STEP II 2016 Solutions

Question 1

Use t_1 and t_2 to represent the value of the parameter t at each of the points P and Q. The equations of the two tangents can therefore be found in terms of t_1 and t_2 and the fact that POQ is a right angle can be used to find a relationship between t_1 and t_2. The point of intersection of the two tangents can therefore be found in terms of just t_1 and this is a pair of parametric equations for the curve that the point of intersection makes.

Substituting the parametric equations for C_1 into the equation for C_2 gives a cubic equation in t^2 which can be solved by inspection to show that there are just two intersections and so the two curves just touch, but do not cross.

Question 2

Substitute $c = a + b$ into the expression to show that $a + b - c$ is a factor. Once this is done, the symmetry shows that $b + c - a$ and $c + a - b$ must also be factors and therefore there is just a constant multiplier that needs to be deduced to obtain the full factorisation of (*).

For part (i), choices of a, b and c need to be made so that

$$a + b + c = x + 1$$

$$a^2 + b^2 + c^2 = \frac{2x^2 + 5}{2}$$

$$a^3 + b^3 + c^3 = \frac{4x^3 + 13}{4}$$

Once these have been identified the solutions to the equation follow from the factorisation already deduced.

Once the substitution $d + e = c$ has been made it is only necessary to identify the parts of the expression which differ from (*) in the first part of the question (which arise from the c^2 and c^3 terms). The factorisation and solution of the equation then follow a similar process to the first part of the question.

Question 3

The differentiation to show the result in part (i) should not present much difficulty, although it is important to show that all of the terms (and no others) are present.

For part (ii) observe that each individual term of $f_n(x)$ has a positive coefficient, so for any positive value of x the value of $f_n(x)$ must be positive.

For part (iii), use the result in part (i) to rewrite $f_n{}'(x)$ in terms of $f_n(x)$ and note that $f_n(a)$ and $f_n(b)$ must be 0. This means that any pair of roots must have a gradient of the same sign, which leads to an argument that there must be another root between the two. As this would lead to an infinite number of roots to a polynomial, there cannot be more than one root.

To establish the number of roots in the two cases consider the behaviour of the graph as $x \to \infty$ and as $x \to -\infty$

Question 4

The equation given can be rewritten as a quadratic in x. The discriminant then establishes the required result. To show the second result, show that $y^2 + 1 \geq (y\cos\theta - \sin\theta)^2$, which can be shown by writing $y\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$ and then this result is a quadratic inequality that leads directly to the next result.

In the case $y = \frac{4+\sqrt{7}}{3}$, careful manipulation of surds shows the required result and so the value of θ must be the value of α obtained in the previous section. Finally, the value of x can be obtained by returning to the original equation and substituting in the values that are known.

Question 5

The binomial expansion for $(1-x)^{-N}$ should be easy enough, it is then required to write the product in terms of factorials so that the expression can be written in terms of $\binom{p}{q}$.

Since the expansion of $(1-x)^{-1}$ involves a coefficient of 1 for every term, the coefficient of x^n in the expansion of $(1-x)^{-1}(1-x)^{-N}$ is simply the sum of the coefficients of all of the terms in the expansion of $(1-x)^{-N}$ up to and including the term in x^n.

The products in the sum on the right-hand side of the result in part (ii) should be recognisable as binomial coefficients in the case where the power is a positive integer, so use

$$(1+x)^p(1+x)^q \equiv (1+x)^{p+q}$$

and compare coefficients as in part (i).

Similarly for part (iii), identify that the result will come from consideration of

$$(1+x)^{N+m}(1+x)^{-m} \equiv (1+x)^N.$$

Question 6

Parts (i) and (ii) only require verification in each of the cases, so simply differentiate the functions given and substitute into the differential equation to confirm that they are solutions. Remember to check as well that the boundary conditions are satisfied.

For part (iii), differentiate the given formula for z and substitute into the differential equation. By observing that the new differential equation is of the same form as (*), but for $2n$ instead of n, the expression for $y_{2n}(x)$ can be established.

For part (iv), again differentiate the given formula, being careful about the application of the chain rule and substitute. Again, by comparing with (*) the final result should be clear.

Question 7

The first result can be shown by using a substitution into the integral, being careful to explain the change of sign when the limits of the integral are switched.

Simple application of knowledge of trigonometric graphs once the substitution has been made can be used to show that twice the integral is equivalent to integrating the function 1 over the interval.

Similarly, the remaining integrals can all be rearranged using standard trigonometric identities and knowledge of logarithms into forms that can be integrated from standard results once the substitution from (*) has been made.

Question 8

The integral required at the start of the question should be a straightforward one to evaluate. When making a sketch to illustrate the result in the second part, ensure that the sum is indicated by a series of rectangles, with the graph of the curve passing through the midpoints of the tops.

In part (i), the integral that would match the sum given results in an answer of 2, so this is the first of the estimates. The remaining estimates arise from using the integral to estimate most of the sum, but taking the first few terms as the exact values (so in each case the integration is taken from a different lower limit).

For part (ii), evaluate the integral for one particular term of the sum and note that it is approximately $\frac{1}{4r^4}$. Finally, using the most accurate estimate for $E\left(\frac{33}{20}\right)$ the sum from $r = 3$ onwards can be calculated and then the first two values of $\frac{1}{r^4}$ can be added to achieve the desired result.

Question 9

The result in part (i) follows from consideration of kinetic energy lost and work done.

In part (ii) apply conservation of momentum to the combined block and bullet after the bullet hits the block. By comparing to the case in part (i) the motion of the bullet until it is at rest relative to the block can be analysed. Once all of the relevant equations of motion have been written down, a series of simultaneous equations will have been found from which the values of b and c can be found.

Question 10

The first requirement will be to find the centre of mass of the triangle. Once this is done a diagram will be very useful and notations will need to be added for various distances, angles and the frictional force. From this diagram the forces can be resolved in two perpendicular directions and moments can be taken. This leads to a series of equations which can be solved to work out the value that the frictional force would have to take to prevent slipping. From this the required result can be established.

Question 11

The particles must collide if they would be in the same position for one particular value of t. Therefore, writing out the equations of motion for the two particles and eliminating the variables that are not needed the required result can be reached.

For the second part, the time of the collision can be found by considering the heights of the bullet and target at time t and noting that these must be equal. Once the value of t has been found, the fact that this must be positive leads to the inequality that is required for the first result.

For the final part, note that gravity affects both the bullet and target in the same way, so if it is ignored then the time of collision (if there is one) will be the same and this is a situation as in part (i). Clearly, in part (i) the two objects must be moving towards each other if there is to be a collision.

Question 12

Replace B with $(B \cup C)$ in the result that you must start with and then observe that $A \cap (B \cup C)$ is the same as $(A \cap B) \cup (A \cap C)$. The corresponding result for four events should be clear, but care must be taken to include all of the possible pairs.

The results for parts (i), (ii) and (iii) should be clear from consideration of arrangements in each case and the result required follows from the generalisation of the result from the start of the question.

The probability that the first card is in the correct position and none of the others is can be established and therefore the probability that exactly one card is in the correct position will be n times that.

Question 13

For part (i) the approximation of the binomial distribution by a normal distribution should be known and the area under the curve (applying a continuity correction) can then be approximated by a rectangle.

The second result follows from a similar approximation and the use of the formula for a probability from the binomial distribution.

Part (iii) follows from an approximation of a Poisson distribution with a normal distribution and again approximating the required area by a rectangle.