Step I, Hints and Answers
June 2005

STEP I, 2005, Hints and Answers

Section A: Pure Mathematics

1 The simplest approach to this question is to use the result that if you have a collection of A identical objects of type a, B identical objects of type b, C identical objects of type c etc, then they can be rearranged in

$$\frac{(A+B+C+...)!}{A!B!C!...} \text{ ways}$$

(i) $43 = 9+9+9+9+7 = 5$ distinct rearrangements, since five objects of which four are 9 can be rearranged in $\frac{5!}{4!1!}$ ways, or

$43 = 9+9+9+8+8 = \frac{5!}{3!2!} = 10$ distinct rearrangements.

(ii) $39 = 9+9+9+9+3 = 5$ distinct rearrangements, or

$39 = 9+9+9+7+5 = 9+9+9+8+4 = 2 \times 20$ distinct rearrangements, or

$39 = 9+9+9+6+6 = 10$ distinct rearrangements, or

$39 = 9+9+8+8+5 = 30$ distinct rearrangements, or

$39 = 9+9+8+7+6 = 60$ distinct rearrangements, or

$39 = 9+9+7+7+7 = 10$ distinct rearrangements, or

$39 = 9+8+8+8+6 = 20$ distinct rearrangements, or

$39 = 9+8+8+7+7 = 30$ distinct rearrangements, or

$39 = 8+8+8+8+7 = 5$ distinct rearrangements

$= 210$ distinct rearrangements.

STEP I, 2005, Hints and Answers

2 $y^2 = 4x \Rightarrow 2yy' = 4 \Rightarrow y' = \dfrac{2}{y}$

Equation of tangent at P: $y - 2p = \dfrac{1}{p}\left(x - p^2\right) \Rightarrow py = x + p^2$.

Equation of tangent at Q: $qy = x + q^2$

Intersect where $qy - q^2 = py - p^2$

$\Rightarrow (q-p)\,y = q^2 - p^2$

$\Rightarrow y = p + q \Rightarrow x = pq$ by substitution. Hence $R\,(pq\,,\ p+q)$.

Equation of normal at P: $y - 2p = -p\left(x - p^2\right) \Rightarrow y + px = 2p + p^3$.

Equation of normal at Q: $y + qx = 2q + q^3$

Intersect where $2p + p^3 - px = 2q + q^3 - qx$

$\Rightarrow x\,(p-q) = 2\,(p-q) + (p-q)\left(p^2 + pq + q^2\right)$ using the identity $p^3 - q^3 \equiv (p-q)\left(p^2 + pq + q^2\right)$

$\Rightarrow x = p^2 + pq + q^2 + 2 \Rightarrow y = -pq\,(p+q)$ by substitution.

But $(1\,,\ 0)$ lies on PQ so the gradient of the line segement from P to $(1\,,\ 0)$ equals the gradient of the line segment from Q to $(1\,,\ 0)$.

$\Rightarrow \dfrac{2p}{p^2 - 1} = \dfrac{2q}{q^2 - 1}$

$\Rightarrow 2pq^2 - 2p = 2qp^2 - 2q \Rightarrow pq\,(q-p) = p - q \Rightarrow pq = -1$

$\Rightarrow S$, where the normals intersect, has coordinates $\left(p^2 + q^2 + 1\,,\ p+q\right)$.

Obviously, SP is perpendicular to PR and QS is perpendicular to QR, because each of these is a tangent-normal pair.

Furthermore, the gradient of PR $\times$ the gradient of $QR = \frac{1}{p} \times \frac{1}{q} = \frac{1}{pq} = -1$, so PR is perpendicular to QR.

Also, the gradient of PS $\times$ the gradient of $QS = -p \times -q = pq = -1$, so PS is perpendicular to QS.

Therefore all four angles are right angles, proving that PSQR is a rectangle. It is not sufficient to consider only the lengths of the sides, since the quadrilateral could be a parallelogram.

STEP I, 2005, Hints and Answers

3 (i) $\dfrac{x}{x-a}+\dfrac{x}{x-b}=1 \Rightarrow x^2-bx+x^2-ax=x^2-ax-bx+ab$

$\Rightarrow x^2=ab$ which has two distinct real solutions since $ab>0$.

