STEP MATHEMATICS 1

2019

Mark Scheme

1	Eqn. of line is $y - k = -\tan\theta(x-1)$ or $y + x\tan\theta = k + \tan\theta$		
	Eqn. of line found with substn. of $y = 0$, $x = 0$ in turn	**M1**	
	$X = \left(1 + k\cot\theta, 0\right)$ and $Y = \left(0, k + \tan\theta\right)$	**A1 A1**	**3**

(i) $A = \frac{1}{2}(OX)(OY) = \frac{1}{2}\left(1 + k\cot\theta\right)\left(k + \tan\theta\right)$

$= \frac{1}{2}\left(k^2\cot\theta + 2k + \tan\theta\right) = \dfrac{1}{2\tan\theta}\left(k + \tan\theta\right)^2$ — **B1 ft** (any correct form)

$\dfrac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{1}{2}\left(-k^2\mathrm{cosec}^2\theta + \sec^2\theta\right)$ — **M1 A1** for the differentiation

or $\dfrac{\mathrm{d}A}{\mathrm{d}\theta} = \dfrac{1}{2\tan\theta}2\left(k + \tan\theta\right)\sec^2\theta - \dfrac{1}{2\tan^2\theta}\sec^2\theta\left(k + \tan\theta\right)^2$

$= \dfrac{\left(k + \tan\theta\right)\sec^2\theta}{2\tan^2\theta}\left\{2\tan\theta - \left(k + \tan\theta\right)\right\} = \dfrac{\sec^2\theta\left(\tan\theta + k\right)\left(\tan\theta - k\right)}{2\tan^2\theta}$

$= 0$ when — **M1** derivate set = 0 and solved

Either $\tan\theta = -k$ ($\Rightarrow$ $A = 0$, but rejected since $\tan\theta > 0$ in given region)
(Not necessary to mention this explicitly)

or $\tan\theta = k \Rightarrow A = 2k$ — **A1** — **5**

(ii) $XY = 1 + k\cot\theta + k + \tan\theta + \sqrt{\left(1 + k\cot\theta\right)^2 + \left(k + \tan\theta\right)^2}$ — **M1** for attempt at XY

$= 1 + k\cot\theta + k + \tan\theta + \left(k + \tan\theta\right)\sqrt{\cot^2\theta + 1}$

NB $XY\sin\theta = k + \tan\theta$ (e.g.) gives XY without distance formula

$L = OX + OY + XY = 1 + \dfrac{k}{\tan\theta} + k + \tan\theta + \left(k\mathrm{cosec}\theta + \sec\theta\right)$ — **M1**

Use of relevant trig. identity to find XY without square-root and with all three sides involved
(No need to justify taking the +ve sq-rt. since given $0 < \theta < \frac{1}{2}\pi$)

$L = 1 + \tan\theta + \sec\theta + k\left(1 + \cot\theta + \mathrm{cosec}\theta\right)$ — **A1** legitimately(**AG**) — **3**

$\dfrac{\mathrm{d}L}{\mathrm{d}\theta} = k\left(-\mathrm{cosec}^2\theta - \mathrm{cosec}\theta\cot\theta\right) + \left(\sec^2\theta + \sec\theta\tan\theta\right)$ — **M1**

$= 0$ when $k = \dfrac{\sec\theta(\sec\theta + \tan\theta)}{\mathrm{cosec}\theta(\mathrm{cosec}\theta + \cot\theta)}$ — **A1**

$= \dfrac{\frac{1}{c}\left(\frac{1}{c} + \frac{s}{c}\right)}{\frac{1}{s}\left(\frac{1}{s} + \frac{c}{s}\right)} = \dfrac{\frac{1}{c^2}\left(1 + s\right)}{\frac{1}{s^2}\left(1 + c\right)} = \dfrac{s^2\left(1 + s\right)}{c^2\left(1 + c\right)}$ — **M1** trig. method for getting k

$= \dfrac{(1 - c)(1 + c)(1 + s)}{(1 - s)(1 + s)(1 + c)}$ — **M1** use of $c^2 + s^2 = 1$ etc.

$= \dfrac{1 - c}{1 - s}$ — **A1** legitimately (**AG**)

Allow the final 3 marks for using the given answer to verify that $\dfrac{\mathrm{d}L}{\mathrm{d}\theta} = 0$ (provided that $\theta = \alpha$ used)

5

Then $L_{\min} = \left(\frac{1-c}{1-s}+\frac{s}{c}\right)\left(1+\frac{c}{s}+\frac{1}{s}\right)$ — **M1** substituting back

Must use correct (given) expression for L

$$= \left(\frac{c-c^2+s-s^2}{c(1-s)}\right)\left(\frac{c+s+1}{s}\right)$$

M1 common denominators

$$= \left(\frac{c+s-1}{c(1-s)}\right)\left(\frac{c+s+1}{s}\right)$$

M1 for dealing with numerator

$(c+s)^2 - 1 = c^2 + s^2 + 2cs - 1$

$$= \frac{2cs}{cs(1-s)} = \frac{2}{1-\sin\alpha}$$

A1 final answer (exactly this)

