



**Cambridge Assessment
Admissions Testing**

Sixth Term Examination Paper [STEP]

Mathematics 2 [9470]

2022

Examiners' Report

Mark Scheme

STEP MATHEMATICS 2

2022

Examiners' Report

STEP 2 Introduction

Candidates appeared to be generally well prepared for most topics within the examination, but there were a few situations in questions where some did not appear to be as proficient in standard techniques as needed. In particular, the method for finding invariant lines required in question 8 and the manipulation of trigonometric functions that were needed in question 10 caused considerable difficulties for some candidates.

An additional issue that occurred at numerous points in the paper relates to the direction in which a deduction is required. It is important that candidates make sure that they know which statement is the one that they should start from as they deduce the other and that it is clear in their solution that the logic has gone in the correct direction. Clarity of solution is also an issue that candidates should be aware of, especially in the situations where the result to be reached has been given. It is important to check that there are no special cases that need to be considered separately, and when dividing by a function it is necessary to confirm that the function cannot be equal to 0 (and in the case of inequalities that the function always has the same sign).

When drawing diagrams and sketching graphs it is useful if significant points that need to be clear are not drawn over the lines on the page as these can be difficult to interpret during the marking process.

Question 1

Most candidates attempted this question. While a few did not recognise that the process of integration by parts applied to one of the terms would then lead to the integral of the other term appearing in the answer in such a way that they could be combined, many candidates were able to show the result in the first part clearly. Many then realised that the second part would follow from a similar process but using two applications of integration by parts.

Part (iii)(a) proved relatively straightforward for many candidates and some who had struggled with the first two parts were able to successfully complete this section. Some candidates struggled to deduce the correct behaviour of the graph, in many cases assuming that the x-axis would be an asymptote on both sides of the graph.

Many realised that part (iii)(b) would follow from application of integration by parts and were able to follow through the process carefully to produce a clear deduction of the required value. In part (iii)(c), while most candidates were able to identify the equation that needed to be solved, many did not notice the link with part (iii)(a), which provided the easiest explanation of why two such integers could not exist. Attempts to justify through other arguments were not often sufficiently convincing to achieve the final mark.

Question 2

Many candidates who attempted this question struggled to achieve high marks as there were several points within this question where the reasoning required careful explanation.

Part (i) was often completed successfully by candidates who recognised that the relationship given could easily be rearranged to show the required result. Approaches which did not recognise this and tried to use the relationship expressed for different triples of terms were unable to make any significant progress. Candidates who attempted to argue that higher powers would not lead to a common difference were generally not successful in showing that the sequence has degree at most one.

The first section of part (ii) was generally well done by those who attempted it, and the relationship with the first part was often seen although in some cases candidates omitted to observe that p was not equal to zero. Candidates were also often able to deduce the formula for the sequence, either by substituting a general form or by looking at the differences between terms.

Many of the candidates who attempted part (iii) recognised that a similar approach to part (ii) would be likely to work. However, many assumed that the required coefficient of n^3 would be a , rather than using a variable so that the correct coefficient could be deduced at a later point. The algebra for this part was more complicated and some struggled to follow through the work accurately. Having reached the point where the correct value of k could be deduced it was then necessary to consider when the new sequence would be of the form in part (ii) and when it would be of the form in part (i) and the analysis of these cases was not always completed fully.

Question 3

There were several good attempts to this question, particularly in the middle section. However, the induction required in the first part of the question caused difficulties for many candidates. In particular, many did not realise which variable it was necessary to perform the induction on.

Attempts to the second part of the question were much better and candidates were generally able to demonstrate a good level of skill in manipulating summations, including changing the variable over which the sum ranges. Unfortunately, some slips in the algebra were seen in several cases, but many candidates were able to reach the given result convincingly.

Many candidates recognised that the left-hand side of the previous result must be equal to zero and were therefore able to show the values of the sums successfully. Some care was needed with the explanation of how the second sum led to certainty about the first six digits of the expansion.

Candidates who had successfully completed part (iii) of the question were often then able to identify the correct approach for the final part. However, the justifications often failed to calculate the value of the error when only considering some of the terms of the sequence.

Question 4

Almost all candidates attempted this question, and many of these did so very successfully. The majority of attempts managed to make some good progress, especially in the first three parts.

Overall, the way in which each of the given modulus forms changed sign was generally well grasped, although explanations were often rather lacking in detail. The graphs to be drawn in parts (i) and (iii) were managed well overall, although in some cases the fact that horizontal lines were drawn along the horizontal lines of the answer booklet meant that the intention was not always clear on the scanned script.

In part (i) some candidates failed to give sufficient explanation to show that the single formula could be written as the given set of three equations. Although a small number of candidates omitted to sketch the graph, most were able to draw the correct three straight line segments.

Part (ii) was well attempted, and many candidates were able to identify that the new function was very closely related to the one in the first part of the question meaning that they were able to write down the correct formula. Those who wrote a general form were almost always able to follow through the process to reach the correct final answer.

The introduction of quadratic terms to the function did not cause too much difficulty for candidates and many were able to deduce the correct equations for the sections of the function in part (iii). While many correctly identified the shape of the two quadratic sections, in several cases the graphs presented were symmetric.

Part (iv) presented more of a challenge, but many candidates who wrote down a general form were able to work through to achieve the correct final answer. A few candidates were able to write down the answer, but in some cases their answer was not verified even though the question explicitly asked for verification that the formula is correct.

Question 5

Many candidates appear to have spent a little time on this question before deciding to concentrate on others, meaning that the marks in general for this question were very low.

In part (i) it was often unclear whether the candidate was using all of the conditions and in many cases the inequalities were written as strict when they should not have been.

In part (ii) most candidates were able to write down a correct formula for the area and part (b) was also generally answered well, although in some cases the candidates appeared to guess the best point, rather than deduce it from the previous part. The incentre was a common incorrect guess for at least one of the parts.

Candidates were generally most successful in part (iii)(a) with many convincing attempts seen. Fewer than half of the scripts progressed beyond this point, however. Those who did attempt part (iii)(b) managed to complete it quite well, although a common mistake was to simply square the answer from part (ii)(b) and claim that it all works out.

There were very few attempts at part (iv), mostly trying to use previous parts or to guess a point. For both of the last two parts, some candidates correctly identified the maximum or minimum, but did not convincingly show where it was attained.

