STEP MATHEMATICS 2

2019

Mark Scheme

STEP II 2019 Mark Scheme

1			$f'(x) = g(x) + (x-p)g'(x)$ Tangent passes through $(a, (a-p)g(a))$ Equation of tangent is $y = \big(g(a) + (a-p)g'(a)\big)(x-a) + (a-p)g(a)$ (or equivalent equation) Substitution of $x = p$ into equation of tangent $y = -(a-p)^2 g'(a)$ Verification that if $g'(a) = 0$, then $y = 0$ If $y = 0$ then $g'(a) = 0$ because $a \neq p$	M1 M1 A1 E1 E1 E1 (AG) (6 marks)
	(i)		$g(x) = A(x-q)(x-r)$ identified $g'(a) = 0 \Rightarrow 2a = r + q$ (legitimately obtained) Gradient of tangent is $g(a) + (a-p)g'(a)$ $= A(a-q)(a-r)$ $= -\frac{1}{4}A(r-q)^2$	M1 A1 (AG) M1 A1 (4 marks)
	(ii)		By symmetry, the gradient of the second tangent is $-\frac{1}{4}A(p-q)^2$ (can be implied) Parallel iff $(p-q)^2 = (q-r)^2$ $\Leftrightarrow q - p = r - q$ since $p < q < r$. Tangent at $x = q$, $y = A(q-p)(q-r)(x-q)$, Meets curve again when $(q-p)(q-r)(x-q) = (x-p)(x-r)(x-q)$ $\Leftrightarrow (q-p)(q-r) = (x-p)(x-r)$ since $x \neq q$ (cancellation must be justified for M1, can be awarded if used correctly on $(x-q)^2(x-p-r+q)$ later) $\Leftrightarrow (x-q)(x-p-r+q) = 0$ $\Leftrightarrow x = p + r - q$ or $x = q$ Therefore there is only one point of intersection between the tangent and the curve if and only if $p + r - q = q$, which is if and only if the tangents are parallel. *One E mark for each direction.*	 B1 M1 A1 E1 M1 M1 M1 A1 E1 E1 (AG) (10 marks)

2			Sketch with areas $\int_0^x f(t)\,dt$, $\int_0^{f(x)} f^{-1}(y)\,dy$ and rectangle correctly identified. (One mark any one)	G1 G1 (2 marks)
	(i)		$g(0)(g(0)^2+1)=0$ factorised $g(0)$ real so $g(0)=0$ (must be justified) $1=(3g(t)^2+1)g'(t)$ $(3g(t)^2+1)>0$ so $g'(t)>0$ $g(2)^3+g(2)-2=0$ $(g(2)-1)(g(2)^2+g(2)+2)=0$ $\Delta=-7<0$ so $g(2)=1$ or g(2)>0 justified $g^{-1}(s)=s^3+s$ $\int_0^2 g(t)dt=2g(2)-\int_0^{g(2)} g^{-1}(s)ds$ $=\frac{5}{4}$	M1 A1 (AG) M1 A1 (AG) M1 A1 B1 M1 A1 (9 marks)
	(ii)		$h(t)=g(t+2)$ so $h(0)=g(2)=1$ and $h'(t)>0$ $(h(8)-2)(h(8)^2+2h(8)+5)=0$ $h(8)=2$ correctly justified $h^{-1}(s)=s^3+s-2$ $\int_0^8 h(t)dt+\int_{h(0)}^{h(8)} h^{-1}(s)ds=16$ (or similar correct equation) $\int_0^8 h(t)dt=16-\int_1^2(s^3+s-2)ds$ $=16-[\frac{s^4}{4}+\frac{s^2}{2}-2s]_1^2$ (integration) $=12\frac{3}{4}$	M1 A1 M1 A1 B1 M1 A1 M1 A1 (9 marks)

