STEP 3 2013 Hints and Solutions

1. The first two results, whilst not necessarily included in current A2 specifications, are standard work. Applying them, $\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\ dx = 2\int_0^1 \frac{1}{(1-a^2)+(t+a)^2} dt$, which can then be evaluated using a change of variable to give $\frac{2}{\sqrt{1-a^2}}\left(\tan^{-1}\frac{1+a}{\sqrt{1-a^2}} - \tan^{-1}\frac{a}{\sqrt{1-a^2}}\right)$. To simplify this to obtain the required result, $\tan\left(\tan^{-1}\frac{1+a}{\sqrt{1-a^2}} - \tan^{-1}\frac{a}{\sqrt{1-a^2}}\right)$ must be simplified using the relevant compound angle formula.

It is fairly straightforward to show that $I_{n+1} + 2I_n = \int_0^{\frac{1}{2}\pi} \sin^n x\, dx$, so applying this for $n = 2, 1, 0$ and applying the main result of the question to evaluate I_0 , gives $I_3 = \left(\frac{9}{4} - \frac{8\sqrt{3}}{9}\right)\pi - 2$

2. It is elegant to multiply by the denominator, then differentiate implicitly, and finally multiply by the same factor again to achieve the desired first result. The general result can be proved by then using induction, or by Leibnitz, if known. The general result can be used alongside the expression for y, and the first derived result with the substitution $x = 0$ to show that the general term of the Maclaurin series for even powers of *x* is zero, and for odd powers of *x* is $\frac{2^{2r}(r!)^2}{(2r+1)!}x^{2r+1}$. Thus, as $y = x + \frac{2^2}{3!}x^3 + \frac{4^2 2^2}{5!}x^5 + \cdots$ the required infinite sum is $\frac{y}{x}$ with $x = \frac{1}{2}$, that is $\frac{2\pi\sqrt{3}}{9}$.

3. The scalar product of p_i with $\sum p_r$, which is of course zero, can be expanded giving $p_i.p_i = 1$ and three products $p_i.p_j$ which are equal by symmetry, giving the required result. Expanding the expression suggested in (i), gives $\sum_{i=1}^4 (p_i.p_i - 2x.p_i + x.x)$, which, bearing in mind that $p_i.p_i = 1$, $x.x = 1$, and that $x.\sum_{i=1}^4 p_i = 0$, gives the correct result. Considering that $p_1.p_2 = -\frac{1}{3}$, $p_2.p_2 = 1$, and that *a* is positive, enables the given values to be found. Similarly $p_1.p_3 = -\frac{1}{3}$, $p_2.p_3 = -\frac{1}{3}$, and $p_3.p_3 = 1$ yields P_3, $P_4 = \left(-\frac{\sqrt{2}}{3}, \pm\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right)$. In (iii), using the logic of (i), $(XP_i)^4 = \left((p_i - x).(p_i - x)\right)^2 = 4(1 - x.p_i)^2$, as required. Expanding this, and using the coordinates of X and those of P_i that have been found,

$$\sum_{i=1}^4 (XP_i)^4 = 16 + 4\left(z^2 + \left(\frac{2\sqrt{2}}{3}x - \frac{1}{3}z\right)^2 + \left(-\frac{\sqrt{2}}{3}x + \frac{\sqrt{2}}{\sqrt{3}}y - \frac{1}{3}z\right)^2 + \left(-\frac{\sqrt{2}}{3}x - \frac{\sqrt{2}}{\sqrt{3}}y - \frac{1}{3}z\right)^2\right)$$

$= 16 + 4\left(\frac{4}{3}x^2 + \frac{4}{3}y^2 + \frac{4}{3}z^2\right) = \frac{64}{3}$ which is sufficient.

4. The initial result is obtained by expanding the brackets and expressing the exponentials in trigonometric form. The (2*n*)th roots of -1 are $e^{i\frac{2m+1}{2n}\pi}$, $-n \le m \le n-1$, which lead to the factors of $z^{2n} + 1$ and these paired using the initial result give the required result. Part (i) follows directly from substituting $z = i$ in the previous result, and as *n* is even, $z^{2n} + 1 = 2$. Using the given factorisation in part (ii), the general result can be simplified by the factor

$z^2 - 2z\cos\frac{n}{2n}\pi + 1 = z^2 + 1$. Again substituting $z = i$, and that $\cos\frac{2n-r}{2n}\pi = -\cos\frac{r}{2n}\pi$ gives the evaluation required.

5. Writing q^nN as $qq^{n-1}N$, and employing the permitted assumption, as p and q are coprime, p divides $q^{n-1}N$. Repetitions of this argument imply finally that p divides N . Letting $N = pQ_1$, $q^nQ_1 = p^{n-1}$. Continuing this argument similarly gives the result $N = kp^n$. As a consequence, $q^nk = 1$, and thus q and k must both be 1 . Thus if $\sqrt[n]{N} = \frac{p}{q}$ where p and q are coprime, it is rational and can be written in lowest terms, then $q^nN = p^n$ and so $q = 1$ and thus $\sqrt[n]{N}$ is an integer. Otherwise, $\sqrt[n]{N}$ cannot be written as $\frac{p}{q}$, that is, it is irrational.

