

STEP MATHEMATICS 3

2021

Mark Scheme

1. (i) $x = -4 \cos^3 t$ so $\frac{dx}{dt} = 12 \cos^2 t \sin t$ **M1**

$y = 12 \sin t - 4 \sin^3 t$ so $\frac{dy}{dt} = 12 \cos t - 12 \sin^2 t \cos t = 12 \cos t (1 - \sin^2 t) = 12 \cos^3 t$ **M1**

So $\frac{dy}{dx} = \frac{12 \cos^3 t}{12 \cos^2 t \sin t} = \cot t$ **A1**

Thus the equation of the normal at $(-4 \cos^3 \varphi, 12 \sin \varphi - 4 \sin^3 \varphi)$ is

$$y - (12 \sin \varphi - 4 \sin^3 \varphi) = -\frac{1}{\cot \varphi} (x - -4 \cos^3 \varphi)$$

M1 A1ft

This simplifies to $x \sin \varphi + y \cos \varphi = 12 \sin \varphi \cos \varphi - 4 \sin^3 \varphi \cos \varphi - 4 \sin \varphi \cos^3 \varphi$

That is $x \sin \varphi + y \cos \varphi = 8 \sin \varphi \cos \varphi$ **A1 (6)**

Alternative simplification $x \tan \varphi + y = 8 \sin \varphi$

For $x = 8 \cos^3 t$, $\frac{dx}{dt} = -24 \cos^2 t \sin t$ and for $y = 8 \sin^3 t$, $\frac{dy}{dt} = 24 \sin^2 t \cos t$

So $\frac{dy}{dx} = \frac{24 \sin^2 t \cos t}{-24 \cos^2 t \sin t} = -\tan t$ **M1 A1ft**

Thus the equation of the tangent to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at $(8 \cos^3 \varphi, 8 \sin^3 \varphi)$ is

$$y - 8 \sin^3 \varphi = -\tan \varphi (x - 8 \cos^3 \varphi)$$

M1

This simplifies to

$$x \sin \varphi + y \cos \varphi = 8 \sin^3 \varphi \cos \varphi + 8 \sin \varphi \cos^3 \varphi = 8 \sin \varphi \cos \varphi (\sin^2 \varphi + \cos^2 \varphi)$$

That is $x \sin \varphi + y \cos \varphi = 8 \sin \varphi \cos \varphi$ as required. **A1 (4)**

Alternative 1

the normal is a tangent to the second curve if it has the same gradient and the point $(8 \cos^3 \varphi, 8 \sin^3 \varphi)$ lies on the normal. **M1**

Gradient working as before **M1A1ft**

Substitution $x \sin \varphi + y \cos \varphi = 8 \sin \varphi \cos^3 \varphi + 8 \sin^3 \varphi \cos \varphi = 8 \sin \varphi \cos \varphi (\sin^2 \varphi + \cos^2 \varphi) = 8 \sin \varphi \cos \varphi$ as required or $x \tan \varphi + y = 8 \sin \varphi \cos \varphi (\sin^2 \varphi + \cos^2 \varphi)$ **A1**

Alternative 2

$$\frac{2}{3} x^{-\frac{1}{3}} + \frac{2}{3} y^{-\frac{1}{3}} \frac{dy}{dx} = 0$$

M1

(ii) $x = \cos t + t \sin t$ so $\frac{dx}{dt} = -\sin t + t \cos t + \sin t = t \cos t$

$y = \sin t - t \cos t$ so $\frac{dy}{dt} = \cos t - \cos t + t \sin t = t \sin t$ **M1**

So $\frac{dy}{dx} = \tan t$ **A1**

Thus the equation of the normal at $(\cos \varphi + \varphi \sin \varphi, \sin \varphi - \varphi \cos \varphi)$ is

$$y - (\sin \varphi - \varphi \cos \varphi) = -\cot \varphi (x - (\cos \varphi + \varphi \sin \varphi))$$

M1 A1ft

This simplifies to $x \cos \varphi + y \sin \varphi = 1$

A1 (5)

Alternatives which can be followed through to perpendicular distance step, or alternative method # are

$x + y \tan \varphi = \sec \varphi$ and $x \cot \varphi + y = \csc \varphi$

The distance of $(0,0)$ from $x \cos \varphi + y \sin \varphi = 1$ is $\left| \frac{-1}{\sqrt{\cos^2 \varphi + \sin^2 \varphi}} \right| = 1$

M1 A1ft

A1

Alternatively, the perpendicular to $x \cos \varphi + y \sin \varphi = 1$ through $(0,0)$ is

$y \cos \varphi - x \sin \varphi = 0$, and these two lines meet at $(\cos \varphi, \sin \varphi)$

M1 A1ft

which is a distance $\sqrt{\cos^2 \varphi + \sin^2 \varphi} = 1$ from $(0,0)$. **A1**

So the curve to which this normal is a tangent is a circle centre $(0,0)$, radius 1 which is thus $x^2 + y^2 = 1$ **M1 A1 (5)**

$$2. (i) \begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - (b - c)x \\ b - (c - a)y \\ c - (a - b)z \end{pmatrix} = \begin{pmatrix} a - a \\ b - b \\ c - c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ as required. } \mathbf{M1 A1^*}$$

As a, b and c are distinct, they cannot all be zero. If M^{-1} exists $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = M^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ which is a contradiction.

$$\text{So, } M^{-1} \text{ does not exist and thus } \det \begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix} = 0, \mathbf{M1}$$

$$\text{i.e. } 1 - xyz + xyz + yz + zx + xy = 0, \text{ (Sarus)}$$

$$\text{or } 1(1 + yz) - x(y - yz) + x(yz + z) = 0 \text{ (by co-factors) } \mathbf{M1}$$

which simplifies to

$$yz + zx + xy = -1 \mathbf{A1^* (5)}$$

$$(x + y + z)^2 \geq 0$$

$$\text{So } x^2 + y^2 + z^2 + 2yz + 2zx + 2xy \geq 0 \mathbf{M1}$$

$$\text{and so } x^2 + y^2 + z^2 \geq 2 \mathbf{A1^* (2)}$$

$$(ii) \begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a - (b + c)x \\ 2b - (c + a)y \\ 2c - (a + b)z \end{pmatrix} = \begin{pmatrix} 2a - 2a \\ 2b - 2b \\ 2c - 2c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

