Step I, Solutions
June 2007

Section A: Pure Mathematics

1 There are many possible ways of answering this question, but it is essential to take a systematic and methodical approach. Here is one such strategy.

Consider the first two digits: there are eight possible sums (from $1 + 0 = 1$ to $4 + 4 = 8$).

If the first two digits sum to 1, there are two choices for the last two digits (1001 and 1010).

If the first two digits sum to 2 (11 and 20), there are three choices for the last two digits (because you can have 02 as well as 11 and 20).

If the first two digits sum to 3 (30, 21, 12), there are four choices for the last two digits.

If the first two digits sum to 4 (four choices), there are five choices for the last two digits.

If the first two digits sum to 5 (only four choices: 14, 23, 32, 41), there are the same four choices for the last two digits (there is no choice of 05 or 50).

If the first two digits sum to 6 (only three choices), there are three choices for the last two digits.

If the first two digits sum to 7 (only two choices), there are two choices for the last two digits.

If the first two digits sum to 8 (only one choice), there is one choice for the last two digits.

Hence there are $(1 \times 2) + (2 \times 3) + (3 \times 4) + (4 \times 5) + (4 \times 4) + (3 \times 3) + (2 \times 2) + (1 \times 1) = 70$ four-digit balanced numbers.

In general, there are $\displaystyle\sum_{r=1}^{k} r\,(r+1) + \sum_{r=k+1}^{2k} (2k+1-r)^2$ possible four-digit balanced numbers.

The second sum simplifies because $(2k+1-[k+1])^2 \equiv k^2$, $(2k+1-[k+2])^2 \equiv (k-1)^2$ and so on until $(2k+1-[2k])^2 \equiv 1$.)

Hence there are $\displaystyle\sum_{r=1}^{k} r\,(r+1) + \sum_{r=1}^{k} r^2$ possible four-digit balanced numbers.

$$\equiv 2\sum_{r=1}^{k} r^2 + \sum_{r=1}^{k} r$$

$$\equiv \frac{2k\,(k+1)\,(2k+1)}{6} + \frac{k\,(k+1)}{2} \text{ using the given identity for } \sum_{r=1}^{k} r^2$$

$$\equiv k\,(k+1)\left(\frac{4k+2+3}{6}\right) \equiv \frac{k\,(k+1)\,(4k+5)}{6}$$

2 (i) $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = 1$

$\Rightarrow A + B = \dfrac{\pi}{4}$, since A and B are both acute.

Now, if $\arctan \dfrac{1}{p} + \arctan \dfrac{1}{q} = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{\frac{1}{p} + \frac{1}{q}}{1 - \frac{1}{p} \times \frac{1}{q}} = 1$, by using the same identity for $\tan(A+B)$

$\Rightarrow \dfrac{q+p}{pq-1} = 1$, multiplying numerator and denominator by pq

$\Rightarrow pq - p - q - 1 = 0$

$\Rightarrow (p-1)(q-1) = 2$ (notice that this is analogous to "completing the square").

Since p and q are positive integers, we require two positive integers whose product is 2: the only choices are 1 and 2.

Hence $p = 2$ and $q = 3$ (or vice-versa: don't forget this second solution).

(ii) If $\arctan \dfrac{1}{r} + \arctan \dfrac{s}{s+t} = \dfrac{\pi}{4}$

$\Rightarrow \dfrac{1}{r} + \dfrac{s}{s+t} = 1 - \left(\dfrac{1}{r} \times \dfrac{s}{s+t}\right)$, by rearranging the expansion of $\tan(A+B) = 1$

$\Rightarrow s + t + rs = r(s+t) - s$, multiplying numerator and denominator by $r(s+t)$

$\Rightarrow s + t = rt - s$

$\Rightarrow 2s = t(r-1)$

$\Rightarrow r = \dfrac{2s}{t} + 1$

Since r is an integer, t must be a factor of $2s$. But the highest common factor of s and t is 1, so t cannot be a factor of s. Hence $t = 1$ or $t = 2$ (in the latter case, s is odd). Therefore either $r = 2s + 1$ or $r = s + 1$.

So, for all positive integer values of s,

$$\arctan \frac{1}{2s+1} + \arctan \frac{s}{s+1} = \frac{\pi}{4}$$

and for all positive odd integer values of s,

$$\arctan \frac{1}{s+1} + \arctan \frac{s}{s+2} = \frac{\pi}{4}$$

You might now like to consider whether the equation

$$\arctan \frac{1}{p} + \arctan \frac{1}{q} + \arctan \frac{1}{r} = \frac{\pi}{4}$$

has any integer solutions; first, you'll need an identity for $\tan(A+B+C)$.

