Step I, Hints and Answers
June 2006

Hints and Answers, STEP I June 2006

Section A: Pure Mathematics

1 Since $182^2 = 33124$ and $183^2 = 33489$, let $n = 182$.

Since $184^2 - 33127 = 729 = 27^2$, let $m = 2$.

Therefore $184^2 - 27^2 = 33127$, so $33127 = (184 - 27) \times (184 + 27) \Rightarrow 33127 = 157 \times 211$.

It is crucial to realise that 157 and 211 are both prime numbers, hence the only other factorization of 33127 is $33127 = 1 \times 33127$.

Therefore $33127 = (16564 - 16563) \times (16564 + 16563) = 16564^2 - 16563^2$, so $m = 16382$.

This question is based on the method of Fermat factorization, which can sometimes be used effectively to factorize integers. Notice how we have factorized a number without ever dividing.

2 A good diagram is crucial here. Notice carefully that you are required to **prove** that the maximum area grazed is $14\pi a^2$, and therefore that **assuming** that this occurs when the goat is tethered to a corner will achieve few marks. Unjustified assumptions are of little value in Mathematics.

Let the goat be tethered a distance x from a corner. Therefore, the goat can graze an area

$$A = \frac{16a^2\pi}{2} + \frac{(4a-x)^2\pi}{4} + \frac{(2a-x)^2\pi}{4} + \frac{(2a+x)^2\pi}{4} + \frac{(x)^2\pi}{4} = \frac{\pi}{4}\left(56a^2 + 4x^2 - 8ax\right)$$

So the area grazed $A = \pi\left[13a^2 + (x-a)^2\right]$. This is minimised when $x = a$, and maximised when $x = 0$ or $2a$ (since $0 \leqslant x \leqslant 2a$), hence $13\pi a^2 \leqslant A \leqslant 14\pi a^2$.

Notice that completing the square is an efficient, easy way of maximising or minimising a quadratic expression. Calculus could also be used.

3 Recall that in this question b, c, p and q are real numbers.

(i) A picture of the graph $y = x^2 + bx + c$ shows that $c < 0$ is a sufficient condition for the roots of $x^2 + bx + c = 0$ to be real and unequal, since it is U-shaped with negative y-intercept. However, $c < 0$ is not a necessary condition; consider for example the equation $x^2 + 5x + 6 = 0$.

(ii) For the equation to have two distinct positive real roots, there must be two distinct real roots ($b^2 - 4c > 0$), they must be of the same sign ($c > 0$), and they must be positive ($b < 0$, consider the turning point of the graph). It is much easier to see this graphically than to try to manipulate the quadratic formula.

(iii) The first two parts of the question suggest that the nature of the roots of an equation can sometimes be ascertained by looking at an appropriate graph: in particular, the location of the turning points and the y-intercept. Therefore we consider the graph $y = x^3 + px + q$, its y-intercept at $(0, q)$, and the derivative $3x^2 + p$. The condition $q < 0$ ensures that the y-intercept is negative.

If $p > 0$ then $3x^2 + p = 0$ has no real solutions, so the cubic curve $y = x^3 + px + q$ has no turning points. Hence if $q < 0$ there will be one positive, real solution of $x^3 + px + q = 0$.

If $p < 0$ then the turning points are at $\left(\sqrt{\frac{-p}{3}}, \frac{2p}{3}\sqrt{\frac{-p}{3}} + q\right)$ and at $\left(-\sqrt{\frac{-p}{3}}, -\frac{2p}{3}\sqrt{\frac{-p}{3}} + q\right)$.