3			$\lvert x_1 + x_2\rvert$ is maximised when both have the same sign, In which case $\lvert x_1 + x_2\rvert = \lvert x_1\rvert + \lvert x_2\rvert$. Thus, $\lvert x_1 + x_2\rvert \le \lvert x_1 + x_2\rvert$ (or by consideration of all four combinations of signs separately) $\lvert x_1 + \cdots + x_{n-1} + x_n\rvert \le \lvert x_1 + \cdots + x_{n-1}\rvert + \lvert x_n\rvert$ $\le \cdots$ $\le \lvert x_1\rvert + \cdots + \lvert x_{n-1}\rvert + \lvert x_n\rvert$ by induction	E1 E1 (2 marks)
	(i)	(a)	$\lvert f(x) - 1\rvert = \lvert a_1 x + \cdots + a_{n-1}x^{n-1} + x^n\rvert$ $\le \lvert a_1 x\rvert + \cdots + \lvert a_{n-1}x^{n-1}\rvert + \lvert x^n\rvert$ $= \lvert a_1\rvert\lvert x\rvert + \cdots + \lvert a_{n-1}\rvert\lvert x\rvert^{n-1} + \lvert x\rvert^n$ $\le A(\lvert x\rvert + \cdots + \lvert x\rvert^{n-1}) + \lvert x\rvert^n$ $\le A(\lvert x\rvert + \cdots + \lvert x\rvert^{n-1} + \lvert x\rvert^n)$ (justified) $= A\frac{\lvert x\rvert(1-\lvert x\rvert^n)}{1-\lvert x\rvert}$ $\le A\frac{\lvert x\rvert}{1-\lvert x\rvert}$ (justified)	 M1 M1 M1 M1 M1 A1 (AG) (6 marks)
		(b)	$1 \le \frac{A\lvert\omega\rvert}{1-\lvert\omega\rvert}$ using $f(\omega) = 0$ $1 \le (A+1)\lvert\omega\rvert$ (with sign of $1 - \lvert\omega\rvert$ justified) $A + 1 \ge 1 \ge \lvert w\rvert$	M1 A1 (AG) B1 (AG) (3 marks)
		(c)	If $\lvert\omega\rvert > 1$, $0 = \omega^n f\left(\frac{1}{\omega}\right)$ $= 1 + a_{n-1}\omega + \cdots + a_1\omega^{n-1} + \omega^n$ Inequalities continue to hold since $\lvert a_i\rvert \le A$ If $\lvert\omega\rvert = 1$, then $1 + A \ge 1 \ge \frac{1}{1+A}$ since $A > 0$	 M1 E1 E1 (3 marks)
	(ii)		$f(x) = x^5 - x^4 - \frac{100}{135}x^3 - \frac{91}{135}x^2 - \frac{126}{135}x + 1$ Use $A = 1$. Integer roots with $\frac{1}{2} \le \lvert\omega\rvert \le 2$ could only be ± 1 or ± 2 $f(\pm 2) \ne 0$ because numerator is odd (or any valid justification) $f(1) = -\frac{182}{135} \ne 0$ $f(1) = 0$ $x = 1$ is the only integer root.	B1 M1 M1 E1 A1 A1 (6 marks)