4

2	$\dfrac{dy}{dx} = \dfrac{\frac{d}{dt}(2t^3)}{\frac{d}{dt}(3t^2)} = \dfrac{6t^2}{6t} = t$ so grad. tgt. at $P(3p^2, 2p^3)$ is p	**M1 A1**	
	Eqn. tgt. at P is then $y - 2p^3 = p(x - 3p^2)$ i.e. $y = px - p^3$	**M1 A1** legitimately (**AG**)	**4**
	$y = px - p^3$ meets $y = qx - q^3$ when $px - p^3 = qx - q^3 \Rightarrow (p - q)x = (p^3 - q^3)$	**M1** equating y's and rearranging for x	
	and since $p \neq q$, $x = p^2 + pq + q^2$, $y = pq(p + q)$	**A1 A1** x, y must be simplified	**3**
	Tgts. perpr. iff $pq = -1 \Rightarrow u = p - \dfrac{1}{p}$, $u^2 = p^2 + \dfrac{1}{p^2} - 2$	**M1 A1** seen or implied	
	and $P_1 = \left(p^2 + \dfrac{1}{p^2} - 1,\ -\left[p - \dfrac{1}{p}\right]\right) = (u^2 + 1, -u)$	**A1** (**AG**) legitimately	**3**
	EITHER $x = y^2 + 1$ meets $\dfrac{x^3}{27} = \dfrac{y^2}{4}$ OR $\left(\dfrac{u^2+1}{3}\right)^3 = t^6 = \left(\dfrac{-u}{2}\right)^2$	**M1**	
	when $4(y^2 + 1)^3 = 27y^2$ $\quad$ $4(u^2 + 1)^3 = 27u^2$		
	$4(v^6 + 3v^4 + 3v^2 + 1) = 27v^2$ $\quad$ ($v = u$ or y)		
	Use of cubic expansion, incl. 1-3-3-1 coefficients	**M1**	
	$4v^6 + 12v^4 - 15v^2 + 4 = 0$		
	$(v^2 + 4)(4v^4 - 4v^2 + 1) = 0$	**M1** attempt to find a factor	
	$(v^2 + 4)(2v^2 - 1)^2 = 0$	**A1** complete factorisation	
	$v^2 \neq -4 \Rightarrow y^2 = \frac{1}{2}$, (OR via $u = \mp\frac{1}{\sqrt{2}}$, $t = \pm\frac{1}{\sqrt{2}}$) $\quad$ $y = \pm\frac{1}{\sqrt{2}}$, $x = \frac{3}{2}$		
	One for each (cartesian) coordinate	**A1 A1**	
	ALT.		
	$u^2 + 1 = 3t^2$ and $-u = 2t^3 \Rightarrow 4t^6 - 3t^2 + 1 = 0$	**M1 A1** Eliminating u	
	$\Rightarrow (t^2 + 1)(2t^2 - 1)^2 = 0$	**M1 A1** attempt to factorise; correct	
	$\Rightarrow t = \pm\frac{1}{\sqrt{2}}$ $\quad$ $y = \pm\frac{1}{\sqrt{2}}$, $x = \frac{3}{2}$	**A1 A1**	**6**
	<u>Graphs:</u>		
	C is a semi-cubical parabola with a cusp at O	**B1**	
	$\widetilde{C}$ is a $\subset$-shaped parabola with vertex at (1, 0)*	**B1** (* apparently, here)	
	Curves meet tangentially	**B1**	
	Key points noted or sketched, esp. (1, 0) and contacts at $x = \frac{3}{2}$	**B1**	
	(Withhold final mark if unclear which curve is which)		**4**

3 **(i)** $I=\int_0^{\pi/4}\frac{1}{1+\sin x}\,dx=\int_0^{\pi/4}\frac{1-\sin x}{\cos^2 x}\,dx=\int_0^{\pi/4}\sec^2 x\,dx-\int_0^{\pi/4}\frac{\sin x}{\cos^2 x}\,dx$

M1 use of $1-s^2=c^2$ and splitting into two integrals

Now $\int_0^{\pi/4}\sec^2 x\,dx=\left[\tan x\right]_0^{\frac{1}{4}\pi}=1$ **B1**

and $\int_0^{\pi/4}\frac{\sin x}{\cos^2 x}\,dx=\int_1^{1/\sqrt{2}}\frac{-1}{c^2}\,dc$ using substn. $c=\cos x$, $dc=-\sin x\,dx$ etc.

M1 (or by "recognition")

$=\left[\frac{1}{c}\right]_1^{\frac{1}{\sqrt{2}}}=\sqrt{2}-1$ **A1**

OR via $\int\frac{\sin x}{\cos^2 x}\,dx=\int\sec x\tan x\,dx=\sec x$ (**M1 A1**)

so that $I=2-\sqrt{2}$ **A1 cao** **5**

(ii) $\int_{\pi/4}^{\pi/3}\frac{1}{1+\sec x}\,dx=\int_{\pi/4}^{\pi/3}\frac{\sec x-1}{\tan^2 x}\,dx$ **M1** use of initial technique

$=\int_{\pi/4}^{\pi/3}\frac{\cos x-\cos^2 x}{\sin^2 x}\,dx=\int_{\pi/4}^{\pi/3}\frac{\cos x}{\sin^2 x}\,dx-\int_{\pi/4}^{\pi/3}\cot^2 x\,dx$ **M1** split appropriately

$=\int_{\sqrt{2}/2}^{\sqrt{3}/2}\frac{1}{s^2}\,ds-\int_{\pi/4}^{\pi/3}\left(\csc^2 x-1\right)dx$ **M1** use of relevant trig. identity

NB $\int\frac{\cos x}{\sin^2 x}\,dx=\int\operatorname{cosec}x\cot x\,dx=-\operatorname{cosec}x$

$=\left[\frac{-1}{s}\right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}}+\left[\cot x+x\right]_{\frac{1}{4}\pi}^{\frac{1}{3}\pi}$ **A1 A1**

$=\left(-\frac{2}{\sqrt{3}}+\frac{2}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{3}}+\frac{\pi}{3}-1-\frac{\pi}{4}\right)$

$=\frac{\pi}{12}+\sqrt{2}-1-\frac{1}{\sqrt{3}}$ **A1 cao** in a suitable, exact form

6

ALT. I

$\int_{\pi/4}^{\pi/3}\frac{1}{1+\sec x}\,dx=\int_{\pi/4}^{\pi/3}\frac{\cos x}{1+\cos x}\,dx$ **M1**

$=\int_{\pi/4}^{\pi/3}\frac{1+\cos x-1}{1+\cos x}\,dx=\int_{\pi/4}^{\pi/3}\left(1-\frac{1}{1+\cos x}\right)dx$ **M1**

$$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right) - J$$

Using the initial technique,

$$J = \int_{\pi/4}^{\pi/3} \frac{1-\cos x}{\sin^2 x}\,dx = \int_{\pi/4}^{\pi/3} \left(\operatorname{cosec}^2 x - \frac{\cos x}{\sin^2 x}\right) dx$$ **M1**

$$= \left[-\cot x\right]_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} - \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{1}{s^2}\,ds$$ **A1 M1** (full substn. attempt)

$$= \frac{-1}{\sqrt{3}} + 1 + \left[\frac{1}{s}\right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = 1 - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{2}}$$

giving $\displaystyle\int_{\pi/4}^{\pi/3} \frac{1}{1+\sec x}\,dx = \frac{\pi}{12} + \sqrt{2} - 1 - \frac{1}{\sqrt{3}}$ **A1** **6**

ALT. II

$$\int_{\pi/4}^{\pi/3} \frac{1}{1+\sec x}\,dx = \int_{\pi/4}^{\pi/3} \frac{\cos x}{1+\cos x}\,dx$$ **M1**

$$= \int_{\pi/4}^{\pi/3} \frac{2\cos^2 \frac{1}{2}x - 1}{2\cos^2 \frac{1}{2}x}\,dx$$ **M1**

$$= \int_{\pi/4}^{\pi/3} \left(1 - \tfrac{1}{2}\sec^2 \tfrac{1}{2}x\right) dx$$ **M1**

$$= x - \tan \tfrac{1}{2}x$$ **A1**

$$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right) - \frac{1}{\sqrt{3}} + \left(\sqrt{2} - 1\right) = \frac{\pi}{12} + \sqrt{2} - 1 - \frac{1}{\sqrt{3}}$$ **M1 A1**

