STEP I, Solutions
June 2008

Question 1

What does it mean to say that a number x is irrational*?*

It means that we cannot write $x = m/n$ where m and n are integers with $n \neq 0$.

Prove by contradiction statements A and B below, where p and q are real numbers.

A: *If pq is irrational, then at least one of p and q is irrational.*

B: *If $p + q$ is irrational, then at least one of p and q is irrational.*

We first prove statement A.

Assume that pq is irrational, but neither p nor q is irrational, so that both p and q are rational. But then pq is the product of two rational numbers, so is rational. This contradicts that assumption that pq is irrational. So statement A is true.

Now for statement B we argue similarly.

Assume that $p + q$ is irrational, but neither p nor q is irrational, so that both p and q are rational. But then $p + q$ is the sum of two rational numbers, so is rational. This contradicts the assumption that $p + q$ is irrational. So statement B is true.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: *If p and q are irrational, then $p + q$ is irrational.*

One example is $p = \sqrt{2}$, $q = -\sqrt{2}$.

If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi + \mathrm{e}$, $\pi - \mathrm{e}$, $\pi^2 - e^2$, $\pi^2 + \mathrm{e}^2$ is rational.

We assume that the five given numbers are, indeed, irrational.

We have $(\pi + \mathrm{e}) + (\pi - \mathrm{e}) = 2\pi$, which is irrational (if p is irrational, then so is $2p$). So by statement B, at least one of $\pi + \mathrm{e}$ and $\pi - \mathrm{e}$ is irrational.

Similarly, $(\pi^2 + \mathrm{e}^2) + (\pi^2 - \mathrm{e}^2) = 2\pi^2$, which is irrational. So by statement B again, at least one of $\pi^2 + \mathrm{e}^2$ and $\pi^2 - \mathrm{e}^2$ is irrational.

Assume that both $\pi + \mathrm{e}$ and $\pi^2 - \mathrm{e}^2$ are rational. Then

$$\pi - \mathrm{e} = \frac{\pi^2 - \mathrm{e}^2}{\pi + \mathrm{e}}$$

would also be rational. But we know that at least one of $\pi + \mathrm{e}$ and $\pi - \mathrm{e}$ is irrational, so $\pi + \mathrm{e}$ and $\pi^2 - \mathrm{e}^2$ cannot both be rational. Similarly, we can't have both $\pi - \mathrm{e}$ and $\pi^2 - \mathrm{e}^2$ rational.

Thus if two of the four numbers are rational, they must be $\pi^2 + \mathrm{e}^2$ and one of $\pi \pm \mathrm{e}$.

Assume that $\pi^2 + \mathrm{e}^2$ and $\pi + \mathrm{e}$ are rational. Then $(\pi + \mathrm{e})^2 = (\pi^2 + \mathrm{e}^2) + 2\mathrm{e}\pi$ is the square of a rational number, so is rational. But then $2\mathrm{e}\pi = (\pi + \mathrm{e})^2 - (\pi^2 + \mathrm{e}^2)$ would be rational, contradicting the irrationality of $\mathrm{e}\pi$. Thus we cannot have both $\pi^2 + \mathrm{e}^2$ and $\pi + \mathrm{e}$ rational.

Similarly, if $\pi^2 + \mathrm{e}^2$ and $\pi - \mathrm{e}$ are both rational, we would have $2\mathrm{e}\pi = (\pi^2 + \mathrm{e}^2) - (\pi - \mathrm{e})^2$ being rational, again a contradiction.

Thus at most one of these four numbers is rational.

Question 2

The variables t and x are related by $t = x + \sqrt{x^2 + 2bx + c}$, where b and c are constants and $b^2 < c$. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t-x}{t+b},$$

and hence integrate $\dfrac{1}{\sqrt{x^2+2bx+c}}$.

We have, differentiating the given expression with respect to x:

$$\begin{aligned}\frac{\mathrm{d}t}{\mathrm{d}x} &= 1 + \frac{2x+2b}{2\sqrt{x^2+2bx+c}}\\ &= 1 + \frac{2(x+b)}{2(t-x)}\\ &= \frac{(t-x)+(x+b)}{t-x}\\ &= \frac{t+b}{t-x}.\end{aligned}$$

The required result follows on taking the reciprocal of both sides:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{\mathrm{d}t/\mathrm{d}x} = \frac{t-x}{t+b}.$$

To find the integral, we use the given substitution for x, yielding:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2bx+c}}\,\mathrm{d}x &= \int \frac{1}{t-x}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t\\ &= \int \frac{1}{t-x}\frac{t-x}{t+b}\,\mathrm{d}t\\ &= \int \frac{1}{t+b}\,\mathrm{d}t\\ &= \ln|t+b| + k\\ &= \ln\bigl|x+b+\sqrt{x^2+2bx+c}\bigr| + k\\ &= \ln\bigl(x+b+\sqrt{x^2+2bx+c}\bigr) + k\end{aligned}$$

with the last line following as $x^2+2bx+c > x^2+2bx+b^2 = (x+b)^2$, so the parenthesised expression is positive.

Verify by direct integration that your result holds also in the case $b^2 = c$ if $x + b > 0$ but that your result does not hold in the case $b^2 = c$ if $x + b < 0$.

With $b^2 = c$, we have the integral

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2bx+b^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{(x+b)^2}}\,\mathrm{d}x\\ &= \int \frac{1}{|x+b|}\,\mathrm{d}x\end{aligned}$$

We now consider the two cases discussed in the question. Firstly, if $x+b>0$, then we have

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2bx+b^2}}\,\mathrm{d}x &= \int \frac{1}{|x+b|}\,\mathrm{d}x\\ &= \int \frac{1}{x+b}\,\mathrm{d}x\\ &= \ln(x+b)+k'\end{aligned}$$

(we don't need absolute value signs as $x+b$ is positive), whereas before we had

$$\begin{aligned}\ln\bigl(x+b+\sqrt{x^2+2bx+c}\bigr)+k &= \ln\bigl(x+b+\sqrt{(x+b)^2}\bigr)+k\\ &= \ln\bigl(x+b+|x+b|\bigr)+k\\ &= \ln\bigl(x+b+(x+b)\bigr)+k\\ &= \ln(2(x+b))+k\\ &= \ln(x+b)+\ln 2+k\end{aligned}$$

Thus our earlier formula works in the case that $b^2=c$ and $x+b>0$, where we take $k'=k+\ln 2$ (which we may do, as they are arbitrary constants).

Next, when $x+b<0$, direct integration yields:

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2bx+b^2}}\,\mathrm{d}x &= \int \frac{1}{|x+b|}\,\mathrm{d}x\\ &= \int -\frac{1}{x+b}\,\mathrm{d}x\\ &= -\ln|x+b|+k'\\ &= -\ln(-(x+b))+k'\end{aligned}$$

as $x+b<0$. But now our earlier formula yields

$$\begin{aligned}\ln\bigl(x+b+\sqrt{x^2+2bx+c}\bigr)+k &= \ln\bigl(x+b+\sqrt{(x+b)^2}\bigr)+k\\ &= \ln\bigl(x+b+|x+b|\bigr)+k\\ &= \ln\bigl(x+b-(x+b)\bigr)+k\\ &= \ln 0+k\end{aligned}$$

which is not even defined. So the earlier result fails to give any answer in the case $b^2=c$ when $x+b<0$.

Question 3

Prove that, if $c \geqslant a$ and $d \geqslant b$, then

$$ab + cd \geqslant bc + ad. \qquad (*)$$

We have

$$\begin{aligned} ab + cd - bc - ad &= (a-c)b + (c-a)d \\ &= (c-a)(d-b) \\ &\geqslant 0 \qquad \text{as } c \geqslant a \text{ and } d \geqslant b, \end{aligned}$$

from which $(*)$ follows immediately.

(i) *If $x \geqslant y$, use $(*)$ to show that $x^2 + y^2 \geqslant 2xy$.*

If, further, $x \geqslant z$ and $y \geqslant z$, use $(*)$ to show that $z^2 + xy \geqslant xz + yz$ and deduce that $x^2 + y^2 + z^2 \geqslant xy + yz + zx$.

Prove that the inequality $x^2 + y^2 + z^2 \geqslant xy + yz + zx$ holds for all x, y and z.

Letting $a = b = y$ and $c = d = x$ in $(*)$, which we can do as $x \geqslant y$, yields $y^2 + x^2 \geqslant yx + yx$, that is

$$x^2 + y^2 \geqslant 2xy. \qquad (1)$$

Next, letting $a = b = z$, $c = x$ and $d = y$ in $(*)$ gives

$$z^2 + xy \geqslant zx + zy \qquad (2)$$

as we wanted.

Now adding the inequalities (1) and (2) gives us

$$x^2 + y^2 + z^2 + xy \geqslant 2xy + yz + zx.$$

Subtracting xy from both sides yields our desired result:

$$x^2 + y^2 + z^2 \geqslant xy + yz + zx. \qquad (3)$$

Finally, we have now proved inequality (3) when $x \geqslant y \geqslant z$, but we need to show that it is true whatever the values of x, y and z. But the inequality is *symmetric* in x, y and z, meaning that rearranging (permuting) the variables in any way does not change the statement. For example, if we swap x and z, we get

$$z^2 + y^2 + x^2 \geqslant zy + yx + xz,$$

which is exactly the same inequality.

So we can assume that the the values of x, y and z we are given satisfy $x \geqslant y \geqslant z$, without changing the statement of the inequality, and we know that the inequality holds in this case.

> ***(ii)*** *Show similarly that the inequality* $\dfrac{s}{t} + \dfrac{t}{r} + \dfrac{r}{s} \geqslant 3$ *holds for all positive* r, s *and* t.

