STEP MATHEMATICS 2

2018

Mark Scheme

Question 1

	Substitute $x = k^{-1}$ into the quartic expression:	**M1**
	$$k^{-4} + ak^{-3} + bk^{-2} + ck^{-1} + 1 = \frac{1 + ak + bk^2 + ak^3 + k^4}{k^4}$$	
	Since k cannot be 0 and the numerator is equal to 0 (since k is a root of the equation), k^{-1} must also be a solution to the equation.	**E1**
(i)	For there to be only one distinct root, the root must be either 1 or -1	
	If the root is 1 then $a = -4, b = 6$	**B1**
	If the root is -1 then $a = 4, b = 6$	**B1**
(ii)	For there to be three distinct roots there must be one repeated root (which must be either 1 or -1).	**E1**
	If the repeated root is $x = 1$ then:	**M1**
	$1 + a + b + a + 1 = 0$	
	Therefore $b = -2a - 2$	**A1 AG**
	If the repeated root is $x = -1$ then:	**M1**
	$1 - a + b - a + 1 = 0$	
	Therefore $b = 2a - 2$	**A1 AG**
(iii)	$b = 2a - 2$ corresponds to the case where the repeated root is -1.	
	$x^4 + ax^3 + bx^2 + ax + 1 = (x + 1)(x^3 + (a - 1)x^2 + (a - 1)x + 1)$	
	$(x + 1)$ is a factor of $(x^3 + (a - 1)x^2 + (a - 1)x + 1)$	
	$x^4 + ax^3 + bx^2 + ax + 1 = (x^2 + 2x + 1)(x^2 + kx + 1)$	**M1**
	Comparing coefficients of x^3:	**A1**
	$a = k + 2$	
	Therefore the other roots are	**M1**
	$\dfrac{(2 - a) \pm \sqrt{(a - 2)^2 - 4}}{2}$	**A1**
	In the case where $b = 2a - 2$:	
	For all three roots to be real, $(a - 2)^2 - 4 > 0$	**M1**
		A1
	$a^2 > 4a = 2b + 4$	
	In the case where $b = -2a - 2$, the quadratic will have $a = k - 2$	**M1**
	Therefore $(a + 2)^2 - 4 > 0$ for three roots	**A1**
	The quadratic factors in the two cases are both of the form $x^2 + kx + 1$. They must have roots that are not ± 1.	**M1** **A1**
	$\frac{k \pm \sqrt{k^2 - 4}}{2} = \pm 1$ if $k^2 - 4 = (k \pm 2)^2$,	
	$(k \pm 2)^2 - (k + 2)(k - 2) = 0$, so $k = \pm 2$.	
	Therefore in neither of the two cases investigated does the quadratic equation have solutions of ± 1	
	Therefore	**A1**
	$(b + 2)^2 = 4a^2$	
	and	
	$a^2 > 2b + 4$	
	Are necessary and sufficient conditions for (*) to have exactly three distinct real roots.	

M1	Substituted correctly.
E1	Conclusion explained fully
B1	Values only need to be stated – there is no need to link them to the value of the root.
B1	Values only need to be stated – there is no need to link them to the value of the root.
	Subtotal: 4
E1	Identify that one of the roots must be 1 or -1
M1	Substitution of $x = 1$
A1	Conclude the first relationship
M1	Substitution of $x = -1$
A1	Conclude the second relationship
	Subtotal: 5
M1	Factorised form
A1	Comparison of coefficient
M1	Application of quadratic formula
A1	Correct roots
	Subtotal: 4
M1	Use of the discriminant
A1	$(a-2)^2 - 4 > 0$ and strictness explained
M1	Follow through same process for second case
A1	$(a+2)^2 - 4 > 0$
M1	Attempt to check that the roots of the quadratic are not equal to ± 1.
A1	Full justification.
A1	Any equivalent expression of the conditions
	Subtotal: 7

Question 2

Sketch showing the curve and chord with the chord entirely below the curve and $f(x_1) < f(x_2)$ **E1**

$tx_1 + (1-t)x_2$ identified as a value in the range (x_1, x_2) **E1**

$\big(tx_1 + (1-t)x_2, tf(x_1) + (1-t)f(x_2)\big)$ identified as the point on the chord. **E1**

If $f''(x) < 0$ for $a < x < b$ then the gradient of the curve $y = f(x)$ must be decreasing as x increases. **E1**

Suppose that a function $f(x)$ satisfies $f''(x) < 0$ for $a < x < b$, but is not concave for $a < x < b$. Then there must be points $x_1 < x_2$ and a value t, $0 < t < 1$ such that

$tf(x_1) + (1-t)f(x_2) > f(tx_1 + (1-t)x_2)$

The gradient at $x = tx_1 + (1-t)x_2$ must be less than the gradient of the chord joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$, and so the curve $y = f(x)$ must continue to have a gradient of this value or less. The curve therefore cannot pass through $(x_2, f(x_2))$. Therefore, it must be the case that a function satisfying $f''(x) < 0$ for $a < x < b$ is concave for $a < x < b$. **E1**

(i) Let $x_1 = \frac{2u+v}{3}, x_2 = \frac{v+2w}{3}$ and $t = \frac{1}{2}$ **M1**

Then, since $f(x)$ is concave for $a < x < b$: **A1**

$\frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right) \le f\left(\frac{u+v+w}{3}\right)$

Setting $x_1 = u, x_2 = v$ and $t = \frac{2}{3}$ gives: **B1**

$\frac{2}{3}f(u) + \frac{1}{3}f(v) \le f\left(\frac{2u+v}{3}\right)$

Similarly, setting $x_1 = v, x_2 = w$ and $t = \frac{1}{3}$ gives: **B1**

$\frac{1}{3}f(v) + \frac{2}{3}f(w) \le f\left(\frac{v+2w}{3}\right)$

Therefore: **M1**

$f\left(\frac{u+v+w}{3}\right) \ge \frac{1}{2}f\left(\frac{2u+v}{3}\right) + \frac{1}{2}f\left(\frac{v+2w}{3}\right)$ **A1 AG**