(ii) $\dfrac{x}{x-a}+\dfrac{x}{x-b}=1+c \Rightarrow 2x^2-(a+b)\,x=(c+1)\left(x^2-(a+b)\,x+ab\right)$

$\Rightarrow (c-1)\,x^2-c\,(a+b)\,x+(c+1)\,ab=0.$

This has one real solution if its discriminant equals 0

$\Rightarrow c^2\,(a+b)^2-4\,(c-1)\,(c+1)\,ab=0$

$\Rightarrow c^2\left[(a+b)^2-4ab\right]=-4ab \Rightarrow c^2=\dfrac{-4ab}{(a-b)^2}=1-\dfrac{a^2+2ab+b^2}{a^2-2ab-b^2}.$

Since a, b and c are real, $c^2\geqslant 0$ and $\left(\dfrac{a+b}{a-b}\right)^2\geqslant 0$. Therefore clearly $0\leqslant c^2\leqslant 1$.

However, $c^2=0\Rightarrow (a+b)^2=(a-b)^2$

$\Rightarrow a+b=\pm\,(a-b)$

$\Rightarrow b=-b$ or $a=-a$

$\Rightarrow b=0$ or $a=0$ neither of which is permitted.

Therefore $0<c^2\leqslant 1$.

4 (a) $\cos\theta=\dfrac{3}{5}\Rightarrow \sin\theta=\pm\dfrac{4}{5}$, since $\sin^2\theta=1-\cos^2\theta$.

Since $\sin\theta$ is negative in the given domain, $\sin 2\theta\equiv 2\cos\theta\sin\theta=2\times\dfrac{3}{5}\times-\dfrac{4}{5}=-\dfrac{24}{25}$

Also, $\cos 3\theta\equiv\cos(\theta+2\theta)\equiv\cos\theta\cos 2\theta-\sin\theta\sin 2\theta$

$=\dfrac{3}{5}\times\left[2\times\left(\dfrac{3}{5}\right)^2-1\right]-\left(-\dfrac{4}{5}\right)\times-\dfrac{24}{25}=-\dfrac{117}{125}$

(b) $\tan(\theta+2\theta)\equiv\dfrac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}\equiv\dfrac{\tan\theta+\frac{2\tan\theta}{1-\tan^2\theta}}{1-\tan\theta\times\frac{2\tan\theta}{1-\tan^2\theta}}=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}.$

So $\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=\dfrac{11}{2}$

$\Rightarrow 6t-2t^3=11-33t^2$ where $t\equiv\tan\theta$

$\Rightarrow 2t^3-33t^2-6t+11=0$

$\Rightarrow (2t-1)\left(t^2-16t-11\right)=0$

$\Rightarrow t=0$ or $t=\dfrac{1}{2}\left(16\pm\sqrt{16^2+44}\right)=8\pm\sqrt{75}$

so $\tan\theta=8+\sqrt{75}$ since $\dfrac{\pi}{4}\leqslant\theta\leqslant\dfrac{\pi}{2}\Rightarrow\tan\theta\geqslant 1$.

STEP I, 2005, Hints and Answers

5 (i) If $k \neq 0$ the given integral $= \left[\dfrac{(x+1)^k}{k}\right]_0^1 = \dfrac{2^k - 1}{k}$

If $k = 0$ then the integral $= [\ln(x+1)]_0^1 = \ln 2$

As k tends to 0, the two integrals become closer in value, so $\dfrac{2^k - 1}{k} \approx \ln 2$ when $k \approx 0$.

You might like to consider whether this is always true: if a sequence of functions converges to a certain limit function, must some definite integral of each function in the sequence converge to the same integral of the limit function? Simple questions such as these have motivated many of the great discoveries in Mathematics.

(ii) The simplest way to integrate $x(x+1)^m$ is to notice that

$$x(x+1)^m \equiv (x+1-1)(x+1)^m \equiv (x+1)^{m+1} - (x+1)^m .$$

This is true for all values of m. Of course, it can be integrated by parts or with a substitution if preferred.

Assume $m \neq -1, -2$. Therefore the given integral equals $\left[\dfrac{(x+1)^{m+2}}{m+2} - \dfrac{(x+1)^{m+1}}{m+1}\right]_0^1$

$= \left(\dfrac{2^{m+2}}{m+2} - \dfrac{2^{m+1}}{m+1}\right) - \left(\dfrac{1}{m+2} - \dfrac{1}{m+1}\right) = \dfrac{m2^{m+1} + 1}{(m+2)(m+1)}$.

