STEP I 2016 Solutions

Question 1

As a starter to the paper, this is a straightforward question in terms of its early demands and involves little more than the need to sort out some clearly-signposted algebra. To begin with, it is clear that, whenever n is odd, the expression (x^n + 1) has (x + 1) as a factor (by the *Factor Theorem*), so that

$$q_n(x) = x^{2n} - x^{2n-1} + x^{2n-2} + \ldots - x^3 + x^2 - x + 1.$$

Examining the $p_n(x)$'s in turn, using the binomial theorem (and *Pascal's Triangle* for the coefficients), gives

$p_1(x) = \left(x^2 + 2x + 1\right) - 3x(1) = x^2 - x + 1$,

$p_2(x) = \left(x^4 + 4x^3 + 6x^2 + 4x + 1\right) - 5x(x^2 + x + 1) = x^4 - x^3 + x^2 - x + 1$, and

$p_3(x) = \left(x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1\right) - 7x(x^2 + x + 1)^2$.

Expanding $(x^2 + x + 1)^2$ is relatively straightforward, and it is relatively easy to obtain the required results.

There are several ways to demonstrate that two given expressions of the given kind are not identically equal. One is to expand them both as polynomials and show that they are not the same. In this case,

$p_4(x) = x^8 - x^7 + x^6 + 2x^5 + 7x^4 + 2x^3 + x^2 - x + 1$ while $q_4(x) = x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$.

Alternatively, one need only show that one corresponding pair of coefficients are not the same – here, the coefficients of (say) x^5 are not equal. However, the simplest thing is to find any one value of x for which the two expressions give different values. It turns out, in fact, that only x = 0 actually does give equal outputs, so almost any chosen value of x would suffice, and the key is then to choose one for which the working involves the minimum of effort, such as $p_4(1) = 2^8 - 9.1.3^3 = 13 \neq q_4(1) = \dfrac{1^9 + 1}{1 + 1} = 1$.

In (ii) (a), the given numerical expression is clearly that for $q_1(x)$ with x = 300. Since p and q are the same thing when n = 1, we instead examine $p_1(300)$, and it becomes clear that if x is 3 times a perfect square (in this case 3×10^2) then we can use the difference-of-two-squares factorisation on $(301)^2 - (3\times10)^2$ to get the answer 271×331.

Part (b) has a similar thing going on, but here we need x to be 7 times a perfect square, and we find that we have $\left[\left(7^7 + 1\right)^3\right]^2 - 7^8\left(7^{14} + 7^7 + 1\right)^2$, which again requires the use of the difference-of-two-squares factorisation and yields $\left[\left(7^7 + 1\right)^3 - \left(7^{18} + 7^{11} + 7^4\right)\right] \times \left[\left(7^7 + 1\right)^3 + \left(7^{18} + 7^{11} + 7^4\right)\right]$ or

$$\left(7^{21} + 3.7^{14} + 3.7^7 + 1 - 7^{18} - 7^{11} - 7^4\right) \times \left(7^{21} + 3.7^{14} + 3.7^7 + 1 + 7^{18} + 7^{11} + 7^4\right),$$

either of which answers would suffice.

Question 2

Once again, this question begins with a very simple instruction, to differentiate a function of *x*, and will clearly involve the use of both the *Product Rule* (twice) and the *Chain Rule* (in order to deal with the log. term and the square-root). In principle, this looks very straightforward, though the key is to be careful not to overlook some aspect of the various processes at play, and then to simplify the resulting expressions in a suitable way. To begin with, one will obtain something that looks quite messy:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left(ax^2+bx+c\right)\frac{1}{x+\sqrt{1+x^2}}\times\left(1+\tfrac{1}{2}\left[1+x^2\right]^{-\frac{1}{2}}.2x\right)+\left(2ax+b\right)\ln\left(x+\sqrt{1+x^2}\right)$$
$$+\ \left(dx+e\right)\left(\tfrac{1}{2}\left[1+x^2\right]^{-\frac{1}{2}}.2x\right)+d\sqrt{1+x^2}$$

and it is easy to be put off; it is especially important not to attempt too much "in your head". You should find this simplifies to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ax^2+bx+c}{\left[x+\sqrt{1+x^2}\right]}\times\frac{\left[\sqrt{1+x^2}+x\right]}{\sqrt{1+x^2}} + \left(2ax+b\right)\ln\left(x+\sqrt{1+x^2}\right) + \frac{x\left(dx+e\right)}{\sqrt{1+x^2}}+d\sqrt{1+x^2}$$

and collecting up terms suitably, and noting that $\frac{1}{\sqrt{1+x^2}} \equiv \frac{\sqrt{1+x^2}}{1+x^2}$ leads to an expression which contains only simple multiples of $\frac{1}{\sqrt{1+x^2}}$ and $\ln\left(x+\sqrt{1+x^2}\right)$; namely

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(a+2d)x^2+(b+e)x+(c+d)}{\sqrt{1+x^2}} + \left(2ax+b\right)\ln\left(x+\sqrt{1+x^2}\right).$$

All the results of the remaining parts of the question can now be deduced by choosing suitable values for the constants *a* to *e*.

