STEP III, Solutions
June 2008

STEP Mathematics III 2008: Solutions

1. Following the hint yields

$$ax^2 + by^2 + (a+b)xy = \frac{1}{3}(x+y)$$

which is $\frac{1}{5} + xy = \frac{1}{3}(x+y)$

The same trick applied to the third equation gives $\frac{1}{7} + \frac{1}{3}xy = \frac{1}{5}(x+y)$.

The two equations can be solved simultaneously for xy and $(x+y)$, giving

$xy = \frac{3}{35}$ and $(x+y) = \frac{6}{7}$

Thus x and y are the roots of the quadratic equation $35z^2 - 30z + 3 = 0$
(x and y are interchangeable).
a and b are then found by substituting back into two of the original equations and the full solution is

$$x = \frac{3}{7} \pm \frac{2}{35}\sqrt{30} = \frac{3}{7} \pm \frac{2}{7}\sqrt{\frac{6}{5}}$$

$$y = \frac{3}{7} \mp \frac{2}{35}\sqrt{30} = \frac{3}{7} \mp \frac{2}{7}\sqrt{\frac{6}{5}}$$

$$a = \frac{1}{2} \mp \frac{\sqrt{30}}{36} = \frac{1}{2} \mp \frac{1}{6}\sqrt{\frac{5}{6}}$$

$$b = \frac{1}{2} \pm \frac{\sqrt{30}}{36} = \frac{1}{2} \pm \frac{1}{6}\sqrt{\frac{5}{6}}$$

2. (i) On the one hand

$\sum_{r=0}^{n}\left[(r+1)^k - r^k\right] = \sum_{r=0}^{n}(r+1)^k - \sum_{r=0}^{n} r^k = \sum_{r=1}^{n+1} r^k - \sum_{r=0}^{n} r^k = (n+1)^k$ whilst expanding binomially yields

$$k\sum_{r=0}^{n} r^{k-1} + \binom{k}{2}\sum_{r=0}^{n} r^{k-2} + \binom{k}{3}\sum_{r=0}^{n} r^{k-3} + \ldots + \binom{k}{k-1}\sum_{r=0}^{n} r^{k-1} + \sum_{r=0}^{n} 1$$

$$= kS_{k-1}(n) + \binom{k}{2}S_{k-2}(n) + \binom{k}{3}S_{k-3}(n) + \ldots + \binom{k}{k-1}S_1(n) + (n+1)$$

and hence the required result.

Applying this in the case $k = 4$ gives

$$4S_3(n) = (n+1)^4 - (n+1) - \binom{4}{2}S_2(n) - \binom{4}{3}S_1(n)$$

which, after substitution of the two given results and factorization, yields the familiar

$$S_3(n) = \frac{1}{4}n^2(n+1)^2$$

The identical process with $k = 5$ results in

$$S_4(n) = \frac{1}{30}n(n+1)\left(6n^3 + 9n^2 + n - 1\right) = \frac{1}{30}n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)$$

(ii) Applying induction, with the assumption that $S_t(n)$ is a polynomial of degree $t+1$ in n for $t < r$ for some r, and then considering (*),

$(n+1)^{r+1} - (n+1)$ is a polynomial of degree $r+1$ in n ,

and each of the terms $-\binom{r+1}{j}S_{r+1-j}(n)$ is a polynomial of degree $r+2-j$ in n where $j \geq 2$, i.e. the degree is $\leq r$. A sum of polynomials of degree $\leq r+1$ in n, is a polynomial of degree $\leq r+1$ in n, and there is a single non-zero term in n^{r+1} from just $(n+1)^{r+1}$ so the degree of the polynomial is not reduced to $< r+1$, i.e. it is $r+1$. (The initial case is true to complete the proof.)

If, $S_k(n) = \sum_{i=0}^{k+1} a_i n^i = \sum_{r=0}^{n} r^k$ then $S_k(0) = a_0 + \sum_{i=1}^{k+1} a_i 0^i = \sum_{r=0}^{0} r^k = 0$ and so $a_0 = 0$

$S_k(1) = \sum_{i=0}^{k+1} a_i 1^i = \sum_{r=0}^{1} r^k = 1$ and so $\sum_{i=0}^{k+1} a_i = 1$ as required.