If $m = -1$, $x(x+1)^{-1} \equiv 1 - (x+1)^{-1}$.

Hence the given integral equals $[x - \ln(x+1)]_0^1 = 1 - \ln 2$.

If $m = -2$, $x(x+1)^{-2} \equiv (x+1)^{-1} - (x+1)^{-2}$.

Hence the given integral equals $\left[\ln(x+1) + (x+1)^{-1}\right]_0^1 = \ln 2 - \frac{1}{2}$

STEP I, 2005, Hints and Answers

6 (i) $PA = 2PB \Rightarrow (x-5)^2 + (y-16)^2 = 4\left((x+4)^2 + (y-4)^2\right)$

$\Rightarrow x^2 + y^2 - 10x - 32y + 281 = 4x^2 + 4y^2 + 32x - 32y + 128$

$\Rightarrow 3x^2 + 3y^2 + 42x - 153 = 0$

$\Rightarrow x^2 + y^2 + 14x - 51 = 0$

$\Rightarrow (x+7)^2 - 49 + y^2 - 51 = 0$

$\Rightarrow (x+7)^2 + y^2 = 100$

which is a circle centre $(-7\,,\ 0)$ with radius 10.

(ii) $QC = k \times QD \Rightarrow (x-a)^2 + y^2 = k^2(x-b)^2 + k^2y^2$

$\Rightarrow x^2\left(k^2-1\right) + y^2\left(k^2-1\right) + x\left(2a - 2k^2b\right) + \left(k^2b^2 - a^2\right) = 0$

If this locus is the same as the locus of P, then the ratios of the coefficients must be the same.

$\Rightarrow \dfrac{2a - 2k^2b}{k^2-1} = 14$ and $\dfrac{k^2b^2 - a^2}{k^2-1} = -51.$

Notice that you **cannot** conclude that $k^2 - 1 = 1$.

$\Rightarrow k^2 = \dfrac{a+7}{b+7}$ and $k^2 = \dfrac{a^2+51}{b^2+51}$

$\Rightarrow \dfrac{a+7}{b+7} = \dfrac{a^2+51}{b^2+51}$

$\Rightarrow (a+7)\left(b^2+51\right) = (b+7)\left(a^2+51\right)$

$\Rightarrow ab^2 - a^2b = 7\left(a^2 - b^2\right) + 51(b-a)$

$\Rightarrow ab(b-a) = 7(a-b)(a+b) + 51(b-a)$

$\Rightarrow ab = 51 - 7(a+b)$ since $a \neq b \Rightarrow a - b \neq 0$

$\Rightarrow ab + 7(a+b) = 51$

$\Rightarrow ab + 7(a+b) + 49 = 51 + 49$

$\Rightarrow (a+7)(b+7) = 100$

STEP I, 2005, Hints and Answers

7 (i) $$\prod_{r=1}^{n}\left(\frac{r+1}{r}\right)=\frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times\ldots\times\frac{n+1}{n}=n+1$$

(ii) $$\prod_{r=2}^{n}\left(\frac{r^2-1}{r^2}\right)=\prod_{r=2}^{n}\left(\frac{r-1}{r}\right)\left(\frac{r+1}{r}\right)=\prod_{r=2}^{n}\left(\frac{r-1}{r}\right)\prod_{r=2}^{n}\left(\frac{r+1}{r}\right)$$
$$=\left[\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\ldots\times\frac{n-1}{n}\right]\times\left[\frac{3}{2}\times\frac{4}{3}\times\frac{5}{4}\times\ldots\times\frac{n+1}{n}\right]=\frac{n+1}{2n}$$

