



**Cambridge Assessment
Admissions Testing**

Sixth Term Examination Paper [STEP]

Mathematics 2 [9470]

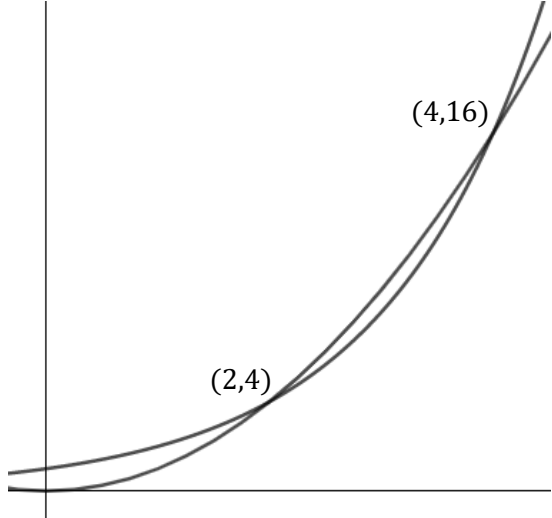
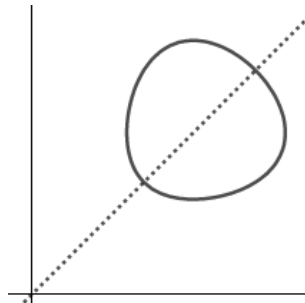
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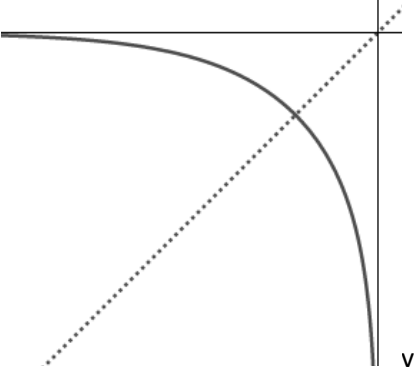
Mark Scheme

			Only penalise missing +c once in parts (i) and (ii)
1(i)	$\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}} dx = \int \frac{(1-u)^2}{u^{\frac{1}{2}}(1-u)^2} du$	M1 A1	Must include attempt at $\frac{du}{dx}$ (or $\frac{dx}{du}$)
	$= 2u^{\frac{1}{2}}$	A1	
	$= 2\left(\frac{x-1}{x}\right)^{\frac{1}{2}} + c$	A1	
1(ii)	Let $x - 2 = s$	M1	
	Then $\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}} dx = \int \frac{1}{s^{\frac{3}{2}}(s+3)^{\frac{1}{2}}} ds$	A1	
	Let $s = \frac{3}{u-1}$	M1 A1	
	$\int \frac{1}{s^{\frac{3}{2}}(s+3)^{\frac{1}{2}}} ds = \int \frac{(u-1)^2}{3^2 u^{\frac{1}{2}}(u-1)^2} du = -\frac{2}{3} u^{\frac{1}{2}}$	A1	
	$= -\frac{2}{3} \left(\frac{s+3}{s}\right)^{\frac{1}{2}} = -\frac{2}{3} \left(\frac{x+1}{x-2}\right)^{\frac{1}{2}} + c$	M1 A1	
1(iii)	Let $x = \frac{1+u}{u}$	M1 A1	Any substitution leading to two algebraic factors in the denominator.

	$\int_2^{\infty} \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}} dx = \int_1^0 \frac{u^2}{(1-u)^{\frac{1}{2}}(3+u)^{\frac{1}{2}}} \cdot \left(\frac{-1}{u^2}\right) du$	M1 A1	If done through a sequence of substitutions: a further substitution leading to a square root of a quadratic as the denominator.
	$= \int_0^1 \frac{1}{(3-2u-u^2)^{\frac{1}{2}}} du$	M1	Expressing the denominator as the square root of a quadratic.
	$= \int_0^1 \frac{1}{(4-(1+u)^2)^{\frac{1}{2}}} du$	M1	
	$= \left[\arcsin\left(\frac{1+u}{2}\right) \right]_0^1$	M1 A1	
	$= \frac{1}{2}\pi - \frac{1}{6}\pi = \frac{1}{3}\pi$	A1	Answer given.

