Question 1

(i) $I_n = \int_0^1 \arctan x \,.\, x^n \,\mathrm{d}x$ — **M1** Use of intgrn. by parts (parts correct way round)

$= \left[\arctan x \,.\, \frac{x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{1}{1+x^2}.\frac{x^{n+1}}{n+1}\,\mathrm{d}x$ — **A1** Correct to here

$= \left(\frac{\pi}{4}.\frac{1}{n+1} - 0\right) - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}\,\mathrm{d}x$

$\Rightarrow\ (n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,\mathrm{d}x$ — **A1** Given Answer legitimately established **3**

Setting $n = 0$, $I_0 = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x$ — **M1** Attempt to solve this using recognition/ substitution

$= \frac{\pi}{4} - \left[\frac{1}{2}\ln\left(1+x^2\right)\right]$ — **M1** Log integral involved

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$ — **A1 CAO** **3**

(ii) $n \to n+2$ in given result:

$(n+3)I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}\,\mathrm{d}x$ — **B1** Noted or used somewhere

$(n+3)I_{n+2} + (n+1)I_n = \frac{\pi}{2} - \int_0^1 \frac{x^{n+1}\left(1+x^2\right)}{1+x^2}\,\mathrm{d}x$ — **M1** Adding and cancelling ready to integrate

$= \frac{\pi}{2} - \frac{1}{n+2}$ — **A1 CAO** **3**

Setting $n = 0$ and then $n = 2$ in this result (or equivalent involving integrals):

$3I_2 + I_0 = \frac{\pi}{2} - \frac{1}{2}$ and $5I_4 + 3I_2 = \frac{\pi}{2} - \frac{1}{4}$ — **M1**

Eliminating I_2 and using value for I_0 to find I_4 — **M1** By subtracting, or equivalent

$I_4 = \frac{1}{20}\left(1 + \pi - 2\ln 2\right)$ — **A1 FT** from their I_0 value **3**

(iii) For $n = 1$, $5I_4 = A - \frac{1}{2}\left(-1 + \frac{1}{2}\right) = A + \frac{1}{4}$

$= \frac{1}{4} + \frac{1}{4}\pi - \frac{1}{2}\ln 2$ — **M1** Comparing formula with found I_4 value

and the result is true for $n = 1$ provided

$A = \frac{1}{4}\pi - \frac{1}{2}\ln 2$ — **A1 FT** from their I_4 value **2**

Assuming $(4k+1)I_{4k+1} = A - \tfrac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r}$ **M1** For a clearly stated induction hypothesis

(or a fully explained "if … then …" at end)

$(4k+5)I_{4k+4} + (4k+3)I_{4k+2} = \frac{\pi}{2} - \frac{1}{4k+4}$ **B1**

$(4k+3)I_{4k+2} + (4k+1)I_{4k} = \frac{\pi}{2} - \frac{1}{4k+2}$ **B1**

Subtracting:

$(4k+5)I_{4k+4} = (4k+1)I_{4k} + \frac{1}{4k+2} - \frac{1}{4k+4}$ **M1**

$= A - \tfrac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r} + \frac{1}{4k+2} - \frac{1}{4k+4}$ **M1** Use of assumed result

$= A - \tfrac{1}{2}\sum_{r=1}^{2k}(-1)^r \frac{1}{r} - \tfrac{1}{2}(-1)^{2k+1}\frac{1}{2k+1} - \tfrac{1}{2}(-1)^{2k+2}\frac{1}{2k+2}$

$= A - \tfrac{1}{2}\sum_{r=1}^{2(k+1)}(-1)^r \frac{1}{r}$ **A1** A clear demonstration of how the two extra terms fit must be given **6**

Question 2

Let $x_n = X$. Then $x_{n+1} = \dfrac{aX-1}{X+b}$ and $x_{n+2} = \dfrac{a\left(\dfrac{aX-1}{X+b}\right)-1}{\left(\dfrac{aX-1}{X+b}\right)+b}$ **M1 A1** Correct, unsimplified

i.e. $x_{n+2} = \dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}$ **M1** Attempt to remove "fractions within fractions"

