STEP MATHEMATICS 2

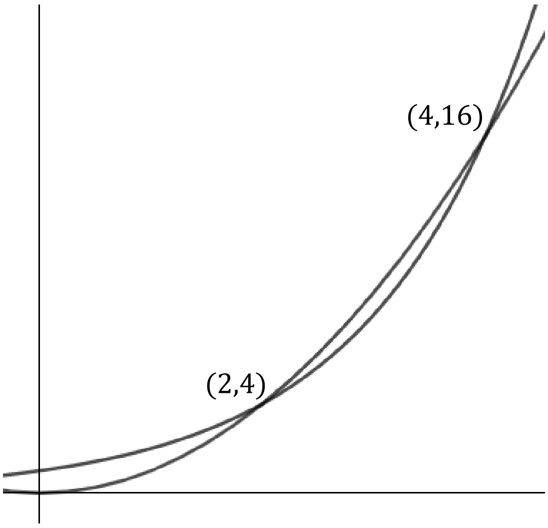
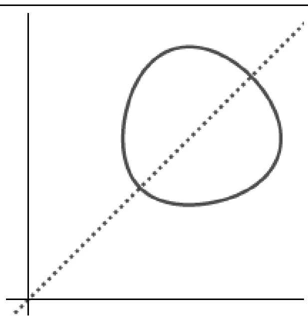
2020

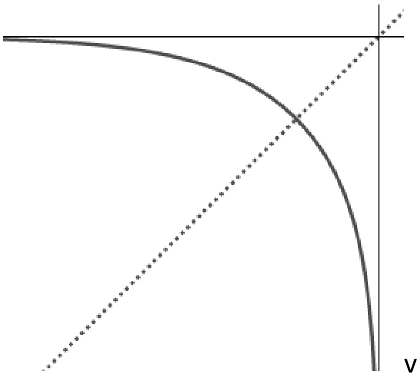
Worked Solutions

STEP 2: BRIEF SOLUTIONS

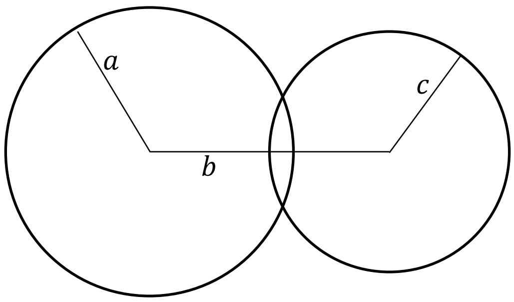
	Only penalise missing +c once in parts (i) and (ii)
1(i)	$\displaystyle\int \frac{1}{x^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}\,\mathrm{d}x = \int \frac{(1-u)^2}{u^{\frac{1}{2}}}\frac{1}{(1-u)^2}\,\mathrm{d}u$ Must include attempt at $\frac{du}{dx}$ (or $\frac{dx}{du}$)
	$= 2u^{\frac{1}{2}}$
	$= 2\left(\frac{x-1}{x}\right)^{\frac{1}{2}} + c$
1(ii)	Let $x - 2 = s$
	Then $\displaystyle\int \frac{1}{(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}\,\mathrm{d}x = \int \frac{1}{s^{\frac{3}{2}}(s+3)^{\frac{1}{2}}}\,\mathrm{d}s$
	Let $s = \dfrac{3}{u-1}$
	$\displaystyle\int \frac{1}{s^{\frac{3}{2}}(s+3)^{\frac{1}{2}}}\,\mathrm{d}s = \int \frac{(u-1)^2}{3^2u^{\frac{1}{2}}}\frac{-3}{(u-1)^2}\,\mathrm{d}u = -\tfrac{2}{3}u^{\frac{1}{2}}$
	$= -\tfrac{2}{3}\left(\frac{s+3}{s}\right)^{\frac{1}{2}} = -\tfrac{2}{3}\left(\frac{x+1}{x-2}\right)^{\frac{1}{2}} + c$
1(iii)	Let $x = \dfrac{1+u}{u}$ Allow substitution leading to two algebraic factors in the denominator.
	$\displaystyle\int_2^{\infty} \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}}\,\mathrm{d}x = \int_1^0 \frac{u^2}{(1-u)^{\frac{1}{2}}(3+u)^{\frac{1}{2}}}\cdot\left(\frac{-1}{u^2}\right)\mathrm{d}u$ If done through a sequence of substitutions: a further substitution leading to a square root of a quadratic as the denominator.
	$\displaystyle= \int_0^1 \frac{1}{(3-2u-u^2)^{\frac{1}{2}}}\,\mathrm{d}u$