Question 6

This was one of the better answered questions on the paper and most candidates produced substantial responses to the question. Part (i)(a) was completed well in general, with most candidates achieving good marks for this section. When considering the second family of curves, some candidates forgot to include the $\frac{dy}{dx}$ on the right-hand side following their differentiation with respect to x . In part (i)(b) many candidates failed to justify their division by $(x^2 - y^2)$ when considering the case where $x \neq y$. Most candidates were however able to show that the tangents do remain perpendicular even in the case where $x = y$.

In part (ii) many candidates failed to recognise that c needed to be eliminated from the differential equation in order to find the second family of curves. This mistake often led to the candidate only being able to achieve one of the marks available in this section.

Many candidates were able to make good progress on part (iii) of the question and many good solutions to this part of the question were seen.

Question 7

Only a small number of candidates attempted this question, and many of those struggled to achieve good marks.

In part (i)(a) many candidates failed to spot the useful way of writing w^5 and were unable to secure any marks. Many of those who achieved few marks overall were able to obtain a mark by writing down an expression for $f(w) - g(w)$. Of those who did score well, many lost a mark for not stating that $g(w) > 0$ when dividing through by it in their inequalities. While there were some successful alternative solutions to part (i)(c), many of those who did not mimic part (i)(b) and tried to deduce the result directly were unsuccessful.

In part (ii)(a) some candidates failed to observe that the coefficients of the polynomial were real when stating that the roots occur in complex conjugate pairs. The remainder of part (ii) was dealt with easily by most candidates, including many of those who had otherwise obtained few parts.

Question 8

There were a large number of attempts at this question, but high scores were rare. One of the major reasons for solutions losing marks was a lack of understanding that the case where one invariant line is the y-axis needs to be addressed separately (as is highlighted in part (i) of the question). Many candidates were, however, able to demonstrate excellent understanding of the geometry involved.

In part (i) many candidates set up a matrix equation to find invariant points rather than invariant lines. Additionally, a significant number of candidates ignored the fact that the invariant lines must go through the origin, meaning that the algebra was more complicated. The special case of one invariant line being the y-axis was often dealt with incorrectly, with many candidates believing that this meant that the other invariant line must be the x-axis.

There were many possible approaches to parts (ii) and (iii), the simplest of which involved the tangent addition formulae for part (ii) and the realisation that the two lines were reflections of one another in the line $y = x$ for part (iii). However, many candidates did not appreciate that the angle between the invariant lines is not necessarily the same as the angle between two vectors representing their directions. Additionally, many solutions did not check carefully that the quantities being used were well defined.

Those candidates who had scored well on parts (ii) and (iii) were often able to link them together and find a suitable matrix. A very small number of candidates produced the relevant calculations but failed to write down a matrix.

Question 9

Many of the attempts at this question failed to achieve good marks. Some candidates struggled to convert the text into a suitable diagram, while others were confused about whether to label the forces that other objects exerted on the plank or the forces exerted by the plank itself. These diagrams then often led to confusion about the directions of the forces in play.

In part (i) there were many attempts that used backwards logic – assuming that there was no frictional force and finding the value of x required for this to be true. There were also a number of candidates who assumed that the frictional force would always be equal to μN , rather than taking this as the maximum.

Part (ii) was often done poorly because it was often not clear that the solution was using the magnitudes of the forces when using $F = \mu N$.

In part (iii) candidates often failed to consider all possible cases for the value of x and it was common to see the first result shown in one of the cases and the second result shown in the other.

Solutions to part (iv) were more successful from the candidates who reached this point.

Question 10

While the first part of the question was answered well in general, candidates did not perform well on the question as a whole.

Many candidates only answered the first part of the question, although some misunderstood and attempted to consider the maximum height reached. Others assigned the sign to acceleration incorrectly in the vertical motion equation.

Many candidates also struggled with the trigonometric functions and failed to identify that trigonometric identities could be applied; this often meant that part (ii) was not possible. The lack of familiarity with trigonometric manipulations caused further marks to be lost later in the question also.

Question 11

Some candidates made good progress with this question. However, a large proportion of attempts did not make much progress and many were unable to score more than one or two marks.

Many candidates were able to correctly identify the way in which the required expression in part (i) should be constructed and they were often successful in putting these together correctly.

In part (ii) most candidates were able to consider the definition of α and demonstrate the inequality required, although some candidates only showed one side of the inequality. Candidate who recognised that differentiation of the expected net loss function was a useful approach were often able to make good progress towards finding the required results for the remainder of this question.

A small number of candidates attempted part (iii) of the question and many of these were able to show the first result. The explanations of the final deduction were often not convincing however.

Question 12

Part (i) of the question was generally well attempted, although some candidates opted to integrate by parts rather than noticing that the integral was simple once the brackets were expanded. A small number of candidates failed to simplify the mean fully.

Part (ii) was found to be difficult. Most candidates attempted to compute an expression for the median rather than comparing the cumulative density function of the mean to $\frac{1}{2}$. Those who did follow the intended approach were generally able to work through the algebra well. Many candidates were also confused about the direction of the logic in this question and instead showed the converse of the required result. In many cases the argument provided was reversible so many of the marks could still be awarded.

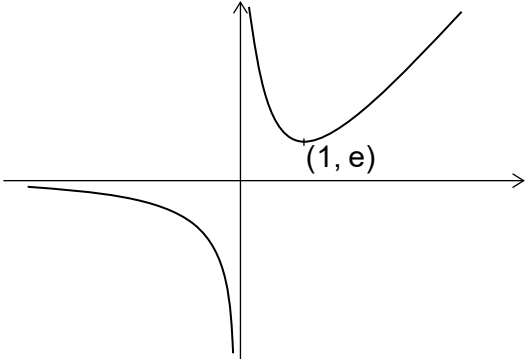
Expansion of the binomial expression was generally done well, but the algebra of the part that followed proved difficult, with most candidates either giving up early on or making mistakes that either rendered the conclusion trivial or impossible to obtain. Only a handful of candidates successfully reached the correct condition and convincingly showed it to be true.

Many candidates did not attempt part (iii). Those who had been successful in part (ii) almost always realised that they needed to consider the cumulative density function of the mode and compare it to $\frac{1}{2}$ and almost all these candidates managed to deduce the argument completely and gain full credit.

