STEP I 2017
Hints and Solutions

Question 1

Part (i)

Using $\tan x = \frac{\sin x}{\cos x}$ and rearranging will give an algebraic fraction where the denominator is u and the numerator is a multiple of $\frac{\mathrm{d}u}{\mathrm{d}x}$, after which the integration can be completed easily.

In a similar way, substituting $\cot x = \frac{\cos x}{\sin x}$ and rearranging will show that

$u = x\cos x - \sin x$ is a sensible choice of substitution to complete the second integration.

Part (ii)

For the first integration, differentiating the denominator of the fraction will show that an appropriate substitution is $u = x\sec^2 x - \tan x$, following which the integration follows easily. The second integration requires some rearrangement of the algebraic fraction: division of the numerator and denominator by $\cos^4 x$ puts it into a form where it should be clear that the same substitution will work for this integration as well.

Question 2

Part (i)

Taking definite integrals of the two sides of the inequality with the lower limit as 1 and the upper limit as x leads directly to the required result in the case $x \geq 1$. To show the result in the case $0 < x \leq 1$, note that $\frac{1}{t} \geq 1$ holds for $0 < t \leq 1$ and perform definite integrals with the lower limit as x and the upper limit as 1.

Part (ii)

Following the same process as for part (i) will lead to the required result.

Part (iii)

Integrating (*) and rearranging leads to one half of the inequality and integrating (**) and rearranging leads to the other half of the inequality. In both cases take limits of 1 and y for the integrals and consider the cases $0 < y < 1$ and $y > 1$ separately.

Question 3

Using implicit differentiation on the equation of the curve gives $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$, which then leads easily to an expression for the gradient of the curve at the point P and so the equation of the tangent can be deduced easily from this point. The equation of the tangent at Q can then be written down as it simply requires changing p to q throughout.

A diagram of the situation described then allows a strategy for the areas of each triangle to be worked out: for RST, one side is vertical and so this length can be used as the base and so the required measurements are easily deduced from the coordinates of the points. There are many ways of deducing the area of OPQ, for example by adding horizontal lines through P and Q and then considering the triangle to be a trapezium with two triangles cut away from it.

Question 4

Part (i)

The function $1 + r + r^2$ is recognisable as a quadratic function and so can be sketched by completing the square to identify the location of its minimum point. Since we know that $|r| < 1$, we can regard this as the domain and find the range of the function to deduce the possible values of p.

In the case where $1 < p < 3$ it can be seen from the graph that there is only one possible value for r as the horizontal line for this value of p only intersects the graph once. Since S is defined in terms of r, the value of S must also be determined uniquely.

In the other case, $S = \frac{1}{1-r}$ can be rearranged to make r the subject and then substituted into the function for p to deduce the required equation.

Part (ii)

Considering a graph of the function for q will again determine the values of q that determine T uniquely and a similar approach by substituting will lead to the quadratic equation required for the last part of the question.

Question 5

A diagram is very helpful in understanding the description in the first paragraph. From such a diagram expressions for the width and height can be identified which will then lead to the formula for the area of the rectangle. Since Q and R are aligned vertically, they must have the same y-coordinate and this can be used to give an expression for s in terms of the other variables.

When performing the differentiation required next, remember that s is a function of x and differentiate the expression for s to find $\frac{\mathrm{d}s}{\mathrm{d}x}$. This can then be substituted into the expression for $\frac{\mathrm{d}A}{\mathrm{d}x}$.

The greatest possible area can be found by setting $\frac{\mathrm{d}A}{\mathrm{d}x} = 0$ and application of trigonometric identities will allow the final result to be shown.

Question 6

Part (i)

Note that if $\mathrm{f}(x)$ does not take any negative values then the value of the integral must be positive (and similarly, if it does not take any positive values then the value of the integral must be negative). The result then follows from this.

Part (ii)

The integral that needs to be considered can be shown to be equal to 0 and so the result from part (i) implies that $(x-\alpha)^2\mathrm{g}(x)$ must have both positive and negative values in the interval. Since $(x-\alpha)^2 > 0$ it must also be the case that $\mathrm{g}(x)$ has both positive and negative values in the interval and so it must also take the value 0 somewhere (by the result that can be assumed in this question).

Substituting the form of $\mathrm{g}(x)$ into the three integrals given in (*) gives a set of simultaneous equations that can be solved to find an example of such a function and it is then straightforward to confirm that there is a 0 by showing that the endpoints of the interval give different signs for the value of $\mathrm{g}(x)$.

Part (iii)

To be able to apply the result from part (ii), it will need to be the case that $\mathrm{g}(x) = \mathrm{h}'(x)$. Once the three integrals have been shown to have values that are consistent with part (ii) the result follows immediately.