$\ge \frac{1}{2}\left(\frac{2}{3}f(u) + \frac{1}{3}f(v)\right) + \frac{1}{2}\left(\frac{1}{3}f(v) + \frac{2}{3}f(w)\right) = \frac{f(u)+f(v)+f(w)}{3}$

(ii) If $f(x) = \sin x$, then $f''(x) = -\sin x$ and $f''(x) < 0$ for $0 < x < \pi$. **B1**

Therefore $f(x)$ is concave for $0 < x < \pi$. **E1**

$0 < A, B, C < \pi$ and $A + B + C = \pi$, therefore, by (i): **M1**

$\sin\frac{\pi}{3} \ge \frac{\sin A + \sin B + \sin C}{3}$

$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$ **A1 AG**

(iii) If $f(x) = \ln(\sin x)$, then $f'(x) = \cot x$ **M1**

$f''(x) = -\operatorname{cosec}^2 x$ **A1**

Therefore $f''(x) < 0$ for $0 < x < \pi$ and so $f(x)$ is concave for $0 < x < \pi$ **E1**

Therefore: **M1**

$\ln\left(\sin\frac{\pi}{3}\right) \ge \frac{\ln(\sin A) + \ln(\sin B) + \ln(\sin C)}{3}$

$3\ln\left(\frac{\sqrt{3}}{2}\right) \ge \ln(\sin A \times \sin B \times \sin C)$

$\sin A \times \sin B \times \sin C \le \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$ **A1 AG**

E1	Sketch must match the case given in the question
E1	Could be explained in text or indicated on the graph (if clearly labelled)
E1	Could be explained in text
E1	Explanation includes reference to the behaviour of the gradient
E1	Fully clear explanation
	Subtotal: 5
M1	Any choice that will lead to $f\left(\frac{u+v+w}{3}\right)$ on RHS
A1	Application of definition of concave.
B1	Any choice that leads to an expression in terms of $f(u)$, $f(v)$ and $f(w)$ on LHS
B1	Any choice that leads to an expression in terms of $f(u)$, $f(v)$ and $f(w)$ on LHS
M1	Combination of previous inequalities
A1	Fully correct derivation
	Subtotal: 6
B1	States second derivative
E1	Concludes that the function is concave
M1	Application of result from (i) (including justification that it can be applied)
A1	Reaches correct inequality
	Subtotal: 4
M1	Differentiation of the correct function
A1	Correct second derivative
E1	Conclusion that the function is concave
M1	Application of result from (i)
A1	Correct manipulation of logarithms to reach given result.
	Subtotal: 5

Question 3

(i)	$f(x) = (1 + \tan x)^{-1}$	**M1**
	$f'(x) = -(1 + \tan x)^{-2} \sec^2 x$	**A1**
	$f'(x) = -\frac{1}{(1+\tan x)^2 \cos^2 x}$	
	$= -\frac{1}{(\sin x+\cos x)^2}$	**M1**
	$= -\frac{1}{\sin^2 x+\cos^2 x+2\sin x\cos x}$	**A1**
	$= -\frac{1}{1+\sin 2x}$	**AG**
	Within the given domain, $0 \leq \sin 2x \leq 1$, so $-1 \leq f'(x) \leq \frac{1}{2}$	**B1**
	Sketch of graph should have the following features:	
	Decreasing function	**G1**
	Points (0,1) and $\left(\frac{\pi}{2}, 0\right)$	**G1**
	Point of inflexion at $x = \frac{\pi}{4}$	**G1**
	All other features correct	**G1**
(ii)	If the point $(x, g(x))$ is rotated through 180 degrees about the point (a, b) then the image will be at the point $(a + (a - x), b + (b - g(x)))$.	**E1**
	Therefore, if the curve has rotational symmetry of order 2 about the point (a, b), then $g(2a - x) = 2b - g(x)$, so $g(x) + g(2a - x) = 2b$	**E1**
	Similarly, if $g(x) + g(2a - x) = 2b$, then any pair of points that are centred horizontally on the point (a, b) will also be centred vertically on the point (a, b),	**E1**
	which means that the curve will have rotational symmetry about that point.	**E1**
	$\int_{-1}^{1} g(x)\, dx = 0$	**B1**
(iii)	Since $\tan\left(\frac{\pi}{2} - x\right) = \cot x$,	**B1**
	$f\left(\frac{\pi}{2} - x\right) = \frac{1}{1+\cot^k x}$	**M1**
	$= \frac{\tan^k x}{\tan^k x+1} = 1 - f(x)$	**M1**
	Therefore $f(x) + f\left(2\left(\frac{\pi}{4}\right) - x\right) = 2\left(\frac{1}{2}\right)$	
	So the curve has rotational symmetry of order 2 about the point $(\frac{\pi}{4}, \frac{1}{2})$	**A1**
	The area under the curve over any interval centred on $x = \frac{\pi}{4}$, will therefore have the same area as a rectangle of the same width and height $\frac{1}{2}$.	**M1**
	Therefore $\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \frac{1}{1+\tan^k x} dx = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \times \frac{1}{2} = \frac{\pi}{12}$	**A1**

M1	Attempt to apply the chain or quotient rule
A1	Correct derivative
M1	Application of an appropriate trigonometric identity to simplify the function
A1	Fully correct simplification
B1	Correct range
	Subtotal: 5
G1	Feature clear on graph
G1	Feature clear on graph
G1	Feature clear on graph
G1	Feature clear on graph
	Subtotal: 4
E1	Identification of required image point
E1	Fully clear explanation
E1	Connection with points centred either horizontally or vertically on the correct value.
E1	Fully clear explanation
B1	Correct value
	Subtotal: 5
B1	Connection with cot, or application of an appropriate trigonometric identity
M1	Appropriate substitution to show rotational symmetry
M1	Correct manipulation to show rotational symmetry
A1	Rotational symmetry shown and point identified
M1	Equivalent area identified
A1	Correct value
	Subtotal: 6