STEP MATHEMATICS 2

2022

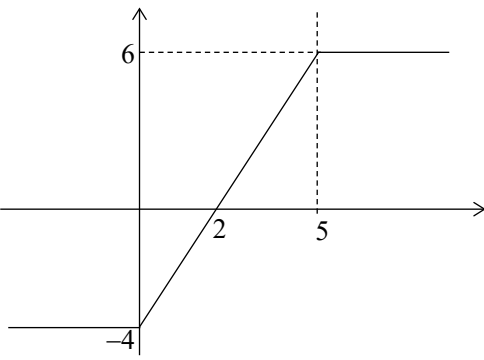
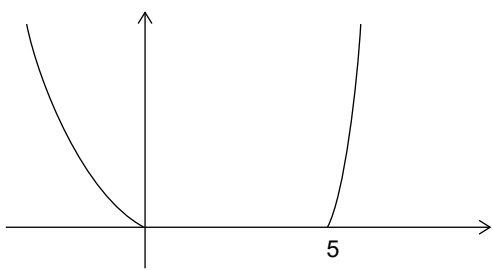
Mark Scheme

| Question | | | Answer | Mark |
|----------|-------|-----|--|------|
| 1 | (i) | | $\int \frac{3x^3}{\sqrt{1+x^3}} dx = u \cdot v - \int u'v dx$ | M1 |
| | | | $\int \frac{3x^3}{\sqrt{1+x^3}} dx = x \cdot k\sqrt{1+x^3} - \int k\sqrt{1+x^3} dx$ | M1 |
| | | | $\int \frac{3x^3}{\sqrt{1+x^3}} dx = x \cdot 2\sqrt{1+x^3} - \int 2\sqrt{1+x^3} dx$ | A1 |
| | | | so $\int 2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}} dx = 2x\sqrt{1+x^3} + c$ | A1 |
| | | | | [4] |
| | (ii) | | $\frac{(x^2 + 2)\sin x}{x^3} = \frac{\sin x}{x} + \frac{2\sin x}{x^3}$ | M1 |
| | | | $\int \frac{2\sin x}{x^3} dx = -\frac{p}{x^2} \cdot \sin x + \int \frac{q \cos x}{x^2} dx$ | M1 |
| | | | $= -\frac{p}{x^2} \cdot \sin x - \frac{r}{x} \cdot \cos x - \int \frac{s \sin x}{x} dx$ | M1 |
| | | | $-\frac{1}{x^2} \cdot \sin x - \frac{1}{x} \cdot \cos x - \int \frac{\sin x}{x} dx$ | |
| | | | so $\int (x^2 + 2) \frac{\sin x}{x^3} dx = -\frac{\sin x + x \cos x}{x^2} + c$ | A1 |
| | | | | [4] |
| | (iii) | (a) | $\frac{dy}{dx} = \frac{(x-1)e^x}{x^2}$ | M1 |
| | | | Therefore there is a stationary point at (1, e). | A1 |
| | | |  | |
| | | | Vertical asymptote at $x = 0$ | G1 |
| | | | Minimum in first quadrant and correct behaviour as $x \rightarrow \infty$ | G1 |
| | | | Correct behaviour as $x \rightarrow -\infty$ | G1 |
| | | | | [5] |

| | | | | |
|--|--|------------|---|------------|
| | | (b) | $\int_a^{2a} \frac{e^x}{x^2} dx = \left[-\frac{p}{x} \cdot e^x \right]_a^{2a} + \int_a^{2a} \frac{qe^x}{x} dx$ | M1 |
| | | | $\int_a^{2a} \frac{e^x}{x^2} dx = \left[-\frac{1}{x} \cdot e^x \right]_a^{2a} + \int_a^{2a} \frac{e^x}{x} dx$ | A1 |
| | | | Therefore for integrals to be equal we need $\left[-\frac{1}{x} \cdot e^x \right]_a^{2a} = 0$ | M1 |
| | | | $-\frac{1}{2a} \cdot e^{2a} + \frac{1}{a} \cdot e^a = 0$ $\frac{1}{2a} \cdot e^a (-e^a + 2) = 0$ | M1 |
| | | | so $a = \ln 2$ | A1 |
| | | | | [5] |
| | | (c) | As before, this means we would need $\left[-\frac{1}{x} \cdot e^x \right]_m^n$ i.e. $\frac{e^n}{n} = \frac{e^m}{m}$ | B1 |
| | | | From the graph in part (iii) (a) this would mean that the smaller of n, m must lie in the range $(0, 1)$. Hence this is not an integer. | E1 |
| | | | | [2] |

| Question | | | Answer | Mark |
|----------|-------|--|---|------|
| 2 | (i) | | $u_{n+2} - u_{n+1} = u_{n+1} - u_n$ | M1 |
| | | | so constant differences. | A1 |
| | | | If $u_n - u_{n-1} = d$, then $u_n = u_1 + (n-1)d$ which is of degree at most 1 | B1 |
| | | | | [3] |
| | (ii) | | $t_{n+1} + p(n+1)^2$ $= \frac{1}{2}(t_{n+2} + p(n+2)^2 + t_n + pn^2) - p$ | M1 |
| | | | so $t_{n+1} = \frac{1}{2}(t_{n+2} + t_n)$ | A1 |
| | | | so t_n has degree at most 1 | A1 |
| | | | Hence since $p \neq 0$, v_n has degree 2. | A1 |
| | | | Taking $v_n = pn^2 + qn + r$, gives: $p + q + r = 0$ $4p + 2q + r = 0$ | M1 |
| | | | so $q = -3p$ | A1 |
| | | | And $r = 2p$ | A1 |
| | | | | [7] |
| | (iii) | | Substitutes $w_n = t_n + kn^3$, so | B1 |
| | | | $t_{n+1} + k(n+1)^3$ $= \frac{1}{2}(t_{n+2} + k(n+2)^3 - t_n + kn^3) - an - b$ | M1 |
| | | | LHS and RHS both give $kn^3 + 3kn^2$ terms | A1 |
| | | | $t_{n+1} = \frac{1}{2}(t_{n+2} + t_n) + (3k - a)n - (b - 3k)$ | A1 |
| | | | Choosing $k = \frac{1}{3}a$ | A1 |
| | | | gives case (ii) (with $p = b - a$) so t_n has degree at most 2 and w_n has degree 3, as $a \neq 0$. | A1 |
| | | | unless $b = a$, when case (i) applies so t_n has degree at most 2 and w_n has degree 3, as $a \neq 0$. | A1 |
| | | | Taking $w_n = \frac{1}{3}an^3 + (b-a)n^2 + qn + r$ gives $b - \frac{2}{3}a + q + r = 0$ $-\frac{4}{3}a + 4b + 2q + r = 0$ | M1 |
| | | | so $q = \frac{2}{3}a - 3b$ | A1 |
| | | | and $r = 2b$ | A1 |
| | | | $w_n = \frac{1}{3}an^3 + (b-a)n^2 + \left(\frac{2}{3}a - 3b\right)n + 2b$ | |
| | | | | [10] |