	$= \int_0^1 \frac{1}{\left(4-(1+u)^2\right)^{\frac{1}{2}}}\,\mathrm{d}u$
	$= \left[\arcsin\left(\frac{1+u}{2}\right)\right]_0^1$
	$= \frac{1}{2}\pi - \frac{1}{6}\pi = \frac{1}{3}\pi$

2(i)	$\frac{1-ky}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kx-1}{x}$
	$\ln\lvert y\rvert - ky = kx - \ln\lvert x\rvert + c$
	Hence, $\ln\lvert xy\rvert = k(x+y) + c$
	$xy = \frac{1}{4}\left[(x+y)^2 - (x-y)^2\right] = Ae^{k(x+y)}$
	C_1 is $(x-y)^2 = (x+y)^2 - 2^{x+y}$
	C_2 is $(x-y)^2 = (x+y)^2 - 2^{x+y+4}$
	In both cases, the equation is invariant under $(x,y) \mapsto (y,x)$, so symmetrical in $y = x$.
2(ii)	
	Graphs: Correct shapes of curves
	Graphs: Intersections at (2,4) and (4,16)
	$(x-y)^2 \geq 0$, so $(x+y)^2 > 2^{x+y}$
	Therefore, $(x+y)$ must lie between 2 and 4
	
	Graph: Symmetry about $y = x$
	Graph: Closed curve lying between $x + y = 3 \pm 1$
	Graph: Passes through (1,1) and (2,2)

2(iii)	Sketches of $y = x^2$ and $y = 2^{x+4}$ $x^2 > 2^{x+4}$ only when $x < -2$.
	
	Graph: Symmetry about $y = x$
	Graph: Passes through (-1,-1)
	Graph: $y \to 0$ as $x \to \infty$, $y \to -\infty$ as $x \to 0$