Question 4

(i)	$\cos x + \cos 4x = 2\cos\frac{5}{2}x\cos\frac{3}{2}x$ and $\cos 2x + \cos 3x = 2\cos\frac{5}{2}x\cos\frac{1}{2}x$	**M1**
	$2\cos\frac{5}{2}x\cos\frac{3}{2}x + 6\cos\frac{5}{2}x\cos\frac{1}{2}x = 0$, so $2\cos\frac{5}{2}x\left(\cos\frac{3}{2}x + 3\cos\frac{1}{2}x\right) = 0$	**M1**
	Therefore $\cos\frac{5}{2}x = 0$ or $\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0$	**A1**
	$\cos\frac{5}{2}x = 0$ gives $x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}$ or $\frac{9\pi}{5}$	**B1 B1**
	If $\cos\frac{3}{2}x + 3\cos\frac{1}{2}x = 0$, then:	**M1**
	$\left(\cos\frac{3}{2}x + \cos\frac{1}{2}x\right) + 2\cos\frac{1}{2}x = 0$	
	$2\cos x\cos\frac{1}{2}x + 2\cos\frac{1}{2}x = 0$	
	$2\cos\frac{1}{2}x\,(\cos x + 1) = 0$	
	$\cos\frac{1}{2}x = 0$ or $\cos x = -1$, both of which give no new solutions to the equation.	**A1**
(ii)	$\cos(x-y) + \cos(x+y) = 2\cos x\cos y$	**M1**
	$2\cos x\cos y - 2\cos^2 x + 1 = 1$	**M1**
	$2\cos x\,(\cos y - \cos x) = 0$	**M1**
	Therefore either $\cos x = \cos y$, which can only be the case if $x = y$ since $0 \le x \le \pi$ and $0 \le y \le \pi$	**E1**
	Or $\cos x = 0$, so $x = \frac{\pi}{2}$	**A1**
(iii)	$2\cos\dfrac{1}{2}(x+y)\cos\dfrac{1}{2}(x-y)$	**M1**
	$-\left(2\cos^2\dfrac{1}{2}(x+y) - 1\right) = \dfrac{3}{2}$	**M1**
	$4\cos^2\dfrac{1}{2}(x+y) - 4\cos\dfrac{1}{2}(x+y)\cos\dfrac{1}{2}(x-y) + 1 = 0$	**M1**
	$\left(2\cos\dfrac{1}{2}(x+y) - \cos\dfrac{1}{2}(x-y)\right)^2 + 1 - \cos^2\dfrac{1}{2}(x-y) = 0$	**M1**
	$\left(2\cos\dfrac{1}{2}(x+y) - \cos\dfrac{1}{2}(x-y)\right)^2 + \sin^2\dfrac{1}{2}(x-y) = 0$	**M1**
	Therefore, since both terms are ≥ 0, they must both be equal to 0.	**M1**
	For $0 \le x \le \pi$ and $0 \le y \le \pi$, $\sin^2\frac{1}{2}(x-y) = 0$ only when $x = y$	**M1**
	Therefore $2\cos x = 1$, so $x = \frac{\pi}{3}$ and $y = \frac{\pi}{3}$	**A1**

M1	Pairing of terms in the equation
M1	Factorisation
A1	Identification of the two cases
B1	One solution identified
B1	Full set of solutions for first case
M1	Manipulation of equation from other case
A1	Justification that this gives no other roots to the equation
	Subtotal: 7
M1	Simplification of sum of cos functions or use of a compound angle formula
M1	Use of $\cos 2x$ identity
M1	Factorisation
E1	Explanation that $x = y$ (must refer to range of values for x and y)
A1	Correct value
	Subtotal: 5
M1	Simplication of sum of first two functions
M1	Use of $\cos 2A$ identity
M1	Simplification to three-term quadratic
M1	Completion of square, or calculation of discriminant
M1	Expression using sin function
M1	Explanation that this implies both equal
M1	Conclusion that $x = y$
A1	Correct solution
	Subtotal: 8

Question 5

(i) n^{th} term of expansion is $\frac{(-1)(-2)\ldots(-n)}{n!}(x)^n$

$$(1+x)^{-1} = \sum_{n=0}^{\infty}(-x)^n$$ **B1**

$$\int (1+x)^{-1}\,dx = \ln(1+x) + c$$ **B1**

E1

$$\int \sum_{n=0}^{\infty}(-x)^n\,dx = -\sum_{n=0}^{\infty}\frac{(-x)^{n+1}}{n+1} = -\sum_{n=1}^{\infty}\frac{(-x)^n}{n}$$ **B1** **AG**

(ii)

$$e^{-ax} = \sum_{n=0}^{\infty}\frac{(-ax)^n}{n!}$$ **B1**

$$\frac{(1-e^{-ax})e^{-x}}{x} = -\left(\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}x^{n-1}\right)e^{-x}$$ **M1**

Let

$$I_n = \int_0^{\infty} x^n e^{-x}\,dx$$

$$u = x^n \qquad \frac{dv}{dx} = e^{-x}$$ **M1** **A1**

$$\frac{du}{dx} = nx^{n-1} \qquad v = -e^{-x}$$

$$I_n = [-x^n e^{-x}]_0^{\infty} + n\int_0^{\infty} x^{n-1}e^{-x}\,dx$$ **M1**

$$I_n = nI_{n-1}$$ **M1**

$$I_0 = \int_0^{\infty} e^{-x}\,dx = [-e^{-x}]_0^{\infty} = 1$$ **M1**

Therefore $I_n = n!$ **A1**

So **A1**

$$\int_0^{\infty} -\left(\sum_{n=1}^{\infty}\frac{(-a)^n}{n!}x^{n-1}\right)e^{-x}\,dx = -\left(\sum_{n=1}^{\infty}\frac{(-a)^n}{n}\right) = \ln(1+a)$$ **AG**