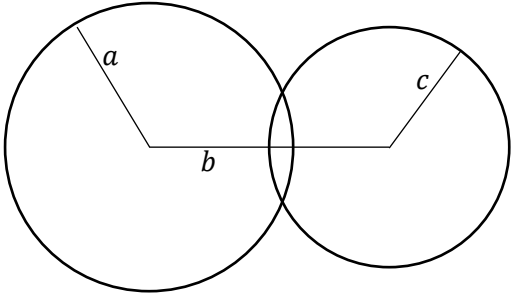
2(i)	$\frac{1-ky}{y} \frac{dy}{dx} = \frac{kx-1}{x}$	M1	
	$\ln y - ky = kx - \ln x + c$	M1 A1	
	Hence, $\ln xy = k(x+y) + c$	M1	
	$xy = \frac{1}{4} \left[(x+y)^2 - (x-y)^2 \right] = Ae^{k(x+y)}$	M1 A1	
	C_1 is $(x-y)^2 = (x+y)^2 - 2^{x+y}$	A1	Answer given.
	C_2 is $(x-y)^2 = (x+y)^2 - 2^{x+y+4}$	A1	
	In both cases, the equation is invariant under $(x, y) \mapsto (y, x)$, so symmetrical in $y = x$.	E1	
			If the form given for C_1 is differentiated to check that it satisfies the differential equation and it is also checked that it passes through (1,1), award 2 marks.

2(ii)			
	Graphs: Correct shapes of curves	G1	
	Graphs: Intersections at (2,4) and (4,16)	G1	
	$(x - y)^2 \geq 0$, so $(x + y)^2 > 2^{x+y}$		
	Therefore, $(x + y)$ must lie between 2 and 4	E1	Explanation of how the graphs show this needed.
			
	Graph: Symmetry about $y = x$	G1	
	Graph: Closed curve lying between $x + y = 3 \pm 1$	G1	
	Graph: Passes through (1,1) and (2,2)	G1	

2(iii)	Sketches of $y = x^2$ and $y = 2^{x+4}$ $x^2 > 2^{x+4}$ only when $x < -2$.	M1 A1	Marks can be awarded for any justification that $x^2 > 2^{x+4}$ only when $x < -2$.
			
	Graph: Symmetry about $y = x$	G1	
	Graph: Passes through (-1,-1)	G1	
	Graph: $y \rightarrow 0$ as $x \rightarrow \infty$, $y \rightarrow -\infty$ as $x \rightarrow 0$	G1	

3(i)	Suppose, $\exists k: 2 \leq k \leq n - 1$ such that $u_{k-1} \geq u_k$, but $u_k < u_{k+1}$	M1	
	Since all of the terms are positive, these imply that $u_k^2 < u_{k-1}u_{k+1}$, so the sequence does not have property L .	E1	Must include that all terms are positive.
	Therefore, if the sequence has property L , once a value k has been reached such that $u_{k-1} \geq u_k$, it must be the case that all subsequent terms also have that property (which is the given definition of unimodality).	E1	
3(ii)	$u_r - \alpha u_{r-1} = \alpha(u_{r-1} - \alpha u_{r-2})$, so $u_r - \alpha u_{r-1} = \alpha^{r-2}(u_2 - \alpha u_1)$	M1 A1	Answer given.
	$u_r^2 - u_{r-1}u_{r+1} = u_r^2 - u_{r-1}(2\alpha u_r - \alpha^2 u_{r-1}) = (u_r - \alpha u_{r-1})^2$ for $r \geq 2$	B1	Answer given.
	The first identity shows that $u_r > 0$ for all r if $u_2 > \alpha u_1 > 0$.	E1	
	Since the right hand side of the second identity is always non-negative, the sequence has property L , and is hence unimodal.	E1	