Notice that $4p^3 + 27q^2 \equiv 27\left(\frac{2p}{3}\sqrt{\frac{-p}{3}} + q\right)\left(-\frac{2p}{3}\sqrt{\frac{-p}{3}} + q\right)$, so that

a) if $4p^3 + 27q^2 < 0$ then the y coordinates of the turning points are of opposite signs, ensuring that there are three real roots, of which two are negative since $q < 0$;

b) if $4p^3 + 27q^2 = 0$ then one of the turning points has y coordinate 0, so the equation has exactly two real roots of opposite signs;

c) if $4p^3 + 27q^2 > 0$ then the y coordinates of the turning points are of the same sign, ensuring that there is one real root, which is positive since $q < 0$;

4 (i) When asked to sketch two graphs on the same axes, it is important to ensure that they look correct relative to each other; in this question, the gradient of each graph needs considering. The x-axis should be measured in radians. A good graph makes the argument obvious: the curves $y = \sin x$ and $y = x$ both pass through the origin, and both have gradient 1 there. Afterwards, $y = \sin x$ has gradient less than 1, and so is always "underneath" the line $y = x$.

(ii) Another graphical argument: the curves look very similar, hence $\dfrac{\sin x}{x} \approx 1$.

The polygon can be split into n isosceles triangle of base $\dfrac{P}{n}$ and vertical height $\dfrac{P}{2n} \div \tan\left(\dfrac{2\pi}{2n}\right)$.

So the area $= n \times \dfrac{1}{2} \times \dfrac{P}{n} \times \dfrac{P}{2n} \div \tan\left(\dfrac{2\pi}{2n}\right) = \dfrac{P^2}{4n\tan\left(\frac{\pi}{n}\right)}$

As instructed, we consider $\dfrac{\mathrm{d}}{\mathrm{d}n}\left[\dfrac{P^2}{4n\tan\left(\frac{\pi}{n}\right)}\right] = \dfrac{P^2}{4\left[n\tan\left(\frac{\pi}{n}\right)\right]^2} \times \left[\tan\left(\dfrac{\pi}{n}\right) - n\dfrac{\pi}{n^2}\sec^2\left(\dfrac{\pi}{n}\right)\right]$

$$= -1 \times \frac{P^2}{4\left[n\tan\left(\frac{\pi}{n}\right)\right]^2} \times \frac{\sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right) - \left(\frac{\pi}{n}\right)}{\cos^2\left(\frac{\pi}{n}\right)}$$

$< -\dfrac{P^2}{4\left[n\tan\left(\frac{\pi}{n}\right)\right]^2} \times \dfrac{\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right) - \left(\frac{\pi}{n}\right)}{\cos^2\left(\frac{\pi}{n}\right)}$ by (i), which tells us that $\sin x < x$.

This derivative is positive, since $\left(\dfrac{\pi}{n}\right)\cos\left(\dfrac{\pi}{n}\right) - \left(\dfrac{\pi}{n}\right) < 0$, hence the area increases.

n large $\Rightarrow \sin\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}$ (from part (ii)), and clearly $\cos\left(\frac{\pi}{n}\right) \approx 1$. Therefore $\tan\left(\frac{\pi}{n}\right) \approx \frac{\pi}{n}$, so the area of the polygon $\approx \frac{P^2}{4\pi}$. The radius of the circumcircle $= \frac{P}{2n} \div \sin\left(\frac{2\pi}{2n}\right) \approx \frac{P}{2n} \times \frac{n}{\pi}$ (since n is large), hence the circumcircle has area $\approx \pi \times \frac{P^2}{4\pi^2}$.

5 (i) $u^2 = 2x + 1 \Rightarrow 2u\, \mathrm{d}u = 2\, \mathrm{d}x$ and $x - 4 = \frac{1}{2}\left(u^2 - 9\right)$

$$\Rightarrow \int \frac{3}{(x-4)\sqrt{2x+1}}\, \mathrm{d}x = \int \frac{6}{(u^2-9)\,u}\, u\, \mathrm{d}u$$

$= \int \frac{1}{u-3} - \frac{1}{u+3}\, \mathrm{d}u$ (splitting the integrand into partial fractions)

$$= \ln(u-3) - \ln(u+3) + K = \ln\left(\frac{\sqrt{2x+1}-3}{\sqrt{2x+1}+3}\right) + K$$

(ii) The similarity of the two integrands suggests a similar substitution.