4	(i)		$\sin\frac{\pi}{9}\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ $=\frac{1}{2}\sin\frac{2\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ $=\frac{1}{8}\sin\frac{8\pi}{9}$ $=\frac{1}{8}\sin\frac{\pi}{9}$ (use of $\sin(\pi-x)=\sin(x)$) $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8}$	B1 M1 M1 A1 (4 marks)
	(ii)		$\sin\left(\frac{x}{2^n}\right)\prod_{k=0}^{n}\cos\left(\frac{x}{2^k}\right)$ $=\frac{1}{2}\sin\left(\frac{x}{2^{n-1}}\right)\prod_{k=0}^{n-1}\cos\left(\frac{x}{2^k}\right)$ $=\cdots$ (convincing use of induction or repeated application) $=\frac{\sin(2x)}{2^{n+1}}$ (induction end point correct) $\prod_{k=0}^{n}\cos\left(\frac{x}{2^k}\right)=\frac{\sin(2x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)}$ $\sum_{k=0}^{n}\log\left(cos\left(\frac{x}{2^k}\right)\right)=\log(\sin(2x))-\log\left(sin\left(\frac{x}{2^n}\right)\right)-\log(2^{n+1})$ $\sum_{k=0}^{n}\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)=-2\cot(2x)+\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)$ (justified with differentiation)	B1 M1 E1 A1 M1 (diff) M1 (division) A1 (AG) (7 marks)
	(iii)		B1 – switch to product starting at 0 M1 – set up as limiting case of product to n M1 – apply small angle for sin A1 – correct answer $\prod_{k=1}^{n}\cos\left(\frac{x}{2^k}\right)=\frac{\sin(2x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)\cos(x)}$ $=\frac{2\sin(x)}{2^{n+1}\sin\left(\frac{x}{2^n}\right)}$ $\sim\frac{\sin(x)}{2^n\times\left(\frac{x}{2^n}\right)}$ $=\frac{\sin(x)}{x}$	 M1 M1 M1 A1 (AG)

			$\sum_{j=2}^{n} \frac{1}{2^{j-2}} \tan\left(\frac{\pi}{2^j}\right)$	M1
			$= \sum_{k=0}^{n} \frac{1}{2^k} \tan\left(\frac{\pi/4}{2^k}\right)$	M1
			$= \lim_{n\to\infty} \left(\frac{1}{2^n} \cot\left(\frac{\pi/4}{2^n}\right) - 2\cot\left(\frac{\pi}{2}\right)\right)$	M1
			$= \lim_{n\to\infty} \left(\frac{1}{2^n \tan\left(\frac{\pi/4}{2^n}\right)}\right)$	M1
			$= \lim_{n\to\infty} \left(\frac{1}{2^n \left(\frac{\pi/4}{2^n}\right)}\right)$	A1
			$= \frac{4}{\pi}$	(9 marks)

5	(i)		Constant iff $a = f(a)$ $\Leftrightarrow a = p + (a - p)a$ $\Leftrightarrow 0 = (a - p)(a - 1)$ $\Leftrightarrow a = p$ or $a = 1$. Period 2 $\Leftrightarrow a = f(f(a))$ $\Leftrightarrow 0 = (a - p)(-1 + 2ap - pa^2 + a^3)$ (factorisation) $\Leftrightarrow 0 = (a - p)(a - 1)(a^2 + (1 - p)a + 1)$ If $a = p$ or $a = 1$, then sequence is constant. The quadratic has solutions when $(p - 1)^2 \geq 4$. If $(p - 1)^2 > 4$, i.e. $p > 3$ or $p < -1$, the solutions are distinct. They are not both $1, p$ since the sum of the roots is $p - 1 \neq p + 1$ So for $p > 3$ or $p < -1$, one of the roots of the quadratic gives a sequence of period 2. If $p = 3$, $a = 1$ so not period 2. If $p = -1$, $a = -1 = p$ so not period 2.	M1 M1 A1 M1 M1 A1 B1 M1 E1 E1 (AG) B1 B1 (12 marks)
	(ii)		No value of a for which the sequence is constant $\Leftrightarrow f(a) = a$ has no solution $\Leftrightarrow f(x) > x$ or $f(x) < x$ for all x But $f(x) > x$ for large x. So cannot have $f(x) < x$ for all x. If no value of a for which sequence constant, then $f(x) > x$ for all x So $f\big(f(x)\big) > f(x) > x$ for all x And hence no solution to $f\big(f(a)\big) = a$. Setting $p = q$, gives (i). Then if $-1 \leq p \leq 3$, there is no period 2 sequence but a constant sequence exists.	 E1 (→) E1 (←) E1 E1 E1 E1 E1 E1 (8 marks)