In (i), choosing $a = d = 0$, $b = 1$, $e = -1$ and $c = 0$ gives $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(0)x^2+(0)x+(0)}{\sqrt{1+x^2}} + (0+1)\ln\left(x+\sqrt{1+x^2}\right)$, so that $\int \ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}x = x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}$ (+ *C*).

In (ii), choosing $a = b = e = 0$ and $c = d = \frac{1}{2}$ gives $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(0+1)x^2+(0)x+(1)}{\sqrt{1+x^2}} + (0+0)\ln\left(x+\sqrt{1+x^2}\right)$, so that $\int \sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)+\frac{1}{2}x\sqrt{1+x^2}$ (+ *C*) .

And in (iii), choosing $a = \frac{1}{2}$, $b = e = 0$, $c = \frac{1}{4}$ and $d = -\frac{1}{4}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(\frac{1}{2}-\frac{1}{2})x^2+(0)x+(\frac{1}{4}-\frac{1}{4})}{\sqrt{1+x^2}} + (x+0)\ln\left(x+\sqrt{1+x^2}\right)$$

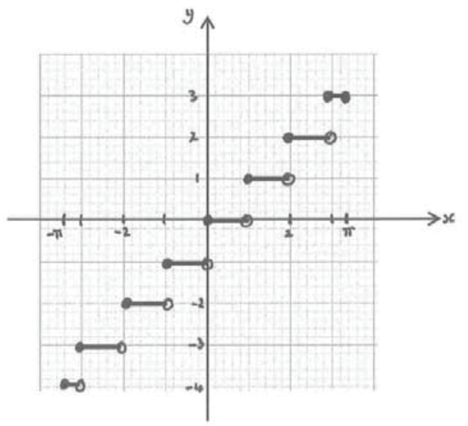
and hence $\int x\ln\left(x+\sqrt{1+x^2}\right)\mathrm{d}x = \left(\frac{1}{2}x^2+\frac{1}{4}\right)\ln\left(x+\sqrt{1+x^2}\right)-\frac{1}{4}x\sqrt{1+x^2}$ (+ *C*).

Question 3

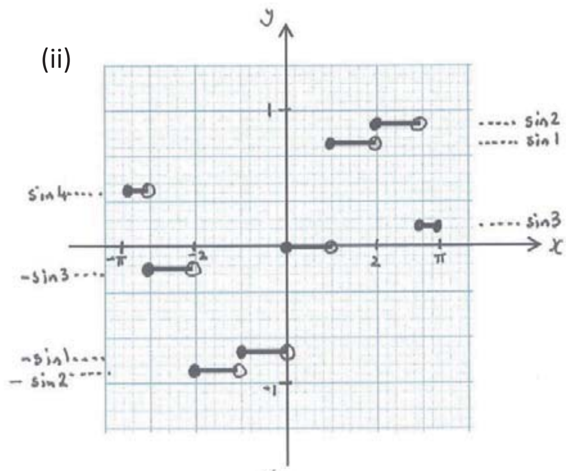
If you have not seen this sort of function before, then it is worthwhile playing around with such things as part of your preparation for the STEPs, which frequently test perfectly simple ideas in contexts that are not a standard part of A-level (or equivalent) courses. Being able to think things through calmly and carefully under examination conditions is an especially high-level skill, but one that can be practised.

In this case, the "integer part" function is a relatively simple one to deal with, as it only changes values when the function it acts on hits an integer value. Before commencing work on this question, note that the "integer part" of a negative number is the one to the left of it (if it lies between integers, of course), and many function-plotting packages are set to "go right" for negative numbers, which is unfortunate. There is also the small matter of how to illustrate the "*y*" values at those points when the "step" occurs … the tradition is to employ a "filled" circle (●) for inclusion and an "open" circle (○) for exclusion. With a bit of care you should find that the four graphs required here look as follows.

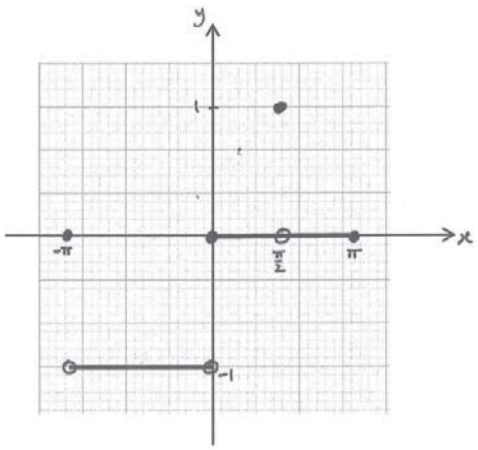
(i)