| Question | | | Answer | Mark |
|----------|-------|--|--|------|
| 3 | (i) | | Base case: $F_n \leq 2^{n-n} F_n$ | B1 |
| | | | For $n \geq 1$, $F_{n-1} \leq F_n$, so if $r \geq n$ and $F_r \leq 2^{r-n} F_n$ | M1 |
| | | | $F_{r+1} \leq 2F_r \leq 2^{(r+1)-n} F_n$ | A1 |
| | | | Logical structure correct, with conclusion. | A1 |
| | | | | [4] |
| | (ii) | | $\sum_{r=1}^n \frac{F_{r+1}}{10^{r-1}} - \sum_{r=1}^n \frac{F_r}{10^{r-1}} - \sum_{r=1}^n \frac{F_{r-1}}{10^{r-1}}$ $= 100 \sum_{r=1}^n \frac{F_{r+1}}{10^{r+1}} - 10 \sum_{r=1}^n \frac{F_r}{10^r} - \sum_{r=1}^n \frac{F_{r-1}}{10^{r-1}}$ | M1 |
| | | | $100 \sum_{r=2}^{n+1} \frac{F_r}{10^r} - 10 \sum_{r=1}^n \frac{F_r}{10^r} - \sum_{r=0}^{n-1} \frac{F_r}{10^r}$ | M1 |
| | | | $= 100(S_n + \dots) - 10S_n - (S_n + \dots)$ | A1 |
| | | | $= 100 \left(\dots + \frac{F_{n+1}}{10^{n+1}} - \frac{F_1}{10} \right) - \left(\dots + F_0 - \frac{F_n}{10^n} \right)$ | A1 |
| | | | | [4] |
| | (iii) | | In (ii), the left hand side is equal to zero, so | M1 |
| | | | $89S_n = 10F_1 + F_0 - \frac{F_n}{10^n} - \frac{F_{n+1}}{10^{n-1}}$ | |
| | | | but $\frac{F_n}{10^n} + \frac{F_{n+1}}{10^{n-1}} \rightarrow 0$ as $n \rightarrow \infty$, from (i) | B1 |
| | | | and $F_0 = 0$, so $S_\infty = \frac{10}{89}$ | A1 |
| | | | $\sum_{r=7}^{\infty} \frac{F_r}{10^r} \leq \frac{F_7}{10^7} \sum_{r=0}^{\infty} \frac{2^r}{10^r} = \frac{13}{10^7 \left(1 - \frac{2}{10}\right)} < 2 \times 10^{-6}$ | M1 |
| | | | $\frac{1}{89} = \frac{1}{10} \left(\frac{1}{10} + \frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \frac{5}{10^5} + \frac{8}{10^6} + \sum_{r=7}^{\infty} \frac{F_r}{10^r} \right)$ $= 0.0112358 + \varepsilon$ | A1 |
| | | | with $\varepsilon < 2 \times 10^{-7}$, so the first six digits of the decimal expansion of $\frac{1}{89}$ are 0.011235 | A1 |
| | | | | [6] |
| | (iv) | | Let $T_n = \sum_{r=1}^n \frac{F_r}{100^r}$ | M1 |
| | | | then $0 = \sum_{r=1}^n \frac{F_{r+1}}{100^{r-1}} - \sum_{r=1}^n \frac{F_r}{100^{r-1}} - \sum_{r=1}^n \frac{F_{r-1}}{100^{r-1}}$ $= 10000 \left(T_n + \frac{F_{n+1}}{100^{n+1}} - \frac{1}{100} \right) - 100T_n - \left(T_n - \frac{F_n}{100^n} \right)$ | M1 |
| | | | $9899T_n = 100 - \frac{F_n}{100^n} - \frac{F_{n+1}}{100^{n-1}}$ | A1 |
| | | | so $T_\infty = \frac{100}{9899}$ and $\frac{1}{9899}$ is the required fraction | A1 |
| | | | as $\frac{1}{9899} = 0.0001010203050813213455 + \varepsilon$ | M1 |
| | | | where $\varepsilon \leq \frac{89}{1 - \frac{2}{100}} \times 10^{-24} < 10^{-22}$ | A1 |
| | | | | [6] |

| Question | | | Answer | Mark |
|----------|------|--|---|----------|
| 4 | (i) | | <p>For $x \leq 0$, $x = -x$ $x - 5 = -(x - 5)$</p> <p>For $0 \leq x \leq 5$ $x = x$ $x - 5 = -(x - 5)$</p> <p>For $5 \leq x$ $x = x$ $x - 5 = x - 5$</p> <p>For $x \leq 0$, $f(x) = -x - (-(x - 5)) + 1 = -4$</p> <p>For $0 \leq x \leq 5$ $f(x) = x - (-(x - 5)) + 1 = 2x - 4$</p> <p>For $5 \leq x$ $f(x) = x - (x - 5) + 1 = 6$</p> | M1 |
| | | |  | A1 |
| | | | | G1 G1 |
| | | | | [4] |
| | (ii) | | Writing $g(x) = a x + b x - 5 + c$ | M1 |
| | | | <p>For $x \leq 0$, $g(x) = -ax + b(-(x - 5)) + c$</p> <p>For $0 \leq x \leq 5$ $g(x) = ax + b(-(x - 5)) + c$</p> <p>For $5 \leq x$ $g(x) = ax + b(x - 5) + c$</p> | M1 |
| | | | Coefficients of x : $-a - b = -1$ $a - b = 3$ $a + b = 1$ | |
| | | | $a = 2, b = -1$ So $c = 5$ | |
| | | | $g(x) = 2 x - x - 5 + 5$ | A1 |
| | | | | [3] |
| | | |  | |
| | | | Convex quadratic shapes of appropriate gradient and without vertex in $(-\infty, 0]$, $[5, \infty)$ | G1 |
| | | | Horizontal section in $[0, 5]$, with discontinuous gradient at endpoints. | G1 |
| | | | Appropriate asymmetry of quadratic parts | G1 |
| | | | | [5] |
| | (iv) | | $k(x) = x^2 - x(x - 5) + \text{linear, constant terms}$ | M1 |
| | | | $k(x) - x^2 + x(x - 5) $ is: | M1 |
| | | | <p>$x \leq 0$: $10x - x^2 + x(x - 5) = 5x$</p> <p>$0 \leq x \leq 5$: $2x^2 - x^2 - x(x - 5) = 5x$</p> <p>$5 \leq x$: $50 - x^2 + x(x - 5) = 50 - 5x$</p> | A1 |
| | | | Set equal to $a + b x - 5 $ | M1 |