(by part (i))

(iii) Let $u = -\ln x$ **M1**

Then $x = e^{-u}$ and $\frac{dx}{du} = -e^{-u}$ **M1**

Change limits:

$x = 1$ becomes $u = 0$

$x = 0$ becomes $u = \infty$ **B1**

$$\int_0^1 \frac{x^p - x^q}{\ln x}\,dx = \int_{\infty}^0 \frac{(e^{-pu} - e^{-qu})}{-u}(-e^{-u})du$$ **M1** **A1**

$$= -\int_0^{\infty}\frac{(1-e^{-qu}) - (1-e^{-pu})}{u}(e^{-u})du$$ **M1**

$$= -\ln(1+q) + \ln(1+p)$$

$$= \ln\left(\frac{1+p}{1+q}\right)$$ **A1**

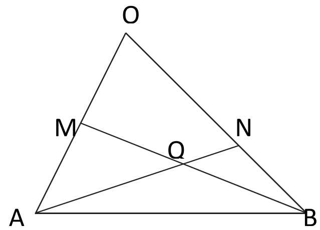
B1	Simplified form
B1	Correct integral
E1	Show that $c = 0$
B1	Correct integration term by term (ensure that signs are dealt with correctly)
	Subtotal: 4
B1	Correct expansion
M1	Substitution into the function to be integrated
M1	Integration by parts
A1	Correct derivative and integral
M1	Completion of integration by parts
M1	Simplification, including substitution of limits
M1	First case evaluated
A1	General result
A1	Fully correct solution
	Subtotal: 9
M1	Selection of appropriate substitution
M1	Differentiation
B1	Limits changed
M1	Substitution applied to the integral
A1	Completed substitution
M1	Rearrangement so that previous result can be applied
A1	Application of previous result (final simplification not needed)
	Subtotal: 7

Question 6

(i)	If $n \geq 5$ then $n! + 5 > 5$ and has 5 as a factor	**E1**
	Therefore the only possible solutions will have $n < 5$	**E1**
	The only pairs are therefore	
	$(2,7)$	**B1**
	$(3,11)$	**B1**
	$(4,29)$	**B1**
(ii)	If $n \geq 7$ then theorem 1 shows that $m > 4n$.	**E1**
	By theorem 2, there is a prime number between $2n$ and m, which must be a factor of $m!$	**E1**
	But that prime cannot be a factor of any of $1!, 3!, \cdots, (2n-1)!$	**E2**
	So it cannot be a factor of $1! \times 3! \times \cdots \times (2n-1)!$	**E1**
	Therefore there is a prime factor on the RHS that does not appear on the LHS.	**E1**
	Therefore the only pairs must have $n < 7$	**E1**
	$n = 1: m = 1$	**B1**
	$n = 2: m = 3$	**B1**
	$n = 3$: LHS=$3! \times 5!$	
	$3! \times 5! = 5! \times 6 = 6!$	
	So $m = 6$	**B1**
	$n = 4$: LHS= $3! \times 5! \times 7!$	**M1**
	$3! \times 5! = 2 \times 3 \times 2 \times 3 \times 4 \times 5 = (2 \times 4) \times (3 \times 3) \times (2 \times 5)$	
	So $m = 10$	**A1**
	$n = 5$: LHS = $3! \times 5! \times 7! \times 9!$	**E1**
	There must be two factors of 7 in the RHS, so $m \geq 14$	
	There will be no way of generating a factor of 11 for the RHS.	
	$n = 6$: LHS = $3! \times 5! \times 7! \times 9! \times 11!$	**E1**
	There must be two factors of 7 in the RHS, so $m \geq 14$	
	There will be no way of generating a factor of 13 for the RHS	**E1**

E1	Identification of common factor of 5
E1	No solutions for high values of n justified
B1	Correct solution
B1	Correct solution
B1	Correct solution
	Subtotal: 5
E1	Significance of theorem 1 explained
E1	Significance of theorem 2 explained
E2	Explicit statement that the prime cannot be a factor is required
E1	Can imply previous mark
E1	Prime factor on one side but not the other clearly explained
E1	Justification that solutions only exist for $n < 7$
	Subtotal: 7
B1	Correct solution
B1	Correct solution
B1	Correct solution
M1	Rearrangement of middle values to create $8 \times 9 \times 10$
A1	Correct value
E1	Explanation that $m \geq 11$
E1	Explanation that $m \geq 13$
E1	Identification that no factor of 13 exists in the LHS – can also be awarded for identifying that no factor of 11 exists in the LHS for previous case
	Subtotal: 8

Question 7



Let k and l be such that $\boldsymbol{m} = k\boldsymbol{a}$ and $\boldsymbol{n} = l\boldsymbol{b}$ **B1**

$\overrightarrow{BM} = k\boldsymbol{a} - \boldsymbol{b}$ **M1**

$\overrightarrow{QM} = \dfrac{\mu}{1+\mu}(k\boldsymbol{a} - \boldsymbol{b})$ **A1**

Similarly: **A1**

$\overrightarrow{QN} = \dfrac{\nu}{1+\nu}(l\boldsymbol{b} - \boldsymbol{a})$

Therefore: **M1**

$\boldsymbol{q} = \overrightarrow{OM} + \overrightarrow{MQ} = k\boldsymbol{a} - \dfrac{\mu}{1+\mu}(k\boldsymbol{a} - \boldsymbol{b})$

$\boldsymbol{q} = \dfrac{k}{1+\mu}\boldsymbol{a} + \dfrac{\mu}{(1+\mu)}\boldsymbol{b}$ **A1**

And: **M1**

$\boldsymbol{q} = \dfrac{l}{1+\nu}\boldsymbol{b} + \dfrac{\nu}{(1+\nu)}\boldsymbol{a}$ **A1**

