STEP I 2011 Solutions

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Question 1

> *(i)* *Show that the gradient of the curve* $\dfrac{a}{x} + \dfrac{b}{y} = 1$*, where* $b \neq 0$*, is* $-\dfrac{ay^2}{bx^2}$.

We begin by differentiating the equation of the curve ($ax^{-1} + by^{-1} = 1$) implicitly with respect to x, to get

$$-ax^{-2} - by^{-2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

so that

$$-\frac{b}{y^2}\,\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x^2},$$

giving our desired result

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ay^2}{bx^2}.$$

An alternative, but more complicated method, is to rearrange the equation first to get y in terms of x before differentiating. We have, on multiplying by xy,

$$ay + bx = xy, \tag{1}$$

so that $(x - a)y = bx$, which gives

$$y = \frac{bx}{x-a}.$$

We can now differentiate this using the quotient rule to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{b(x-a) - bx.1}{(x-a)^2} = \frac{-ab}{(x-a)^2}.$$

The challenge is now to rewrite this in the form required. We can rearrange equation (1) to get $(x - a)y = bx$, so that $(x - a) = bx/y$. Substituting this into our expression for the derivative then gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{ab}{(bx/y)^2} = -\frac{aby^2}{b^2x^2} = -\frac{ay^2}{bx^2}$$

as required.

> *The point (p, q) lies on both the straight line $ax + by = 1$ and the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $ab \neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p = \pm q$.*

Rearranging the equation of the straight line $ax + by = 1$ as $y = -(\frac{a}{b})x + \frac{1}{b}$ shows that its gradient is $-a/b$.

Then using the above result for the gradient of the curve, we require that

$$-\frac{aq^2}{bp^2} = -\frac{a}{b},$$

so $q^2/p^2 = 1$, that is $p^2 = q^2$ or $p = \pm q$.

> *Show further that either $(a - b)^2 = 1$ or $(a + b)^2 = 1$.*

Since (p, q) lies on both the straight line and the curve, it must satisfy both equations, so

$$ap + bq = 1 \qquad \text{and} \qquad \frac{a}{p} + \frac{b}{q} = 1.$$

Now if $p = q$, then the first equation gives $(a + b)p = 1$ and the second gives $(a + b)/p = 1$, and multiplying these gives $(a + b)^2 = 1$.

Alternatively, if $p = -q$, then the first equation gives $(a - b)p = 1$ and the second equation gives $(a - b)/p = 1$, and multiplying these now gives $(a - b)^2 = 1$.

> *(ii) Show that if the straight line $ax + by = 1$, where $ab \neq 0$, is a normal to the curve $\dfrac{a}{x} - \dfrac{b}{y} = 1$, then $a^2 - b^2 = \frac{1}{2}$.*

We can find the derivative of this curve as above. A slick alternative is to notice that this is identical to the above curve, but with b replaced by $-b$, so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{ay^2}{bx^2}.$$

The gradient of the straight line is $-a/b$ as before, so as this line is normal to the curve at the point (p, q), say, we have

$$\frac{aq^2}{bp^2}\left(-\frac{a}{b}\right) = -1$$

as perpendicular gradients multiply to -1; thus $a^2q^2/b^2p^2 = 1$, or $a^2q^2 = b^2p^2$.

We therefore deduce that $aq = \pm bp$, which we can divide by $pq \neq 0$ to get $\frac{a}{p} = \pm\frac{b}{q}$.

Now since (p, q) lies on both the straight line and the curve, we have, as before,

$$ap + bq = 1 \qquad \text{and} \qquad \frac{a}{p} - \frac{b}{q} = 1.$$

Now if $\frac{a}{p} = \frac{b}{q}$, the second equation would become $0 = 1$, which is impossible. So we must have $\frac{a}{p} = -\frac{b}{q}$, giving

$$\frac{a}{p} + \frac{a}{p} = 1,$$

so that $\frac{a}{p} = -\frac{b}{q} = \frac{1}{2}$, giving $p = 2a$ and $q = -2b$.

Substituting this into the equation of the straight line yields

$$a.2a + b.(-2b) = 1,$$

so that $a^2 - b^2 = \frac{1}{2}$ as required.

Question 2

The number E is defined by $E = \displaystyle\int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x\,.$

Show that

$$\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x = \mathrm{e} - 1 - E,$$

and evaluate $\displaystyle\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x$ *in terms of* e *and* E.

Approach 1: Using polynomial division

Using polynomial division or similar, we find that we can write

$$\frac{x}{1+x} = 1 - \frac{1}{1+x}.$$

Therefore our first integral becomes

$$\begin{aligned}
\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x &= \int_0^1 \Big(1 - \frac{1}{1+x}\Big)\mathrm{e}^x\,\mathrm{d}x \\
&= \int_0^1 \mathrm{e}^x\,\mathrm{d}x - \int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x \\
&= \big[\mathrm{e}^x\big]_0^1 - E \\
&= \mathrm{e} - 1 - E,
\end{aligned}$$

as required.

We can play the same trick with the second integral, as

$$\frac{x^2}{1+x} = x - 1 + \frac{1}{1+x},$$

so that

$$\begin{aligned}
\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x &= \int_0^1 \Big(x - 1 + \frac{1}{1+x}\Big)\mathrm{e}^x\,\mathrm{d}x \\
&= \int_0^1 x\mathrm{e}^x\,\mathrm{d}x - \int_0^1 \mathrm{e}^x\,\mathrm{d}x + \int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x.
\end{aligned}$$

Now we can use integration by parts for the first integral to get

$$\begin{aligned}
\int_0^1 x\mathrm{e}^x\,\mathrm{d}x &= \big[x\mathrm{e}^x\big]_0^1 - \int_0^1 \mathrm{e}^x\,\mathrm{d}x \\
&= \mathrm{e} - (\mathrm{e} - 1) \\
&= 1.
\end{aligned}$$

Therefore

$$\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x = 1 - (\mathrm{e} - 1) + E = 2 - \mathrm{e} + E.$$

Approach 2: Substitution

We can substitute $u = 1 + x$ to simplify the denominator in the integral. This gives us

$$\begin{aligned}\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x &= \int_1^2 \frac{(u-1)\mathrm{e}^{u-1}}{u}\,\mathrm{d}u \\ &= \int_1^2 \mathrm{e}^{u-1} - \frac{\mathrm{e}^{u-1}}{u}\,\mathrm{d}u.\end{aligned}$$

The first part of this integral can be easily dealt with. The second part needs the reverse substitution to be applied, replacing u by $1 + x$, giving

$$\left[\mathrm{e}^{u-1}\right]_1^2 - \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x = \mathrm{e} - 1 - E.$$

This is essentially identical to the first approach. The second integral follows in the same way.

Approach 3: Integration by parts

Integration by parts is trickier for this integral, as it is not obvious how to break up our integral. We use the parts formula as written in the formula booklet: $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$.

There are several ways which work (and many which do not). Here is a relatively straightforward approach. For the first integral, we take

$$u = \frac{x}{1+x} \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$$

so that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{(1+x)^2} \qquad \text{and} \qquad v = \mathrm{e}^x.$$

We then get

$$\begin{aligned}\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x &= \left[\frac{x\mathrm{e}^x}{1+x}\right]_0^1 - \int_0^1 \frac{\mathrm{e}^x}{(1+x)^2}\,\mathrm{d}x \\ &= \tfrac{1}{2}\mathrm{e} - \int_0^1 \frac{\mathrm{e}^x}{(1+x)^2}\,\mathrm{d}x.\end{aligned}$$

The difficulty is now integrating the remaining integral. We again use parts, this time taking

$$u = \mathrm{e}^x \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{(1+x)^2}$$

so that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x \qquad \text{and} \qquad v = -\frac{1}{1+x}.$$

This gives

$$\begin{aligned}\int_0^1 \frac{\mathrm{e}^x}{(1+x)^2}\,\mathrm{d}x &= \left[-\frac{\mathrm{e}^x}{1+x}\right]_0^1 - \int_0^1 -\frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x \\ &= -\tfrac{1}{2}\mathrm{e} + 1 + E.\end{aligned}$$

Combining this result with the first result then gives

$$\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x = \tfrac{1}{2}\mathrm{e} - (-\tfrac{1}{2}\mathrm{e} + 1 + E) = \mathrm{e} - 1 - E.$$

For the second integral, we use a similar procedure, this time taking

$$u = \frac{x^2}{1+x} \qquad \text{and} \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$$

so that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{2x+x^2}{(1+x)^2} \qquad \text{and} \qquad v = \mathrm{e}^x.$$

We then get

$$\begin{aligned}\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x &= \left[\frac{x^2\mathrm{e}^x}{1+x}\right]_0^1 - \int_0^1 \frac{(2x+x^2)\mathrm{e}^x}{(1+x)^2}\,\mathrm{d}x \\ &= \tfrac{1}{2}\mathrm{e} - \int_0^1 \frac{(2x+x^2)\mathrm{e}^x}{(1+x)^2}\,\mathrm{d}x.\end{aligned}$$

The integral in the last step can be handled in several ways; the easiest is to write

$$\frac{2x+x^2}{(1+x)^2} = \frac{x^2+2x+1-1}{(1+x)^2} = 1 - \frac{1}{(1+x)^2}$$

and then use the earlier calculation of $\int \mathrm{e}^x/(1+x)^2\,\mathrm{d}x$ to get

$$\begin{aligned}\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x &== \tfrac{1}{2}\mathrm{e} - \left(\int_0^1 \mathrm{e}^x - \frac{\mathrm{e}^x}{(1+x)^2}\,\mathrm{d}x\right) \\ &= \tfrac{1}{2}\mathrm{e} - [\mathrm{e}^x]_0^1 + (-\tfrac{1}{2}\mathrm{e} + 1 + E) \\ &= -\mathrm{e} + 1 + 1 + E \\ &= 2 - \mathrm{e} + E.\end{aligned}$$

Evaluate also, in terms of E and e *as appropriate:*

(i) $\displaystyle\int_0^1 \frac{\mathrm{e}^{\frac{1-x}{1+x}}}{1+x}\,\mathrm{d}x\,;$

This integral looks to be of a vaguely similar form, but with a more complicated exponential part. We therefore try the substitution $u = \frac{1-x}{1+x}$ and see what we get.

If $u = \dfrac{1-x}{1+x}$, then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-(1+x)-(1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2},$$

so that $\frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{2}(1+x)^2$. Also, when $x=0$, $u=1$, and when $x=1$, $u=0$.

We can also rearrange $u = \dfrac{1-x}{1+x}$ to get

$$\begin{aligned} && (1+x)u &= 1-x \\ \text{so} && ux + x &= 1-u \\ \text{giving} && x &= \frac{1-u}{1+u}. \end{aligned}$$

Thus

$$\begin{aligned} \int_0^1 \frac{\mathrm{e}^{\frac{1-x}{1+x}}}{1+x}\,\mathrm{d}x &= \int_1^0 \frac{\mathrm{e}^u}{1+x}\bigl(-\tfrac{1}{2}(1+x)^2\bigr)\,\mathrm{d}u \\ &= \int_0^1 \tfrac{1}{2}\mathrm{e}^u(1+x)\,\mathrm{d}u && \text{reversing the limits} \\ &= \int_0^1 \tfrac{1}{2}\mathrm{e}^u\left(1+\frac{1-u}{1+u}\right)\mathrm{d}u \\ &= \int_0^1 \tfrac{1}{2}\mathrm{e}^u\left(\frac{2}{1+u}\right)\mathrm{d}u \\ &= \int_0^1 \frac{\mathrm{e}^u}{1+u}\,\mathrm{d}u \\ &= E. \end{aligned}$$

Evaluate also in terms of E and e as appropriate:

(ii) $\displaystyle\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x$

Again we have a different exponent, so we try substituting $u = x^2$, so that $x = \sqrt{u}$, while $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and so $\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2x}$. Also the limits $x = 1$ and $x = \sqrt{2}$ become $u = 1$ and $u = 2$, giving us

$$\begin{aligned}\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x &= \int_1^2 \frac{\mathrm{e}^u}{x}\,\frac{1}{2x}\,\mathrm{d}u \\ &= \int_1^2 \frac{\mathrm{e}^u}{2u}\,\mathrm{d}u.\end{aligned}$$

This is very similar to what we are looking for, except that it has the wrong limits and a denominator of $2u$ rather than $u+1$ or perhaps $2(u+1)$. So we make a further substitution: $u = v + 1$, so that $v = u - 1$ and $\mathrm{d}u/\mathrm{d}v = 1$, giving us

$$\begin{aligned}\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x &= \int_1^2 \frac{\mathrm{e}^u}{2u}\,\mathrm{d}u \\ &= \int_0^1 \frac{\mathrm{e}^{v+1}}{2(v+1)}\,\mathrm{d}v \\ &= \frac{\mathrm{e}}{2}\int_0^1 \frac{\mathrm{e}^v}{v+1}\,\mathrm{d}v \\ &= \frac{\mathrm{e}E}{2},\end{aligned}$$

where on the penultimate line we have written $\mathrm{e}^{v+1} = \mathrm{e}.\mathrm{e}^v$ and so taken out a factor of $\mathrm{e}/2$.

