STEP 3 2012 Hints and Solutions

1. The stem integrates to give $z = y^n \left(\frac{dy}{dx}\right)^2$ and for part (i) using $n = 1$, the stem gives $\frac{dz}{dx} = \sqrt{y}\frac{dy}{dx}$ which can solved for *z* using the initial conditions, and the integrated stem is a first order differential equation for *y*, which when solved, again with the initial conditions, produces the required result. Part (ii) follows the same pattern, with $n = -2$ instead, which has solution $y = e^{\frac{1}{2}x^2}$.

2. The simplification in the opening is $\left(1 - x^{2^{n+1}}\right)$, obtained by repeated use of the difference of two squares. A simple algebraic rearrangement, followed by taking a limit, the logarithm of both expressions, and differentiation produces the other three results in part (i). Part (ii) can be obtained by replacing x by x^3 in $\ln(1-x) = -\sum_{r=0}^{\infty} \ln\left(1 + x^{2^r}\right)$ from part (i), factorising the difference and the sums of cubes and subtracting that part (i) result before differentiating. An alternative is to replicate part (i) using instead the product

$(1 + x + x^2)(1 - x + x^2)(1 - x^2 + x^4)(1 - x^4 + x^8) \dots \left(1 - x^{2^n} + x^{2^{n+1}}\right)$, but then a little extra care is required with the rearrangement, and consideration of the limit.

3. The two parabolas, with vertices oriented in the direction of the positive axes, touch in the third quadrant in case (a), and in the first quadrant in the other three cases. In case (b), there are intersections in the second and third quadrants, in (c) in the third and fourth, and in (d), the trickiest case, they are in the third quadrant and in the first, between the touching point and the vertex of the parabola on the *x* axis. The first result of part (ii)is obtained by eliminating y between the two equations, the second by differentiating and equating gradients, and the third, by eliminating x^4 , $(a^4 = aa^3)$, from the first result using the second.

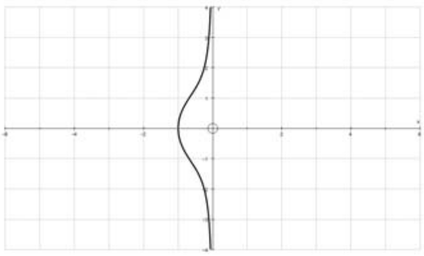
The cases that arise are $a = 1 , = 21 , \frac{k}{m} = \frac{21}{12} < 2$ (d), $a = \frac{-1+\sqrt{13}}{2}$, $k = \frac{13}{2}\sqrt{13} - \frac{5}{2}$, $\frac{k}{m} < 2$ (d), and $a = \frac{-1-\sqrt{13}}{2}$, $k = \frac{-13}{2}\sqrt{13} - \frac{5}{2}$, $k < 0$ (a).

4. Writing $\frac{n+1}{n!}$ as $\frac{n}{n!} + \frac{1}{n!}$ and then cancelling the first fraction, give exponential series. Similarly, $(n+1)^2$ can be written as $n(n-1) + 3n + 1$, and $(2n-1)^3$ as $8n(n-1)(n-2) + 12n(n-1) + 2n - 1$, the latter giving the result $21e + 1$. Using partial fractions, $\frac{n^2+1}{(n+1)(n+2)}$ can be written as $1 + \frac{2}{n+1} - \frac{5}{n+2}$, the first term giving a GP, and the other two, log series. The result for part (ii) is thus $12 - 16\ln 2$.

5. For non-integer rational points, it makes sense to use values of $\cos\theta$ and $\sin\theta$ based on Pythagorean triples such as 3, 4, 5 or 5, 12, 13, The technique for (i) (b) can be used for (ii) (a), merely by changing the value of m, whereas for (ii) (b), a slightly more involved expression is needed such as $x = a\cos\theta + b\sqrt{m}\sin\theta$, $y = a\sin\theta - b\sqrt{m}\cos\theta$. For (ii) (c), there are two alternatives that work sensibly, $= a\cosh\theta + \sqrt{m}\sinh\theta$, $y = a\sinh\theta + \sqrt{m}\cosh\theta$
or $x = a\sec\theta + \sqrt{m}\ \tan\theta$, $y = a\tan\theta + \sqrt{m}\ \sec\theta$.