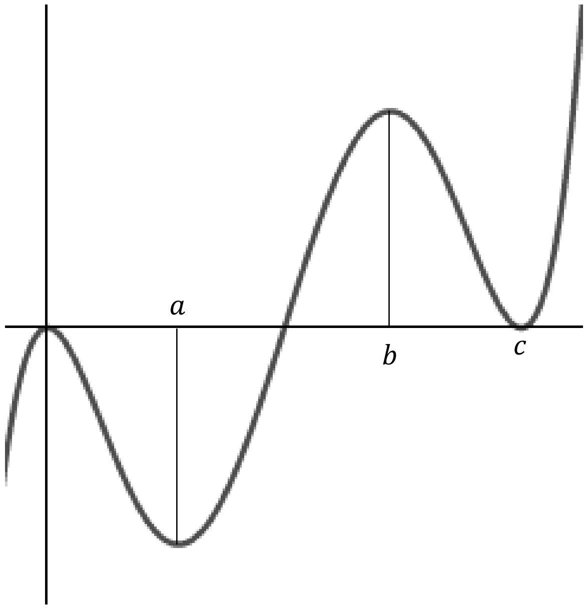
3(i)	Suppose, $\exists k: 2 \le k \le n-1$ such that $u_{k-1} \ge u_k$, but $u_k < u_{k+1}$
	Since all of the terms are positive, these imply that $u_k^2 < u_{k-1}u_{k+1}$, so the sequence does not have property *L*.
	Therefore, if the sequence has property *L*, once a value k has been reached such that $u_{k-1} \ge u_k$, it must be the case that all subsequent terms also have that property (which is the given definition of unimodality).
3(ii)	$u_r - \alpha u_{r-1} = \alpha\left(u_{r-1} - \alpha u_{r-2}\right)$, so $u_r - \alpha u_{r-1} = \alpha^{r-2}\left(u_2 - \alpha u_1\right)$
	$u_r^2 - u_{r-1}u_{r+1} = u_r^2 - u_{r-1}\left(2\alpha u_r - \alpha^2 u_{r-1}\right) = \left(u_r - \alpha u_{r-1}\right)^2$ for $r \geqslant 2$
	The first identity shows that $u_r > 0$ for all *r* if $u_2 > au_1 > 0$.
	Since the right hand side of the second identity is always non-negative, the sequence has property *L*, and is hence unimodal.
3(iii)	$u_1 = (2-1)\alpha^{1-1} + 2(1-1)\alpha^{1-2} = 1$, which is correct. $u_2 = (2-2)\alpha^{2-1} + 2(2-1)\alpha^{2-2} = 2$, which is correct.
	Suppose that: $u_{k-2} = (4-k)\alpha^{k-3} + 2(k-3)\alpha^{k-4}$, and $u_{k-1} = (3-k)\alpha^{k-2} + 2(k-2)\alpha^{k-3}$.
	$u_k = 2\alpha\left((3-k)\alpha^{k-2} + 2(k-2)\alpha^{k-3}\right) - \alpha^2((4-k)\alpha^{k-3} + 2(k-3)\alpha^{k-4})$ $= \alpha^{k-1}(6-2k-4+k) + \alpha^{k-2}(4k-8-2k+6)$ $= \alpha^{k-1}(2-k) + 2\alpha^{k-2}(k-1)$ which is the correct expression for u_k
	Hence, by induction $\mathrm{u_r} = (2-\mathrm{r})\alpha^{\mathrm{r}-1} + 2(\mathrm{r}-1)\alpha^{\mathrm{r}-2}$
	$u_r - u_{r+1} = \left((2-r)\alpha^{r-1} + 2(r-1)\alpha^{r-2}\right) - \left((1-r)\alpha^r + 2r\alpha^{r-1}\right)$
	$= \alpha^{r-2}\left(2(r-1) + (2-3r)\alpha + (r-1)\alpha^2\right)$
	$= \dfrac{\alpha^{r-2}}{N^2}\left(2N^2(r-1) + (2-3r)N(N-1) + (r-1)(N-1)^2\right)$ $= \dfrac{\alpha^{r-2}}{N^2}\left((r-1) + rN - N^2\right)$
	when $r = N$, $u_N - u_{N+1} = \dfrac{\alpha^{r-2}(N-1)}{N^2} > 0$
	when $r = N-1$, $u_{N-1} - u_N = \dfrac{-2\alpha^{r-2}}{N^2} < 0$
	so u_r is largest when $r = N$

4(i)	The straight line distance between two points must be less than the length of any other rectilinear path between the points.
4(ii)	
	Diagram showing two circles and straight line joining their centres. Length of line and radii of circles are a, b and c in some order.
	Either statement that the straight line is the longest of the lengths, or explanation that one circle cannot be contained inside the other.
	Explanation that the circles must meet.
4(iii)	**(A)** If $a + b > c$ then $(a + 1) + (b + 1) > c + 2 > c + 1$ *et cycl.*, so $a + 1$, $b + 1$, $c + 1$ can always form the sides of a triangle.
	(B) If $a = b = c = 1$ we have 1, 1, 1 which can form the sides of a triangle.
	If $a = 1, b = c = 2$ we have $\frac{1}{2}$, 1, 2 which cannot form the sides of a triangle.
	Therefore, $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$ can sometimes, but not always form the sides of a triangle.
	(C) If $p \geq q \geq r$ then $\lvert p - q\rvert + \lvert q - r\rvert = p - q + q - r = p - q = \lvert p - r\rvert$
	So two of $\lvert p - q\rvert$, $\lvert q - r\rvert$, $\lvert p - r\rvert$ will always sum to the third, so they never form the sides of a triangle.
	(D) If $a + b > c$ then $a^2 + bc + b^2 + ca = a^2 + b^2 - 2ab + c(a + b) + 2ab$
	$= (a - b)^2 + c(a + b) + 2ab > c^2 + ab$ *et cycl.* so $a^2 + bc$, $b^2 + ca$, $c^2 + ab$ can always form the sides of a triangle.
4(iv)	Since $a + b > a \text{ and } b$, $\frac{f(a)}{a} > \frac{f(a+b)}{a+b}$ and $\frac{f(b)}{b} > \frac{f(a+b)}{a+b}$
	Since $c < a + b$, $f(c) < f(a + b)$
	Thus $f(a) + f(b) > \frac{af(a)}{a+b} + \frac{bf(b)}{a+b} = f(a + b) > f(c)$ *et cycl.* So $f(a)$, $f(b)$ and $f(c)$ can form the sides of a triangle.