We begin by assuming that $r \geqslant s \geqslant t > 0$, and set $c = r$, $a = s$, $d = 1/s$ and $b = 1/r$. Then by our assumption, we see that $c \geqslant a$ and $d \geqslant b$, so $(*)$ gives

$$\frac{s}{r} + \frac{r}{s} \geqslant 1 + 1. \qquad (4)$$

Now set $c = s$, $a = t$, $d = 1/t$ and $b = 1/r$, and note that $c \geqslant a$ and $d \geqslant b$, so that $(*)$ gives

$$\frac{t}{r} + \frac{s}{t} \geqslant 1 + \frac{s}{r}. \qquad (5)$$

Adding the inequalities (4) and (5) gives

$$\frac{s}{r} + \frac{r}{s} + \frac{t}{r} + \frac{s}{t} \geqslant 1 + 1 + 1 + \frac{s}{r},$$

so that

$$\frac{r}{s} + \frac{t}{r} + \frac{s}{t} \geqslant 1 + 1 + 1, \qquad (6)$$

as we want.

Now this inequality is true for $r \geqslant s \geqslant t$, and by exactly the same argument it will also hold if $s \geqslant t \geqslant r$ or $t \geqslant r \geqslant s$, by cycling the variables.

If $r \geqslant t \geqslant s$, then we have to start again, but the argument is almost identical.

We set $c = r$, $a = t$, $d = 1/t$ and $b = 1/r$. Then by our assumption, we see that $c \geqslant a$ and $d \geqslant b$, so $(*)$ gives

$$\frac{t}{r} + \frac{r}{t} \geqslant 1 + 1. \qquad (7)$$

We now set $c = r$, $a = s$, $d = 1/s$ and $b = 1/t$, and note that $c \geqslant a$ and $d \geqslant b$, so that $(*)$ gives

$$\frac{t}{r} + \frac{s}{t} \geqslant 1 + \frac{s}{r}. \qquad (8)$$

Once again, adding these two inequalities gives inequality (6), and by cycling the variables, it is also true if $t \geqslant s \geqslant r$ or $s \geqslant r \geqslant t$.

We have now shown (6) to be true for all six possible orderings of r, s and t, so it is true for all possible (positive) values of r, s and t.

Question 4

A function $\mathrm{f}(x)$ *is said to be* convex *in the interval* $a < x < b$ *if* $\mathrm{f}''(x) \geqslant 0$ *for all* x *in this interval.*

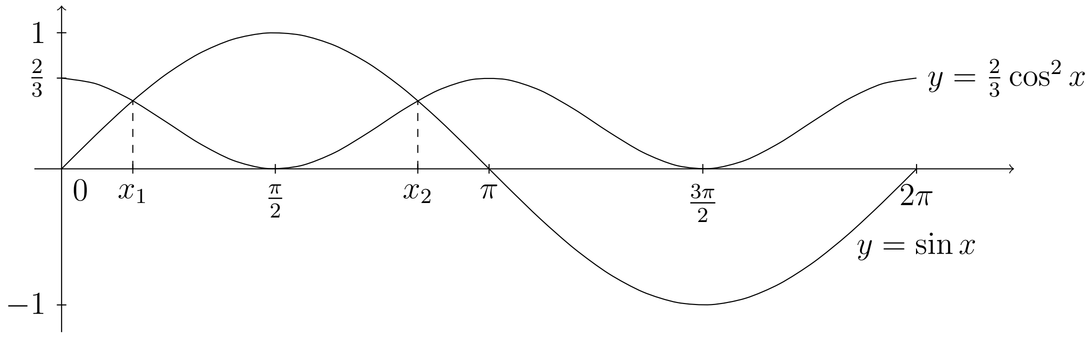
(i) *Sketch on the same axes the graphs of* $y = \frac{2}{3}\cos^2 x$ *and* $y = \sin x$ *in the interval* $0 \leqslant x \leqslant 2\pi$.

The function $\mathrm{f}(x)$ *is defined for* $0 < x < 2\pi$ *by*

$$\mathrm{f}(x) = \mathrm{e}^{\frac{2}{3}\sin x}.$$

Determine the intervals in which $\mathrm{f}(x)$ *is convex.*

We note that $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ from the double angle formula, so that the graph of $\frac{2}{3}\cos^2 x$ is a translated, stretched version of $y = \cos 2x$.



Now we have

$$\begin{aligned}
\mathrm{f}(x) &= \mathrm{e}^{\frac{2}{3}\sin x} \\
\mathrm{f}'(x) &= (\tfrac{2}{3}\cos x)\mathrm{e}^{\frac{2}{3}\sin x} \\
\mathrm{f}''(x) &= (\tfrac{4}{9}\cos^2 x - \tfrac{2}{3}\sin x)\mathrm{e}^{\frac{2}{3}\sin x} \\
&= \tfrac{2}{3}(\tfrac{2}{3}\cos^2 x - \sin x)\mathrm{e}^{\frac{2}{3}\sin x}.
\end{aligned}$$

As $\mathrm{e}^{\frac{2}{3}\sin x} > 0$ for all x, we have

$$\mathrm{f}''(x) \geqslant 0 \qquad \text{if and only if} \qquad \tfrac{2}{3}\cos^2 x - \sin x \geqslant 0.$$

But we have just drawn a graph of the two functions $y = \frac{2}{3}\cos^2 x$ and $y = \sin x$, so we see that $\mathrm{f}(x) \geqslant 0$ when $0 \leqslant x \leqslant x_1$ and when $x_2 \leqslant x \leqslant 2\pi$, where x_1 and x_2 are the x-coordinates of the points of intersection of the two graphs. So all we need to do is to solve the equation

$$\tfrac{2}{3}\cos^2 x - \sin x = 0$$

to determine the values of x_1 and x_2.

Using $\cos^2 x = 1 - \sin^2 x$ and then factorising gives

$$\begin{aligned} & \tfrac{2}{3}\cos^2 x = \sin x \\ \iff\quad & \tfrac{2}{3}(1-\sin^2 x) = \sin x \\ \iff\quad & 2 - 2\sin^2 x = 3\sin x \\ \iff\quad & 2\sin^2 x + 3\sin x - 2 = 0 \\ \iff\quad & (2\sin x - 1)(\sin x + 2) = 0 \\ \iff\quad & \sin x = \tfrac{1}{2} \end{aligned}$$

Therefore $x_1 = \frac{\pi}{6}$ and $x_2 = \frac{5\pi}{6}$, and the function $\mathrm{f}(x)$ is convex in the intervals $0 < x < \frac{\pi}{6}$ and $\frac{5\pi}{6} < x < 2\pi$.

(ii) *The function* $\mathrm{g}(x)$ *is defined for* $0 < x < \frac{1}{2}\pi$ *by*

$$\mathrm{g}(x) = \mathrm{e}^{-k\tan x}.$$

If $k = \sin 2\alpha$ *and* $0 < \alpha < \pi/4$*, show that* $\mathrm{g}(x)$ *is convex in the interval* $0 < x < \alpha$*, and give one other interval in which* $\mathrm{g}(x)$ *is convex.*

We have

$$\begin{aligned} \mathrm{g}(x) &= \mathrm{e}^{-k\tan x} \\ \mathrm{g}'(x) &= -k\sec^2 x.\mathrm{e}^{-k\tan x} \\ \mathrm{g}''(x) &= (k^2\sec^4 x - 2k\sec^2 x\tan x)\mathrm{e}^{-k\tan x} \\ &= k\sec^2 x(k\sec^2 x - 2\tan x)\mathrm{e}^{-k\tan x}. \end{aligned}$$

Therefore $g''(x) \geqslant 0$ when $k\sec^2 x - 2\tan x \geqslant 0$, that is, when $k\tan^2 x - 2\tan x + k \geqslant 0$. We can solve the equality $k\tan^2 x - 2\tan x + k = 0$ using the quadratic formula:

$$\begin{aligned} \tan x &= \frac{2 \pm \sqrt{4-4k^2}}{2k} \\ &= \frac{1 \pm \sqrt{1-k^2}}{k} \\ &= \frac{1 \pm \sqrt{1-\sin^2 2\alpha}}{\sin 2\alpha} && \text{substituting } k = \sin 2\alpha \\ &= \frac{1 \pm \cos 2\alpha}{\sin 2\alpha}. \end{aligned}$$

Thus we have two possibilities:

$$\tan x = \frac{1+\cos 2\alpha}{\sin 2\alpha} = \frac{2\cos^2\alpha}{2\sin\alpha\cos\alpha} = \cot\alpha = \tan(\tfrac{\pi}{2} - \alpha)$$

or

$$\tan x = \frac{1-\cos 2\alpha}{\sin 2\alpha} = \frac{2\sin^2\alpha}{2\sin\alpha\cos\alpha} = \tan\alpha.$$

Then, since $0 < \alpha < \frac{\pi}{4}$ and $0 < x < \frac{\pi}{2}$, we have $x = \alpha$ or $x = \frac{\pi}{2} - \alpha$.

It follows, since $k > 0$ and $\alpha < \frac{\pi}{2} - \alpha$, that $\mathrm{g}(x)$ is convex, that is, $\mathrm{g}''(x) > 0$, when $0 < x < \alpha$ or when $\frac{\pi}{2} - \alpha < x < \frac{\pi}{2}$. (In more detail, when $x \approx 0$, $\mathrm{g}''(x) \approx k > 0$, and when $x \approx \frac{\pi}{2}$, $\mathrm{g}''(x) \approx k\tan^2 x > 0$.)

Question 5

The polynomial $\mathrm{p}(x)$ *is given by*

$$x^n + \sum_{r=0}^{n-1} a_r x^r \,,$$

where $a_0, a_1, \ldots, a_{n-1}$ *are fixed real numbers and* $n \geqslant 1$*. Let* M *be the greatest value of* $|\mathrm{p}(x)|$ *for* $|x| \leqslant 1$*. Then* Chebyshev's theorem *states that* $M \geqslant 2^{1-n}$*.*

(i) *Prove Chebyshev's theorem in the case* $n = 1$ *and verify that Chebyshev's theorem holds in the following cases:*

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}$;

(b) $\mathrm{p}(x) = x^3 - x$.

In the case $n = 1$, Chebyshev's theorem states:

> Let $\mathrm{p}(x)$ be the polynomial $x + a_0$, and let M be the greatest value of $|\mathrm{p}(x)|$ for $|x| \leqslant 1$. Then $M \geqslant 1$.

If $a_0 > 0$, then when $x = 1$, $p(1) = 1 + a_0 > 1$, so $M > 1$.

If $a_0 < 0$, then when $x = -1$, $p(-1) = -1 + a_0 < -1$, so $|p(-1)| > 1$ and $M > 1$.