(iii) $$\prod_{r=1}^{n}\left(\cos\frac{2\pi}{n}+\sin\frac{2\pi}{n}\cot\frac{(2r-1)\pi}{n}\right)\text{ where } n \text{ is even}$$
$$=\prod_{r=1}^{n}\left(\cos\frac{2\pi}{n}+\frac{\sin\frac{2\pi}{n}\cos\frac{(2r-1)\pi}{n}}{\sin\frac{(2r-1)\pi}{n}}\right)$$
$$=\prod_{r=1}^{n}\left(\frac{\cos\frac{2\pi}{n}\sin\frac{(2r-1)\pi}{n}+\sin\frac{2\pi}{n}\cos\frac{(2r-1)\pi}{n}}{\sin\frac{(2r-1)\pi}{n}}\right)$$
$$=\prod_{r=1}^{n}\left(\frac{\sin\left[\frac{(2r-1)\pi}{n}+\frac{2\pi}{n}\right]}{\sin\frac{(2r-1)\pi}{n}}\right)$$
$$=\prod_{r=1}^{n}\left(\frac{\sin\frac{(2r+1)\pi}{n}}{\sin\frac{(2r-1)\pi}{n}}\right)=\frac{\sin\frac{3\pi}{n}}{\sin\frac{\pi}{n}}\times\frac{\sin\frac{5\pi}{n}}{\sin\frac{3\pi}{n}}\times\frac{\sin\frac{7\pi}{n}}{\sin\frac{5\pi}{n}}\times\ldots\times\frac{\sin\frac{(2n+1)\pi}{n}}{\sin\frac{(2n-1)\pi}{n}}$$
$$=\frac{\sin\frac{(2n+1)\pi}{n}}{\sin\frac{\pi}{n}}=\frac{\sin\left(2\pi+\frac{\pi}{n}\right)}{\sin\frac{\pi}{n}}=\frac{\sin\frac{\pi}{n}}{\sin\frac{\pi}{n}}=1\text{ using the periodicity of }\sin x.$$

You should consider why it was necessary to require that n was even.

STEP I, 2005, Hints and Answers

8 $y^2 = x^k \mathrm{f}(x) \Rightarrow 2yy' = x^k \mathrm{f}'(x) + kx^{k-1}\mathrm{f}(x)$

$\Rightarrow 2xyy' = x^{k+1}\mathrm{f}'(x) + kx^k\mathrm{f}(x) = x^{k+1}\mathrm{f}'(x) + ky^2$

(i) $k = 1 \Rightarrow 2xyy' = x^2\mathrm{f}'(x) + y^2$

$\Rightarrow x^2\mathrm{f}'(x) + y^2 = y^2 + x^2 - 1$

$\Rightarrow \mathrm{f}'(x) = 1 - \dfrac{1}{x^2}$

$\Rightarrow \mathrm{f}(x) = x + \dfrac{1}{x} + c$

$\Rightarrow y^2 = x^2 + 1 + cx$ since $y^2 = x\mathrm{f}(x)$.

But $x = 1, y = 2 \Rightarrow y^2 = 4 = 1 + 1 + c \Rightarrow c = 2$

$\Rightarrow y^2 = x^2 + 2x + 1$, which is the pair of straight lines $y = \pm(x+1)$.

(ii) Since $2xyy' = 2\dfrac{\ln x}{x} - y^2$ let $k = -1$.

$\Rightarrow 2xyy' = -y^2 + x^0\mathrm{f}'(x) = 2\dfrac{\ln x}{x} - y^2$

$\Rightarrow \mathrm{f}'(x) = 2\dfrac{\ln x}{x}$

$\Rightarrow \mathrm{f}(x) = (\ln x)^2 + c$

$\Rightarrow y^2 = \dfrac{(\ln x)^2 + 1}{x}$ since $y = 1$ when $x = 1$, and $y^2 = \dfrac{\mathrm{f}(x)}{x}$.

STEP I, 2005, Hints and Answers

Section B: Mechanics

9 Let the centre of mass of the rod be a distance x from A and let the tension in the string attached at B be S. Let moments be taken about A.

$\Rightarrow S + T = W$ and $Wx = 3Sl$.

Therefore, $Wx = 3l\,(W - T)$

Let the upward force supplied by the pivot be Q, and let moments be taken about A.

$\Rightarrow T + Q = W$ and $Ql + 3Tl = Wx$.

$\Rightarrow l\,(W - T) + 3Tl = 3l\,(W - T) \Rightarrow 5Tl = 2Wl \Rightarrow 5T = 2W$.