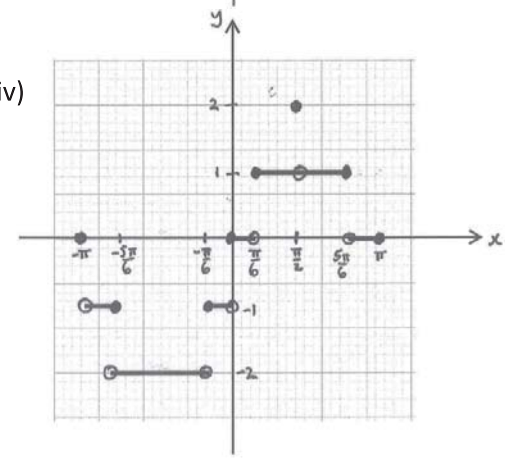
(ii)



(iii)



(iv)



Question 4

As with Q2, this begins with a simple instruction to differentiate; again, the idea is that you will tidy up the final answer for later reference. Using the *Quotient* and *Chain Rules* on $y = \frac{z}{\sqrt{1+z^2}}$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{\sqrt{1+z^2}.1 - z.\tfrac{1}{2}\left(1+z^2\right)^{-\frac{1}{2}}.2z}{\left(\sqrt{1+z^2}\right)^2}$$

which simplifies to $\frac{1}{\left(1+z^2\right)^{\frac{3}{2}}}$.

The given expression in part (ii) initially appears to be quite awful, until you realise that writing, for instance, $z = \frac{\mathrm{d}y}{\mathrm{d}x}$ turns $\frac{\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)}{\left\{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right\}^{\frac{3}{2}}} = \kappa$ into $\frac{\frac{\mathrm{d}z}{\mathrm{d}x}}{\left(1+z^2\right)^{\frac{3}{2}}} = \kappa$, and this can now be seen to be a standard "separable variables" first-order differential equation: $\int \frac{\mathrm{d}z}{\left(1+z^2\right)^{\frac{3}{2}}} = \int \kappa\,\mathrm{d}x$. Using (i)'s result then gives $\frac{z}{\sqrt{1+z^2}} = \kappa(x+c)$ (where the usual "+ *c*" has been incorporated into a slightly more helpful form here).

Re-arranging this for *z* or z^2 leads to $z^2 = \kappa^2(x+c)^2(z^2+1) \Rightarrow z = \pm\frac{u}{\sqrt{1-u^2}}$ where (again) the more complicated-looking term has been given a new label, which is a simple but effective device to make what to do next more obvious: here, $u = \kappa(x + c)$.

We now substitute back for $z = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}.\frac{\mathrm{d}u}{\mathrm{d}x}$ and use the *Chain Rule* (e.g.) with $\frac{\mathrm{d}u}{\mathrm{d}x} = \kappa$ to obtain another "separable variables" first-order differential equation, $\kappa\frac{\mathrm{d}y}{\mathrm{d}u} = \pm\frac{u}{\sqrt{1-u^2}}$ or $\int \kappa\,\mathrm{d}y = \pm\int\frac{u}{\sqrt{1-u^2}}\mathrm{d}u$. At this point, you should be able to see that $\int\frac{u}{\sqrt{1-u^2}}\mathrm{d}u$ can be integrated (by "recognition", "reverse chain rule" or a substitution) to give $-\sqrt{1-u^2}$. Substituting for *u* then gives $ky + d = \mp\sqrt{1-\kappa^2(x+c)^2}$ and squaring both sides leads towards a circle equation $(\kappa y + d)^2 = 1 - \kappa^2(x+c)^2$ or $(x+c)^2 + \left(y + \frac{d}{\kappa}\right)^2 = \left(\frac{1}{\kappa}\right)^2$, which is the equation of a circle, with centre $\left(-c, -\frac{d}{\kappa}\right)$ and a radius which is the reciprocal of the curvature .

Question 5

This sort of situation is relatively common in '*Maths Challenges*' and the usual approach is to join all the circles' centres to the points of tangential contact and then form some right-angled triangles by considering (here) the horizontal line through B's centre. Because of the well-known (GCSE) *Circle Theorem* "tangent perpendicular to radius" result, it is the case that each of *AB*, *BC* and *CA* is a straight line. This enables us to use *Pythagoras' Theorem*:

$$PR = PQ + QR \Rightarrow \sqrt{(a+c)^2-(a-c)^2} = \sqrt{(b+a)^2-(b-a)^2} + \sqrt{(c+b)^2-(c-b)^2}$$

which simplifies to $\sqrt{4ac} = \sqrt{4ab} + \sqrt{4bc}$ and, upon division throughout by $\sqrt{4abc}$, gives the required answer $\dfrac{1}{\sqrt{b}} = \dfrac{1}{\sqrt{c}} + \dfrac{1}{\sqrt{a}}$.

There are many ways to approach the next result, but it should be clear that each will, at some stage, require the replacement of the *b*'s with *a*'s and *c*'s (or equivalent). The most direct route would be to examine the LHS and RHS of (**) separately, and then show that they match up. This would look like:

LHS = $2\left(\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}\right) = \dfrac{2}{a^2}+\dfrac{2}{c^2}+2\left(\dfrac{1}{a^2}+\dfrac{4}{a\sqrt{ac}}+\dfrac{6}{ac}+\dfrac{4}{c\sqrt{ac}}+\dfrac{1}{c^2}\right) = \dfrac{4}{a^2}+\dfrac{12}{ac}+\dfrac{4}{c^2}+\dfrac{8}{a\sqrt{ac}}+\dfrac{8}{c\sqrt{ac}}$.