Finally, if $a_0 = 0$, then $p(x) = x$, so $|p(x)| = |x|$. It follows that $|p(x)| = |x| \leqslant 1$ when $|x| \leqslant 1$ and $p(1) = 1$, so $M = 1$.

Thus in all cases $M \geqslant 1$.

Now to verify the theorem in the specified cases. The obvious approach is to find the maximum absolute value of the function over the interval. It is important to verify that the function's maximum absolute value really is at least 2^{1-n} in both cases.

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}$ is a quadratic whose minimum value is at $x = 0$. So we only need to consider the value of $\mathrm{p}(0)$ and the values of $\mathrm{p}(x)$ at the endpoints of the interval: $\mathrm{p}(-1)$ and $\mathrm{p}(1)$. We have $\mathrm{p}(0) = -\frac{1}{2}$, $\mathrm{p}(-1) = \mathrm{p}(1) = \frac{1}{2}$, so $-\frac{1}{2} \leqslant \mathrm{p}(x) \leqslant \frac{1}{2}$, and hence $|\mathrm{p}(x)| \leqslant \frac{1}{2} = 2^{-1}$ with the maximum value taken on by $|p(x)|$ being $\frac{1}{2}$.

In this case, where $n = 2$, Chebyshev's theorem states that $M \geqslant 2^{-1}$. Since we have $M = 2^{-1}$ in this case, Chebyshev's theorem holds.

(b) Given $\mathrm{p}(x) = x^3 - x$ we first look for stationary points. We have $\mathrm{p}'(x) = 3x^2 - 1$ so there are stationary points at $x = \pm 1/\sqrt{3}$. We thus evaluate $\mathrm{p}(x)$ at these points and at the endpoints $x = \pm 1$. We have $\mathrm{p}(-1) = \mathrm{p}(1) = 0$, $\mathrm{p}(-1/\sqrt{3}) = 2/3\sqrt{3}$ and $\mathrm{p}(1/\sqrt{3}) = -2/3\sqrt{3}$. Hence $|\mathrm{p}(x)| \leqslant 2/3\sqrt{3}$, so that $M = 2/3\sqrt{3}$.

As $n = 3$, we wish to show that $M \geqslant \frac{1}{4}$. But $M^2 = \frac{4}{27} > \frac{4}{64} = \frac{1}{16}$, so $M > \frac{1}{4}$ as required.

A second approach, which is simpler and more direct, is to observe that all we need to do is to find *some* value of x in the interval $-1 \leqslant x \leqslant 1$ for which $|p(x)| \geqslant 2^{1-n}$, for then we know that the *maximum* value of $|p(x)|$ in this interval will be at least that. In (a), we have $|p(1)| = \frac{1}{2} \geqslant 2^{-1}$ and in (b), $|p(\frac{1}{2})| = |\frac{1}{8} - \frac{1}{2}| = \frac{3}{8} \geqslant 2^{-2}$. So we are done.

> ***(ii)*** *Use Chebyshev's theorem to show that the curve $y = 64x^5+25x^4-66x^3-24x^2+3x+1$ has at least one turning point in the interval $-1 \leqslant x \leqslant 1$.*

Let $\mathrm{p}(x) = x^5 + \frac{1}{64}(25x^4 - 66x^3 - 24x^2 + 3x + 1) = y/64$. The turning points of $\mathrm{p}(x)$ are the same as the turning points of y. Then if we let M be the greatest value of $|\mathrm{p}(x)|$ for $|x| \leqslant 1$, we have $M \geqslant 2^{-4} = \frac{1}{16}$. Let x_0 be the value of x in the interval $-1 \leqslant x \leqslant 1$ for which $|p(x_0)| = M$, i.e., where $|p(x)|$ takes its maximum value. (If there is more than one such point, choose any of them to be x_0.)

Now $p(-1) = \frac{1}{64}$ and $p(1) = \frac{3}{64}$, and so $|p(1)| < \frac{1}{16}$ and $|p(-1)| < \frac{1}{16}$. But since $|p(x_0)| \geqslant \frac{1}{16}$, we cannot have $x_0 = \pm 1$, so $-1 < x_0 < 1$.

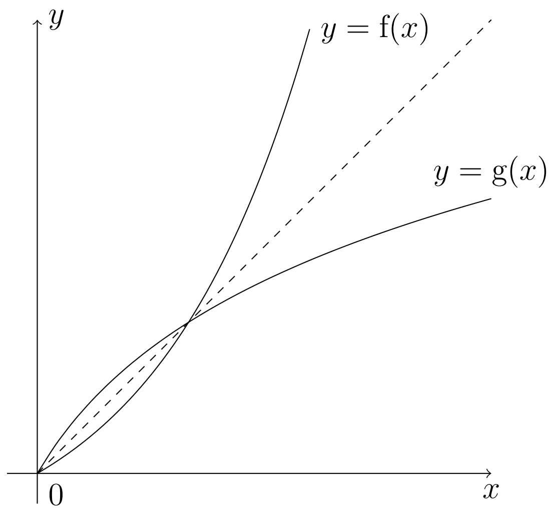
Then x_0 must be a local maximum or a local minimum: if $p(x_0) \geqslant \frac{1}{16}$ then it is at the the greatest value of $p(x)$ in the interval $-1 \leqslant x \leqslant 1$ (and is not at an endpoint); similarly, if $p(x_0) \leqslant -\frac{1}{16}$, then x_0 is at the least value. Either way, x_0 is a turning point as we wanted.

Question 6

The function f *is defined by*

$$\mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \qquad x \geqslant 0,$$

and the function g *is the inverse function to* f, *so that* $\mathrm{g}(\mathrm{f}(x)) = x$. *Sketch* $\mathrm{f}(x)$ *and* $\mathrm{g}(x)$ *on the same axes.*



Verify, by evaluating each integral, that

$$\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^k \mathrm{g}(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}}+1)},$$

where $k = \dfrac{1}{\sqrt{\mathrm{e}}+1}$, *and explain this result by means of a diagram.*

We find $\mathrm{g}(x)$, the inverse of $\mathrm{f}(x)$, as follows, noting that $y = \mathrm{f}(x)$ if and only if $x = \mathrm{g}(y)$:

$$\begin{aligned} & y = \mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1} \\ \iff \quad & (\mathrm{e}-1)y = \mathrm{e}^x - 1 \\ \iff \quad & \mathrm{e}^x = (\mathrm{e}-1)y + 1 \\ \iff \quad & x = \ln\big((\mathrm{e}-1)y + 1\big) \end{aligned}$$

so that $\mathrm{g}(x) = \ln\big((\mathrm{e}-1)x + 1\big)$.

Now we can evaluate the integrals. We have

$$\begin{aligned}\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}} \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}\,\mathrm{d}x \\ &= \frac{1}{\mathrm{e}-1}\Big[\mathrm{e}^x - x\Big]_0^{\frac{1}{2}} \\ &= \frac{1}{\mathrm{e}-1}\big((\mathrm{e}^{\frac{1}{2}} - \tfrac{1}{2}) - (1-0)\big) \\ &= \frac{1}{\mathrm{e}-1}\big(\mathrm{e}^{\frac{1}{2}} - \tfrac{3}{2}\big) \\ &= \frac{2\sqrt{\mathrm{e}} - 3}{2(\mathrm{e}-1)}.\end{aligned}$$

For $\mathrm{g}(x)$, we can either use substitution and the standard result $\int \ln x\,\mathrm{d}x = x\ln x - x + c$ or integration by parts, effectively deriving the result. We demonstrate both methods.

Using substitution, we set $u = (\mathrm{e}-1)x+1$. When $x = 0$, $u = 1$, and when $x = k = 1/(\sqrt{\mathrm{e}}+1)$, we can easily calculate that $u = (\sqrt{\mathrm{e}} - 1) + 1 = \sqrt{\mathrm{e}}$. Finally, $\mathrm{d}u/\mathrm{d}x = \mathrm{e} - 1$. Thus

$$\begin{aligned}\int_0^k \mathrm{g}(x)\,\mathrm{d}x &= \int_0^k \ln\big((\mathrm{e}-1)x+1\big)\,\mathrm{d}x \\ &= \int_1^{\sqrt{\mathrm{e}}} \ln u \,\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u \\ &= \int_1^{\sqrt{\mathrm{e}}} \frac{1}{\mathrm{e}-1}\,\ln u\,\mathrm{d}u \\ &= \frac{1}{\mathrm{e}-1}\Big[u\ln u - u\Big]_1^{\sqrt{\mathrm{e}}} \\ &= \frac{1}{\mathrm{e}-1}\big((\sqrt{\mathrm{e}}\ln\sqrt{\mathrm{e}} - \sqrt{\mathrm{e}}) - (\ln 1 - 1)\big) \\ &= \frac{1}{\mathrm{e}-1}\big(1 - \tfrac{1}{2}\sqrt{\mathrm{e}}\big) \\ &= \frac{2-\sqrt{\mathrm{e}}}{2(\mathrm{e}-1)}.\end{aligned}$$

Alternatively we can use integration by parts. We first note that

$$\begin{aligned}\ln((\mathrm{e}-1)k+1) &= \ln\left(\frac{\mathrm{e}-1}{\sqrt{\mathrm{e}}+1} + 1\right) \\ &= \ln\big((\sqrt{\mathrm{e}}-1)+1\big) \\ &= \ln(\sqrt{\mathrm{e}}) \\ &= \tfrac{1}{2}.\end{aligned}$$