A1 Correct, simplified **4**

(i) If $x_{n+1} = x_n$ then $aX - 1 = X^2 + bX$ **M1**

$\Rightarrow 0 = X^2 - (a-b)X + 1$ **A1**

If $x_{n+2} = x_n$ then

$(a^2-1)X - (a+b) = (a+b)X^2 + (b^2-1)X$ **M1**

$\Rightarrow 0 = (a+b)\{X^2 - (a-b)X + 1\}$ **M1 A1** Factorisation

and so, for $x_{n+2} = x_n$ but $x_{n+1} \neq x_n$

we must have $a + b = 0$ **A1** **Given Answer** fully justified & clearly stated

(No marks for setting $b = -a$, for instance, and showing sufficiency)

For "comparing coefficients" approach (must be all 3 terms) max. 3/4 **6**

(ii) $x_{n+4} = \dfrac{(a^2-1)x_{n+2}-(a+b)}{(a+b)x_{n+2}+(b^2-1)}$ **M1** Use of the two-step result from earlier

$= \dfrac{(a^2-1)\left[\dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}\right]-(a+b)}{(a+b)\left[\dfrac{(a^2-1)X-(a+b)}{(a+b)X+(b^2-1)}\right]+(b^2-1)}$ **A1** Correct, unsimplified, in terms of X

If $x_{n+4} = x_n$ then **M1** Equating

$(a^2-1)^2X - (a+b)(a^2-1) - (a+b)^2X - (a+b)(b^2-1)$ **A1** LHS correct

$= (a+b)(a^2-1)X^2 - (a+b)^2X + (a+b)(b^2-1)X^2 + (b^2-1)^2X$ **A1** RHS correct

$\Rightarrow 0 = (a+b)(a^2+b^2-2)X^2 - [(a^2-1)^2 - (b^2-1)^2]X + (a+b)(a^2+b^2-2)$

M1 Good attempt to simplify

$\Rightarrow 0 = (a+b)(a^2+b^2-2)\{X^2 - (a-b)X + 1\}$ **M1** Factorisation attempt

A1 A1 Partial; complete

and the sequence has period 4 if and only if

$a^2 + b^2 = 2,\ a + b \neq 0,\ X^2 - (a-b)X + 1 \neq 0$ **B1** **CAO** Correct final statement

[Ignore any discussion or confusion regarding issues of necessity and sufficiency]

NB Some candidates may use the one-step result repeatedly and get to x_{n+4} via x_{n+3}:

$x_{n+3} = \dfrac{(a^3-2a-b)X-(a^2+ab+b^2-1)}{(a^2++ab+b^2-1)X-(a+2b+b^3)}$ and $x_{n+4} = \dfrac{ax_{n+3}-1}{x_{n+3}+b}$ starts the process; then as above.

10

ALT. Consider the two-step sequence $\{\ldots, x_n, x_{n+2}, x_{n+4}, \ldots\}$ given by (assuming $a + b \neq 0$)

$$x_{n+2} = \frac{\left(\dfrac{a^2-1}{a+b}\right)X - 1}{X + \left(\dfrac{b^2-1}{a+b}\right)} \equiv \frac{AX-1}{X+B},$$ which is clearly of exactly the same form as before.

Then $x_{n+4} = x_n$ if and only if $a + b \neq 0$, $X^2 - (a-b)X + 1 \neq 0$ (from $x_{n+4} \neq x_{n+2}$ and $x_{n+4} \neq x_n$ as before), together with the condition $A + B = 0$ (also from previous work); i.e. $\dfrac{a^2-1}{a+b} + \dfrac{b^2-1}{a+b} = 0$, which is equivalent to $a^2 + b^2 - 2 = 0$ since $a + b \neq 0$.