3 There are many ways of tackling this question; here is one:

$$\cos^4\theta - \sin^4\theta \equiv \left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right) \equiv \cos 2\theta$$

$$\cos^4\theta + \sin^4\theta \equiv \left(\cos^2\theta + \sin^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta \equiv 1 - \tfrac{1}{2}\sin^2 2\theta.$$

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta - \sin^4\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \cos 2\theta \, \mathrm{d}\theta = \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 0.$

Also $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta + \sin^4\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 1 - \frac{1}{2}\sin^2 2\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 1 - \frac{1}{4}\left(1 - \cos 4\theta\right) \, \mathrm{d}\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{3}{4} + \frac{\cos 4\theta}{4} \, \mathrm{d}\theta = \left[\frac{3\theta}{4} + \frac{\sin 4\theta}{16}\right]_0^{\frac{\pi}{2}} = \frac{3\pi}{8}.$$

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \cos^4\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sin^4\theta \, \mathrm{d}\theta = \frac{3\pi}{16}$

Adapting this idea:

$$\cos^6\theta - \sin^6\theta \equiv \left(\cos^2\theta - \sin^2\theta\right)^3 + 3\cos^2\theta\sin^2\theta\left(\cos^2\theta - \sin^2\theta\right) \equiv \cos^3 2\theta + \frac{3}{4}\sin^2 2\theta\cos 2\theta$$

$$\cos^6\theta + \sin^6\theta \equiv \left(\cos^2\theta + \sin^2\theta\right)^3 - 3\cos^2\theta\sin^2\theta\left(\cos^2\theta + \sin^2\theta\right) \equiv 1 - \frac{3}{4}\sin^2 2\theta$$

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta - \sin^6\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \cos^3 2\theta + \frac{3}{4}\sin^2 2\theta\cos 2\theta \, \mathrm{d}\theta$

$$= \int_0^{\frac{\pi}{2}} \left(1 - \sin^2 2\theta\right)\cos 2\theta + \frac{3}{4}\sin^2 2\theta\cos 2\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \cos 2\theta - \frac{1}{4}\sin^2 2\theta\cos 2\theta \, \mathrm{d}\theta$$

$$= \left[\frac{\sin 2\theta}{2} - \frac{\sin^3 2\theta}{24}\right]_0^{\frac{\pi}{2}} = 0.$$

Also $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta + \sin^6\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 1 - \frac{3}{4}\sin^2 2\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 1 - \frac{3}{8}\left(1 - \cos 4\theta\right) \, \mathrm{d}\theta$

$$= \int_0^{\frac{\pi}{2}} \frac{5}{8} + \frac{3\cos 4\theta}{8} \, \mathrm{d}\theta = \left[\frac{5\theta}{8} + \frac{3\sin 4\theta}{32}\right]_0^{\frac{\pi}{2}} = \frac{5\pi}{16}.$$

Hence $\displaystyle\int_0^{\frac{\pi}{2}} \cos^6\theta \, \mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \sin^6\theta \, \mathrm{d}\theta = \frac{5\pi}{32}$

You might like consider how you would prove that $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2k}\theta - \sin^{2k}\theta \, \mathrm{d}\theta = 0$ for all k, without having to derive a new identity for each value of k.

4 The beginning of this question is made much easier if you are able to factorise by inspection i.e. by looking at a product and filling in the missing terms by deduction rather than using a formal method such as long division.

For example, to factorise $x^3 + 2x + 12$ you should first see that $x = -2$ is a root of the expression (because $-8 - 4 + 12 = 0$) hence $x + 2$ is a factor (using the factor theorem), then think along the lines: $x^3 + 2x + 12 \equiv (x + 2) \times$ (a quadratic expression), and the coefficient of x^3 is 1 so the coefficient of x^2 must be 1; but that will create a term $2x^2$ which you don't want so there needs to be a $-2x$ term in the quadratic (which will generate a $-2x^2$ term when the brackets are expanded); but then there will be a $-4x$ term and you want $2x$ overall so you need a final term of 6 to create an additional $6x$; and $2 \times 6 = 12$ so the constant term will be correct. This takes time to write out, but is very quick to do in your head.