The M is for a method to find $\tan\left(\frac{1}{8}\pi\right)$ **6**

(iii) $\displaystyle\int_0^{\pi/3} \frac{1}{(1+\sin x)^2}\,dx = \int_0^{\pi/3} \frac{(1-\sin x)^2}{\cos^4 x}\,dx$ **M1** multg. nr. & dr. by $(1 - \sin x)^2$

$$= \int_0^{\pi/3} \frac{1 - 2\sin x + \sin^2 x}{\cos^4 x}\,dx = \int_0^{\pi/3} \frac{2 - 2\sin x - \cos^2 x}{\cos^4 x}\,dx$$ **M1**

$$= \int_0^{\pi/3} 2\sec^4 x\,dx - \int_0^{\pi/3} \frac{2\sin x}{\cos^4 x}\,dx - \int_0^{\pi/3} \sec^2 x\,dx$$ **A1**

Must be separated into individually integrable forms *

Now $\displaystyle\int_0^{\pi/3} \frac{2\sin x}{\cos^4 x}\,dx = -2\int_1^{1/2} \frac{-1}{c^4}\,dc = \left[\frac{2}{3c^3}\right]_1^{\frac{1}{2}} = \frac{14}{3}$ **M1 A1**

and $\displaystyle\int_0^{\pi/3} \sec^2 x\,dx = \left[\tan x\right]_0^{\frac{1}{3}\pi} = \sqrt{3}$ **- -** Rewarded in final answer mark

$$K = \int_0^{\pi/3} \sec^4 x\,dx = \int_0^{\pi/3} \sec^2 x.\sec^2 x\,dx = \left[\sec^2 x.\tan x\right]_0^{\frac{1}{3}\pi} - \int_0^{\pi/3} 2\sec^2 x.\tan^2 x\,dx$$

M1 A1 use of *integration by parts*

$K = 4\sqrt{3} - 0 - 2\int_0^{\pi/3} \sec^2 x\left(\sec^2 x - 1\right)\mathrm{d}x$ **M1** ‘recognition’ attempt with *loop*

$= 4\sqrt{3} - 2K + 2\left[\tan x\right]_0^{\frac{1}{3}\pi} = 4\sqrt{3} - 2K + 2\sqrt{3}$

$\Rightarrow K = 2\sqrt{3}$ **- -** Rewarded in final answer mark

giving $\int_0^{\pi/3} \dfrac{1}{(1+\sin x)^2}\,\mathrm{d}x = 3\sqrt{3} - \frac{14}{3}$ **A1** **9**

***** NB $\int\left(\dfrac{1+s^2}{c^4}\right)\mathrm{d}x = \int\left(\dfrac{c^2+2s^2}{c^4}\right)\mathrm{d}x = \int \sec^2 x\,\mathrm{d}x + \int 2\sec^2 x\tan^2 x\,\mathrm{d}x = \tan x + \frac{2}{3}\tan^3 x$

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4 **(i)** $\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$ by (e.g.) squaring and comparing terms in m, n

M1 A1 (i.e. $m = n = 1$)

NB A0 for $m = n = -1$ also **2**

(ii) Existence of four roots α, β, γ, δ means we must have

$f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= (x^2 - [\alpha+\beta]\,x + \alpha\beta)(x^2 - [\gamma+\delta]\,x + \gamma\delta)$ — **E1** Justifying factorisation into quadratics

Since $\alpha+\beta+\gamma+\delta = 0$ from coefft. of x^3 in $f(x)$

it follows that $\alpha+\beta = -(\gamma+\delta)$ — **B1**

Comparing the other coeffts. of $f(x)$ — **M1** (or by multiplying out)

$pq = -2$

$s(p-q) = -12$ — **A1** for at least 2 correct

$p + q - s^2 = -10$ — **A1** for 3rd correct

Use of $p + q = s^2 - 10 \Rightarrow (p+q)^2 = (s^2-10)^2$

and $(p-q)^2 = (p+q)^2 - 4pq = \left(\dfrac{12}{s}\right)^2$ — **M1**

to get an eqn. in s only — **A1** (correct unsimplified)

$\Rightarrow\ s^2(s^2-10)^2 + 8s^2 - 144 = 0$ — **A1** (**AG**) legitimately obtained **8**

$(s^2-10)^3 + 10(s^2-10)^2 + 8(s^2-10) - 64 = 0$ — **M1** attempt at cubic in (s^2-10)

i.e. $u^3 + 10u^2 + 8u - 64 = 0 \Rightarrow (u-2)(u+4)(u+8) = 0$ — **M1** finding one factor

A1 complete linear factorisation

$\Rightarrow\ s^2 - 10 = 2, -4, -8 \Rightarrow s^2 = 12, 6, 2$ — **A1** (i.e. $\Rightarrow s = \pm 2\sqrt{3}, \pm\sqrt{6}, \pm\sqrt{2}$)

ALT.

$s^2(s^4 - 20s^2 + 100) + 8s^2 - 144 = 0$

$\Rightarrow\ s^6 - 20s^4 + 108s^2 - 144 = 0$ — **M1** attempt at cubic in s^2

$\Rightarrow\ (s^2-2)(s^2-6)(s^2-12) = 0$ — **M1** finding one factor

A1 complete linear factorisation

$\Rightarrow s^2 = 12, 6, 2$ — **A1** **4**

$s = \sqrt{2}$, $p = -4-3\sqrt{2}$, $q = -4+3\sqrt{2}$

(Note that taking the –ve sq.rt. simply swaps the brackets)

Or $s = \sqrt{6}$, $p = -2-\sqrt{6}$, $q = -2+\sqrt{6}$

Or $s = 2\sqrt{3}$, $p = 1-\sqrt{3}$, $q = 1+\sqrt{3}$

Candidates told to use the smallest value of s^2, so the working *should* proceed as follows:-

$f(x) = \left(x^2 + x\sqrt{2} - 4 - 3\sqrt{2}\right)\left(x^2 - x\sqrt{2} - 4 + 3\sqrt{2}\right) = 0$ — **M1**

(using $s = \sqrt{2}$, $t = -\sqrt{2}$, $p = -4-3\sqrt{2}$, $q = -4+3\sqrt{2}$)

Using quadratic formula on each factor:

$x = \dfrac{-\sqrt{2} \pm \sqrt{18+12\sqrt{2}}}{2},\ \dfrac{\sqrt{2} \pm \sqrt{18-12\sqrt{2}}}{2}$ — **M1 A1** (**A** for 2 correct discriminants)

$$= \frac{-\sqrt{2} \pm \sqrt{6}\sqrt{3+2\sqrt{2}}}{2}, \frac{\sqrt{2} \pm \sqrt{6}\sqrt{3-2\sqrt{2}}}{2}$$

$$= \frac{-\sqrt{2} \pm \sqrt{6}\left(1+\sqrt{2}\right)}{2}, \frac{\sqrt{2} \pm \sqrt{6}\left(\sqrt{2}-1\right)}{2}$$