3(iii)	$u_1 = (2 - 1)\alpha^{1-1} + 2(1 - 1)\alpha^{1-2} = 1$, which is correct. $u_2 = (2 - 2)\alpha^{2-1} + 2(2 - 1)\alpha^{2-2} = 2$, which is correct.	M1	Confirm both cases.
	Suppose that: $u_{k-2} = (4 - k)\alpha^{k-3} + 2(k - 3)\alpha^{k-4}$, and $u_{k-1} = (3 - k)\alpha^{k-2} + 2(k - 2)\alpha^{k-3}$.		
	$u_k = 2\alpha \left((3 - k)\alpha^{k-2} + 2(k - 2)\alpha^{k-3} \right) - \alpha^2 \left((4 - k)\alpha^{k-3} + 2(k - 3)\alpha^{k-4} \right)$ $= \alpha^{k-1}(6 - 2k - 4 + k) + \alpha^{k-2}(4k - 8 - 2k + 6)$ $= \alpha^{k-1}(2 - k) + 2\alpha^{k-2}(k - 1)$ which is the correct expression for u_k	M1 A1	
	Hence, by induction $u_r = (2 - r)\alpha^{r-1} + 2(r - 1)\alpha^{r-2}$	E1	Requires both statement of induction hypotheses and conclusion.
	$u_r - u_{r+1} = \left((2 - r)\alpha^{r-1} + 2(r - 1)\alpha^{r-2} \right) - \left((1 - r)\alpha^r + 2r\alpha^{r-1} \right)$	M1	
	$= \alpha^{r-2} \left(2(r - 1) + (2 - 3r)\alpha + (r - 1)\alpha^2 \right)$	A1	
	$= \frac{\alpha^{r-2}}{N^2} \left(2N^2(r - 1) + (2 - 3r)N(N - 1) + (r - 1)(N - 1)^2 \right)$ $= \frac{\alpha^{r-2}}{N^2} \left((r - 1) + rN - N^2 \right)$	M1 A1	
	when $r = N$, $u_N - u_{N+1} = \frac{\alpha^{r-2}(N - 1)}{N^2} > 0$	M1 A1	
	when $r = N - 1$, $u_{N-1} - u_N = \frac{-2\alpha^{r-2}}{N^2} < 0$	A1	
	so u_r is largest when $r = N$	E1	Answer given.

4(i)	The straight line distance between two points must be less than the length of any other rectilinear path between the points.	E1	
4(ii)			
	Diagram showing two circles and straight line joining their centres. Length of line and radii of circles are a , b and c in some order.	B1	
	Either statement that the straight line is the longest of the lengths, or explanation that one circle cannot be contained inside the other.	E1	
	Explanation that the circles must meet.	E1	
4(iii)	<p>(A) If $a + b > c$ then $(a + 1) + (b + 1) > c + 2 > c + 1$ <i>et cycl.</i>, so $a + 1$, $b + 1$, $c + 1$ can always form the sides of a triangle.</p>	M1 A1	If it is not made clear that the result applies for cyclic permutations, do not award A1 .
	<p>(B) If $a = b = c = 1$ we have 1, 1, 1 which can form the sides of a triangle.</p>	B1	
	If $a = 1$, $b = c = 2$ we have $\frac{1}{2}$, 1, 2 which cannot form the sides of a triangle.	B1	
	Therefore, $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{a}$ can sometimes, but not always form the sides of a triangle.	E1	
	<p>(C) If $p \geq q \geq r$ then $p - q + q - r = p - q + q - r = p - r = p - r$</p>	M1 A1	
	So two of $ p - q $, $ q - r $, $ p - r $ will always sum to the third, so they never form the sides of a triangle.	E1	
	<p>(D) If $a + b > c$ then $a^2 + bc + b^2 + ca = a^2 + b^2 - 2ab + c(a + b) + 2ab$</p>	M1 A1	

	$= (a - b)^2 + c(a + b) + 2ab > c^2 + ab$ <i>et cycl.</i> so $a^2 + bc, b^2 + ca, c^2 + ab$ can always form the sides of a triangle.	M1 A1	Penalise not referencing cyclic permutations, but only if not penalised for (A)
4(iv)	Since $a + b > a$ and $b, \frac{f(a)}{a} > \frac{f(a+b)}{a+b}$ and $\frac{f(b)}{b} > \frac{f(a+b)}{a+b}$	M1	
	Since $c < a + b, f(c) < f(a + b)$	M1	
	Thus $f(a) + f(b) > \frac{af(a)}{a+b} + \frac{bf(b)}{a+b} = f(a + b) > f(c)$ <i>et cycl.</i> So $f(a), f(b)$ and $f(c)$ can form the sides of a triangle.	M1 A1	Penalise not referencing cyclic permutations. Answer given.