RHS = $\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)^2 = \left(\dfrac{1}{a}+\left\{\dfrac{1}{a}+\dfrac{2}{\sqrt{ac}}+\dfrac{1}{c}\right\}+\dfrac{1}{c}\right)^2 = \left(\dfrac{2}{a}+\dfrac{2}{\sqrt{ac}}+\dfrac{2}{c}\right)^2 = 4\left(\dfrac{1}{a}+\dfrac{1}{\sqrt{ac}}+\dfrac{1}{c}\right)^2$

$= 4\left(\dfrac{1}{a^2}+\dfrac{3}{ac}+\dfrac{1}{c^2}+\dfrac{2}{a\sqrt{ac}}+\dfrac{2}{c\sqrt{ac}}\right)$, and these are clearly the same.

Working in the other direction is trickier, but not much more so, and it is again helpful to re-label the variables to make things *look* simpler, especially if we can somehow remove the need for everything to appear as a fraction. So, following an initial observation that

$$2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{2}{ab}+\frac{2}{bc}+\frac{2}{ca}$$

we could write $x = \dfrac{1}{\sqrt{a}}$, $y = \dfrac{1}{b}$, $z = \dfrac{1}{\sqrt{c}}$, so that we are now trying to prove that

$$x^4+y^4+z^4 = 2x^2y^2+2y^2z^2+2z^2x^2.$$

(Although it is not essential to do this at this stage, it is often the case that folks forget to do it at the end if they don't; and that is to consider the given conditions $b < c < a$, which translate to $y > z > x$.)

Now, completing the square: $\left(x^2+z^2-y^2\right)^2 = 4x^2z^2 \Leftrightarrow x^2+z^2-y^2 = \pm 2xz \Leftrightarrow (z \mp x)^2 = y^2$, and there are the four cases to consider: $y = x - z$, $y = z - x$, $y = x + z$ or $y = -x - z$. Consideration of the above conditions on *x*, *y*, *z* then shows that only $y = x + z$ is suitable, and so $\dfrac{1}{\sqrt{b}} = \dfrac{1}{\sqrt{c}} + \dfrac{1}{\sqrt{a}}$, as required.

Question 6

This is a fairly straightforward vectors question that involves little beyond working with the vector equation of a line. To begin with, you are required to explain a couple of introductory results that rely on the fact that two (non-zero) vectors are multiples of each other if and only if they are parallel. Thus, $OX \parallel OA \Rightarrow \mathbf{x} = m\mathbf{a}$, with $0 < m < 1$, since X is between O and A; and $BC \parallel OA \Rightarrow \mathbf{c} - \mathbf{b} = k\mathbf{a}$ and so $\mathbf{c} = k\mathbf{a} + \mathbf{b}$, with $k < 0$ since BC is in the opposite direction to OA.

Then, lines OB and AC have vector equations $\mathbf{r} = \beta\,\mathbf{b}$ and $\mathbf{r} = \mathbf{a} + \alpha\,(\mathbf{c} - \mathbf{a})$ respectively, for some scalar parameters α and β. Replacing $\mathbf{c}$ by $k\mathbf{a} + \mathbf{b}$ and equating the two $\mathbf{r}$'s for the point of intersection then gives $\beta\,\mathbf{b} = \mathbf{a} + \alpha\,(k\mathbf{a} + \mathbf{b} - \mathbf{a})$. Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, we can equate terms to find that $1 - \alpha + \alpha k = 0$ and $\alpha = \beta$. Solving leads to $\alpha = \beta = \frac{1}{1-k}$, so that $\mathbf{d} = \frac{1}{1-k}\mathbf{b}$.

In an exactly similar way, we then have $Y = XD \cap BC \Rightarrow m\mathbf{a} + \alpha\left(\frac{1}{1-k}\mathbf{b} - m\mathbf{a}\right) = \mathbf{b} + \beta k\,\mathbf{a}$. (Note that there is no reason why we have to use different symbols for the scalar parameters each time, as they are of no actual significance in relation to the problem.) Equating coefficients $\Rightarrow m - \alpha m - \beta k = 0$ and $\frac{\alpha}{1-k} = 1$, so that $\mathbf{y} = km\mathbf{a} + \mathbf{b}$.