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel: **M1**

$\dfrac{k}{1+\mu} = \dfrac{\nu}{(1+\nu)}$

Therefore **A2 AG**

$k = \dfrac{(1+\mu)\nu}{1+\nu}$

So

$\boldsymbol{m} = \dfrac{(1+\mu)\nu}{1+\nu}\boldsymbol{a}$

$\overrightarrow{AN} = \dfrac{(1+\nu)\mu}{1+\mu}\boldsymbol{b} - \boldsymbol{a}$ **B1**

Therefore: **M1**

$\overrightarrow{OL} = \dfrac{(1+\mu)\nu}{1+\nu}\boldsymbol{a} + p\left(\dfrac{(1+\nu)\mu}{1+\mu}\boldsymbol{b} - \boldsymbol{a}\right)$ **A1**

For some value of p

Since $\overrightarrow{OL}$ is parallel to $\boldsymbol{b}$, the coefficient of $\boldsymbol{a}$ must be 0 **M1**

$\dfrac{(1+\mu)\nu}{1+\nu} - p = 0$ **M1**

Therefore **M1**

$\overrightarrow{OL} = \dfrac{(1+\mu)\nu}{1+\nu}\boldsymbol{a} + \dfrac{(1+\mu)\nu}{1+\nu}\left(\dfrac{(1+\nu)\mu}{1+\mu}\boldsymbol{b} - \boldsymbol{a}\right) = \nu\mu\boldsymbol{b}$

So $\lambda = \mu\nu$ **A2**

$\mu\nu < 1$ means that L lies on OB. **E1**

B1	Diagram
M1	Method to work out $\overrightarrow{BM}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$
A1	Expression for $\overrightarrow{QM}$
A1	Expression for $\overrightarrow{QN}$
M1	Find an expression for $\boldsymbol{q}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$
A1	Correct expression
M1	Find a second expression for $\boldsymbol{q}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$
A1	Correct expression
M1	Equate coefficients of $\boldsymbol{a}$
A2	Reach given expression for $\boldsymbol{m}$
	Subtotal: 11
B1	Find expression for $\overrightarrow{AN}$
M1	Form an equation of the line on which L lies.
A1	Correct equation
M1	Identify that the component in the direction of $\boldsymbol{a}$ must be 0
M1	Correct equation for p
M1	Substitution back into equation of line
A2	Correct relationship
E1	Correct explanation
	Subtotal: 9

Question 8

(i)

$$\frac{dv}{dt} = \frac{1}{2}y^{-\frac{1}{2}} \times \frac{dy}{dt}$$ **M1**

$$\frac{dy}{dt} = 2v\frac{dv}{dt}$$ **A1**

$$2v\frac{dv}{dt} = \alpha v - \beta v^2$$

$$\frac{dv}{dt} = \frac{1}{2}(\alpha - \beta v)$$ **M1**

$$\int \frac{1}{\alpha - \beta v}dv = \int \frac{1}{2}dt$$ **M1**

$$-\frac{1}{\beta}\ln|\alpha - \beta v| = \frac{1}{2}t + c$$ **M1**

$$\alpha - \beta v = Ae^{-\frac{1}{2}\beta t}$$ **M1**

$$v = \frac{1}{\beta}\left(\alpha - Ae^{-\frac{1}{2}\beta t}\right)$$ **A1**

$$y = \frac{1}{\beta^2}\left(\alpha - Ae^{-\frac{1}{2}\beta t}\right)^2$$

$$y_1 = \frac{\alpha^2}{\beta^2}\left(1 - e^{-\frac{1}{2}\beta t}\right)^2$$ **A1**

(ii)

Use the substitution $v = y^{\frac{1}{3}}$: **M1**

$$\frac{dv}{dt} = \frac{1}{3}y^{-\frac{2}{3}} \times \frac{dy}{dt}$$

$$3v^2\frac{dv}{dt} = \alpha v^2 - \beta v^3$$

$$\frac{dv}{dt} = \frac{1}{3}\alpha - \frac{1}{3}\beta v$$ **A1**

$$\int \frac{1}{\alpha - \beta v}dv = \int \frac{1}{3}dt$$

$$-\frac{1}{\beta}\ln|\alpha - \beta v| = \frac{1}{3}t + c$$

$$\alpha - \beta v = Ae^{-\frac{1}{3}\beta t}$$

$$v = \frac{1}{\beta}\left(\alpha - Ae^{-\frac{1}{3}\beta t}\right)$$ **A1**

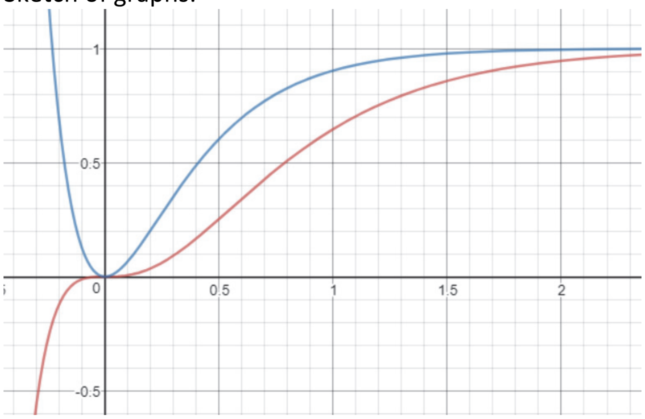
$$y = \frac{1}{\beta^3}\left(\alpha - Ae^{-\frac{1}{3}\beta t}\right)^3$$

$$y_2 = \frac{\alpha^3}{\beta^3}\left(1 - e^{-\frac{1}{3}\beta t}\right)^3$$ **A1**

(iii) If $\alpha = \beta$: **B1**

$y_1(x) = \left(1 - e^{-\frac{1}{2}\beta x}\right)^2$ and $y_2(x) = \left(1 - e^{-\frac{1}{3}\beta x}\right)^3$

Sketch of graphs:



Ignore anything to left of y-axis.