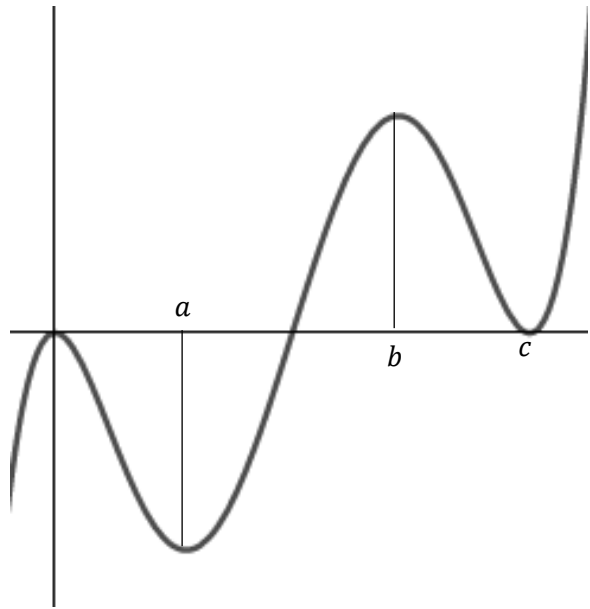
5(i)	$x - q(x) = \sum_{r=0}^{n-1} a_r \times 10^r - \sum_{r=0}^{n-1} a_r = \sum_{r=0}^{n-1} a_r \times (10^r - 1)$	M1	
	$10^r \geq 1 \quad \forall r$, so $x - q(x)$ is non-negative	A1	Answer given.
	$9 (10^r - 1) \quad \forall r$	A1	Answer given.
5(ii)	$x - 44q(x) = 44(x - q(x)) = 43x$	M1	
	So it is a multiple of 9 iff $43x$ is.	A1	
	$(43, 9) = 1$, so $x - 44q(x)$ is a multiple of 9 iff x is	A1	Answer given.
	If x has n digits, $q(x) \leq 9n$	E1	
	Since $x = 44q(x)$, $x \leq 396n$. Any n digit number must be at least 10^{n-1} .	E1	Answer given
	These inequalities cannot be simultaneously true for $n \geq 5$ ($396 \times 5 < 10^4$). Therefore $n \leq 4$.	E1	Answer given.
	Since $x - 44q(x) = 0$, which is a multiple of 9, x is a multiple of 9.	M1	
	$q(x)$ is an integer and $x = 44q(x)$, so x is a multiple of 44. Since $(9, 44) = 1$, x must be a multiple of $44 \times 9 = 396$.	A1	
	So $x = 396k$ and therefore (by the result above) $k \leq 4$.	M1	
	Checking: Only $k = 2$ works.	A1	Must be clear that there are no other solutions to get this mark.

5(iii)	$x - 107q(q(x)) = 0 = 107(x - q(x)) + 107(q(x) - q(q(x))) - 106x$	M1 A1	
	$(x - q(x))$ and $(q(x) - q(q(x)))$ are both divisible by 9 (by part (i)) and so x is divisible by 9	A1	
	$x = 107q(q(x))$ and so is divisible by 107, and so is divisible by 963. So $x = 963k$ for some k .	M1	
	If x has n digits, then $q(x) \leq 9n$. By (i), $q(q(x)) \leq q(x) \leq 9n$. So $x \leq 963n$ and $x \geq 10^{n-1}$ which implies that $n \leq 4$ and so $k \leq 4$	M1 A1	Establishing that $n \leq 4$ Linking to the value of k .
	Checking: Only $k = 1$ works.	A1	

6(i)	Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$	M1 A1	
	so $\text{Tr}(\mathbf{M}^2) = a^2 + d^2 + 2bc = (a+d)^2 - 2(ad-bc)$	A1	Answer given.
6(ii)	Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2 = \begin{pmatrix} a\tau - \delta & b\tau \\ c\tau & d\tau - \delta \end{pmatrix}$, where $\tau = \text{Tr}(\mathbf{M})$ and $\delta = \text{Det}(\mathbf{M})$.		
	Thus $\mathbf{M}^2 = \pm \mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = \mp 1$ or $b = c = 0$ and $a^2 = d^2 = \pm 1$	M1 A1 A1	
	If $b = c = 0$ and $a = d = \pm 1$, then $\mathbf{M} = \pm \mathbf{I}$	E1	
	If $b = c = 0$ and $a = -d = \pm 1$, then $\tau = 0$ and $\delta = -1$	E1	
	Thus $\mathbf{M}^2 = +\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = -1$.	E1	Answer given.
	Thus $\mathbf{M}^2 = -\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = +1$.	E1	Answer given.
6(iii)	Part (ii) implies $\text{Det}(\mathbf{M}^2) = -1$, if $\mathbf{M}^4 = \mathbf{I}$, but $\mathbf{M}^2 \neq \pm \mathbf{I}$.	E1	
	However, $\text{Det}(\mathbf{M}^2) = \text{Det}(\mathbf{M})^2$, so this is impossible.	E1	
	Clearly $\mathbf{M}^2 = \pm \mathbf{I} \Rightarrow \mathbf{M}^4 = \mathbf{I}$	E1	
	Part (ii) implies that $\mathbf{M}^4 = -\mathbf{I} \Leftrightarrow \text{Tr}(\mathbf{M}^2) = 0$ and $\text{Det}(\mathbf{M}^2) = 1$	E1	
	so from (i) $\Leftrightarrow \text{Tr}(\mathbf{M})^2 = 2\text{Det}(\mathbf{M})$ and $\text{Det}(\mathbf{M}) = \pm 1$	M1 A1	
	so $\Leftrightarrow \text{Tr}(\mathbf{M}) = \pm\sqrt{2}$ and $\text{Det}(\mathbf{M}) = 1$.	A1 A1	
	Any example, for instance a matrix satisfying the conditions for any of $\mathbf{M}^2 = \mathbf{I}$, $\mathbf{M}^2 = -\mathbf{I}$, $\mathbf{M}^4 = -\mathbf{I}$, which is not a rotation or reflection.	M1 A1	