Then we can evaluate our integral as follows:

$$
\begin{aligned}
\int_0^k \mathrm{g}(x)\,\mathrm{d}x &= \int_0^k 1.\ln((\mathrm{e}-1)x+1)\,\mathrm{d}x \\
&= \big[x\ln((\mathrm{e}-1)x+1)\big]_0^k - \int_0^k x\left(\frac{\mathrm{e}-1}{(\mathrm{e}-1)x+1}\right)\mathrm{d}x \\
&= k\ln((\mathrm{e}-1)k+1) - \int_0^k \frac{(\mathrm{e}-1)x+1-1}{(\mathrm{e}-1)x+1}\,\mathrm{d}x \\
&= \tfrac{1}{2}k - \int_0^k 1 - \frac{1}{(\mathrm{e}-1)x+1}\,\mathrm{d}x \\
&= \tfrac{1}{2}k - \left[x - \frac{1}{\mathrm{e}-1}\ln((\mathrm{e}-1)x+1)\right]_0^k \\
&= \tfrac{1}{2}k - \left(\left(k - \frac{1}{\mathrm{e}-1}\ln((\mathrm{e}-1)k+1)\right) - (0-\ln 1)\right) \\
&= \tfrac{1}{2}k - \left(k - \frac{1}{2(\mathrm{e}-1)}\right) \\
&= \frac{1}{2(\mathrm{e}-1)} - \tfrac{1}{2}k \\
&= \frac{1}{2(\mathrm{e}-1)} - \frac{1}{2(\sqrt{\mathrm{e}}+1)} \\
&= \frac{1}{2(\mathrm{e}-1)} - \frac{\sqrt{\mathrm{e}}-1}{2(\mathrm{e}-1)} \\
&= \frac{2-\sqrt{\mathrm{e}}}{2(\mathrm{e}-1)}
\end{aligned}
$$

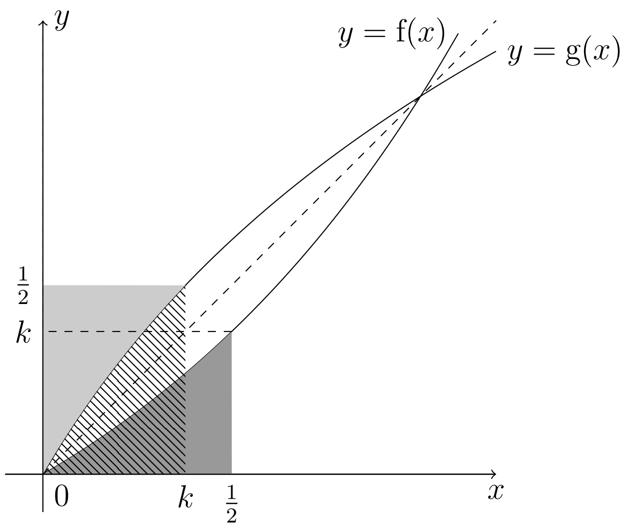
as we found using the substitution method.

We therefore have

$$
\begin{aligned}
\int_0^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_0^k \mathrm{g}(x)\,\mathrm{d}x &= \frac{2\sqrt{\mathrm{e}}-3}{2(\mathrm{e}-1)} + \frac{2-\sqrt{\mathrm{e}}}{2(\mathrm{e}-1)} \\
&= \frac{\sqrt{\mathrm{e}}-1}{2(\mathrm{e}-1)} \\
&= \frac{1}{2(\sqrt{\mathrm{e}}+1)}
\end{aligned}
$$

as we wanted.

Finally, to explain this result with the aid of a diagram, we want to fill in the two areas indicated by the integrals on our sketch of the functions above. It would be useful to know the value of $\mathrm{f}(\frac{1}{2})$ for this purpose: it is $(\mathrm{e}^{1/2}-1)/(\mathrm{e}-1) = 1/(\sqrt{\mathrm{e}}+1) = k$, which is very convenient. Since $\mathrm{g}(x)$ is the inverse of $\mathrm{f}(x)$, it follows likewise that $\mathrm{g}(k) = \frac{1}{2}$. We can now sketch the areas on our graph.

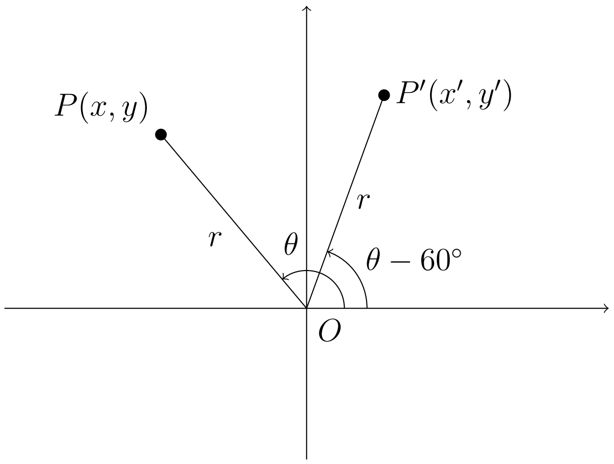


In this sketch, the dark shaded area is the integral $\int_0^{1/2} \mathrm{f}(x)\,\mathrm{d}x$ and the striped area is the integral $\int_0^k \mathrm{g}(x)\,\mathrm{d}x$. We have reflected the dark shaded area in the line $y = x$ to get the light shaded area also shown. It is now clear that the shaded and striped areas add to give a $\frac{1}{2} \times k$ rectangle, so the area is $k/2 = 1/2(\sqrt{\mathrm{e}} + 1)$, as we found.

Question 7

The point P has coordinates (x, y) with respect to the origin O. By writing $x = r\cos\theta$ and $y = r\sin\theta$, or otherwise, show that, if the line OP is rotated by $60°$ clockwise about O, the new y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$. What is the new y-coordinate in the case of an anti-clockwise rotation by $60°$?

The situation described is illustrated in the following diagram, where OP' is the image of OP under the specified rotation, P' having coordinates (x', y').



Then we have

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and

$$y' = r\sin(\theta - 60°)$$

We use the compound angle formula for sine to get

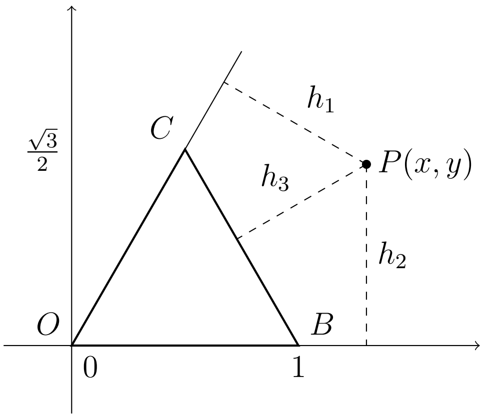
$$\begin{aligned} y' &= r(\sin\theta\cos 60° - \cos\theta\sin 60°) \\ &= \tfrac{1}{2}r\sin\theta - \tfrac{\sqrt{3}}{2}r\cos\theta \\ &= \tfrac{1}{2}y - \tfrac{\sqrt{3}}{2}x \\ &= \tfrac{1}{2}(y - \sqrt{3}\,x). \end{aligned}$$

Likewise, if the rotation is by $60°$ anticlockwise, we replace the $\theta - 60°$ by $\theta + 60°$ and repeat the above to get

$$y' = \tfrac{1}{2}(y + \sqrt{3}\,x).$$

An equilateral triangle OBC has vertices at O, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, respectively. The point P has coordinates (x, y). The perpendicular distance from P to the line through C and O is h_1; the perpendicular distance from P to the line through O and B is h_2; and the perpendicular distance from P to the line through B and C is h_3.

Show that $h_1 = \frac{1}{2}\left|y - \sqrt{3}\,x\right|$ and find expressions for h_2 and h_3.



Clearly $h_2 = |y|$. (We need to take the absolute value as P might lie under the x-axis.)

For h_1, consider rotating the entire shape by $60°$ clockwise about O. This will rotate OC to the x-axis, and the perpendicular from P to OC will become vertical. The transformed y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$, as we deduced earlier, so $h_1 = \frac{1}{2}\big|y - \sqrt{3}\,x\big|$.

Finally, for h_3, we start by rotating anticlockwise by $60°$ around O. Then P ends up with y-coordinate $\frac{1}{2}(y + \sqrt{3}\,x)$ and the side BC ends up lying along the line $y = \frac{\sqrt{3}}{2}$; subtracting these then gives $h_3 = \frac{1}{2}\big|y + \sqrt{3}\,x - \sqrt{3}\big|$.

[An alternative argument is to translate the whole diagram by one unit to the left first, so that B moves to the origin and P moves to $(x-1, y)$. Then rotating around the new B gives P the new y-coordinate of $\frac{1}{2}(y + \sqrt{3}(x-1))$, which is the same as before.]

Show that $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or in the triangle OBC.

We have

$$\begin{aligned} h_1 + h_2 + h_3 &= \tfrac{1}{2}\big(\big|2y\big| + \big|y - \sqrt{3}\,x\big| + \big|y + \sqrt{3}\,x - \sqrt{3}\big|\big) \\ &= \tfrac{1}{2}\big(\big|2y\big| + \big|\sqrt{3}\,x - y\big| + \big|\sqrt{3} - \sqrt{3}\,x - y\big|\big). \end{aligned}$$

Using the triangle inequality, we then have

$$\begin{aligned} h_1 + h_2 + h_3 &= \tfrac{1}{2}\big(\big|2y\big| + \big|\sqrt{3}\,x - y\big| + \big|\sqrt{3} - \sqrt{3}\,x - y\big|\big) \\ &\geqslant \tfrac{1}{2}\big|(2y) + (\sqrt{3}\,x - y) + (\sqrt{3} - \sqrt{3}\,x - y)\big| \\ &= \tfrac{1}{2}\sqrt{3}, \end{aligned}$$

with equality if and only if all of the bracketed terms are $\geqslant 0$ or all of the bracketed terms are $\leqslant 0$.

If all of the terms are negative or zero, then $2y \leqslant 0$, so $y \leqslant 0$. And $\sqrt{3}\,x - y \leqslant 0$ implies that $x \leqslant y/\sqrt{3} \leqslant 0$. But then we must have $\sqrt{3} - \sqrt{3}\,x - y > 0$, which is impossible. So we cannot have all three terms negative or zero.

Therefore, if we have equality, we must have all three terms positive or zero. But $2y \geqslant 0$ if and only if P lies on or above the x-axis, that is, on or above the line OB. Similarly, $\sqrt{3}\,x - y \geqslant 0$ if and only if $y \leqslant \sqrt{3}\,x$, which is true if and only if P lies on or below the line OC (which has equation $y = \sqrt{3}\,x$). Finally, $\sqrt{3} - \sqrt{3}\,x - y \geqslant 0$ if and only if

$y \leqslant \sqrt{3} - \sqrt{3}\,x$, which is true if and only if P lies on or below the line BC (which has equation $y = \sqrt{3} - \sqrt{3}\,x$).