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<table>
<tr><td>6</td><td>(i)</td><td></td><td>If $y = mx + c$,
Then the differential equation becomes
$m = mx + c + x + 1$
$m = -1, c = -2$
$y = -x - 2$

$\frac{dy}{dx} = 0 \Rightarrow y + x + 1 = 0 \Rightarrow y = -x - 1$

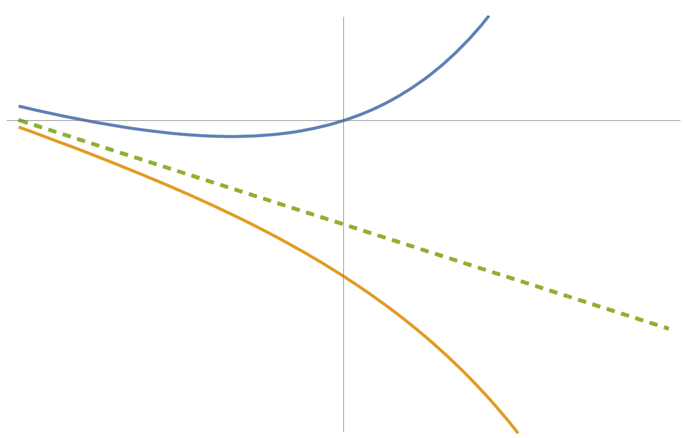
$y = y_3(x)$ cannot cross the line $y = -x - 2$.
So if $y_3(0) < -2$, it cannot reach the line $y = -x - 1$ and hence has no stationary points.

At a stationary point,
$\frac{d^2y}{dx^2} = \frac{dy}{dx} + 1 = y + x + 2 = 1 > 0$ so minimum

$\frac{dY}{dx} = Y + 2$
$\log(Y + 2) = x + c$
$Y = -2 + Ae^x$
$y = -x - 2 + Ae^x$

$y(0) = 0 \Rightarrow A = 2$
$y(0) = -3 \Rightarrow A = -1$ (attempt at both)

So $y = -x - 2 + 2e^x$
So $y = -x - 2 - e^x$ (both)


Curves tending to asymptote to the left
Curve above line through origin tending to ∞
Curve below line tending to −∞</td><td>M1

A1

E1 (AG)

E1

E1

M1
M1

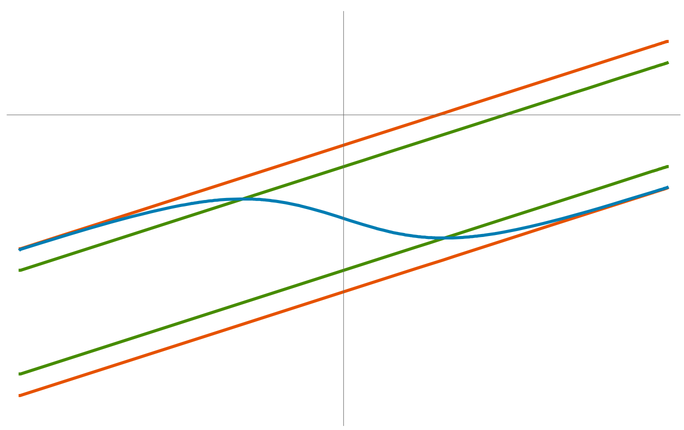
A1

M1

G1
G1
G1

(12 marks)</td></tr>
<tr><td></td><td>(ii)</td><td></td><td>If $y = mx + c$,
Then the differential equation becomes
$m = (mx + c)^2 + 4(mx + c) + x^2 - 4x - 2x(mx + c) + 3$
$0 = (m^2 - 2x + 1)x^2 + (2mc + 4m - 4 - 2c) + c^2 + 4c + 3 - m$

From x^2: $m = 1$
From x: $2mc + 4m - 4 - 2c = 2c + 4 - 4 - 2c = 0$</td><td></td></tr>
</table>

