2y/dx^2 is positive when $x = -2/3$ and negative when $x = 1/2$, so that C has a minimum at $(-2/3,\ -8192/729)$ and a maximum at $(1/2,\ 243/64)$. (Note that it is unnecessary to determine a simplified version of d^2y/dx^2 before inserting values of x.)

The argument $d^2y/dx^2 = 0$ at $x = 2 \Rightarrow C$ has a point of inflexion at $(2, 0)$ is false. In fact, in the neighbourhood of this point, $y \approx 6(x - 2)^4$, so that it is obvious that C has a minimum there.

The sketch of C must have correct overall shape, location and orientation, and also show correct forms at (0,0), (2,0) and at ∞.

(i) This sketch may be deduced from that of C. It has symmetry about the x - axis and no part of it appears in the region $-1 < x < 0$.

(ii) This sketch may also be deduced from that of C. It has symmetry about the y - axis and no part of it appears in the region $y < 0$.

Q4 It is important to realise at the outset that α is a constant defined by a and b and that β is a constant defined by a, b and w. Variable angles θ/ϕ are needed to define the orientation of the rod/table in the general situation.

(i) Clearly, for all $\theta \in (0,\ \pi/2)$, it is necessary that $f(\theta) \geq L$, where $f(\theta) = a\csc\theta + b\sec\theta$. Setting $f'(\theta) = 0$ will then lead to the required result.

(ii) Here, for all $\phi \in (0,\ \pi/2)$, it is necessary that $y \geq l$, where y is such that $b = (y-x)\cos\phi + w\sin\phi$ and x is such that $a = x\sin\phi + w\cos\phi$. (Other formulations are possible.) Elimination of x leads to $y = a\csc\phi + b\sec\phi - 2w\csc 2\phi$

Setting $y'(\phi) = 0$ plus some further working will then produce the required result.

Q5 Using the integration by parts rule it is easy to establish the results $\int_0^\pi x\sin x\, dx = \pi$ and $\int_0^\pi x\cos x\, dx = -2$.

• Write $\sin(x+t) = \sin x\cos t + \sin t\cos x$ and the result $f(t) = t + A\sin t + B\cos t$, where A and B are as defined in the question, follows immediately.

• Hence write $t + A\sin t + B\cos t = t + \int_0^\pi (x + A\sin x + B\cos x)\sin(x+t)\, dx$ $(***)$ so that as

$\int_0^\pi x\sin(x+t)\, dx = \ldots = \pi\cos t - 2\sin t$,

$\int_0^\pi \sin x\sin(x+t)\, dx = \ldots = (\pi/2)\cos t$,

$\int_0^\pi \cos x\sin(x+t)\, dx = \ldots = (\pi/2)\sin t$,

then, by considering the coefficients of $\cos t$ and $\sin t$ on both sides of (***), it follows that

$A = -2 + (\pi/2)B,\ B = \pi + (\pi/2)A \Rightarrow A = -2,\ B = 0$.

Alternatively, equations for A and B can be obtained by putting $t = 0$ and $t = \pi/2$ in (***).

Q6 From the data it follows that the component of $\mathbf{b}$ in the direction of $\mathbf{a}$ is $3\mathbf{a}$.

Hence $\mathbf{p} = 4\mathbf{a}$ and $\mathbf{q} = \mathbf{b} - 3\mathbf{a}$.

• Again from the data, it follows that $(\mathbf{c}.\mathbf{a})\mathbf{a} = -2\mathbf{a}$ and

$|\mathbf{q}|^2 = \mathbf{b}.\mathbf{b} - 6\mathbf{a}.\mathbf{b} + 9\mathbf{a}.\mathbf{a} = 25 - 18 + 9 = 16 \Rightarrow |\mathbf{q}| = 4$, so that

$\left[(\mathbf{c}.\mathbf{q})/|\mathbf{q}|^2\right]\mathbf{q} = (1/2)\mathbf{b} - (3/2)\mathbf{a}$.

Thus $\mathrm{P} = 2\mathrm{a}$, $\mathrm{Q} = -(9/2)\mathrm{a} + (3/2)\mathrm{b}$, $\mathrm{R} = (7/2)\mathrm{a} - (1/2)\mathrm{b} + \mathrm{c}$.

Q7 Good sketch graphs of $y = x$ and $y = 2\sin x$, in the same diagram and over the interval $[0, \pi]$, will readily show that the equation $f(x) = 0$ has exactly one root in the interval $[\pi/2, \pi]$.

• $f(3\pi/4) = \sqrt{2} - 3\pi/4$ has the same sign as $2 - 9\pi^2/16 \approx 2 - 45/8 = -29/8 < 0$. Hence as $f(\pi/2) = 2 - \pi/2 > 0$ and $f(\pi) = -\pi < 0$, then $I_1 = [\pi/2,\ 3\pi/4]$.