B1

M1

A1

$$\text{As } a, b \text{ and } c \text{ are positive, they cannot all be zero. Thus as } \begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\text{as in part (i), } \det \begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix} = 0,$$

$$\text{i.e. } 8 - xyz - xyz - 2yz - 2zx - 2xy = 0, \text{ that is } \mathbf{M1 A1}$$

$$xyz + yz + zx + xy = 4 \mathbf{A1^* (6)}$$

$$(x + 1)(y + 1)(z + 1) = xyz + zx + xy + x + y + z + 1 = 4 + x + y + z + 1 > 5$$

M1

A1

because as a, b , and c are all positive, so are x, y and z . **E1**

$$\text{Thus } \left(\frac{2a}{b+c} + 1\right) \left(\frac{2b}{c+a} + 1\right) \left(\frac{2c}{a+b} + 1\right) > 5$$

Multiplying by $(b + c)(c + a)(a + b)$, all three factors of which are positive, gives

$(2a + b + c)(a + 2b + c)(a + b + c) > 5(b + c)(c + a)(a + b)$ as required. **A1* (4)**

$x = \frac{2a}{b+c} > \frac{2a}{a+b+c}$ as a, b, and c are positive, and similarly both, $y > \frac{2b}{a+b+c}$ and $z > \frac{2c}{a+b+c}$

M1

Thus $4 + x + y + z + 1 > 4 + \frac{2a}{a+b+c} + \frac{2b}{a+b+c} + \frac{2c}{a+b+c} + 1 = 4 + \frac{2(a+b+c)}{a+b+c} + 1 = 7$

dM1

and thus following the argument used to obtain the previous result

$(2a + b + c)(a + 2b + c)(a + b + c) > 7(b + c)(c + a)(a + b)$ as required.

A1* (3)

3. (i)

$$\begin{aligned}\frac{1}{2} (I_{n+1} + I_{n-1}) &= \frac{1}{2} \int_0^\beta (\sec x + \tan x)^{n+1} + (\sec x + \tan x)^{n-1} dx \\ &= \frac{1}{2} \int_0^\beta (\sec x + \tan x)^{n-1} ((\sec x + \tan x)^2 + 1) dx\end{aligned}$$

M1

$$\begin{aligned}&= \frac{1}{2} \int_0^\beta (\sec x + \tan x)^{n-1} (\sec^2 x + 2 \sec x \tan x + \tan^2 x + 1) dx \\ &= \int_0^\beta (\sec x + \tan x)^{n-1} (\sec^2 x + \sec x \tan x) dx\end{aligned}$$

M1

$$= \left[\frac{1}{n} (\sec x + \tan x)^n \right]_0^\beta = \frac{1}{n} ((\sec \beta + \tan \beta)^n - 1)$$

M1 A1

***A1 (5)**

as required.

$$\begin{aligned}\frac{1}{2} (I_{n+1} + I_{n-1}) - I_n &= \frac{1}{2} (I_{n+1} - 2I_n + I_{n-1}) \\ &= \frac{1}{2} \int_0^\beta (\sec x + \tan x)^{n+1} - 2(\sec x + \tan x)^n + (\sec x + \tan x)^{n-1} dx\end{aligned}$$

M1

$$= \frac{1}{2} \int_0^\beta (\sec x + \tan x)^{n-1} ((\sec x + \tan x) - 1)^2 dx$$

M1 A1

$$((\sec x + \tan x) - 1)^2 > 0 \text{ for all } x > 0$$

$\sec x \geq 1$ for $0 \leq x < \frac{\pi}{2}$ and hence for $0 \leq x < \beta$ and similarly $\tan x \geq 0$, and thus also $(\sec x + \tan x)^{n-1} > 0$.

E1

Therefore, $\frac{1}{2} (I_{n+1} + I_{n-1}) - I_n > 0$, **A1**

and so $I_n < \frac{1}{2} (I_{n+1} + I_{n-1}) = \frac{1}{n} ((\sec \beta + \tan \beta)^n - 1)$ as required. **M1 *A1 (7)**

Alternative 1: it has already been shown that

$$\begin{aligned}\frac{1}{2} (I_{n+1} + I_{n-1}) &= \int_0^\beta (\sec x + \tan x)^{n-1} (\sec^2 x + \sec x \tan x) dx \\ &= \int_0^\beta \sec x (\sec x + \tan x)^n dx\end{aligned}$$

which is greater than I_n as the expression being integrated is greater than $(\sec x + \tan x)^n$ because $\sec x > 0$ over this domain.

Alternative 2:-

$$\begin{aligned}I_{n+1} - I_n &= \int_0^\beta (\sec x + \tan x)^n (\sec x + \tan x - 1) dx \\ I_n - I_{n-1} &= \int_0^\beta (\sec x + \tan x)^{n-1} (\sec x + \tan x - 1) dx\end{aligned}$$

M1 A1 A1

For $0 < x < \beta$, $\sec x > 1$, $\tan x > 0$ so $\sec x + \tan x > 1$ **E1** and thus $I_{n+1} - I_n > I_n - I_{n-1}$ **A1**

and so $I_n \leq \frac{1}{2} (I_{n+1} + I_{n-1}) = \frac{1}{n} ((\sec \beta + \tan \beta)^n - 1)$ **M1 *A1 (7)**

$$\begin{aligned}\text{(ii)} \quad \frac{1}{2} (J_{n+1} + J_{n-1}) &= \frac{1}{2} \int_0^\beta (\sec x \cos \beta + \tan x)^{n+1} + (\sec x \cos \beta + \tan x)^{n-1} dx \\ &= \frac{1}{2} \int_0^\beta (\sec x \cos \beta + \tan x)^{n-1} ((\sec x \cos \beta + \tan x)^2 + 1) dx\end{aligned}$$

M1

$$\begin{aligned}&= \frac{1}{2} \int_0^\beta (\sec x \cos \beta + \tan x)^{n-1} (\sec^2 x \cos^2 \beta + 2 \sec x \cos \beta \tan x + \tan^2 x + 1) dx \\ &= \frac{1}{2} \int_0^\beta (\sec x \cos \beta + \tan x)^{n-1} (\sec^2 x (1 - \sin^2 \beta) + 2 \sec x \cos \beta \tan x + \tan^2 x + 1) dx \\ &= \int_0^\beta (\sec x \cos \beta + \tan x)^{n-1} ((\sec^2 x + \sec x \cos \beta \tan x) - \sec^2 x \sin^2 \beta) dx\end{aligned}$$

M1

$$\int_0^\beta (\sec x \cos \beta + \tan x)^{n-1} (\sec^2 x + \sec x \cos \beta \tan x) dx = \left[\frac{1}{n} (\sec x \cos \beta + \tan x)^n \right]_0^\beta$$