Factorising complex algebraic expressions by inspection is much easier than any other method: you should work carefully through the reasoning that tells you that

$$x^3 - 3xbc + b^3 + c^3 \equiv (x + b + c)\left(x^2 + b^2 + c^2 - xb - xc - bc\right).$$

Hence $2Q(x) \equiv 2x^2 + 2b^2 + 2c^2 - 2xb - 2xc - 2bc \equiv (x - b)^2 + (x - c)^2 + (b - c)^2$.

When we are told that k is a root of both equations, we can deduce that

$$ak^2 + bk + c = 0$$

and also that

$$bk^2 + ck + a = 0.$$

There are now a number of possible steps: an obvious one might be to use the quadratic formula to try to find an expression for k in terms of a, b and c. But notice that the expression we are asked to derive is still in terms of k, so any step which eliminates k is unlikely to be correct.

Instead, multiplying the first equation by b, the second by a and subtracting yields

$$\left(ac - b^2\right)k = bc - a^2.$$

Multiplying the first equation by c, the second by b and subtracting yields

$$\left(ac - b^2\right)k^2 = ab - c^2.$$

$$\Rightarrow \left(\frac{bc - a^2}{ac - b^2}\right)^2 = \frac{ab - c^2}{ac - b^2}$$

$$\Rightarrow \left(ac - b^2\right)\left(ab - c^2\right) = \left(bc - a^2\right)^2$$

$$\Rightarrow a^2bc - ab^3 - ac^3 + b^2c^2 = b^2c^2 + a^4 - 2a^2bc$$

$$\Rightarrow a^3 - 3abc + b^3 + c^3 = 0$$

$\Rightarrow (a + b + c)\left(a^2 + b^2 + c^2 - ab - ac - bc\right) = 0$, using the factorisation from the beginning of the question.

Hence either $a^2 + b^2 + c^2 - ab - ac - bc = 0$

$\Rightarrow (a-b)^2 + (a-c)^2 + (b-c)^2 = 0$

$\Rightarrow a = b = c$ (you should consider what can be deduced if a sum of squares is zero)

hence the equations are identical (and each reduces to $x^2 + x + 1 = 0$)

or $a + b + c = 0$

$\Rightarrow k = 1$.

You might like to consider what would happen if $ac = b^2$.

5 To tackle this question, a big, clear diagram is essential!

(i) Let the side length of the octahedron be $2k$.

Then the sloping "height" of a triangular face is $k\sqrt{3}$.

Also, the vertical height of the whole octahedron is $2k\sqrt{2}$.

Therefore, by the cosine rule, $8k^2 = 3k^2 + 3k^2 - 2 \times k\sqrt{3} \times k\sqrt{3} \times \cos A$.

Hence $A = \arccos\left(-\frac{2k^2}{6k^2}\right) = \arccos\left(-\frac{1}{3}\right)$.

(ii) The centre of each face is on any median of the equilateral triangle that is the face, and the centre is two-thirds of the way along the median from any vertex.

This is a quotable fact, but can be worked out from a diagram, using the fact that the centre of an equilateral triangle is equidistant from the three vertices: the centre divides the median in the ratio 1: $\cos 60°$.

The feet of the two medians from the apex of the octahedron in two adjacent triangles are $k\sqrt{2}$ apart.

Therefore, by similarity, adjacent centres of the triangular faces are $\frac{2}{3} \times k\sqrt{2}$ apart.

Therefore, the volume of the cube (whose vertices are the centres of the faces) is

$$\left(\frac{2}{3} \times k\sqrt{2}\right)^3 = \frac{16k^3\sqrt{2}}{27}$$

and the volume of the octahedron is

$$2 \times \frac{4k^2 \times k\sqrt{2}}{3} = \frac{8k^3\sqrt{2}}{3}$$

Hence the ratio of the volume of the octahedron to the volume of the cube is $9 : 2$

6 **(i)** Since $x^2 - y^2 \equiv (x-y)(x+y)$, the given equation reduces to $d(x+y) = d^3$.

Hence $x + y = d^2$, and also we are given that $x - y = d$.

Therefore $x = \frac{1}{2}\left(d^2 + d\right)$ and $y = \frac{1}{2}\left(d^2 - d\right)$.

For the second equation, let $x = \sqrt{m}$ and $y = \sqrt{n}$ and let $d = 6$ (any choice of $d \geqslant 6$ will work).

Then $x = 21$ and $y = 15$, so $m = 441$ and $n = 225$.

You might like to check: $441 - 225 = 216 = (21 - 15)^3$.

(ii) It helps here to know that $x^3 - y^3 \equiv (x-y)\left(x^2 + xy + y^2\right)$.

