Section A: Pure Mathematics

1 (i) The quickest method is to use Pascal's triangle:

$$\left(3+2\sqrt{5}\right)^3 = 27 + 3\,(3)^2\left(2\sqrt{5}\right) + 3\,(3)\left(2\sqrt{5}\right)^2 + \left(2\sqrt{5}\right)^3$$

$$= 27 + 54\sqrt{5} + 180 + 40\sqrt{5} = 207 + 94\sqrt{5}$$

but for a small power such as 3 it is easy to compute $\left(3+2\sqrt{5}\right)^2 = 29 + 12\sqrt{5}$ and then multiply the answer by $3+2\sqrt{5}$. You might like to consider how to calculate e.g. $\left(3+2\sqrt{5}\right)^{11}$ efficiently, using the fact that $u^{11} = \left(\left(u^2\right)^2\right)^2 \times u^2 \times u$.

(ii) If $\sqrt[3]{99-70\sqrt{2}} = c - d\sqrt{2}$ then $99 - 70\sqrt{2} = \left(c - d\sqrt{2}\right)^3$.

By Pascal's triangle $\Rightarrow 99 - 70\sqrt{2} = c^3 - 3c^2d\sqrt{2} + 3c\left(d\sqrt{2}\right)^2 - \left(d\sqrt{2}\right)^3$.

Equating rational and irrational coefficients $\Rightarrow c^3 + 6cd^2 = 99$ and $3c^2d + 2d^3 = 70$. These equations are hard to solve algebraically, but the question tells you that c and d are **positive integers**, and since you can see that $c^3 < 99$ and $2d^3 < 70$ there aren't many values of c and d to try. You should quickly find that $c = 3$ and $d = 2$: remember to check that these values satisfy both equations.

(iii) Using the quadratic formula $x^3 = \dfrac{198 \pm \sqrt{198^2 - 4}}{2}$.

Notice how the discriminant is the difference of two squares:

$198^2 - 2^2 = (198-2)(198+2) = 196 \times 200 = 4 \times 49 \times 2 \times 100 = (2 \times 7 \times 10)^2 \times 2.$

$$\Rightarrow x^3 = \frac{198 \pm (2 \times 7 \times 10)\sqrt{2}}{2}$$

$$\Rightarrow x^3 = 99 \pm 70\sqrt{2}$$

$\Rightarrow x = 3 \pm 2\sqrt{2}$ using the answer to part (ii).

2 (i) Your graph should look like a staircase: if $0 \leqslant x < 1$ then $\sqrt{[x]} = 0$; if $1 \leqslant x < 2$ then $\sqrt{[x]} = 1$; if $2 \leqslant x < 3$ then $\sqrt{[x]} = \sqrt{2} \approx 1.4$; if $3 \leqslant x < 4$ then $\sqrt{[x]} = \sqrt{3} \approx 1.7$ etc.

Since a definite integral can be interpreted as the area between a graph and the x axis, it can be evaluated by adding up the areas of the rectangles under the graph, therefore

$$\int_0^a \sqrt{[x]}\,\mathrm{d}x = \left(1 \times \sqrt{0}\right) + \left(1 \times \sqrt{1}\right) + \left(1 \times \sqrt{2}\right) + \ldots + \left(1 \times \sqrt{a-1}\right) = \sum_{r=0}^{a-1} \sqrt{r}$$

(ii) A graph would be very helpful, as it will show the rectangles which need to be summed:

$$\int_0^a 2^{[x]}\,\mathrm{d}x = (1 \times 1) + (1 \times 2) + (1 \times 4) + \ldots + \left(1 \times 2^{a-1}\right).$$

This is the sum of a geometric progression, and so the definite integral $= 2^a - 1$.

(iii) If a is positive but not an integer, then the graph of $y = 2^{[x]}$ from 0 to a will be a staircase of rectangles each of width 1 from 0 to $[a]$, and then a thinner rectangle of width $a - [a]$. The rectangles of width 1 have total area $2^{[a]} - 1$ using your answer to part (ii), and the final rectangle has height $2^{[a]}$ and so has area $(a - [a]) \times 2^{[a]}$. Therefore, the definite integral $= \left(2^{[a]} - 1\right) + (a - [a]) \times 2^{[a]}$.