7(i)	$ w - 1 ^2 = \left \frac{1-ti}{1+ti} \right ^2 = \frac{(1-ti)(1+ti)}{(1+ti)(1-ti)} = 1$, which is independent of t .	M1 A1	
	Points on the line $Re(z) = 3$ have the form $z = 3 + ti$ and the points satisfying $ w - 1 = 1$ lie on a circle with centre 1.	E1	Answer given.
	If $z = p + ti$, then $ w - c ^2 = \left \frac{2 - (p-2)c - cti}{(p-2) + ti} \right ^2 = \frac{(2 - (p-2)c)^2 + c^2 t^2}{(p-2)^2 + t^2}$	M1 M1 A1	Choice of form for z .
	which is independent of t when $(2 - (p-2)c)^2 = c^2 (p-2)^2$	M1	
	which is when $c = \frac{1}{p-2}$. Thus the circle has centre at $\frac{1}{p-2}$ and radius $\frac{1}{ p-2 }$.	A1 A1	Centre Radius
	$w = \frac{2}{(p-2) + ti} = \frac{2(p-2) - 2ti}{(p-2)^2 + t^2},$	M1	
	so $\text{Im}(w) > 0$ when $t < 0$; that is, for those z on V with negative imaginary part.	A1	

7(ii)	<p>If $z = t + qi$ then</p> $ w - ci ^2 = \left \frac{2 + cq - (t-2)ci}{(t-2) + qi} \right ^2 = \frac{c^2(t-2)^2 + (cq+2)^2}{(t-2)^2 + q^2}$	<p>M1 M1 M1 A1</p>	<p>Choice of form for z. Recognises the centre is on imaginary axis.</p>
	<p>which is independent of t when $(cq+2)^2 = c^2q^2$</p>	M1	
	<p>which is when $c = -\frac{1}{q}$ so the circle has centre $-\frac{1}{q}i$ A1 and radius $\sqrt{c^2} = \frac{1}{ q }$ A1.</p>	<p>A1 A1</p>	<p>Centre Radius</p>
	$w = \frac{2}{(t-2) + qi} = \frac{2(t-2) - 2qi}{(t-2)^2 + q^2},$	M1	
	<p>so $\text{Re}(w) > 0$ when $t > 2$; that is, for those z on H with real part greater than 2.</p>	A1	

8(i)			
	Graph: Zeroes at $x = 0, c$ and one other point (h : label not required) in (a, b) .	G1	
	Graph: Turning points at $x = 0, a, b, c$.	G1	
	Graph: Quintic shape with curve below axis in $(0, h)$ and above axis in (h, c)	G1	
	The area conditions give $F(0) = F(c) = 0$. $F'(x) = f(x)$, so $F'(0) = F'(a) = F'(b) = F'(c) = 0$	E1	
	Since f is a quartic and the coefficient of x^4 is 1, F must be a quintic and the coefficient of x^5 is $\frac{1}{5}$. $F(0) = F'(0) = 0$ and $F(c) = F'(c) = 0$, so F must have double roots at $x = 0$ and c . So $F(x)$ must have the given form.	E1	Explanation must be clear that the double roots are deduced from the fact that $F(x) = F'(x) = 0$ at those points.
	$F(x) + F(c - x) = \frac{1}{5}x^2(x - c)^2[(x - h) + (c - x - h)]$ $= \frac{1}{5}x^2(c - x)^2(c - 2h)$	M1 A1	

