STEP MATHEMATICS 2

2018

Hints and Solutions

Question 1

The first result can be shown by substituting k^{-1} into the quartic expression.

It then follows for part (i) that the only way to achieve one distinct root is for that root to be either 1 or -1. In either case the factorised form of the quartic can then be considered to find the values of a and b.

Similarly, for three distinct roots, there must be one pair (k, k^{-1}) along with either 1 or -1. Substitution of 1 or -1 into the quartic then leads to the required relationships.

For part (iii) note that the case $b = 2a - 2$ corresponds to the case where there is a root of -1 and it can be seen that it must be a repeated root. The other factor is therefore a quadratic, which can then be solved.

Finally, the conditions for there to be three roots can be found by considering the discriminant of the quadratic (and the corresponding one for the other case). It is also necessary to confirm that this quadratic does not repeat the root of either 1 or -1 depending on which case is being considered.

Question 2

The point with x-coordinate $tx_1 + (1-t)x_2$ is a point within the range (x_1, x_2) (for $0 < t < 1$), and $tf(x_1) + (1-t)f(x_2)$ is the y-coordinate of a point on the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Therefore, for any position between the two endpoints on the curve, the inequality is comparing the point on the curve with the point on the chord joining the two points. The sketch therefore needs to show the chord entirely below the curve. The final part of the introductory section can be shown either with a proof by contradiction or by arguing that $f''(x) < 0$ means that the gradient is always decreasing within the interval.

Part (i) requires choosing values for x_1 and x_2 so that the inequality can be applied and then applying this process multiple times to reach the required result. In each case the choice of x_1 and x_2 need to be made so that they lie within the range for which applying the inequality is valid.

Parts (ii) and (iii) both follow from the result of part (i), but it is important to check that the function being used is concave in the relevant range which needs to be stated clearly. In part (ii) the result follows immediately, whereas the final part requires some manipulation of logarithms to reach the final form of the relationship.

Question 3

The differentiation for the first part of the question can be achieved by applying the chain rule. Consideration of the sine function within the interval then allows the range to be determined. Consideration of the gradient function then allows the graph to be sketched - it can be seen from the symmetry of the sine function that $f'(x) = f'\left(\frac{1}{2}\pi\right) - x$, which means that the graph must have rotational symmetry about the point where $x = \frac{1}{4}\pi$.

A sketch of a rotationally symmetric function is a helpful way of demonstrating the second part as it allows the distances that must be equal to be identified clearly. It is important to show clearly that the result works in both the if and only if directions. For the final question in this part, note that the sections of the graph above the axis must exactly match those below the axis, so the area must be 0.

For part (iii), begin by showing that the equation from part (ii) holds for this function. Once the rotational symmetry has been demonstrated it follows that the area of any interval with the centre of rotation in the centre will be equal to the area of a rectangle over the same interval passing through the centre of rotation.

Question 4

For the first part, note that two of the terms in the left-hand side of the equation have a coefficient of 1 and two have a coefficient of 3. Applying the given identity to each of these pairs gives a common factor of $\cos\frac{5x}{2}$. The equation can therefore be factorised and then another application of the given identity will allow the full set of roots to be found.

The identity given at the start of the question can be applied to the first two terms of the left-hand side of the equation in part (ii) and the double angle formula can be applied to the $\cos 2x$. This then leads to an equation that can easily be factorised to show the required result. The range of possible values needs to be considered when considering the case where $\cos x = \cos y$.

For the final part a similar process to part (ii) can be used to create a quadratic function of $\cos\frac{1}{2}(x+y)$. Completing the square or considering a discriminant then allows the solutions to be found.

Question 5

For the first part of the question note that $\ln(1+x)$ can be obtained by integrating $(1+x)^{-1}$ and so the required expansion can be found by integrating the binomial expansion term by term. Note also, that the integration produces a constant, which needs to be shown to be 0.

In part (ii), the series expansion of e^{ax} can be obtained by adjusting the series expansion of e^{x}. To evaluate the integral, substitute the series expansion for the e^{ax}, but leave the e^{-x} unchanged. The integration can then be completed term by term.

For part (iii) note that a substitution of $u = -\ln x$ will transform the integral into one that can be expressed in terms of the integral in part (ii), which then allows the result to follow.

Question 6

For the first part, note that 5 will certainly be a factor of the left-hand side in any case where $n \geq 5$. It then remains to check the other cases one at a time.

For part (ii), first explain how the two theorems show that there will not be any solutions if $n \geq 7$, by showing that $m > 4n$ and so there must be a prime factor of the right-hand side that cannot exist in the product on the left-hand side.

The remaining cases then need to be checked one at a time, noting that the individual numbers within the product can be split into their prime factorisation and then combined differently to form the right-hand side.