For (ii), using the same logic as in part (i), as b^a divides a^ad^b , b^a divides d^b, so $d^b = kb^a$, for some k . Likewise, $a^a = k'c^b$, for some integer k' , and thus $k'k = 1$, so $k = k' = 1$, and $d^b = b^a$. If p is a prime factor of d , then p divides d^b , and so b^a too. Writing $b^a = bb^{a-1}$, using the logic of the very first part of the question, if p does not divide b , p divides b^{a-1}, and repetition of this argument leads to a contradiction. So p is a prime factor of b . p^{mb} and p^{na} is the highest power of p that divides $d^b = b^a$. So $mb = na$, and $b = \frac{na}{m}$. So p^n divides na , but as a and b are coprime, p^n divides n and thus $p^n \leq n$. By the given result, this means $p = 1$, and as b is only divisible by 1 , $b = 1$. If r is a positive rational $\frac{a}{b}$, such that $r^r = \frac{c}{d}$ is rational, then $a^ad^b = b^ac^b$ so $b = 1$ and r is a positive integer.

6. The opening result is the triangle inequality applied to OW, OZ, and WZ where OW and OZ are represented by the complex numbers w and z.
Part (i) relies on using $|z - w|^2 = (z - w)(z - w)^*$, $(z - w)^* = (z^* - w^*)$, $|zw| = |z|\,|w|$, and substituting $wz^* + zw^* = (E - 2|zw|)$. Having obtained the desired equation , the reality of E is apparent from the reality of the other terms and its non-negativity is obtained from the opening result of the question. Part (ii) relies on the same principles as part (i).
The inequality can be most easily obtained by squaring it, and substituting for both numerator and denominator on the left hand side using parts (i) and (ii), and algebraic rearrangement leads to $E(1 - |z|^2)(1 - |w|^2) \geq 0$ which is certainly true. The argument is fully reversible as $|z| > 1$, and $|w| > 1$, $|zw^*| > 1$, and so $1 - zw^* \neq 0$ so the division is permissible, and the square rooting of the inequality causes no problem as the quantities are positive. The working follows identically if $|z| < 1$, and $|w| < 1$.

7. As $\frac{dE}{dx} = 2\,\frac{dy}{dx}\left(\frac{d^2y}{dx^2} + y^3\right)$ is zero for all x , $E(x)$ is constant, and $E(x) = \frac{1}{2}$ using the initial conditions. The deduction follows from the non-negativity of $\left(\frac{dy}{dx}\right)^2$. In part (ii), it can be shown that $\frac{dE}{dx} = -2x\left(\frac{dv}{dx}\right)^2 \leq 0$ for $x \geq 0$, and as initially $E(x) = \frac{10}{3}$, the deduction for $\cosh v(x)$ follows in the same way as that in part (i). In part (iii), the choice of $E(x)$ relies on $2\int(w\cosh w + 2\sinh w)dw$ so $E(x) = \left(\frac{dw}{dx}\right)^2 + 2\,(w\sinh w + \cosh w)$. Then $\frac{dE}{dx} = -2\left(\frac{dw}{dx}\right)^2(5\cosh x - 4\sinh x - 3) = -2\left(\frac{dw}{dx}\right)^2\frac{e^{-x}}{2}\,(e^x - 3)^2$, and initially $E(x) = \frac{5}{2}$.
The final result can be deduced as in the previous parts, with the additional consideration that $w\sinh w \geq 0$.

8. The sum is evaluated by recognising that it is a geometric progression with common ratio $e^{2i\pi/n}$ which may be summed using the standard formula and as $1 - e^{2i\pi/n} \neq 0$, the denominator

is non-zero so the sum is zero. By simple trigonometry, $s = d - r\cos\theta$. As $r = ks$, $r = \frac{kd}{1+k\cos\theta}$. Thus $l_j = \frac{kd}{1+k\cos\theta} + \frac{kd}{1+k\cos(\theta+\pi)}$ where $\theta = \alpha + (j-1)\pi/n$. Simplifying, $l_j = \frac{2kd}{1-k^2\cos^2\theta}$. The summation of the reciprocals of this expression is simply found using a double angle formula and then by expressing the trigonometric terms as the real part of the sum at the start of the question.

9. The volume is obtained as a volume of revolution $V = \int_x^R \pi(R^2 - t^2)dt$ which gives the result. Similarly, Newton's 2nd law gives $\frac{4}{3}\pi R^3 \rho_s \ddot{x} = V\rho g - \frac{4}{3}\pi R^3 \rho_s g$ which simplifies to the required result. Substituting $x = \frac{1}{2}R$ when $\ddot{x} = 0$ gives $\rho_s = \frac{5}{32}\rho$. Substituting $x = \frac{1}{2}R + y$ yields $\frac{5}{8}R^3\ddot{y} = g\left(-\frac{9}{4}R^2 y + \frac{3}{2}Ry^2 + y^3\right)$, so for small y this approximates to SHM with period $\frac{\pi}{3}\sqrt{\frac{10R}{g}}$.