			From 1: $c^2 + 4c + 2 = 0 \Rightarrow c = -2 \pm \sqrt{2}$ Any of these equations Correct values of m and c Solutions: $y = x - 2 \pm \sqrt{2}$ $\frac{dy}{dx} = (y-x)^2 + 4(y-x) + 3$ (writing as a function of $y - x$) $= (y - x + 3)(y - x + 1)$ Stationary pts: $y = x - 1$ or $y = x - 3$ Between these lines the gradient is negative. (Correctly justified) So stationary points on $y = x - 1$ are maxima and stationary on $y = x - 3$ are minima.  Curve does not intersect other solutions Curve has stationary points on correct lines	 M1 A1 M1 A1 A1 A1 G1 G1 (8 marks)

7	(i)		$\boldsymbol{a}\cdot(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c})=0$ $\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{c}=-1$ <u>and cyclic permutations</u> $\boldsymbol{a}\cdot\boldsymbol{b}=-\frac{1}{2}$ legitimately obtained $\cos\theta=-\frac{1}{2}$ where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ $\theta=120°$ Similarly, the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $120°$. Justification of equilateral triangle by sketch or otherwise ABC is equilateral	M1 M1 A1 M1 A1 M1 M1 A1 (8 marks)
	(ii)		$\boldsymbol{a_1}\cdot\boldsymbol{a_2}+\boldsymbol{a_1}\cdot\boldsymbol{a_3}+\boldsymbol{a_1}\cdot\boldsymbol{a_4}=-1$ <u>and cyclic permutations</u> Linear combination of these equations $\boldsymbol{a_1}\cdot\boldsymbol{a_2}=\boldsymbol{a_3}\cdot\boldsymbol{a_4}$ (legitimately obtained)	M1 M1 A1 (AG) (3 marks)
		(a)	Angles $\angle A_1OA_2=\angle A_3OA_4$ By symmetry, $\angle A_2OA_3=\angle A_4OA_1$ The $\boldsymbol{a_i}$ are distinct and unit length so no angles are zero (accept justification by sketch) $A_1A_2A_3A_4$ is a rectangle	M1 M1 A1 (3 marks)
		(b)	$(A_1A_2)^2=(\boldsymbol{a_1}-\boldsymbol{a_2})^2$ $=\boldsymbol{a_1^2}+\boldsymbol{a_1^2}-2\boldsymbol{a_1}\cdot\boldsymbol{a_2}$ $=2-2\boldsymbol{a_1}\cdot\boldsymbol{a_2}$ By symmetry, $\boldsymbol{a_1}\cdot\boldsymbol{a_2}=\boldsymbol{a_1}\cdot\boldsymbol{a_3}=\boldsymbol{a_1}\cdot\boldsymbol{a_4}$ So $\boldsymbol{a_1}\cdot\boldsymbol{a_2}=-\frac{1}{3}$ So $(A_1A_2)^2=\frac{8}{3}$ $A_1A_2=\frac{2\sqrt{2}}{\sqrt{3}}$	 M1 M1 M1 A1 M1 A1 (6 marks)