M1

$$= \frac{1}{n} ((1 + \tan \beta)^n - \cos^n \beta)$$

A1

$$\int_0^\beta (\sec x \cos \beta + \tan x)^{n-1} \sec^2 x \sin^2 \beta \, dx > 0$$

by a similar argument to part (i), namely $\sec^2 x \sin^2 \beta > 0$ for any x , and $\sec x \cos \beta + \tan x > 0$ as $\sec x > 0$ and $\tan x \geq 0$ for $0 \leq x < \beta < \frac{\pi}{2}$ **E1**

Hence $\frac{1}{2} (J_{n+1} + J_{n-1}) < \frac{1}{n} ((1 + \tan \beta)^n - \cos^n \beta)$ **A1**

But

$$\frac{1}{2} (J_{n+1} + J_{n-1}) - J_n = \frac{1}{2} \int_0^\beta (\sec x \cos \beta + \tan x)^{n-1} ((\sec x \cos \beta + \tan x) - 1)^2 \, dx > 0$$

M1

as before, and thus $J_n < \frac{1}{2} (J_{n+1} + J_{n-1}) < \frac{1}{n} ((1 + \tan \beta)^n - \cos^n \beta)$ as required. ***A1 (8)**

4. (i)

$m \cdot a = \frac{1}{2}(a + b) \cdot a = \frac{1}{2}(1 + a \cdot b) = m \cos \alpha$ where α is the non-reflex angle between a and m

$m \cdot b = \frac{1}{2}(a + b) \cdot b = \frac{1}{2}(1 + a \cdot b) = m \cos \beta$ where α is the non-reflex angle between b and m

M1 A1

Thus $\cos \alpha = \cos \beta$ and so $\alpha = \beta$ as for $0 \leq \tau \leq \pi$, there is only one value of τ for any given value of $\cos \tau$. **E1 (3)**

(ii) $a_1 \cdot c = (a - (a \cdot c)c) \cdot c = a \cdot c - a \cdot c c \cdot c = 0$ as required. ***B1**

$a \cdot c = \cos \alpha$, $b \cdot c = \cos \beta$, $a \cdot b = \cos \theta$

$a_1 = a - (a \cdot c)c$ and $b_1 = b - (b \cdot c)c$

$$\begin{aligned} |a_1|^2 &= a_1 \cdot a_1 = (a - (a \cdot c)c) \cdot (a - (a \cdot c)c) = a \cdot a - 2a \cdot c a \cdot c + a \cdot c a \cdot c c \cdot c \\ &= 1 - 2 \cos^2 \alpha + \cos^2 \alpha = \sin^2 \alpha \end{aligned}$$

M1

and so, as α is acute, $|a_1| = \sin \alpha$ as required. ***A1**

$$\begin{aligned} a_1 \cdot b_1 &= (a - (a \cdot c)c) \cdot (b - (b \cdot c)c) = a \cdot b - 2(a \cdot c)(b \cdot c) + (a \cdot c)(b \cdot c)(c \cdot c) \\ &= \cos \theta - \cos \alpha \cos \beta \end{aligned}$$

M1 A1

but also, $a_1 \cdot b_1 = \sin \alpha \sin \beta \cos \varphi$

B1 M1

and hence,

$$\cos \varphi = \frac{\cos \theta - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}$$

as required.

***A1 (8)**

(iii) $m_1 = m - (m \cdot c)c = \frac{1}{2}(a + b) - \left(\frac{1}{2}(a + b) \cdot c\right)c = \frac{1}{2}(a_1 + b_1)$ **B1**

m_1 bisects the angle between a_1 and b_1 if and only if

$$\frac{m_1 \cdot a_1}{\sin \alpha} = \frac{m_1 \cdot b_1}{\sin \beta}$$

M1

Thus, multiplying through by $2 \sin \alpha \sin \beta$,

$$(a_1 + b_1) \cdot a_1 \sin \beta = (a_1 + b_1) \cdot b_1 \sin \alpha$$

A1

$$(\sin^2 \alpha + a_1 \cdot b_1) \sin \beta = (\sin^2 \beta + a_1 \cdot b_1) \sin \alpha$$

M1 A1

So

$$(a_1 \cdot b_1 - \sin \alpha \sin \beta)(\sin \alpha - \sin \beta) = 0$$

A1

and thus, $\sin \alpha = \sin \beta$ in which case $\alpha = \beta$ as both angles are acute, ***A1**

or $\cos \theta - \cos \alpha \cos \beta = \sin \alpha \sin \beta$, meaning that $\cos \theta = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos(\alpha - \beta)$

M1 *A1 (9)

5. (i) The curves meet when $a + 2 \cos \theta = 2 + \cos 2\theta$

That is, $a + 2 \cos \theta = 2 + 2 \cos^2 \theta - 1$ or as required, **B1** $2 \cos^2 \theta - 2 \cos \theta + 1 - a = 0$

The curves touch if this quadratic has coincident roots, **M1** i.e. if $4 - 8(1 - a) = 0 \Rightarrow a = \frac{1}{2}$, ***A1** or if $\cos \theta = \pm 1$, **M1** in which cases $a = 1$ **A1** or $a = 5$. **A1 (6)**

Alternatively, for the curves to touch, they must have the same gradient, so differentiating,

$$-2 \sin \theta = -2 \sin 2\theta = -4 \sin \theta \cos \theta$$

M1

in which case, either $\sin \theta = 0$ giving $\cos \theta = \pm 1$, **M1** in which cases $a = 1$ **A1** or $a = 5$, **A1** or $\cos \theta = \frac{1}{2}$ in which case $a = \frac{1}{2}$. ***A1 (6)**

(ii) If $a = \frac{1}{2}$ then at points where they touch, $\cos \theta = \frac{1}{2}$ so $\theta = \pm \frac{\pi}{3}$ and thus $\left(\frac{3}{2}, \pm \frac{\pi}{3}\right)$. **M1A1**

$r = a + 2 \cos \theta$ is symmetrical about the initial line which it intercepts at $\left(\frac{5}{2}, 0\right)$ and has a cusp at $\left(0, \pm \cos^{-1}\left(-\frac{1}{4}\right)\right)$. It passes through $\left(\frac{1}{2}, \pm \frac{\pi}{2}\right)$ and only exists for

$$-\cos^{-1}\left(-\frac{1}{4}\right) < \theta < \cos^{-1}\left(-\frac{1}{4}\right).$$