Therefore the equation $x^3 - y^3 = (x-y)^4$ reduces to $x^2 + xy + y^2 = d^3$.

We know that $x^2 - 2xy + y^2 = d^2$ since $x - y = d$.

Hence, subtracting these two, $3xy = d^3 - d^2$.

This result can also be deduced by simplifying $x^3 - (x-d)^3 = d^4$ (using $x - y = d$).

Since $d = x - y$

$\Rightarrow 3x(x-d) = d^3 - d^2$

$\Rightarrow 3x^2 - 3dx - \left(d^3 - d^2\right) = 0$

$$\Rightarrow x = \frac{3d \pm \sqrt{9d^2 + 12\left(d^3 - d^2\right)}}{6} = \frac{3d \pm d\sqrt{12d - 3}}{6}$$

$$\Rightarrow 2x = d \pm d\sqrt{\frac{4d-1}{3}}$$

For x to be integer we need $\dfrac{4d-1}{3}$ to be a perfect square.

If $d = 1$ then either $x = 0$ (not permitted) or $x = 1$ which implies $y = 0$ (not permitted).

So let $d = 7$ (for example), since $4 \times 7 - 1 = 27 = 3 \times 3^2$ ($d = 17$ also works).

$\Rightarrow 2x = 7 \pm 7\sqrt{9}$

Therefore $x = 14$ and hence $y = x - d = 7$ (choosing the positive values).

You might like to check: $14^3 - 7^3 = (2 \times 7)^3 - 7^3 = 7 \times 7^3 = 7^4 = (14 - 7)^4$.

7 When answering this question, it is useful to remember that the distance D between points with position vectors $\mathbf{p}$ and $\mathbf{q}$ is evaluated as $D = |\mathbf{p} - \mathbf{q}|$.

Hence if $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ then $D = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2}$

If you tackle a question such as this using a known formula (there is one for the minimum distance between two skew lines, using the vector product), you need to be careful with the subsequent algebra: if you cancel k from the numerator and denominator of a fraction, you are excluding the possibility that k might equal 0.

(i) $D^2 = (3 - 2\lambda + \mu)^2 + (-2 - 2\lambda + 2\mu)^2 + (7 + 3\lambda - 2\mu)^2$

$\equiv 17\lambda^2 + 9\mu^2 - 24\lambda\mu - 30\mu + 38\lambda + 62$

$\equiv (3\mu - 4\lambda - 5)^2 + (\lambda - 1)^2 + 36.$

The minimum value of D^2 is therefore 6, achieved by setting $\lambda = 1$ and $\mu = 3$.

When $\lambda = 1$ and $\mu = 3$ the points $(3, 2, -1)$ and $(7, 4, 3)$ are the minimal distance apart.

(ii) The strategy in part **(ii)** must be to adapt the strategy in part **(i)**:

$D^2 = (1 + 4k\beta)^2 + (-\alpha + \beta - \beta k)^2 + (-7 - 3\beta k)^2$

$\equiv 1 + 16\beta^2k^2 + 8k\beta + \alpha^2 + \beta^2 + \beta^2k^2 - 2\alpha\beta + 2\alpha\beta k - 2\beta^2 k + 49 + 9\beta^2k^2 + 42\beta k$

$\equiv 50 + 26\beta^2k^2 + 50\beta k + \alpha^2 + \beta^2 - 2\alpha\beta + 2\alpha\beta k - 2\beta^2 k$

$\equiv (\alpha - \beta + \beta k)^2 + (5\beta k + 5)^2 + 25.$

This is a useful simplification because the second bracket is only in terms of β and k.

Hence if $k \neq 0$ then the minimum distance between the two lines is 5 (when $\beta = -\frac{1}{k}$ and $\alpha = 1 - \frac{1}{k}$).

But if $k = 0$ then the minimum distance is $\sqrt{50}$ (when $\alpha = \beta$), and the lines are parallel: look at the two direction vectors.

You might like to consider what happens geometrically to the two lines as k tends to zero: notice that there is a "jump" from a minimum separation of 5 to a minimum separation of $\sqrt{50}$.

8 Consider $\mathrm{f}(x) = ax^3 - 6ax^2 + (12a + 12)\,x - (8a + 16)$.

Since $\mathrm{f}(2) = 8$ and $\mathrm{f}'(2) = 12$, the curve $y = \mathrm{f}(x)$ touches $y = x^3$ at $(2\,,8)$: notice that both calculations are necessary to prove that the curves **touch**.