8(ii)	Let A be the (positive) area enclosed by the curve between 0 and a . The maximum turning point of $F(x)$ occurs at $x = b$, with $F(b) = A$. The minimum turning point of $F(x)$ occurs at $x = a$, with $F(a) = -A$.	E1	
	Therefore $F(x) \geq -A$, with equality iff $x = a$. So $F(b) + F(x) \geq 0$, with equality iff $x = a$.	E1	Answer given.
	$F(a) + F(x) \leq 0$, with equality iff $x = b$.	B1	
	Since $F(b) + F(c - b) = \frac{1}{5}b^2(c - b)^2(c - 2h)$, either $c > 2h$, or $c = 2h$ and $c - b = a$.	M1 A1	Answer given.
	Also, $F(a) + F(c - a) = \frac{1}{5}a^2(c - a)^2(c - 2h)$, so either $c < 2h$, or $c = 2h$ and $c - a = b$.	M1 A1	
	Thus $c = a + b$ and $c = 2h$.	E1	Answer given.
8(iii)	$F(x) = \frac{1}{10}x^2(x - c)^2(2x - c)$ So $f(x) = \frac{1}{5}x(x - c)(5x^2 - 5xc + c^2)$	M1 A1	
	The roots of the quadratic factor must be a and b .		
	$f(x) = \frac{1}{5}(5x^4 - 10cx^3 + 6c^2x^2 - c^3x)$ $f'(x) = \frac{1}{5}(20x^3 - 30cx^2 + 12c^2)$ $f''(x) = \frac{1}{5}(60x^2 - 60cx + 12c^2) = \frac{12}{5}(5x^2 - 5cx + c^2)$	M1 A1	
	Therefore $f''(x) = 0$ at $x = a$ and $x = b$ and so $(a, 0)$ and $(b, 0)$ are points of inflection.	E1	

9	If the particles collide at time t : $Vt + Ut \cos \theta = d$, and $h - \frac{1}{2}gt^2 = Ut \sin \theta - \frac{1}{2}gt^2$ (or $h = Ut \sin \theta$)	M1 M1	
	Therefore, $d \sin \theta - h \cos \theta = Vt \sin \theta + Ut \sin \theta \cos \theta - Ut \sin \theta \cos \theta$ $= \frac{vh}{U}$	M1 A1	Answer given.
9(i)	Dividing the previous result by $d \cos \theta$ gives: $\tan \theta - \frac{h}{d} = \frac{vh}{Ud \cos \theta} > 0$	M1	
	Since $\tan \beta = \frac{h}{d}$, $\tan \theta > \tan \beta$ and so $\theta > \beta$	A1	Answer given.
9(ii)	The height of collision must be non-negative, so $Ut \sin \theta - \frac{1}{2}gt^2 \geq 0$.	M1	
	So $U \sin \theta \geq \frac{1}{2}gt = \frac{1}{2}g \frac{h}{U \sin \theta}$ or $(U \sin \theta)^2 \geq \frac{gh}{2}$ Therefore $U \sin \theta \geq \sqrt{\frac{gh}{2}}$.	M1 A1	Answer given.
9(iii)	$d \sin \theta - h \cos \theta$ can be written as $\sqrt{d^2 + h^2} \sin(\theta - \beta)$	M1 A1 A1	Amplitude β
	So $d \sin \theta - h \cos \theta < \sqrt{d^2 + h^2}$ (since $\theta > \beta$)	A1	
	Therefore, $\frac{vh}{U} < \sqrt{d^2 + h^2}$ or $\sin \beta = \frac{h}{\sqrt{d^2 + h^2}} < \frac{U}{v}$	M1 A1	Answer given.

	The height at which the particles collide is: $h - \frac{1}{2}gt^2 = h - \frac{gh^2}{2U^2 \sin^2 \theta}$		
	$h - \frac{gh^2}{2U^2 \sin^2 \theta} > \frac{1}{2}h \text{ iff } U^2 \sin^2 \theta > gh$	M1 A1	
	The vertical velocity of the particle fired from B at the point of collision is: $U \sin \theta - gt = U \sin \theta - \frac{gh}{U \sin \theta}$		
	$U \sin \theta - \frac{gh}{U \sin \theta} > 0 \text{ iff } U^2 \sin^2 \theta > gh$	M1 A1	
	Since both cases have the same condition: The particles collide at a height greater than $\frac{1}{2}h$ if and only if the particle projected from B is moving upwards at the time of collision.	E1	Answer given.