M1 using **(i)**'s result

$$x = \frac{-\sqrt{2}+\sqrt{6}+2\sqrt{3}}{2}, \frac{-\sqrt{2}-\sqrt{6}-2\sqrt{3}}{2}, \frac{\sqrt{2}+\sqrt{6}-2\sqrt{3}}{2}, \frac{\sqrt{2}-\sqrt{6}+2\sqrt{3}}{2}$$

A1 any two correct

A1 all four (& no extras) **6**

However,

$f(x) = \left(x^2 + x\sqrt{6} - 2 - \sqrt{6}\right)\left(x^2 - x\sqrt{6} - 2 + \sqrt{6}\right) = 0$

(using $s = \sqrt{6}$, $t = -\sqrt{6}$, $p = -2-\sqrt{6}$, $q = -2+\sqrt{6}$)

with discriminants $14+4\sqrt{6} = 2\left(7+2\sqrt{6}\right) = \left[\sqrt{2}\left(1+\sqrt{6}\right)\right]^2$ and $14-4\sqrt{6}$ etc.

and $f(x) = \left(x^2 + x\sqrt{12} + 1 - \sqrt{3}\right)\left(x^2 - x\sqrt{12} + 1 + \sqrt{3}\right) = 0$

(using $s = 2\sqrt{3}$, $t = -2\sqrt{3}$, $p = 1-\sqrt{3}$, $q = 1+\sqrt{3}$)

with discriminants $8+4\sqrt{3} = 2\left(4+2\sqrt{3}\right) = \left[\sqrt{2}\left(1+\sqrt{3}\right)\right]^2$ and $8-4\sqrt{3}$ etc.

5 **(i)** If $\overrightarrow{PQ} = \overrightarrow{SR}$ then $PQRS$ is a parallelogram **B1**

If $\overrightarrow{PQ} = \overrightarrow{SR}$ and $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{PS}\right|$ then $PQRS$ is a rhombus **B1** **2**

$$\overrightarrow{PQ} = \begin{pmatrix} 1-p \\ q \\ 0 \end{pmatrix}, \quad \overrightarrow{PR} = \begin{pmatrix} r-p \\ 1 \\ 1 \end{pmatrix}, \quad \overrightarrow{PS} = \begin{pmatrix} -p \\ s \\ 1 \end{pmatrix}, \quad \overrightarrow{QR} = \begin{pmatrix} r-1 \\ 1-q \\ 1 \end{pmatrix}, \quad \overrightarrow{QS} = \begin{pmatrix} -1 \\ s-q \\ 1 \end{pmatrix} \text{ and } \overrightarrow{RS} = \begin{pmatrix} -r \\ s-1 \\ 0 \end{pmatrix} \quad \textbf{(*)}$$

(ii) Diagonal PR has eqn. $\mathbf{r} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} r-p \\ 1 \\ 1 \end{pmatrix}$; diagonal QS has eqn. $\mathbf{r} = \begin{pmatrix} 1 \\ q \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ s-q \\ 1 \end{pmatrix}$

M1 Good attempt at both eqns.

Diagonals intersect iff

$p + \lambda(r-p) = 1 - \mu$, $\lambda = q + \mu(s-q)$, $\lambda = \mu$ **M1**

Setting $\mu = \lambda \Rightarrow p + \lambda(r-p) = 1 - \lambda$, $\lambda = q + \lambda(s-q)$ and equating for λ

M1

$\lambda = \dfrac{1-p}{r-p+1} = \dfrac{q}{1-s+q} \Rightarrow 1 - s + q - p + ps - pq = rq - pq + q$

$\Rightarrow 1 - s - p + ps = rq \Rightarrow (1-s)(1-p) = rq$ **A1** legitimately (**AG**) **4**

ALT.

Taking any 3 ('*independent*') vectors from **(*)** and showing them linearly dependent (consistently)

(ii)(a) Then $\overrightarrow{PQ} = \overrightarrow{SR}$ iff $1 - p = r$ **and** $q = 1 - s$

i.e. iff $p + r = 1$ **and** $q + s = 1$ **B1**

Centroid G has $\mathbf{g} = \left(\frac{1}{4}(p+r+1), \frac{1}{4}(q+s+1), \frac{1}{2}\right)$ **B1**

while the centre of the unit cube is at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

These are the same point iff $p + r = 1$ **and** $q + s = 1$ **B1**

Final mark *not* to be awarded unless a correct "iff" argument has been made **3**

(ii) (b) We have $p + r = 1$ **and** $q + s = 1$ as before

and $\sqrt{(1-p)^2 + q^2} = \sqrt{p^2 + s^2 + 1}$ using $\overrightarrow{PS}$ from **(*)** **B1**

$\Rightarrow 1 - 2p + p^2 + q^2 = p^2 + s^2 + 1$ **M1** solving for p

$\Rightarrow p = \dfrac{q^2 - s^2}{2} = \dfrac{(q-s)(q+s)}{2} = \dfrac{q-s}{2}$ using $q + s = 1$

Then $q + s = 1$ and $q - s = 2p \Rightarrow q = \frac{1}{2} + p$, $r = 1 - p$, $s = \frac{1}{2} - p$

All found in terms of p **M1 A1** **4**

$\overrightarrow{PQ}=\begin{pmatrix}1-p\\ \frac{1}{2}+p\\ 0\end{pmatrix}$, $\overrightarrow{PR}=\begin{pmatrix}1-2p\\ 1\\ 1\end{pmatrix}$, $\overrightarrow{RQ}=\begin{pmatrix}p\\ p-\frac{1}{2}\\ -1\end{pmatrix}$ **B1** must all be in terms of p (**ft**)

so $PQ^2=2p^2-p+\frac{5}{4}$, $PR^2=4p^2-4p+3$

and $RQ^2=2p^2-p+\frac{5}{4}$ **M1** three lengths attempted

Then by the Cosine Rule,

$$\cos PQR=\frac{PQ^2+RQ^2-PR^2}{2.PQ.RQ}=\frac{2p-\frac{1}{2}}{2\left(2p^2-p+\frac{5}{4}\right)}$$ **M1** (rearranged into cos = … form)

$$=\frac{4p-1}{5-4p+8p^2}$$ **A1** (**AG**) legitimately obtained **4**

ALT. $\overrightarrow{RQ}=\begin{pmatrix}1-r\\ q-1\\ -1\end{pmatrix}=\begin{pmatrix}p\\ p-\frac{1}{2}\\ -1\end{pmatrix}$ **B1** must be all in terms of p (possibly later on)

$$\cos PQR=\frac{\overrightarrow{PQ}\bullet\overrightarrow{RQ}}{\left|\overrightarrow{PQ}\right|\left|\overrightarrow{RQ}\right|}=\frac{(1-p)p+\left(p+\frac{1}{2}\right)\left(p-\frac{1}{2}\right)+0}{\sqrt{(1-p)^2+\left(p+\frac{1}{2}\right)^2}\sqrt{p^2+\left(p-\frac{1}{2}\right)^2+1}}$$