It is also possible to evaluate this integral more directly by substituting $u = x^2 - 1$, so that $x^2 = u + 1$. The details are left to the reader.

Question 3

> *Prove the identity*
> $$4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)=\sin 3\theta\,. \tag{$*$}$$

We make use of two of the factor formulæ:

$$\begin{aligned}2\sin A\sin B &= \cos(A-B)-\cos(A+B)\\ 2\sin A\cos B &= \sin(A+B)+\sin(A-B)\end{aligned}$$

(These can be derived by expanding the right hand sides using the addition formulæ, and then collecting like terms.)

Then initially taking $A=\frac{1}{3}\pi-\theta$ and $B=\frac{1}{3}\pi+\theta$ and using the first of the factor formulæ gives

$$\begin{aligned}4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta) &= 2\sin\theta\bigl(\cos(-2\theta)-\cos(\tfrac{2}{3}\pi)\bigr)\\ &= 2\sin\theta(\cos 2\theta+\tfrac{1}{2})\\ &= 2\sin\theta\cos 2\theta+\sin\theta.\end{aligned}$$

We now use the second factor formula with $A=\theta$ and $B=2\theta$ to simplify this last expression to

$$\bigl(\sin 3\theta+\sin(-\theta)\bigr)+\sin\theta=\sin 3\theta,$$

as required.

An alternative approach is to expand the second and third terms on the left hand side using the addition formulæ, giving:

$$\begin{aligned}&4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)\\ &\quad= 4\sin\theta(\sin\tfrac{1}{3}\pi\cos\theta-\cos\tfrac{1}{3}\pi\sin\theta)(\sin\tfrac{1}{3}\pi\cos\theta+\cos\tfrac{1}{3}\pi\sin\theta)\\ &\quad= 4\sin\theta\bigl(\tfrac{\sqrt{3}}{2}\cos\theta-\tfrac{1}{2}\sin\theta\bigr)\bigl(\tfrac{\sqrt{3}}{2}\cos\theta+\tfrac{1}{2}\sin\theta\bigr)\\ &\quad= 4\sin\theta(\tfrac{3}{4}\cos^2\theta-\tfrac{1}{4}\sin^2\theta)\\ &\quad= 3\sin\theta\cos^2\theta-\sin^3\theta,\end{aligned}$$

while

$$\begin{aligned}\sin 3\theta &= \sin(2\theta+\theta)\\ &= \sin 2\theta\cos\theta+\cos 2\theta\sin\theta\\ &= 2\sin\theta\cos^2\theta+(\cos^2\theta-\sin^2\theta)\sin\theta\\ &= 3\sin\theta\cos^2\theta-\sin^3\theta.\end{aligned}$$

Thus the required identity holds.

> *(i)* *By differentiating* (∗), *or otherwise, show that*
> $$\cot \tfrac{1}{9}\pi - \cot \tfrac{2}{9}\pi + \cot \tfrac{4}{9}\pi = \sqrt{3}\,.$$

We can differentiate a product of several terms using the product rule repeatedly. In general, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(uvwt\dots) = \frac{\mathrm{d}u}{\mathrm{d}x}vwt\dots + u\frac{\mathrm{d}v}{\mathrm{d}x}wt\dots + uv\frac{\mathrm{d}w}{\mathrm{d}x}t\dots + \cdots$$

In our case, we are differentiating a product of three terms, and we get

$$4\cos\theta\sin(\tfrac{1}{3}\pi - \theta)\sin(\tfrac{1}{3}\pi + \theta) - 4\sin\theta\cos(\tfrac{1}{3}\pi - \theta)\sin(\tfrac{1}{3}\pi + \theta) + 4\sin\theta\sin(\tfrac{1}{3}\pi - \theta)\cos(\tfrac{1}{3}\pi + \theta) = 3\cos 3\theta.$$

Now we are aiming to get an expressing involving cot, so we divide this result by (∗) to get

$$\cot\theta - \cot(\tfrac{1}{3}\pi - \theta) + \cot(\tfrac{1}{3}\pi + \theta) = 3\cot 3\theta.$$

We now let $\theta = \frac{1}{9}\pi$ to get

$$\cot\tfrac{1}{9}\pi - \cot\tfrac{2}{9}\pi + \cot\tfrac{4}{9}\pi = 3\cot\tfrac{1}{3}\pi = 3/\sqrt{3} = \sqrt{3},$$

and we are done.

> *(ii)* *By setting* $\theta = \frac{1}{6}\pi - \phi$ *in* (∗), *or otherwise, obtain a similar identity for* $\cos 3\theta$ *and deduce that*
> $$\cot\theta\cot(\tfrac{1}{3}\pi - \theta)\cot(\tfrac{1}{3}\pi + \theta) = \cot 3\theta\,.$$

Setting $\theta = \frac{1}{6}\pi - \phi$ in (∗) as instructed gives

$$4\sin(\tfrac{1}{6}\pi - \phi)\sin(\tfrac{1}{6}\pi + \phi)\sin(\tfrac{1}{2}\pi - \phi) = \sin 3(\tfrac{1}{6}\pi - \phi).$$

To get cosines from this expression, we will need to use the identity $\sin(\frac{1}{2}\pi - x) = \cos x$. So we rewrite this as

$$4\sin\bigl(\tfrac{1}{2}\pi - (\tfrac{1}{3}\pi + \phi)\bigr)\sin\bigl(\tfrac{1}{2}\pi - (\tfrac{1}{3}\pi - \phi)\bigr)\sin(\tfrac{1}{2}\pi - \phi) = \sin(\tfrac{1}{2}\pi - 3\phi)$$

which allows us to apply our identity to get

$$4\cos(\tfrac{1}{3}\pi + \phi)\cos(\tfrac{1}{3}\pi - \phi)\cos\phi = \cos 3\phi,$$

which is a similar identity for $\cos 3\phi$. Replacing ϕ by θ and reordering the terms in the product gives

$$4\cos\theta\cos(\tfrac{1}{3}\pi - \theta)\cos(\tfrac{1}{3}\pi + \theta) = \cos 3\theta.$$

Now dividing this identity by (∗) gives our desired identity for cot:

$$\cot\theta\cot(\tfrac{1}{3}\pi - \theta)\cot(\tfrac{1}{3}\pi + \theta) = \cot 3\theta. \qquad (\dagger)$$

(Note that there is no factor of 4 in this expression.)

Show that

$$\operatorname{cosec} \tfrac{1}{9}\pi - \operatorname{cosec} \tfrac{5}{9}\pi + \operatorname{cosec} \tfrac{7}{9}\pi = 2\sqrt{3}\,.$$

As before, we differentiate the expression (†) which we have just derived to get

$$\begin{aligned} -\operatorname{cosec}^2\theta \cot(\tfrac{1}{3}\pi - \theta)\cot(\tfrac{1}{3}\pi + \theta) + \\ \cot\theta \operatorname{cosec}^2(\tfrac{1}{3}\pi - \theta)\cot(\tfrac{1}{3}\pi + \theta) - \\ \cot\theta\cot(\tfrac{1}{3}\pi - \theta)\operatorname{cosec}^2(\tfrac{1}{3}\pi + \theta) = -3\operatorname{cosec}^2 3\theta. \end{aligned}$$

When we negate this identity and then divide it by (†), we will have lots of cancellation and we will be left with terms of the form $\operatorname{cosec}^2 x/\cot x$. Now

$$\frac{\operatorname{cosec}^2 x}{\cot x} = \frac{1}{\sin^2 x}.\frac{\sin x}{\cos x} = \frac{1}{\sin x\cos x} = \frac{2}{\sin 2x} = 2\operatorname{cosec} 2x,$$

so that the division gives us

$$2\operatorname{cosec} 2\theta - 2\operatorname{cosec} 2(\tfrac{1}{3}\pi - \theta) + 2\operatorname{cosec} 2(\tfrac{1}{3}\pi + \theta) = 6\operatorname{cosec} 6\theta.$$

To get the requested equality, we halve this identity and set $2\theta = \frac{1}{9}\pi$ so that

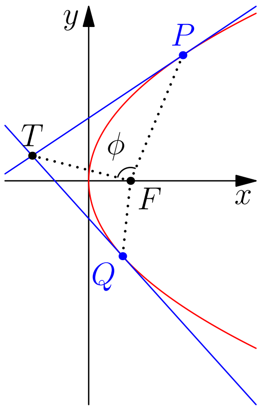
$$\operatorname{cosec} \tfrac{1}{9}\pi - \operatorname{cosec} \tfrac{5}{9}\pi + \operatorname{cosec} \tfrac{7}{9}\pi = 3\operatorname{cosec} \tfrac{1}{3}\pi = 3.\tfrac{2}{\sqrt{3}} = 2\sqrt{3}$$

as required.

Question 4

> *The distinct points P and Q, with coordinates $(ap^2, 2ap)$ and $(aq^2, 2aq)$ respectively, lie on the curve $y^2 = 4ax$. The tangents to the curve at P and Q meet at the point T. Show that T has coordinates $\big(apq, a(p+q)\big)$. You may assume that $p \neq 0$ and $q \neq 0$.*

We begin by sketching the graph (though this may be helpful, it is not required):



The equation of the curve is $y^2 = 4ax$, so we can find the gradient of the curve by implicit differentiation:

$$2y\,\frac{\mathrm{d}y}{\mathrm{d}x} = 4a,$$

and thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y},$$

as long as $y \neq 0$. (Alternatively, we could write $x = y^2/4a$ and then work out $\mathrm{d}x/\mathrm{d}y = 2y/4a$; taking reciprocals then gives us the same result.)

Therefore the tangent at the point P with coordinates $(ap^2, 2ap)$ has equation

$$y - 2ap = \frac{2a}{2ap}(x - ap^2),$$

which can easily be rearranged to give

$$x - py + ap^2 = 0.$$

Since $y = 0$ would require $p = 0$, we can ignore this case, as we are assuming that $p \neq 0$. [In fact, if $y = p = 0$, we can look at the reciprocal of the gradient, $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y}{2a}$, and this is zero, so the line is vertical. In this case, our equation gives $x = 0$, which is, indeed, a vertical line, so our equation works even when $p = 0$.]

Thus the tangent through P has equation $x - py + ap^2 = 0$ and the tangent through Q has equation $x - qy + aq^2 = 0$ likewise.

We solve these equations simultaneously to find the coordinates of T. Subtracting them gives

$$(p - q)y - a(p^2 - q^2) = 0.$$

Since $p \neq q$, we can divide by $p - q$ to get

$$y - a(p+q) = 0,$$

so $y = a(p+q)$ and therefore $x = py - ap^2 = ap(p+q) - ap^2 = apq$.

Thus T has coordinates $\big(apq, a(p+q)\big)$, as wanted.