To find the other intersection point, let $\mathrm{f}(x) = x^3$

$\Rightarrow (a-1)\,x^3 - 6ax^2 + (12a+12)\,x - (8a+16) = 0$

Substituting $x = 2$: $8(a-1) - 24a + 2(12a+12) - (8a+16) \equiv 0$

$\Rightarrow (x-2)\left[(a-1)\,x^2 - (4a+2)\,x + (4a+8)\right] = 0$ (notice that factorising by inspection is much easier than using a method such as long division)

Substituting $x = 2$ into the quadratic factor: $4(a-1) - 2(4a+2) + (4a+8) \equiv 0$.

$\Rightarrow (x-2)\,(x-2)\left[(a-1)\,x - (2a+4)\right] = 0$

So the other intersection point has coordinates $\left(\dfrac{2a+4}{a-1}\,,\left[\dfrac{2a+4}{a-1}\right]^3\right)$

(i) When $a = 2$, $\dfrac{2a+4}{a-1} = 8$.

Hence the two graphs touch at $(2\,,8)$ and intersect at $(8\,,512)$.

$y = 2x^3 - 12x^2 + 36x - 32$ has no turning points: consider the derivative $6x^2 - 24x + 36 = 0$.

(ii) When $a = 1$, $\dfrac{2a+4}{a-1}$ is undefined.

Hence the two graphs touch at $(2\,,8)$, and do not intersect elsewhere.

$y = x^3 - 6x^2 + 24x - 24$ has no turning points: consider the derivative $3x^2 - 12x + 24 = 0$.

(iii) When $a = -2$, $\dfrac{2a+4}{a-1} = 0$.

Hence the two graphs touch at $(2\,,8)$ and intersect at $(0\,,0)$.

$y = -2x^3 + 12x^2 - 12x$ turns when $x = 2 \pm \sqrt{2} \approx 3.4$ and 0.6: consider the derivative $-6x^2 + 24x - 12 = 0$.

It is essential that each sketch shows these features clearly, though the graphs need not be to scale.

Section B: Mechanics

9 A big, clear diagram is very helpful here - for you and also for the examiner!

Resolving parallel and perpendicular to the plane in both cases:

$X\cos\theta + \mu R = W\sin\theta \qquad R = X\sin\theta + W\cos\theta$

and

$kX\cos\theta = \mu R + W\sin\theta \qquad R = kX\sin\theta + W\cos\theta.$

$$\Rightarrow \frac{\cos\theta + \mu\sin\theta}{k\cos\theta - k\mu\sin\theta} = \frac{\sin\theta - \mu\cos\theta}{\mu\cos\theta + \sin\theta}$$

$$\Rightarrow \sin\theta\cos\theta + \mu\cos^2\theta + \mu\sin^2\theta + \mu^2\sin\theta\cos\theta = k\cos\theta\sin\theta - k\mu\sin^2\theta - k\mu\cos^2\theta + k\mu^2\sin\theta\cos\theta$$

$$\Rightarrow (k-1)\left(1+\mu^2\right)\sin\theta\cos\theta = \mu(k+1)$$

Since $\sin\theta\cos\theta \equiv \frac{1}{2}\sin 2\theta \leqslant \frac{1}{2}$

$$\Rightarrow \frac{\mu(k+1)}{(k-1)\left(1+\mu^2\right)} \leqslant \frac{1}{2}$$

$$\Rightarrow k\left(1+\mu^2\right) - 2\mu k \geqslant 1 + \mu^2 + 2\mu$$

$$\Rightarrow k\left(1 - 2\mu + \mu^2\right) \geqslant 1 + 2\mu + \mu^2$$

$$\Rightarrow k \geqslant \frac{(1+\mu)^2}{(1-\mu)^2}$$

Of course, the same result can be derived by initially resolving horizontally and vertically. In this case, the four equations are

$X + \mu R\cos\theta = R\sin\theta \qquad W = R\cos\theta + \mu R\sin\theta$

and

$kX = \mu R\cos\theta + R\sin\theta \qquad W = R\cos\theta - \mu R\sin\theta.$

From these, $\dfrac{X}{W} = \dfrac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} = \dfrac{1}{k}\left(\dfrac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin\theta}\right)$ which is equivalent to the above argument.

10 The displacement-time graph should have straight lines with equations:

$s = d - ut$ which is the path of the Norman army;

$s = xt$ which is the path of the Saxon horseman riding towards the Norman army;

$s = d - yt$ which is the path of the Norman horseman riding towards the Saxon army.