Putting these together shows that $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or inside the triangle OBC.

Question 8

(i) *The gradient y' of a curve at a point (x, y) satisfies*

$$(y')^2 - xy' + y = 0. \qquad (*)$$

By differentiating $(*)$ with respect to x, show that either $y'' = 0$ or $2y' = x$.

Hence show that the curve is either a straight line of the form $y = mx + c$, where $c = -m^2$, or the parabola $4y = x^2$.

Differentiating $(*)$ with respect to x, using the chain rule and product rule, gives

$$2y'y'' - y' - xy'' + y' = 0,$$

which, on cancelling terms and factorising, yields

$$(2y' - x)y'' = 0,$$

so either $y'' = 0$ or $2y' - x = 0$.

We now solve these two differential equations. Firstly, by integrating $y'' = 0$ twice with respect to x, we get $y' = a$, so $y = ax + b$ (where a and b are constants). Substituting this back into $(*)$ gives

$$a^2 - x.a + (ax + b) = 0,$$

so

$$a^2 + b = 0,$$

which gives us the straight line $y = mx + c$ with $m = a$ and $c = b = -a^2 = -m^2$.

In the other case, $y' = \frac{1}{2}x$, which on integrating gives $y = \frac{1}{4}x^2 + c$. Substituting this back into $(*)$ gives

$$(\tfrac{1}{2}x)^2 - x(\tfrac{1}{2}x) + (\tfrac{1}{4}x^2 + c) = 0,$$

so $c = 0$ and $y = \frac{1}{4}x^2$, or $4y = x^2$ as required.

(ii) *The gradient y' of a curve at a point (x, y) satisfies*

$$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0. \qquad (\dagger)$$

Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.

Differentiating $(\dagger)$ with respect to x, using the chain rule and product rule, gives:

$$2x(y')^2 + (x^2 - 1).2y'y'' - 2yy' - 2xy'y' - 2xyy'' + 2yy' = 0,$$

which, on cancelling terms, yields

$$2(x^2 - 1)y'y'' - 2xyy'' = 0.$$

Finally, factorising brings us to our desired conclusion:

$$\big((x^2-1)y'-xy\big)y''=0,$$

so either $y''=0$ or $y'(x^2-1)-xy=0$.

We now solve these two equations. Again, integrating $y''=0$ twice gives $y'=a$, so $y=ax+b$. Substituting this back into (†) gives:

$$(x^2-1).a^2-2x(ax+b).a+(ax+b)^2-1=0,$$

which is equivalent to

$$a^2x^2-a^2-2a^2x^2-2abx+a^2x^2+2abx+b^2-1=0,$$

so

$$-a^2+b^2-1=0,$$

or $b^2=a^2+1$. Thus the equation is satisfied by straight lines $y=mx+c$ where $c^2=m^2+1$.

In the other case, $y'(x^2-1)-xy=0$.

It looks as though we could solve this by separating the variables to give:

$$\int\frac{1}{y}\,\mathrm{d}y=\int\frac{x}{x^2-1}\,\mathrm{d}x,$$

so $\ln y=\frac{1}{2}\ln|x^2-1|+c$. Doubling and exponentiating then gives

$$y^2=C|x^2-1|.$$

To determine C and if its value depends upon whether $|x|<1$ or $|x|>1$, we ought to try substituting back into (†). However, it is simpler to substitute in the result $y'(x^2-1)=xy$ directly into (†).

Substituting $y'=xy/(x^2-1)$ back into (†) gives

$$(x^2-1)\left(\frac{xy}{x^2-1}\right)^2-2xy\left(\frac{xy}{x^2-1}\right)+y^2-1=0.$$

Expanding brackets then gives

$$\frac{x^2y^2}{x^2-1}-\frac{2x^2y^2}{x^2-1}+y^2-1=0$$

so

$$-\frac{x^2y^2}{x^2-1}+y^2-1=0.$$

Multiplying by x^2-1 gives

$$-x^2y^2+(y^2-1)(x^2-1)=0$$

so

$$-x^2y^2+y^2x^2-y^2-x^2+1=0$$

or

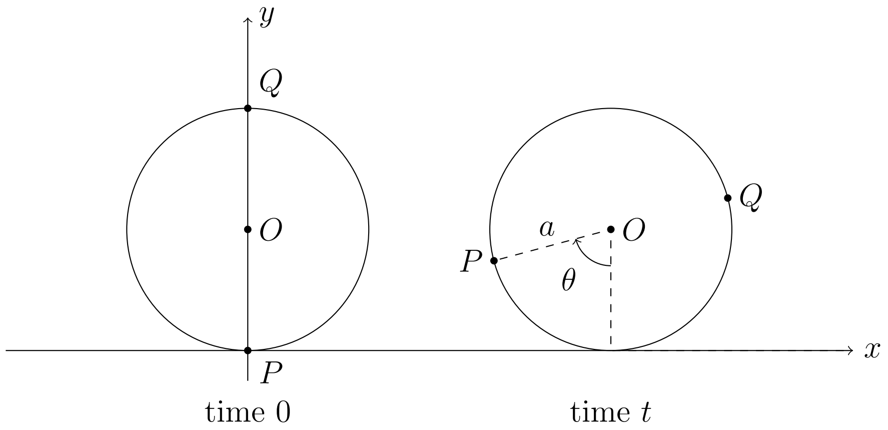
$$-x^2-y^2+1=0,$$

and we therefore deduce that the only other possible solution is the circle $x^2+y^2=1$.

We must finally check that the circle does, in fact, satisfy $y'=xy/(x^2-1)$. Differentiating $x^2+y^2=1$ with respect to x gives $2x+2yy'=0$, so $y'=-x/y=-xy/y^2=-xy/(1-x^2)$, as required.

Question 9

Two identical particles P and Q, each of mass m, are attached to the ends of a diameter of a light thin circular hoop of radius a. The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially, P is in contact with the table. At time t, the hoop has rotated through an angle θ. Write down the position at time t of P, relative to its starting point, in cartesian coordinates, and determine its speed in terms of a, θ and $\dot{\theta}$. Show that the total kinetic energy of the two particles is $2ma^2\dot{\theta}^2$.



The diagram shows the hoop rolling to the right, indicating the positions of the hoop at time 0 and time t.

Taking the origin to be at the initial position of P, the coordinates of the centre of the hoop at time t are $(a\theta, a)$, since the hoop has rolled a distance $a\theta$. Therefore P has coordinates $(a(\theta - \sin\theta), a(1 - \cos\theta))$ and position vector

$$\mathbf{r}_P = a(\theta - \sin\theta)\mathbf{i} + a(1 - \cos\theta)\mathbf{j}.$$

The velocity vector of P is then

$$\begin{aligned}
\dot{\mathbf{r}}_P &= \frac{\mathrm{d}}{\mathrm{d}t}\big(a(\theta - \sin\theta)\big)\mathbf{i} + \frac{\mathrm{d}}{\mathrm{d}t}\big(a(1 - \cos\theta)\big)\mathbf{j} \\
&= a\left(\frac{\mathrm{d}\theta}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\mathbf{i} - a\frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t}\mathbf{j} \\
&= a(\dot{\theta} - \cos\theta.\dot{\theta})\mathbf{i} + a\sin\theta.\dot{\theta}\,\mathbf{j} \\
&= a\dot{\theta}\big((1 - \cos\theta)\mathbf{i} + \sin\theta\,\mathbf{j}\big),
\end{aligned}$$

and hence P has speed v_P given by

$$\begin{aligned}
v_P^2 &= (a\dot{\theta})^2\big((1 - \cos\theta)^2 + (\sin\theta)^2\big) \\
&= (a\dot{\theta})^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta) \\
&= (a\dot{\theta})^2(2 - 2\cos\theta) \\
&= (a\dot{\theta})^2.4\sin^2\tfrac{1}{2}\theta \\
&= (2a\dot{\theta}\sin\tfrac{1}{2}\theta)^2,
\end{aligned}$$

so that $v_P = 2a|\dot{\theta}\sin\frac{1}{2}\theta|$.

Similarly, the coordinates of Q are $(a(\theta+\sin\theta), a(1+\cos\theta))$, so Q has position vector

$$\mathbf{r}_Q = a(\theta+\sin\theta)\mathbf{i} + a(1+\cos\theta)\mathbf{j}.$$

Arguing as before, the velocity vector of Q is then

$$\begin{aligned}\dot{\mathbf{r}}_Q &= a(\dot{\theta}+\cos\theta.\dot{\theta})\mathbf{i} - a\sin\theta.\dot{\theta}\,\mathbf{j} \\ &= a\dot{\theta}\big((1+\cos\theta)\mathbf{i} - \sin\theta\,\mathbf{j}\big)\end{aligned}$$

so that Q has speed v_Q given by

$$\begin{aligned}v_Q^2 &= (a\dot{\theta})^2\big((1+\cos\theta)^2 + (\sin\theta)^2\big) \\ &= (a\dot{\theta})^2(1+2\cos\theta+\cos^2\theta+\sin^2\theta) \\ &= (a\dot{\theta})^2(2+2\cos\theta) \\ &= (a\dot{\theta})^2.4\cos^2\tfrac{1}{2}\theta.\end{aligned}$$

Adding $v_P^2 = (a\dot{\theta})^2.4\sin^2\frac{1}{2}\theta$ to this gives the total kinetic energy as

$$\begin{aligned}\tfrac{1}{2}mv_P^2 + \tfrac{1}{2}mv_Q^2 &= \tfrac{1}{2}m(v_P^2+v_Q^2) \\ &= \tfrac{1}{2}m\big((a\dot{\theta})^2.4\sin^2\tfrac{1}{2}\theta + (a\dot{\theta})^2.4\cos^2\tfrac{1}{2}\theta\big) \\ &= \tfrac{1}{2}m(a\dot{\theta})^2.4 \\ &= 2ma^2\dot{\theta}^2\end{aligned}$$

as required.

Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.