> *The point F has coordinates $(a, 0)$ and ϕ is the angle TFP. Show that*
>
> $$\cos\phi = \frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}}$$
>
> *and deduce that the line FT bisects the angle PFQ.*

In the triangle TFP, we can use the cosine rule to find $\cos\phi$:

$$TP^2 = TF^2 + PF^2 - 2.TF.PF.\cos\phi,$$

so that

$$\cos\phi = \frac{TF^2 + PF^2 - TP^2}{2.TF.PF}.$$

Now using Pythagoras to find the distance between two points given their coordinates, we obtain

$$\begin{aligned}
TF^2 &= \big(a(pq-1)\big)^2 + \big(a(p+q)\big)^2 \\
&= a^2(p^2q^2 - 2pq + 1 + p^2 + 2pq + q^2) \\
&= a^2(p^2q^2 + p^2 + q^2 + 1) \\
&= a^2(p^2+1)(q^2+1) \\
FP^2 &= \big(a(p^2-1)\big)^2 + (2ap)^2 \\
&= a^2(p^4 - 2p^2 + 1 + 4p^2) \\
&= a^2(p^4 + 2p^2 + 1) \\
&= a^2(p^2+1)^2 \\
TP^2 &= \big(a(pq-p^2)\big)^2 + \big(a(p+q-2p)\big)^2 \\
&= \big(ap(q-p)\big)^2 + \big(a(q-p)\big)^2 \\
&= a^2(p^2+1)(q-p)^2
\end{aligned}$$

Thus

$$\begin{aligned}
TF^2 + FP^2 - TP^2 &= a^2(p^2+1)(q^2+1+p^2+1-q^2+2pq-p^2) \\
&= 2a^2(1+p^2)(1+pq)
\end{aligned}$$

so that

$$\begin{aligned}
\cos\phi &= \frac{2a^2(1+p^2)(1+pq)}{2a^2\sqrt{(p^2+1)(q^2+1)(p^2+1)^2}} \\
&= \frac{1+pq}{\sqrt{(p^2+1)(q^2+1)}}
\end{aligned}$$

as we wanted.

An alternative approach is to use vectors and dot products to find $\cos\phi$. We have

$$\overrightarrow{FP}.\overrightarrow{FT} = FP.FT.\cos\phi$$

(where the dot on the left hand side is the dot product, but on the right is ordinary multiplication), so we need only find the lengths FP, FT as above and the dot product. The dot product is

$$\begin{aligned}
\overrightarrow{FP}.\overrightarrow{FT} = \begin{pmatrix} ap^2 - a \\ 2ap - 0 \end{pmatrix} \cdot \begin{pmatrix} apq - a \\ a(p+q) - 0 \end{pmatrix} &= (ap^2 - a)(apq - a) + 2ap.a(p+q) \\
&= a^2(p^2 - 1)(pq - 1) + 2a^2(p^2 + pq) \\
&= a^2(p^3q - p^2 - pq + 1 + 2p^2 + 2pq) \\
&= a^2(p^3q + p^2 + pq + 1) \\
&= a^2(p^2(pq + 1) + pq + 1) \\
&= a^2(p^2 + 1)(pq + 1)
\end{aligned}$$

Therefore we deduce

$$\begin{aligned}
\cos\phi &= \frac{\overrightarrow{FP}.\overrightarrow{FT}}{FP.FT} \\
&= \frac{a^2(p^2+1)(pq+1)}{a(p^2+1).a\sqrt{(p^2+1)(q^2+1)}} \\
&= \frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}}
\end{aligned}$$

as required.

Now to show that the line FT bisects the angle PFQ, it suffices to show that ϕ is equal to the angle TFQ (see the sketch above).

Now we can find $\cos(\angle TFQ)$ by using the above formula and swapping every p and q in it, as this will swap the roles of P and Q.

But swapping every p and q does not change the formula, so $\cos(\angle TFQ) = \cos(\angle TFP)$, and so $\angle TFQ = \angle TFP$ as both angles are strictly less than $180°$ and cosine is one-to-one in this domain.

Thus the line FT bisects the angle PFQ, as required.

Question 5

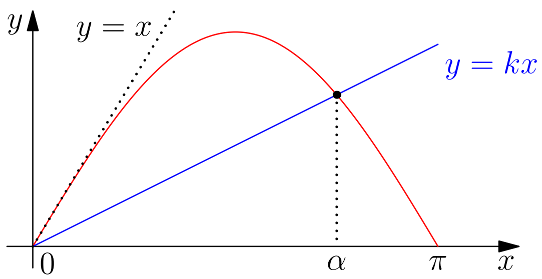
Given that $0 < k < 1$, show with the help of a sketch that the equation

$$\sin x = kx \qquad (*)$$

has a unique solution in the range $0 < x < \pi$.

We sketch the graph of $y = \sin x$ in the range $0 \leqslant x \leqslant \pi$ along with the line $y = kx$.

Now since $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, the gradient of $y = \sin x$ at $x = 0$ is 1, so the tangent at $x = 0$ is $y = x$. We therefore also sketch the line $y = x$.



It clear that there is at most one intersection of $y = kx$ with $y = \sin x$ in the interval $0 < x < \pi$, and since $0 < k < 1$, there is exactly one, as the gradient is positive and less that that of $y = \sin x$ at the origin. (If $k \leqslant 0$, there would be no intersections in this range as kx would be negative or zero; if $k \geqslant 1$, the only intersection would be at $x = 0$.)

Let

$$I = \int_0^\pi \left|\sin x - kx\right| \mathrm{d}x\,.$$

Show that

$$I = \frac{\pi^2 \sin\alpha}{2\alpha} - 2\cos\alpha - \alpha\sin\alpha\,,$$

where α is the unique solution of $(*)$.

It is a pain to work with absolute values (the "modulus function"), so we split the integral into two integrals: in the interval $0 \leqslant x \leqslant \alpha$, $\sin x - kx \geqslant 0$, and in the interval $\alpha \leqslant x \leqslant \pi$, $\sin x - kx \leqslant 0$. So

$$\begin{aligned}
I &= \int_0^\pi \left|\sin x - kx\right| \mathrm{d}x \\
&= \int_0^\alpha \left|\sin x - kx\right| \mathrm{d}x + \int_\alpha^\pi \left|\sin x - kx\right| \mathrm{d}x \\
&= \int_0^\alpha \sin x - kx \,\mathrm{d}x + \int_\alpha^\pi -\sin x + kx \,\mathrm{d}x \\
&= \left[-\cos x - \tfrac{1}{2}kx^2\right]_0^\alpha + \left[\cos x + \tfrac{1}{2}kx^2\right]_\alpha^\pi \\
&= (-\cos\alpha - \tfrac{1}{2}k\alpha^2) - (-\cos 0 - 0) + (\cos\pi + \tfrac{1}{2}k\pi^2) - (\cos\alpha + \tfrac{1}{2}k\alpha^2) \\
&= -2\cos\alpha - k\alpha^2 + \tfrac{1}{2}k\pi^2
\end{aligned}$$

$$= -2\cos\alpha - \alpha\sin\alpha + \frac{\pi^2 \sin\alpha}{2\alpha}$$

where the last line follows using $k\alpha = \sin\alpha$ so that $k = (\sin\alpha)/\alpha$, and we have reached the desired result.

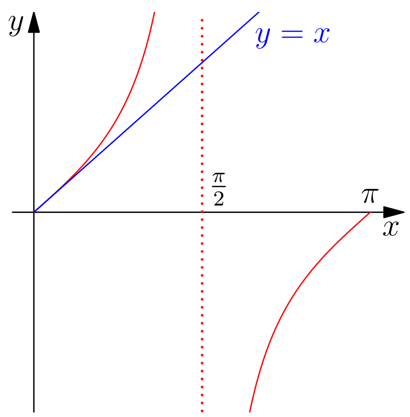
> *Show that I, regarded as a function of α, has a unique stationary value and that this stationary value is a minimum. Deduce that the smallest value of I is*
>
> $$-2\cos\frac{\pi}{\sqrt{2}}\,.$$

We differentiate I to find its stationary points. We have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}\alpha} &= \frac{\pi^2}{2}\left(\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2}\right) + 2\sin\alpha - \sin\alpha - \alpha\cos\alpha \\ &= (\sin\alpha - \alpha\cos\alpha) - \frac{\pi^2}{2\alpha^2}(\sin\alpha - \alpha\cos\alpha) \\ &= \left(1 - \frac{\pi^2}{2\alpha^2}\right)(\sin\alpha - \alpha\cos\alpha)\end{aligned}$$

so $\frac{\mathrm{d}I}{\mathrm{d}\alpha} = 0$ if and only if $2\alpha^2 = \pi^2$ or $\sin\alpha = \alpha\cos\alpha$. The former condition gives $\alpha = \pm\pi/\sqrt{2}$, while the latter condition gives $\tan\alpha = \alpha$.

A quick sketch of the tan graph (see below) shows that $\tan\alpha = \alpha$ has no solutions in the range $0 < \alpha < \pi$ (though $\alpha = 0$ is a solution); the sketch uses the result that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$, so the tangent to $y = \tan x$ at $x = 0$ is $y = x$.



Thus the only solution in the required range is $\alpha = \pi/\sqrt{2}$ (and note that $\pi/\sqrt{2} < \pi$).

To ascertain whether it is a maximum, a minimum or a point of inflection, we could either look at the values of I or $\mathrm{d}I/\mathrm{d}\alpha$ at this point and either side or we could consider the second derivative.

Either way, we will eventually have to work out the value of I when $\alpha = \pi/\sqrt{2}$, so we will do so now:

$$
\begin{aligned}
I &= \frac{\pi^2 \sin(\pi/\sqrt{2})}{2\pi/\sqrt{2}} - 2\cos\frac{\pi}{\sqrt{2}} - \frac{\pi}{\sqrt{2}}\sin\frac{\pi}{\sqrt{2}} \\
&= \frac{\pi}{\sqrt{2}}\sin\frac{\pi}{\sqrt{2}} - 2\cos\frac{\pi}{\sqrt{2}} - \frac{\pi}{\sqrt{2}}\sin\frac{\pi}{\sqrt{2}} \\
&= -2\cos\frac{\pi}{\sqrt{2}}.
\end{aligned}
$$

Approach 1: Using values either side

The function I is not well-defined when $\alpha = 0$, but if we know that $\frac{\sin\alpha}{\alpha} \to 1$ as $\alpha \to 0$, we can deduce that as $\alpha \to 0$, $I \to -2 + \frac{\pi^2}{2} > \frac{5}{2} > 2$ (using $\pi > 3$).

When $\alpha = \pi$, we have $I = 2$.

Since at $\alpha = \pi/\sqrt{2}$, we have $I = -2\cos(\pi/\sqrt{2}) < 2$, this must be the minimum value of I.

Alternatively, if we wish to consider the value of $\mathrm{d}I/\mathrm{d}\alpha$, we need to know the sign of $\sin\alpha - \alpha\cos\alpha$ near $\alpha = \pi/\sqrt{2}$. Now since $\frac{\pi}{2} < \alpha < \pi$, $\cos\alpha < 0$ and so this expression is positive. Therefore for α slightly less that $\pi/\sqrt{2}$, $\mathrm{d}I/\mathrm{d}\alpha < 0$ and for $\alpha > \pi/\sqrt{2}$, $\mathrm{d}I/\mathrm{d}\alpha > 0$, so that $\alpha = \pi/\sqrt{2}$ is a (local) minimum.