3. $$\frac{dy}{dx} = \frac{b\cos\theta}{-a\sin\theta}$$

So the line ON is $y = \dfrac{a\sin\theta}{b\cos\theta}x$

SP is $y = \dfrac{b\sin\theta}{a(\cos\theta + e)}(x + ae)$

Solving simultaneously by substituting for *x* to find the *y* coordinate of *T*,

$$y = \frac{b\sin\theta}{a(\cos\theta + e)}\left(\frac{b\cos\theta}{a\sin\theta}y + ae\right)$$

and using $b^2 = a^2\left(1 - e^2\right)$ to eliminate a^2 gives the required result.

Then the *x* coordinate of *T* is $\dfrac{b^2\cos\theta}{a(1 + e\cos\theta)}$.

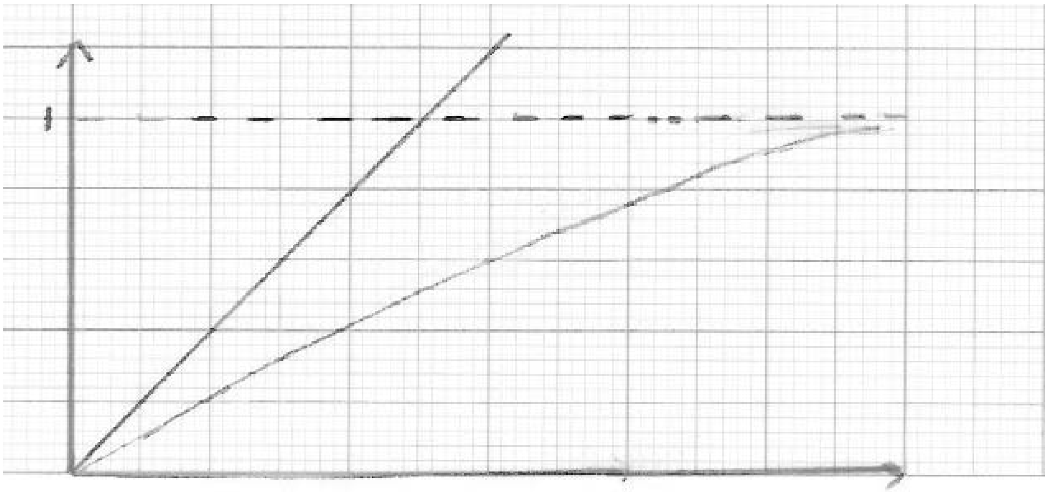
Eliminating θ using $\sec\theta + e = \dfrac{b^2}{ax}$ and $\tan\theta = \dfrac{by}{ax}$,

(x, y) satisfies $\left(\dfrac{b^2}{ax} - e\right)^2 = 1 + \left(\dfrac{by}{ax}\right)^2$

and again using $b^2 = a^2\left(1 - e^2\right)$, this time to eliminate b^2, gives, following simplifying algebra

$(x + ae)^2 + y^2 = a^2$, as required.

4. (i)



The graph of $z = y$ has gradient 1 and passes through the origin.

The graph of $z = \tanh\left(\frac{y}{2}\right)$ which has gradient $\frac{1}{2}\operatorname{sec}h^2\left(\frac{y}{2}\right) \leq \frac{1}{2}$ for $y \geq 0$ also passes though the origin and is asymptotic to $z = 1$.

Thus $y \geq \tanh\left(\frac{y}{2}\right)$ for $y \geq 0$.

If $x = \cosh y$, then $\sqrt{\frac{x-1}{x+1}} = \sqrt{\frac{\cosh y - 1}{\cosh y + 1}} = \sqrt{\frac{2\sinh^2\left(\frac{y}{2}\right)}{2\cosh^2\left(\frac{y}{2}\right)}} = \tanh\left(\frac{y}{2}\right)$

and as $y \geq \tanh\left(\frac{y}{2}\right)$ for $y \geq 0$, $ar\cosh x \geq \sqrt{\frac{x-1}{x+1}}$ for $x \geq 1$.