Question 7

Part (i)

Applying the cosine rule to the triangle CMA leads to the required result and the corresponding result for CL is easy to deduce.

Part (ii)

Applying the cosine rule to triangle CLM leads directly to the first result required. Since the formula is symmetric in a, b and c, it must be the case that all three sides of LMN are equal and so the triangle is equilateral. The final result of this part can be shown by expressing the area of LMN in terms of the length of one side.

Part (iii)

The equivalence of the conditions follows by applying the formula for $\cos(A - B)$ and rearranging.

The final deduction will follow by combining the results from parts (ii) and (iii).

Question 8

Following the standard procedure for proof by induction establishes the first result.

Part (i)

The required fact about the sequence b_n can be proven by induction, and then division of (*) by b_n^2 will show that as $n \to \infty$, c_n will approach the root of a quadratic equation and this equation can be solved to give the required result.

Part (ii)

By considering $c_{n+1} - c_n$ it can be shown that the sequence is decreasing. Hence c_n must therefore be greater than the limit of the sequence for all n. Rearranging this equation then leads to the required inequality.

Finally, calculating the first few terms of the sequences leads to the required approximations.

Question 9

Part (i)

By considering the horizontal speed it is possible to find the amount of time before the particle passes through P. Considering the vertical speed then allows a relationship between u and α to be deduced. Differentiating with respect to α and rearranging then leads to the first result.

The relation between α and β can then be deduced by considering the relationship between the graphs of $y = \tan x$ and $y = \cot x$.

Part (ii)

Considering the horizontal and vertical components of the velocity as the particle passes through P allows the tangent of the angle at which it is travelling to be deduced. Graphical considerations can then be used to deduce the relationship between this angle and α.

Question 10

Part (i)

Standard procedures considering conservation of momentum and the coefficient of restitution can be used to show the relationship between each of the velocities in terms of the variables required.

Part (ii)

It is clear that each particle must be involved in at least one collision. To show that there are no more than two collisions it is required to show that each particle will still be moving faster than the one behind it following the second collision.

Part (iii)

When $e = \lambda$ all of the particles will have the same velocity following the collision and it can be seen that the kinetic energies will form a geometric progression and so the sum to infinity can be calculated and then the fraction of kinetic energy lost deduced.

Part (iv)

In this case the particles can be seen to come to rest eventually and so all of the kinetic energy is lost.

Question 11

Adding the forces to the diagram and then resolving parallel and perpendicular to the slope gives two equations relating all of the forces and lengths. Taking moments about the centre of the rod gives a further equation. Solving these equations and using the relationship between frictional and reaction forces then allows an equation in terms of tangents of the appropriate variables to be found. Rearranging then allows the formula for $\tan(A+B)$ to be obtained and therefore the required result can be shown to be the only solution to the equation that matches the situation described.

Question 12

Part (i)

The probability that any one participant does not pick the winning number is $1-\frac{1}{N}$ and since the participants' choices are independent, the probability that no participant picks the winning number is this value raised to the power N. The profit is then £cN in the case where no one picks the winning number and £$(cN-J)$ if someone does pick the winning number. Applying the given approximations then allows the result to be shown.

Part (ii)

The first relationship that is needed follows from the fact that the probability that one of the numbers will be chosen must be 1. The expected profit can then be calculated by following a similar procedure as in part (i).

In the case described at the end of this part the information along with the relationship between a, b and γ is enough to deduce the values of a and b and that the organiser expects to make a profit if $2Nc=J$.

Question 13

It should be clear that the first slice of the loaf of bread can never be the second of two slices to make a sandwich and so the value of s_1 can be deduced. To explain the first of the two equations note that to use the r^{th} slice of bread for toast means that the previous slice must have been either the second slice of a sandwich or used to make toast (the total probability of this is $(s_{r-1} + t_{r-1})$). This must then be multiplied by the probability of using this next slice to make toast. Since it is not possible to follow this reasoning for the first slice of toast, the equation can only be valid for values of 2 or greater. Similarly, it cannot be valid for the final slice as in that case it must be used for toast.

The fact that the second equation begins $1 - \cdots$ suggests that a sensible approach is to begin with identifying all of the cases that prevent the r^{th} slice being used for toast.

To show the next result, first eliminate the bracketed expressions from the two equations to get an equation relating s_r and t_r. Substituting this into the second equation then gives the required result.

The formula for s_r can then easily be proven by induction and the relationship between s_r and t_r used to find the corresponding expression for t_r.

Finding expressions for the final terms of the sequences can now be achieved by finding the appropriate equations to relate the use of the final slice of bread to the previous one and substituting the expressions for s_{n-1} and t_{n-1} into these.