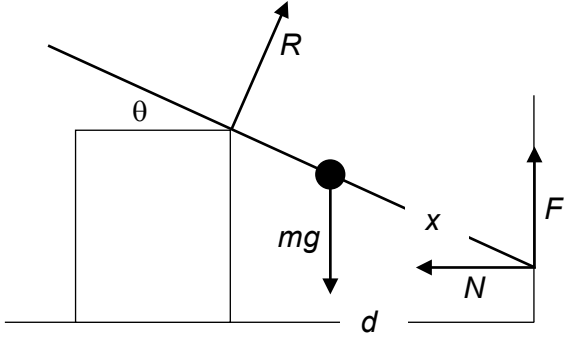
| | | | | |
|--|--|--|---|------------|
| | | | Determine by substitution necessary values of a and b | M1 |
| | | | $a = 25$ and $b = -5$ | A1 |
| | | | Verification that these are sufficient | A1 |
| | | | Thus $k(x) = x^2 - x(x - 5) + 25 - 5 x - 5 $ | A1 |
| | | | | [8] |

| Question | | | Answer | Mark |
|----------|-------|-----|--|------|
| 5 | (i) | | As z, y non-negative and $a > b, c$: $ay \geq by$ and $az \geq cz$ | B1 |
| | | | | [1] |
| | (ii) | (a) | $\Delta = \frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz$ | B1 |
| | | (b) | By (i), $(x + y + z) \geq \frac{2\Delta}{a}$ | M1 |
| | | | $\frac{2\Delta}{a}$ is the minimum value | A1 |
| | | | [as this lower bound is attained at] $\left(\frac{2\Delta}{a}, 0, 0\right)$. | A1 |
| | | | | [4] |
| | (iii) | (a) | Correct number of terms for expansions of any two of: $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$ $(bx - ay)^2 + (cy - bz)^2 + (az - cx)^2$ $(ax + by + cz)^2$ | M1 |
| | | | Fully correct expansions. | A1 |
| | | | Given result fully shown. | A1 |
| | | | | [3] |
| | (iii) | (b) | By (iii), $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$ $= (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2 + (2\Delta)^2$ | M1 |
| | | | so the minimum value of $x^2 + y^2 + z^2$ is $\frac{4\Delta^2}{a^2 + b^2 + c^2}$ | A1 |
| | | | This occurs when $bx = ay, cy = bz$ and $az = cx$ | M1 |
| | | | so when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \lambda$, say, where $\lambda > 0$. | M1 |
| | | | Then $\Delta = \frac{1}{2}a(a\lambda) + \frac{1}{2}b(b\lambda) + \frac{1}{2}c(c\lambda)$ | M1 |
| | | | so $\lambda = \frac{2\Delta}{a^2 + b^2 + c^2}$ | A1 |
| | | | minimum at $(a\lambda, b\lambda, c\lambda)$ with this value of λ . | A1 |
| | | | | [7] |
| | (iv) | | $(ax + by + cz)^2 \geq (cx + cy + cz)^2$ | M1 |
| | | | $= c^2(x + y + z)^2 \geq c^2(x^2 + y^2 + z^2)$ | M1 |
| | | | so $x^2 + y^2 + z^2 \leq \frac{4\Delta^2}{c^2}$ | M1 |
| | | | Maximum of $\frac{4\Delta^2}{c^2}$ | A1 |
| | | | at $\left(0, 0, \frac{2\Delta}{c}\right)$. | A1 |
| | | | | [5] |

| Question | | | Answer | Mark |
|----------|-------|-----|--|-----------|
| 6 | (i) | (a) | Differentiating implicitly with respect to x gives $2x + 2y \frac{dy}{dx} = 2a$ so, by substitution, $x^2 + y^2 = x \left(2x + 2y \frac{dy}{dx} \right)$ | B1 |
| | | | For second family: $2x + 2y \frac{dy}{dx} = 2b \frac{dy}{dx}$ | M1 |
| | | | so $y \left(2x + 2y \frac{dy}{dx} \right) = (x^2 + y^2) \frac{dy}{dx}$ | A1 |
| | | | $(x^2 - y^2) \frac{dy}{dx} = 2xy$ | A1 |
| | | | | [4] |
| | | (b) | The product of the gradients at points (x, y) where the curves meet is $\frac{y^2 - x^2}{2xy} \times \frac{2xy}{x^2 - y^2} = -1$, provided $x \neq y$, So the tangents to the curves at these points are perpendicular | B1 |
| | | | At (c, c) , for the first family of curves: $2c^2 \frac{dy}{dx} = 0$ and so $\frac{dy}{dx} = 0$. For the second family of curves: $2c^2 \frac{dx}{dy} = 0$ and so $\frac{dx}{dy} = \infty$. | M1 |
| | | | and the tangents to the circles $(x - c)^2 + y^2 = c^2$ and $(y - c)^2 + x^2 = c^2$ at this point are $y = c$ and $x = c$. | A1 |
| | | | which are indeed perpendicular. | A1 |
| | | | | [4] |
| | (ii) | | First family $\frac{dy}{dx} = \frac{c}{x}$ | M1 |
| | | | so $x \ln x \frac{dy}{dx} = y$ | A1 |
| | | | so orthogonal family has $y \frac{dy}{dx} = -x \ln x$ | A1 |
| | | | solving differential equation by separating variables | M1 |
| | | | $\int -x \ln x \, dx = -\frac{1}{2}x^2 \ln x - \int -\frac{1}{2}x^2 \cdot \frac{1}{x} \, dx$ | M1 |
| | | | $= -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2$ | A1 |
| | | | so $\frac{1}{2}y^2 = -\frac{1}{2}x^2 \ln x + \frac{1}{4}x^2 + c$ | A1 |
| | | | | [7] |
| | (iii) | | If two curves, with parameters k_1, k_2 meet, require $4k_1(x + k_1) = 4k_2(x + k_2)$ | M1 |
| | | | so $x = -(k_1 + k_2)$ | A1 |
| | | | $y^2 = -4k_1k_2$ | A1 |
| | | | for any curve, $2y \frac{dy}{dx} = 4k$ | M1 |
| | | | so the gradients of the two curves satisfy $\left. \frac{dy}{dx} \right _1 \cdot \left. \frac{dy}{dx} \right _2 = \frac{2k_1}{y} \cdot \frac{2k_2}{y} = -1$ | A1 CSO |
| | | | | [5] |