Next, $Z = OY \cap AB \Rightarrow (1 - \alpha)\mathbf{a} + \alpha\mathbf{b} = \beta\big(km\mathbf{a} + \mathbf{b}\big)$, and so $1 - \alpha - km\beta = 0$ and $\alpha = \beta \left(= \frac{1}{1+km}\right)$, giving $\mathbf{z} = \left(\frac{km}{1+km}\right)\mathbf{a} + \left(\frac{1}{1+km}\right)\mathbf{b}$; and $T = DZ \cap OA \Rightarrow \alpha\,\mathbf{a} = \frac{1}{1-k}\mathbf{b} + \beta\left(\frac{km}{1+km}\mathbf{a} + \frac{1}{1+km}\mathbf{b} - \frac{1}{1-k}\mathbf{b}\right)$, whence $\alpha = \frac{\beta km}{1+km}$ and $0 = \frac{1-\beta}{1-k} + \frac{\beta}{1+km}$, so that $\mathbf{t} = \left(\frac{m}{1+m}\right)\mathbf{a}$.

Notice that all that has been done so far is to work out the position vectors of the points created as the intersections of various pairs of lines. If this is difficult to visualize, then a diagram should be drawn first.

All that remains is to set up the lengths of the various line segments of interest. If we call $OA = a$, then it follows that $OX = ma$, $OT = \left(\frac{m}{1+m}\right)a$, $TX = \left(\frac{m^2}{1+m}\right)a$, $TA = \left(\frac{1}{1+m}\right)a$ and $XA = (1 - m)a$. (Note that the shrewd solver would simply take, w.l.o.g., the value of a to be 1, as it is an arbitrary length and cancels throughout any of the working that follows in order to obtain the two given answers,

$$\frac{1}{OT} = \frac{1}{a}\left(1 + \frac{1}{m}\right) = \frac{1}{OA} + \frac{1}{OX} \text{ and } OT.OA = \left(\frac{m}{1+m}\right)a^2 = (ma).\left(\frac{1}{1+m}\right)a = OX.TA.$$

Question 7

Firstly, $S \cup T$ = the set of all positive odd numbers; and $S \cap T = \phi$.

Next, we must show that the product of two different elements of S is also an element of S, and then do a similar thing for two different elements of T. This involves noting that

$$(4a + 1)(4b + 1) = 4(4ab + a + b) + 1,$$

which is an element of S; and

$$(4a + 3)(4b + 3) = 4(4ab + 3a + 3b + 2) + 1,$$

which is, in fact, in S rather than T.

The result of part (iii) essentially requires proof by contradiction, so we first suppose that t is an element of T, and that all of t's prime factors are in S. Noting that there are no even factors, we can write

$$t = (4a + 1)\ (4b + 1)\ (4c + 1) \ldots (4n + 1).$$

But we have already noted that the product of any pair of elements of S will always yield another element of S, and hence (inductively, I suppose), $t = 4\{ \ldots \ldots \ldots\} + 1$ is always in S. Which contradicts the assumption that t is an element of T.

For part (iv) (a), we note that an element of T is either T-prime or T-composite. If it is the latter, then it can be expressed as a product of T-primes. However, we have already established that every pair of factors in T multiply to give an element of S, as do every pair of elements of T. So, after every pairing-up of this element's factors, there must be an odd one left over to multiply by in order to give an element of T. Hence, altogether, there is an odd number of them.

For the final part of the question, we are required to find non-primes in S that are products of elements of T that can be "re-grouped" suitably. One such example involves the numbers 9, 21, 33 and 77, each of which is both S-prime and a product of elements in T:

$9 = 3 \times 3$, $21 = 3 \times 7$, $33 = 3 \times 11$ and $77 = 7 \times 11$; with 3, 7 and 11 in T.

This leads to the example $9 \times 77 = 21 \times 33$ (= 693). Of course, the main purpose of the question is to demonstrate the existence of perfectly reasonable number-sets (in this case, S), having the standard properties of multiplication, yet for which the "unique factorisation" principle no longer holds. This is a very important principle relating to prime numbers within the set of positive integers, which you have been taught (quite rightly) to assume, but that I imagine you have never had reason to think that it might not necessarily hold in this case, or in other similar situations.

Question 8

This is a simple idea involving terms of series. The most important thing here is to be sure that you can justify the form of the n^{th} term for any given series. To begin with, it is helpful to realise that the *Binomial Theorem* applied to the negative integers, gives the following results:

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots$, with the coefficient of x^n being 1;

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots$, with the coefficient of x^n being $(n + 1)$; and

$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \tfrac{1}{2}(n+1)(n+2)x^n + \ldots$, having the *triangular numbers* as coefficients; etc.

Having established these results, it is easy to show that

$$0 + x + 2x^2 + 3x^3 + \ldots + nx^n + \ldots = x(1-x)^{-2}$$

and that

$$x(1-x)^{-3} = x\left(1 + 3x + 6x^2 + 10x^3 + \ldots + \tfrac{1}{2}n(n+1)x^{n-1} + \ldots\right) = 0 + x + 3x^2 + 6x^3 + \ldots + \tfrac{1}{2}n(n+1)x^n + \ldots$$

has coefficient of x^n equal to $u_n = \tfrac{1}{2}n^2 + \tfrac{1}{2}n$.

Using these first two results: $2 \times (2^{\text{nd}}) - (1^{\text{st}})$ gives $\dfrac{2x}{(1-x)^3} - \dfrac{x}{(1-x)^2}$ with $u_n = n^2$.