To determine x:

$$\frac{5T}{2}x = 3l\left(\frac{5T}{2} - T\right) \Rightarrow \frac{5x}{2} = 3l \times \frac{3}{2} \Rightarrow x = \frac{9l}{5}$$

Let the upwards reaction force acting on the rod from the ground be R, and let the frictional force acting on the rod be F. It is essential **not** to assert that the frictional force is μR: this would only be true in limiting equilibrium. A common error is to assume that $F = \mu R$ rather than $F \leqslant \mu R$.

$\Rightarrow R + \dfrac{T}{2}\cos\theta = W$ and $\dfrac{T}{2}\sin\theta = F$.

Also, taking moments about B, $\dfrac{T}{2} \times 3l = W \times \left(3l - \dfrac{9l}{5}\right)\cos\theta = \dfrac{5T}{2} \times \dfrac{6l}{5}\cos\theta$

$\Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \theta = 60^\circ$

$\Rightarrow R = W - \dfrac{T}{2} \times \dfrac{1}{2} = \dfrac{5T}{2} - \dfrac{T}{4} = \dfrac{9T}{4}$

and $F = \dfrac{T}{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{T\sqrt{3}}{4} \leqslant \mu R$

$\Rightarrow \mu \geqslant \dfrac{\sqrt{3}}{9}$

STEP I, 2005, Hints and Answers

10 Let the velocity of mass A be x after it collides with B.

Let the velocity of mass B be v after it collides with A.

Let the velocity of mass B be y after it collides with C.

Let the velocity of mass C be w after it collides with B.

Using the fact that linear momentum is conserved in collisions, and Newton's Law of Restitution (in the form $e \times$ approach speed = separation speed)

$\Rightarrow au + 0 = ax + bv$ and $eu = v - x$

$\Rightarrow au + aeu = bv + av$ since $x = v - eu$

$\Rightarrow v = \dfrac{au(1+e)}{a+b}$

Also, $au - beu = ax + bx \Rightarrow x = \dfrac{u(a-be)}{a+b}$

Similarly, to find y:

$bv + 0 = by + cw$ and $ev = w - y$

$\Rightarrow bv + bev = cw + bw \Rightarrow w = \dfrac{bv(1+e)}{b+c} = \dfrac{abu(1+e)^2}{(a+b)(b+c)}$

Also, $bv - cev = by + cy \Rightarrow y = \dfrac{v(b-ce)}{b+c}$

(i) If the masses a, b and c are such that $\dfrac{a}{b} = \dfrac{b}{c} = e \Rightarrow a = be$ and $b = ce$

$\Rightarrow x = y = 0$ and $w = \dfrac{abu(1+e)^2}{(be+b)(ce+c)} = \dfrac{abu}{bc} = e^2u$

i.e. A and B are at rest and C has velocity e^2u.

(ii) If the masses a, b and c are such that $\dfrac{b}{a} = \dfrac{c}{b} = e \Rightarrow b = ae$ and $c = be$

$\Rightarrow x = \dfrac{u(a-ae^2)}{a+ae} = u(1-e)$

and $v = \dfrac{au(1+e)}{a+ae} = u \Rightarrow y = \dfrac{u(b-be^2)}{b+be} = u(1-e)$

and $w = \dfrac{abu(1+e)^2}{(a+ae)(b+be)} = u$

i.e. C has velocity u and A and B each has velocity $u(1-e)$.

STEP I, 2005, Hints and Answers

11 (i) $\mathbf{r} = 0 \Rightarrow \sin 2t = 0$ and $\cos t = 0 \Rightarrow t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

(ii) $\mathbf{r} = \begin{pmatrix} \sin 2t \\ 2\cos t \end{pmatrix} \Rightarrow \mathbf{v} = \begin{pmatrix} 2\cos 2t \\ -2\sin t \end{pmatrix}$

If $\mathbf{r}$ is perpendicular to $\mathbf{v}$ then $\begin{pmatrix} \sin 2t \\ 2\cos t \end{pmatrix} \cdot \begin{pmatrix} 2\cos 2t \\ -2\sin t \end{pmatrix} = 0$

$\Rightarrow 2\sin 2t \cos 2t - 4\sin t \cos t = 0$

$\Rightarrow 2\sin 2t\,(\cos 2t - 1) = 0$

$\Rightarrow t = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$ (recall that $t < 2\pi$).