Consider the hoop as a single system. The only external forces on the hoop are gravity and the normal reaction. Both of these are vertical, while the hoop only moves in a horizontal direction. Therefore, no work is done on the hoop, so that GPE + KE is constant.

The gravitational potential energy of P and Q together, taking the centre of the hoop as potential energy zero, gives $mga(-\cos\theta) + mga(+\cos\theta) = 0$. So the GPE of the system is constant, meaning that the kinetic energy is also constant.

Since the total kinetic energy is $2ma^2\dot{\theta}^2$, it follows that $\dot{\theta}$ is constant, that is, the hoop rolls with the constant speed $a\dot{\theta}$.

Question 10

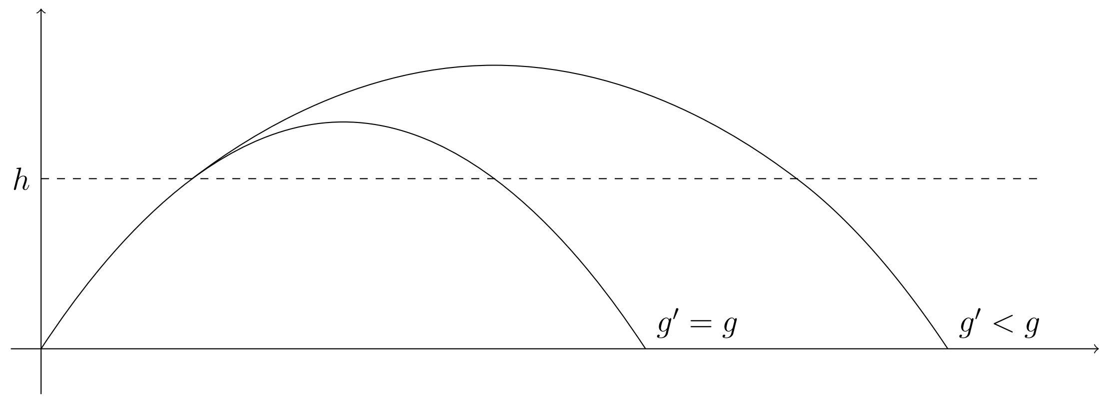
On the (flat) planet Zog, the acceleration due to gravity is g up to height h above the surface and g' at greater heights. A particle is projected from the surface at speed V and at an angle α to the surface, where $V^2 \sin^2 \alpha > 2gh$. Sketch, on the same axes, the trajectories in the cases $g' = g$ and $g' < g$.

We know that the path of a projectile is parabolic when the gravity is constant. When the gravity is less, the parabola will be "bigger", as the projectile will travel higher before returning to the ground, but the horizontal component of velocity will be unaffected.

So our sketch will consist of a parabola for the case $g = g'$ and a pair of parabolas joined at height h in the case $g' < g$. Note that the velocity does not change suddenly at height h, so the curve will be "smooth" at this point. (Technically, it has a continuous first derivative.)

We need to check that the particle does reach height h before we draw a sketch. The vertical component of velocity is initially $V \sin \alpha$. If the gravity were a constant g, then the maximum height reached would be s, where the formula $v^2 = u^2 + 2as$ gives us $0^2 = V^2 \sin^2 \alpha - 2gs$, so $s = V^2 \sin^2 \alpha / 2g > h$, so the particle does reach height greater than h.

So here, then, is the sketch:



Show that the particle lands a distance d from the point of projection given by

$$d = \left(\frac{V - V'}{g} + \frac{V'}{g'} \right) V \sin 2\alpha,$$

where $V' = \sqrt{V^2 - 2gh \operatorname{cosec}^2 \alpha}$.

We note that the horizontal speed is a constant $V \cos \alpha$, as there is no horizontal component of acceleration. Therefore the distance travelled is this times the time travelled. By symmetry, we find the time taken to reach the highest point on the trajectory, and then double it to find the total time.

First part: below height h.

We use the "suvat" equations to determine the time taken and the vertical speed at height h. Taking upwards as positive, we have $s = h$, $u = V \sin \alpha$, $a = -g$. Then $v^2 = u^2 + 2as$ gives

$v^2 = V^2 \sin^2 \alpha - 2gh$ and $v = u + at$ gives

$$\begin{aligned} t &= \frac{V \sin \alpha - \sqrt{V^2 \sin^2 \alpha - 2gh}}{g} \\ &= \left(\frac{V - \sqrt{V^2 - 2gh \operatorname{cosec}^2 \alpha}}{g} \right) \sin \alpha \\ &= \left(\frac{V - V'}{g} \right) \sin \alpha, \end{aligned}$$

writing $V' = \sqrt{V^2 - 2gh \operatorname{cosec}^2 \alpha}$.

Second part: above height h.

This time, $u = \sqrt{V^2 \sin^2 \alpha - 2gh} = V' \sin \alpha$, $v = 0$ and $a = -g'$, so $v = u + at$ gives

$$t = \frac{V' \sin \alpha}{g'}.$$

Therefore, the total time taken to reach the highest point is

$$\left(\frac{V - V'}{g} \right) \sin \alpha + \frac{V' \sin \alpha}{g'} = \left(\frac{V - V'}{g} + \frac{V'}{g'} \right) \sin \alpha.$$

Finally, we need to multiply this by 2 to get the total time taken and then by $V \cos \alpha$ to get the distance travelled, giving the distance

$$\begin{aligned} d &= 2 \left(\frac{V - V'}{g} + \frac{V'}{g'} \right) \sin \alpha . V \cos \alpha \\ &= \left(\frac{V - V'}{g} + \frac{V'}{g'} \right) V \sin 2\alpha, \end{aligned}$$

using $2 \sin \alpha \cos \alpha = \sin 2\alpha$.

Question 11

A straight uniform rod has mass m. Its ends P_1 and P_2 are attached to small light rings that are constrained to move on a rough circular wire with centre O fixed in a vertical plane, and the angle P_1OP_2 is a right angle. The rod rests with P_1 lower than P_2, and with both ends lower than O. The coefficient of friction between each of the rings and the wire is μ. Given that the rod is in limiting equilibrium (i.e., on the point of slipping at both ends), show that

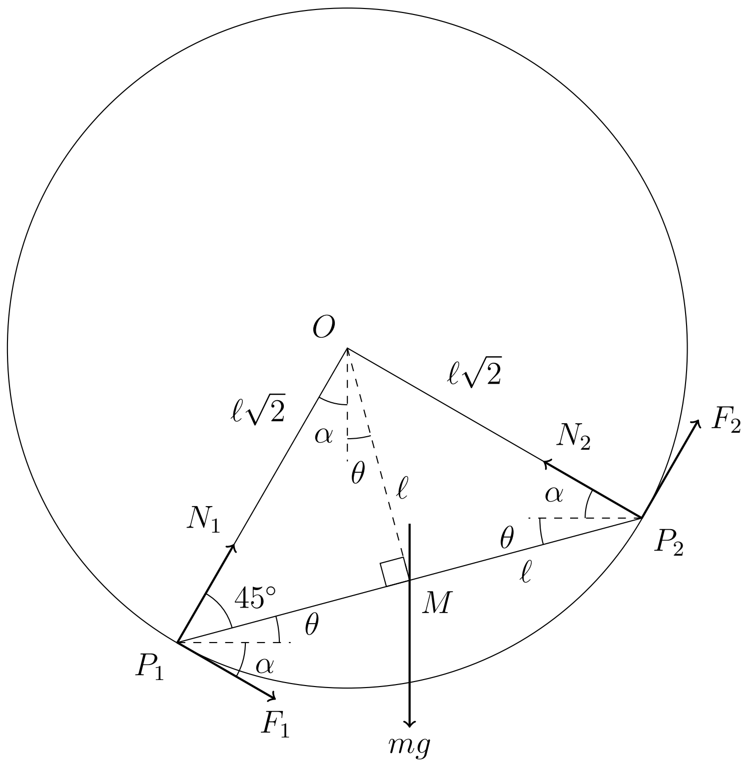
$$\tan\alpha = \frac{1-2\mu-\mu^2}{1+2\mu-\mu^2},$$

where α is the angle between P_1O and the vertical ($0<\alpha<45°$).
Let θ be the acute angle between the rod and the horizontal. Show that $\theta = 2\lambda$, where λ is defined by $\tan\lambda = \mu$ and $0<\lambda<22.5°$.

We present two methods for solving the problem. The first is a standard method using resolution of forces; the second is to use standard results about three forces acting on a large body. We specify that the length of the rod is 2ℓ, so that the radius of the circle is $\ell\sqrt{2}$.

Method 1: Resolving all the forces

We begin by drawing a clear sketch of the situation.



Note that as the rod is in limiting equilibrium, both of the frictional forces act to prevent it from slipping towards the horizontal, and $F_1=\mu N_1$, $F_2=\mu N_2$. Also, we see that $\theta = 45°-\alpha$, as the angle OP_1P_2 is $45°$ (the triangle being isosceles), so $0<\theta<45°$.

We now resolve the forces in two directions. We could resolve in any two directions, but so that we can exclude mg from at least one of the equations, we choose to resolve horizontally and vertically. Another sensible choice would have been to resolve along the directions of OP_1 and OP_2.

$$\mathscr{R}(\uparrow) \qquad N_1 \cos\alpha - \mu N_1 \sin\alpha + N_2 \sin\alpha + \mu N_2 \cos\alpha - mg = 0 \tag{1}$$
$$\mathscr{R}(\rightarrow) \qquad N_1 \sin\alpha + \mu N_1 \cos\alpha - N_2 \cos\alpha + \mu N_2 \sin\alpha = 0 \tag{2}$$

We also need to take moments. There are four obvious places about which we can take moments: O, P_1, P_2 and M. For completeness, we show what happens if we calculate moments about all four points; clearly only one of these is necessary.

$$\mathscr{M}(\widehat{O}) \qquad mg.\ell\sin(45^\circ - \alpha) - F_1.\ell\sqrt{2} - F_2.\ell\sqrt{2} = 0 \tag{3}$$
$$\mathscr{M}(\widehat{P_1}) \qquad mg.\ell\cos(45^\circ - \alpha) - N_2.\ell\sqrt{2} - F_2.\ell\sqrt{2} = 0 \tag{4}$$
$$\mathscr{M}(\widehat{P_2}) \qquad N_1.\ell\sqrt{2} - F_1.\ell\sqrt{2} - mg.\ell\cos(45^\circ - \alpha) = 0 \tag{5}$$
$$\mathscr{M}(\widehat{M}) \qquad N_1.\ell/\sqrt{2} - F_1.\ell/\sqrt{2} - N_2.\ell/\sqrt{2} - F_2.\ell/\sqrt{2} = 0 \tag{6}$$

Our task is now to eliminate everything to find an expression for $\tan\alpha$ in terms of μ. We can use any one of the equations (3)–(6) to do this, but (6) appears to be the easiest to work with. (With the others, we would have to use a compound angle formula such as $\sin(45^\circ - \alpha) = \sin 45^\circ \cos\alpha - \cos 45^\circ \sin\alpha = \frac{1}{\sqrt{2}}(\cos\alpha - \sin\alpha)$.)