3 (i) If you substitute $x = 3$ into the given expression, it equals $27 - 45 + 18y + 3y^2 - 24y - 3y^2 + 18 + 6y = 0$, so by the Factor Theorem $x - 3$ is a factor of it.

If you expand $(x-3)(x+ay+b)(x+cy+d)$ you should get $x^3 + (a+c)x^2y + acxy^2 + (bc+ad-3a-3c)xy + (b+d-3)x^2 + (bd-3b-3d)x - 3(a+c)xy - 3acy^2 - 3bd$. Then equating coefficients:

$a + c = 2$

$b + d - 3 = -5$

$ac = 1$

$bd = 0$

$bc + ad - 3a - 3c = -8$

$bd - 3b - 3d = 6$

$bc + ad = -2$

The first and third equations tell you that $a = c = 1$ (since you are given that a, b, c and d are integers), and the second and fourth equations tell you that $b = 0,\ d = -2$ or $b = -2,\ d = 0$. Therefore, the given expression factorises as $(x-3)(x+y-2)(x+y)$.

A neat idea is to factorise the given expression (by division or inspection) as
$(x-3)\left(x^2 + y^2 + 2xy - 2x - 2y\right)$
and then solve the quadratic factor as an equation ($= 0$) in x with coefficients in y using the quadratic formula: $x = \dfrac{-(2y-2) \pm \sqrt{(2y-2)^2 - 4\left(y^2 - 2y\right)}}{2} = \dfrac{-2y + 2 \pm \sqrt{4}}{2}$
$\Rightarrow x = -y + 2$ or $x = -y$, so the factors are $x - (-y+2)$ and $x - (-y)$.

(ii) Since $6y^3 - y^2 - 21y + 2x^2 + 12x - 4xy + x^2y - 5xy^2 + 10$ has no term in x^3, there must be a linear factor with no x term i.e. of the form $py + q$. Checking a few values of y shows that when $y = -2$, the given expression $= 0 \Rightarrow (y+2)(x+ay+b)(x+cy+d)$ is a factorisation. Notice how you shouldn't assume a factorisation of the form $(x \pm 6y + b)(x \pm y + d)$ as it might be of the form $(x \pm 3y + b)(x \pm 2y + d)$, but it is reasonable to assume that the coefficient of x in both factors is 1, since that will ensure a term $2x^2$.

The claimed factorisation is $(y+2)\left(x^2 + acy^2 + (a+c)xy + (b+d)x + (ad+bc)y + bd\right)$ so equating coefficients implies that

$ac = 6$

$ad + bc + 2ac = -1$

$bd + 2(ad + bc) = -21$

$b + d = 6$

$b + d + 2(a + c) = -4$

$a + c = -5$

$bd = 5$

The first and sixth equations tell you that $a = -2$ and $c = -3$ or vice versa, and the fourth and seventh equations tell you that $b = 5$ and $d = 1$ or vice versa. You need to use the second equation, which tells you that $ad + bc = -13$, to determine that the factorisation is $(y+2)(x-2y+1)(x-3y+5)$.

It may be instructive to find these factors using the quadratic formula, as described at the end of part (i).

4 The derivative of $\sec x$ is $\sec x \tan x$, which can be determined by differentiating $(\cos x)^{-1}$.

(i) Using the given substitution, and remembering not only to write the limits in terms of t but also to write dx in terms of dt,

$$\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec t \tan t}{\sec^3 t\sqrt{\sec^2 t - 1}}\,\mathrm{d}t = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec t \tan t}{\sec^3 t\sqrt{\tan^2 t}}\,\mathrm{d}t = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2 t\,\mathrm{d}t$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos 2t + 1}{2}\,\mathrm{d}t = \left[\frac{\sin 2t}{4} + \frac{t}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$$

using the identities $1 + \tan^2 t \equiv \sec^2 t$ and $\cos 2t \equiv 2\cos^2 t - 1$.

(ii) Since $(x+1)(x+3) = x^2 + 4x + 3 = (x+2)^2 - 1$, this integral in part (ii) looks similar to the integral in part (i).

So let $x + 2 = \sec t \Rightarrow \displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\,\mathrm{d}x = \int \frac{\sec t \tan t}{\sec t\sqrt{(\sec t - 1)(\sec t + 1)}}\,\mathrm{d}t$

$$= \int \frac{\sec t \tan t}{\sec t\sqrt{\sec^2 t - 1}}\,\mathrm{d}t = \int 1\,\mathrm{d}t = t + c = \operatorname{arcsec}(x+2) + c$$

(iii) Since $x^2 + 4x - 5 = (x+2)^2 - 9$, let $x + 2 = 3\sec t$.