8	(i)		$f(\boldsymbol{M}) = f(\boldsymbol{MI}) = f(\boldsymbol{M})f(\boldsymbol{I})$ $\Rightarrow f(\boldsymbol{I}) = 1$ since $f(\boldsymbol{M}) \neq 0$	M1 A1 (AG) (2 marks)
	(ii)		$f(J)^2 = f(J^2)$ $= f(I) = 1$ $\Rightarrow f(J) = -1$ since $f(J) \neq 1$ $f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ $= -f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ legitimately obtained $f\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right) = f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ $= -f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right)$ legitimately obtained	M1 M1 A1 M1 A1 (AG) M1 A1 (AG) (7 marks)
	(iii)		Using first equality in previous part (or otherwise correctly justified) $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = -f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right)$ $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = 0$ $f\left(\begin{pmatrix} a & b \\ ka & kb \end{pmatrix}\right) = f\left(\boldsymbol{K}\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right)$ $= 0$	 M1 M1 M1 A1 (AG) (4 marks)
	(iv)		$\boldsymbol{K}^{-1}\boldsymbol{PK} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ $f(\boldsymbol{K})f(\boldsymbol{K}^{-1}) = F(\boldsymbol{I}) = 1 \Rightarrow f(\boldsymbol{K}^{-1}) = f(\boldsymbol{K})^{-1}$ $f(\boldsymbol{K}^{-1}\boldsymbol{PK}) = f(\boldsymbol{K}^{-1})f(\boldsymbol{PK})$ (must use two stages) $= f(\boldsymbol{K}^{-1})f(\boldsymbol{P})f(\boldsymbol{K})$ $= f(\boldsymbol{P})$ $\boldsymbol{P}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $f(\boldsymbol{P}^2) = f(\boldsymbol{P}) \Rightarrow f(\boldsymbol{P}) = 0$ or 1 $\boldsymbol{P}^{-1}$ exists so $f(\boldsymbol{P})f(\boldsymbol{P}^{-1}) = 1 \Rightarrow f(\boldsymbol{P}) \neq 0$	B1 M1 M1 A1 (AG)? B1 M1 A1 (7 marks)

9	(i)		$\boldsymbol{r} = \begin{pmatrix} ut\cos\alpha \\ ut\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix}$ $\boldsymbol{r}^2 = u^2t^2\cos^2\alpha + u^2t^2\sin^2\alpha - ugt^3\sin\alpha + \frac{1}{4}g^2t^4$ $= u^2t^2 - ugt^3\sin\alpha + \frac{1}{4}g^2t^4$ $\frac{d}{dt}(\boldsymbol{r}^2) = 2u^2t - 3ugt^2\sin\alpha + g^2t^3$ $= t(2u^2 - 3ugt\sin\alpha + g^2t^2)$ $= t(2u^2 - \frac{9}{4}u^2\sin^2\alpha + (gt - \frac{3}{2}u\sin\alpha)^2)$ If $\sin\alpha < \frac{2\sqrt{2}}{3}$, then $2u^2 - \frac{9}{4}u^2\sin^2\alpha > 0$ and distance is always increasing. If $\sin\alpha > \frac{2\sqrt{2}}{3}$, then distance is decreasing at $t = \frac{3u}{2g}\sin\alpha$ Landing occurs at $t = \frac{2u}{g}\sin\alpha$, which is later (Or imagine falls through ground. Distance increasing while underground, so any decrease must be above ground)	M1 M1 A1 M1 A1 M1 M1 M1 A1 (AG) M1 E1 (11 marks)
	(ii)		$\boldsymbol{r} = \begin{pmatrix} ut\cos\alpha + vt \\ ut\sin\alpha - \frac{1}{2}gt^2 \end{pmatrix}$ $PQ^2 = (u\cos\alpha + v)^2t^2 + u^2t^2\sin^2\alpha - ugt^3\sin\alpha + \frac{1}{4}g^2t^4$ $\frac{d}{dt}(PQ^2) = 2t(u\cos\alpha + v)^2 + 2u^2t\sin^2\alpha + 2tu^2\sin^2\alpha - 3ugt^2\sin\alpha + g^2t^3$ $= t\left(2(u\cos\alpha + v)^2 + 2u^2\sin^2\alpha - \frac{9}{4}u^2\sin^2\alpha + \left(gt - \frac{3}{2}u\sin\alpha\right)^2\right)$ $= t\left(2(u\cos\alpha + v)^2 - \frac{1}{4}u^2\sin^2\alpha + \left(gt - \frac{3}{2}u\sin\alpha\right)^2\right)$ If $2\sqrt{2}v > (\sin\alpha - 2\sqrt{2}\cos\alpha)u$, then $8(u\cos\alpha + v)^2 > u^2\sin^2\alpha$ So PQ is increasing for all t.	B1 M1 A1 M1 M1 M1 A1 M1 A1 (AG) (9 marks)