Recalling that $F_1 = \mu N_1$ and $F_2 = \mu N_2$, equation (6) then gives us

$$N_1 - \mu N_1 = N_2 + \mu N_2$$

so

$$N_1(1-\mu) = N_2(1+\mu), \tag{7}$$

or equivalently

$$\frac{N_1}{N_2} = \frac{1+\mu}{1-\mu}. \tag{8}$$

$$\tag{9}$$

We can also solve equations (1) and (2) simultaneously to get the results

$$N_1 = \frac{mg(\cos\alpha - \mu\sin\alpha)}{1+\mu^2}$$

and

$$N_2 = \frac{mg(\sin\alpha + \mu\cos\alpha)}{1+\mu^2},$$

so

$$\frac{N_1}{N_2} = \frac{\cos\alpha - \mu\sin\alpha}{\sin\alpha + \mu\cos\alpha}.$$

Equating this expression with equation (8) yields

$$\frac{\cos\alpha - \mu\sin\alpha}{\sin\alpha + \mu\cos\alpha} = \frac{1+\mu}{1-\mu}.$$

Dividing the numerator and denominator of the left hand side by $\cos\alpha$ then gives

$$\frac{1-\mu\tan\alpha}{\tan\alpha+\mu}=\frac{1+\mu}{1-\mu}.$$

A simple rearrangement of this then yields our desired result:

$$\tan\alpha=\frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}.$$

(Alternatively, one could use (7) to substitute for N_2 in equation (2), after writing $F_1=\mu N_1$ and $F_2=\mu N_2$. Factorising then gives

$$N_1(1+\mu)(\sin\alpha+\mu\cos\alpha)-N_1(1-\mu)(\cos\alpha-\mu\sin\alpha)=0.$$

On dividing by $N_1\cos\alpha$ and expanding the brackets, we end up with an expression in $\tan\alpha$ which we can again rearrange to reach our desired conclusion.)

Now if $\tan\lambda=\mu$ with $0<\lambda<22.5^\circ$, we have (recalling that $\theta=45^\circ-\alpha$)

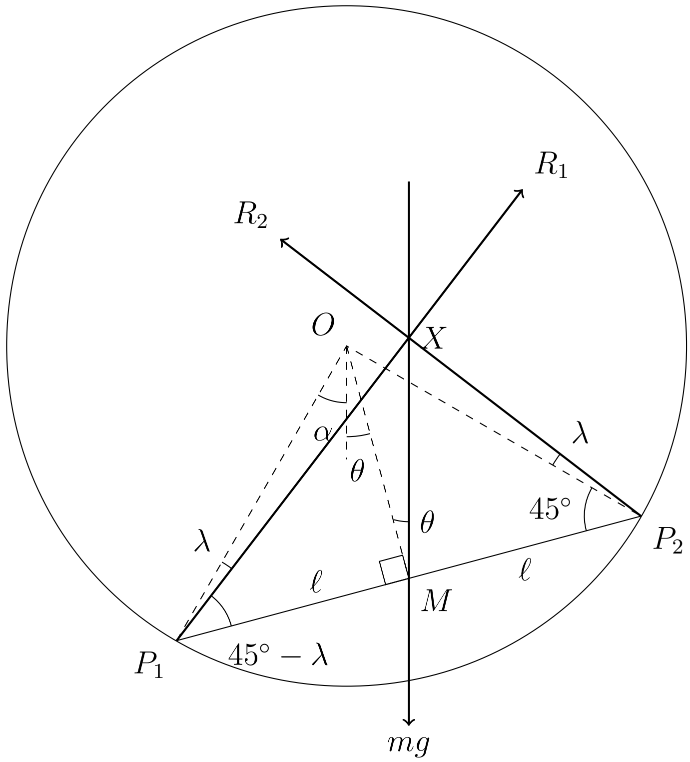
$$\begin{aligned}\tan\theta&=\tan(45^\circ-\alpha)\\&=\frac{\tan 45^\circ-\tan\alpha}{1+\tan 45^\circ\tan\alpha}\\&=\frac{1-\tan\alpha}{1+\tan\alpha}\qquad\text{as }\tan 45^\circ=1\\&=\frac{1-\frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}}{1+\frac{1-2\mu-\mu^2}{1+2\mu-\mu^2}}\\&=\frac{(1+2\mu-\mu^2)-(1-2\mu-\mu^2)}{(1+2\mu-\mu^2)+(1-2\mu-\mu^2)}\\&=\frac{4\mu}{2-2\mu^2}\\&=\frac{2\mu}{1-\mu^2}\\&=\frac{2\tan\lambda}{1-\tan^2\lambda}\\&=\tan 2\lambda.\end{aligned}$$

Then since $0<\theta<45^\circ$, it follows that $\theta=2\lambda$ as required.

Method 2: Three forces on a large body theorem

We recall the theorem that if three forces act on a large body in equilibrium, and they are not all parallel, then they are concurrent (i.e., they all pass through a single point). We combine the normal reaction, N, and friction, F, at each end of the rod into a single reaction force, R. This acts at an angle ϕ to the normal, where $\tan\phi=F/N$. In our case, $F=\mu N$ as the rod is on the point of slipping, so $\tan\phi=\mu$. But the question defines λ to be the acute angle such that $\tan\lambda=\mu$, from which it follows that $\phi=\mu$.

We now redraw the diagram showing only the total reaction forces, which we call R_1 and R_2 in this context, and we ensure that R_1, R_2 and the weight mg pass through a single point X.



The rest of the question is now pure trigonometry. We apply the sine rule to the triangles P_1MX and P_2MX, first noting that

$$\angle MXP_1 = 180° - (45° - \lambda) - (90° + \lambda) = 45° - \theta + \lambda$$

and

$$\angle MXP_2 = 180° - (90° - \theta) - (45° + \lambda) = 45° + \theta - \lambda$$

to get

$$\frac{\ell}{\sin(45° - \theta + \lambda)} = \frac{MX}{\sin(45° - \lambda)} \tag{10}$$

$$\frac{\ell}{\sin(45° + \theta - \lambda)} = \frac{MX}{\sin(45° + \lambda)}\,. \tag{11}$$

Then using the identity $\sin(90° - x) = \cos x$ twice, once with $x = 45° - \theta + \lambda$ and then with $x = 45° + \lambda$, we deduce from equation (11) that

$$\frac{\ell}{\cos(45° - \theta + \lambda)} = \frac{MX}{\cos(45° - \lambda)}\,. \tag{12}$$

We can now divide equation (12) by (10) to get

$$\tan(45° - \theta + \lambda) = \tan(45° - \lambda). \tag{13}$$

It follows immediately that $45° - \theta + \lambda = 45° - \lambda$ as both angles are acute, so $\theta = 2\lambda$, which answers the second part of the question.

This result then leads us to conclude that

$$\tan\theta = \tan 2\lambda = \frac{2\tan\lambda}{1 - \tan^2\lambda} = \frac{2\mu}{1 - \mu^2}.$$

Finally, as we know that $\alpha = 45^\circ - \theta$, we can deduce that

$$\begin{aligned}
\tan\alpha &= \tan(45^\circ - \theta) \\
&= \frac{\tan 45^\circ - \tan\theta}{1 + \tan 45^\circ \tan\theta} \\
&= \frac{1 - \tan\theta}{1 + \tan\theta} \qquad \text{as } \tan 45^\circ = 1 \\
&= \frac{1 - \frac{2\mu}{1-\mu^2}}{1 + \frac{2\mu}{1-\mu^2}} \\
&= \frac{(1-\mu^2) - 2\mu}{(1-\mu^2) + 2\mu} \\
&= \frac{1 - 2\mu - \mu^2}{1 + 2\mu - \mu^2},
\end{aligned}$$

and we are done.

Question 12

In this question, you may use without proof the results:

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1) \qquad \text{and} \qquad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

The independent random variables X_1 and X_2 each take values 1, 2, …, N, each value being equally likely. The random variable X is defined by

$$X = \begin{cases} X_1 & \text{if } X_1 \geqslant X_2 \\ X_2 & \text{if } X_2 \geqslant X_1. \end{cases}$$

(i) *Show that* $\mathrm{P}(X = r) = \dfrac{2r-1}{N^2}$ *for* $r = 1, 2, \ldots, N$.