5(i)	$x - q(x) = \sum_{r=0}^{n-1} a_r \times 10^r - \sum_{r=0}^{n-1} a_r = \sum_{r=0}^{n-1} a_r \times (10^r - 1)$
	$10^r \geq 1 \quad \forall r$, so $x - q(x)$ is non-negative
	$9 \mid (10^r - 1) \quad \forall r$
5(ii)	$x - 44q(x) = 44\big(x - q(x)\big) = 43x$
	So it is a multiple of 9 iff $43x$ is.
	$(43,9) = 1$, so $x - 44q(x)$ is a multiple of 9 iff x is
	If x has n digits, $q(x) \leq 9n$
	Since $x = 44q(x)$, $x \leq 396n$. Any n digit number must be at least 10^{n-1}.
	These inequalities cannot be simultaneously true for $n \geq 5$ $(396 \times 5 < 10^4)$. Therefore $n \leq 4$.
	Since $x - 44q(x) = 0$, which is a multiple of 9, x is a multiple of 9.
	$q(x)$ is an integer and $x = 44q(x)$, so x is a multiple of 44. Since $(9,44) = 1$, x must be a multiple of $44 \times 9 = 396$.
	So $x = 396k$ and therefore (by the result above) $k \leq 4$.
	Checking: Only $k = 2$ works.
5(iii)	$x - 107q\big(q(x)\big) = 0 = 107\big(x - q(x)\big) + 107\left(q(x) - q\big(q(x)\big)\right) - 106x$
	$(x - q(x))$ and $(q(x) - q\big(q(x)\big))$ are both divisible by 9 (by part (i)) and so x is divisible by 9
	$x = 107q(q(x))$ and so is divisible by 107, and so is divisible by 963. So $x = 963k$ for some k.
	If x has n digits, then $q(x) \leq 9n$. By (i), $q\big(q(x)\big) \leq q(x) \leq 9n$. So $x \leq 963n$ and $x \geq 10^{n-1}$ which implies that $n \leq 4$ and so $k \leq 4$
	Checking: Only $k = 1$ works.

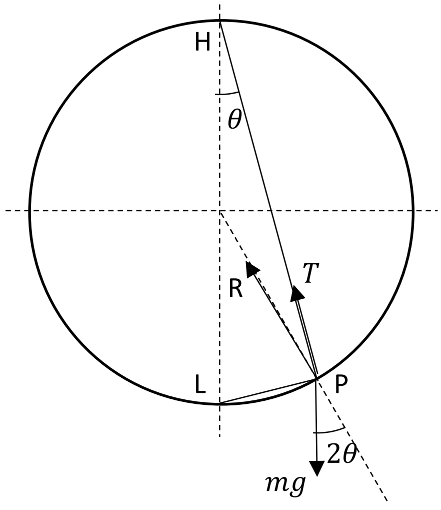
6(i)	Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$
	so Tr($\mathbf{M}^2$) = $a^2 + d^2 + 2bc = (a + d)^2 - 2(ad - bc)$
6(ii)	Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then $\mathbf{M}^2 = \begin{pmatrix} a\tau - \delta & b\tau \\ c\tau & d\tau - \delta \end{pmatrix}$, where τ = Tr($\mathbf{M}$) and δ = Det($\mathbf{M}$).
	Thus $\mathbf{M}^2 = \pm\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = \mp 1$ or $b = c = 0$ and $a^2 = d^2 = \pm 1$
	If $b = c = 0$ and $a = d = \pm 1$, then $\mathbf{M} = \pm\mathbf{I}$
	If $b = c = 0$ and $a = -d = \pm 1$, then $\tau = 0$ and $\delta = -1$
	Thus $\mathbf{M}^2 = +\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = -1$.
	Thus $\mathbf{M}^2 = -\mathbf{I} \Leftrightarrow \tau = 0$ and $\delta = +1$.
6(iii)	Part **(ii)** implies Det($\mathbf{M}^2$) = – 1, if $\mathbf{M}^4 = \mathbf{I}$, but $\mathbf{M}^2 \neq \pm\mathbf{I}$.
	However, Det($\mathbf{M}^2$) = Det($\mathbf{M}$)^2 , so this is impossible.
	Clearly $\mathbf{M}^2 = \pm\mathbf{I} \Rightarrow \mathbf{M}^4 = \mathbf{I}$
	Part **(ii)** implies that $\mathbf{M}^4 = -\mathbf{I} \Leftrightarrow$ Tr($\mathbf{M}^2$) = 0 and Det($\mathbf{M}^2$) = 1
	so from **(i)** $\Leftrightarrow$ Tr($\mathbf{M}$)^2 = 2Det($\mathbf{M}$) and Det($\mathbf{M}$) = ± 1
	so $\Leftrightarrow$ Tr($\mathbf{M}$) = $\pm\sqrt{2}$ and Det($\mathbf{M}$) = 1.
	Any example, for instance a matrix satisfying the conditions for any of $\mathbf{M}^2 = \mathbf{I}$, $\mathbf{M}^2 = -\mathbf{I}$, $\mathbf{M}^4 = -\mathbf{I}$, which is not a rotation or reflection.

