STEP I - Hints and Solutions

Question 1

The relationships for p and q can be obtained by substituting the coordinates of the three known points into the equation of the line, or by using the formula for calculating the gradient from a pair of points.

The formula for the sum of the distances is then easy to find and differentiation with respect to m will allow the minimum value to be found.

Similarly, the distance for the second part can be written as $\sqrt{p^2 + q^2}$ and again differentiation can be used to find the value of m for which the minimum occurs. Since the minimum value of $\sqrt{p^2 + q^2}$ occurs at the same value of m as the minimum value of $(p^2 + q^2)$, the differentiation can be simplified by just differentiating $(p^2 + q^2)$ with respect to m. It is then a simple matter to write this answer in a form similar to that of the first part.

Question 2

To sketch the graph it is important to know where the stationary points are. Either by considering the graph of $y = x^2 - 3$ or by differentiating it can be seen that there are two minima and one maximum.

The equation in the second part can be rearranged to show that the solutions correspond to the intersections of the graph of $y = (x^2 - 3)^2$ and a straight line. The case requiring care is the two solution case as this must include the straight line which touches the two minima.

For the next part of the question, differentiation of the equation twice leads quickly to the two possible values of x. Both cases then need to be considered, but it should be clear that one graph is a reflection of the other in the y-axis, so the sets of values for b will be the same for both cases.

The final graph should clearly have a minimum for some negative value of x. $\frac{d^2y}{dx^2}$ will still be 0 at $x = \pm 1$, so there will be two points of inflexion.

Question 3

The sketch of the graph, including a chord and tangent should not cause much difficulty. Adding the line $x = b$ should show that the area under the graph lies between two triangles, both with a base of length b and with heights $\sin b$ and b. Integrating the function between the two limits and then rearranging will give the correct relationship.

For the second part of the question a different diagram is needed, this time showing the area under a curve contained within a trapezium and with a trapezium contained within it. The differentiation and integration of $y = a^x$ will produce the $\ln a$ expressions required in the final answer and so the vertical lines $x = 0$ and $x = 1$ can be used to define the regions.

Question 4

The equations of the tangents at P and Q should be easy to find and then the solution of simultaneous equations will give the required coordinates for T. Similarly, the equations for the normal should be easy to find, but it is more difficult to find simplified expressions for the solution to the simultaneous equations (which are useful for the final part of the question). The factorisation of the difference between two cubes, $a^3 - b^3 \equiv (a - b)(a^2 + ab + b^2)$ will be useful for avoiding a lot of algebraic manipulation. If this expression is simplified correctly then the final result becomes straightforward.

Question 5

The most obvious approaches to the first integration are to integrate by parts or to use the identity $\sin 2x \equiv 2 \sin x \cos x$ and then make a substitution. The second integration can also be evaluated by integrating by parts, but the identity for $\cos 2x$ is not as useful.

For the third integration it is necessary to rewrite the integral in a form from which the previous results can be applied. The first point to note is that the expression within the logarithm is not a simple cosine function and so the first step to making the expression similar to those used previously is to rewrite it in the form $R\cos(x - \alpha)$. Once this is done, the substitution of $u = x - \frac{1}{4}\pi$, with some knowledge of the relationship between sine and cosine graphs should reduce the integral to a combination of the two previous ones.

Question 6

By writing down expressions for the height of the pole using the tangent of each of the angles of elevation, the problem is quickly reduced to a two dimensional problem about lines within a circle. The simplest way to tackle this is to observe that the triangles in the diagram are similar to each other, but approaches working with various right angled triangles also lead to the correct solution.

The proof of the identity should be straight forward for those familiar with the commonly used trigonometric relationships and the inequality is then easily found by considering the consequence of the constraint placed on *p* and *q*. Given that cot is a decreasing function in the required range of values, the final result follows easily.

Question 7

By substituting x for all of the terms in the recurrence relation the result for the first part should follow easily. For the second part, the two values x and y can be substituted in both orders into the relation, giving two equations in x and y. Solving these two equations then leads to the correct set of possibilities.

Following the same principle, the substitution of x , y and z into the relation can be done in three different ways, leading to three simultaneous equations. Solving these equations gives an equation in p and q , which can be solved to give the two different cases. The case where $p+q=1$ is easy to check, and for the case where $(p-q)^2+(p+1)^2+(q+1)^2=0$ it should be noticed that this can only occur if $p=q=-1$.

Question 8

The new differential equation follows quite easily once the substitution given has been followed. It is an example of a differential equation that can be solved by separating the variables and so by evaluating the two integrals that are reached the required form can be obtained.

The same substitution will also reduce the second differential equation to a simpler form and it is again an example that can be solved by separating the variables. It should be clear that partial fractions are an appropriate method for evaluating the integral that is required.

Question 9

The first part of the question is a straightforward calculation involving the equations for motion under uniform acceleration. It is important to explain the reason for choosing the positive square root however. The final result follows from correct use of an identity for $\cos 2\theta$.

It is also easy to find an expression for the range once the time has been calculated and further application of an identity for $\cos 2\theta$ will give an expression for this in terms of c . Differentiation with respect to c will then give the result that the maximum value occurs when $c=\frac{1}{5}$. The final part of the question can be solved by substituting the appropriate values of c into the formula for the range.

Question 10

Although the equation looks complicated, calculations for the time that it takes the stone to drop and the time for the sound to return allow the first relationship to be deduced quite easily. The second relationship can be shown by simplifying the expression for T and showing that it is equal to $\sqrt{\frac{2D}{g}}$. This can then be rearranged to give $D=\frac{1}{2}gT^2$. The final part of the question is then a substitution of the values given into the formula to obtain the estimate.

Question 11

While the diagram may look a little more complicated than the standard questions on this topic, the first section of this question requires the usual steps to establish a pair of simultaneous equations. The difference in the second part of the question is that the acceleration cannot be assumed to be constant (as the pulley in the middle of the diagram is able to move) and the important extra relationship that is needed is the relationship between the accelerations at the three points (the two particles and the pulley).

Question 12

The probabilities for a failure in each year long period need to be calculated by evaluating the integral and from these it is possible to construct a tree diagram from which the probability can be calculated. The final part of the question is simply the calculation of a conditional probability. As always with conditional probability the important step is to deduce which two probabilities need to be calculated.

Question 13

There are a number of ways to approach this problem. The most obvious is to work out how many possibilities there are for each number of digits. A clear method for categorising these is needed to work out the number of possibilities in each case. For example, if there are 4 different digits then there are five choices for the digits to be used and four choices for the digit to be repeated. There are then ten choices for the positions of the repeated digits and 3! choices for the order of the remaining digits. This gives 1200 altogether.