$r = 2 + \cos 2\theta$ is symmetrical about both the initial line, and its perpendicular. It passes through $(3, 0)$, $(3, \pi)$, and $\left(1, \pm \frac{\pi}{2}\right)$

Sketch **G6 (8)**

(iii) If $a = 1$, then the curves meet where $2 \cos^2 \theta - 2 \cos \theta = 0$, i.e. $\cos \theta = 1$ at $(3, 0)$ where they touch, and $\cos \theta = 0$ at $\left(1, \pm \frac{\pi}{2}\right)$

$r = a + 2 \cos \theta$ is symmetrical about the initial line which it intercepts at $(3, 0)$ and has a cusp at $\left(0, \pm \cos^{-1}\left(-\frac{1}{2}\right)\right) = \left(0, \pm \frac{2\pi}{3}\right)$. It passes through $\left(1, \pm \frac{\pi}{2}\right)$ and only exists for

$$-\frac{2\pi}{3} < \theta < \frac{2\pi}{3}.$$

Sketch **G3**

If $a = 5$, then the curves meet where $2 \cos^2 \theta - 2 \cos \theta - 4 = 0$, i.e. only $\cos \theta = -1$ at $(3, \pi)$ where they touch, as $\cos \theta \neq 2$.

$r = a + 2 \cos \theta$ is symmetrical about the initial line which it intercepts at $(7, 0)$ and $(3, \pi)$. It also passes through $\left(5, \pm \frac{\pi}{2}\right)$.

Sketch **G3 (6)**

6. (i)

$$\begin{aligned}
 f_{\alpha}(x) &= \tan^{-1} \left(\frac{x \tan \alpha + 1}{\tan \alpha - x} \right) \\
 f'_{\alpha}(x) &= \frac{1}{1 + \left(\frac{x \tan \alpha + 1}{\tan \alpha - x} \right)^2} \frac{(\tan \alpha - x) \tan \alpha + (x \tan \alpha + 1)}{(\tan \alpha - x)^2} \\
 &\quad \text{M1 A1} \\
 &= \frac{\tan^2 \alpha + 1}{(\tan \alpha - x)^2 + (x \tan \alpha + 1)^2} \\
 &= \frac{\sec^2 \alpha}{\tan^2 \alpha + x^2 + x^2 \tan^2 \alpha + 1} = \frac{\sec^2 \alpha}{\sec^2 \alpha (1 + x^2)} = \frac{1}{1 + x^2} \\
 &\quad \text{M1} \qquad \text{M1} \qquad \text{*A1 (5)}
 \end{aligned}$$

as required.

Alternative

$$\begin{aligned}
 f_{\alpha}(x) &= \tan^{-1} \left(\frac{x \tan \alpha + 1}{\tan \alpha - x} \right) \\
 &= \tan^{-1} \left(\frac{x + \cot \alpha}{1 - x \cot \alpha} \right) \\
 &= \tan^{-1} \left(\frac{\tan(\tan^{-1} x) + \tan\left(\frac{\pi}{2} - \alpha\right)}{1 - \tan(\tan^{-1} x) \tan\left(\frac{\pi}{2} - \alpha\right)} \right) \\
 &\quad \text{M1 A1} \\
 &= \tan^{-1} \left(\tan \left(\tan^{-1} x + \frac{\pi}{2} - \alpha \right) \right) \\
 &\quad \text{M1}
 \end{aligned}$$

$$= \tan^{-1} x + \frac{\pi}{2} - \alpha \text{ if this is less than } \frac{\pi}{2}, \text{ i.e. if } x < \tan \alpha$$

$$\text{or } = \tan^{-1} x - \frac{\pi}{2} - \alpha \text{ if } x > \tan \alpha \quad \text{M1}$$

$$\text{So } f'_{\alpha}(x) = \frac{d}{dx} (\tan^{-1} x) = \frac{1}{1+x^2} \quad \text{*A1 (5)}$$

$$\text{Thus } f_{\alpha}(x) = \tan^{-1} x + c$$

$$f_{\alpha}(0) = \tan^{-1} \left(\frac{1}{\tan \alpha} \right) = \tan^{-1}(\cot \alpha) = \frac{\pi}{2} - \alpha$$

$$f_{\alpha}(x) = 0 \text{ when } x = -\cot \alpha$$

There is a discontinuity at $x = \tan \alpha$, with $f_{\alpha}(x)$ approaching $\frac{\pi}{2}$ from below and $-\frac{\pi}{2}$ from above.

$$\text{As } x \rightarrow \pm\infty, f_{\alpha}(x) \rightarrow \tan^{-1}(-\tan \alpha) = -\alpha$$

So $f_\alpha(x) = \tan^{-1} x + \frac{\pi}{2} - \alpha$ for $x < \tan \alpha$ and $f_\alpha(x) = \tan^{-1} x - \frac{\pi}{2} - \alpha$ for $x > \tan \alpha$

Sketch **G1 G1 G1 (3)**

$$y = f_\alpha(x) - f_\beta(x) =$$

$$\left(\frac{\pi}{2} - \alpha\right) - \left(\frac{\pi}{2} - \beta\right) = \beta - \alpha \text{ for } x < \tan \alpha$$

$$\left(-\frac{\pi}{2} - \alpha\right) - \left(\frac{\pi}{2} - \beta\right) = \beta - \alpha - \pi \text{ for } \tan \alpha < x < \tan \beta$$

$$\text{and } \left(-\frac{\pi}{2} - \alpha\right) - \left(-\frac{\pi}{2} - \beta\right) = \beta - \alpha \text{ for } x > \tan \beta$$

Sketch **G1 G1 G1 (3)**

(ii) $g(x) = \tanh^{-1}(\sin x) - \sinh^{-1}(\tan x)$

$$g'(x) = \frac{1}{1 - \sin^2 x} \cos x - \frac{1}{\sqrt{1 + \tan^2 x}} \sec^2 x$$

M1 A1

A1

$$= \frac{\cos x}{\cos^2 x} - \frac{\sec^2 x}{|\sec x|} = \sec x - \frac{\sec^2 x}{-\sec x} = 2 \sec x$$

M1

***A1 (5)**

as required, for $\sec x < 0$, i.e. for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

(For $\sec x > 0$, $g'(x) = 0$)

Sketch **G1 G1 G1 G1 (4)**

7.

$$z = \frac{e^{i\theta} + e^{i\varphi}}{e^{i\theta} - e^{i\varphi}}$$

$$= \frac{\cos \theta + i \sin \theta + \cos \varphi + i \sin \varphi}{\cos \theta + i \sin \theta - \cos \varphi - i \sin \varphi}$$