7(i)	$\lvert w-1\rvert^2 = \left\lvert\dfrac{1-ti}{1+ti}\right\rvert^2 = \dfrac{(1-ti)(1+ti)}{(1+ti)(1-ti)} = 1$, which is independent of t.
	Points on the line $Re(z) = 3$ have the form $z = 3 + ti$ and the points satisfying $\lvert w-1\rvert = 1$ lie on a circle with centre 1.
	If $z = p + ti$, then $\lvert w-c\rvert^2 = \left\lvert\dfrac{2-(p-2)c-cti}{(p-2)+ti}\right\rvert^2 = \dfrac{(2-(p-2)c)^2+c^2t^2}{(p-2)^2+t^2}$
	which is independent of t when $(2-(p-2)c)^2 = c^2(p-2)^2$
	which is when $c = \dfrac{1}{p-2}$. Thus the circle has centre at $\dfrac{1}{p-2}$ and radius $\dfrac{1}{\lvert p-2\rvert}$.
	$w = \dfrac{2}{(p-2)+ti} = \dfrac{2(p-2)-2ti}{(p-2)^2+t^2}$,
	so $\mathrm{Im}(w) > 0$ when $t < 0$; that is, for those z on V with negative imaginary part.
7(ii)	If $z = t + qi$ then $\lvert w-ci\rvert^2 = \left\lvert\dfrac{2+cq-(t-2)ci}{(t-2)+qi}\right\rvert^2 = \dfrac{c^2(t-2)^2+(cq+2)^2}{(t-2)^2+q^2}$
	which is independent of t when $(cq+2)^2 = c^2q^2$
	which is when $c = -\dfrac{1}{q}$ so the circle has centre $-\dfrac{1}{q}i$ **A1** and radius $\sqrt{c^2} = \dfrac{1}{\lvert q\rvert}$ **A1**.
	$w = \dfrac{2}{(t-2)+qi} = \dfrac{2(t-2)-2qi}{(t-2)^2+q^2}$,
	so $\mathrm{Re}(w) > 0$ when $t > 2$; that is, for those z on H with real part greater than 2.

8(i)	
	Graph: Zeroes at $x = 0, c$ and one other point (h: label not required) in (a, b).
	Graph: Turning points at $x = 0, a, b, c$.
	Graph: Quintic shape with curve below axis in $(0, h)$ and above axis in (h, c)
	The area conditions give $F(0) = F(c) = 0$. $F'(x) = f(x)$, so $F'(0) = F'(a) = F'(b) = F'(c) = 0$
	Since f is a quartic and the coefficient of x^4 is 1, F must be a quintic and the coefficient of x^5 is $\frac{1}{5}$. $F(0) = F'(0) = 0$ and $F(c) = F'(c) = 0$, so F must have double roots at $x = 0$ and c. So $F(x)$ must have the given form. [Explanation must be clear that the double roots are deduced from the fact that $\mathrm{F(x) = F'(x) = 0}$ at those points.]
	$F(x) + F(c - x) = \frac{1}{5}x^2(x - c)^2[(x - h) + (c - x - h)]$ $= \frac{1}{5}x^2(c - x)^2(c - 2h)$
8(ii)	Let A be the (positive) area enclosed by the curve between 0 and a. The maximum turning point of $F(x)$ occurs at $x = b$, with $F(b) = A$. The minimum turning point of $F(x)$ occurs at $x = a$, with $F(a) = -A$.
	Therefore $F(x) \geq -A$, with equality iff $x = a$. So $F(b) + F(x) \geq 0$, with equality iff $x = a$.
	$F(a) + F(x) \leq 0$, with equality iff $x = b$.
	Since $F(b) + F(c - b) = \frac{1}{5}b^2(c - b)^2(c - 2h)$, either $c > 2h$, or $c = 2h$ and $c - b = a$.
	Also, $F(a) + F(c - a) = \frac{1}{5}a^2(c - a)^2(c - 2h)$, so either $c < 2h$, or $c = 2h$ and $c - a = b$.
	Thus $c = a + b$ and $c = 2h$.
8(iii)	$F(x) = \frac{1}{10}x^2(x - c)^2(2x - c)$ So $f(x) = \frac{1}{5}x(x - c)(5x^2 - 5xc + c^2)$