$u^2 = \mathrm{e}^x + 1 \Rightarrow 2u\, \mathrm{d}u = \mathrm{e}^x\, \mathrm{d}x$; also, $x = \ln 8 \Rightarrow u = 3$ and $x = \ln 3 \Rightarrow u = 2$.

$$\Rightarrow \int_{\ln 3}^{\ln 8} \frac{2}{\mathrm{e}^x\sqrt{\mathrm{e}^x+1}}\, \mathrm{d}x = \int_2^3 \frac{2}{(u^2-1)\,u}\,\frac{2u}{u^2-1}\, \mathrm{d}u = \int_2^3 \frac{4}{(u^2-1)^2}\, \mathrm{d}u$$

Let $\frac{4}{(u^2-1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2}$

$\Rightarrow 4 = A(u-1)(u+1)^2 + B(u+1)^2 + C(u+1)(u-1)^2 + D(u-1)^2$

Let $u = 1 \Rightarrow B = 1$. Let $u = -1 \Rightarrow D = 1$

Comparing the coefficients of $u^3 \Rightarrow 0 = A + C$.

Comparing the coefficients of $u^0 \Rightarrow 4 = -A + B + C + D$

$\Rightarrow A = -1,\ C = 1$

So the integral $= \left[\ln(u+1) - \ln(u-1) - \frac{1}{u-1} - \frac{1}{u+1}\right]_2^3 = \frac{7}{12} + \ln\frac{2}{3}$

Alternatively, the identity $\left[\frac{2}{u^2-1}\right]^2 \equiv \left[\frac{1}{u-1} - \frac{1}{u+1}\right]^2$ can be used to split the integrand into partial fractions.

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6 (i) The assertion that "$(a\,,\,b)$ lies on the curve $x^2-2y^2=1$" is equivalent to stating that $a^2-2b^2=1$. Since $(3a+4b)^2-2\,(2a+3b)^2 \equiv 9a^2+24ab+16b^2-2\left(4a^2+12ab+9b^2\right) \equiv a^2-2b^2$, the point $(3a+4b\,,\,2a+3b)$ lies on the curve $x^2-2y^2=1$ if $(a\,,\,b)$ does.

(ii) We are told that $Ma^2-Nb^2=1$, and we want to find M and N so that $M\,(5a+6b)^2-N\,(4a+5b)^2=1=Ma^2-Nb^2$.

Since $M\,(5a+6b)^2-N\,(4a+5b)^2 \equiv a^2\,(25M-16N)+ab\,(60M-40N)+b^2\,(36M-25N)$, we could let $25M-16N=M$, $60M-40N=0$ and $36M-25N=-N$.

These are all equivalent to $3M=2N$, so let $M=2$ and $N=3$.

(iii) We require $(Pa+Qb)^2-3\,(Ra+Sb)^2 \equiv a^2-3b^2=1$.

$\Rightarrow P^2-3R^2=1$

and $2PQ=6RS$

and $Q^2-3S^2=-3$

The first of these equations suggests letting $P=2$ and $R=1$. Then the second equation reduces to $4Q=6S$ which suggests letting $Q=3$ and $S=2$. These values are consistent with the third equation.

Therefore $(2a+3b\,,\,a+2b)$ is the simplest solution.

$(7a+12b\,,\,4a+7b)$ is another solution, since $7^2-3\times 4^2=1, 7\times 12=3\times 7\times 4$, and $12^2-3\times 7^2=-3$.

Equations of the form $x^2-dy^2=1$ are called Pell equations; the techniques for solving them (which underlie this question) are explained in most undergraduate textbooks on Number Theory. It might be interesting to consider the equation $x^2-4y^2=1$: what happens when an argument similar to (iii) is pursued?