M2 use of the scalar product (correct vectors)

$$=\frac{p-p^2+p^2-\frac{1}{4}}{\sqrt{(1-p)^2+\left(p+\frac{1}{2}\right)^2}\sqrt{p^2+\left(p-\frac{1}{2}\right)^2+1}}=\frac{p-\frac{1}{4}}{\sqrt{\frac{5}{4}-p+2p^2}\sqrt{\frac{5}{4}-p+2p^2}}$$

$$=\frac{4p-1}{5-4p+8p^2}$$ **A1** legitimately (**AG**) **4**

For a square, adjacent sides perpr. $\Rightarrow p=\frac{1}{4}, q=\frac{3}{4}, r=\frac{3}{4}, s=\frac{1}{4}$ **B1**

Side-length is $\left|\overrightarrow{PQ}\right|=\sqrt{\frac{5}{4}-p+2p^2}=\sqrt{\frac{5}{4}-\frac{1}{4}+\frac{2}{16}}=\frac{3}{2\sqrt{2}}$ **B1**

$\frac{3}{2\sqrt{2}}>\frac{21}{20}\Leftarrow\frac{1}{\sqrt{2}}>\frac{7}{10}\Leftarrow 10>7\sqrt{2}\Leftarrow 100>98$ **B1** or equivalent

(penalise incorrect direction of the logic) **3**

ALT. (final two marks)

Side-length is $\sqrt{\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^2}$ **B1** $=\sqrt{\frac{9}{8}}=\sqrt{\frac{450}{400}}>\sqrt{\frac{441}{400}}=\frac{21}{20}$ **B1**

6	**(i)** $9x^2 - 12x\cos\theta + 4 \equiv (3x - 2\cos\theta)^2 + 4 - 4\cos^2\theta$	**M1**	
	$\geq 4\sin^2\theta$ with equality when $x = \frac{2}{3}\cos\theta$	**A1 B1**	**3**
	(the value of x giving the minimum may appear later on)		

$12x^2\sin\theta - 9x^4 \equiv 4\sin^2\theta - (3x^2 - 2\sin\theta)^2$	**M1**	
$\leq 4\sin^2\theta$ with equality when $x^2 = \frac{2}{3}\sin\theta$	**A1 B1**	
(the value of x giving the maximum may appear later on)		**3**
ALT. $y = 12x^2\sin\theta - 9x^4 \Rightarrow \dfrac{dy}{dx} = 24x\sin\theta - 36x^3$	M1	
$\dfrac{dy}{dx} = 0$ when $x^2 = \frac{2}{3}\sin\theta$, $y = 4\sin^2\theta$	B1 both	
(ignore consideration of $x = 0$; this clearly does not give a max.)		
$\dfrac{d^2y}{dx^2} = 24\sin\theta - 108x^2 = -48\sin\theta < 0$ for $0 < \theta < \pi \Rightarrow$ maximum		
	A1 must justify MAX. if using calculus	

$9x^4 + (9 - 12\sin\theta)x^2 - 12x\cos\theta + 4 = 0$		
$\Leftrightarrow 9x^2 - 12x\cos\theta + 4 = 12x^2\sin\theta - 9x^4$	**B1**	
These two functions meet only at $4\sin^2\theta$ when $x^2 = \frac{4}{9}\cos^2\theta = \frac{2}{3}\sin\theta$		
	E1 explained	
$\frac{4}{9}(1 - s^2) = \frac{2}{3}s \Rightarrow 0 = 2s^2 + 3s - 2 = (2s - 1)(s + 2)$	**M1** creating and solving a quadratic	
$\Rightarrow \sin\theta = \frac{1}{2}$, $x^2 = \frac{1}{3}$		
$\Rightarrow (x, \theta) = \left(\pm\dfrac{1}{\sqrt{3}}, \dfrac{\pi}{6}\right), \left(\pm\dfrac{1}{\sqrt{3}}, \dfrac{5\pi}{6}\right)$	**A1** at least two correct solutions	
Checking for extraneous solutions, we find that only		
$\left(\dfrac{1}{\sqrt{3}}, \dfrac{\pi}{6}\right)$ and $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{5\pi}{6}\right)$ are valid solutions	**B1**	**5**

(ii)	Vertical Asymptote $x = \theta$	**B1** stated or shown on graph	
	$y = \dfrac{x(x-\theta) + \theta(x-\theta) + \theta^2}{x - \theta} = x + \theta + \dfrac{\theta^2}{x - \theta}$		
	$\Rightarrow$ Oblique Asymptote $y = x + \theta$	**B1** stated or shown on graph	
	(NB OAs aren't on-syllabus so allow $y \to \pm\infty$ as $x \to \pm\infty$)		
	$\dfrac{dy}{dx} = 1 - \dfrac{\theta^2}{(x-\theta)^2} = 0$ when …	**M1** method for finding TPs	
	$(x-\theta)^2 = \theta^2 \Rightarrow x = 0, y = 0$ or $x = 2\theta, y = 4\theta$	**A1** stated or shown on graph	
	From graph, $\dfrac{x^2}{x-\theta} \leq 0$ or $\dfrac{x^2}{x-\theta} \geq 4\theta$	**B1** graph must be correct	**5**

Since $4\theta > 0$, we have $\dfrac{x^2}{4\theta(x-\theta)} \le 0$ or $\dfrac{x^2}{4\theta(x-\theta)} \ge 1$

so we have $\dfrac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} \le 0$ or $\dfrac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} \ge 1$

However, it is clear that $(0 \le)\ \dfrac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} \le 1$ **B1**

(since numerator ≤ 1 and denominator ≥ 1)

The 0 case occurs if and only if $x = 0$ (on LHS) but, since $\sin\theta \ne 0$,

the RHS is then non-zero (as cos 0 = 1)

$\dfrac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} = 1 \Leftrightarrow$ both numerator & denominator are 1 **M1**

$\Leftrightarrow \sin^2\theta = 1$ **and** $\cos^2 x = 1$ **M1**

$\Leftrightarrow \theta = \dfrac{\pi}{2},\ x = \pi$ **A1** **4**

ALT. For the 1 case, we want $\sin^2\theta\cos^2 x = 1 + \cos^2\theta\sin^2 x$ when $x = 2\theta$

(from previous bit) **B1**

Thus $\sin^2\theta\cos^2 2\theta - \cos^2\theta\sin^2 2\theta = 1$

$\Rightarrow (\sin\theta\cos 2\theta - \cos\theta\sin 2\theta)(\sin\theta\cos 2\theta + \cos\theta\sin 2\theta) = 1$

M1

$\Rightarrow (-\sin\theta)(\sin 3\theta) = 1$ **or** $(4\sin^2\theta + 1)(\sin^2\theta - 1) = 0$

This can only occur when either $\sin\theta = 1$ and $\sin 3\theta = -1$

or $\sin\theta = -1$ and $\sin 3\theta = 1$ **M1**

Since $0 < \theta < \pi$, this is only satisfied when $\theta = \dfrac{\pi}{2},\ x = \pi$ **A1**

7 **(i) Step 1:** If a is not divisible by 3 then it is either one more than, or one less than, a multiple of 3.