10	(i)		Correct diagram Moments about A: $Wa\cos\theta\,(1+2k) = 2aT\sin 2\theta$ If $2k+1 > 4\sin\theta$ then $2T\sin 2\theta > W\cos\theta\,(4\sin\theta) = 2W\sin 2\theta$ Since $\sin 2\theta > 0$, $T > W$ and so the string will break.	B2 M1 M1 A1 A1 (AG) (6 marks)
	(ii)		Resolving forces vertically: $R = ((k+1)W - T\sin\theta)$ Resolving horizontally, ring will slip if: $T\cos\theta > \mu\big((k+1)W - T\sin\theta\big)$ (= max value for friction) Moments about A: $W(2k+1) = 4T\sin\theta$ $\mu\big((k+1)W - T\sin\theta\big) = \mu\left(\frac{4(k+1)}{2k+1} - 1\right)T\sin\theta$ $\mu\left(\frac{2k+3}{2k+1}\right)T\sin\theta$ If $2k+1 > (2k+3)\mu\tan\theta$, then $\mu\left(\frac{2k+3}{2k+1}\right)\sin\theta < \cos\theta$ So the ring will slip.	 M1 M1 M1 A1 M1 A1 (AG) (6 marks)
	(iii)		Attempt to solve breaking inequality for k Breaks at $k = \frac{4\sin\theta - 1}{2}$ Attempt to solve slipping inequality for k Slips at $k = \frac{3\mu\tan\theta - 1}{2(1-\mu\tan\theta)}$ If ring slips before it breaks: $\frac{3\mu\tan\theta - 1}{2(1-\mu\tan\theta)} < \frac{4\sin\theta - 1}{2}$ (for A1, do not allow >) Confirming that inequality is being multiplied by a positive quantity. $3\mu\tan\theta - 1 < 4\sin\theta - 1$ $\mu < \frac{2\cos\theta}{2\sin\theta + 1}$	M1 A1 B1 M1 A1 E1 M1 A1 (AG) (8 marks)

11	(i)		In both cases, award the M mark if all possible values of n_2 for at least 3 values of n_1 are identified. $\boldsymbol{n_3 = 9}$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 8$ $n_1 = 3; n_2 = 8, 7$ $n_1 = 4; n_2 = 8, 7, 6$ $n_1 = 5; n_2 = 8, 7, 6$ $n_1 = 6; n_2 = 8, 7$ $n_1 = 7; n_2 = 8$ $n_1 = 8; n_2$ has no options $\text{Total} = (1 + 2 + 3) \times 2 = 12$ $\boldsymbol{n_3 = 10}$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 9$ $n_1 = 3; n_2 = 9, 8$ $n_1 = 4; n_2 = 9, 8, 7$ $n_1 = 5; n_2 = 9, 8, 7, 6$ $n_1 = 6; n_2 = 9, 8, 7$ $n_1 = 7; n_2 = 9, 8$ $n_1 = 8; n_2 = 9$ $n_1 = 0; n_2$ has no options $\text{Total} = (1 + 2 + 3 + 4) \times 2 - 4 = 16$ $\boldsymbol{n_3 = 2n + 1}$ Total ways $= \big(1 + \cdots + (n - 1)\big) \times 2$ (method mark may be implicit) $= (n - 1)n$ $\boldsymbol{n_3 = 2n}$ Total ways $= \big(1 + \cdots + (n - 1)\big) \times 2 - (n - 1)$ (method mark may be implicit) $= (n - 1)^2$	M1 M1 A1 (both totals correct) M1 A1 M1 A1 (7 marks)
	(ii)		Total number of pairs is $\binom{N-1}{2} = \frac{1}{2}(N - 1)(N - 2)$ Justification for using first part of question $\boldsymbol{N = 2n + 1}$ $\text{Prob} = \frac{(n-1)n}{\frac{1}{2}(2n)(2n-1)} = \frac{n-1}{2n-1}$ $\boldsymbol{N = 2n}$ $\text{Prob} = \frac{(n-1)^2}{\frac{1}{2}(2n-1)(2n-2)} = \frac{n-1}{2n-1}$	M1 B1 A1 (AG) A1 (4 marks)