7 This question requires familiarity with the notation $|x|$, which can be defined as: $|x|\equiv x$ if $x\geqslant 0$, and $|x|\equiv -x$ if $x<0$.

(i) The graph of $y=\operatorname{cosec}x$ has asymptotes $x=0$ and $x=\pi$. Both this graph and the line $y=\frac{2}{\pi}x$ pass through the point $\left(\frac{\pi}{2}\,,\,1\right)$, so the equation $x\sin x=\frac{\pi}{2}$ (which is equivalent to $\frac{2}{\pi}x=\operatorname{cosec}x$) has two solutions for $0<x<\pi$. The smaller of these is $\frac{\pi}{2}$ and the larger is defined in the question to be α.

The two graphs should make it clear that $\operatorname{cosec}x\leqslant\frac{2}{\pi}x$ for $\frac{\pi}{2}\leqslant x\leqslant\alpha$, hence in the same domain $\frac{\pi}{2}\leqslant x\sin x \Rightarrow x\sin x-\frac{\pi}{2}\geqslant 0$. By a similar argument, $x\sin x-\frac{\pi}{2}<0$ for $\alpha<x\leqslant\pi$. This analysis enables us to remove correctly the modulus in the integrand:

$$\int_{\frac{\pi}{2}}^{\pi}\left|x\sin x-\frac{\pi}{2}\right|\,\mathrm{d}x=\int_{\frac{\pi}{2}}^{\alpha}x\sin x-\frac{\pi}{2}\,\mathrm{d}x+\int_{\alpha}^{\pi}\frac{\pi}{2}-x\sin x\,\mathrm{d}x$$

$$=\left[\sin x-x\cos x-\frac{\pi x}{2}\right]_{\frac{\pi}{2}}^{\alpha}+\left[-\sin x+x\cos x+\frac{\pi x}{2}\right]_{\alpha}^{\pi}$$

$$= \left(\sin\alpha - \alpha\cos\alpha - \frac{\pi\alpha}{2}\right) - \left(1 - \frac{\pi^2}{4}\right) + \left(-\pi + \frac{\pi^2}{2}\right) - \left(-\sin\alpha + \alpha\cos\alpha + \frac{\pi\alpha}{2}\right)$$

$$= 2\sin\alpha + \frac{3\pi^2}{4} - \alpha\pi - 2\alpha\cos\alpha - \pi - 1$$

(ii) As suggested by the first part of the question, a careful sketch of the graph of $y = \left||\mathrm{e}^x - 1| - 1\right|$ is sensible. It should show that

if $x \leqslant 0$ then $\left||\mathrm{e}^x - 1| - 1\right| \equiv \mathrm{e}^x$

if $0 < x \leqslant \ln 2$ then $\left||\mathrm{e}^x - 1| - 1\right| \equiv 2 - \mathrm{e}^x$

if $x > \ln 2$ then $\left||\mathrm{e}^x - 1| - 1\right| \equiv \mathrm{e}^x - 2$.

Hence the area in the domain $0 \leqslant x \leqslant \ln 2$ is

$$\int_0^{\ln 2} 2 - \mathrm{e}^x \,\mathrm{d}x = (2\ln 2 - 2) - (0 - 1) = \ln 4 - 1$$

8 (i) The volume of $OABC = \frac{1}{3}\times$ the area of triangle $OAB \times OC = \frac{1}{6}abc$.

(ii) Using the scalar product with vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$,

$$\sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \cos\theta = \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix} = c^2 \Rightarrow \cos\theta = \frac{c^2}{\sqrt{a^2+c^2}\sqrt{b^2+c^2}}$$

The cosine rule ($AB^2 = AC^2 + BC^2 - 2 \times AC \times BC \times \cos\theta$) will also yield this result.