| Question | | | Answer | Mark |
|----------|------|-----|--|------|
| 7 | (i) | (a) | $w^5 = -\frac{w+n}{nw+1}$ | M1 |
| | | | $ w^5 = \left \frac{w+n}{nw+1} \right = \sqrt{\left(\frac{w+n}{nw+1} \right) \overline{\left(\frac{w+n}{nw+1} \right)}}$ | M1 |
| | | | $= \sqrt{\frac{w+n}{nw+1} \frac{\bar{w}+n}{n\bar{w}+1}} = \sqrt{\frac{w\bar{w}+n(w+\bar{w})+n^2}{n^2w\bar{w}+n(w+\bar{w})+1}}$ | A1 |
| | | | which gives the required result, as $w + \bar{w} = 2 \operatorname{Re}(w)$ | A1 |
| | | | | [4] |
| | | (b) | $f(w) - g(w) = (n^2 - 1)(1 - w ^2)$ | M1 |
| | | | and $n > 1$, so if $ w < 1$, $f(w) - g(w) > 0$ | A1 |
| | | | but since $f(w)$ and $g(w)$ are both positive (each is the square of the magnitude of a complex number) $f(w) > g(w) > 0$ | A1 |
| | | | so $\frac{f(w)}{g(w)} > 1$ and so $ w = \sqrt[10]{\left \frac{f(w)}{g(w)} \right } > 1$ # Hence $ w \geq 1$ | A1 |
| | | | | [4] |
| | | (c) | if $ w > 1$, $f(w) - g(w) < 0$ | M1 |
| | | | so $\frac{f(w)}{g(w)} < 1$ | A1 |
| | | | so $ w = \sqrt[10]{\left \frac{f(w)}{g(w)} \right } < 1$ #. Hence $ w \leq 1$ | A1 |
| | | | and, in combination with (b), this gives $ w = 1$ | A1 |
| | | | | [4] |
| | (ii) | (a) | Since the coefficients of $h(z)$ are real, but none of the roots is purely real, the six roots occur in conjugate pairs | B1 |
| | | | Suppose $p \pm iq$ are roots; then quadratic factor of $(z - p - iq)(z - p + iq) = (z^2 - 2pz + p^2 + q^2)$ with $2p$ real and $p^2 + q^2 = z ^2 = 1$ by (i)(c) | M1 |
| | | | Hence the algebraic factors are as stated, and the only remaining possibility is a numerical factor, which must be n by comparison of the z^6 term | A1 |
| | | | | [3] |
| | | (b) | $a_1 + a_2 + a_3$ is the sum of all six roots, so equal to $-\frac{1}{n}$ | B1 |
| | | | | [1] |
| | | (c) | The coefficient of z^3 in h is $-a_1a_2a_3 - 2a_1 - 2a_2 - 2a_3$ | M1 |
| | | | which must be zero so $a_1a_2a_3 = \frac{2}{n}$ | A1 |
| | | | | [2] |
| | | (d) | The sum of a_1, a_2, a_3 is negative, so they cannot all be positive, but their product is positive, so exactly two of them are negative hence exactly four roots of the equation have negative real part | B1 |
| | | | | [2] |

| Question | | | Answer | Mark |
|----------|-------|--|--|------|
| 8 | (i) | | If neither parallel to the y -axis, their gradients satisfy $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ m \end{pmatrix}$ with $\lambda \neq 0$ | M1 |
| | | | eliminating λ from $c + dm = \lambda m$, $a + bm = \lambda$ | M1 |
| | | | $\Leftrightarrow c + dm = m(a + bm)$ | A1 |
| | | | If $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is invariant, then $b = 0$ | M1 |
| | | | and the gradient of the other line satisfies $(a - d)m = c$ | A1 |
| | | | | [5] |
| | (ii) | | If $b \neq 0$, and the angle θ between the lines is 45° then $\cos^2 \theta = \frac{1}{2}$, so using the scalar product | B1 |
| | | | $\left(\begin{pmatrix} 1 \\ m_1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ m_2 \end{pmatrix} \right)^2 = \frac{1}{2}(1 + m_1^2)(1 + m_2^2)$ | B1 |
| | | | so $(1 + m_1 m_2)^2 + 4m_1 m_2 = (m_1 + m_2)^2$ | M1 |
| | | | so $\left(1 - \frac{c}{b}\right)^2 - 4\frac{c}{b} = \frac{(a-d)^2}{b^2}$ | A1 |
| | | | If $b = 0$, the condition is $\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} a-d \\ c \end{pmatrix} \right)^2 = \frac{1}{2}((a-d)^2 + c^2)$ | M1 |
| | | | so $c^2 = (a-d)^2$ as required | A1 |
| | | | | [6] |
| | (iii) | | If $b \neq 0$, the angles with $y = x$ are equal iff $\begin{pmatrix} 1 \\ m_1 \end{pmatrix}, \begin{pmatrix} 1 \\ m_2 \end{pmatrix}$ make equal angles with $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ | B1 |
| | | | so $\frac{\left(\begin{pmatrix} 1 \\ m_1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^2}{2(1+m_1^2)} = \frac{\left(\begin{pmatrix} 1 \\ m_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)^2}{2(1+m_2^2)}$ | M1 |
| | | | $(1 + m_2^2)(1 + m_1)^2 = (1 + m_1^2)(1 + m_2)^2$ so $(m_1 - m_2)(1 - m_1 m_2) = 0$ | A1 |
| | | | but $m_1 \neq m_2$ so requirement is $m_1 m_2 = 1$ | B1 |
| | | | which is $b + c = 0$ | A1 |
| | | | If $b = 0$, require $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ also invariant | M1 |
| | | | so $c = 0$, which is the same condition | A1 |
| | | | | [7] |
| | (iv) | | Require $c = -b$ and $(a - d)^2 = 8b^2$ | M1 |
| | | | so e.g. $\begin{pmatrix} 2\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \sqrt{2} & 1 \\ -1 & -\sqrt{2} \end{pmatrix}, \begin{pmatrix} \sqrt{2} & -2 \\ 2 & 5\sqrt{2} \end{pmatrix}$ etc | A1 |
| | | | | [2] |