Therefore $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\,\mathrm{d}x = \int \frac{3\sec t \tan t}{3\sec t\sqrt{9(\sec t)^2 - 9}}\,\mathrm{d}t$

$$= \int \frac{1}{3}\,\mathrm{d}t = \frac{t}{3} + c = \frac{1}{3}\operatorname{arcsec}\left(\frac{x+2}{3}\right) + c$$

5 Using the formula for the kth term of an arithmetic progression:

A: value of kth term $= 5k + 1$

B: value of kth term $= 5k + 2$

C: value of kth term $= 5k + 3$

D: value of kth term $= 5k + 4$

E: value of kth term $= 5k$

or equivalent.

Therefore, the sum of any term in B and any term in C can be written $(5k + 2) + (5n + 3) = 5(k + n + 1)$ which is a term in E. Notice that it is important to use k and n, to ensure that you are not only adding corresponding terms in B and C.

Similarly, the square of any term in B can be written $(5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4$ which is a term in D, and the square of any term in C can be written $(5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4$ which is also a term in D.

(i) Since $(5k)^2 = 5(5k^2)$, $(5k + 1)^2 = 5(5k^2 + 2k) + 1$ and $(5k + 4)^2 = 5(5k^2 + 8k + 3) + 1$, x^2 must be in E, A or D (including the results from above) i.e. $x^2 = 5n$, $5n + 1$ or $5n + 4$..

But $(5n + 1) + 5y = 5(n + y) + 1$, $(5n + 4) + 5y = 5(n + y) + 4$, and $5n + 5y = 5(n + y)$.

So x^2 + a term in E cannot be a term in C.

(ii) You know that x^2 must be in E, A or D, so $x^4 = (x^2)^2$ must be a term in A or E, as all terms in E square to give a term in E, and all terms in A and D square to give terms in A.

Therefore $2y^4$ must be a term in B or E.

No pair of terms from A, B or E can add to give a term in D (consider the "+4"), so the equation $x^4 + 2y^4 = 26\ 081\ 974$ has no solution for integer x and y.

6 The line joining A to the midpoint of BC has equation $y - q_1 = \left(\dfrac{\frac{q_2+q_3}{2} - q_1}{\frac{p_2+p_3}{2} - p_1}\right)(x - p_1)$

$$\Rightarrow y - q_1 = \left(\frac{q_2 + q_3 - 2q_1}{p_2 + p_3 - 2p_1}\right)(x - p_1)$$

The line joining B to the midpoint of AC has equation $y - q_2 = \left(\dfrac{q_1 + q_3 - 2q_2}{p_1 + p_3 - 2p_2}\right)(x - p_2)$

To find where the two lines meet you need to solve simultaneously these two equations. The many subscripts make it difficult to do this correctly, but the algebra required is not advanced. You should find that the lines meet at $\left(\dfrac{p_1 + p_2 + p_3}{3}, \dfrac{q_1 + q_2 + q_3}{3}\right)$.

You then need to verify that these coordinates satisfy the equation of the line joining C to the midpoint of AB, which is $y - q_3 = \left(\dfrac{q_1 + q_2 - 2q_3}{p_1 + p_2 - 2p_3}\right)(x - p_3)$

If AH is perpendicular to BC then $\dfrac{q_2 + q_3}{p_2 + p_3} \times \dfrac{q_3 - q_2}{p_3 - p_2} = -1$ since the product of the gradients of perpendicular lines is -1.

$$\Rightarrow q_3^2 - q_2^2 = -\left(p_3^2 - p_2^2\right) \Rightarrow p_2^2 + q_2^2 = p_3^2 + q_3^2$$

If the line BH intersects the line AC at right angles $\Rightarrow p_1^2 + q_1^2 = p_3^2 + q_3^2$. The instruction "write down" in the question suggests that very little work is needed to do so: see how this answer is structurally identical to the previous answer.