• $x = \sin 5\pi/8 \Rightarrow 2x\sqrt{1-x^2} = \sin 3\pi/4 = 1/\sqrt{2} \Rightarrow 8x^4 - 8x^2 + 1 = 0\,(*) \Rightarrow x^2 = 1/2 + 1/(2\sqrt{2}) \approx 0.85$. $(**)$. The sign of $f(5\pi/8)$ is the same as that of $4x^2 - 25\pi^2/64 \approx 17/5 - 125/32 = -81/625 < 0$. Hence $I_2 = [\pi/2,\ 5\pi/8]$.

• A good approximation to $x = \sin 9\pi/16$ may also be obtained in a similar way. In fact, it will be found that $f(9\pi/16) > 0$ so that $I_3 = [9\pi/16,\ 5\pi/8]$.

Q8(i) Integration leads to the general solution $t = A - \ln(1-x)$ and $x(0) = 0 \Rightarrow A = 0$. Thus $x = 1 - e^{-t}$.

(ii) Obviously, $(1-x)^{1/2} < (1+x)^{1/2}$ for all $x \in (0, 1]$. Hence multiplying this inequality through by $(1-x)^{1/2}$ leads immediately to the required result.

Arguments which go in the wrong direction, e.g., $1 - x < (1-x^2)^{1/2} \Rightarrow \ldots \Rightarrow x - x^2 > 0$, etc., are invalid. It may be possible to salvage them by replacing $'\Rightarrow'$ by $'\Leftarrow'$.

In the case $n = 2$, the substitution $x = \sin y$ will lead to $t = y + B$ and hence to $t = \sin^{-1}(x) + B$ as the general solution. In particular, $x(0) = 0 \Rightarrow B = 0 \Rightarrow x = \sin t$.

Note that the question does not allow the use of the standard form $\int (1-x^2)^{-1/2}\,dx = \sin^{-1}(x) +$ an arbitrary constant, without proof.

(iii) If G_n is the graph of x for $0 \le x \le 1$, then the given inequality shows that the gradient of G_3 is greater than the gradient of G_2 for each x in this interval. (The inequality of (ii) shows that the same is true of G_2 in relation to G_1.) These considerations will help to clarify ideas when drawing sketches of G_n for $n = 1, 2, 3$ in the same diagram. In particular, the sketch of G_3 should make it clear that once x reaches the value 1 it remains there.

Q9 For each of the two given situations, it is essential that a properly annotated diagram consistent with a possible state of equilibrium is supplied.

In the first situation, taking moments about the point of contact of the hemisphere with the floor leads to

$$mgr\cos\alpha = Mg\,(p\sin\alpha - q\cos\alpha) \Rightarrow \tan\alpha = (Mq + mr)/Mp.$$

A similar argument applied to the second situation leads to

$$mgr\cos\beta = Mg\,(p\sin\beta + q\cos\beta) \Rightarrow \tan\beta = (mr - Mq)/Mp.$$

It is then easy to see that

$$\tan(\alpha+\beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\tan\beta) = 2mMrp/\left[M^2(p^2+q^2) - m^2r^2\right].$$

If the sense of the rotation is taken into account then β should be changed to $-\beta$.

Q10 If the retardation of the particles when moving up the plane is $a_1\,ms^{-2}$, then $4g(\sqrt{3}/2)(1/5\sqrt{3})+2g = 4a_1 \Rightarrow a_1 = 6$, so that P comes to rest after 1 second at D where $AD = 3\,m$.

If the acceleration of P down the slope is $a_2\,ms^{-2}$, then $-4g(\sqrt{3}/2)(1/5\sqrt{3}) + 2g = 4a_2 \Rightarrow a_2 = 4$.

Hence if P and Q meet at time τ, then $3 - 2(\tau-1)^2 = 6(\tau - T) - 3(\tau - T)^2$

$\Rightarrow \ldots \Rightarrow \tau^2 - (2+6T)\tau + 3T^2 + 6T + 1 = 0 \Rightarrow \ldots \Rightarrow \tau = 1 + (3-\sqrt{6})T$.

Note that the condition $T < 1 + \sqrt{3/2}$ ensures that the collision takes place before P returns to A.

(ii) A possible solution is first to show that $T = 1 + \sqrt{2/3} \Rightarrow \tau = 2$.

Hence as $v_P(2) = 4\ ms^{-2}$, $v_Q(2) = 2\sqrt{6}\ ms^{-2}$ then the total KE at $t = 2$ of P and Q = 80 j.

Further, gain in PE at $t = 2$ since start of motion = 40 j so that energy lost due to friction = 144 - 80 - 40 = 24 j.

Alternatively, and more directly, the work done against friction up to the moment of collision = frictional force opposing motion of P (or Q) $\times 6$ j $= 4 \times 6 = 24$ j.

Q11 (i) At full engine power, the equation of motion of A is $Pv^{1/2} - kv = m(dv/dt)$.

The result $\int 1/(Pv^{1/2} - kv)\,dv = -(2/k)\ln(P - kv^{1/2})$ + constant, together with use of the condition $v(0) = 0$, followed by some algebra will lead to $v_A = (P^2/k^2)\left(1 - e^{-kt/2m}\right)^2$ (*), where v_A is the velocity of A at time t.