| Question | | | Answer | Mark |
|----------|-------|--|--|------|
| 9 | | | | |
| | | |  | |
| | | | Diagram: correct location of plank, prism, wall | G1 |
| | | | and all forces | G1 |
| | | | For equilibrium: $F + R \cos \theta - mg = 0$ | B1 |
| | | | and $R \sin \theta - N = 0$ | B1 |
| | | | and $R \cdot d \sec \theta = mgx \cos \theta$ | B1 |
| | | | so $R = \frac{mgx \cos^2 \theta}{d}$, $N = \frac{mgx \sin \theta \cos^2 \theta}{d}$ | M1 |
| | | | $F = mg \left(1 - \frac{x \cos^3 \theta}{d} \right)$ | A1 |
| | | | | [7] |
| | (i) | | so if $x = d \sec^3 \theta$, $F = 0$ | B1 |
| | | | | [1] |
| | (ii) | | If $x > d \sec^3 \theta$, F is negative so necessary that | M1 |
| | | | $mg \left(\frac{x \cos^3 \theta}{d} - 1 \right) \leq \mu \frac{mgx \sin \theta \cos^2 \theta}{d}$ | |
| | | | $\mu \geq \frac{x \cos^3 \theta - d}{x \sin \theta \cos^2 \theta}$ | A1 |
| | | | If $x < d \sec^3 \theta$, F is positive so necessary that | M1 |
| | | | $mg \left(1 - \frac{x \cos^3 \theta}{d} \right) \leq \mu \frac{mgx \sin \theta \cos^2 \theta}{d}$ | |
| | | | so $\mu \geq \frac{d \sec^3 \theta - x}{x \tan \theta}$ | A1 |
| | | | | [4] |
| | (iii) | | If $\frac{x}{d} \geq \sec^3 \theta$ then $\frac{x}{d} \geq \frac{\sec^3 \theta}{1 + \mu \tan \theta}$ | B1 |
| | | | if $\frac{x}{d} < \sec^3 \theta$ require $\mu \geq \frac{d \sec^3 \theta - x}{x \tan \theta}$ | M1 |
| | | | so $\mu x \tan \theta + x \geq d \sec^3 \theta$ | A1 |
| | | | When $\mu < \cot \theta$, if $\frac{x}{d} \leq \sec^3 \theta$, then $\frac{x}{d} \leq \frac{\sec^3 \theta}{1 - \mu \tan \theta}$ | B1 |
| | | | if $\frac{x}{d} > \sec^3 \theta$ require $\mu \geq \frac{x - d \sec^3 \theta}{x \tan \theta}$ | M1 |
| | | | $d \sec^3 \theta \geq x - \mu x \tan \theta$ | A1 |
| | | | | [6] |
| | (iv) | | Now require $x < d \sec \theta$, so $\sec \theta > \frac{\sec^3 \theta}{1 + \mu \tan \theta}$, by the first inequality in (iii) | M1 |
| | | | so $\mu \tan \theta > \sec^2 \theta - 1 = \tan^2 \theta$ | A1 |
| | | | | [2] |

| Question | | | Answer | Mark |
|----------|-------|--|---|----------|
| 10 | (i) | | $h = ut \sin \alpha - \frac{1}{2}gt^2$ and $s = ut \cos \alpha$ | B1 |
| | | | so $h = \frac{us}{u \cos \alpha} \sin \alpha - \frac{1}{2}g \frac{s^2}{u^2 \cos^2 \alpha}$ or $t = \frac{s}{u \cos \alpha}$ | B1 |
| | | | | [2] |
| | (ii) | | require $y \tan \theta = \tan \alpha \sqrt{x^2 + y^2} - \frac{g}{2u^2}(x^2 + y^2)(1 + \tan^2 \alpha)$ | M1 |
| | | | or real solutions to $a \tan^2 \alpha - b \tan \alpha + c = 0$ with $a = \frac{g}{2u^2}(x^2 + y^2)$, $b = \sqrt{x^2 + y^2}$, $c = \frac{g}{2u^2}(x^2 + y^2) + y \tan \theta$ | A1 |
| | | | so $(x^2 + y^2) \geq 4 \frac{g}{2u^2}(x^2 + y^2) \left(\frac{g}{2u^2}(x^2 + y^2) + y \tan \theta \right)$ | M1 |
| | | | so $\frac{u^4}{g^2} \geq x^2 + y^2 + \frac{2yu^2}{g} \tan \theta$ | A1 |
| | | | $\frac{u^4}{g^2} + \frac{u^4}{g^2} \tan^2 \theta \geq x^2 + \left(y + \frac{u^2}{g} \tan \theta \right)^2$ | A1 |
| | | | | [5] |
| | (iii) | | If $x = 0$, the condition can be written as $\left(y + \frac{u^2 \tan \theta}{g} \right)^2 \leq \frac{u^4}{g^2} \sec^2 \theta \dots$ | M1 |
| | | | $-\frac{u^2}{g} \tan \theta \pm \frac{u^2}{g} \sec \theta$ | A1 A1 |
| | | | | |
| | | | distance up plane $d = y \sec \theta$ satisfies $d \leq \frac{u^2}{g} \sec \theta (\sec \theta - \tan \theta)$ | M1 |
| | | | so greatest d is $\frac{u^2}{g} \frac{1 - \sin \theta}{\cos^2 \theta} = \frac{u^2}{g} \frac{1 - \sin \theta}{1 - \sin^2 \theta}$ | A1 |
| | | | also, $d \geq -\frac{u^2(1 + \sin \theta)}{g \cos^2 \theta} = -\frac{u^2}{g(1 - \sin \theta)}$ so greatest distance down slope is $\frac{u^2}{g(1 - \sin \theta)}$ | A1 |
| | | | | [6] |
| | (iv) | | If $y = 0$, the condition can be written as $x^2 \leq \frac{u^4}{g^2}$ | M1 |
| | | | so the length of road is $\frac{2u^2}{g}$ | A1 |
| | | | If the gun is moved a distance r up the slope, the condition is derived by substituting $y - r \cos \theta$ for y | M1 |
| | | | so $x^2 + \left(y - r \cos \theta + \frac{u^2 \tan \theta}{g} \right)^2 \leq \frac{u^4}{g^2} (1 + \tan^2 \theta)$ | A1 |
| | | | so when $y = 0$, $x^2 \leq \frac{u^4}{g^2} (1 + \tan^2 \theta) - \left(\frac{u^2 \tan \theta}{g} - r \cos \theta \right)^2$ | M1 |
| | | | which is maximised by $r = \frac{u^2}{g} \tan \theta \sec \theta$ | A1 |
| | | | with length of road reached $\frac{2u^2}{g} \sec \theta$ | A1 |
| | | | | [7] |