Therefore, if AH is perpendicular to BC and BH is perpendicular to AC

$$\Rightarrow p_2^2 + q_2^2 = p_3^2 + q_3^2 \text{ and } p_1^2 + q_1^2 = p_3^2 + q_3^2$$

$$\Rightarrow p_1^2 + q_1^2 = p_2^2 + q_2^2$$

$$\Rightarrow q_2^2 - q_1^2 = -\left(p_2^2 - p_1^2\right) \Rightarrow \frac{q_2 + q_1}{p_2 + p_1} \times \frac{q_2 - q_1}{p_2 - p_1} = -1$$

$\Rightarrow CH$ is perpendicular to AB

This question aims to show some interesting properties of triangles. The line joining a vertex of a triangle to the midpoint of the opposite side (e.g. A to the midpoint of BC) is called a median: you have proved that the three medians of any triangle are concurrent, i.e. they all meet at the same point with coordinates $\left(\dfrac{p_1 + p_2 + p_3}{3}, \dfrac{q_1 + q_2 + q_3}{3}\right)$. This point is called the centroid.

The line through a vertex which is perpendicular to the opposite side is called an altitude: the three altitudes of any triangle are concurrent, and they meet at a point called the orthocentre. In this question the triangle has been set up so that the orthocentre has simple coordinates $(p_1 + p_2 + p_3\,,\ q_1 + q_2 + q_3)$. You should consider how this has been achieved: what geometric fact must be true if the coordinates of A, B and C satisfy $p_1^2 + q_1^2 = p_2^2 + q_2^2 = p_3^2 + q_3^2$?

7 (i) The relationship $a_0 = 2$, $a_1 = 7$ and $a_n - 7a_{n-1} + 10a_{n-2} = 0$ is called a recurrence relationship: given the values of a_0 and a_1 you can calculate that $a_2 = 29$, $a_3 = 133$ and so on. Therefore $\mathrm{f}(x) = 2 + 7x + 29x^2 + 133x^3 + \ldots$

$\Rightarrow \mathrm{f}(x) - 7x\mathrm{f}(x) + 10x^2\mathrm{f}(x) =$

$\left(2 + 7x + 29x^2 + 133x^3 + \ldots\right) -$

$\left(14x + 49x^2 + 203x^3 + 931x^4 + \ldots\right) +$

$\left(20x^2 + 70x^3 + 290x^4 + 1330x^5 + \ldots\right) = 2 - 7x$

because all the other terms cancel.

More formally, $\mathrm{f}(x) - 7x\mathrm{f}(x) + 10x^2\mathrm{f}(x) = \sum\limits_{r=0}^{\infty} a_r x^r - \sum\limits_{r=0}^{\infty} 7a_r x^{r+1} + \sum\limits_{r=0}^{\infty} 10a_r x^{r+2}$

$= 2 + 7x + \sum\limits_{r=0}^{\infty} a_{r+2}x^{r+2} - 14x - \sum\limits_{r=0}^{\infty} 7a_{r+1}x^{r+2} + \sum\limits_{r=0}^{\infty} 10a_r x^{r+2}$ (if this is unclear, expand the first few terms of each of the sigmas).

$= 2 - 7x + \sum\limits_{r=0}^{\infty} \left(a_{r+2} - 7a_{r+1} + 10a_r\right) x^{r+2}$

$= 2 - 7x + \sum\limits_{r=0}^{\infty} (0)\, x^{r+2}$

because of the given relationship $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geqslant 2$.

$\Rightarrow \mathrm{f}(x) = \dfrac{7 - 2x}{1 - 7x + 10x^2} = \dfrac{7 - 2x}{(1 - 2x)(1 - 5x)} = \dfrac{1}{1 - 2x} + \dfrac{1}{1 - 5x}$ by partial fractions.

Since $(1 - 2x)^{-1} = 1 + 2x + 4x^2 + 8x^3 + \ldots + 2^n x^n + \ldots$

and $(1 - 5x)^{-1} = 1 + 5x + 25x^2 + 125x^3 + \ldots + 5^n x^n + \ldots$

the coefficient of x^n is $2^n + 5^n$, therefore $a_n = 2^n + 5^n$.

(ii) In this part of the question you haven't been given the precise relationship between the terms of the sequence b_n, so you need to determine that first.