	The roots of the quadratic factor must be a and b.
	$f(x) = \frac{1}{5}(5x^4 - 10cx^3 + 6c^2x^2 - c^3x)$ $f'(x) = \frac{1}{5}(20x^3 - 30cx^2 + 12c^2)$ $f''(x) = \frac{1}{5}(60x^2 - 60cx + 12c^2) = \frac{12}{5}(5x^2 - 5cx + c^2)$
	Therefore $f''(x) = 0$ at $x = a$ and $x = b$ and so $(a, 0)$ and $(b, 0)$ are points of inflection.

9	If the particles collide at time t: $Vt + Ut\cos\theta = d$, and $h - \frac{1}{2}gt^2 = Ut\sin\theta - \frac{1}{2}gt^2$ (or $h = Ut\sin\theta$)
	Therefore, $d\sin\theta - h\cos\theta = Vt\sin\theta + Ut\sin\theta\cos\theta - Ut\sin\theta\cos\theta$ $= \frac{Vh}{U}$
9(i)	Dividing the previous result by $d\cos\theta$ gives: $\tan\theta - \frac{h}{d} = \frac{Vh}{Ud\cos\theta} > 0$
	Since $\tan\beta = \frac{h}{d}$, $\tan\theta > \tan\beta$ and so $\theta > \beta$
9(ii)	The height of collision must be non-negative, so $Ut\sin\theta - \frac{1}{2}gt^2 \geq 0$.
	So $U\sin\theta \geq \frac{1}{2}gt = \frac{1}{2}g\frac{h}{U\sin\theta}$ or $(U\sin\theta)^2 \geq \frac{gh}{2}$ Therefore $U\sin\theta \geq \sqrt{\frac{gh}{2}}$.
9(iii)	$d\sin\theta - h\cos\theta$ can be written as $\sqrt{d^2+h^2}\sin(\theta-\beta)$
	So $d\sin\theta - h\cos\theta < \sqrt{d^2+h^2}$ (since $\theta > \beta$)
	Therefore, $\frac{Vh}{U} < \sqrt{d^2+h^2}$ or $\sin\beta = \frac{h}{\sqrt{d^2+h^2}} < \frac{U}{V}$
	The height at which the particles collide is: $h - \frac{1}{2}gt^2 = h - \frac{gh^2}{2U^2\sin^2\theta}$
	$h - \frac{gh^2}{2U^2\sin^2\theta} > \frac{1}{2}h$ iff $U^2\sin^2\theta > gh$
	The vertical velocity of the particle fired from B at the point of collision is: $U\sin\theta - gt = U\sin\theta - \frac{gh}{U\sin\theta}$
	$U\sin\theta - \frac{gh}{U\sin\theta} > 0$ iff $U^2\sin^2\theta > gh$
	Since both cases have the same condition: The particles collide at a height greater than $\frac{1}{2}h$ if and only if the particle projected from B is moving upwards at the time of collision.