10(ii)	Energy: $\frac{1}{2}mv^2 - mga \cos 2\theta + \frac{\lambda}{2l}(2a \cos \theta - l)^2 = \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2$	M1 A1 A1	Potential Elastic
	If the particle comes to rest when $\theta = \beta$: $-mga(2\cos^2 \beta - 1) + \frac{\lambda}{2l}(2a \cos \beta - l)^2 = \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2$	M1	
	$a\lambda \cos^2 \beta \left(\frac{2(a\lambda - mgl)}{\lambda l} \right) - 2a\lambda \cos \beta = \frac{1}{2}mu^2 - 2mga + \frac{2\lambda a^2}{l} - 2a\lambda$	A1	
	Therefore, $\cos^2 \beta - 2 \cos \alpha \cos \beta = \frac{mu^2}{2a\lambda} \cos \alpha + 1 - 2 \cos \alpha$	M1 A1	Introduces α
	Adding $\cos^2 \alpha$ to both sides: $(\cos \alpha - \cos \beta)^2 = (1 - \cos \alpha)^2 + \frac{mu^2}{2a\lambda} \cos \alpha$	A1	Answer given
	For this to occur, $\cos \beta > 0$:	M1	
	$\cos^2 \alpha > (1 - \cos \alpha)^2 + \frac{mu^2}{2a\lambda} \cos \alpha$	A1	
	And so, $u^2 < \frac{2a\lambda}{m}(2 - \sec \alpha)$	A1	Answer given

11(i)	<p>If the game has not ended after $2n$ turns, then the sequence has either been n repetitions of HT or n repetitions of TH.</p> <p>So $P(\text{Game has not finished after } 2n \text{ turns}) = 2(pq)^n$.</p> <p>So the probability that the game never ends is $\lim_{n \rightarrow \infty} 2(pq)^n = 0$.</p>	E1	Answer given.
	Sequence that follows the first H will be k repetitions of TH , followed by H , where $k \geq 0$.	M1	
	So $P(A \text{ wins} \text{first toss is } H) = \sum_{k=0}^{\infty} (pq)^k p = \frac{p}{1-pq}$	A1	Answer given
	$P(A \text{ wins} \cap \text{first toss is } H) = p \times \frac{p}{1-pq}$	M1	Use of conditional probability
	If first toss is a tail then the sequence that follows would be k repetitions of HT followed by HH .		
	So $P(A \text{ wins} \text{first toss is } T) = \frac{p^2}{1-pq}$		
	$P(A \text{ wins} \cap \text{first toss is } T) = \frac{p^2 q}{1-pq}$		
	Therefore $P(A \text{ wins}) = \frac{p^2(1+q)}{1-pq}$	A1	
11(ii)	<p>Following a first toss of H:</p> <p>A wins with HH</p> <p>or $(HT$ followed by any sequence where A wins after first toss was T)</p> <p>or $(T$ followed by any sequence where A wins after first toss was T)</p>	E1	
	<p>The probabilities of these cases are:</p> <p>p^2</p> <p>$pq P(A \text{ wins} \text{the first toss is a tail})$</p> <p>$q P(A \text{ wins} \text{the first toss is a tail})$</p>	E1	
	<p>Therefore:</p> <p>$P(A \text{ wins} \text{the first toss is a head}) = p^2 + (q + pq)P(A \text{ wins} \text{the first toss is a tail})$</p>	E1	Answer given.