Approach 2: Using the second derivative

We have

$$
\begin{aligned}
\frac{\mathrm{d}^2 I}{\mathrm{d}\alpha^2} &= \frac{\pi^2}{\alpha^3}(\sin\alpha - \alpha\cos\alpha) + \left(1 - \frac{\pi^2}{2\alpha^2}\right)(\cos\alpha - \cos\alpha + \alpha\sin\alpha) \\
&= \frac{\pi^2}{\alpha^3}(\sin\alpha - \alpha\cos\alpha) + \left(1 - \frac{\pi^2}{2\alpha^2}\right)\alpha\sin\alpha
\end{aligned}
$$

Now when $\alpha = \pi/\sqrt{2}$, so that $\pi^2/2\alpha^2 = 1$, we have

$$
\frac{\mathrm{d}^2 I}{\mathrm{d}\alpha^2} = \frac{\pi^2}{\alpha^3}(\sin\alpha - \alpha\cos\alpha).
$$

Since $\alpha = \frac{\pi}{2}\sqrt{2} > \frac{\pi}{2}$, we have $\sin\alpha > 0$ and $\cos\alpha < 0$, so $\frac{\mathrm{d}^2 I}{\mathrm{d}\alpha^2} > 0$ and I has a local minimum at this value of α.

Question 6

> *Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.*

Using the formula in the formula book for the binomial expansion, we find that the x^r term is

$$\begin{aligned}\binom{-3}{r}(-x)^r &= \frac{(-3)(-4)(-5)\dots(-3-r+1)}{r!}(-1)^r x^r \\ &= \frac{3.4.5.\cdots.(r+2)}{r!}x^r \\ &= \frac{3.4.5.\cdots.(r+2)}{1.2.3.4.5.\cdots.r}x^r \\ &= \frac{(r+1)(r+2)}{1.2}x^r\end{aligned}$$

so the coefficient of x^r is $\frac{1}{2}(r+1)(r+2)$. But the argument as we've written it assumes that $r \geqslant 2$ (as we've left ourselves with "1.2" in the denominator), so we need to check that this this also holds for $r=0$ and $r=1$. But this is easy, as $\binom{-3}{0}(-1)^0 = 1 = \frac{1}{2}\times 1\times 2$ and $\binom{-3}{1}(-1)^1 = 3 = \frac{1}{2}\times 2\times 3$.

Alternatively, we could have argued

$$\frac{3.4.5.\cdots.(r+2)}{r!}x^r = \frac{1.2.3.4.5.\cdots.(r+2)}{1.2.r!}x^r = \frac{(r+1)(r+2)}{1.2}x^r$$

and this would have dealt with the cases $r=0$ and $r=1$ automatically, as we are not implicitly assuming that $r \geqslant 2$.

> (i) *Show that the coefficient of x^r in the expansion of*
>
> $$\frac{1-x+2x^2}{(1-x)^3}$$
>
> *is r^2+1 and hence find the sum of the series*
>
> $$1+\frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\frac{26}{32}+\frac{37}{64}+\frac{50}{128}+\cdots.$$

We have

$$\frac{1-x+2x^2}{(1-x)^3} = (1-x+2x^2)(a_0+a_1x+a_2x^2+\cdots+a_rx^r+\dots)$$

where $a_r = \frac{1}{2}(r+1)(r+2)$. Thus

$$\begin{aligned}\frac{1-x+2x^2}{(1-x)^3} &= a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots \\ &\qquad - a_0x - a_1x^2 - \cdots - a_{r-1}x^r - \cdots \\ &\qquad\qquad + 2a_0x^2 + \cdots + 2a_{r-2}x^r + \cdots \\ &= a_0 + (a_1-a_0)x + (a_2-a_1+2a_0)x^2 + \cdots \\ &\qquad + (a_r - a_{r-1} + 2a_{r-2})x^r + \cdots\end{aligned}$$

Thus the coefficient of x^r for $r \geqslant 2$ is

$$\begin{aligned} a_r - a_{r-1} + 2a_{r-2} &= \tfrac{1}{2}(r+1)(r+2) - \tfrac{1}{2}r(r+1) + (r-1)r \\ &= \tfrac{1}{2}(r^2 + 3r + 2 - r^2 - r + 2r^2 - 2r) \\ &= \tfrac{1}{2}(2r^2 + 2) \\ &= r^2 + 1 \end{aligned}$$

as required. Also, the coefficient of x^0 is $a_0 = 1 = 0^2 + 1$ and the coefficient of x^1 is $a_1 - a_0 = 3 - 1 = 2 = 1^2 + 1$, so the formula $r^2 + 1$ holds for these two cases as well. Therefore, the coefficient of x^r is $r^2 + 1$ for all $r \geqslant 0$.

Now we can sum our series: it is

$$\begin{aligned} 1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \cdots &= \frac{0^2+1}{2^0} + \frac{1^2+1}{2^1} + \frac{2^2+1}{2^2} + \cdots + \frac{r^2+1}{2^r} + \cdots \\ &= (0^2+1) + (1^2+1)(\tfrac{1}{2}) + \cdots + (r^2+1)(\tfrac{1}{2})^r + \cdots \\ &= \frac{1 - \frac{1}{2} + 2(\frac{1}{2})^2}{(1-\frac{1}{2})^3} \\ &= \frac{1}{(\frac{1}{8})} \\ &= 8. \end{aligned}$$

(ii) Find the sum of the series

$$1 + 2 + \frac{9}{4} + 2 + \frac{25}{16} + \frac{9}{8} + \frac{49}{64} + \cdots .$$

The denominators look like powers of 2, so we will rewrite the terms using powers of 2:

$$1 + 2 + \frac{9}{4} + 2 + \frac{25}{16} + \frac{9}{8} + \cdots = \frac{1}{1} + \frac{4}{2} + \frac{9}{4} + \frac{16}{8} + \frac{25}{16} + \frac{36}{32} + \frac{49}{64} + \cdots$$

and it is clear that the general term is $r^2/2^{r-1}$, starting with the term where $r = 1$.

We can rewrite this in terms of the series found in part (i) by writing

$$\frac{r^2}{2^{r-1}} = 2 \cdot \frac{r^2}{2^r} = 2 \cdot \frac{r^2+1}{2^r} - 2 \cdot \frac{1}{2^r},$$

so our series becomes

$$\begin{aligned} &2\left(\frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \cdots\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) \\ &\quad = 2\left(1 + \frac{2}{2} + \frac{5}{4} + \frac{10}{8} + \frac{17}{16} + \cdots\right) - 2\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots\right) \\ &\quad = 2 \cdot 8 - 2 \cdot 2 \\ &\quad = 12, \end{aligned}$$

where on the second line, we have introduced the term corresponding to $r = 0$, and on the penultimate line, we have used the result from (i) and the sum of the infinite geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1/(1 - \frac{1}{2}) = 2$.

An alternative approach is to begin with the result of part (i) and to argue as follows.

We have

$$\begin{aligned}\frac{1-x+2x^2}{(1-x)^3} &= \sum_{r=0}^{\infty}(r^2+1)x^r \\ &= \sum_{r=0}^{\infty} r^2x^r + \sum_{r=0}^{\infty} x^r \\ &= x\sum_{r=0}^{\infty} r^2x^{r-1} + \sum_{r=0}^{\infty} x^r.\end{aligned}$$

But our required sum is $\sum_{r=0}^{\infty} r^2(\frac{1}{2})^{r-1}$, so we put $x=\frac{1}{2}$ into this result and get

$$\frac{1-\frac{1}{2}+2(\frac{1}{2})^2}{(1-\frac{1}{2})^3} = \tfrac{1}{2}\sum_{r=0}^{\infty} r^2(\tfrac{1}{2})^{r-1} + \sum_{r=0}^{\infty}(\tfrac{1}{2})^r.$$

The last term on the right hand side is our geometric series, summing to 2. The left hand side evaluates to 8, and so we get

$$8 = \tfrac{1}{2}\sum_{r=0}^{\infty} r^2(\tfrac{1}{2})^{r-1} + 2.$$

Thus our series sums to 12, as before.

A third approach is to observe that the series can be written as $\sum_{r=0}^{\infty}(r+1)^2x^r = \sum_{r=0}^{\infty}(r^2+2r+1)x^r$ with $x=\frac{1}{2}$, then to look for a polynomial $\mathrm{p}(x)$ of degree at most 2 such that the coefficient of x^r in the expansion of $\mathrm{p}(x)/(1-x)^3$ is exactly r^2+2r+1, using methods like those in part (i). (The polynomial needs to be of degree at most 2 so that the terms are also correct for $r=0$ and $r=1$ in addition to the general term being correct.) This turns out to give $\mathrm{p}(x)=x+1$, so that the sum is $(\frac{1}{2}+1)/(1-\frac{1}{2})^3=12$.

Question 7

In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ *when* $|x|$ *is small.*

The height of the water in a tank at time t *is* h. *The initial height of the water is* H *and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.*

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, *where* α *is a constant greater than 1, the height remains constant. Show that*

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(\alpha^2 H - h),$$

for some positive constant k. *Deduce that the time* T *taken for the water to reach height* αH *is given by*

$$kT = \ln\left(1 + \frac{1}{\alpha}\right)$$

and that $kT \approx \alpha^{-1}$ *for large values of* α.

Since the tank has constant cross-sectional area, the volume of water within the tank is proportional to the height of the water.

Therefore we have the height increasing at a rate $a - bh$, where a is the rate of water flowing in divided by the cross-sectional area, and b is a constant of proportionality representing the rate of water leaking out. In other words, we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = a - bh.$$

Now, when $h = \alpha^2 H$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, so $a - b\alpha^2 H = 0$, or $a = b\alpha^2 H$, giving

$$\frac{\mathrm{d}h}{\mathrm{d}t} = b\alpha^2 H - bh = b(\alpha^2 H - h).$$

Hence if we write $k = b$, we have our desired equation.

We can now solve this by separating variables to get

$$\int \frac{1}{\alpha^2 H - h}\,\mathrm{d}h = \int k\,\mathrm{d}t$$

so that

$$-\ln(\alpha^2 H - h) = kt + c.$$

At $t = 0$, $h = H$, so

$$-\ln(\alpha^2 H - H) = c,$$

which finally gives us

$$kt = \ln(\alpha^2 H - H) - \ln(\alpha^2 H - h).$$

Now at time T, $h = \alpha H$, so that

$$\begin{aligned} kT &= \ln(\alpha^2 H - H) - \ln(\alpha^2 H - \alpha H) \\ &= \ln\left(\frac{\alpha^2 H - H}{\alpha^2 H - \alpha H}\right) \\ &= \ln\left(\frac{\alpha^2 - 1}{\alpha^2 - \alpha}\right) \\ &= \ln\left(\frac{\alpha + 1}{\alpha}\right) \\ &= \ln\left(1 + \frac{1}{\alpha}\right) \end{aligned}$$

as required.

When α is large, so that $\frac{1}{\alpha}$ is small, this is

$$\begin{aligned} kT &= \ln\left(1 + \frac{1}{\alpha}\right) \\ &\approx \frac{1}{\alpha} - \frac{1}{2\alpha^2} \\ &\approx \frac{1}{\alpha}. \end{aligned}$$

> *(ii)* *Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by*
>
> $$cT' = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha \ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$
>
> *for some positive constant c and that $cT' \approx \sqrt{H}$ for large values of α.*

We proceed just as in part (i).

This time we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = a - b\sqrt{h},$$

where a and b are some constants. Now, when $h = \alpha^2 H$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 0$, so $a - b\sqrt{\alpha^2 H} = 0$, which yields $a = b\alpha\sqrt{H}$. We thus have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = b\alpha\sqrt{H} - b\sqrt{h} = b(\alpha\sqrt{H} - \sqrt{h}).$$

So if this time we write $c = b$, we have our desired differential equation.

We again solve this by separating variables to get

$$\int \frac{1}{\alpha\sqrt{H} - \sqrt{h}}\,\mathrm{d}h = \int c\,\mathrm{d}t.$$

To integrate the left hand side, we use the substitution $u = \sqrt{h}$, so that $h = u^2$ and $\frac{\mathrm{d}h}{\mathrm{d}u} = 2u$. This gives us

$$\int \frac{1}{\alpha\sqrt{H} - u} \cdot 2u\,\mathrm{d}u = ct.$$

We divide the numerator by the denominator to get

$$\begin{aligned}
ct &= \int \frac{-2(\alpha\sqrt{H}-u)+2\alpha\sqrt{H}}{\alpha\sqrt{H}-u}\,\mathrm{d}u \\
&= \int -2+\frac{2\alpha\sqrt{H}}{\alpha\sqrt{H}-u}\,\mathrm{d}u \\
&= -2u-2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H}-u\bigr)+c' \\
&= -2\sqrt{h}-2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H}-\sqrt{h}\bigr)+c'
\end{aligned}$$

where c' is a constant.