$n = 2 \Rightarrow b_2 = pb_1 + qb_0 \Rightarrow 40 = 10p + 5q$

$n = 3 \Rightarrow b_3 = pb_2 + qb_1 \Rightarrow 100 = 40p + 10q$

$\Rightarrow p = 1,\ q = 6 \Rightarrow b_n - b_{n-1} - 6b_{n-2} = 0$

Part (a) suggests consider $\mathrm{g}(x) - x\mathrm{g}(x) - 6x^2\mathrm{g}(x) =$

$\left(5 + 10x + 40x^2 + 100x^3 + \ldots\right) -$

$\left(5x + 10x^2 + 40x^3 + 100x^4 + \ldots\right) -$

$\left(30x^2 + 60x^3 + 240x^4 + 600x^5 + \ldots\right) = 5 + 5x$

This is sufficient, but you should try to write a convincing argument using sigma notation as above.

$\Rightarrow \mathrm{g}(x) = \dfrac{5 + 5x}{1 - x - 6x^2} = \dfrac{5 + 5x}{(1 + 2x)(1 - 3x)} = \dfrac{1}{1 + 2x} + \dfrac{4}{1 - 3x}$

$\Rightarrow \mathrm{g}(x) = \left(1 - 2x + 4x^2 - 8x^3 + \ldots + (-2)^n x^n + \ldots\right) + 4\left(1 + 3x + 9x^2 + 27x^3 + \ldots 3^n x^n + \ldots\right)$

$\Rightarrow b_n = (-2)^n + 4(3)^n$

8 (i) It is important to realise that it is not sufficient to choose a specific value of x_0 e.g. $x_0 = 4$, evaluate $x_1 = 4.4$ and then claim that the sequence is increasing for all n and for all $x_0 > 3$. When asked to prove that $a > b$, a good strategy may be to consider $a - b$, and try to see why $a - b$ is positive.

In this problem, therefore, consider

$$x_{n+1} - x_n = \frac{x_n^2 + 6}{5} - x_n = \frac{x_n^2 - 5x_n + 6}{5} = \frac{(x_n - 3)(x_n - 2)}{5}$$

which will be positive for any $x_n > 3$ (or any $x_n < 2$).

Let $n = 0$: you are told that $x_0 > 3$, so $x_1 - x_0 > 0$

$\Rightarrow x_1 > x_0 \Rightarrow x_1 > 3$ since $x_0 > 3$

$\Rightarrow x_2 - x_1 > 0 \Rightarrow x_2 > x_1$

$\Rightarrow x_2 > 3$ since $x_1 > 3$

$\Rightarrow x_3 - x_2 > 0 \Rightarrow x_3 > x_2$ and so on.

Therefore $x_{n+1} > x_n$ for all n.

(ii) As in part (i), consider

$$y_{n+1} - y_n = 5 - \frac{6}{y_n} - y_n = \frac{5y_n - 6 - y_n^2}{y_n} = \frac{-(y_n - 2)(y_n - 3)}{y_n}$$

which is positive for any y_n between 2 and 3.

Let $n = 0$: $2 < y_0 < 3 \Rightarrow y_1 - y_0 > 0 \Rightarrow y_1 > y_0$.

Also $3 - y_1 = 3 - \left(5 - \dfrac{6}{y_0}\right) = \dfrac{6}{y_0} - 2 > 0$ since $y_0 < 3$.

$\Rightarrow 3 - y_1 > 0 \Rightarrow y_1 < 3$

Therefore $2 < y_0 < y_1 < 3$, so $y_2 - y_1 > 0 \Rightarrow y_2 > y_1$.

Also $3 - y_2 = 3 - \left(5 - \dfrac{6}{y_1}\right) = \dfrac{6}{y_1} - 2 > 0$ since $y_1 < 3$.

So $2 < y_1 < y_2 < 3$ and so on.

Therefore, $y_{n+1} > y_n$ for all n, but $y_n < 3$ for all n.

Both of these arguments are most easily written using Mathematical Induction: if you are familiar with this method of proof, then you might like to prove these two results in this way.

Section B: Mechanics

9 At the maximum height H, the vertical velocity $= 0$.

Therefore $0 = (u\sin\theta)^2 - 2gH \Rightarrow H = \dfrac{u^2\sin^2\theta}{2g}$

At the point P, $ut\cos\theta = d$ and $ut\sin\theta - \frac{1}{2}gt^2 = \frac{1}{2}d$

$$\Rightarrow \frac{d}{2} = u\sin\theta \times \frac{d}{u\cos\theta} - \frac{g}{2}\left(\frac{d}{u\cos\theta}\right)^2 = d\tan\theta - \frac{gd^2\sec^2\theta}{2u^2}$$

$$\Rightarrow u^2 = \frac{gd\sec^2\theta}{2\tan\theta - 1} \quad \text{so} \quad \frac{gd\sec^2\theta}{2\tan\theta - 1} \times \frac{\sin^2\theta}{2g} < \frac{9d}{10}$$

because the projectile doesn't hit the roof of the tunnel and therefore its maximum height is less than $\frac{9}{10}d$.