(iii) $\begin{pmatrix} \sin 2t \\ 2\cos t \end{pmatrix}$ is parallel to $\begin{pmatrix} 2\cos 2t \\ -2\sin t \end{pmatrix} \Rightarrow \frac{\sin 2t}{2\cos t} = \frac{2\cos 2t}{-2\sin t}$

$\Rightarrow -2\sin t \sin 2t = 4\cos t \cos 2t$

$\Rightarrow -\sin^2 t \cos t = \cos t \cos 2t$

$\Rightarrow \cos t\left(\cos 2t + \sin^2 t\right) = 0$

$\Rightarrow \cos t = 0 \Rightarrow \mathbf{r} = \mathbf{0}$ or $1 - 2\sin^2 t + \sin^2 t = 0$

$\Rightarrow 1 = \sin^2 t \Rightarrow \cos^2 t = 0 \Rightarrow \mathbf{r} = \mathbf{0}$

(iv) The distance of the particle from the origin $= \sqrt{\sin^2 2t + 4\cos^2 t}$

$= \sqrt{4\sin^2 t \cos^2 t + 4\cos^2 t}$

$= \sqrt{8\cos^2 t - 4\cos^4 t}$

Using differentiation to find the maximum or minimum distance

$\Rightarrow \frac{16\cos t \sin t - 16\cos^3 t \sin t}{2\sqrt{8\cos^2 t - 4\cos^4 t}} = 0$

$\Rightarrow \cos t \sin t\left(1 - \cos^2 t\right) = 0$

$\Rightarrow \sin t = 0 \Rightarrow$ distance $= 2$

or $\cos t = 0 \Rightarrow$ distance $= 0$

or $\cos^2 t = 1 \Rightarrow$ distance $= 2$.

The path of the particle is a "figure of 8".

Alternatively, $\sqrt{\sin^2 2t + 4\cos^2 t}$

$= \sqrt{4\sin^2 t \cos^2 t + 4\cos^2 t}$

$= \sqrt{4\cos^2 t\left(\sin^2 t + 1\right)}$

$= 2\sqrt{\left(1 - \sin^2 t\right)\left(1 + \sin^2 t\right)}$

$= 2\sqrt{1 - \sin^4 t} \leqslant 2$

STEP I, 2005, Hints and Answers

Section C: Probability and Statistics

12 Venn diagrams are very helpful when answering this question. This question **cannot** be answered using a tree diagram: you are not told the probability that a hobbit wears a hat given that he wears a pipe, so you cannot draw a tree with 0.4 on a branch subsequent to 0.7 (or vice versa: you are not told the probability that a hobbit smokes a pipe given that he wears a hat).

(a) Since $P\,(\text{pipe but no hat}) = p$, we can state that $P\,(\text{pipe and hat}) = 0.7 - p$ and $P\,(\text{hat but no pipe}) = 0.4 - (0.7 - p) = p - 0.3$. Clearly therefore, $p \geqslant 0.3$.

Furthermore, $P\,(\text{pipe}) + P\,(\text{hat}) = 1.1 \Rightarrow P\,(\text{pipe and hat}) \geqslant 0.1$.

Therefore $0.7 - p \geqslant 0.1 \Rightarrow p \leqslant 0.6$

(b) $P\,(\text{wizard wears hat, cloak and ring}) = 0.1$ and $P\,(\text{wizard wears none}) = 0.05$

Let $x = P\,(\text{wizard wears hat and cloak but not ring})$.

Let $y = P\,(\text{wizard wears hat and ring but not cloak})$.

Let $z = P\,(\text{wizard wears ring and cloak but not hat})$.

Therefore none of the following can be negative:

$P\,(\text{wizard wears only hat}) = 0.6 - x - y$

$P\,(\text{wizard wears only cloak}) = 0.7 - x - z$

$P\,(\text{wizard wears only ring}) = 0.3 - y - z$

Since $0.1 + 0.05 + (0.6 - x - y) + (0.7 - x - z) + (0.3 - y - z) + x + y + z = 1$

$\Rightarrow x + y + z = 0.75 = P\,(\text{wizard wears exactly two items})$

$P\,(\text{wizard wears hat but not ring given that he is wearing a cloak}) = q = \dfrac{x}{0.8}$

To determine the range of x, let $x = 0.6 - k$

Then $P\,(\text{hat only}) = k - y$ and $P\,(\text{cloak only}) = k - z + 0.1$; remember that neither of these can be negative.