Both curves have a horizontal asymptote $y = 1$ **G1**

Both curves have gradient 0 as they pass through the origin **G1**

Both functions have decreasing gradient. **G1**

For positive values of x:

$0 > e^{-\frac{1}{3}\beta x} > e^{-\frac{1}{2}\beta x}$ **E1**

Therefore

$\left(1 - e^{-\frac{1}{3}\beta x}\right) < \left(1 - e^{-\frac{1}{2}\beta x}\right) < 1$ **E1**

$\left(1 - e^{-\frac{1}{3}\beta x}\right)^3 < \left(1 - e^{-\frac{1}{3}\beta x}\right)^2 < \left(1 - e^{-\frac{1}{2}\beta x}\right)^2$ **E1**

So the graph of y_2 should lie below the graph of y_1 **G1**

M1	Relationship between $\frac{dy}{dt}$ and $\frac{dv}{dt}$ (accept $dy = 2vdv$)
A1	Correct differentiation
M1	Substitution completed and simplified
M1	Variables separated
M1	Integration completed
M1	Logarithm removed
A1	Rearranged so that v is subject
A1	Formula for y and boundary condition applied (must be in the form $y = \cdots$)
	Subtotal: 8
M1	Correct substitution chosen and applied (could use $v = y^{\frac{2}{3}}$)
A1	Simplified differential equation reached
A1	Solution rearranged so that v is the subject
A1	Formula for y and boundary condition applied (must be in the form $y = \cdots$)
	Subtotal: 4
B1	Simplified expressions found for the case $\alpha = \beta$
G1	Asymptote must be indicated (accept if not explicit, but $y \to 1$ seen and clear from shape)
G1	Zero gradient through origin must be clear
G1	General shape away from origin correct. Accept any increasing function with decreasing gradient
E1	Comparison of exponential functions
E1	Comparison of the functions that will be raised to a power
E1	Correctly deduced relationship between the two graphs
G1	$y_1 > y_2$. Must have at least one of the E marks awarded to receive this mark.
	Subtotal: 8

Question 9

When A reaches the ground for the first time B will be at a height of $9h$ above P. **B1**

For the motion until A reaches the ground: **M1**

$u = 0, a = g, s = 8h$

$v^2 = u^2 + 2as$

$v^2 = 16gh$

Therefore $v = 4\sqrt{gh}$ **A1**

A rebounds with a speed of $2\sqrt{gh}\ ms^{-1}$ **A1**

The velocity of B relative to A for the subsequent motion will be $6\sqrt{gh}$ **B1**

The particles will therefore collide after $\frac{9h}{6\sqrt{gh}} = \frac{3h}{2\sqrt{gh}}\ s$ **M1** **A1**

For particle A:

$u = -2\sqrt{gh}, a = g, t = \dfrac{3h}{2\sqrt{gh}}$

$s = ut + \dfrac{1}{2}at^2 = -2\sqrt{gh}\left(\dfrac{3h}{2\sqrt{gh}}\right) + \dfrac{1}{2}g\left(\dfrac{3h}{2\sqrt{gh}}\right)^2$ **M1**

$s = -3h + \dfrac{9h}{8} = -\dfrac{15}{8}h$ **A1** **AG**

So the collision occurs a distance of $\frac{15}{8}h$ above P.

$v = u + at = -2\sqrt{gh} + g\left(\dfrac{3h}{2\sqrt{gh}}\right)$ **M1**

$v = -\dfrac{1}{2}\sqrt{gh}$ **A1**

$u_A = \dfrac{1}{2}\sqrt{gh}$

The velocity of B will be **M1**

$-\dfrac{1}{2}\sqrt{gh} + 6\sqrt{gh} = \dfrac{11}{2}\sqrt{gh}$

$u_B = \dfrac{11}{2}\sqrt{gh}$ **A1**

To hit the ground the second time with speed $4\sqrt{gh}$:

$v = 4\sqrt{gh}, a = g, s = \frac{15}{8}h$

$v^2 = u^2 + 2as$

$16gh = u^2 + \frac{15}{4}gh$ **M1**

$u^2 = \frac{49}{4}gh$ **A1**

$u = \frac{7}{2}\sqrt{gh}$ (since $u > -\frac{1}{2}\sqrt{gh}$) **E1**

Conservation of momentum for collision between the beads: **M1**

$m\left(-\frac{1}{2}\sqrt{gh}\right) + m\left(\frac{11}{2}\sqrt{gh}\right) = m\left(\frac{7}{2}\sqrt{gh}\right) + mv$

where v is the velocity of B after the collision.

$v = \frac{3}{2}\sqrt{gh}$ **A1**

$e = \dfrac{\frac{7}{2}\sqrt{gh} - \frac{3}{2}\sqrt{gh}}{\frac{11}{2}\sqrt{gh} - \left(-\frac{1}{2}\sqrt{gh}\right)} = \frac{1}{3}$ **M1** **A1**

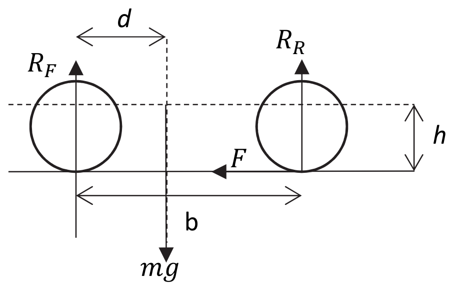
B1	May be implied by later work
M1	Application of correct formula
A1	Correct value for velocity
A1	Correct rebound speed
B1	May be implied by later work
M1	Application of correct formula
A1	Correct time
M1	Application of correct formula
A1	Correct solution
	Subtotal: 9
M1	Application of correct formula
A1	Correct speed
M1	Application of correct formula
A1	Correct speed
	Subtotal: 4
M1	Application of correct formula
A1	Reach two possible values
E1	Select correct value
M1	Apply conservation of momentum
A1	Find velocity of B after collision
M1	Apply correct formula
A1	Correct value
	Subtotal: 7