$\sqrt{\frac{x-1}{x+1}} = \sqrt{\frac{x-1}{x+1}}\sqrt{\frac{x-1}{x-1}} = \frac{x-1}{\sqrt{x^2-1}}$ for $x > 1$, and (*) is obtained.

(ii) By parts $\int ar\cosh x dx = x ar\cosh x - \sqrt{x^2-1} + c$

and $\int \frac{x-1}{\sqrt{x^2-1}} dx = \sqrt{x^2-1} - ar\cosh x + c'$

Thus $\int_1^x ar\cosh x dx \geq \int_1^x \frac{x-1}{\sqrt{x^2-1}} dx$ for $x > 1$ gives

$x ar\cosh x - \sqrt{x^2-1} \geq \sqrt{x^2-1} - ar\cosh x$ for $x > 1$, which rearranges to give result

(iii) Integrating (ii) similarly gives $x ar\cosh x - \sqrt{x^2-1} \geq 2\left(\sqrt{x^2-1} - ar\cosh x\right)$ for $x > 1$, which also can be rearranged as desired.

5. There are a number of correct routes to proving the induction, though the simplest is to consider $\left(\left(T_{k+1}(x)\right)^2 - T_k(x)T_{k+2}(x)\right) - \left(\left(T_k(x)\right)^2 - T_{k-1}(x)T_{k+1}(x)\right)$

For $f(x) = 0$, $\left(T_n(x)\right)^2 - T_{n-1}(x)T_{n+1}(x) = 0$

and so $\dfrac{T_{n+1}(x)}{T_n(x)} = \dfrac{T_n(x)}{T_{n-1}(x)}$ provided that neither denominator is zero, leading to
$\dfrac{T_n(x)}{T_{n-1}(x)} = \dfrac{T_1(x)}{T_0(x)} = r(x)$,
and so $\dfrac{T_n(x)}{T_{n-1}(x)} \times \dfrac{T_{n-1}(x)}{T_{n-2}(x)} \times \ldots \times \dfrac{T_1(x)}{T_0(x)} = \left(r(x)\right)^n$
Thus $T_n(x) = \left(r(x)\right)^n T_0(x)$

Substituting this result into (*) for $n = 1$,
$\left(\left(r(x)\right)^2 - 2xr(x) + 1\right)T_0(x) = 0$, and as $T_0(x) \neq 0$, solving the quadratic gives
$r(x) = x \pm \sqrt{x^2 - 1}$

6. (i) Differentiating $y = p^2 + 2xp$ with respect to x gives
$p = 2p\dfrac{dp}{dx} + 2x\dfrac{dp}{dx} + 2p$ which can be rearranged suitably.
The differential equation $\dfrac{dx}{dp} + \dfrac{2}{p}x = -2$ has an integrating factor p^2 and integrating will give the required general solution.
Substituting $x = 2, p = -3$, leads to $A = 0$, i.e. $p = -\dfrac{3}{2}x$ which can be substituted in the original equation and so $y = -\dfrac{3}{4}x^2$.

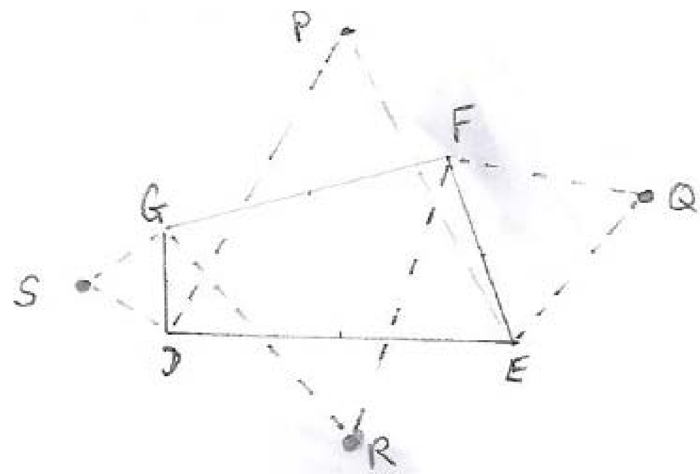
(ii) The same approach as in part (i) generates $\dfrac{dx}{dp} + \dfrac{2}{p}x = -\dfrac{(\ln p + 1)}{p}$,
which with the same integrating factor has general solution
$x = -\dfrac{1}{4} - \dfrac{1}{2}\ln p + Bp^{-2}$
and particular solution
$x = -\dfrac{1}{2}\ln p - \dfrac{1}{4}$