A completely different approach for the last part of the question is to write $x^2 - y^2 = (x+y)(x-y) = 7$ and to choose $x + y = a + b\sqrt{2}$ with nearly any choices of rational *a* and *b* possible. Then, as $x - y = \frac{7}{x+y}$, and numbers of the form $a + b\sqrt{2}$ are a field over the operations $\times$ and $+$, $x - y$ has to be of the correct form, and then solving for *x* and *y*, they likewise have to be of the required form.
Some possible solutions are
(i)(a) $(1,0)$ and $\left(\frac{3}{5}, \frac{4}{5}\right)$, (b) $(1,1)$, $1 + m$, and $\left(\frac{7}{5}, \frac{1}{5}\right)$ (using $m = 1$, $\cos\theta = \frac{3}{5}$)
(ii)(a) $(1, \sqrt{2})$ and $\left(\frac{3}{5} + \frac{4}{5}\sqrt{2}, \frac{4}{5} - \frac{3}{5}\sqrt{2}\right)$ (using $m = 2$, $\cos\theta = \frac{3}{5}$), (b) $\left(\frac{9}{5} + \frac{4}{5}\sqrt{2}, \frac{12}{5} - \frac{3}{5}\sqrt{2}\right)$ (using $a = 3$, $b = 1$, $\cos\theta = \frac{3}{5}$), (c) $\left(\frac{39}{5} + \frac{12}{5}\sqrt{2}, \frac{36}{5} + \frac{13}{5}\sqrt{2}\right)$ (using $m = 2$, $a = 3$ and $\cosh\theta = \frac{13}{5}$, $\sinh\theta = \frac{12}{5}$ or $\sec\theta = \frac{13}{5}$, $\tan\theta = \frac{12}{5}$)

6. Substituting $x + iy$ for *z* in the quadratic equation and equating real and imaginary parts yields the first two results, the imaginary gives two situations, one as required and the other substituted into the real gives the second. The first of these two results substituted into the real gives a circle radius 1, centre the origin, whilst the second gives the real axis without the origin. The second quadratic equation succumbs to the same approach giving the real axis without the origin (again), and a circle centre $(-1,0)$ radius 1 also omitting the origin. The same approach in the third case yields the real axis with $p = \frac{-x^2 \pm \sqrt{x^4 - 8x}}{2x}$ and considering the discriminant, $x < 0$ and $x \geq 2$. On the other hand, $p = -2x$ produces $y^2 = -\frac{x^3+1}{x}$



7. The second order differential equation $\ddot{y} + 7\dot{y} + 6y = 0$ may be obtained by differentiating the first equation, then substituting for $\dot{z}$ using the other equation, and then doing likewise for z using the first again. The solutions for y and z follow in the usual manner. Parts (i) and (ii) yield $z_1(t) = 2e^{-t} - 2e^{-6t}$ and $z_2(t) = \frac{c(e^6-1)}{(e^5-1)} e^{-t} - \frac{ce^5(e-1)}{(e^5-1)} e^{-6t}$ respectively. Part (iii) merely requires the sum to be expanded and two geometric progressions emerge so that $c = \frac{2e}{e-1} \frac{(e^5-1)}{(e^6-1)}$.