An alternative way of doing this step is to use the substitution $v=\alpha\sqrt{H}-\sqrt{h}$, so that $h=(\alpha\sqrt{H}-v)^2=\alpha^2H-2\alpha v\sqrt{H}+v^2$ and $\frac{\mathrm{d}h/\mathrm{d}v}{=}-2\alpha\sqrt{H}+2v$. This gives us

$$\begin{aligned}
ct &= \int \frac{1}{v}(-2\alpha\sqrt{H}+2v)=ct \\
&= \int \frac{-2\alpha\sqrt{H}}{v}+2\,\mathrm{d}v \\
&= -2\alpha\sqrt{H}\ln v+2v+c' \\
&= -2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H}-\sqrt{h}\bigr)+2\bigl(\alpha\sqrt{H}-\sqrt{h}\bigr)+c'
\end{aligned}$$

where c' is again a constant.

At $t=0$, $h=H$, so

$$c'=2\sqrt{H}+2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H}-\sqrt{H}\bigr).$$

Now at time T', $h=\alpha H$, so that

$$\begin{aligned}
cT' &= -2\sqrt{\alpha H}-2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H}-\sqrt{\alpha H}\bigr)+2\sqrt{H}+2\alpha\sqrt{H}\ln\bigl(\alpha\sqrt{H}-\sqrt{H}\bigr) \\
&= 2\sqrt{H}(1-\sqrt{\alpha})+2\alpha\sqrt{H}\ln\left(\frac{\alpha\sqrt{H}-\sqrt{H}}{\alpha\sqrt{H}-\sqrt{\alpha H}}\right) \\
&= 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\ln\left(\frac{(\sqrt{\alpha}+1)(\sqrt{\alpha}-1)}{\sqrt{\alpha}(\sqrt{\alpha}-1)}\right)\right) \\
&= 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\ln\left(\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}}\right)\right) \\
&= 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\ln\left(1+\frac{1}{\sqrt{\alpha}}\right)\right)
\end{aligned}$$

as required.

When α is large, $1/\sqrt{\alpha}$ is small, so this gives

$$\begin{aligned}
cT' &= 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\ln\left(1+\frac{1}{\sqrt{\alpha}}\right)\right) \\
&\approx 2\sqrt{H}\left(1-\sqrt{\alpha}+\alpha\left(\frac{1}{\sqrt{\alpha}}-\frac{1}{2\alpha}\right)\right) \\
&\approx 2\sqrt{H}\bigl(1-\sqrt{\alpha}+\sqrt{\alpha}-\tfrac{1}{2}\bigr) \\
&\approx \sqrt{H}.
\end{aligned}$$

Question 8

> (i) *The numbers m and n satisfy*
> $$m^3 = n^3 + n^2 + 1\,. \qquad (*)$$
>
> (a) *Show that $m > n$. Show also that $m < n+1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n+1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.*

As $n^2 \geqslant 0$, we have

$$\begin{aligned} m^3 &= n^3 + n^2 + 1 \\ &\geqslant n^3 + 1 \\ &> n^3 \end{aligned}$$

so $m > n$ as the function $\mathrm{f}(x) = x^3$ is strictly increasing.

Now

$$\begin{aligned} m < n+1 &\iff m^3 < (n+1)^3 \\ &\iff n^3 + n^2 + 1 < n^3 + 3n^2 + 3n + 1 \\ &\iff 0 < 2n^2 + 3n \end{aligned}$$

so $m < n+1$ if and only if $2n^2 + 3n > 0$.

Combining these two conditions, $n < m$ always, and $m < n+1$ if and only if $2n^2 + 3n > 0$, so $n < m < n+1$ unless $2n^2 + 3n \leqslant 0$.

Now $2n^2 + 3n = 2n(n + \frac{3}{2}) \leqslant 0$ if and only if $-\frac{3}{2} \leqslant n \leqslant 0$, so $n < m < n+1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.

> (b) *Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m,n) = (1,0)$ and $(m,n) = (1,-1)$.*

If solution to $(*)$ has both m and n integer, we cannot have $n < m < n+1$, as there is no integer strictly between two consecutive integers. We therefore require $-\frac{3}{2} \leqslant n \leqslant 0$, so $n = -1$ or $n = 0$.

If $n = -1$, then $m^3 = 1$, so $m = 1$.

If $n = 0$, then $m^3 = 1$, so $m = 1$.

Thus the only integer solutions are $(m,n) = (1,0)$ and $(m,n) = (1,-1)$.

> (ii) *Find all integer solutions of the equation*
> $$p^3 = q^3 + 2q^2 - 1\,.$$

We try a similar argument here. We start by determining whether $p > q$:

$$\begin{aligned} p > q &\iff p^3 > q^3 \\ &\iff q^3 + 2q^2 - 1 > q^3 \\ &\iff 2q^2 > 1 \\ &\iff q^2 > \tfrac{1}{2} \end{aligned}$$

so that $p > q$ unless $q^2 \leqslant \frac{1}{2}$, and $q^2 \leqslant \frac{1}{2}$ if and only if $-\frac{1}{\sqrt{2}} \leqslant q \leqslant \frac{1}{\sqrt{2}}$.

We now determine the conditions under which $p < q + 1$:

$$\begin{aligned} p < q + 1 &\iff p^3 < (q+1)^3 \\ &\iff q^3 + 2q^2 - 1 < q^3 + 3q^2 + 3q + 1 \\ &\iff 0 < q^2 + 3q + 2 \end{aligned}$$

so $p < q+1$ unless $q^2 + 3q + 2 \leqslant 0$. This condition becomes $(q+1)(q+2) \leqslant 0$, so $-2 \leqslant q \leqslant -1$.

Thus $q < p < q + 1$ unless $-\frac{1}{\sqrt{2}} \leqslant q \leqslant \frac{1}{\sqrt{2}}$ or $-2 \leqslant q \leqslant -1$.

If p and q are both integers, this then limits us to three cases: $q = 0$, $q = -1$ and $q = -2$.

If $q = 0$, then $p^3 = -1$, so $p = -1$.

If $q = -1$, then $p^3 = 0$, so $p = 0$.

If $q = -2$, then $p^3 = -1$, so $p = -1$.

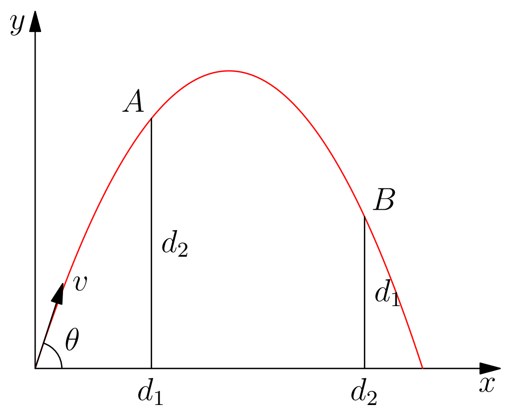
Hence there are three integer solutions: $(p, q) = (-1, 0)$, $(p, q) = (-1, -2)$ and $(p, q) = (0, -1)$.

Question 9

A particle is projected at an angle θ above the horizontal from a point on a horizontal plane. The particle just passes over two walls that are at horizontal distances d_1 and d_2 from the point of projection and are of heights d_2 and d_1, respectively. Show that

$$\tan\theta = \frac{d_1^2 + d_1 d_2 + d_2^2}{d_1 d_2}\,.$$

We draw a sketch of the situation:



We let the speed of projection be v and the time from launch be t. We resolve the components of velocity to find the position (x, y) at time t:

$$\mathscr{R}(\rightarrow) \qquad x = (v\cos\theta)t \tag{1}$$
$$\mathscr{R}(\uparrow) \qquad y = (v\sin\theta)t - \tfrac{1}{2}gt^2 \tag{2}$$

At A (distance d_1 from the point of projection), we find

$$(v\cos\theta)t = d_1$$
$$(v\sin\theta)t - \tfrac{1}{2}gt^2 = d_2$$

so that

$$t = \frac{d_1}{v\cos\theta}$$

giving

$$\frac{v\sin\theta}{v\cos\theta}\,d_1 - \frac{\frac{1}{2}gd_1^2}{v^2\cos^2\theta} = d_2,$$

so that

$$d_2 = d_1\tan\theta - \frac{gd_1^2}{2v^2\cos^2\theta}.$$

This can be rearranged to get

$$\frac{gd_1^2}{2v^2\cos^2\theta} = d_1\tan\theta - d_2. \tag{3}$$

(An alternative is to first eliminate t from equations (1) and (2) first to get

$$y = x\tan\theta - \frac{gx^2}{2v^2\cos^2\theta} \tag{4}$$

and then substitute $x = d_1$ and $y = d_2$ into this formula.)

Likewise, at B we get (on swapping d_1 and d_2):

$$\frac{gd_2^2}{2v^2\cos^2\theta} = d_2\tan\theta - d_1. \tag{5}$$

Multiplying (3) by d_2^2 gives the same left hand side as when we multiply (5) by d_1^2, so that

$$(d_1\tan\theta - d_2)d_2^2 = (d_2\tan\theta - d_1)d_1^2.$$

Expanding this gives

$$d_1d_2^2\tan\theta - d_2^3 = d_1^2d_2\tan\theta - d_1^3.$$

Collecting terms gives:

$$(d_1d_2^2 - d_1^2d_2)\tan\theta = d_2^3 - d_1^3,$$

and we can factorise this (recalling that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$) to get

$$d_1d_2(d_2 - d_1)\tan\theta = (d_2 - d_1)(d_2^2 + d_2d_1 + d_1^2).$$

Dividing by $d_1d_2(d_2 - d_1) \neq 0$ gives us our desired result:

$$\tan\theta = \frac{d_1^2 + d_1d_2 + d_2^2}{d_1d_2}.$$

Find (and simplify) an expression in terms of d_1 and d_2 only for the range of the particle.

The range can be found by determining where $y = 0$, so $(v\sin\theta)t - \frac{1}{2}gt^2 = 0$. This has solutions $t = 0$ (the point of projection) and $t = (2v/g)\sin\theta$. At this point,

$$\begin{aligned} x = (v\cos\theta)t &= \frac{2v^2\sin\theta\cos\theta}{g} \\ &= \frac{2v^2\cos^2\theta}{g}\tan\theta. \end{aligned}$$

We have written $\sin\theta = \cos\theta\tan\theta$ because equation (3) gives us a formula for the fraction part of this expression: we get

$$x = \frac{d_1^2}{d_1\tan\theta - d_2}\tan\theta.$$

Alternatively, using equation (4), we can solve for $y = 0$ to get $x = 0$ or

$$\tan\theta = \frac{gx}{2v^2\cos^2\theta}.$$

Since $x = 0$ at the start, the other solution gives the range. Using equation (3) to write

$$\frac{g}{2v^2\cos^2\theta} = \frac{d_1\tan\theta - d_2}{d_1^2},$$

we deduce that

$$x = \frac{d_1^2\tan\theta}{d_1\tan\theta - d_2}$$

as before.