Again, substitution of ln p (and p) in the original equation leads to the solution which is $y = -\dfrac{1}{2}e^{-2x-\frac{1}{2}}$

7. The starting point $c-a=\frac{1}{2}\left(1+i\sqrt{3}\right)(b-a)$ leads to the given result.

Interchanging a and b gives $2c=(a+b)+i\sqrt{3}(a-b)$ if A, B, C are described clockwise.

(i) The clue to this is the phrase “can be chosen” and a sketch demonstrates that a pair of the equilateral triangles need to be clockwise, and the other pair anti-clockwise



Applying the results in the stem of the question to this configuration,

$$2p=(d+e)+i\sqrt{3}(e-d)$$
$$2q=(e+f)+i\sqrt{3}(e-f)$$
$$2r=(f+g)+i\sqrt{3}(g-f)$$
$$2s=(g+d)+i\sqrt{3}(g-d)$$

and so $2PS=(g-e)+i\sqrt{3}(g-e)=-2RQ$, PSQR is a parallelogram.
(The pairs could have been chosen with opposite parity leading to very similar working.)

(ii) Supposing LMN is clockwise, U is the centroid of equilateral triangle LMH, V of MNJ, and W of NLK, then
$3u=l+m+h$ where $2h=(l+m)+i\sqrt{3}(m-l)$ with similar results for *v* and *w*.

Both $6w$, and $3\left[(u+v)+i\sqrt{3}(u-v)\right]$ can be shown to equal $3(n+l)+i\sqrt{3}(l-n)$ and so UVW is a clockwise equilateral triangle.

8. (i) $p=-\frac{1}{2}$

$\left(1+px\right)S=\frac{1}{3}x$ with all other terms cancelling and so $S=\frac{1}{3}x\Big/\left(1-\frac{1}{2}x\right)=\frac{2x}{3(2-x)}$

Using the sum of a GP

$$S_{n+1} = \frac{1}{3}x + \frac{1}{6}x^2 + \frac{1}{12}x^3 + \ldots + \frac{1}{3\times 2^n}x^{n+1} = \frac{\frac{1}{3}x\left(1-\frac{x^{n+1}}{2^{n+1}}\right)}{\left(1-\frac{x}{2}\right)}$$

Alternatively $S_{n+1} = S - (a_{n+2}x^{n+2} + \ldots) = S - \frac{1}{2^{n+1}}x^{n+1}S$

$$= \left(1-\frac{x^{n+1}}{2^{n+1}}\right)\frac{2x}{3(2-x)}$$

(ii) Using similar working to part (i)

$18 + 8p + 2q = 0$

$37 + 18p + 8q = 0$

so $p = -\frac{5}{2}, q = 1$

and so $\left(1 + px + qx^2\right)T = 2 + 3x$

giving $T = (2+3x)\Big/\left(1-\frac{5}{2}x + x^2\right) = \frac{4+6x}{2-5x+2x^2} = \frac{4+6x}{(2-x)(1-2x)}$

By partial fractions $T = \frac{14}{3}(1-2x)^{-1} - \frac{8}{3}\left(1-\frac{x}{2}\right)^{-1}$

and so $T_{n+1} = \frac{14}{3}\left(1 + 2x + (2x)^2 + \ldots + (2x)^n\right) - \frac{8}{3}\left(1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \ldots + \left(\frac{x}{2}\right)^n\right)$

$$= \frac{14}{3}\frac{\left(1-(2x)^{n+1}\right)}{1-2x} - \frac{8}{3}\frac{\left(1-\left(\frac{x}{2}\right)^{n+1}\right)}{1-\frac{x}{2}}$$