Question 10

At time t the string will have a length of $a + ut$	**M1**
The speed of the point on the string will therefore be $\frac{xu}{a+ut}$	**A1**
$\frac{dx}{dt} = \frac{xu}{a+ut} + v$	**B1**
$\frac{d}{dt}\left(\frac{x}{a+ut}\right) = \frac{(a+ut)\frac{dx}{dt} - xu}{(a+ut)^2}$	**M1** **A1**
$= \frac{xu + v(a+ut) - xu}{(a+ut)^2} = \frac{v}{a+ut}$	**M1** **A1 AG**
$\frac{x}{a+ut} = \int \frac{v}{a+ut}\,dt$	**M1**
$\frac{x}{a+ut} = \frac{v}{u}\ln\lvert C(a+ut)\rvert$	**A1**
At $t = 0$, $x = 0$:	
$0 = \frac{v}{u}\ln aC$	**M1**
Therefore $\mathrm{C} = \frac{1}{a}$	**A1**
At $t = T$, $x = a + uT$:	
$\frac{a+uT}{a+uT} = \frac{v}{u}\ln\left\lvert\frac{1}{a}(a+uT)\right\rvert$	**M1**
$1 + \frac{uT}{a} = e^k$	
where $k = u/v$.	
$uT = a(e^k - 1)$	**A1 AG**
For the journey back:	
$\frac{dx}{dt} = \frac{xu}{a+ut} - v$	**M1**
$\frac{d}{dt}\left(\frac{x}{a+ut}\right) = -\frac{v}{a+ut}$	**M1**
Therefore	
$\frac{x}{a+ut} = -\frac{v}{u}\ln\lvert C(a+ut)\rvert$	**A1**
At $t = T$, $x = a + uT$:	
$\frac{a+uT}{a+uT} = -\frac{v}{u}\ln\lvert C(a+uT)\rvert$	**M1**
Therefore:	
$C(a+uT) = e^{-k}$	**A1**
Solve for $x = 0$:	
$0 = -\frac{v}{u}\ln\lvert C(a+ut)\rvert$	**M1**
Therefore	
$C(a+ut) = 1$	
$e^{-k}(a+ut) = a + uT$	
$t = \frac{(a+uT)e^k - a}{u}$	
Therefore the time for the journey back is:	
$$\frac{(a+uT)e^k - a}{u} - \frac{a\left(e^k - 1\right)}{u} = Te^k$$	**A1** **CAO**

M1	Expression for length at time t
A1	Correct speed
B1	Correct differential equation
M1	Use of quotient rule
A1	Correctly completed
M1	Substitution
A1	Result verified
	Subtotal: 7
M1	Method for solving the differential equation
A1	Correctly integrated
M1	Substitute boundary condition
A1	Correct value for constant
M1	Substitute for end point
A1	Simplified
	Subtotal: 6
M1	New differential equation
M1	Correct new differential
A1	Correct solution to differential equation
M1	Substitute for start of journey back
A1	Correct constant
M1	Solve for time of return
A1	Find time for return journey.
	Subtotal: 7

Question 11



Taking moments about the centre of mass:	**M1**
$R_F d + Fh = R_R(b-d)$	**A1**
	A1
$F = \dfrac{R_R(b-d) - R_F d}{h}$	**M1**
	A1
At the time when the front wheel loses contact with the ground:	**B1**
$R_F = 0$ and $R_R = mg$	
$F = \dfrac{mg(b-d)}{h}$	
Maximum possible frictional force is μmg	**B1**
Therefore if	**E1**
$\mu mg < \dfrac{mg(b-d)}{h}$	**AG**
then the rear wheel will have slipped before this point.	
i.e. if	
$\mu < \dfrac{b-d}{h}$	
At the moment before the rear wheel slips, friction will take its maximum value	**B1**
$\dfrac{R_R(b-d) - R_F d}{h} = \mu R_R$	**M1**
Resolving vertically:	**M1**
$R_F + R_R = mg$	
$R_R b - mgd = \mu h R_R$	**M1**
$R_R = \dfrac{mgd}{b - \mu h}$	**A1**
Therefore	
$F = \dfrac{\mu mgd}{b - \mu h}$	
Newton's second law:	**M1**
$F = ma$	
Therefore	**A1**
$a = \dfrac{\mu dg}{b - \mu h}$	

The front wheel would lose contact with the road when $R_F = 0$:

The acceleration is given by **E1**

$$a = \frac{R_R b - mgd}{mh}$$ **E1**

Therefore a increases as R_R increases and R_F decreases

So the maximum acceleration is at the moment when the front wheel would be about to leave the ground **E1**

At this point $F = \frac{mg(b-d)}{h}$ and so **A1**

$$a = \frac{g(b-d)}{h}$$

B1	Forces all identified
M1	Taking moments
A1	All clockwise moments correct
A1	All anticlockwise moments correct
M1	Rearrange to make F the subject
A1	Correct form
B1	Identify reaction forces for this case
B1	Identify maximum possible value for F
E1	Explanation that rear wheel would have slipped
	Subtotal: 9
B1	Maximum value used
M1	Substituted into equation
M1	Resolve forces vertically (may be seen earlier)
M1	Eliminate R_F
A1	Correct reaction force
M1	Substitute into frictional force and apply Newton's second law
A1	Correct value for a
	Subtotal: 7
E1	Use of formula for the acceleration
E1	Identify that higher accelerations have higher reaction at the rear
E1	Identify moment when maximum acceleration occurs
A1	Correct value
	Subtotal: 4

Question 12

(i) I will win if there are h consecutive heads and lose otherwise. **M1**

$P(h\ consecutive\ heads) = p^h \left[= \left(\frac{N}{N+1}\right)^h\right]$

$Expected\ winnings = p^h h \left[= \left(\frac{N}{N+1}\right)^h h\right]$ **A1**

Let E_h be the expected winnings when the value h is chosen. **M1**

$$\frac{E_{h+1}}{E_h} = \left(\frac{N}{N+1}\right)\left(\frac{h+1}{h}\right) = \frac{Nh+N}{Nh+h}$$ **A1**