Question 7

It is very useful to draw a diagram to represent the situation described at the start of this question. Defining the vectors $\boldsymbol{m}$ and $\boldsymbol{n}$ as scalar multiples of $\boldsymbol{a}$ and $\boldsymbol{b}$ (using two new unknowns) allows the position vector of Q to be written in two different ways. Since the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, the coefficients of these vectors can be equated and this then leads to the correct expression for $\boldsymbol{m}$.

A similar process then leads to an expression for the position vector of L in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$, but since L lies on OB, the coefficient of $\boldsymbol{a}$ must be 0.

It then follows that $\lambda\mu < 1$ means that L lies on the segment OB.

Question 8

Making the substitution given reduces the differential equation into one for which it is easy to separate the variables. The two sides can then be integrated to find a general solution to the equation and then the boundary condition can be applied to find the required solution.

The differential equation in part (ii) is similar to the one from part (i), so a similar substitution should work (using a cube root rather than square root). The same process can then be followed as in part (i) to solve this differential equation.

In part (iii), the information that $\alpha = \beta$ can be used to simplify the equations being considered and then it can be seen that the two curves will both approach an asymptote at $y = 1$. We also know that the curves both pass through the origin and the differential equations show that the curves should both have gradient 0 at the origin. All that remains is to deduce the relative positions of the two curves by considering the behaviour of the exponential function.

Question 9

Since the two particles are released from rest the distance between them will remain constant until A reaches the ground. The height of B above the ground at this point can therefore be calculated.

Application of the uniform acceleration formulae will therefore give the speeds of A and B at the moment A hits the ground and the coefficient of restitution can then be used to calculate the speed with which A rebounds. Since they both continue to move under gravity the speed of B relative to A will remain constant and therefore the time until they collide can be calculated. Once the time is known one of the uniform acceleration formulae can then be used to determine the height at which the collision happens and the two speeds at this time.

For the final part another application of the uniform acceleration formulae can be used to find the velocity of A immediately after the collision. Conservation of momentum can then be used to find the velocity of B, although care needs to be taken at this stage with the signs of the terms. Finally, the velocities before and after the collision can be used to calculate the coefficient of restitution.

Question 10

Finding an expression for the length of the string at time t allows the speed of the point on the string to be determined. The differential equation can then be set up by adding the speed of the ant to the speed of the point on the string. The next result can then be verified by applying the quotient rule to perform the differentiation.

Once the differential equation has been verified, integration leads to a relationship between x and t, which then leads to the required result.

For the journey back, the differential equation needs to be changed so that the speed of the ant is subtracted rather than added. The differential equation can then be rewritten in a manner similar to the first part of the question and solved.

Question 11

A diagram representing the situation will help to ensure that the correct calculations are performed. In particular it is important to note that the frictional force will be acting in the direction of motion of the motorbike. Taking moments about the centre of mass as instructed and then setting the reaction at the front wheel of the motorbike to 0 for the case when the front wheel loses contact with the ground gives the maximum possible frictional force for this motion. Comparing this to μR then gives the first inequality.

When the rear wheel is about to slip the frictional force will be taking its maximum value. Substituting this into the equation found by taking moments and resolving forces vertically then allows the value of this frictional force to be found. Newton's second law then gives the acceleration.

For the final part, first show that the maximum acceleration is at the moment when the front wheel would be about to leave the ground. The value of the frictional force at this point can then be found and the acceleration can then be deduced.

Question 12

First note that the only winning sequence is h heads in a row and the probability of this can be found easily. The expected winnings can then be expressed as a function of h (E_h). By considering the value of $\frac{E_{h+1}}{E_h}$ it can be shown that the expected winnings increases until $h = N$, remain the same for the next case and then decreases thereafter.

For the second part there are multiple sequences that lead to a win. Begin with a sequence of h heads and then consider adding tails to any of the h positions before those heads (only 1 tail can be placed in each position). The number of ways of winning with a total of t tails in the sequence can therefore be seen to be $\binom{h}{t}$. The sum of these probabilities can then be seen to be a binomial expansion and can therefore be simplified. An expression for the expected winnings can therefore be found. The case where $N = 2$ leads to a function of the form of the previous part, so the point at which the maximum value occurs can be written down immediately. Taking logarithms of this maximum value allows it to be shown that the value is very close to 3^1.

Question 13

The probabilities in the first part of the question can most easily be deduced by using a tree diagram.

For the second part, note that the probabilities at B must be equal to the probabilities at D by the symmetry of the problem. The sum of the four probabilities for any value of n must be equal to 1. Therefore, it is possible to deduce a recurrence relation for B_n and see that this remains at a constant value. With the values of B_n and D_n known recurrence relations for A_n and C_n can be found. These recurrence relations can be related to geometric sequences in order to find the formula for the general term.