There are several ways to proceed with part (ii) (a); the simplest being to note that

$$a + akx + ak^2x^2 + ak^3x^3 + \ldots + ak^nx^n + \ldots = a + kx\left(a + akx + ak^2x^2 + ak^3x^3 + \ldots + ak^nx^n + \ldots\right)$$

$$= a + kx\,\mathrm{f}(x)$$

and so $\mathrm{f}(x) = a\left(\dfrac{1}{1-kx}\right)$.

For part (ii) (b), you should note that the given *second*-order recurrence relation (i.e. *two* preceding terms are involved) requires two starting terms before things get going systematically, so it is best to split off the first two terms of the series before attempting to make use of this defining feature.

Writing $\mathrm{f}(x) = 0 + x + \sum_{n=2}^{\infty} u_n x^n = 0 + x + \sum_{n=2}^{\infty}(u_{n-1} + u_{n-2})x^n = x + x\sum_{n=2}^{\infty} u_{n-1}x^{n-1} + x^2\sum_{n=2}^{\infty} u_{n-2}x^{n-2}$

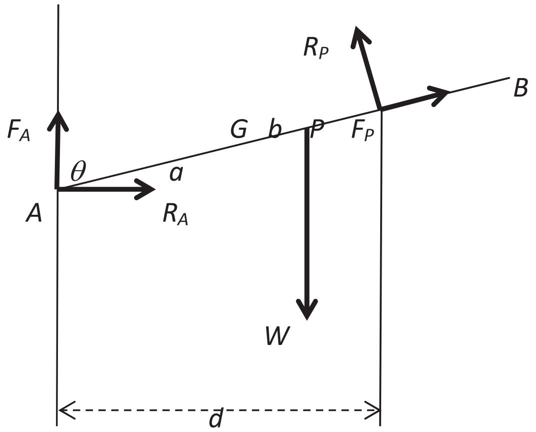
Note: we are trying to re-create f(x) on the right-hand side

$$= x + x\sum_{n=1}^{\infty} u_n x^n + x^2\sum_{n=0}^{\infty} u_n x^n = x + x\{\mathrm{f}(x) - 0\} + x^2\mathrm{f}(x)$$

so that $\mathrm{f}(x) = \dfrac{x}{1-x-x^2}$.

Question 9

As with all *Statics* problems, it is so important to have a clear diagram, suitably marked with all the relevant forces (and preferably in the correct directions). The given diagram appears to give the situation when the horizontal rail is more than half-way along the rod; in this configuration, the rod will slip so that the end *A* slides down the wall, and to the left relative to the contact at *P* (as shown below).



The next essential part of any solution is to write down all the key statements before attempting to process them in some way. The guidance "resolve twice and take moments" is surely a stock part of every mechanics teacher's repertoire, and it is sound advice. It is customary to resolve in two perpendicular directions (vertically and horizontally here) and to find some point about which to take moments that minimises the "clutter" of subsequent algebra-and-trigonometry (*A* has been chosen here). The additional use of the *Friction Law* for, on this occasion, the case of limiting equilibrium at the two points of contact is also needed. This gives us the following "ingredients" for use in a solution.

Friction Law $F_A = \lambda R_A$ and $F_P = \mu R_P$.

Res.↑ $W = F_A + R_P \sin\theta + F_P \cos\theta$

Res.→ $R_A = R_P \cos\theta - F_P \sin\theta$

↻A $W\,a \sin\theta = R_P\,(a + b)$

Eliminating the *F*'s from the two resolving statements gives

$$W = \lambda R_A + R_P \sin\theta + \mu R_P \cos\theta \quad \text{and} \quad R_A = R_P \cos\theta - \mu R_P \sin\theta$$

and introducing *d* in the moments statement (noting that it needs to appear in the answer) gives

$$W\,a \sin^2\theta = R_P\,d.$$

Since this last equation involves just the forces *W* and R_P it makes sense to eliminate R_A next, to get

$$W = R_P\left(\lambda\cos\theta - \lambda\mu\sin\theta + \sin\theta + \mu\cos\theta\right).$$

Finally, dividing these last two leads to the given answer,

$$d\operatorname{cosec}^2\theta = a\left([\lambda+\mu]\cos\theta + [1-\lambda\mu]\sin\theta\right).$$

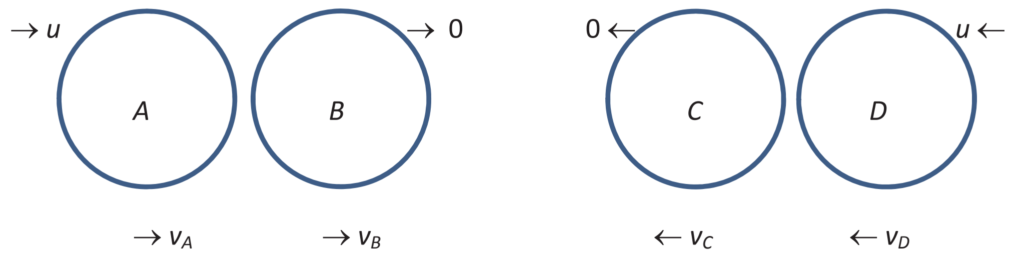
For the case when *P* is less than half-way along the rod, which will now slide up the wall at *A* (etc.), we can simply write $F_A \to -F_A$; $F_P \to -F_P$; $a + b \to a - b$, or just switch the signs of λ and μ, as everything else remains unchanged. The corresponding result is thus shown to be

$$d\operatorname{cosec}^2\theta = a\left(-[\lambda+\mu]\cos\theta + [1-\lambda\mu]\sin\theta\right).$$