	(iii)		$\text{Prob} = \sum_{n=1}^{M} \frac{n-1}{2n-1} \times \mathbb{P}(\text{largest rod is } 2n+1) + \sum_{n=1}^{M} \frac{n-1}{2n-1} \times \mathbb{P}(\text{largest rod is } 2n)$	M1 A1 (ft)
			$= \sum_{n=1}^{M} \frac{n-1}{2n-1}\left(\frac{\binom{2n}{2}}{\binom{2M+1}{3}} + \frac{\binom{2n-1}{2}}{\binom{2M+1}{3}}\right)$	
			$= \frac{6}{(2M+1)(2M)(2M-1)} \cdot \frac{1}{2}\sum_{n=1}^{M} \frac{n-1}{2n-1}(2n(2n-1) + (2n-1)(2n-2))$ (Use of formula for binomial coefficients with factorials cancelled)	M1 A1
			$= \frac{3}{M(2M+1)(2M-1)} \sum_{n=1}^{M} (n-1)\,(2n-1)$	M1
			Use of $\sum_1^K k^2 = \frac{1}{6}K(K+1)(2K+1)$ to simplify above	M1
			$= \frac{3}{M(2M+1)(2M-1)}\left(\frac{1}{3}M(M+1)(2M+1) - 3 \times \frac{1}{2}M(M+1) + M\right)$	M1
			$= \frac{1}{2(2M+1)(2M-1)}(4M^2 - 3M - 1)$	
			$= \frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$	M1 A1 (9 marks)

12	(i)		$\mu = \int_0^1 nx^n\,dx = \frac{n}{n+1}$	M1 A1
			$\mathbb{E}(X^2) = \int_0^1 nx^{n+1}\,dx = \frac{n}{n+2}$	M1
			$\sigma^2 = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 = \frac{n}{(n+1)^2(n+2)}$	M1 A1 (AG)
				(5 marks)
	(ii)		$\text{LQ} = \frac{1}{2}, \text{UQ} = \frac{\sqrt{3}}{2}$	M1
			$\text{IQR} = \frac{\sqrt{3}-1}{2}$	A1
			$2\sigma = \frac{\sqrt{2}}{3}$	B1
			Squaring IQR and 2σ	M1
			Comparing $\sqrt{3}$ with a rational number… …by squaring both sides	M1 M1
			Argument correct	A1
				(7 marks)
	(iii)		$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$	A1
			$+\frac{n(n-1)\cdots(n-k+1)}{k!}x^k + \cdots$	A1
			$\text{LQ} = \left(\frac{1}{4}\right)^{1/n}$ and $\text{Median} = \left(\frac{1}{2}\right)^{1/n}$	B1
			$\left(\frac{1}{\mu}\right)^n = \left(1+\frac{1}{n}\right)^n > 1 + n\left(\frac{1}{n}\right) = 2$	M1
			So $\mu < \left(\frac{1}{2}\right)^{1/n}$	A1
			$\left(\frac{1}{\mu}\right)^n = \left(1+\frac{1}{n}\right)^n < 1 + n\left(\frac{1}{n}\right) + \frac{n^2}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{n^k}{k!}\left(\frac{1}{n}\right)^k + \cdots$	M1
			$< 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} + \cdots$ < 4	M1
			So $\mu > \left(\frac{1}{4}\right)^{1/n}$	A1
				(8 marks)