We can now simply substitute in our formula for $\tan\theta$, simplify a little, and we will be done:

$$\begin{aligned}
x &= \frac{d_1^2\left(\dfrac{d_1^2+d_1d_2+d_2^2}{d_1d_2}\right)}{d_1\left(\dfrac{d_1^2+d_1d_2+d_2^2}{d_1d_2}\right)-d_2} \\
&= \frac{d_1(d_1^2+d_1d_2+d_2^2)}{(d_1^2+d_1d_2+d_2^2)-d_2^2} \\
&= \frac{d_1(d_1^2+d_1d_2+d_2^2)}{d_1^2+d_1d_2} \\
&= \frac{d_1^2+d_1d_2+d_2^2}{d_1+d_2}
\end{aligned}$$

Alternative approach

An entirely different approach to the whole question is as follows. We know that the path of the projectile is a parabola. Taking axes as in the above sketch, the path passes through the three points $(0,0)$, (d_1,d_2) and (d_2,d_1). If the equation of the curve is $y=ax^2+bx+c$, then this gives three simultaneous equations:

$$\begin{aligned}
0 &= 0a+0b+c \\
d_2 &= d_1^2a+d_1b+c \\
d_1 &= d_2^2a+d_2b+c.
\end{aligned}$$

The first gives $c=0$, and we can then solve the other two equations to get a and b. This gives

$$\begin{aligned}
a &= \frac{d_2^2-d_1^2}{d_1^2d_2-d_2^2d_1} = -\frac{d_1+d_2}{d_1d_2} \\
b &= \frac{d_1^3-d_2^3}{d_1^2d_2-d_2^2d_1} = \frac{d_1^2+d_1d_2+d_2^2}{d_1d_2}
\end{aligned}$$

Then the gradient is given by $\mathrm{d}y/\mathrm{d}x=2ax+b$, so at $x=0$, the gradient $\mathrm{d}y/\mathrm{d}x=b$, which gives us $\tan\theta$ (as the gradient is the tangent of angle made with the x-axis). The range is given by solving $y=0$, so $x(ax+b)=0$, giving $x=-b/a=(d_1^2+d_1d_2+d_2^2)/(d_1+d_2)$ as before.

Question 10

A particle, A, is dropped from a point P which is at a height h above a horizontal plane. A second particle, B, is dropped from P and first collides with A after A has bounced on the plane and before A reaches P again. The bounce and the collision are both perfectly elastic. Explain why the speeds of A and B immediately before the first collision are the same.

Assume they collide at height $H < h$. The perfectly elastic bounce means that there was no loss of energy, so A has the same total energy at height H on its upwards journey as it did when travelling downwards. We can work out the speeds at the point of collision, calling them v_A and v_B for A and B respectively. We write M for the mass of A and m for the mass of B (as in the next part of the question). We have, by conservation of energy

$$MgH + \tfrac{1}{2}Mv_A^2 = Mgh$$
$$mgH + \tfrac{1}{2}mv_B^2 = mgh$$

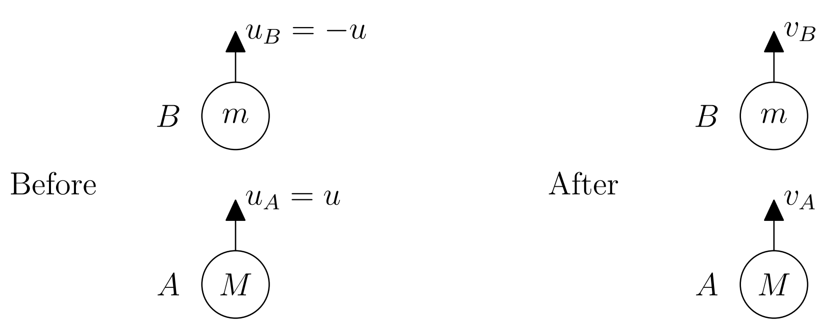
so that $v_A^2 = 2(gh - gH)$ and $v_B^2 = 2(gh - gH)$, so $|v_A| = |v_B|$ and the speeds of A and B are the same.

The masses of A and B are M and m, respectively, where $M > 3m$, and the speed of the particles immediately before the first collision is u. Show that both particles move upwards after their first collision and that the maximum height of B above the plane after the first collision and before the second collision is

$$h + \frac{4M(M-m)u^2}{(M+m)^2 g}.$$

*This begins as a standard collision of particles question, and so I will repeat the advice from the 2010 mark scheme: **ALWAYS** draw a diagram for collisions questions; you will do yourself (and the marker) no favours if you try to keep all of the directions in your head, and you are very likely to make a mistake. My recommendation is to always have all of the velocity arrows pointing in the same direction. In this way, there is no possibility of getting the signs wrong in the Law of Restitution: it **always** reads $v_1 - v_2 = e(u_2 - u_1)$ or $\dfrac{v_1 - v_2}{u_2 - u_1} = e$, and you only have to be careful with the signs of the given velocities. The algebra will then keep track of the directions of the unknown velocities for you.*

A diagram showing the first collision is as follows.



Then Conservation of Momentum gives

$$Mu_A + mu_B = Mv_A + mv_B$$

so

$$Mu - mu = Mv_A + mv_B,$$

and Newton's Law of Restitution gives

$$v_B - v_A = 1(u_A - u_B)$$

(using $e = 1$ as the collision is perfectly elastic). Substituting $u_A = u$ and $u_B = -u$ gives

$$Mv_A + mv_B = (M - m)u \tag{1}$$

$$v_B - v_A = 2u. \tag{2}$$

Then solving these equations (by (1) $- m \times$ (2) and (1) $+ M \times$ (2)) gives

$$v_A = \frac{(M - 3m)u}{M + m} \tag{3}$$

$$v_B = \frac{(3M - m)u}{M + m}. \tag{4}$$

To show that both particles move upwards after their first collision, we need to show that $v_A > 0$ and $v_B > 0$. From equation (3) and $M > 3m$ (given in the question), we see that $v_A > 0$; from equation (4) and $3M - m > 9m - m > 0$ (as $M > 3m$), we see that $v_B > 0$. Thus both particles move upwards after their first collision.

To find the maximum height of B between the two collisions, we begin by finding the maximum height that would be achieved by B following the first collision assuming that there is no second collision. We then explain why the second collision occurs during B's subsequent downward motion and deduce that it reaches that maximum height between the collisions.

The kinetic energy (KE) of B before the first collision is $\frac{1}{2}mu^2$ and after the first collision is

$$\tfrac{1}{2}mv_B^2 = \tfrac{1}{2}m\left(\frac{3M - m}{M + m}\right)^2 u^2,$$

so that B has a gain in KE of $\frac{1}{2}mv_B^2 - \frac{1}{2}mu^2$. When B is again at height h above the plane, which is where it was dropped from, it now has this gain as its KE. (This is because the KE just before the first collision has come from the loss of GPE; when the particle is once again at height h, this original KE ($\frac{1}{2}mu^2$) has been converted back into GPE.)

The particle B can therefore rise by a further height of H, where

$$mgH = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mu^2 = \tfrac{1}{2}mu^2\left(\left(\frac{3M - m}{M + m}\right)^2 - 1\right),$$

so

$$\begin{aligned} H &= \frac{u^2}{2g}\left(\frac{9M^2 - 6Mm + m^2}{(M + m)^2} - \frac{M^2 + 2Mm + m^2}{(M + m)^2}\right) \\ &= \frac{u^2}{2g}\left(\frac{8M^2 - 8Mm}{(M + m)^2}\right) \\ &= \frac{4u^2}{g}\left(\frac{M(M - m)}{(M + m)^2}\right). \end{aligned}$$

Thus the maximum height reached by B after the first collision, assuming that the second collision occurs after B has started falling is

$$h + H = h + \frac{4M(M-m)u^2}{(M+m)^2 g}.$$

Finally, we have to explain why A does not catch up with B before B begins to fall. But this is easy: B initially has a greater upward velocity than A (as $v_B - v_A = 2u > 0$), so the height of B is always greater than the height of A. Therefore they can only collide again after A has bounced on the ground and is in its ascent while B is in its descent.

Alternative approach: using constant acceleration and "suvat"

An alternative approach is to use the formulæ for constant acceleration ("suvat"), as follows.

Just before collision, B has speed u, so the height H of B at this point is given by the "suvat" equation $v^2 = u^2 + 2as$, taking positive to be downwards:

$$u^2 = 0^2 + 2g(h - H),$$

giving $H = h - u^2/2g$.

Immediately after the collision, B has velocity upwards given by equation (4) above. At the maximum height, h_{max}, the speed of B is zero, so we can determine the maximum height using $v^2 = u^2 + 2as$ again; this time, we take positive to be upwards, so $a = -g$:

$$0^2 = \left(\frac{(3M-m)u}{M+m}\right)^2 - 2g(h_{\text{max}} - H).$$

(Note that $h_{\text{max}} - H > 0$.)

Rearranging this gives

$$\begin{aligned}
h_{\text{max}} &= H + \frac{1}{2g}\left(\frac{(3M-m)u}{M+m}\right)^2 \\
&= h - \frac{u^2}{2g} + \frac{u^2}{2g}\left(\frac{(3M-m)}{M+m}\right)^2 \\
&= h + \frac{u^2}{2g}\left(\frac{(3M-m)^2 - (M+m)^2}{(M+m)^2}\right) \\
&= h + \frac{u^2}{2g}\left(\frac{8M^2 - 8Mm}{(M+m)^2}\right) \\
&= h + \frac{4M(M-m)u^2}{(M+m)^2 g}
\end{aligned}$$

as required.

Question 11

A thin non-uniform bar AB of length $7d$ has centre of mass at a point G, where $AG = 3d$. A light inextensible string has one end attached to A and the other end attached to B. The string is hung over a smooth peg P and the bar hangs freely in equilibrium with B lower than A. Show that

$$3\sin\alpha = 4\sin\beta\,,$$

where α and β are the angles PAB and PBA, respectively.

We begin by drawing a diagram of the situation, showing the forces involved (the tension in the string, which is the same at A and B since the peg is smooth, and the weight of the bar acting through G). Clearly $BG = 4d$, which we have shown as well.

We have indicated the angles α, β and ϕ as defined in the question, and have also introduced the angle θ as angle AGP. The point M is the foot of the perpendicular from P to AB, which is used in some of the methods of solution.

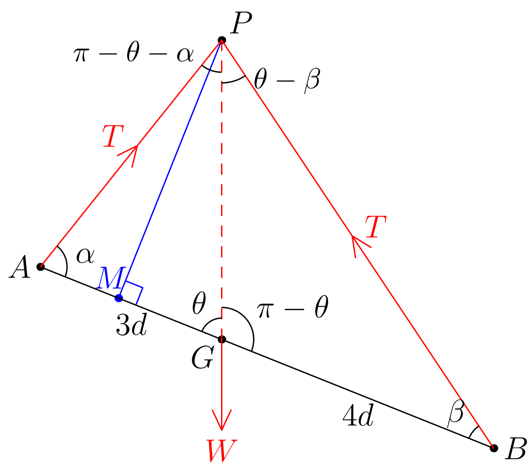


Diagram 1: Using θ for angle between rod and vertical

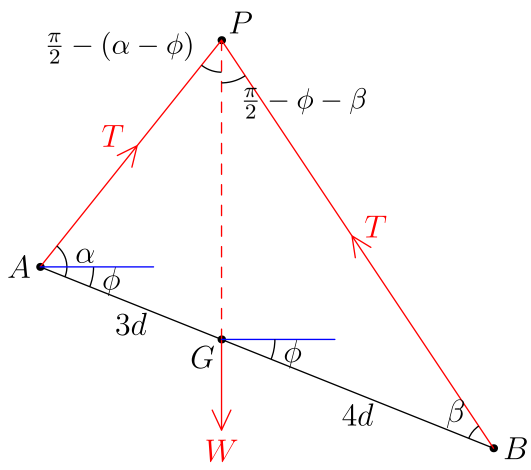


Diagram 2: Using ϕ for angle between rod and horizontal

Note that we have drawn the sketch with the weight passing through P. This must be the case: both tensions pass through P and the system is in equilibrium. So taking moments around P shows that W times the distance of the line of force of W from P must be zero, so that W acts through P.

The simplest way of showing that $3\sin\alpha = 4\sin\beta$ is to take moments about G:

$$\mathscr{M}(\widehat{G}) \qquad T.3d\sin\alpha - T.4d\sin\beta = 0$$

so that $3\sin\alpha = 4\sin\beta$.