The Saxon horseman meets the Norman army when $xt = d - ut$,

i.e. at the point with coordinates $\left(\dfrac{d}{u+x}, \dfrac{dx}{u+x}\right)$.

Therefore his return path has equation $s - \dfrac{dx}{u+x} = -x\left(t - \dfrac{d}{u+x}\right) \Rightarrow s = -xt + \dfrac{2dx}{u+x}$

The Norman horseman meets the Saxon army when $s = 0$ i.e. when $t = \dfrac{d}{y}$.

Therefore his return path has equation $s = yt - d$

The subsequent analysis assumes that the two horsemen meet for the first time as they approach the opposing army, and meet for the second time as they retreat from the opposing army. If so, the horsemen first meet when $xt = d - yt \Rightarrow t = \dfrac{d}{x+y} \Rightarrow s = \dfrac{dx}{x+y}$ since $s = xt$.

They next meet when $-xt + \dfrac{2dx}{u+x} = yt - d$

$$\Rightarrow \frac{2dx}{u+x} + d = t\,(x+y) \Rightarrow t = \frac{d\,(u+3x)}{(x+y)\,(u+x)}$$

$$\Rightarrow s = -\frac{dx\,(u+3x)}{(x+y)\,(u+x)} + \frac{2dx}{u+x} = \frac{dx\,(2y-x-u)}{(u+x)\,(x+y)}.$$

Of course, this need not happen:

(i) If the Norman horseman rides quickly, he will meet the Saxon horseman for the second time $\left(\text{when}\, xt = yt - d \Rightarrow t = \dfrac{d}{y-x}\right)$ before the Saxon has reached the Norman army for the first time. If so, $\dfrac{d}{y-x} < \dfrac{d}{u+x} \Rightarrow u + x < y - x \Rightarrow u < y - 2x$

(ii) If the Saxon horseman rides quickly, he will meet the Norman horseman for the second time $\left(\text{when } -xt + \dfrac{2dx}{u+x} = d - yt \Rightarrow t = \dfrac{d\,(x-u)}{(u+x)\,(x-y)}\right)$ before the Norman has reached the Saxon army for the first time.

If so, $\dfrac{d\,(x-u)}{(u+x)\,(x-y)} < \dfrac{d}{y} \Rightarrow y\,(x-u) < (u+x)\,(x-y)$

$\Rightarrow 2xy < ux + x^2 \Rightarrow 2y - x < u$

11 Again, a big, clear diagram is very helpful here.

The acceleration up the slope is $-\frac{g}{2}$, hence the velocity at height $\frac{L}{2}$ is $\sqrt{u^2 - gL}$.

Then the particle travels freely as a projectile until it hits the floor, with a displacement of D horizontally and $-\frac{L}{2}$ vertically.

Hence $-\frac{L}{2} = \left(\sqrt{u^2 - gL}\right) t \sin 30^\circ - \frac{g}{2}t^2$ and $D = \left(\sqrt{u^2 - gL}\right) t \cos 30^\circ$

$\Rightarrow -L = t\sqrt{u^2 - gL} - gt^2$ and $2D = t\sqrt{3}\sqrt{u^2 - gL}$

$$\Rightarrow -L = \frac{2D}{\sqrt{3}} - \frac{4gD^2}{3\left(u^2 - gL\right)}$$

$$\Rightarrow 4gD^2 - 2D\sqrt{3}\left(u^2 - gL\right) - 3L\left(u^2 - gL\right) = 0$$

$$\Rightarrow 8gD\frac{\mathrm{d}D}{\mathrm{d}L} - 2\frac{\mathrm{d}D}{\mathrm{d}L}\sqrt{3}\left(u^2 - gL\right) + 2Dg\sqrt{3} - 3\left(u^2 - gL\right) + 3Lg = 0$$

$$\Rightarrow \frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}\left(u^2 - gL\right)}.$$

Since $R = D + \frac{L\sqrt{3}}{2} \Rightarrow \frac{\mathrm{d}R}{\mathrm{d}L} = \frac{\mathrm{d}D}{\mathrm{d}L} + \frac{\sqrt{3}}{2}$

Hence $\frac{\mathrm{d}R}{\mathrm{d}L} = 0 \Rightarrow \frac{\mathrm{d}D}{\mathrm{d}L} = -\frac{\sqrt{3}}{2}$.