We have $X = r$ when either $X_1 = r$ and $X_2 < r$, or $X_2 = r$ and $X_1 < r$ or $X_1 = X_2 = r$. Therefore

$$\begin{aligned} \mathrm{P}(X = r) &= \mathrm{P}(X_1 = r \cap X_2 < r) + \mathrm{P}(X_2 = r \cap X_1 < r) + \mathrm{P}(X_1 = X_2 = r) \\ &= \frac{1}{N}.\frac{r-1}{N} + \frac{1}{N}.\frac{r-1}{N} + \frac{1}{N}.\frac{1}{N} \\ &= \frac{2r-1}{N^2}\,. \end{aligned}$$

Alternatively, one can argue as follows:

$$\begin{aligned} \mathrm{P}(X = r) &= \mathrm{P}(X_1 = r \cap X_2 \leqslant r) + \mathrm{P}(X_2 = r \cap X_1 \leqslant r) - \mathrm{P}(X_1 = X_2 = r) \\ &= \frac{1}{N}.\frac{r}{N} + \frac{1}{N}.\frac{r}{N} - \frac{1}{N}.\frac{1}{N} \\ &= \frac{2r-1}{N^2}\,. \end{aligned}$$

(ii) *Find an expression for the expectation, μ, of X and show that $\mu = 67.165$ in the case $N = 100$.*

By the definition of expectation, we have

$$
\begin{aligned}
\mu = \mathrm{E}(X) &= \sum_{r=1}^{N} r.\mathrm{P}(X=r) \\
&= \sum_{r=1}^{N} \frac{r(2r-1)}{N^2} \\
&= \frac{1}{N^2} \sum_{r=1}^{N} (2r^2 - r) \\
&= \frac{1}{N^2} \big(\tfrac{2}{6} N(N+1)(2N+1) - \tfrac{1}{2} N(N+1) \big) \quad \text{using the given results} \\
&= \frac{\tfrac{1}{6} N(N+1)(4N+2-3)}{N^2} \\
&= \frac{(N+1)(4N-1)}{6N} \,.
\end{aligned}
$$

In the case $N = 100$, this is $101 \times 399/600 = 13\,433/200 = 67.165$ as required.

> ***(iii)*** *The median, m, of X is defined to be the integer such that $\mathrm{P}(X \geqslant m) \geqslant \frac{1}{2}$ and $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$. Find an expression for m in terms of N and give an explicit value for m in the case $N = 100$.*

We have

$$
\begin{aligned}
\mathrm{P}(X \leqslant k) &= \sum_{r=1}^{k} \frac{2r-1}{N^2} \\
&= \frac{1}{N^2} \big(2.\tfrac{1}{2} k(k+1) - k \big) \\
&= \frac{k^2}{N^2}
\end{aligned}
$$

so that

$$
\mathrm{P}(X \geqslant k) = 1 - \mathrm{P}(X \leqslant k-1) = 1 - \frac{(k-1)^2}{N^2} \,.
$$

We are looking for the value of m which makes $\mathrm{P}(X \geqslant m) \geqslant \frac{1}{2}$ and $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$. The first condition gives

$$
\begin{aligned}
& 1 - \frac{(m-1)^2}{N^2} \geqslant \tfrac{1}{2} \\
\iff \quad & \frac{(m-1)^2}{N^2} \leqslant \tfrac{1}{2} \\
\iff \quad & (m-1)^2 \leqslant \tfrac{1}{2} N^2 \\
\iff \quad & m - 1 \leqslant \tfrac{N}{\sqrt{2}} \\
\iff \quad & m \leqslant \tfrac{N}{\sqrt{2}} + 1.
\end{aligned}
$$

The second condition, $\mathrm{P}(X \leqslant m) \geqslant \frac{1}{2}$, yields

$$\begin{aligned} & \frac{m^2}{N^2} \geqslant \tfrac{1}{2} \\ \iff \quad & m^2 \geqslant \tfrac{1}{2}N^2 \\ \iff \quad & m \geqslant \tfrac{N}{\sqrt{2}}\,. \end{aligned}$$

So we have $N/\sqrt{2} \leqslant m \leqslant (N/\sqrt{2}) + 1$, thus m is the smallest integer greater than $N/\sqrt{2}$ (which is not itself an integer as $\sqrt{2}$ is irrational). The smallest integer greater than or equal to a number x is called the *ceiling* of x, and is denoted by $\lceil x \rceil$, so we can state our result as $m = \lceil N/\sqrt{2} \rceil$.

In the case $N = 100$, $m = \lceil 100/\sqrt{2} \rceil = \lceil 70.7\ldots \rceil = 71$.

(iv) *Show that when N is very large,*

$$\frac{\mu}{m} \approx \frac{2\sqrt{2}}{3}\,.$$

We have formulæ for μ from part (ii) and for m from part (iii). Therefore we have, for large N,

$$\begin{aligned} \frac{\mu}{m} &= \frac{(N+1)(4N-1)}{6N} \Bigg/ \lceil N/\sqrt{2} \rceil \\ &\approx \frac{(N+1)(4N-1)}{6N(N/\sqrt{2})} \\ &= \frac{4N^2 + 3N - 1}{6N^2/\sqrt{2}} \\ &= \frac{4 + \frac{3}{N} - \frac{1}{N^2}}{6/\sqrt{2}} \\ &\approx \frac{4}{6/\sqrt{2}} \\ &= \frac{2\sqrt{2}}{3}\,. \end{aligned}$$

Question 13

Three married couples sit down at a round table at which there are six chairs. All of the possible seating arrangements of the six people are equally likely.

(i) *Show that the probability that each husband sits next to his wife is $\frac{2}{15}$.*

We call the couples H_1 and W_1, H_2 and W_2, H_3 and W_3. We seat H_1 arbitrarily, leaving $5! = 120$ ways of seating the remaining five people. If each husband sits next to his wife, then there are two seats in which W_1 can sit, each of which leaves four consecutive seats for the other two couples.

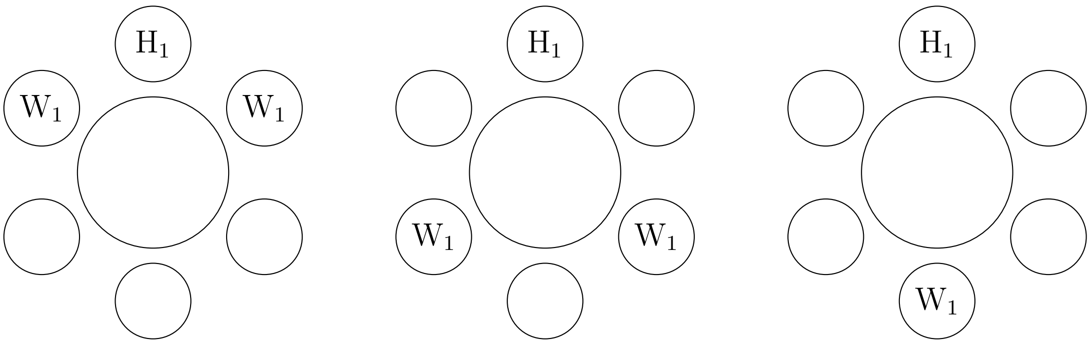
There are four choices for who to sit immediately next to W_1, and that forces the following seat as well (being the spouse of that person).

Next, there are two choices for who to seat the other side of H_1, and the final person must sit next to their spouse.

So there are $2 \times 4 \times 2 = 16$ ways to have all of the husbands sitting next to their wives, with a probability of $\frac{16}{120} = \frac{2}{15}$.

(ii) *Find the probability that exactly two husbands sit next to their wives.*

Assume to begin with that H_1 and W_1 are separated and the other two husbands are seated next to their wives. Once H_1 has been seated, there are five possible positions for W_1, as shown in the following diagrams (there are two possibilities shown in each of the first two):



Clearly the first two possibilities do not work: in the first, H_1 and W_1 are next to each other. In the second, whoever sits between H_1 and W_1 will be separated from their spouse. So W_1 must sit opposite H_1, one couple sits to the right of H_1 and the other couple to his left. There are two choices for which couple sits to the right of H_1, and two choices for whether the husband or wife sits next to H_1; similarly there are two choices for whether the husband or wife of the third couple sits next to H_1. So in all, there are $2 \times 2 \times 2 = 8$ ways to seat the couples with H_1 and W_1 separated and the other couples together.

Similarly, there are 8 ways with couple 2 separated and 8 ways with couple 3 separated, so there are $3 \times 8 = 24$ ways in total.

Thus the probability is $\frac{24}{120} = \frac{3}{15} = \frac{1}{5}$.

> ***(iii)*** *Find the probability that no husband sits next to his wife.*

Method 1: First find the probability of exactly one husband sitting next to his wife.

Let us assume that H_3 and W_3 are the only pair next to each other. Then in the above diagrams, the left hand one fails as H_1 and W_1 are together. The right hand one also fails, as if H_3 and W_3 are together, H_2 and W_2 must also be. So the only valid configuration is the middle one, with either H_2 or W_2 between H_1 and W_1 and the other partner on the other side of either H_1 or W_1.

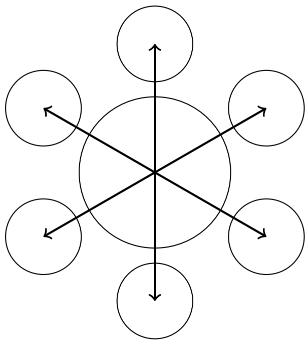
There are two choices for where W_1 will sit, two choices for which of H_2 or W_2 will sit between H_1 and W_1, two choices for where the other partner will sit, and two choices for which way round H_3 and W_3 will sit, giving $2 \times 2 \times 2 \times 2 = 16$ possibilities.

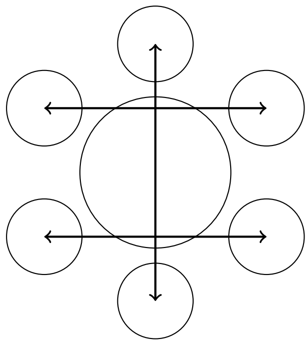
Finally, we need to multiply this by three, as any one of the three couples could be the adjacent one, giving $3 \times 16 = 48$ possibilities, and hence a probability of $\frac{48}{120} = \frac{2}{5} = \frac{6}{15}$.

Thus the probability that no husband sits next to his wife is $1 - \frac{2}{15} - \frac{3}{15} - \frac{6}{15} = \frac{4}{15}$.

Method 2: Find the probability directly.

If no husband sits next to his wife, there are two possible configurations as shown in these diagrams (the arrows join husbands with their wives):





The number of ways of arranging the first case (fixing H_1 at the top as usual) is $2 \times 2 \times 2 = 8$, as their are two ways of choosing which couple sits in which of the diagonal pairs of seats, two ways of couple 2 sitting and two ways of couple 3 sitting.

For the second case, which is no longer totally symmetrical between the three couples, if H_1 sits in the top seat, there are again 8 ways of seating the other two couples. As there are three choices for which couple sits opposite each other, there are $3 \times 8 = 24$ ways in all.

Thus in total there are $8 + 24 = 32$ ways, giving a total probability of $\frac{32}{120} = \frac{4}{15}$.