10(i)	
	Diagram showing necessary forces and angles
	$T = \frac{\lambda(2a\cos\alpha - l)}{l}$
	Resolving tangentially: $T\sin\alpha - mg\sin 2\alpha = 0$
	Therefore $\sin\alpha\left(\frac{\lambda}{l}(2a\cos\alpha - l) - 2mg\cos\alpha\right) = 0$
	Since $\sin\alpha > 0$, $2a\lambda\cos\alpha - \lambda l - 2mgl\cos\alpha = 0$ $\cos\alpha = \frac{\lambda l}{2(a\lambda - mgl)}$
	$\cos\alpha < 1$, so $\lambda l < 2(a\lambda - mgl)$ Therefore $\lambda(2a - l) > 2mgl$
	Since $2a - l > 0$, $\lambda > \frac{2mgl}{2a-l}$
10(ii)	Energy: $\frac{1}{2}mv^2 - mga\cos 2\theta + \frac{\lambda}{2l}(2a\cos\theta - l)^2 = \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2$
	If the particle comes to rest when $\theta = \beta$: $-mga\,(2\cos^2\beta - 1) + \frac{\lambda}{2l}(2a\cos\beta - l)^2 = \frac{1}{2}mu^2 - mga + \frac{\lambda}{2l}(2a - l)^2$
	a$\lambda\cos^2\beta\left(\frac{2(a\lambda - mgl)}{\lambda l}\right) - 2a\lambda\cos\beta = \frac{1}{2}mu^2 - 2mga + \frac{2\lambda a^2}{l} - 2a\lambda$
	Therefore, $\cos^2\beta - 2\cos\alpha\cos\beta = \frac{mu^2}{2a\lambda}\cos\alpha + 1 - 2\cos\alpha$
	Adding $\cos^2\alpha$ to both sides: $(\cos\alpha - \cos\beta)^2 = (1 - \cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha$
	For this to occur, $\cos\beta > 0$:
	$\cos^2\alpha > (1 - \cos\alpha)^2 + \frac{mu^2}{2a\lambda}\cos\alpha$
	And so, $u^2 < \frac{2a\lambda}{m}(2 - \sec\alpha)$

11(i)	If the game has not ended after $2n$ turns, then the sequence has either been n repetitions of HT or n repetitions of TH. So $P(Game\ has\ not\ finished\ after\ 2n\ turns) = 2(pq)^n$. So the probability that the game never ends is $\lim_{n\to\infty} 2(pq)^n = 0$.
	Sequence that follows the first H will be k repetitions of TH, followed by H, where $k \geq 0$.
	So $P(A\ wins\|first\ toss\ is\ H) = \sum_{k=0}^{\infty}(pq)^k p = \frac{p}{1-pq}$
	$P(A\ wins \cap first\ toss\ is\ H) = p \times \frac{p}{1-pq}$
	If first toss is a tail then the sequence that follows would be k repetitions of HT followed by HH.
	So $P(A\ wins\|first\ toss\ is\ T) = \frac{p^2}{1-pq}$
	$P(A\ wins \cap first\ toss\ is\ T) = \frac{p^2 q}{1-pq}$
	Therefore $P(A\ wins) = \frac{p^2(1+q)}{1-pq}$
11(ii)	Following a first toss of H: A wins with HH or (HT followed by any sequence where A wins after first toss was T) or (T followed by any sequence where A wins after first toss was T)
	The probabilities of these cases are: p^2 $pq\ P(A \text{ wins } \| \text{the first toss is a tail})$ $q\ P(A \text{ wins } \| \text{the first toss is a tail})$
	Therefore: $P(\text{A wins } \| \text{ the first toss is a head}) = p^2 + (q + pq)P(\text{A wins } \| \text{ the first toss is a tail})$