The area of triangle ABC will be $\frac{1}{2} \times \sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \sin\theta$

$$= \frac{1}{2} \times \sqrt{a^2+c^2}\sqrt{b^2+c^2} \times \sqrt{1 - \left(\frac{c^2}{\sqrt{a^2+c^2}\sqrt{b^2+c^2}}\right)^2} \text{ (because } \sin^2\theta \equiv 1 - \cos^2\theta)$$

$$= \frac{1}{2} \times \sqrt{(a^2+c^2)(b^2+c^2) - c^4}$$

$$= \frac{1}{2} \times \sqrt{a^2b^2 + b^2c^2 + c^2a^2}$$

So $\frac{1}{3} \times \left(\frac{1}{2} \times \sqrt{a^2b^2 + b^2c^2 + c^2a^2}\right) \times d = \frac{1}{6}abc \Rightarrow \frac{1}{d^2} = \frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2b^2c^2}$

which simplifies to the stated result.

A similar result is true for the right-angled triangle PQR, in which X is the foot of the perpendicular from the right-angle Q to the hypotenuse PR: $\frac{1}{PQ^2} + \frac{1}{QR^2} = \frac{1}{QX^2}$

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Section B: Mechanics

9 It is very difficult to answer this question without a clear diagram and the use of consistent notation. It is strongly recommended (though not essential) that the motion of the three objects be considered separately.

Before the string breaks, let the acceleration of the system be a, and let T and S be the tensions in the strings. Therefore (using Newton's Second Law on each object):

$xg - T = xa \qquad T - S = 4a \qquad S - yg = ya.$

Their sum $\Rightarrow a = \dfrac{g(x-y)}{x+y+4} = \dfrac{g(6-2y)}{10}$ since $x + y = 6$. The block takes time $t_1 = \sqrt{\dfrac{2d}{a}} = \sqrt{\dfrac{20d}{g(6-2y)}}$ to travel d. At this point it has velocity $v = \sqrt{2ad} = \sqrt{\dfrac{gd(6-2y)}{5}}$.

When the string over pulley P breaks, let the new acceleration be f and let the tension in the string over pulley Q be S_1.

Newton's Second Law tells us that $-S_1 = 4f$ and $S_1 - yg = yf \quad \Rightarrow f = \dfrac{-yg}{y+4}$

Therefore it takes $t_2 = -\sqrt{\dfrac{gd(6-2y)}{5}} \div \dfrac{-yg}{y+4} = \sqrt{\dfrac{d(6-2y)}{5g}}\left(\dfrac{y+4}{y}\right)$ to come to rest.

Hence the total time $T = t_1 + t_2 = \sqrt{\dfrac{d}{5g}}\ \mathrm{f}(y)$, where $\mathrm{f}(y)$ is as given.

To minimise T, set $\dfrac{\mathrm{d}f}{\mathrm{d}y} = 0 \Rightarrow \dfrac{10}{(6-2y)^{\frac{3}{2}}} + \left[\left(1+\dfrac{4}{y}\right)\dfrac{-1}{\sqrt{6-2y}} + \sqrt{6-2y}\left(\dfrac{-4}{y^2}\right)\right] = 0$

$\Rightarrow 10 - \left(1+\dfrac{4}{y}\right)(6-2y) - (6-2y)^2\left(\dfrac{4}{y^2}\right) = 0$

$\Rightarrow (6-2y)\left(y^2+4y+4(6-2y)\right) = 10y^2$

$\Rightarrow -2y^3 + 4y^2 - 72y + 144 = 0$

$\Rightarrow y^3 - 2y^2 + 36y - 72 = 0$

$\Rightarrow (y-2)\left(y^2+36\right) = 0$

$\Rightarrow y = 2$

Notice that the value of y does not depend on d: whenever the string is cut, a 2:1 division of the total mass of 6 kg will result in the shortest time taken.

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10 Since $x = Vt\cos 45^\circ$ and $y = Vt\sin 45^\circ - \frac{1}{2}gt^2$, we derive $y = x - \dfrac{gx^2}{V^2}$ as the cartesian equation of the trajectory. Next we consider the intersection of this parabola and the line $y = x\tan\alpha + b$. This occurs when $x\tan\alpha + b = x - \dfrac{gx^2}{V^2} \Rightarrow gx^2 + V^2x\,(-1+\tan\alpha) + bV^2 = 0$.