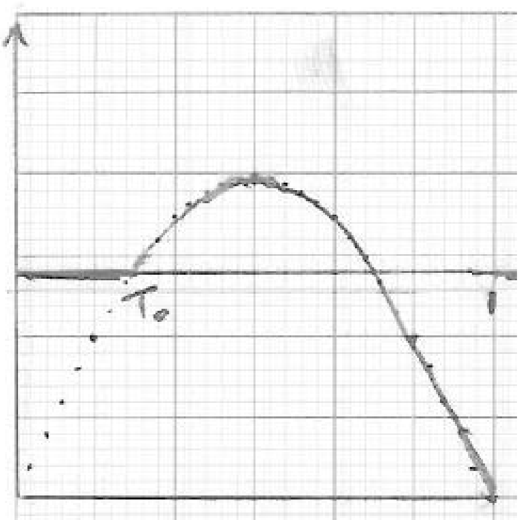
Section B: Mechanics

9. When the particle starts to move, friction is limiting and so

$mg\sin\pi T_0 - \mu mg = 0$

i.e. $\mu = \sin\pi T_0$

(i)

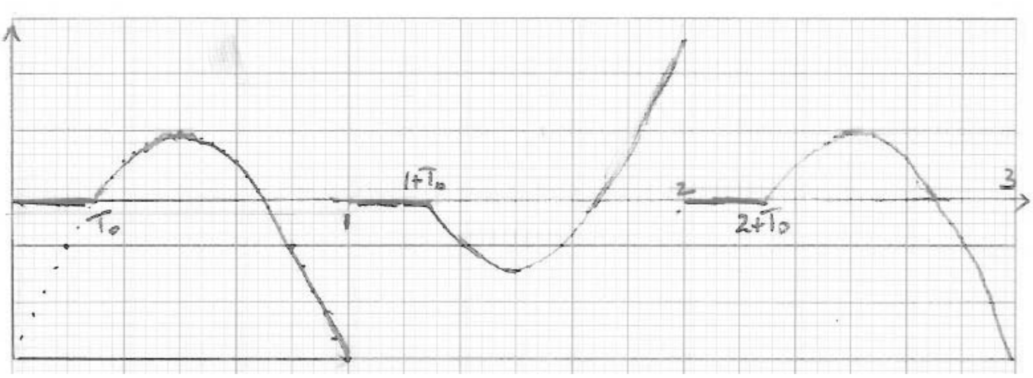


When the particle comes to rest, the area under the acceleration-time graph is zero

i.e. $\int_{T_0}^{1} g\sin \pi t - \mu_o g dt = 0$

Completing the manipulation and eliminating T_0 using the relation at the start of the question renders the required result.

(ii)



In the case $\mu = \mu_0$, the motion is periodic with period 2, the particle is stationary in intervals $(0, T_0), (1, 1+T_0), (2, 2+T_0) \ldots$, reversing its direction of motion after times 1, 2, 3, … , and returning to its starting point at time 2 (and 4,6….)

In the case $\mu = 0$, the motion is simple harmonic motion (period 2) superimposed on uniform motion , the particle instantaneously comes to rest at time 2, 4, …but otherwise always moves in the positive x direction.

($x = \frac{g}{\pi^2}(\pi t - \sin \pi t)$)

10. Considering the rth short string $T_r = mg + T_{r-1}$

Also we have $T_r = \dfrac{\lambda x_r}{l}$, and $T_1 = mg$

Thus $T_r = rmg$ and so the total length is given by $\sum_1^n (l + x_r) = \sum_1^n \left(l + \dfrac{rmgl}{\lambda}\right)$

$= nl + \dfrac{mgl}{\lambda}\dfrac{n(n+1)}{2}$

The elastic energy stored is $\sum_1^n \dfrac{\lambda x_r^{\,2}}{2l} = \sum_1^n \lambda\left(\dfrac{lmg}{\lambda}\right)^2 \dfrac{r^2}{2l} = \dfrac{m^2 g^2 l}{12\lambda} n(n+1)(2n+1)$

For the uniform heavy rope, we let $M = nm$, $L_0 = nl$, and consider the limit as $n \to \infty$