$$\Rightarrow -\frac{2\sqrt{3}gD - 3(u^2 - 2gL)}{8gD - 2\sqrt{3}\left(u^2 - gL\right)} = -\frac{\sqrt{3}}{2}.$$

$$\Rightarrow -4gD\sqrt{3} + 3\left(u^2 - gL\right) + 2Dg\sqrt{3} - 3\left(u^2 - 2gL\right) = 0$$

$$\Rightarrow 3Lg = 2Dg\sqrt{3}$$

$$\Rightarrow 2D = L\sqrt{3}$$

$$\Rightarrow R = 2D = L\sqrt{3}.$$

Substituting this into $4gD^2 - 2D\sqrt{3}\left(u^2 - gL\right) - 3L\left(u^2 - gL\right) = 0$ we deduce that:

$$3gL^2 - 3L\left(u^2 - gL\right) - 3L\left(u^2 - gL\right) = 0$$

$$\Rightarrow gL = 2u^2 - 2gL$$

$$\Rightarrow 3gL = 2u^2$$

$$\Rightarrow R = \frac{2u^2}{g\sqrt{3}} \quad \left(= L\sqrt{3}\right)$$

Section C: Probability and Statistics

12 A tree diagram is very helpful when answering each part of this question.

(i) In the first case, $P(\text{the first sweet is red}) = \dfrac{a}{N}$

and $P(\text{the second sweet is red}) = \dfrac{a(a-1)}{N(N-1)} + \dfrac{(N-a)a}{N(N-1)} = \dfrac{aN-a}{N(N-1)} = \dfrac{a}{N}$

(ii) In the second case, $P(\text{the first sweet is red}) = \dfrac{pa}{N} + \dfrac{qb}{N}$

and $P(\text{the second sweet is red})$

$$= \frac{pa \times p(a-1)}{N(N-1)} + \frac{pa \times q(b+1)}{N(N+1)} + \frac{qb \times p(a+1)}{N(N+1)} + \frac{qb \times q(b-1)}{N(N-1)}$$

$$+ \frac{p(N-a) \times pa}{N(N-1)} + \frac{p(N-a) \times qb}{N(N+1)} + \frac{q(N-b) \times pa}{N(N+1)} + \frac{q(N-b) \times qb}{N(N-1)}$$

$$= \frac{pa}{N(N-1)}\Big[p(a-1) + p(N-a)\Big] + \frac{qb}{N(N-1)}\Big[q(b-1) + q(N-b)\Big]$$

$$+ \frac{pq}{N(N+1)}\Big[a(b+1) + b(a+1) + b(N-a) + a(N-b)\Big]$$

$$= \frac{pa}{N(N-1)}\Big[Np - p\Big] + \frac{qb}{N(N-1)}\Big[Nq - q\Big]$$

$$+ \frac{pq}{N(N+1)}\Big[Na + Nb + a + b\Big]$$

$$= \frac{p^2a}{N} + \frac{q^2b}{N} + \frac{pq(a+b)}{N}$$

$$= \frac{(p+q)(pa+qb)}{N}$$

$$= \frac{pa}{N} + \frac{qb}{N}$$

13 There are two ways of writing the answer to each part: one way is to think of a tree diagram, and then to combine the appropriate fractions; the other way is to count the number of "successful" combinations and divide by the number of "possible" combinations. Of course, these two approaches give the same numerical answer! You should make sure that you understand each version of the solution.

The probability that

(i) all four discs taken are numbered

$$= \frac{5}{11} \times \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{{}^5C_4}{{}^{11}C_4} = \frac{1}{66}$$

(ii) all four discs taken are numbered given that the disc numbered "3" is taken first

$$= \frac{\frac{1}{11} \times \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8}}{\frac{1}{11}} = \frac{{}^4C_3}{{}^{10}C_3} = \frac{1}{30}$$

(iii) exactly two numbered discs are taken, given that the disc numbered "3" is taken first

$$= \frac{\frac{1}{11} \times \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}}{\frac{1}{11}} \times 3 = \frac{{}^4C_1 \times {}^6C_2}{{}^{10}C_3} = \frac{1}{2}$$

because there are three equally likely outcomes once "3" has been taken first: "numbered, blank, another blank" or "blank, numbered, another blank" or "blank, another blank, numbered".