If this quadratic equation has only one solution then the line and the curve will touch. This will happen if the discriminant $(b^2 - 4ac) = 0$, i.e. $V^4\,(-1+\tan\alpha)^2 = 4gbV^2$

$\Rightarrow V\,(-1+\tan\alpha) = \pm 2\sqrt{gb}$

Since the particle cannot reach the roof if $\alpha \geqslant 45^\circ$ (consider the gradient of the parabolic path of projection at the origin), it can be seen that necessarily $\tan\alpha < 1$. Therefore $-1+\tan\alpha < 0$, so the negative square root must be chosen: $V\,(-1+\tan\alpha) = -2\sqrt{gb}$

If the condition for touching is satisfied, then this will occur where $x = \dfrac{-V^2\,(-1+\tan\alpha)}{2g}$ (using the quadratic formula with the discriminant equal to 0) at time $t = \dfrac{x\sqrt{2}}{V} = \dfrac{-V\,(-1+\tan\alpha)}{g\sqrt{2}}$.

To answer the last part of the question, a clear diagram of Q, and the forces acting on it, is recommended. The initial horizontal velocity of Q is $U\cos\alpha$, and its horizontal acceleration is $-g\cos\alpha\sin\alpha$ (caused by the horizontal component of the normal reaction).

Given that the particles touch at time $t = \dfrac{-V\,(-1+\tan\alpha)}{g\sqrt{2}}$, it is required that

$$U\cos\alpha \times \frac{-V\,(-1+\tan\alpha)}{g\sqrt{2}} - \frac{1}{2} \times -g\cos\alpha\sin\alpha \times V^2\left(\frac{-1+\tan\alpha}{g\sqrt{2}}\right)^2 = \frac{-V^2\,(-1+\tan\alpha)}{2g}$$

(this is because $s = ut + \frac{1}{2}at^2$ must be equal for both particles in the horizontal direction).

$$\Rightarrow U\cos\alpha + \frac{1}{2} \times \cos\alpha\sin\alpha \times V\left(\frac{-1+\tan\alpha}{\sqrt{2}}\right) = \frac{V}{\sqrt{2}}$$

$$\Rightarrow 2\sqrt{2}U\cos\alpha = 2V - V\cos\alpha\sin\alpha \times (-1+\tan\alpha)$$

$$\Rightarrow 2\sqrt{2}U\cos\alpha = V\left(2 + \cos\alpha\sin\alpha - \sin^2\alpha\right)$$

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11 It is essential to realise that we cannot consider each individual collision, since we do not know the masses of particles A_1 to A_{n-2}. We must consider the conservation of momentum and of kinetic energy overall.

(i) If only one particle were moving after all collisions have taken place, it would have to be A_n with velocity v.

This would require $mu = \lambda mv$ (conservation of momentum) and $mu^2 = \lambda mv^2$ (conservation of kinetic energy).

Hence $u^2 = \lambda^2 v^2$ and $u^2 = \lambda v^2 \Rightarrow \lambda^2 = \lambda \Rightarrow \lambda = 1$ or 0, neither of which is permitted.

(ii) Let the final speeds of A_{n-1} and A_n be v and w respectively, where both are positive.

This scenario requires $mu = mv + \lambda mw$ (conservation of momentum) and $mu^2 = mv^2 + \lambda mw^2$ (conservation of kinetic energy).

Therefore $(v + \lambda w)^2 = v^2 + \lambda w^2$

$\Rightarrow 2v\lambda w + \lambda^2 w^2 = \lambda w^2$

$\Rightarrow 2v = w(1 - \lambda)$ which implies that $v < 0$ since $\lambda > 1$. This contradicts the supposition of the question: if A_{n-1} moves backwards then it will not be the only particle moving other than A_n.