Therefore $\frac{E_{h+1}}{E_h} > 1$ if $h < N$ **M1**

And $\frac{E_{h+1}}{E_h} < 1$ if $h > N$ **M1**

So as h increases, the values of E_h increase until $h = N$, the value then remains the same for $h = N + 1$ and decreases thereafter. **A1**

So I can maximise my winnings by choosing $h = N$

(ii) Possible sequences that lead to a win are:

All heads: Probability: $\left(\frac{N}{N+1}\right)^h$

There are h positions available (one before each of the heads) where at most one tail can be placed. **B1**

1 tail can be placed in any of the h positions, so the probability of a sequence containing just one tail is $\binom{h}{1}\left(\frac{N}{N+1}\right)^h\left(\frac{1}{N+1}\right)^1$ **M1**

Similarly, for any other number of tails, $t \le h$, the probability of a winning sequence containing that number of tails will be $\binom{h}{t}\left(\frac{N}{N+1}\right)^h\left(\frac{1}{N+1}\right)^t$ **M1**

Therefore the probability that I win is **M1**

$$\sum_{t=0}^{h}\binom{h}{t}\left(\frac{N}{N+1}\right)^h\left(\frac{1}{N+1}\right)^t = \left(\frac{N}{N+1}\right)^h\sum_{t=0}^{h}\binom{h}{t}\left(\frac{1}{N+1}\right)^t$$ **A1**

As the sum in the expression on the right is a binomial expansion it can be rewritten as $\left(\frac{1}{N+1}+1\right)^h$ **M1** **A1**

The probability that I win is therefore

$$\left(\frac{N}{N+1}\right)\left(\frac{1}{N+1}+1\right)^h = \frac{N^h(1+N+1)^h}{(N+1)^{2h}} = \frac{N^h(N+2)^h}{(N+1)^{2h}}$$

So my expected winnings are $\frac{hN^h(N+2)^h}{(N+1)^{2h}}$ **A1 AG**

In the case $N = 2$, the expected winnings are $h\left(\frac{8}{9}\right)^h$

The maximum value is when $h = 8$ or $h = 9$ and has a value of $\frac{8^9}{9^8}$ **B1**

$\log_3\left(\frac{8^9}{9^8}\right) = 9\log_3 8 - 8\log_3 9$ **M1**

$= 27\log_3 2 - 16$ **M1**

$\approx 27(0.63) - 16 = 1.01$ **M1**

Therefore $\frac{8^9}{9^8} \approx 3^{1.01} \approx 3$ **A1 AG**

M1	Attempt to find a probability of a sequence of heads followed by a tail
A1	Correct expected value
M1	Consideration of how expected value changes with h
A1	Correct expression
M1	Justification that the expected value increases with h while $h < N$
M1	Justification that the expected value decreases with h while $h > N$
A1	Conclusion that winnings can be maximised if $h = N$
	Subtotal: 7
B1	Identifies a strategy for considering all winning sequences
M1	Correct probability for one case, could be seen as part of full sum
M1	Generalised to any case
M1	Expression as a sum and restatement so that binomial can be identified
A1	Fully correct expression
M1	Identification of binomial expansion
A1	Correct simplification
A1	Fully justified expression for expected winnings
	Subtotal: 8
B1	Identification of maximum value for expected winnings
M1	Takes logs and simplifies
M1	Further simplification
M1	Applies given approximation
A1	Concludes given estimate for expected winnings
	Subtotal: 5

Question 13

(i)	$A_1 = \frac{1}{2}, C_1 = 0$	**B1**
	$B_1 = \frac{1}{4}, D_1 = \frac{1}{4}$	**B1**
	$A_2 = \frac{1}{2} \times \frac{1}{2} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{3}{8}$	**M1** **M1** **A1**
	$B_2 = D_2 = \frac{1}{2} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{2} = \frac{1}{4}$	**M1** **A1**
	$C_2 = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{1}{8}$	**M1** **A1**
(ii)	$B_{n+1} = \frac{1}{2}B_n + \frac{1}{4}(A_n + C_n)$	**M1**
	$A_n + B_n + C_n + D_n = 1$	**M1**
	$B_n = D_n$ (by symmetry)	**M1**
	Therefore $A_n + C_n = 1 - 2B_n$	**M1**
	$B_{n+1} = \frac{1}{4}$ and so $B_n = D_n = \frac{1}{4}$ for all n.	**A1**
	$A_{n+1} = \frac{1}{2}A_n + \frac{1}{4}(B_n + D_n) = \frac{1}{2}A_n + \frac{1}{8}$	**M1** **A1**
	$A_{n+1} - \frac{1}{4} = \frac{1}{2}\left(A_n - \frac{1}{4}\right)$	**M1**
	Therefore $\left(A_n - \frac{1}{4}\right)$ is a geometric sequence with common ratio $\frac{1}{2}$	**M1**
	$A_n = \frac{1}{4} + \left(\frac{1}{2}\right)^{n+1}$	**A1**
	$C_n = \frac{1}{4} - \left(\frac{1}{2}\right)^{n+1}$	**A1**

B1	Both values correct
B1	Both values correct
M1	One of the three cases identified in calculation or a tree diagram drawn to show all cases
M1	All three cases correctly identified
A1	Correct value
M1	Correct calculation
A1	Correct value
M1	Correct calculation
A1	Correct value
	Subtotal: 9
M1	Recurrence relation for B_n (or D_n) found
M1	Statement that probabilities add up to 1
M1	Identification of symmetry in problem or a recurrence relation to identify this relationship
M1	Combination so that A_n and C_n can be eliminated
A1	Correct value
M1	Recurrence relation for A_n
A1	Correct relation having substituted for B_n and D_n
M1	Appropriate method to find A_n
M1	Identification of geometric sequence
A1	Correct expression for A_n (must be simplified)
A1	Correct expression for C_n (must be simplified)
	Subtotal: 11