M1

$$= \frac{2 \cos \frac{\theta + \varphi}{2} \cos \frac{\theta - \varphi}{2} + 2i \sin \frac{\theta + \varphi}{2} \cos \frac{\theta - \varphi}{2}}{-2 \sin \frac{\theta + \varphi}{2} \sin \frac{\theta - \varphi}{2} + 2i \cos \frac{\theta + \varphi}{2} \sin \frac{\theta - \varphi}{2}}$$

M1 A1 A1

$$= \frac{2 \cos \frac{\theta - \varphi}{2} \left(\cos \frac{\theta + \varphi}{2} + i \sin \frac{\theta + \varphi}{2} \right)}{2 \sin \frac{\theta - \varphi}{2} \left(i \cos \frac{\theta + \varphi}{2} - \sin \frac{\theta + \varphi}{2} \right)}$$

$$= -i \cot \frac{\theta - \varphi}{2}$$

$$= i \cot \frac{\varphi - \theta}{2}$$

***A1 (5)**

as required.

Alternatively,

$$z = \frac{e^{i\theta} + e^{i\varphi}}{e^{i\theta} - e^{i\varphi}} = \frac{e^{i\left(\frac{\theta-\varphi}{2}\right)} + e^{-i\left(\frac{\theta-\varphi}{2}\right)}}{e^{i\left(\frac{\theta-\varphi}{2}\right)} - e^{-i\left(\frac{\theta-\varphi}{2}\right)}} = \frac{2 \cos \frac{\theta - \varphi}{2}}{2i \sin \frac{\theta - \varphi}{2}} = -i \cot \frac{\theta - \varphi}{2} = i \cot \frac{\varphi - \theta}{2}$$

M1

M1 A1 A1

***A1 (5)**

$$|z| = \left| \cot \frac{\theta - \varphi}{2} \right|$$

M1 A1

$$|\arg z| = \frac{\pi}{2}$$

[or $\arg z = \frac{\pi}{2}$ or $\frac{3\pi}{2}$]

M1 A1 (4)

(ii) Let $a = e^{i\alpha}$ and $b = e^{i\beta}$ **M1** then $x = a + b = e^{i\alpha} + e^{i\beta}$ and $AB = b - a = e^{i\beta} - e^{i\alpha}$

$$\arg x - \arg AB = \arg \frac{x}{AB} = \arg \frac{e^{i\alpha} + e^{i\beta}}{e^{i\beta} - e^{i\alpha}}$$

so using (i), $|\arg x - \arg AB| = \frac{\pi}{2}$ **A1** and thus OX and AB are perpendicular, since $x = a + b \neq 0$ and $a \neq b$ as A and B are distinct. **E1 (3)**

Alternative:- $0, a, a + b, b$ define a rhombus OAXB as $|a| = |b| = 1$. Diagonals of a rhombus are perpendicular (and bisect one another).

(iii) $h = a + b + c$ so $AH = a + b + c - a = b + c$ and $BC = c - b$ and thus

$$\frac{AH}{BC} = \frac{b + c}{c - b}$$

B1

as $c - b \neq 0$

From (ii),

$$\left| \arg \frac{AH}{BC} \right| = \frac{\pi}{2}$$

so BC is perpendicular to AH **E1**

unless $b + c = 0$ **E1** in which case $h = a$ **E1 (4)**

(iv) $p = a + b + c$ $q = b + c + d$ $r = c + d + a$ $s = d + a + b$

The midpoint of AQ is $\frac{a+q}{2} = \frac{a+b+c+d}{2}$ and so by its symmetry it is also the midpoint of BR, CS, and DP, **B1 E1**

and thus ABCD is transformed to PQRS by a rotation of π radians about midpoint of AQ. **E1 B1 (4)**

Alternatively, ABCD is transformed to PQRS by an enlargement scale factor -1, centre of enlargement midpoint of AQ.

8. (i) Suppose $x_k \geq 2 + 4^{k-1}(a - 2)$ for some particular integer k (and this is positive as $a > 2$)

E1

$$\begin{aligned} \text{Then } x_{k+1} &= x_k^2 - 2 \geq [2 + 4^{k-1}(a - 2)]^2 - 2 = 4 + 4^k(a - 2) + 4^{2k-2}(a - 2)^2 - 2 \\ &= 2 + 4^k(a - 2) + 4^{2k-2}(a - 2)^2 \\ &> 2 + 4^k(a - 2) \end{aligned}$$

M1 A1

which is the required result for $k + 1$.

For $n = 1$, $2 + 4^{n-1}(a - 2) = 2 + a - 2 = a$ so in this case, $x_n = 2 + 4^{n-1}(a - 2)$ **B1** and thus by induction $x_n \geq 2 + 4^{n-1}(a - 2)$ for positive integer n . **E1 (5)**

(ii) If $|x_k| \leq 2$, then $0 \leq |x_k|^2 \leq 4$, so $-2 \leq |x_k|^2 - 2 \leq 2$, that is $-2 \leq x_{k+1} \leq 2$. **M1A1**

If $|a| \leq 2$, $|x_1| \leq 2$ and thus by induction $-2 \leq x_n \leq 2$, that is $x_n \rightarrow \infty$ **E1**

Whether $a = \pm \alpha$, x_2 would equal the same value, namely $\alpha^2 - 2$. **E1**

So to consider $|a| \geq 2$, we only need consider $a > 2$ to discuss the behaviour of all terms after the first. Therefore, from part (i), we know $x_n \geq 2 + 4^{n-1}(|a| - 2)$ for $n \geq 2$, and thus $x_n \rightarrow \infty$ as $n \rightarrow \infty$; **B1** hence we have shown $x_n \rightarrow \infty$ as $n \rightarrow \infty$ if and only if $|a| \geq 2$. **(5)**

(iii)

$$\begin{aligned} y_k &= \frac{Ax_1x_2 \cdots x_k}{x_{k+1}} \\ y_{k+1} &= \frac{Ax_1x_2 \cdots x_{k+1}}{x_{k+2}} = \frac{x_{k+1}^2}{x_{k+2}} y_k \end{aligned}$$

M1

Suppose that

$$y_k = \frac{\sqrt{x_{k+1}^2 - 4}}{x_{k+1}}$$

for some positive integer k , **E1** then

$$y_{k+1} = \frac{x_{k+1}^2 \sqrt{x_{k+1}^2 - 4}}{x_{k+2} x_{k+1}} = \frac{x_{k+1} \sqrt{x_{k+1}^2 - 4}}{x_{k+2}}$$