(iii) Let the final velocities of A_{n-2}, A_{n-1} and A_n be p, q and r respectively, where all are positive, and $p < q < r$. Also, we must let the mass of A_{n-2} be km, since we do not know what it is.

This scenario requires $mu = kmp + mq + \lambda mr$ (conservation of momentum) and $mu^2 = kmp^2 + mq^2 + \lambda mr^2$ (conservation of kinetic energy).

$\Rightarrow (kp + q + \lambda r)^2 = kp^2 + q^2 + \lambda r^2$

$\Rightarrow k^2p^2 + q^2 + \lambda^2 r^2 + 2kpq + 2\lambda qr + 2\lambda kpr = kp^2 + q^2 + \lambda r^2$

$\Rightarrow 2kpq + 2\lambda kpr - kp^2 + k^2p^2 = \lambda r^2 - \lambda^2 r^2 - 2\lambda qr$

$\Rightarrow kp\,[2q + 2\lambda r - p(1-k)] = r^2(\lambda - \lambda^2) - 2\lambda qr$

Since $\lambda > 1$, the RHS is negative.

But $q > p \Rightarrow 2q - p(1-k) > 2p - p(1-k) \equiv p + pk > 0$.

Hence the RHS < 0 but the LHS > 0: a contradiction.

(iv) The two particles must be A_0 and A_n, with velocities x and y respectively.

Therefore, $mu = mx + \lambda my$ (conservation of momentum) and $mu^2 = mx^2 + \lambda my^2$ (conservation of kinetic energy).

$\Rightarrow (u - \lambda y)^2 = u^2 - \lambda y^2$

$\Rightarrow \lambda^2 y^2 - 2u\lambda y = -\lambda y^2$

$\Rightarrow \lambda y - 2u = -y$

$\Rightarrow y = \dfrac{2u}{1+\lambda} \Rightarrow x = \lambda y - u = u\left(\dfrac{2}{1+\lambda} - 1\right) = u\left(\dfrac{1-\lambda}{1+\lambda}\right).$

Note that x is negative, as is required.

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Section C: Probability and Statistics

12 There are many arguments to derive the first result: the neatest is probably to argue that if there is no road from Oxtown to Camville then the third road must be blocked (with probability p), and also on **both** of the other two roads it is **not** the case that **both** of the sections are **un**blocked. Therefore,

$$P\left(\text{no road from Oxtown to Camville}\right) = p\left(1-(1-p)^2\right)^2 = p\left(2p-p^2\right)^2 = p^3\left(2-p\right)^2.$$

It is crucial to recognise that the second paragraph is asking for a conditional probability:

P (the chosen road isn't blocked and the others are, given that the chosen road isn't blocked)

$$= \frac{\frac{1}{3}(1-p)\left(2p-p^2\right)^2 + \frac{2}{3}p\left(2p-p^2\right)(1-p)^2}{\frac{1}{3}(1-p)+\frac{2}{3}(1-p)^2} = \frac{\left(2p-p^2\right)(1-p)\left[\left(2p-p^2\right)+2p(1-p)\right]}{(1-p)\left[1+2(1-p)\right]}$$

$$= \frac{\left(2p-p^2\right)\left(4p-3p^2\right)}{(3-2p)} = \frac{p^2(2-p)(4-3p)}{(3-2p)}$$

Notice that when $p = 1$ this probability equals 1; but if $p = 1$ there will be no route from Oxtown to Camville at all! As p tends to 1 this probability tends to 1: as blocked roads become more and more certain, then if you do (just) manage to get to Camville it's very likely that the other roads would have been blocked. The resolution to this apparent paradox is that when $p = 1$ you're conditioning on an event of zero probability: notice the cancelled factor of $1-p$, which requires $p \neq 1$. The violation of this explains the apparent contradiction.