Note that it is not necessary to consider $x_{n+4} \neq x_{n+3}$ since if $x_{n+4} = x_{n+3} = X$ then the sequence would be constant.

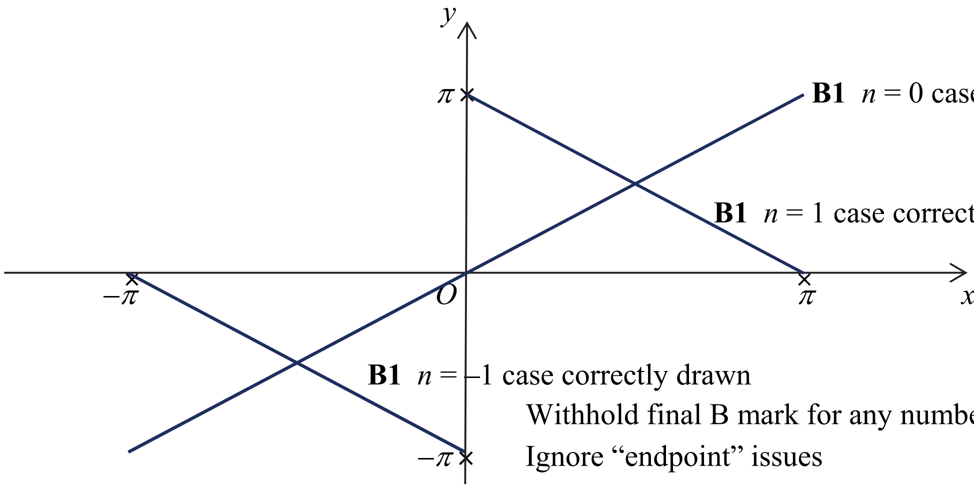
Question 3

(i) $\sin y = \sin x \;\Rightarrow\; y = n\pi + (-1)^n x$

$n = -1$: $\quad y = -\pi - x$ **B1**

$n = 0$: $\quad y = x$ **B1**

$n = 1$: $\quad y = \pi - x$ **B1** Withhold final B mark for any number of extra eqns.



B1 $n = 0$ case correctly drawn

B1 $n = 1$ case correctly drawn

B1 $n = -1$ case correctly drawn

Withhold final B mark for any number of extra lines

Ignore "endpoint" issues

6

(ii) $\sin y = \frac{1}{2}\sin x \;\Rightarrow\; \cos y \dfrac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\cos x$ **M1** Implicit diffn. attempt (or equivalent)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x}{2\cos y}$$ **A1** Correct

$$= \frac{\cos x}{2\sqrt{1-\frac{1}{4}\sin^2 x}} \quad \text{or} \quad \frac{\cos x}{\sqrt{4-\sin^2 x}}$$ **A1** Correct and in terms of x only

3

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\left(4-\sin^2 x\right)^{\frac{1}{2}}.-\sin x - \cos x.\frac{1}{2}\left(4-\sin^2 x\right)^{-\frac{1}{2}}.-2\sin x\cos x}{4-\sin^2 x}$$

M1 For use of the *Quotient Rule* (or equivalent)

M1 For use of the *Chain Rule* for d/dx(denominator)

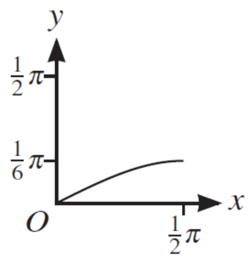
A1

$$= \frac{-\sin x\left(4-\sin^2 x\right)+\cos^2 x.\sin x}{\left(4-\sin^2 x\right)^{\frac{3}{2}}}$$ **M1** Method for getting correct denominator

$$= \frac{\sin x\left\{\cos^2 x - 4 + \sin^2 x\right\}}{\left(4-\sin^2 x\right)^{\frac{3}{2}}}$$