As $x_{k+2} = x_{k+1}^2 - 2$, $x_{k+1} = \sqrt{x_{k+2} + 2}$, and $\sqrt{x_{k+1}^2 - 4} = \sqrt{x_{k+2} - 2}$,

and thus,

$$y_{k+1} = \frac{\sqrt{x_{k+2} + 2} \sqrt{x_{k+2} - 2}}{x_{k+2}} = \frac{\sqrt{x_{k+2}^2 - 4}}{x_{k+2}}$$

M1 A1

which is the required result for $k + 1$.

$$y_1 = \frac{Ax_1}{x_2}$$

and also we wish to have

$$y_1 = \frac{\sqrt{x_2^2 - 4}}{x_2}$$

M1

then $Ax_1 = \sqrt{x_2^2 - 4}$, that is $A^2x_1^2 = x_2^2 - 4$, and as $x_1 = a$, $x_2 = x_1^2 - 2 = a^2 - 2$

so

$A^2a^2 = (a^2 - 2)^2 - 4 = a^4 - 4a^2$, $A^2 = a^2 - 4$, and thus $a = \sqrt{A^2 + 4}$, as $a \neq 0$ nor $-\sqrt{A^2 + 4}$ because $a > 2$. **A1 E1**

So as the result is true for y_1 , and we have shown it to be true for y_{k+1} if it is true for y_k , it is true by induction for all positive integer n that

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

E1 (8)

As $a > 2$ from (ii) $x_n \rightarrow \infty$ as $n \rightarrow \infty$ **M1** and thus using result just proved, $y_n \rightarrow 1$ as $n \rightarrow \infty$, i.e. the sequence converges. ***A1 (2)**

9.

Using the sine rule, from triangle PQR

$$\frac{PR}{\sin \theta} = \frac{PQ}{\sin \left(\frac{2\pi}{3} - \varphi \right)}$$

M1 A1

From triangle PQC

$$\frac{PQ}{\sin \frac{\pi}{3}} = \frac{a - x}{\sin \left(\frac{2\pi}{3} - \theta \right)}$$

A1

From triangle PBR

$$\frac{PR}{\sin \frac{\pi}{3}} = \frac{x}{\sin \varphi}$$

A1

Eliminating PR and PQ between these three equations

$$x \sin \frac{\pi}{3} \sin \left(\frac{2\pi}{3} - \varphi \right) \sin \left(\frac{2\pi}{3} - \theta \right) = \sin \varphi \sin \theta (a - x) \sin \frac{\pi}{3}$$

M1 A1

Hence

$$x \left(\frac{\sqrt{3}}{2} \cos \varphi + \frac{1}{2} \sin \varphi \right) \left(\frac{\sqrt{3}}{2} \cos \theta + \frac{1}{2} \sin \theta \right) = (a - x) \sin \varphi \sin \theta$$

giving

$$(\sqrt{3} \cot \varphi + 1)(\sqrt{3} \cot \theta + 1)x = 4(a - x)$$

as required.

M1 *A1 (8)

If the ball has speed v_1 moving from P to Q, speed v_2 moving from Q to R, and speed v_3 moving from R to P,

then CLM at Q parallel to CA gives $v_1 \cos \left(\frac{2\pi}{3} - \theta \right) = v_2 \cos \frac{\pi}{3}$ and NELI perpendicular to CA gives $e v_1 \sin \left(\frac{2\pi}{3} - \theta \right) = v_2 \sin \frac{\pi}{3}$, and dividing these gives $e \tan \left(\frac{2\pi}{3} - \theta \right) = \tan \frac{\pi}{3}$

M1 A1

and similarly,

CLM at R parallel to AB gives $v_2 \cos \frac{\pi}{3} = v_3 \cos \varphi$ and NELI perpendicular to AB gives

$e v_2 \sin \frac{\pi}{3} = v_3 \sin \varphi$, and dividing these gives $e \tan \frac{\pi}{3} = \tan \varphi$. **A1**

$e \tan \left(\frac{2\pi}{3} - \theta \right) = \tan \frac{\pi}{3}$ yields $e \frac{-\sqrt{3} - \tan \theta}{1 - \sqrt{3} \tan \theta} = \sqrt{3}$ **M1** which simplifies to

$$e(\sqrt{3} + \tan \theta) = \sqrt{3}(\sqrt{3} \tan \theta - 1), \text{ or in turn, } (3 - e) \tan \theta = \sqrt{3}(1 + e) \text{ and so}$$

$$\cot \theta = \frac{(3-e)}{\sqrt{3}(1+e)} \quad \text{A1}$$

$$e \tan \frac{\pi}{3} = \tan \varphi \text{ yields } \cot \varphi = \frac{1}{e\sqrt{3}} \quad \text{A1}$$

Substituting these two expressions into the first result of the question,

$$\left(\frac{1}{e} + 1\right) \left(\frac{(3-e)}{(1+e)} + 1\right) x = 4(a-x)$$

M1

This simplifies to

$$x \frac{1+e}{e} \frac{4}{1+e} = 4(a-x)$$

that is

$$x = e(a-x)$$

so

$$x = \frac{ae}{1+e}$$

as required.

***A1 (8)**

To continue the motion at P, then similarly to before, the third impact gives $e \tan\left(\frac{2\pi}{3} - \varphi\right) = \tan \theta$

M1

So

$$\tan \theta = e \frac{-\sqrt{3} - \tan \varphi}{1 - \sqrt{3} \tan \varphi} = e \frac{\sqrt{3}(e+1)}{3e-1}$$

and thus, using the previously found result for $\cot \theta$

$$\frac{(3-e)}{\sqrt{3}(1+e)} = \frac{3e-1}{\sqrt{3}(e+1)e}$$

M1 A1

That is $e(3-e) = 3e-1$, that is $e^2 = 1$ and as $e \geq 0$, $e = 1$ (and not -1) ***B1 (4)**

10. (i) At time t , the point where the string is tangential to the cylinder, **M1** say T is at $(a \cos \theta, a \sin \theta)$, **A1** the piece of string that remains straight is of length $b - a\theta$, **M1**, the vector representing the string is thus $(b - a\theta) \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$ **dM1 A1** so the particle is at the point $(a \cos \theta - (b - a\theta) \sin \theta, a \sin \theta + (b - a\theta) \cos \theta)$. **M1 A1 (7)**

$$\dot{x} = -a\dot{\theta} \sin \theta - (b - a\theta)\dot{\theta} \cos \theta + a\dot{\theta} \sin \theta = -(b - a\theta)\dot{\theta} \cos \theta$$

$$\dot{y} = a\dot{\theta} \cos \theta - (b - a\theta)\dot{\theta} \sin \theta - a\dot{\theta} \cos \theta = -(b - a\theta)\dot{\theta} \sin \theta$$