Question 10

When it comes to it, collisions questions involve only the use of two (sometimes three, if energy considerations are involved) principles. In this case, the principles of *Conservation of Linear Momentum* (CLM) and *Newton's (Experimental) Law of Restitution* (NEL or NLR). Diagrams are also quite important in these types of questions, but largely to enable the solver to be clear about what directions are being taken as positive: remember that velocity and momentum are vectorial in nature. If you are ever unsure about which direction any of the objects (particles or spheres, etc.) will be moving in after the collision has taken place, then just have them all going the same way; this makes it much easier to interpret negative signs in any later answers.

To begin with, there are two separate collisions, between A/B and C/D, as shown below.

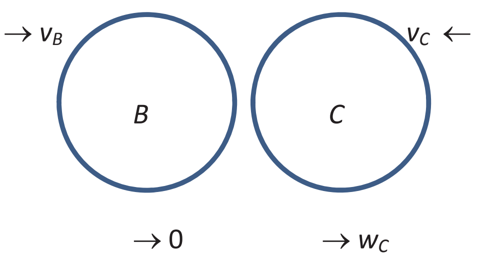


For collision A/B **CLM**: $m(\lambda u = \lambda v_A + v_B)$ and **NEL**: $eu = v_B - v_A$
and for collision C/D **CLM**: $m(u = v_C + v_D)$ and **NEL**: $eu = v_C - v_D$
Solving each pair of equations, separately, gives

$$v_A = \frac{\lambda - e}{\lambda + 1}u\,,\ v_B = \frac{\lambda(1+e)}{\lambda + 1}u\,,\ v_C = \tfrac{1}{2}(1+e)u \text{ and } v_D = \tfrac{1}{2}(1-e)u\,;$$

even though some of these turn out not to be required (though they may be later on, of course).

There is then a further collision between B/C:



CLM: $m(v_B - v_C) = m\, w_C$ and **NEL**: $e(v_B + v_C) = w_C$

Now, substituting previous answers in terms of e and u, and identifying for e, leads to the required answer $e = \dfrac{\lambda - 1}{3\lambda + 1}$. To justify the following condition on e, note that

$$e = \frac{\lambda - 1}{3\lambda + 1} = \tfrac{1}{3}\left(\frac{3\lambda - 3}{3\lambda + 1}\right) = \tfrac{1}{3}\left(\frac{3\lambda + 1 - 4}{3\lambda + 1}\right) = \tfrac{1}{3} - \frac{\tfrac{4}{3}}{3\lambda + 1} < \tfrac{1}{3}$$

since the term being subtracted is positive. (It is not sufficient simply to show that $e \to \tfrac{1}{3}$.)

Finally, using $w_C = \dfrac{(1+e)(\lambda-1)}{2(\lambda+1)}u$ from previous work, and equating this to v_D, gives

$$\frac{(1+e)(\lambda-1)}{2(\lambda+1)}u = \tfrac{1}{2}(1-e)u\,;$$

and substituting for e (e.g.) enables us to solve for λ and then find e: $\lambda = \sqrt{5}+2$, $e = \sqrt{5}-2$.

Question 11

Projectiles questions are often straightforward in principle, but can often require both careful thought and some tricky trigonometric manipulation. Occasionally, there is the additional requirement to maximise or minimise something-or-other using calculus.
To begin with, it is useful to know (or to be able to derive quickly) the *Trajectory Equation* of the parabola: $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$. Setting $y = -h$ and re-arranging then gives $\dfrac{gx^2}{u^2} = 2h\cos^2\alpha + 2x\sin\alpha\cos\alpha$, and it should be fairly clear from the given answer that a little bit of work using the double-angle formulae will lead to the required result $\dfrac{gx^2}{u^2} = h(1+\cos 2\alpha) + x\sin 2\alpha$.