10. The initial result can be obtained in a number of different ways, but probably use of the parallel axes rule is the simplest. Conserving angular momentum about P,

$mu(a+x) = mv(a+x) + \frac{1}{3}M(a^2 + 3x^2)\omega$ where v is the velocity of the particle after impact, and ω is the angular velocity of the beam after the impact, and by Newton's experimental law of impact $(a+x)\omega - v = eu$. **Eliminating** v between these two equations gives the quoted expression for ω . Substituting $m = 2M$, for maximum ω , $\frac{d\omega}{dx} = 0$. This gives a quadratic equation, with solutions $x = -\frac{1}{3}a$ and $x = -\frac{5}{3}a$. The latter is not feasible and the former can be shown to generate a maximum which equates to the given result.

11. As the distance from the vertex to the centre of the equilateral triangle is a , the extended length of each spring is $\frac{a}{\cos\theta}$ giving the tension in each as $kmg\frac{\left(\frac{a}{\cos\theta} - a\right)}{a}$ which simplifies to the given result. Resolving vertically $3T\sin\theta = 3mg$, and using the result for T , substituting $\theta = \frac{\pi}{6}$, and rationalising the denominator gives the required value for k . Conserving energy, when $\theta = \frac{\pi}{3}$, gravitational potential energy is $-3mga\tan\frac{\pi}{3}$, elastic potential energy is $\frac{3}{2}kmg\frac{\left(\frac{a}{\cos\frac{\pi}{3}} - a\right)^2}{a} = \frac{3}{2}kmga\left(\frac{1}{\cos\frac{\pi}{3}} - 1\right)^2$, whereas when $\theta = \frac{\pi}{6}$, gravitational potential energy is $-3mga\tan\frac{\pi}{6}$, elastic potential energy is $\frac{3}{2}kmga\left(\frac{1}{\cos\frac{\pi}{6}} - 1\right)^2$, and kinetic energy is $\frac{3}{2}mV^2$ hence giving V^2 .

12. $P(X_1 = 1) = \frac{a}{n}$, so $E(X_1) = \frac{a}{n}$. There are $\frac{n!}{a!b!}$ arrangements of the As and Bs, and the number of arrangements with a B in the $(k-1)$ th place and an A in the k th place is $\frac{(n-2)!}{(a-1)!(b-1)!}$, so $P(X_k = 1) = \frac{ab}{n(n-1)}$ for $2 \le k \le n$, and $E(X_i) = \frac{ab}{n(n-1)}$ if $i \ne 1$. These combine to give $E(S)$ correctly.

$X_1 X_j = 1$ only if the first letter is an A, the $(j-1)$ th letter is a B, and the j th letter is an A. This has probability $\frac{(n-3)!}{(a-2)!(b-1)!} / \frac{n!}{a!b!}$ giving $E(X_1 X_j)$ correctly.

$X_i X_j = 1$ only if the $(i-1)$ th letter is a B, and the i th letter is an A, the $(j-1)$ th letter is a B, and the j th letter is an A which has probability $\frac{(n-4)!}{(a-2)!(b-2)!} / \frac{n!}{a!b!}$ **so** $E(X_i X_j) = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$, and thus $\sum_{j=i+2}^{n} E(X_i X_j) = (n-i-1)\frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$ and so

$\sum_{i=2}^{n-2}\left(\sum_{j=i+2}^{n} E(X_i X_j)\right) = \sum_{i=2}^{n-2}\left((n-i-1)\frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}\right)$ which yields the required result.

$S^2 = \sum_{i=1}^{n} {X_i}^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} 2X_i X_j$ so $E(S^2) = \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)}$ which can be used to obtain $Var(S)$ correctly.

13. integrating $0 \le f(t) \le M$ between limits of 0 and x gives the result of (a) (i), and integrating the left hand side by parts yields part (ii). As $kF(y)f(y)$ is a probability density function, $\int_0^1 k\,F(y)f(y)dy = 1$, which can be evaluated using the result of (a) (ii) with $2g(x) = k$ and so $k = 2$. $E(Y^n) = \int_0^1 y^n\,2F(y)f(y)dy \le \int_0^1 y^n\,2Myf(y)dy = 2M\mu_{n+1}$ and using (a) (ii) , $E(Y^n) = \int_0^1 y^n\,2F(y)f(y)dy = 1 - n\int_0^1 y^{n-1}\big(F(y)\big)^2 dy$, as $\int_0^1 y^{n-1}\big(F(y)\big)^2 dy \le \int_0^1 y^{n-1}My\,F(y)dy = M\int_0^1 y^n\,F(y)dy$, integration by parts gives $\int_0^1 y^n\,F(y)dy = \frac{1}{n+1} - \frac{1}{n+1}\mu_{n+1}$. Part (iii) is derived from part (ii) by rearranging

$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \le 2M\mu_{n+1}$ and making μ_{n+1} the subject, then translating $n+1$ to n .