$\Rightarrow 10\sec^2\theta\sin^2\theta < 18\,(2\tan\theta - 1)$

$\Rightarrow 5\tan^2\theta - 18\tan\theta + 9 < 0$

$\Rightarrow (5\tan\theta - 3)\,(\tan\theta - 3) < 0$

$\Rightarrow \dfrac{3}{5} < \tan\theta < 3$

$\Rightarrow \arctan\frac{3}{5} < \theta < \arctan 3$

10 (i) A good velocity time graph makes this question much easier to understand.

Since $v^2 = u^2 + 2(3a)(d_1)$ and $0 = v^2 - 2a(d_2)$,

$\Rightarrow 2ad_2 = u^2 + 6ad_1$

$\Rightarrow 6ad_1 < 2ad_2$ because $u^2 > 0$

$\Rightarrow 3d_1 < d_2$

(ii) A velocity time graph is **very** helpful here.

The particle takes time $\dfrac{-u}{3a}$ to accelerate from initial velocity $u < 0$ to rest, in which time it travels a **distance** of $\dfrac{u^2}{6a}$. You should not calculate this distance using $v^2 = u^2 + 2as$ because s is the displacement of the particle not the distance it travels, but instead should look at the (positive) area of the relevant triangle on the graph.

Then the particle travels a distance D where $v^2 = 0 + 2(3a)(D)$, so that $D = \dfrac{v^2}{6a}$.

So $d_1 = \dfrac{u^2 + v^2}{6a}$.

Still $d_2 = \dfrac{v^2}{2a}$, so $3d_1 = \dfrac{u^2 + v^2}{2a} > \dfrac{v^2}{2a} = d_2$.

Also, $v > |u| \Rightarrow v^2 > u^2$, so $3d_1 = \dfrac{u^2 + v^2}{2a} < \dfrac{v^2 + v^2}{2a} = 2d_2$

If $u > 0$ then the total time taken by the journey $= \dfrac{v - u}{3a} + \dfrac{v}{a} = \dfrac{4v - u}{3a} = \dfrac{4v - |u|}{3a}$ since $u > 0 \Rightarrow u = |u|$.

Therefore, average speed $= \dfrac{\text{total distance}}{\text{total time}} = \dfrac{\frac{v^2-u^2}{6a} + \frac{v^2}{2a}}{\frac{4v-|u|}{3a}} = \dfrac{4v^2 - u^2}{2(4v - |u|)}$.

If $u < 0$ then total time taken by the journey $= \dfrac{v + |u|}{3a} + \dfrac{v}{a} = \dfrac{4v + |u|}{3a}$.

Note the use of $|u|$ to ensure that you are adding two positive quantities: the expression $v + u$ is actually a subtraction if u is negative.

$$\Rightarrow \text{average speed} = \frac{\frac{v^2+u^2}{6a} + \frac{v^2}{2a}}{\frac{4v+|u|}{3a}} = \frac{4v^2 + u^2}{2(4v + |u|)}.$$

As in question 8, you can consider the difference between these two:

$$\frac{4v^2 - u^2}{2(4v - |u|)} - \frac{4v^2 + u^2}{2(4v + |u|)} = \frac{(4v^2 - u^2)(4v + |u|) - (4v^2 + u^2)(4v - |u|)}{2(4v - |u|)(4v + |u|)}$$

$$= \frac{-8u^2v + 8v^2|u|}{2(4v - |u|)(4v + |u|)} = \frac{8v|u|(v - |u|)}{2(4v - |u|)(4v + |u|)} > 0 \text{ since } v > |u|.$$

11 A good diagram is essential here. Let AB be the left ladder. Let the normal reaction at A be R, and the frictional resistance be $F_1 \leqslant \mu R$: recall $F_1 = \mu R$ only if the friction is limiting e.g. the ladder AB is about to slip - and you don't know that this is the case. Let the normal reaction at C be S, and the frictional resistance be $F_2 \leqslant \mu S$. Let $AB = BC = l$.