For $a = 3k \pm 1$, $a^2 = 9k^2 \pm 6k + 1$

$= 3(3k^2 \pm 2k) + 1$ **B1** (both shown 1 more than a multiple of 3) **1**

Step 3: $\left(\sqrt{2}+\sqrt{3}\right)^2 = 5 + 2\sqrt{6}$ **B1**

$\left(\sqrt{2}+\sqrt{3}\right)^4 = 49 + 20\sqrt{6}$ **M1** and relating back to a, b

$\left(\frac{a}{b}\right)^4 = 10\left(\frac{a}{b}\right)^2 - 1$ **A1**

$\times$ by b^4 and rearranging gives $a^4 + b^4 = 10a^2b^2$

A1 legitimately (**AG**) **4**

ALT. $a = \left(\sqrt{2}+\sqrt{3}\right)b \Rightarrow a^2 = \left(5+2\sqrt{6}\right)b^2$ **B1**

$\Rightarrow a^2 + b^2 = \left(6+2\sqrt{6}\right)b^2$ **M1** adding b^2 to both sides

$= 2\sqrt{3}\left(\sqrt{3}+\sqrt{2}\right)b^2 = 2\sqrt{3}ab$ **A1**

$\Rightarrow \left(a^2+b^2\right)^2 = 12a^2b^2 \Rightarrow a^4 + b^4 = 10a^2b^2$ **A1** legitimately (**AG**)

Step 4: If $a = 3k$ then $b^4 = 90k^2b^2 - 81k^4 = 3(30k^2b^2 - 27k^4)$

Explanation that $3 \mid \text{RHS} \Rightarrow 3 \mid \text{LHS} \Rightarrow 3 \mid b$

E1 must be thorough **1**

Step 5: Since hcf(a, b) = 1, a can't be a multiple of 3 (from previous working)

B1

So both a^2 and $a^4 \equiv 1 \pmod 3$ **M1** any suitable wording

giving $1 + b^4 \equiv b^2 \pmod 3$ **A1**

Each case $b^2 \equiv 0$, $b^2 \equiv 1$ gives $\Rightarrow\Leftarrow$ **E1** carefully explained **4**

(ii) If a is not a multiple of 5, it is $5k \pm 1$ or $5k \pm 2$ **B1**

Squaring gives $a^2 \equiv \pm 1$ (and $a^4 \equiv 1$) **B1**

$\left(\sqrt{6}+\sqrt{7}\right)^2 = 13 + 2\sqrt{42}$

and $\left(\sqrt{6}+\sqrt{7}\right)^4 = 337 + 52\sqrt{42}$ **M1** and relating back to a, b

so that $\left(\frac{a}{b}\right)^4 = 26\left(\frac{a}{b}\right)^2 - 1$ **A1**

$\times$ by b^4 and rearranging gives $a^4 + b^4 = 26a^2b^2$ **A1** legitimately (**AG**)

Now if $a = 5k$ then $b^4 = 650k^2b^2 - 625k^4 = 5(130k^2b^2 - 125k^4)$

so if a is a multiple of 5 then b is also **E1**

Since a, b co-prime, this doesn't happen

so $a^4 + b^4 = 26a^2b^2$ becomes **M1** considering this mod 5

$1 + b^4 \equiv \pm b^2$ **A1**

Each case $b^2 \equiv 0$, $b^2 \equiv \pm 1$ gives $\Rightarrow\Leftarrow$ **E1** carefully explained/demonstrated **9**

Cannot work with divisibility by 3 in this case since $26 \equiv 2 \pmod 3$ and $a^4 + b^4 = 26a^2b^2$

could be from $1 + 1 \equiv 2.1.1$ (e.g.) **E1** must demonstrate the failing case explicitly
1

8 (i) Set $u = 2t$, $du = 2dt$ in $f(x) = \int_1^x \sqrt{\frac{t-1}{t+1}}\,dt$	**M1** choice of substitution	
$t = 1,\ u = 2$ and $t = \frac{1}{2}x,\ u = x$	**A1** limits correctly sorted	
Then $f\left(\frac{1}{2}x\right) = \int_2^x \sqrt{\frac{\frac{u}{2}-1}{\frac{u}{2}+1}}\,.\frac{1}{2}du = \frac{1}{2}\int_2^x \sqrt{\frac{u-2}{u+2}}\,du$	**M1 A1** full substitution attempted; correct	
and $\int_2^x \sqrt{\frac{u-2}{u+2}}\,du = 2\,f\left(\frac{1}{2}x\right)$	**A1** legitimately (**AG**)	
A 'backwards' verification approach equally ok		**5**

(ii) Set $u = v + 2$, $du = dv$	**M1** choice of substitution, using **(i)**	
$u = 2,\ v = 0$ and $u = x + 2,\ v = x$	**A1** limits correctly sorted	
Then $2\,f\left(\frac{x+2}{2}\right) = \int_0^x \sqrt{\frac{v}{v+4}}\,dv$	**A1**	**3**

ALT. (from the beginning)

Set $u + 2 = 2t$, $du = 2\,dt$	M1 choice of substitution
$t = 1,\ u = 0$ and $t = \frac{1}{2}x + 1,\ u = x$	
Then $f\left(\frac{1}{2}x+1\right) = \int_1^{\frac{1}{2}x+1} \sqrt{\frac{t-1}{t+1}}\,dt = \int_0^x \sqrt{\frac{\frac{u+2}{2}-1}{\frac{u+2}{2}+1}}\,.\frac{1}{2}du$	M1 full substitution
$= \frac{1}{2}\int_0^x \sqrt{\frac{u}{u+4}}\,du$	
and $2\,f\left(\frac{1}{2}x+1\right) = \int_0^x \sqrt{\frac{u}{u+4}}\,du$	A1