M1 A1

Thus the speed is $\sqrt{((b - a\theta)\dot{\theta} \cos \theta)^2 + ((b - a\theta)\dot{\theta} \sin \theta)^2} = (b - a\theta)\dot{\theta}$ as required. **M1 A1 (4)**

(ii) The only horizontal force on the particle is the tension in the string, which is perpendicular to the velocity at any time, so kinetic energy is conserved. **E1** Therefore,

$$\frac{1}{2}m((b - a\theta)\dot{\theta})^2 = \frac{1}{2}mu^2$$

M1

and so, as $(b - a\theta)\dot{\theta}$ and u are both positive $(b - a\theta)\dot{\theta} = u$ ***A1 (3)**

(iii) The tension in the string, using instantaneous circular motion, at time t is

$$\frac{mu^2}{(b - a\theta)}$$

M1 A1

As $(b - a\theta)\dot{\theta} = u$, integrating with respect to t ,

$$b\theta - \frac{a\theta^2}{2} = ut + c$$

M1

but when $t = 0$, $\theta = 0$ so $c = 0$. **M1 A1**

Thus, $b\theta - \frac{a\theta^2}{2} = ut$

i.e.

$$\theta^2 - \frac{2b\theta}{a} + \frac{b^2}{a^2} = \frac{b^2}{a^2} - \frac{2ut}{a} = \frac{b^2 - 2aut}{a^2}$$

Alternatively, integrating $(b - a\theta)\dot{\theta} = u$ with respect to t ,

$$-\frac{(b - a\theta)^2}{2a} = ut + k$$

M1

When $t = 0, \theta = 0$ so $k = -\frac{b^2}{2a}$ **M1 A1**

$$\frac{(b - a\theta)^2}{2a} = \frac{b^2}{2a} - ut = \frac{b^2 - 2aut}{2a}$$

Thus, taking positive roots,

$$\frac{b - a\theta}{a} = \frac{\sqrt{b^2 - 2aut}}{a}$$

Hence, the tension is

$$\frac{mu^2}{\sqrt{b^2 - 2aut}}$$

***A1 (6)**

11. (i)

$$P(Y = n) = P(n \leq X < n + 1) = \int_n^{n+1} \lambda e^{-\lambda x} dx = [-e^{-\lambda x}]_n^{n+1} = -e^{-\lambda(n+1)} + e^{-\lambda n}$$

M1

M1

$$= (1 - e^{-\lambda})e^{-\lambda n}$$

as required.

***A1 (3)**

(ii)

$$P(Z < z) = \sum_{r=0}^{\infty} P(r \leq X < r + z) = \sum_{r=0}^{\infty} \int_r^{r+z} \lambda e^{-\lambda x} dx = \sum_{r=0}^{\infty} [-e^{-\lambda x}]_r^{r+z}$$

M1

M1

$$= \sum_{r=0}^{\infty} (-e^{-\lambda(r+z)} + e^{-\lambda r}) = \sum_{r=0}^{\infty} (1 - e^{-\lambda z})e^{-\lambda r}$$

M1 A1

$$= (1 - e^{-\lambda z}) \frac{1}{1 - e^{-\lambda}} = \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}$$

using sum of an infinite GP with magnitude of common ratio less than one.

M1 *A1 (6)

$$(iii) \text{ As } P(Z < z) = \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}, \quad f_Z(z) = \frac{d}{dz} \left(\frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}} \right) = \frac{\lambda e^{-\lambda z}}{1 - e^{-\lambda}} \quad \textbf{M1}$$

so

$$E(Z) = \int_0^1 z \frac{\lambda e^{-\lambda z}}{1 - e^{-\lambda}} dz = \frac{1}{1 - e^{-\lambda}} \left\{ [-ze^{-\lambda z}]_0^1 + \int_0^1 e^{-\lambda z} dz \right\}$$

M1

M1

$$= \frac{1}{1 - e^{-\lambda}} \left\{ -e^{-\lambda} - \left[\frac{e^{-\lambda z}}{\lambda} \right]_0^1 \right\} = \frac{1}{1 - e^{-\lambda}} \left\{ -e^{-\lambda} - \frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda} \right\}$$

A1

$$= \frac{1}{\lambda} - \frac{e^{-\lambda}}{1 - e^{-\lambda}}$$

or alternatively

$$\frac{1}{\lambda} \frac{(1 - (\lambda + 1)e^{-\lambda})}{1 - e^{-\lambda}}$$

A1 (5)

(iv)

$$P(Y = n \text{ and } z_1 < Z < z_2) = P(n + z_1 < X < n + z_2)$$

$$= \int_{n+z_1}^{n+z_2} \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x} \right]_{n+z_1}^{n+z_2} = -e^{-\lambda(n+z_2)} + e^{-\lambda(n+z_1)} = e^{-\lambda n} (e^{-\lambda z_1} - e^{-\lambda z_2})$$

M1

A1

$$P(Y = n \text{ and } z_1 < Z < z_2) = e^{-\lambda n} (e^{-\lambda z_1} - e^{-\lambda z_2})$$

$$= (1 - e^{-\lambda}) e^{-\lambda n} \left(\frac{1 - e^{-\lambda z_2}}{1 - e^{-\lambda}} - \frac{1 - e^{-\lambda z_1}}{1 - e^{-\lambda}} \right)$$

M1 A1

$$= P(Y = n) \times P(z_1 < Z < z_2) \quad \mathbf{M1}$$

so Y and Z are independent. **E1 (6)**

12. (i)

$$P(X_{12} = 1) = \frac{1}{6}, P(X_{12} = 0) = \frac{5}{6}, P(X_{23} = 1) = \frac{1}{6}, P(X_{23} = 0) = \frac{5}{6}$$

If $X_{23} = 1$, then players 2 and 3 score the same as one another. In that case, $X_{12} = 1$ would mean that player 1 also obtained that same score so $P(X_{12} = 1|X_{23} = 1) = \frac{1}{6} = P(X_{12} = 1)$.