The easiest strategy is to take moments about B, and therefore not worry about the nature of the reaction force acting there between the ladders.

Taking moments about B of the forces acting on BC

$$\Rightarrow S \times \frac{l}{2} = 4W \times \frac{l}{4} + F_2 \times \frac{l\sqrt{3}}{2}$$

Taking moments about B of the forces acting on AB

$$\Rightarrow R \times \frac{l}{2} = W \times \frac{l}{4} + 7W \times \frac{2l}{3} \times \frac{1}{2} + F_2 \times \frac{l\sqrt{3}}{2}$$

Therefore, $S = 2W + F_2\sqrt{3}$ and $6R = 31W + 6F_1\sqrt{3}$.

Resolving vertically all forces acting on the system $\Rightarrow R + S = 12W$.

Resolving horizontally all forces acting on the system $\Rightarrow F_1 = F_2$.

$$\Rightarrow S - 2W = \frac{6R - 31W}{6}$$

$$\Rightarrow 6S = 6R - 19W$$

$$\Rightarrow 72W - 6R = 6R - 19W$$

$$\Rightarrow R = \frac{91W}{12} \text{ and } S = \frac{53W}{12}$$

$$\Rightarrow F_1 = \frac{29W\sqrt{3}}{36} = F_2$$

BC slips as $S < R$, and $\mu = \dfrac{F_2}{S} = \dfrac{29\sqrt{3}}{159}$.

If the reaction between the two ladders at the hinge B is included, it is best to consider a horizontal and vertical component of the reaction force acting on each ladder. There is no reason to assume that the reaction force will act purely horizontally: not all the forces act symmetrically on the ladders. The horizontal components will be equal in magnitude but in opposite directions, and the vertical components will be similar. Let the reaction at B acting on AB be X horizontally (to the left) and Y upwards; let the reaction acting on BC be X to the right and Y downwards. It would be instructive now, for example, to take moments about A and C, and compare the merits of the two solutions.

Section C: Probability and Statistics

12 A tree diagram is very helpful, if only to clarify the language of the question.

$$P\,(\text{made by } A \mid \text{perfect}) = \frac{P\,(\text{made by } A \text{ and perfect})}{P\,(\text{perfect})} = \frac{2}{5}$$

$$\Rightarrow \frac{\lambda p}{\lambda p + (1-\lambda)\,q} = \frac{2}{5} \Rightarrow 5\lambda p = 2\lambda p + 2\,(1-\lambda)\,q$$

$$\Rightarrow \lambda = \frac{2q}{3p+2q}$$

Now, $P\,(\text{made by } A \mid \text{declared to be perfect}) = \dfrac{2}{5}$

$$\Rightarrow \frac{\frac{3}{4}\lambda p + \frac{1}{4}\lambda\,(1-p)}{\frac{3}{4}\lambda p + \frac{1}{4}\lambda\,(1-p) + \frac{3}{4}\,(1-\lambda)\,q + \frac{1}{4}\,(1-\lambda)\,(1-q)} = \frac{2}{5}$$

since those that are declared to be perfect are either those that are perfect and have been correctly assigned (with probability $\frac{3}{4}$) or those that are imperfect but have been incorrectly assigned (with probability $\frac{1}{4}$).

$$\Rightarrow 15\lambda p + 5\lambda\,(1-p) = 6\lambda p + 2\lambda\,(1-p) + 6\,(1-\lambda)\,q + 2\,(1-\lambda)\,(1-q)$$

$$\Rightarrow \lambda = \frac{4q+2}{6p+4q+5}$$

13 (i) $P\,(\text{maximum of 3 numbers} \leqslant 0.8) = P(\textbf{all} \text{ three numbers} \leqslant 0.8) = 0.8^3 = 0.512$

Similarly, $\mathrm{F}(x) =$ the cumulative distribution function of $X = P\,(X \leqslant x) = x^3$

$\Rightarrow \mathrm{f}(x) =$ the probability density function of $X = \mathrm{F}'(x) = 3x^2$

$$\Rightarrow E\,(X) = \int_0^1 3x^3 \,\mathrm{d}x = \frac{3}{4}.$$

A similar argument would work to calculate $E\,(Y)$, where Y is the minimum of the numbers drawn.