An alternative approach is to apply the sine rule to the triangles PAG and PBG and resolve horizontally; this, though, is a somewhat longer-winded method.

> *Given that $\cos\beta = \frac{4}{5}$ and that α is acute, find in terms of d the length of the string and show that the angle of inclination of the bar to the horizontal is* $\arctan\frac{1}{7}$.

From $\cos\beta = \frac{4}{5}$, we deduce $\sin\beta = \frac{3}{5}$, and hence $\sin\alpha = \frac{4}{3}\sin\beta = \frac{4}{5}$. Thus $\cos\alpha = \pm\frac{3}{5}$, and since α is acute, $\cos\alpha = \frac{3}{5}$.

There are numerous ways of finding the length of the string, l, in terms of d. We present a few approaches here.

Approach 1: Show that $\angle APB = \frac{\pi}{2}$

To find the length of the string, we first find the angle APB.

One method is to note that $\angle APB = \pi - \alpha - \beta$, so

$$\begin{aligned}\sin(\pi - \alpha - \beta) &= \sin(\alpha + \beta)\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \tfrac{4}{5}\times\tfrac{4}{5} + \tfrac{3}{5}\times\tfrac{3}{5}\\ &= 1,\end{aligned}$$

so $\angle APB = \frac{\pi}{2}$ and the triangle APB is right-angled at P.

Alternatively, as $\sin\alpha = \cos\beta$, we must have $\alpha + \beta = \frac{\pi}{2}$, so $\angle APB = \frac{\pi}{2}$.

Thus $AP = AB\cos\alpha = 7d.\frac{3}{5} = \frac{21}{5}d$ and $BP = AB\sin\alpha = 7d.\frac{4}{5} = \frac{28}{5}d$, so the string has length $(\frac{21}{5} + \frac{28}{5})d = \frac{49}{5}d$.

Approach 2: Trigonometry with the perpendicular from P

In the triangle APB, we draw a perpendicular from P to AB, meeting AB at M. Then $PM = AP\sin\alpha = BP\sin\beta$. (This can also be shown directly by applying the sine rule: $AP/\sin\beta = BP/\sin\alpha$.)

We also have $AB = AM + BM = AP\cos\alpha + BP\cos\beta = 7d$.

Now using our known values of $\sin\alpha$, etc., these equations become $\frac{4}{5}AP = \frac{3}{5}BP$ so that $BP = \frac{4}{3}AP$, and $\frac{3}{5}AP + \frac{4}{5}BP = 7d$.

Combining these gives $\frac{3}{5}AP + \frac{16}{15}AP = 7d$, so $AP = \frac{21}{5}d$ and hence $BP = \frac{28}{5}d$ and $l = AP + BP = \frac{49}{5}d$.

Approach 3: Cosine rule

We apply the cosine rule to the triangle APB, and we write $x = AP$ and $y = BP$ for simplicity. This gives us

$$\begin{aligned}x^2 &= AB^2 + y^2 - 2AB.y\cos\beta \qquad \text{or}\\ y^2 &= AB^2 + x^2 - 2AB.x\cos\alpha.\end{aligned}$$

There are different ways of continuing from here. The most straightforward is probably to begin by showing that $BP = \frac{4}{3}AP$ or $y = \frac{4}{3}x$ as in Approach 2. This then simplifies the two equations to give

$$\begin{aligned}(\tfrac{3}{4}y)^2 &= (7d)^2 + y^2 - 2(7d).y.\tfrac{4}{5} \qquad \text{or}\\ (\tfrac{4}{3}x)^2 &= (7d)^2 + x^2 - 2(7d).x.\tfrac{3}{5}.\end{aligned}$$

We can then expand and rearrange these quadratics to get

$$\begin{aligned}\tfrac{7}{16}y^2 - \tfrac{56}{5}dy + 49d^2 &= 0 \qquad \text{or}\\ \tfrac{7}{9}x^2 + \tfrac{42}{5}dx - 49d^2 &= 0.\end{aligned}$$

Dividing by 7 and clearing fractions (multiplying by 80 and 45 respectively) gives

$$\begin{aligned}5y^2 - 128dy + 560d^2 &= 0 \qquad \text{or}\\ 5x^2 + 54dx - 315d^2 &= 0.\end{aligned}$$

These quadratics turn out to factorise as

$$\begin{aligned}(y - 20d)(5y - 28d) &= 0 \qquad \text{or}\\ (x + 15d)(5x - 21d) &= 0.\end{aligned}$$

The first equation gives two possibilities: $y = 20d$ or $y = \frac{28}{5}d$, whereas the second only gives one: $x = \frac{21}{5}d$.

For the first equation, $y = 20d$ would imply $x = \frac{3}{4}y = 15d$, but then we would have

$$\cos\alpha = \frac{AB^2 + x^2 - y^2}{2AB.x} = \frac{49d^2 + 225d^2 - 400d^2}{14d.15d} < 0,$$

which is not possible as α is acute.

So we must have $x = \frac{21}{5}d$ and $y = \frac{28}{5}d$, and hence $l = x + y = \frac{49}{5}d$.

Approach 4: Sine rule

Using the sine rule on the triangle APB, we have

$$\frac{AB}{\sin(\pi - \alpha - \beta)} = \frac{AP}{\sin\beta} = \frac{BP}{\sin\alpha}.$$

We use $\sin(\pi - \phi) = \sin\phi$ and the addition (compound angle) formula to write

$$\begin{aligned}\sin(\pi - \alpha - \beta) &= \sin(\alpha + \beta)\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\\ &= \tfrac{4}{5} \times \tfrac{4}{5} + \tfrac{3}{5} \times \tfrac{3}{5}\\ &= 1,\end{aligned}$$

so that the sine rule becomes

$$\frac{7d}{1} = \frac{AP}{3/5} = \frac{BP}{4/5},$$

giving $AP = \frac{21}{5}d$, $BP = \frac{28}{5}d$ and hence $l = AP + BP = \frac{49}{5}d$.

We now find ϕ, the angle of inclination of the bar to the horizontal. Referring to the above diagrams, we have $\phi = \frac{\pi}{2} - \theta$, so $\tan\phi = \cot\theta$.

Here again are several approaches to this problem.

Approach 1: Resolving forces horizontally and vertically

We resolve horizontally to get

$$\mathscr{R}(\rightarrow) \qquad T\sin(\pi-\theta-\alpha) - T\sin(\theta-\beta) = 0.$$

Therefore we get

$$\sin(\theta+\alpha) = \sin(\theta-\beta).$$

We now use the addition formula for sine to expand these, and then substitute in our values for $\sin\alpha$, etc., giving:

$$\sin\theta\cos\alpha + \cos\theta\sin\alpha = \sin\theta\cos\beta - \cos\theta\sin\beta$$

so

$$\tfrac{3}{5}\sin\theta + \tfrac{4}{5}\cos\theta = \tfrac{4}{5}\sin\theta - \tfrac{3}{5}\cos\theta$$

giving

$$\tfrac{7}{5}\cos\theta = \tfrac{1}{5}\sin\theta.$$

Dividing by $\cos\theta$ now gives $\cot\theta = \frac{1}{7}$, hence the angle made with the horizontal is given by $\tan\phi = \frac{1}{7}$, yielding $\phi = \arctan\frac{1}{7}$ as required.

Alternatively, using ϕ instead of θ in the original equations, we get

$$\mathscr{R}(\rightarrow) \qquad T\sin(\tfrac{\pi}{2}+\phi-\alpha) - T\sin(\tfrac{\pi}{2}-\phi-\beta) = 0,$$

which simplifies (on dividing by $T \neq 0$ and using $\sin(\frac{\pi}{2}-x) = \cos x$) to

$$\cos(\alpha-\phi) - \cos(\phi+\beta) = 0.$$

The rest of the argument follows as before.

Approach 2: Resolving forces parallel and perpendicular to the rod

We resolve parallel to the rod to get

$$\mathscr{R}(\searrow) \qquad T\cos\alpha + W\cos\theta - T\cos\beta = 0$$

and perpendicular to the rod to get

$$\mathscr{R}(\nearrow) \qquad T\sin\alpha - W\sin\theta + T\sin\beta = 0.$$

Rearranging these gives:

$$\begin{aligned} W\cos\theta &= -T\cos\alpha + T\cos\beta \\ W\sin\theta &= T\sin\alpha + T\sin\beta. \end{aligned}$$

Dividing these equations gives

$$\begin{aligned} \cot\theta &= \frac{-\cos\alpha + \cos\beta}{\sin\alpha + \sin\beta} \\ &= \frac{-\frac{3}{5}+\frac{4}{5}}{\frac{4}{5}+\frac{3}{5}} \\ &= \tfrac{1}{7}. \end{aligned}$$

Since $\tan\phi = \cot\theta$, as we noted above, we have $\tan\phi = \frac{1}{7}$, so $\phi = \arctan\frac{1}{7}$ as required.

Approach 3: Dropping a perpendicular

In the triangle APB, we draw a perpendicular from P to AB, meeting AB at M. Then $PM = AP\sin\alpha = \frac{21}{5}d.\frac{4}{5} = \frac{84}{25}d$ and $AM = AP\cos\alpha = \frac{21}{5}d.\frac{3}{5} = \frac{63}{25}d$. As $AG = 3d$, it follows that $MG = AG - AM = \frac{12}{25}d$.

Then (see the diagram above) we have $\tan\theta = PM/MG = \frac{84}{25}d/\frac{12}{25}d = 7$ so that $\cot\theta = \tan\phi = \frac{1}{7}$, giving $\phi = \arctan\frac{1}{7}$ as required.

Approach 4: PG bisects $\angle APB$

As in approach 1 above, we resolve horizontally (using diagram 2) to get

$$T\sin(\tfrac{\pi}{2} - (\alpha - \phi) - T\sin(\tfrac{\pi}{2} - \phi - \beta) = 0.$$

Since both of the angles involved here are acute (as the triangle BGP has an obtuse angle at G), they must be equal, giving $\alpha - \phi = \phi + \beta$, so that $2\phi = \alpha - \beta$.

Hence we have

$$\begin{aligned}\cos 2\phi &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \tfrac{3}{5}.\tfrac{4}{5} + \tfrac{4}{5}.\tfrac{3}{5} \\ &= \tfrac{24}{25}.\end{aligned}$$

We therefore deduce using $\cos 2\phi = 2\cos^2\phi - 1$ that $\cos^2\phi = \frac{49}{50}$ and $\sin^2\phi = \frac{1}{50}$. It follows that $\tan^2\phi = \sin^2\phi/\cos^2\phi = \frac{1}{49}$, giving $\tan\phi = \frac{1}{7}$ (the positive root as ϕ is acute) or $\phi = \arctan\frac{1}{7}$ as required.

Question 12

I am selling raffle tickets for £1 per ticket. In the queue for tickets, there are m people each with a single £1 coin and n people each with a single £2 coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

(i) In the case $n = 1$ and $m \geqslant 1$, find the probability that I am able to sell one ticket to each person in the queue.

I can sell one ticket to each person as long as I have a £1 coin when the single person with a £2 coin arrives, which will be the case as long as they are not the first person in the queue. Thus the probability is

$$1 - \frac{1}{m+1} = \frac{m}{m+1}.$$

(ii) By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geqslant 2$ is $\dfrac{m-1}{m+1}$.

This time, I can sell to all the people as long as I have one £1 coin when the first £2 coin is given to me and I have received at least two £1 coins (in total) by the time the second £2 coin is offered.

So we consider the first three people in the queue and the coin they bring; in the table below, "any" means that either coin could be offered at this point. (This called also be represented as a tree diagram, of course.) The probabilities in black are those of success, the ones in red are for the cases of failure. Only one or the other of these needs to be calculated.