	Similarly, following first toss of T : A wins with (H followed by any sequence where A wins after first toss was H) or (TH followed by any sequence where A wins after first toss was H)	M1	
	Therefore: $P(A \text{ wins} \mid \text{the first toss is a tail}) = (p + pq)P(A \text{ wins} \mid \text{the first toss is a head})$	A1	
	So $P(A \mid H \text{ first}) = p^2 + (q + pq)(p + pq)P(A \mid H \text{ first})$ $P(A \mid H \text{ first}) = \frac{p^2}{1-(p+pq)(q+pq)}$	M1	
	And $P(A \mid T \text{ first}) = (p + pq)(p^2 + (q + pq)P(A \mid T \text{ first}))$ $P(A \mid T \text{ first}) = \frac{p^2(p+pq)}{1-(p+pq)(q+pq)}$	M1	
	So $P(A \text{ wins}) = p \times \frac{p^2}{1-(p+pq)(q+pq)} + q \times \frac{p^2(p+pq)}{1-(p+pq)(q+pq)} = \frac{p^2(1-q^3)}{1-(1-p^2)(1-q^2)}$	A1	Answer given
11(iii)	Let W be the event that A wins the game. $P(W \mid H \text{ first}) = p^{a-1} + (1 + p + p^2 + \dots + p^{a-2})qP(W \mid T \text{ first})$	M1 A1	
	$P(W \mid T \text{ first}) = (1 + q + q^2 + \dots + q^{b-2})pP(W \mid H \text{ first})$	A1	
	$P(W \mid H \text{ first}) = \frac{p^{a-1}}{1-(1-p^{a-1})(1-q^{b-1})}$	M1	
	$P(W \mid T \text{ first}) = \frac{p^{a-1}(1-q^{b-1})}{1-(1-p^{a-1})(1-q^{b-1})}$	M1	
	Therefore: $P(W) = \frac{p^{a-1}(1-q^b)}{1-(1-p^{a-1})(1-q^{b-1})}$	A1	
	If $a = b = 2$, $P(W) = \frac{p(1-q^2)}{1-(1-p)(1-q)} = \frac{p^2(1+q)}{1-pq}$ as expected.	B1	

12(i)	For the biased die: $P(R_1 = R_2) = \sum_{i=1}^n \left(\frac{1}{n} + \varepsilon_i\right)^2$	M1	
	$P(R_1 = R_2) = \frac{1}{n^2} \sum_{i=1}^n 1 + \frac{2}{n} \sum_{i=1}^n \varepsilon_i + \sum_{i=1}^n \varepsilon_i^2$	M1 A1	
	$\sum_{i=1}^n \varepsilon_i = 0$, so $P(R_1 = R_2) = \frac{1}{n} + \sum_{i=1}^n \varepsilon_i^2$	M1 A1	
	For a fair die, $P(R_1 = R_2) = \frac{1}{n}$ and $\sum_{i=1}^n \varepsilon_i^2 > 0$, so it is more likely with the biased die.	E1	
(ii)	$P(R_1 > R_2) = \frac{1}{2}(1 - P(R_1 = R_2))$	M1	
	Therefore, the value of $P(R_1 > R_2)$ if the die is possibly biased is $\leq P(R_1 > R_2)$ if the die is fair.	M1	
	Let $T = \sum_{r=1}^n x_r$ and, for each i , let $p_i = \frac{x_i}{T}$ Then $\sum_{i=1}^n p_i = 1$, so we can construct a biased n -sided die with $P(X = i) = p_i$	M1	
	$P(R_1 > R_2) = \sum_{i=2}^n \sum_{j=1}^{i-1} p_i p_j$	M1 A1 A1	
	For a fair die: $P(R_1 > R_2) = \frac{n-1}{2n}$	A1	
	Therefore $\sum_{i=2}^n \sum_{j=1}^{i-1} \frac{x_i x_j}{T^2} \leq \frac{n-1}{2n}$ and so $\sum_{i=2}^n \sum_{j=1}^{i-1} x_i x_j \leq \frac{n-1}{2n} \left(\sum_{i=1}^n x_i \right)^2$	A1	Answer given.

(iii)	For the biased die: $P(R_1 = R_2 = R_3) = \sum_{i=1}^n \left(\frac{1}{n} + \varepsilon_i\right)^3$	M1	
	$= \sum_{i=1}^n \frac{1}{n^3} + \sum_{i=1}^n \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^n \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^n \varepsilon_i^3$	A1	
	Therefore $P(R_1 = R_2 = R_3 \text{ biased}) - P(R_1 = R_2 = R_3 \text{ fair}) = \sum_{i=1}^n \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^n \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^n \varepsilon_i^3$ $= \sum_{i=1}^n \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^n \varepsilon_i^3 \text{ (since } \sum_{i=1}^n \varepsilon_i = 0)$	A1	
	$= \sum_{i=1}^n \frac{3\varepsilon_i^2}{n} + \varepsilon_i^3 = \sum_{i=1}^n \varepsilon_i^2 \left(\frac{3}{n} + \varepsilon_i\right)$		
	But $\varepsilon_i \geq -\frac{1}{n}$ (since $p_i \geq 0$), so this sum must be positive.	M1 A1	
	Therefore, $P(R_1 = R_2 = R_3)$ must be greater for the biased die.	E1	

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