$L = \lim\left(L_0 + \dfrac{M}{n}\dfrac{g}{\lambda}\dfrac{L_0}{n}\dfrac{n(n+1)}{2}\right) = L_0\left(1 + \dfrac{Mg}{2\lambda}\right)$

and the elastic energy stored is

$\lim\left(\dfrac{m^2 g^2 l}{12\lambda} n(n+1)(2n+1)\right) = \lim\left(\dfrac{M^2 g^2 L_0}{12\lambda}\dfrac{n(n+1)(2n+1)}{n^3}\right) = \dfrac{M^2 g^2 L_0}{6\lambda}$

and eliminating M using the result just found for L we obtain $\dfrac{2\lambda(L - L_0)^2}{3L_0}$

11. If the resistance couple (constant) is L, then using $L = I\alpha$ for the second phase of the motion, $L = \dfrac{I\omega_0}{T}$ and rotational kinetic energy used up doing work against the couple in the second phase gives

$$\frac{1}{2} I\omega_0^{\,2} = L \times n_2 \times 2\pi$$

Hence, eliminating L and simplifying gives the first result.

If the particle descends a distance h in the first phase of motion, then $h = 2\pi r n_1$. If the particle has speed v at the end of the first phase, then $v = r\omega_0$

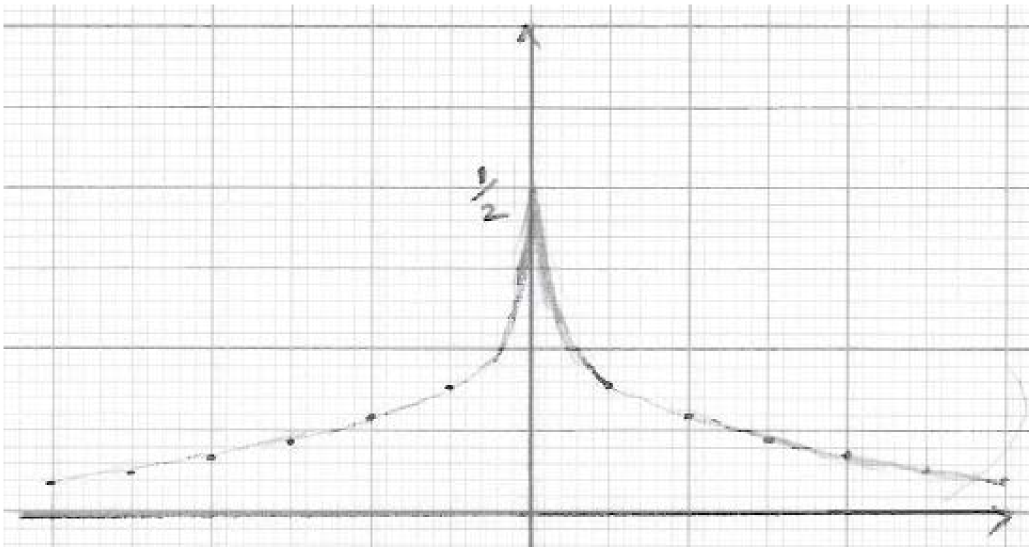
and using the work-energy principle,

$mgh - L \times n_1 \times 2\pi = \dfrac{1}{2} I\omega_0^{\,2} + \dfrac{1}{2} mv^2$

Hence, eliminating h, v and $\omega_0^{\,2}$ obtains the second result.

Section C: Probability and Statistics

12.



$$M_x(\theta) = \int_{-\infty}^{\infty} e^{\theta x} f(x)dx = \int_{-\infty}^{0} \frac{1}{2} e^{x(1+\theta)} dx + \int_{0}^{\infty} \frac{1}{2} e^{-x(1-\theta)} dx$$

$$= \frac{1}{2(1+\theta)}\left[e^{x(1+\theta)}\right]_{-\infty}^{0} - \frac{1}{2(1-\theta)}\left[e^{-x(1-\theta)}\right]_{0}^{\infty} \qquad = \frac{1}{2(1+\theta)} + \frac{1}{2(1-\theta)} \quad \text{(requiring } |\theta| < 1\text{)}$$