13 (i) Given the information in the question, we can assume that the number of diamonds in $100N$ grams of chocolate is distributed Poisson $(0.1N)$. Therefore, the probability of there being no diamonds in $100N$ grams of chocolate is $\mathrm{e}^{-0.1N}$, and the expected number of diamonds in $100N$ grams of chocolate is $0.1N$

$$P\left(\text{I have no diamonds}\right) = \frac{1}{6}\left(\mathrm{e}^{-0.1}+\mathrm{e}^{-0.2}+\ldots+\mathrm{e}^{-0.6}\right) = \frac{\mathrm{e}^{-0.1}}{6}\left(\frac{1-\mathrm{e}^{-0.6}}{1-\mathrm{e}^{-0.1}}\right)$$

A tree diagram might make this calculation more clear: there are six alternative branches, representing the possible scores on a die, at the end of each of which are two branches, representing either finding no diamonds or finding some.

A similar "expectation" tree diagram explains the argument that I expect to find $\frac{1}{6}(0.1+0.2+\ldots+0.6) = 0.35$ diamonds. These are examples of conditional expectations: $0.2 = E$ (the number of diamonds I find **given that** I roll a 2), analogous to $\mathrm{e}^{-0.2} = P$ (I find no diamonds **given that** I roll a 2). The idea of conditional expectation is developed considerably in undergraduate mathematics.

Hints and Answers, STEP I June 2006

(ii) P (I have no diamonds)

$$= \left(\frac{1}{6}\right)\left(e^{-0.1}\right) + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)\left(e^{-0.2}\right) + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)\left(e^{-0.3}\right) + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)\left(e^{-0.4}\right) + \ldots$$

(because if I score my first 6 on the rth roll, then the previous $r-1$ rolls have all scored "not 6")

$$= \frac{\left(\frac{1}{6}\right)e^{-0.1}}{1-\left(\frac{5}{6}\right)e^{-0.1}}$$

$$= \frac{e^{-0.1}}{6-5e^{-0.1}}$$ using the formula for the sum to infinity of a geometric progression.

The question suggests we recall that $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$, because a similar argument to that used in part (i) tells us that the number of diamonds I expect to find is

$$\left(\frac{1}{6}\right)0.1 + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right)0.2 + \left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)0.3 + \left(\frac{5}{6}\right)^3\left(\frac{1}{6}\right)0.4 + \ldots$$

$$= \left(\frac{1}{6}\right)0.1\left[1 + 2\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + 4\left(\frac{5}{6}\right)^3 + \ldots\right]$$

$$= \left(\frac{1}{6}\right)0.1\left(1-\frac{5}{6}\right)^{-2} = 0.6 \text{ diamonds.}$$

14 (i) $P\,(\text{red sweet is first drawn as } r\text{th sweet}) = \left(\dfrac{n}{n+1}\right)^{r-1}\dfrac{1}{n+1}$

$$P \text{ maximal} \Rightarrow \frac{dP}{dn} = 0 \Rightarrow \frac{(n+1)^r (r-1)\, n^{r-2} - n^{r-1} r\,(n+1)^{r-1}}{(n+1)^{2r}} = 0$$

$\Rightarrow (n+1)(r-1) - nr = 0$

$\Rightarrow n = r - 1$

Notice that we can use this result to estimate n: if the rth sweet is red for the first time, it is sensible (because it makes the observed event most likely) to estimate that there are $r-1$ blue sweets in the bag.

(ii) $P\,(\text{red sweet is first drawn as } r\text{th sweet}) =$

$$\left(\frac{n}{n+1}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\ldots\left(\frac{n-r+2}{n-r+3}\right)\left(\frac{1}{n-r+2}\right) = \frac{1}{n+1}$$

The value of this probability decreases as n increases. Therefore, to maximise this probability, n needs to be as small as possible. Since r sweets have been chosen from the bag, this implies that $n+1 \geqslant r$: there must have been more sweets in the bag initially than are chosen during this procedure. Hence the minimum n is $r-1$.