If $X_{23} = 1$, $X_{12} = 0$ would mean that player 1 obtained a different score so

$$P(X_{12} = 0|X_{23} = 1) = \frac{5}{6} = P(X_{12} = 0)$$

If $X_{23} = 0$, then players 2 and 3 score differently to one another. In that case, $X_{12} = 1$ would mean that player 1 also obtained the same score as player 2 so $P(X_{12} = 1|X_{23} = 0) = \frac{1}{6} = P(X_{12} = 1)$

If $X_{23} = 0$, $X_{12} = 0$ would mean that player 1 obtained a different score to player 2 so

$$P(X_{12} = 0|X_{23} = 0) = \frac{5}{6} = P(X_{12} = 0)$$

Hence X_{12} is independent of X_{23} . **M1 A1 (2)**

Alternatively,

$X_{12} \quad X_{23}$

1 1 requires players 2 and 3 to both score same as player 1 so

$$P(X_{12} = 1 \text{ and } X_{23} = 1) = \frac{1}{36} = \frac{1}{6} \times \frac{1}{6} = P(X_{12} = 1) \times P(X_{23} = 1)$$

1 0 requires player 2 to score the same as player 1, and player 3 score differently so

$$P(X_{12} = 1 \text{ and } X_{23} = 0) = \frac{5}{36} = \frac{1}{6} \times \frac{5}{6} = P(X_{12} = 1) \times P(X_{23} = 0)$$

0 1 requires players 2 and 3 to score the same as one another, and player 1 score differently so

$$P(X_{12} = 0 \text{ and } X_{23} = 1) = \frac{5}{36} = \frac{5}{6} \times \frac{1}{6} = P(X_{12} = 0) \times P(X_{23} = 1)$$

0 0 requires both player 1 and 3 to score differently to player 2 so

$$P(X_{12} = 0 \text{ and } X_{23} = 0) = \frac{25}{36} = \frac{5}{6} \times \frac{5}{6} = P(X_{12} = 0) \times P(X_{23} = 0)$$

Hence X_{12} is independent of X_{23} . **M1 A1 (2)**

If total score is T, then

$$T = \sum_{i < j} X_{ij}$$

M1

so

$$E(T) = E\left(\sum_{i < j} X_{ij}\right) = \sum_{i < j} E(X_{ij}) = {}^nC_2 E(X_{12}) = {}^nC_2 \left(1 \times \frac{1}{6} + 0 \times \frac{5}{6}\right) = \frac{n(n-1)}{12}$$

M1

A1

$$Var(T) = Var\left(\sum_{i < j} X_{ij}\right) = \sum_{i < j} Var(X_{ij}) = {}^nC_2 Var(X_{12}) = {}^nC_2 \left(1^2 \times \frac{1}{6} + 0^2 \times \frac{5}{6} - \frac{1^2}{6}\right)$$

M1

$$= \frac{5n(n-1)}{72}$$

A1 (5)

(ii)

$$\begin{aligned} Var(Y_1 + Y_2 + \dots + Y_m) &= E((Y_1 + Y_2 + \dots + Y_m)^2) - [E(Y_1 + Y_2 + \dots + Y_m)]^2 \\ &= E(Y_1^2 + Y_2^2 + \dots + Y_m^2 + 2Y_1Y_2 + 2Y_1Y_3 + \dots + 2Y_{m-1}Y_m) - [E(Y_1) + E(Y_2) + \dots + E(Y_m)]^2 \\ &= E\left(\sum_{i=1}^m Y_i^2\right) + 2E\left(\sum_{i=1}^{m-1} \sum_{j=i+1}^m Y_iY_j\right) - (0 + 0 + \dots + 0)^2 \\ &= \sum_{i=1}^m E(Y_i^2) + 2 \sum_{i=1}^{m-1} \sum_{j=i+1}^m E(Y_iY_j) \end{aligned}$$

M1 *A1 (2)

(iii)

$$P(Z_{12} = 1) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

If $Z_{23} = 1$ then player 2 has rolled an even score and player 3 has scored the same so, in this case, for $Z_{12} = 1$, require player 1 to roll the score that player has so $P(Z_{12} = 1 | Z_{23} = 1) = \frac{1}{6}$.

Therefore, $P(Z_{12} = 1) \neq P(Z_{12} = 1 | Z_{23} = 1)$ and thus Z_{12} and Z_{23} are not independent.

Alternatively,

$$P(Z_{12} = 1) = \frac{1}{12}, P(Z_{23} = 1) = \frac{1}{12}$$

For $Z_{12} = 1$ and $Z_{23} = 1$ we require all three players to score the same even number so

$$P(Z_{12} = 1 \text{ and } Z_{23} = 1) = \frac{3}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{72} \neq \frac{1}{12} \times \frac{1}{12} = P(Z_{12} = 1) \times P(Z_{23} = 1)$$

and thus they are not independent. **M1 A1 (2)**

Using part (ii), let $Y_1 = Z_{12}$, let $Y_2 = Z_{13}$, ... let $Y_m = Z_{(n-1)n}$

(and with $m = {}^nC_2 = \frac{n(n-1)}{2}$).

$P(Z_{12} = 1) = \frac{1}{12}, P(Z_{12} = -1) = \frac{1}{12}, P(Z_{12} = 0) = \frac{5}{6}$ so $E(Z_{12}) = 0$ and $E(Z_{12}^2) = \frac{1}{6}$ and likewise for all other Z (Y!). **B1** **B1**

If total score is U , then

$$U = \sum_{i < j} Z_{ij}$$

so

$$E(U) = E\left(\sum_{i < j} Z_{ij}\right) = \sum_{i < j} E(Z_{ij}) = 0$$

B1

which means we can apply the result of (ii).

If $Z_{12} = 1$ then $Z_{13} = 1$ or $Z_{13} = 0$

If $Z_{12} = -1$ then $Z_{13} = -1$ or $Z_{13} = 0$

Otherwise $Z_{12} = 0$

So $E(Z_{12}Z_{13}) = 1 \times 1 \times \frac{1}{72} + -1 \times -1 \times \frac{1}{72} = \frac{1}{36}$ **M1 A1**

So

$$Var(U) = \frac{n(n-1)}{2} \times \frac{1}{6} + 2 \times n \times {}^{n-1}C_2 \times \frac{1}{36} = \frac{n(n-1)}{12} + \frac{n(n-1)(n-2)}{36}$$

M1

M1 A1

$$= \frac{n(n-1)}{36} (3 + (n-2)) = \frac{n(n-1)(n+1)}{36} = \frac{n(n^2-1)}{36}$$

***A1 (9)**

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