$$= \left(1-\theta^2\right)^{-1}$$

$$Var(X) = M_x{}''(0) - \left(M_x{}'(0)\right)^2$$

$$M_x{}'(\theta) = 2\theta\left(1-\theta^2\right)^{-2},\; M_x{}''(\theta) = 2\left(1-\theta^2\right)^{-2} + 8\theta\left(1-\theta^2\right)^{-3}$$

and so $M_x{}'(0) = 0$, $M_x{}''(0) = 2$, $Var(X) = 2$

Or alternatively, $M_x(\theta) = E\left(e^{\theta X}\right) = E\left(1 + \theta X + \frac{1}{2}\theta^2 X^2 + ...\right)$

$$= \left(1-\theta^2\right)^{-1} = 1 + \theta^2 + \theta^4 + ...$$

and so $E(X) = 0$, $E\left(\frac{1}{2}X^2\right) = 1$, $Var(X) = E\left(X^2\right) - \left(E(X)\right)^2 = 2$

If $T = Y/\sqrt{2n}$, then $M_T(\theta) = E\left(e^{\theta T}\right) = E\left(e^{\theta\sum\left(X_i/\sqrt{2n}\right)}\right) = \prod_{i=1}^{n} E\left(e^{\frac{\theta}{\sqrt{2n}}X_i}\right) = \left(1 - \frac{\theta^2}{2n}\right)^{-n}$

$$\log\left(M_T(\theta)\right) = -n\log\left(1-\frac{\theta^2}{2n}\right) = -n\left[-\frac{\theta^2}{2n}-\frac{\theta^4}{8n^2}-\frac{\theta^6}{24n^3}-.....\right] = \frac{\theta^2}{2}+\frac{\theta^4}{8n}+\frac{\theta^6}{24n^2}+.....$$

Thus as $n\to\infty$, $\log\left(M_T(\theta)\right)\to\frac{\theta^2}{2}$, and so $M_T(\theta)\to\exp\left(\frac{\theta^2}{2}\right)$

$P\left(|Y|\geq 25\right) = 0\cdot 05$ and $P\left(\left|Y/\sqrt{2n}\right|\geq 1\cdot 96\right) = 0\cdot 05$ and so

$$25 = 1\cdot 96\sqrt{2n}$$
$$2n = \frac{25^2}{1\cdot 96^2}\approx\frac{625}{4}$$
$$n\approx\frac{625}{8}\approx 78$$

13. P(1 ring created at first step) = $\frac{1}{2n-1}$,

P(0 rings created at first step) = $\frac{2n-2}{2n-1}$

E(number of rings created at first step) = $\frac{1}{2n-1}\times 1+\frac{2n-2}{2n-1}\times 0 = \frac{1}{2n-1}$

Regardless of what happens at first step, after the first step there $2n-2$ free ends. Similarly after second step $2n-4$ free ends regardless, etc.

E(number of rings at end of process) = $\frac{1}{2n-1}+\frac{1}{2n-3}+\frac{1}{2n-5}+\frac{1}{2n-7}+......+\frac{1}{1}$

Var(number of rings at end of process) =

$$\frac{1}{2n-1}-\left(\frac{1}{2n-1}\right)^2+\frac{1}{2n-3}-\left(\frac{1}{2n-3}\right)^2+\frac{1}{2n-5}-\left(\frac{1}{2n-5}\right)^2+\frac{1}{2n-7}-\left(\frac{1}{2n-7}\right)^2+......+\frac{1}{1}-\left(\frac{1}{1}\right)^2$$

(as numbers of rings created at each step are independent)

$$=\frac{2(n-1)}{(2n-1)^2}+\frac{2(n-2)}{(2n-3)^2}+\frac{2(n-3)}{(2n-5)^2}+......+\frac{2}{3^2}$$

For $n = 40000$, E(number of rings created) = $1+\frac{1}{3}+\frac{1}{5}+.....+\frac{1}{79999}$

$$=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+.....+\frac{1}{80000}-\left(\frac{1}{2}+\frac{1}{4}+.....+\frac{1}{80000}\right)$$
$$\approx\ln 80000-\frac{1}{2}\ln 40000$$
$$= 2\ln 20$$
$$\approx 6$$