(iii) Set $u = at + b$, $du = a\,dt$	**M1** choice of substitution	
$t = 1,\ u = 5$ and $t = \frac{x-b}{a},\ u = x$	**A1** limits correctly sorted	
Then $f\left(\frac{x-b}{a}\right) = \int_1^{\frac{x-b}{a}} \sqrt{\frac{t-1}{t+1}}\,dt = \int_5^x \sqrt{\frac{\frac{u-b}{a}-1}{\frac{u-b}{a}+1}}\,\frac{1}{a}du$	**M1** full substitution	
$= \frac{1}{a}\int_5^x \sqrt{\frac{u-(a+b)}{u+(a-b)}}\,du$	**A1** correct	
We need $a + b = 5$ and $a - b = 1 \Rightarrow a = 3, b = 2$	**M1** method for determining a, b	
giving $3f\left(\frac{x-2}{3}\right) = \int_5^x \sqrt{\frac{u-5}{u+1}}\,du$	**A1**	
Might also be done using two substitutions (split marks **3 + 3** if fully correct)		**6**

(iv) Set $y = u^2$, $dy = 2u\,du$	**M1** choice of substitution	
$u = 1,\ y = 1$ and $u = 2,\ y = 4$	**A1** limits correctly sorted	

Then $\int_1^2 \sqrt{\dfrac{u^2}{u^2+4}}\, u \,\mathrm{d}u = \int_1^4 \sqrt{\dfrac{y}{y+4}}\, \dfrac{1}{2}\,\mathrm{d}y$ **M1** full substitution

$= \dfrac{1}{2}\int_0^4 \sqrt{\dfrac{y}{y+4}}\,\mathrm{d}y - \dfrac{1}{2}\int_0^1 \sqrt{\dfrac{y}{y+4}}\,\mathrm{d}y$ **M1** dealing with lower limit

$= \mathrm{f}\left(\dfrac{4+2}{2}\right) - \mathrm{f}\left(\dfrac{1+2}{2}\right)$ using **(ii)** **M1** use of **(ii)**

$= \mathrm{f}(3) - \mathrm{f}(\tfrac{3}{2})$ **A1** **6**

For those interested, $\mathrm{f}(x) = \sqrt{x^2-1} - 2\sinh^{-1}\sqrt{\dfrac{x-1}{2}}$ or equivalent involving log forms

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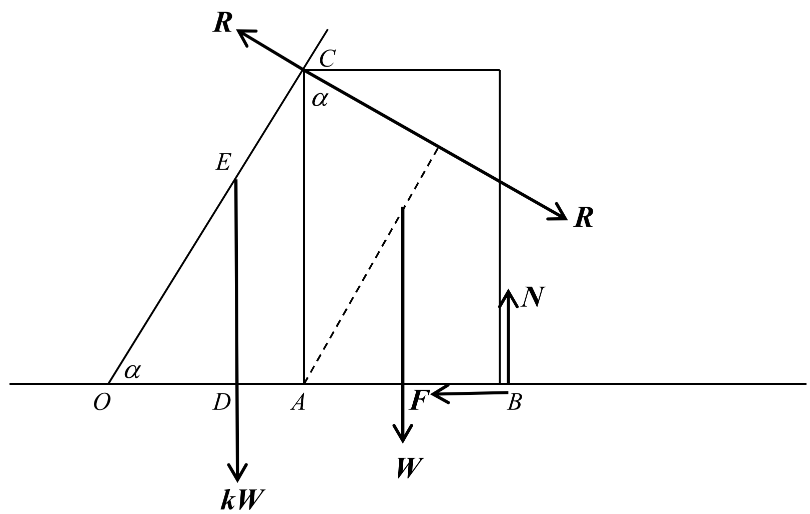
9 Diagram at the moment of toppling:-



Note: There could also be ↑ and → components of the contact force at O, but these can be ignored

$OA = AB = b$ and $AC = h \Rightarrow OC = \sqrt{b^2 + h^2}$

$DE = \lambda h$ and $OD = \lambda b$

(i) $\circlearrowleft O$ for ladder: $kW.\lambda b = R\sqrt{b^2+h^2}$ **M1 A1**

$\Rightarrow R = k\lambda W \dfrac{b}{\sqrt{b^2+h^2}} = k\lambda W \cos\alpha$ **M1 A1** legitimately (**AG**) **4**

(ii) Resolve ↑ for box: $N = W + R\cos\alpha$ **M1 A1**

$\circlearrowleft A$ for box: $W.\tfrac{1}{2}b + R.h\sin\alpha = N.b$ **M1 A1**

$\Rightarrow \tfrac{1}{2}b\,W + hk\lambda W\cos\alpha\sin\alpha = b(W + k\lambda W\cos^2\alpha)$ **M1 M1** substituting for R, N

$\Rightarrow \tfrac{1}{2}b + hk\lambda\cos\alpha\sin\alpha = b + bk\lambda\cos^2\alpha$

$(\times 2) \Rightarrow b\tan\alpha\,.\,2k\lambda\cos\alpha\sin\alpha = b + 2bk\lambda\cos^2\alpha$ **M1** substituting $h = b\tan\alpha$

$(\div b) \Rightarrow 2k\lambda\sin^2\alpha = 1 + 2k\lambda\cos^2\alpha$

$\Rightarrow 0 = 1 + 2k\lambda\cos 2\alpha$ **A1** (since $c^2 - s^2 = \cos 2\alpha$) legitimately (**AG**) **8**

(iii) Resolve → for box: $F = R\sin\alpha$ **B1**

Friction Law: $F \le \mu N$ **B1** *used*, not just stated (**B0** for $F = \mu N$)

$\Rightarrow \mu \ge \dfrac{R\sin\alpha}{W + R\cos\alpha}$ **M1** substituting for F and N

$\Rightarrow \mu \ge \dfrac{k\lambda W\cos\alpha\sin\alpha}{W + k\lambda W\cos^2\alpha}$ **M1** substituting for R

$\Rightarrow \mu \ge \dfrac{k\lambda 2\sin\alpha\cos\alpha}{2 + k\lambda 2\cos^2\alpha} = \dfrac{k\lambda\sin 2\alpha}{2 + k\lambda(1+\cos 2\alpha)}$ **M1** use of double-angle formulae

Using (ii)'s result, $1 = -2k\lambda\cos 2\alpha \Rightarrow 2 = -4k\lambda\cos 2\alpha$ **M1 A1** using (ii)'s result

$$\Rightarrow \qquad \mu \geq \frac{k\lambda \sin 2\alpha}{-4k\lambda \cos 2\alpha + k\lambda + k\lambda \cos 2\alpha}$$

$$(\div\, k\lambda) \Rightarrow \qquad \mu \geq \frac{\sin 2\alpha}{-4\cos 2\alpha + 1 + \cos 2\alpha} = \frac{\sin 2\alpha}{1 - 3\cos 2\alpha}$$ **A1** legitimately (**AG**) **8**