	Similarly, following first toss of T: A wins with (H followed by any sequence where A wins after first toss was H) or (TH followed by any sequence where A wins after first toss was H)
	Therefore: $P(\text{A wins} \mid \text{the first toss is a tail}) = (p + pq)P(\text{A wins} \mid \text{the first toss is a head})$
	So $P(A \mid H\ first) = p^2 + (q + pq)(p + pq)P(A \mid H\ first)$ $P(A \mid H\ first) = \frac{p^2}{1-(p+pq)(q+pq)}$
	And $P(A \mid T\ first) = (p + pq)(p^2 + (q + pq)P(A \mid T\ first))$ $P(A \mid T\ first) = \frac{p^2(p+pq)}{1-(p+pq)(q+pq)}$
	So $P(A\ wins) = p \times \frac{p^2}{1-(p+pq)(q+pq)} + q \times \frac{p^2(p+pq)}{1-(p+pq)(q+pq)} = \frac{p^2(1-q^3)}{1-(1-p^2)(1-q^2)}$
11(iii)	Let W be the event that A wins the game. $P(W \mid H\ first) = p^{a-1} + (1 + p + p^2 + \cdots + p^{a-2})qP(W \mid T\ first)$
	$P(W \mid T\ first) = \left(1 + q + q^2 + \cdots + q^{b-2}\right)pP(W \mid H\ first)$
	$P(W \mid H\ first) = \frac{p^{a-1}}{1-(1-p^{a-1})(1-q^{b-1})}$
	$P(W \mid T\ first) = \frac{p^{a-1}(1-q^{b-1})}{1-(1-p^{a-1})(1-q^{b-1})}$
	Therefore: $P(W) = \frac{p^{a-1}(1-q^b)}{1-(1-p^{a-1})(1-q^{b-1})}$
	If $a = b = 2$, $P(W) = \frac{p(1-q^2)}{1-(1-p)(1-q)} = \frac{p^2(1+q)}{1-pq}$ as expected.

12(i)	For the biased die: $P(R_1 = R_2) = \sum_{i=1}^{n} \left(\frac{1}{n} + \varepsilon_i\right)^2$
	$P(R_1 = R_2) = \frac{1}{n^2} \sum_{i=1}^{n} 1 + \frac{2}{n} \sum_{i=1}^{n} \varepsilon_i + \sum_{i=1}^{n} {\varepsilon_i}^2$
	$\sum_{i=1}^{n} \varepsilon_i = 0$, so $P(R_1 = R_2) = \frac{1}{n} + \sum_{i=1}^{n} {\varepsilon_i}^2$
	For a fair die, $P(R_1 = R_2) = \frac{1}{n}$ and $\sum_{i=1}^{n} {\varepsilon_i}^2 > 0$, so it is more likely with the biased die.
(ii)	$P(R_1 > R_2) = \frac{1}{2}\big(1 - P(R_1 = R_2)\big)$
	Therefore, the value of $P(R_1 > R_2)$ if the die is possibly biased is $\leq P(R_1 > R_2)$ if the die is fair.
	Let T= $\sum_{r=1}^{n} x_r$ and, for each i, let $p_i = \frac{x_i}{T}$ Then $\sum_{i=1}^{n} p_i = 1$, so we can construct a biased n-sided die with $P(X = i) = p_i$
	$P(R_1 > R_2) = \sum_{i=2}^{n} \sum_{j=1}^{i-1} p_i p_j$
	For a fair die: $P(R_1 > R_2) = \frac{n-1}{2n}$
	Therefore $\sum_{i=2}^{n} \sum_{j=1}^{i-1} \frac{x_i x_j}{T^2} \leq \frac{n-1}{2n}$ and so $$\sum_{i=2}^{n} \sum_{j=1}^{i-1} x_i x_j \leq \frac{n-1}{2n} \left(\sum_{i=1}^{n} x_i \right)^2$$
(iii)	For the biased die: $P(R_1 = R_2 = R_3) = \sum_{i=1}^{n} \left(\frac{1}{n} + \varepsilon_i\right)^3$
	$= \sum_{i=1}^{n} \frac{1}{n^3} + \sum_{i=1}^{n} \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n} {\varepsilon_i}^3$
	Therefore $$P(R_1 = R_2 = R_3\ biased) - P(R_1 = R_2 = R_3\ fair) = \sum_{i=1}^{n} \frac{3\varepsilon_i}{n^2} + \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n} {\varepsilon_i}^3$$ $= \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \sum_{i=1}^{n} {\varepsilon_i}^3$ (since $\sum_{i=1}^{n} \varepsilon_i = 0$)
	$= \sum_{i=1}^{n} \frac{3\varepsilon_i^2}{n} + \varepsilon_i^3 = \sum_{i=1}^{n} \varepsilon_i^2 \left(\frac{3}{n} + \varepsilon_i\right)$
	But $\varepsilon_i \geq -\frac{1}{n}$ (since $p_i \geq 0$), so this sum must be positive.
	Therefore, $P(R_1 = R_2 = R_3)$ must be greater for the biased die.