Differentiating w.r.t. $\alpha \Rightarrow \dfrac{\mathrm{d}}{\mathrm{d}\alpha}\left(\dfrac{gx^2}{u^2}\right) = h(-2\sin 2\alpha) + \left(x.2\cos 2\alpha + \sin 2\alpha.\dfrac{\mathrm{d}x}{\mathrm{d}\alpha}\right)$. Notice that there is actually no need to re-arrange for $\dfrac{\mathrm{d}}{\mathrm{d}\alpha}(\ldots)$ since we require both derivative terms to disappear. This immediately leads to $x = h\tan 2\alpha$; and, substituting back, we get $\dfrac{gh^2\tan^2 2\alpha}{u^2} = h(1+\cos 2\alpha) + h\tan 2\alpha\sin 2\alpha$. By cancelling one of the h's and (e.g.) writing all trig. terms in $c = \cos 2\alpha$ then yields

$$\frac{gh\left(1-c^2\right)}{u^2c^2} = 1 + c + \frac{1-c^2}{c} \Rightarrow gh - ghc^2 = u^2\left(c^2 + c^3 + c - c^3\right).$$

As a quadratic in c : $0 = \left(u^2 + gh\right)c^2 + u^2c - gh = \left[\left(u^2+gh\right)c - gh\right](c+1)$ … note that it is always worth trying to factorise before deploying the messier *quadratic formula*.
We now have $\cos 2\alpha = \dfrac{gh}{u^2+gh}$, and substituting $x = h\tan 2\alpha$ and $y = -h$ in $\Delta^2 = x^2 + y^2$, then gives

$$\Delta^2 = h^2\sec^2 2\alpha \quad \text{i.e. } \Delta = h\sec 2\alpha$$

so that $\Delta = h.\dfrac{u^2+gh}{gh} = \dfrac{u^2}{g} + h$, as required.

Question 12

Bizarrely, this probability question concerning tossing fair coins has all of its answers equal to $\frac{1}{2}$. As one might imagine, such an answer can be obtained in very many ways indeed and so the key is to make your working really clear as to how the answer is arrived at. Merely writing down a whole load of fractions on the page is really not very enlightening. There needs to be some (visible) systematic approach to counting cases, followed by the numerical work that goes with it. For instance, in part (i), we could break down the answer into each possible value for *A* and the values of *B* that could then go with it.

e.g. p(*A* = 0).p(*B* = 1,2 or 3) + p(*A* = 1).p(*B* = 2 or3) + p(*A* = 2).p(*B* = 3) = $\frac{1}{4}\times\frac{7}{8}+2\times\frac{1}{4}\times\frac{4}{8}+\frac{1}{4}\times\frac{1}{8}$ = $\frac{1}{2}$.

For part (ii), one should by now have been able to "see" how the results arise, and can appeal to a "similar" process; e.g. $\frac{1}{8}\times\left(\frac{4+6+4+1}{16}\right)+\frac{3}{8}\times\left(\frac{6+4+1}{16}\right)+\frac{3}{8}\times\left(\frac{4+1}{16}\right)+\frac{1}{8}\times\left(\frac{1}{16}\right)$ … notice the appearance of the binomial coefficients for counting the relevant numbers of *B*'s possible values. This gives $\frac{1}{128}\left(15+3\times11+3\times5+1\right)=\frac{1}{128}\left(64\right)$ = $\frac{1}{2}$.

In part (iii), you should note that, when each of them has tossed *n* coins,

p(*B* has more Hs) = p(*A* has more Hs) = p_2

and that p($A_H = B_H$) = p_1 . Thus $p_1 + 2p_2 = 1$.

Next, considering what happens when *B* tosses the extra coin,

p(*B* has more Hs) = p(*B* already has more Hs) × p(*B* gets T)

+ p(*B* already has more, or equal, Hs) × p(*B* gets H)

= $p_2\times\frac{1}{2}+(p_1+p_2)\times\frac{1}{2}$ = $\frac{1}{2}(p_1+2p_2)=\frac{1}{2}$.

Question 13

For the i-th e-mail, the pdf is $f_i(t) = \lambda e^{-\lambda t}$. Integrating this gives the cdf: $F_i(t) = -e^{-\lambda t} + C$, and $F(0) = 0 \Rightarrow C = 1$.

Then, for n e-mails sent simultaneously,

$F(t) = P(T \le t) = 1 - P(\text{all } n \text{ take longer than } t) = 1 - \left(e^{-\lambda t}\right)^n$,

(using the product of n independent probabilities)

$= 1 - \lambda e^{-\lambda nt}$.

Differentiating this then gives the required pdf of T, $f(t) = n\lambda e^{-\lambda nt}$.

Finding an expected value is a standard integration process: $E(T) = \int_0^\infty t \times n\lambda e^{-\lambda nt}\, dt$, and this requires the use of integration by parts: $E(T) = \left[-te^{-\lambda nt}\right]_0^\infty + \int_0^\infty n\lambda e^{-\lambda nt}\, dt = [0] + \left[\frac{-e^{-\lambda nt}}{\lambda n}\right]_0^\infty = \frac{1}{n\lambda}$.

For the very final part, one *could* go again through the route of pdfs and cdfs, but it should be obvious that the waiting-time for the 2nd email is simply the 1st from the remaining $(n - 1)$ … with expected arrival time $\frac{1}{(n-1)\lambda}$, giving a total waiting time of $\frac{1}{n\lambda} + \frac{1}{(n-1)\lambda} = \frac{1}{\lambda}\left(\frac{1}{n} + \frac{1}{(n-1)}\right)$.