10 $x = ut\sin\alpha$ $\quad y = ut\cos\alpha - \frac{1}{2}gt^2$	**B1** both		
Setting $t = \dfrac{x}{u\sin\alpha}$ and substituting into y formula	**M1**		
$\Rightarrow y = x\cot\alpha - \frac{1}{2}g\dfrac{x^2}{u^2\sin^2\alpha}$			
$= x\cot\alpha - \frac{1}{2}g\dfrac{x^2}{u^2}\left(1+\cot^2\alpha\right)$	**M1** use of $\operatorname{cosec}^2\alpha = 1 + \cot^2\alpha$		
Setting $x = h\tan\beta$ and $y = h$	**M1**		
$\Rightarrow h = ch\tan\beta - \dfrac{gh^2}{2u^2}\tan^2\beta\left(1+c^2\right) \Rightarrow$ (since $h \neq 0$) $1 = c\tan\beta - \dfrac{gh}{2u^2}\tan^2\beta\left(1+c^2\right)$			
$\times\ k = \dfrac{2u^2}{gh} \Rightarrow k = ck\tan\beta - (1+c^2)\tan^2\beta$	**M1** use of k		
$\div\tan^2\beta \quad \Rightarrow k\cot^2\beta = ck\cot\beta - 1 - c^2$			
$\Rightarrow c^2 - ck\cot\beta + 1 + k\cot^2\beta = 0$	**A1** legitimately (**AG**)	**6**	

(i) Considering this quadratic in c:		
sum of roots: $\cot\alpha_1 + \cot\alpha_2 = k\cot\beta$	**B1** must be clearly shown (**AG**)	
product of roots: $\cot\alpha_1\cot\alpha_2 = 1 + k\cot^2\beta$	**B1** stated or used somewhere	
$\cot(\alpha_1+\alpha_2) = \dfrac{1}{\tan(\alpha_1+\alpha_2)} = \dfrac{1-\tan\alpha_1\tan\alpha_2}{\tan\alpha_1+\tan\alpha_2}$	**M1** tan/cot($A + B$) result	
$= \dfrac{\cot\alpha_1\cot\alpha_2 - 1}{\cot\alpha_1+\cot\alpha_2}$	**M1** everything in terms of cots	
$= \dfrac{1+k\cot^2\beta-1}{k\cot\beta} = \cot\beta$	**A1**	
and it follows that $\alpha_1 + \alpha_2 = \beta$ ($\because \beta, \alpha_1, \alpha_2$ all acute)	**E1** (**AG**) must be justified	**6**
Still considering the quadratic in c:		
For real c, discriminant $\Delta = (k\cot\beta)^2 - 4(1 + k\cot^2\beta) \geq 0$	**M1** considering discriminant	
$\Rightarrow (k^2 - 4k)\cot^2\beta \geq 4 \Rightarrow k^2 - 4k \geq 4\tan^2\beta$		
$\Rightarrow (k-2)^2 \geq 4\tan^2\beta + 4 = 4\sec^2\beta$	**M1** completing the square and trig. identity used	
$\Rightarrow \quad k \geq 2(1+\sec\beta)$ (… ignore $k \leq$ –ve thing)	**A1** legitimately (**AG**)	**3**

(ii) $\dot{y} = u\cos\alpha - gt \Rightarrow t = \dfrac{u\cos\alpha}{g}$ at max. height	**M1** stated or used in y-formula	
$\Rightarrow H = \dfrac{u^2\cos^2\alpha}{2g}$	**A1** (give M1 A1 if result correctly quoted)	
$h \leq H \Rightarrow 2gh \leq u^2\cos^2\alpha$	**M1** comparing h with H	
$\Rightarrow 2 \times \dfrac{2u^2}{k} \leq u^2\cos^2\alpha$	**M1** substituting for k	
$\Rightarrow k \geq 4\sec^2\alpha$	**A1** (**AG**) legitimately obtained	**5**

11 **(i)**	P(HH) = p^2 P(TT) = q^2 P(TH or HT) = $2pq$	**B1** seen at any stage	
	P(first n–1 rounds indecisive) = $(2pq)^{n-1}$		
	$\Rightarrow$ Decision at round n = $(2pq)^{n-1} \times$ P(HH or TT)	**M1**	
	$= (2pq)^{n-1}(p^2 + q^2)$	**A1** legitimately (**AG**)	**3**
	Let d = P(decision on or before n^{th} round)		
	= 1 – P(decision after n^{th} round)	**M1** with working	
	$= 1 - \left\{(2pq)^n(p^2+q^2)+(2pq)^{n+1}(p^2+q^2)+(2pq)^{n+2}(p^2+q^2)+\ldots\right\}$		
	$= 1 - (2pq)^n(p^2+q^2)\left\{1+(2pq)+(2pq)^2+\ldots\right\}$		
	$= 1 - (2pq)^n(p^2+q^2)\times\dfrac{1}{1-2pq}$	**M1** use of S_∞(GP)	
	since $p^2+q^2=(p+q)^2-2pq=1-2pq$		
	$= 1-(2pq)^n$	**A1**	
	Now $\sqrt{pq}\le\frac{1}{2}(p+q)=\frac{1}{2}$ by the AM-GM inequality	**M1 A1** method; correct	
	or via $\left(\sqrt{p}-\sqrt{q}\right)^2\ge 0 \;\Rightarrow\; p+q-2\sqrt{pq}\ge 0$ etc.		
	or via $pq=p(1-p)\le\frac{1}{4}$ by calculus/completing the square	**E1** inequality concluded	
	and $d=1-(2pq)^n=1-2^n\left(\sqrt{pq}\right)^{2n}\ge 1-2^n\left(\frac{1}{2}\right)^{2n}=1-\dfrac{1}{2^n}$	**A1** legitimately (**AG**)	**7**
(ii)	P(decision at 1^{st} round) = p^3+q^3 or $1-3pq$	**B1**	
	P(decision at 2^{nd} round) = $3p^2q.p^2 + 3q^2p.q^2$	**M1** Good attempt at two cases	
	So overall prob. is P = $p^3 + q^3 + 3p^4q + 3pq^4$	**A1**	**3**
	P = $p^3 + (1-p)^3 + 3(p^4 - p^5) + 3p(1-p)^4$	**M1** a polynomial in p only	
	$= 1 - 9p^2 + 18p^3 - 9p^4$	**A1**	
	$\dfrac{\text{dP}}{\text{d}p} = -18p + 54p^2 - 36p^3$	**M1**	
	$= -18p(2p-1)(p-1)$	**M1** and set to zero	
	giving $p = 0, \frac{1}{2}, 1$	**A1**	
	Since P is a positive cubic, 0 and 1 give maxima, while $\frac{1}{2}$ gives a (local) minimum	**E1** justification	
	So, on [0, 1], $p=\frac{1}{2}$ and $P_{\min}=\frac{7}{16}$	**A1**	**7**