8. $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ so both expressions in part (i) equal -1. Considering either $[(F_n F_{n+3} - F_{n+1}F_{n+2}) - (F_{n-2}F_{n+1} - F_{n-1}F_n)]$ or $(F_n F_{n+3} - F_{n+1}F_{n+2}) + (F_{n-1}F_{n+2} - F_n F_{n+1})$, and applying the recurrence relation, both can be found to be zero. \so the given expression is shown to be 1 if *n* is odd and -1 if *n* is even. The tan compound angle formula enables the proof in part (iii) to be completed once the initial recurrence relation and the result from part (ii) have been applied to the expression obtained following algebraic simplification. Rearranging the result and substituting into the required sum gives, by the method of differences, $\sum_{r=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2r+1}}\right) = \frac{\pi}{4}$

9. Eliminating T_1, T_2 , a , and α between the five equations $m_1 g - T_1 = m_1 a$, $T_1 r - T_2 r = I\alpha$, $T_2 - m_2 g = m_2 a$, $\alpha = \frac{a}{r}$, and $P + Mg = Mg + T_1 + T_2$ yields the required result for I . The only difference in the second part is that the second equation becomes $T_1 r - T_2 r - C = I\alpha$, and so the same elimination yields $I = \frac{((m_1+m_2)P - 4m_1 m_2 g)r^2 - (m_1 - m_2)Cr}{(m_1+m_2)g - P}$, which can be seen to be smaller than that in part (i). The first four equations in this second part give $m_1 g - m_2 g - \frac{C}{r} = \left(m_1 + m_2 + \frac{I}{r^2}\right) a$, and so as $a > 0$, the required result follows.

10. After motion commences, the next at rest position has the string at $\frac{\pi}{3}$ to the vertical. Conserving energy between the two at rest positions gives $\lambda = 3mg$. Conserving energy for the general position and resolving radially, bearing in mind that the angle of the radius to the vertical is twice the angle of the string to the vertical, and using a double angle formula gives the required result. The discriminant being negative or completing the square demonstrates that the reaction force is always positive.

11. Various approaches can be used to find the energy terms. If potential energy zero level is taken to be at P, then the initial potential energy is $-2Mg\frac{L}{2}$. When the particle has fallen a distance *x*, the kinetic energy of the particle is $\frac{1}{2}mv^2$, the potential energy of the particle is $-mgx$, the potential energy of the part of the stationary piece of string of length *x* is $-\frac{x}{2L}2Mg\frac{x}{2}$, the potential energy of the remaining piece of (doubled up) string is $-\left(1 - \frac{x}{2L}\right)2Mg\left(x + \frac{1}{2}\left(L - \frac{1}{2}x\right)\right)$, and the kinetic energy of the shorter moving piece is $\frac{1}{2}\frac{L - \frac{1}{2}x}{2L}\,2M\,v^2$. This yields the first result and differentiation of it yields the second. As $0 < x \leq 2L$, $ML - \frac{Mx}{4} \geq \frac{ML}{2}$, as $Mgx > 0$ and the denominator is twice a square $\frac{Mgx\left(mL + ML - \frac{Mx}{4}\right)}{2\left(mL + ML - \frac{xM}{2}\right)^2} > 0$, the final result follows.

12. The sketched region contained by AB, and the two line segments connecting A and B to the centre of the triangle. A simple approach for the pdf is via $P(X > x) \propto \left(\frac{1}{3} - x\right)^2$, finding the constant of proportionality and then differentiating to give the required result. $E(X) = \frac{1}{9}$.

For the second part, $g(x) = 192\left(\frac{1}{4} - x\right)^2$, and so $E(X) = \frac{1}{16}$.

13. That $P(z < Z < z + \delta z | a < Z < b) = \frac{P(z<Z<z+\delta Z)}{P(a<Z<b)}$ yields the part (i) result. Similarly, that $E(X|X > 0) = E(\sigma Z + \mu | \sigma Z + \mu > 0)$ yields the first result of part (ii), and that

$m = E(|X|) = E(X|X > 0)\,P(X > 0) + E(-X|X < 0)\,P(X < 0)$ yields the second.

Using $Var(X) = E(X^2) - \mu^2 = \sigma^2$, and so

$Var(|X|) = E(|X|^2) - (E|X|)^2 = E(X^2) - m^2 = \mu^2 + \sigma^2 - m^2$