1st	2nd	3rd	Success?	Probability
£1	£1	any	yes	$\dfrac{m}{m+2} \times \dfrac{m-1}{m+1} = \binom{m}{2} \Big/ \binom{m+2}{2}$
£1	£2	£1	yes	$\dfrac{m}{m+2} \times \dfrac{2}{m+1} \times \dfrac{m-1}{m} = \binom{m-1}{1} \Big/ \binom{m+2}{2}$
£1	£2	£2	no	$\dfrac{m}{m+2} \times \dfrac{2}{m+1} \times \dfrac{1}{m} = 1 \Big/ \binom{m+2}{2}$
£2	any	any	no	$\dfrac{2}{m+2} = \binom{m+1}{1} \Big/ \binom{m+2}{2}$

To determine the probabilities in the table, there are two approaches. The first is to find the probability that the kth person brings the specified coin given the previous coins which have been brought; this is the most obvious method when this is drawn as a tree diagram. The second approach is to count the number of possible ways of arranging the remaining coins and to divide it by the total number of possible arrangements of the $m + 2$ coins, which is $\binom{m+2}{2} = \frac{1}{2}(m+2)(m+1)$.

Therefore the probability of success is

$$\frac{m(m-1)}{(m+2)(m+1)} + \frac{2(m-1)}{(m+2)(m+1)} = \frac{(m+2)(m-1)}{(m+2)(m+1)} = \frac{m-1}{m+1}.$$

Alternatively, we could calculate the probability of failure (adding up the probabilities in red) and subtract from 1 to get

$$1 - \frac{2}{(m+2)(m+1)} - \frac{2}{m+2} = 1 - \frac{2}{m+2} . \frac{1+(m+1)}{m+2} = 1 - \frac{2}{m+2} = \frac{m-1}{m+2}.$$

(iii) Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geqslant 3$ is $\dfrac{m-2}{m+1}$.

This time, it turns out that we need to consider the first five people in the queue to distinguish the two cases which begin with £1, £1, £2, £2; the rest of the method is essentially the same as in part (ii).

1st	2nd	3rd	4th	5th	Success?	Probability
£1	£1	£1	any	any	yes	$\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{m-2}{m+1} = \binom{m}{3} \Big/ \binom{m+3}{3}$
£1	£1	£2	£1	any	yes	$\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{3}{m+1} \times \frac{m-2}{m} = \binom{m-1}{2} \Big/ \binom{m+3}{3}$
£1	£1	£2	£2	£1	yes	$\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{3}{m+1} \times \frac{2}{m} \times \frac{m-2}{m-1} = \binom{m-2}{1} \Big/ \binom{m+3}{3}$
£1	£1	£2	£2	£2	no	$\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{3}{m+1} \times \frac{2}{m} \times \frac{1}{m-1} = 1 \Big/ \binom{m+3}{3}$
£1	£2	£1	£1	any	yes	$\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-1}{m+1} \times \frac{m-2}{m} = \binom{m-1}{2} \Big/ \binom{m+3}{3}$
£1	£2	£1	£2	£1	yes	$\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-1}{m+1} \times \frac{2}{m} \times \frac{m-2}{m-1} = \binom{m-2}{1} \Big/ \binom{m+3}{3}$
£1	£2	£1	£2	£2	no	$\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-1}{m+1} \times \frac{2}{m} \times \frac{1}{m-1}$

$$= 1 \Big/ \binom{m+3}{3}$$

£1	£2	£2	any	any	no	$\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{2}{m+1} = \binom{m}{1} \Big/ \binom{m+3}{3}$
£2	any	any	any	any	no	$\frac{3}{m+3} = \binom{m+2}{2} \Big/ \binom{m+3}{3}$

(Alternatively, the four cases beginning £1, £2 can be regarded as £1, £2 followed by 2 people with £2 coins and $m-1$ people with £1 coins, bringing us back into the case of part (ii). So the probability of success in these cases is $\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-2}{m}$, where the final fraction comes from the result of part (ii).)

Therefore the probability of success is

$$\begin{aligned}
&\frac{1}{(m+3)(m+2)(m+1)}\big(m(m-1)(m-2) + 3(m-1)(m-2) + \\
&\qquad 6(m-2) + 3(m-1)(m-2) + 6(m-2)\big) \\
&= \frac{m-2}{(m+3)(m+2)(m+1)}\big(m(m-1) + 6(m-1) + 12\big) \\
&= \frac{m-2}{(m+3)(m+2)(m+1)}(m^2 + 5m + 6) \\
&= \frac{m-2}{m+1}.
\end{aligned}$$

Similarly, the probability of success can be calculated by considering the probability of failure: the probability of success is therefore

$$\begin{aligned}
1 &- \frac{6}{(m+3)(m+2)(m+1)} - \frac{6}{(m+3)(m+2)(m+1)} - \frac{6m}{(m+3)(m+2)(m+1)} - \frac{3}{m+3} \\
&= 1 - \frac{12 + 6m + 3(m+1)(m+2)}{(m+3)(m+2)(m+1)} \\
&= 1 - \frac{3m^2 + 15m + 18}{(m+3)(m+2)(m+1)} \\
&= 1 - \frac{3(m+2)(m+3)}{(m+3)(m+2)(m+1)} \\
&= 1 - \frac{3}{m+1} \\
&= \frac{m-2}{m+1}.
\end{aligned}$$

There seems to be a pattern in these results, and one might conjecture that the probability of being able to sell one ticket to each person in the general case $m \geqslant n$ is $\frac{m+1-n}{m+1}$. This turns out to be correct, though the proof uses significantly different ideas from those used above.

Question 13

In this question, you may use without proof the following result:

$$\int \sqrt{4-x^2}\,\mathrm{d}x = 2\arcsin(\tfrac{1}{2}x) + \tfrac{1}{2}x\sqrt{4-x^2} + c\,.$$

A random variable X has probability density function f *given by*

$$\mathrm{f}(x) = \begin{cases} 2k & -a \leqslant x < 0 \\ k\sqrt{4-x^2} & 0 \leqslant x \leqslant 2 \\ 0 & \textit{otherwise,} \end{cases}$$

where k and a are positive constants.

(i) Find, in terms of a, the mean of X.

We know that $\int_{-\infty}^{\infty} \mathrm{f}(x)\,\mathrm{d}x = 1$, so we begin by performing this integration to determine k.

We have

$$\begin{aligned}\int_{-a}^{0} 2k\,\mathrm{d}x + \int_0^2 k\sqrt{4-x^2}\,\mathrm{d}x &= \big[2kx\big]_{-a}^{0} + k\big[2\arcsin\tfrac{x}{2} + \tfrac{1}{2}x\sqrt{4-x^2}\,\big]_0^2 \\ &= 2ak + k\big((2\arcsin 1 + 0) - (2\arcsin 0 + 0)\big) \\ &= 2ak + k\pi \\ &= k(2a+\pi) \\ &= 1,\end{aligned}$$

so $k = 1/(2a+\pi)$.

We can now work out the mean of X; we work in terms of k until the very end to avoid ugly calculations. We can integrate the expression $x\sqrt{4-x^2} = kx(4-x^2)^{\frac{1}{2}}$ either using inspection (as we do in the following) or the substitution $u = 4-x^2$, giving $\mathrm{d}u/\mathrm{d}x = -2x$, so that the integral becomes $k\int_4^0 -\frac{1}{2}u^{\frac{1}{2}}\,\mathrm{d}u = k\big[-\frac{1}{3}u^{\frac{3}{2}}\big]_4^0 = \frac{8}{3}k$.

$$\begin{aligned}\mathrm{E}(X) &= \int_{-\infty}^{\infty} x\mathrm{f}(x)\,\mathrm{d}x \\ &= \int_{-a}^{0} 2kx\,\mathrm{d}x + \int_0^2 kx\sqrt{4-x^2}\,\mathrm{d}x \\ &= \big[kx^2\big]_{-a}^{0} + \big[-\tfrac{k}{3}(4-x^2)^{\frac{3}{2}}\big]_0^2 \\ &= (0-ka^2) + (0 - (-\tfrac{k}{3}\times 4^{\frac{3}{2}})) \\ &= -ka^2 + \tfrac{8}{3}k \\ &= \frac{\frac{8}{3}-a^2}{2a+\pi} \\ &= \frac{8-3a^2}{3(2a+\pi)}.\end{aligned}$$

> *(ii)* *Let d be the value of X such that $\mathrm{P}(X > d) = \frac{1}{10}$. Show that $d < 0$ if $2a > 9\pi$ and find an expression for d in terms of a in this case.*

We have $d < 0$ if and only if $\mathrm{P}(X > 0) < \mathrm{P}(X > d) = \frac{1}{10}$, so we consider $\mathrm{P}(X > 0)$. Using the above integration (or noting that X is uniform for $x < 0$), we have

$$\begin{aligned}\mathrm{P}(X > 0) &= 1 - \mathrm{P}(X < 0)\\ &= 1 - 2ak\\ &= 1 - \frac{2a}{2a+\pi}\\ &= \frac{\pi}{2a+\pi}.\end{aligned}$$

Therefore $\mathrm{P}(X > 0) < \frac{1}{10}$ if and only if

$$\frac{\pi}{2a+\pi} < \frac{1}{10};$$

that is $10\pi < 2a + \pi$, or $2a > 9\pi$. Putting these together gives $d < 0$ if and only if $2a > 9\pi$.

In this case, as $d < 0$, we have

$$\mathrm{P}(X > d) = 1 - \mathrm{P}(X < d) = 1 - 2k(d - (-a)),$$

so $1 - 2k(d + a) = \frac{1}{10}$, so $d + a = \frac{9}{10}/2k$, giving

$$\begin{aligned}d &= \frac{9}{20k} - a\\ &= \frac{9(2a+\pi)}{20} - a\\ &= \frac{9\pi - 2a}{20}.\end{aligned}$$

Note that, since $2a > 9\pi$, this gives us $d < 0$ as we expect.

An alternative approach is to calculate the cumulative distribution function first. We have

$$\mathrm{F}(x) = \begin{cases} 0 & x < -a \\ 2k(x+a) & -a \leqslant x < 0 \\ k\big(2a + 2\arcsin\frac{1}{2}x + \frac{1}{2}x\sqrt{4-x^2}\big) & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

(though only the part with $-a \leqslant x \leqslant 0$ is actually needed).

Then we solve $\mathrm{F}(d) = \frac{9}{10}$. If it turns out that $d < 0$, then we have $\mathrm{F}(d) = 2k(d+a) = \frac{9}{10}$, which rearranges to give $d = (9\pi - 2a)/20$ as above. Now if $2a > 9\pi$, then $(9\pi - 2a)/20 < 0$ so that $F((9\pi - 2a)/20) = \frac{9}{10}$ and $d < 0$, as required.

(iii) Given that $d = \sqrt{2}$, find a.

We note that now $d > 0$, so we have to integrate to find a explicitly. We get

$$\begin{aligned}
\mathrm{P}(X > \sqrt{2}) = \int_{\sqrt{2}}^{2} k\sqrt{4-x^2}\,\mathrm{d}x &= \left[2\arcsin\tfrac{x}{2} + \tfrac{1}{2}x\sqrt{4-x^2}\,\right]_{\sqrt{2}}^{2} \\
&= k\big((2\arcsin 1 + 0) - (2\arcsin\tfrac{\sqrt{2}}{2} + \tfrac{\sqrt{2}}{2}\sqrt{4-2})\big) \\
&= k(\pi - \tfrac{\pi}{2} - 1) \\
&= k(\tfrac{\pi}{2} - 1) \\
&= \frac{\frac{\pi}{2} - 1}{2a + \pi} \\
&= \tfrac{1}{10}.
\end{aligned}$$

Thus $10(\frac{\pi}{2} - 1) = 2a + \pi$, so that $2a = 4\pi - 10$, giving our desired result: $a = 2\pi - 5$.

Alternatively, one could calculate $\mathrm{P}(X < \sqrt{2})$ in the same manner and find a such that this equals $\frac{9}{10}$.

As a check, it is clear that $2a = 4\pi - 10 < 9\pi$, so $d \geqslant 0$ from part (ii).