Therefore $0.95 = (k - y) + (0.6 - k) + (k - z + 0.1) + 0.1 + y + z + (0.3 - y - z)$

$\Rightarrow 0.95 = 1.1 + k - (y + z) \Rightarrow k = y + z - 0.15$

Since $k - y \geqslant 0 \Rightarrow z \geqslant 0.15$, and since $k - z + 0.1 \geqslant 0 \Rightarrow y - 0.05 \geqslant 0 \Rightarrow y \geqslant 0.05$

Therefore $0.2 \leqslant y + z \leqslant 0.3$

$\Rightarrow 0.05 \leqslant k \leqslant 0.15$

$\Rightarrow 0.45 \leqslant x \leqslant 0.55$

$\Rightarrow \dfrac{9}{16} \leqslant q \leqslant \dfrac{11}{16}$

STEP I, 2005, Hints and Answers

13 Although X is not Normally distributed, you can picture it as something similar: a sketch of a (symmetrical) graph will help you.

(a) Analogously to a Normal distribution:

$P\left(\mu - \frac{1}{2}\sigma \leqslant X \leqslant \mu + \sigma\right) = a - (1 - b) = a + b - 1$

$P\left(X \leqslant \mu + \frac{1}{2}\sigma \mid X \geqslant \mu - \frac{1}{2}\sigma\right) = \dfrac{b - (1-b)}{b} = \dfrac{2b-1}{b}.$

(b) A tree diagram is useful: the first pair of branches indicates the type of milk (skimmed with probability 0.6 or full-fat with probability 0.4), and then branches to represent the volume of the carton being more or less than the stated amounts.

(i) It is important to recognise that you are being asked for a **conditional** probability.

$P\,(\text{volume} > 500 \text{ ml given that volume} < 505 \text{ ml})$

$= \dfrac{0.6\,(b - 0.5) + 0.4\,(a - b)}{0.6b + 0.4a}$

$= \dfrac{0.4a + 0.2b - 0.3}{0.6b + 0.4a}$

(ii) The stated information is saying that $0.6b + 0.4a = 0.7$ and $\dfrac{0.4 \times \frac{1}{2}}{0.6b + 0.4 \times \frac{1}{2}} = \dfrac{1}{3}$

(again, notice the language of a **conditional** probability in the second statement).

Therefore $b = \dfrac{2}{3}$ and $a = \dfrac{3}{4}$.

STEP I, 2005, Hints and Answers

14 (i) $m + P(0 \leqslant X < \infty) = 1 \Rightarrow k = 1 - m.$

(ii) Since the cumulative distribution function of X is $k(1 - e^{-x})$, the probability density function of X is ke^{-x}, the derivative of the cdf.

$$\Rightarrow E(X) = -1 \times m + (1-m)\int_0^{\infty} xe^{-x}\,\mathrm{d}x = -m + (1-m) \times 1 = 1 - 2m$$

(iii) $$\operatorname{var}(X) = \left[(-1)^2 \times m + (1-m)\int_0^{\infty} x^2 e^{-x}\,\mathrm{d}x\right] - (1-2m)^2$$

$$= m + (1-m) \times 2 - \left(1 - 4m + 4m^2\right) = 1 + 3m - 4m^2$$

Let the median value of X be T.

$$\Rightarrow k\left(1 - e^{-T}\right) = \frac{1}{2} - m$$

$$\Rightarrow 1 - e^{-T} = \frac{1-2m}{2-2m}$$

$$\Rightarrow e^{-T} = \frac{1}{2-2m}$$

$$\Rightarrow -T = -\ln(2-2m)$$

$$\Rightarrow T = \ln(2-2m)$$

(iv) $$E\left(\sqrt{|X|}\right) = m\sqrt{|-1|} + (1-m)\int_0^{\infty} \sqrt{x}e^{-x}\,\mathrm{d}x$$

$$= m + (1-m)\int_0^{\infty} ue^{-u^2}\,2u\,\mathrm{d}u \text{ using } u^2 = x$$

$$= m + 2(1-m)\frac{\sqrt{\pi}}{4} \text{ using the given result}$$

$$= m + (1-m)\frac{\sqrt{\pi}}{2}$$