(ii) If H_0 is true, then $P\,(X \leqslant 0.8) = 0.8^N$.

We require the minimum N such that $0.8^N < 0.05$ since then H_0 will be rejected.

$$\Rightarrow \left(\frac{2^3}{10}\right)^N < \frac{1}{20} \Rightarrow 2^{3N} < \frac{10^N}{20} \Rightarrow 2^{3N} < \frac{10^N}{2 \times 10}$$

$\Rightarrow 2^{3N+1} < 10^{N-1} \Rightarrow 2^{3N+1} < \left(2^{\frac{10}{3}}\right)^{N-1}$ using the given approximation $10^3 \approx 2^{10}$

$$\Rightarrow 3N + 1 < \frac{10\,(N-1)}{3} \Rightarrow 9N + 3 < 10N - 10$$

$\Rightarrow N > 13$

$\Rightarrow N = 14$

If $a = 0.8$ then $P\,(H_0 \text{ is rejected}) = 1$

If $a = 0.9$ then $P\,(H_0 \text{ is rejected}) = \left(\frac{8}{9}\right)^{14}$

14 (i) The first pirate will have some gold if he does not take a lead coin first.

$$\Rightarrow P\,(\text{first pirate has some gold }) = \frac{n}{n+2}$$

(ii) There are at least two different approaches that can be used to answer this part. The easiest argument, if you are familiar with it, is to see this as a problem about the number of ways of rearranging a selection of objects.

You may recall that there are $m!$ ways of rearranging m different objects in a line. Here there are $n+2$ coins which can be rearranged in $(n+2)!$ ways, but the second pirate wants the two lead coins **not** to be adjacent. It is easier to count the number of ways to rearrange the $n+2$ coins so that the two lead ones **are** adjacent: there are $(n+1)! \times 2!$ ways of doing this, because you treat the two adjacent lead coins as a single item and therefore rearrange the remaining $n+1$ coins, but the two lead coins can be swapped around $2!$ ways so that there are twice as many rearrangements.

Hence there are $(n+2)! - (n+1)! \times 2!$ ways of rearranging the $n+2$ coins so that the lead coins are not adjacent.

$$\Rightarrow P\,(\text{the second pirate has some gold}) = \frac{(n+2)! - 2!\,(n+1)!}{(n+2)!} = \frac{(n+1)!\,(n+2-2!)}{(n+2)!}$$

$$= \frac{(n+1)! \times n}{(n+2) \times (n+1)!} = \frac{n}{n+2}$$

An alternative argument is the following. If the second pirate has some gold coins, then the coins must have been taken from the chest in the order LG or GLG or $GGLG$ or $GGGLG$ etc, until $GGG...GGLGL$, where G represents taking a gold coin and L represents taking a lead coin. You should consider why the orders LGG, $LGGG$, $LGGGG$ etc do not need to be listed: a tree diagram will be helpful.

$$\text{Therefore, } P\,(\text{the second pirate has some gold}) = \left(\frac{2}{n+2} \times \frac{n}{n+1}\right) + \left(\frac{n}{n+2} \times \frac{2}{n+1} \times \frac{n-1}{n}\right) +$$

$$\left(\frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{2}{n} \times \frac{n-2}{n-1}\right) + \ldots + \left(\frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n} \times \ldots \times \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1}\right)$$

$$= \frac{2}{(n+2)\,(n+1)}\left[n + (n-1) + (n-2) + \ldots + 1\right]$$

$$= \frac{2}{(n+2)\,(n+1)}\left[\frac{n\,(n+1)}{2}\right]$$

$$= \frac{n}{n+2}$$

(iii) If all three pirates have some gold, then a gold coin was chosen first, a gold coin was chosen last, and when the remaining n coins were chosen the lead coins were not adjacent.

$$\Rightarrow P\,(\text{all three pirates have some gold}) = \frac{n}{n+2} \times \frac{n-1}{n+1} \times \frac{n-2}{n}$$

because the probability that the last coin is gold must equal the probability that the second coin is gold: once the second coin has been drawn it can be renamed as the last coin.

The third fraction has been deduced by adapting the answer from part (ii).

$$\text{This simplifies to } \frac{n-1}{n+1} \times \frac{n-2}{n+2} = \frac{n^2-3n+2}{n^2+3n+2}$$