| Question | | | Answer | Mark |
|----------|-------|--|---|------|
| 11 | (i) | | Expected net loss is $q^T(\dots)$ | M1 |
| | | | $(\dots(1 - q^{N-T}) - \dots q^{N-T})$ | M1 |
| | | | $= q^T(D(1 - q^{N-T}) - (N - T)q^{N-T})$ | A1 |
| | | | | [3] |
| | (ii) | | all variables non-negative and $N \geq T, D > 0$, so denominator positive so $\alpha \geq 0$. | B1 |
| | | | $N(N - T + D) - DT = (N + D)(N - T) > 0$, so $\alpha < 1$ | B1 |
| | | | $\frac{d}{dq}[\text{expected net loss}] = 0$ | M1 |
| | | | $TDq^{T-1} - N(N - T + D)q^{N-1} = 0$ | A1 |
| | | | $N(N - T + D)q^{T-1}(\alpha - q^{N-T}) = 0$ | A1 |
| | | | hence $q = \alpha^{\frac{1}{N-T}}$ determines exactly one value of q with $0 \leq q < 1$ for which the expected net loss is stationary | A1 |
| | | | $\frac{d^2}{dq^2}[\text{expected net loss}]$ $= T(T - 1)Dq^{T-2} - N(N - 1)(N - T + D)q^{N-2}$ | M1 |
| | | | at the root $= N(N - T + D)q^{T-2}((T - 1)\alpha - (N - 1)q^{N-T})$ | M1 |
| | | | $= N(N - T + D)q^{N-2}((T - 1) - (N - 1))$ at the root | M1 |
| | | | but $-N(N - T)(N - T + D)q^{N-2} < 0$, so maximum | A1 |
| | | | The maximum net loss is $q^T(D - \alpha(N - T + D))$ | M1 |
| | | | $= \frac{Dq^T}{N}(N - T)$ but $q^T = (q^{N-T})^{\frac{T}{N-T}} = \alpha^k$ | A1 |
| | | | | [12] |
| | (iii) | | The expected loss is an increasing function of T if the expected net loss with one extra stick tested is larger than that without the extra stick | M1 |
| | | | so when $[Dq^{T+1} - q^N(N - (T + 1) + D)] - [Dq^T - q^N(N - T + D)]$ $= q^T(q^{N-T} - Dp) > 0$ for all T | A1 |
| | | | which is the case if $q^{N-T} > Dp$ | A1 |
| | | | As p tends to zero, the left hand side of this expression tends to 1, and the right hand side to 0 hence there exists $\beta > 0$ such that, for all $p < \beta$, the expected net loss is an increasing function of T | E1 |
| | | | Thus for $p < \beta$, testing no sticks minimises the expected net loss. | E1 |
| | | | | [5] |

| Question | | | Answer | Mark |
|----------|-------|--|--|------|
| 12 | (i) | | $\int_0^1 kx^n(1-x) dx = \left[k \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right) \right]_0^1 = \frac{k}{(n+1)(n+2)}$ | B1 |
| | | | $\mu = \int_0^1 kx^{n+1}(1-x) dx = \left[k \left(\frac{x^{n+2}}{n+2} - \frac{x^{n+3}}{n+3} \right) \right]_0^1$ | M1 |
| | | | $= \frac{n+1}{n+3}$ | A1 |
| | | | | [3] |
| | (ii) | | μ less than the median if $\int_0^\mu kx^n(1-x) dx < \frac{1}{2}$ | M1 |
| | | | so if $k \left(\frac{\mu^{n+1}}{n+1} - \frac{\mu^{n+2}}{n+2} \right) < \frac{1}{2}$ | A1 |
| | | | $2 \left((n+2) - \frac{(n+1)^2}{n+3} \right) < \left(\frac{n+3}{n+1} \right)^{n+1}$ | A1 |
| | | | $(n+2) - \frac{(n+1)^2}{n+3} = \frac{3n+5}{n+3} = 3 - \frac{4}{n+3}$ and $\frac{n+3}{n+1} = 1 + \frac{2}{n+1}$ | A1 |
| | | | [The terms of the expansion are all positive, so the inequality holds if] $1 + (n+1) \frac{2}{n+1} + \frac{(n+1)n}{2} \left(\frac{2}{n+1} \right)^2$ $+ \frac{(n+1)n(n-1)}{6} \left(\frac{2}{n+1} \right)^3 > 6 - \frac{8}{n+3}$ | M1 |
| | | | expansion gives $1 + 2 + \frac{2n}{n+1} + \frac{4n(n-1)}{3(n+1)^2}$ | A1 |
| | | | [so if] $6n(n+1)(n+3) + 4n(n-1)(n+3)$ $> 9(n+3)(n+1)^2 - 24(n+1)^2$ | M1 |
| | | | $2n(n+3)(5n+1) > 3(3n+1)(n^2 + 2n + 1)$ | A1 |
| | | | $n^3 + 11n^2 - 9n - 3 > 0$ | A1 |
| | | | [so if] $(n-1)(n^2 + 12n + 3) > 0$ | A1 |
| | | | which is certainly the case if $n > 1$ | A1 |
| | | | | [11] |
| | (iii) | | The mode m satisfies $f'(m) = k(nm^{n-1} - (n+1)m^n) = 0$ | M1 |
| | | | so $m = \frac{n}{n+1}$ | A1 |
| | | | $\int_0^m kx^n(1-x) dx = \left[k \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right) \right]_0^m$ | M1 |
| | | | $= 2 \left(\frac{n}{n+1} \right)^{n+1}$ | A1 |
| | | | but the given result implies | M1 |