To obtain v_B, the velocity of B at time t, substitute $2m$ for m and $2P$ for P in (*). Thus $v_B = (4P^2/k^2)\left(1 - e^{-kt/4m}\right)^2$

(ii) $9v_A = 4v_B \Rightarrow 9\left(1 - e^{-kt/2m}\right)^2 = 16\left(1 - e^{-kt/4m}\right)^2 \Rightarrow 9\left(1 + e^{-kt/4m}\right)^2 = 16$

$\Rightarrow \ldots \Rightarrow e^{-kt/4m} = 1/3 \Rightarrow v_A = 64P^2/81k^2$ and $v_B = 16P^2/9k^2$.

(iii) The equation of motion of A is now $m(dv_A/dt) = -kv_A$, where t is now measured from the instant at which the engine of A is switched off. Since the velocity of A at the start of this phase of the motion is $64P^2/81k^2$, then subsequently $v_A = (64P^2/81k^2)e^{-kt/m}$. By a similar argument the result $v_B = (16P^2/9k^2)e^{-kt/2m}$ will be obtained. Elimination of t will then lead to $k^2v_B^2 = 4P^2v_A$.

Q12 This question generates seven separate tasks and so it is especially important to set out responses in an orderly way.

- The sketch is unimodal and falls entirely in the first quadrant of the $x-y$ plane. In particular, $y'(0+) > 0$ and y is asymptotic to $y = 0$ as $x \to \infty$.

• For $f(x)$, the constant k is determined by $\int_0^a kxe{-}x^2\,dx = 1 \Rightarrow \ldots \Rightarrow k = 2a/(1-e^{-a})$.

• For the mode, note first that $f'(x) = k\left[1-2ax^2\right]e^{-2ax^2}$ which is zero when $x = \sqrt{1/2a}$.

As $a < 1/2 \Rightarrow x = \sqrt{1/2a} > 1$ and $f'(x) > 0$ for any $x \in [0,1]$, then in this case m= 1.

On the other hand, $a \geq 1/2 \Rightarrow \sqrt{1/2a} \in [0,1]$ in which case $m = \sqrt{1/2a}$.

• To determine h, set $F(h) = 1/2$, where $F(x) = \int_0^x f(y)\,dy$. This leads to $k/2a - (k/2a)e^{-ah^2} = 1/2 \Rightarrow \ldots \Rightarrow h = \sqrt{(1/a)\ln[2/(1+e^{-a})]}$.

• $a > -\ln\left(2e^{-1/2}-1\right) \Rightarrow \ldots \Rightarrow e^{1/2} < 2/(1+e^{-a}) \Rightarrow \ldots \Rightarrow h > m$.

• $e > 1 \Rightarrow e^{-1/2} < 1 \Rightarrow 2e^{-1/2} - 1 < e^{-1/2} \Rightarrow \ln(2e^{-1/2}-1) < -1/2 \Rightarrow -\ln(2e^{-1/2}-1) > 1/2$.

• $P(X > m|X < h)P(X < h) = P(X > m \cap X < h) \Rightarrow P(X > m|X < h) = [1/2 - F(1/\sqrt{2a})]/(1/2) = 1 - (k/a)\left[1-e^{-1/2}\right] = \ldots = (2e^{-1/2} - e^{-a} - 1)/(1-e^{-a})$.

Q13 If W_n pounds is the gain from draw n, then $E(W_{n+1}) = (b-r-n)/(b-n)\times 1 + r/(b-n)\times(-n)$ which is zero if $n = (b-r)/(r+1) = \xi$, say.

• W_{n+1} increases as n increases for $n < \xi$, and W_{n+1} decreases as n increases for $n > \xi$. Hence W_n maximum when $n = [\xi] + 1 = n_c$, say, so that optimal stopping n is n_c.

• For $r = 1$ and b even, $n_c = b/2$, in which case $P(\text{first } n_c - 1 \text{ draws are all white}) = (b - n_c + 1)/b = 1/2 + 1/b$.

Thus expected total reward $= (1/2 - 1/b) \times 0 + (1/2 + 1/b)\,[(b/2)/((b/2)+1)] \times n_c = \ldots = b/4$ pounds.

• For $r = 1$ and b odd, $n_c = b/2 + 1/2$ so that now $P(\text{first } n_c - 1 \text{ draws are all white}) = 1/2 + 1/b$.

Hence expected total reward $= (1/2 + 1/2b) \times [(b/2 - 1/2)/(b + 1/2)] \times (b+1)/2 = \ldots = (b^2-1)/4b$ pounds.

Q14 The introductory result may be explained by means of a diagram. Alternatively, replacing B by $B \cup C$ in $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ will lead to the displayed result almost immediately.

• $P_r = P(\text{at least one pudding contains no sixpence}) = 3\left[(2/3)^r - (1/3)^r\right]$.

• $P_5 = 31/81 > 1/3$, $P_6 = 7/27 < 1/3 \Rightarrow \min(r) = 6$.

• With $r = 6$, let A be the event that each pudding contains ≥ 1 sixpences and let B be the event that each pudding contains 2 sixpences. Then,

$P(A) = 1 - 7/27 = 20/27$,

$P(A \cap B) = P(B) = \ldots = 10/81$,

$P(B|A) = (10/81)/(20/27) = 1/6$.