(iv) exactly two numbered discs are taken, given that the disc numbered "3" is taken

$$= \frac{\frac{1}{11} \times \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}}{1 - \frac{10}{11} \times \frac{9}{10} \times \frac{8}{9} \times \frac{7}{8}} \times 12 = \frac{{}^1C_1 \times {}^4C_1 \times {}^6C_2}{{}^{11}C_4 - {}^{10}C_4} = \frac{1}{2}$$

because there are twelve equally likely rearrangements of "3, another number, blank, another blank". Notice how P (disc numbered "3" taken) has been calculated on the denominator.

(v) exactly two numbered discs are taken, given that a numbered disc is taken first

$$= \frac{\frac{5}{11} \times \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}}{\frac{5}{11}} \times 3 = \frac{{}^5C_1 \times {}^4C_1 \times {}^6C_2}{{}^5C_1 \times {}^{10}C_3} = \frac{1}{2}$$

because there are three equally likely outcomes once a numbered disc has been taken first.

(vi) exactly two numbered discs are taken, given that a numbered disc is taken

$$= \frac{\frac{5}{11} \times \frac{4}{10} \times \frac{6}{9} \times \frac{5}{8}}{1 - \frac{6}{11} \times \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8}} \times 6 = \frac{{}^5C_2 \times {}^6C_2}{{}^{11}C_4 - {}^6C_4} = \frac{10}{21}$$

because there are six equally likely rearrangements of "number, another number, blank, another blank". Notice how P (a numbered disc is taken) has been calculated on the denominator.

14 $y = (x+1)\,\mathrm{e}^{-x} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\,(x+1)\,\mathrm{e}^{-x} + \mathrm{e}^{-x} \equiv -x\mathrm{e}^{-x}$

Hence the graph turns (and crosses the y-axis) at $(0\,,1)$, and it crosses the x-axis at $(-1\,,0)$. As x tends to ∞, y tends to 0 from above; as x tends to $-\infty$, y also tends to $-\infty$.

(i) $\mathrm{P}\,(X \geqslant 2) = 1 - p$

$\Rightarrow \mathrm{P}\,(X = 0) + \mathrm{P}\,(X = 1) = p$

$\Rightarrow (1+\lambda)\,\mathrm{e}^{-\lambda} = p.$

By considering the graphs $y = (x+1)\,\mathrm{e}^{-x}$ and $y = p$, we can see that this equation has a unique solution (for $0 < p < 1$).

(ii) $\mathrm{P}\,(X = 1) = q$

$\Rightarrow \lambda\mathrm{e}^{-\lambda} = q.$

The structure of part **(i)** suggests we consider the graphs $y = x\mathrm{e}^{-x}$ and $y = q$.

$y = x\mathrm{e}^{-x} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{-x} - x\mathrm{e}^{-x} \equiv (1-x)\,\mathrm{e}^{-x}$

Hence the graph of $y = x\mathrm{e}^{-x}$ passes through $(0\,,\,0)$ and turns at $\left(1\,,\mathrm{e}^{-1}\right)$.

As x tends to ∞, y tends to 0 from above; as x tends to $-\infty$, y also tends to $-\infty$.

Hence the equation $\lambda\mathrm{e}^{-\lambda} = q$ will have a unique solution when $\lambda = 1$ and $q = \mathrm{e}^{-1}$.

(iii) $\mathrm{P}\,(X = 1\,|\,X \leqslant 2) = r$

$$\Rightarrow \frac{\lambda\mathrm{e}^{-\lambda}}{\mathrm{e}^{-\lambda} + \lambda\mathrm{e}^{-\lambda} + \frac{\lambda^2}{2!}\mathrm{e}^{-\lambda}} = r$$

$$\Rightarrow \frac{2\lambda}{2 + 2\lambda + \lambda^2} = r$$

The structure of parts **(i)** and **(ii)** suggests we consider the graphs $y = \dfrac{2x}{2 + 2x + x^2}$ and $y = r$.

The graph $y = \dfrac{2x}{2 + 2x + x^2}$ passes through $(0\,,\,0)$.

It turns when $2\left(2 + 2x + x^2\right) - 2x\,(2 + 2x) = 0 \Rightarrow 4 - 2x^2 = 0 \Rightarrow x = \pm\sqrt{2}$.

As x tends to ∞, y tends to 0 from above; as x tends to $-\infty$, y tends to 0 from below.

Hence the equation $\Rightarrow \dfrac{2\lambda}{2 + 2\lambda + \lambda^2} = r$ will have a unique solution when $\lambda = \sqrt{2}$ and $r = \dfrac{2\sqrt{2}}{2 + 2\sqrt{2} + 2} = \dfrac{\sqrt{2}}{2 + \sqrt{2}} = \sqrt{2} - 1.$