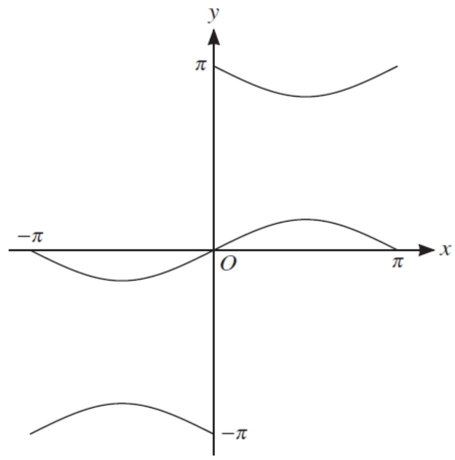
$$= \frac{-3\sin x}{\left(4-\sin^2 x\right)^{\frac{3}{2}}}$$ **A1** **Given Answer** correctly obtained from $c^2 + s^2 = 1$

5



Initially, $\frac{dy}{dx} = \frac{1}{2}$ at (0, 0) increasing to a maximum at $(\frac{\pi}{2}, \frac{\pi}{6})$ since $\frac{d^2y}{dx^2} < 0$

B1 (Gradient and coordinate details unimportant unless graphs look silly as a result)



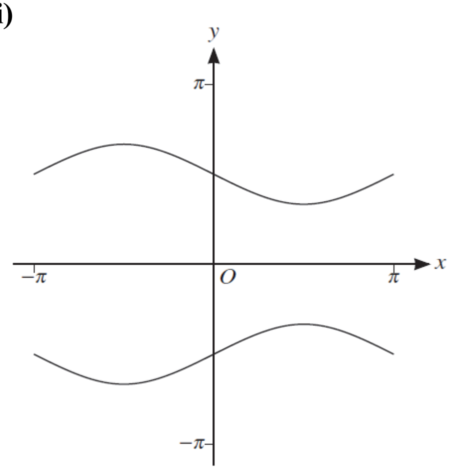
B1 Reflection symmetry in $x = \frac{\pi}{2}$

B1 Rotational symmetry about O

B1 Reflection symmetry in $y = \pm\frac{\pi}{2}$

4

(iii)



B1 RHS correct

B1 LHS correct

2

Question 4

(i) Setting f(x) = 1 in **(*)** gives

$$\left(\int_a^b \mathrm{g}(x)\,\mathrm{d}x\right)^2 \le \left(\int_a^b 1\,\mathrm{d}x\right)\left(\int_a^b [\mathrm{g}(x)]^2\,\mathrm{d}x\right)$$ **B1** Clearly stated

Let g(x) = e^x : $\left(\int_a^b \mathrm{e}^x\,\mathrm{d}x\right)^2 \le (b-a)\left(\int_a^b \mathrm{e}^{2x}\,\mathrm{d}x\right)$ **M1**

$\Rightarrow\ \left(\mathrm{e}^b - \mathrm{e}^a\right)^2 \le (b-a).\tfrac{1}{2}\left(\mathrm{e}^{2b} - \mathrm{e}^{2a}\right)$

$\Rightarrow\ \left(\mathrm{e}^b - \mathrm{e}^a\right)^2 \le (b-a).\tfrac{1}{2}\left(\mathrm{e}^b - \mathrm{e}^a\right)\left(\mathrm{e}^b + \mathrm{e}^a\right)$

$\Rightarrow\ \mathrm{e}^b - \mathrm{e}^a \le \tfrac{1}{2}(b-a)\left(\mathrm{e}^b + \mathrm{e}^a\right)$ **A1**

Choosing $a = 0$ and $b = t$ gives **M1**

$\mathrm{e}^t - 1 \le \tfrac{1}{2}t\left(\mathrm{e}^t + 1\right) \ \Rightarrow\ \dfrac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1} \le \tfrac{1}{2}t$ **A1** **Given Answer** legitimately obtained **5**

(ii) Setting f(x) = x, $a = 0$ and $b = 1$ in **(*)** gives

$$\left(\int_0^1 x\,\mathrm{g}(x)\,\mathrm{d}x\right)^2 \le \left(\int_0^1 x^2\,\mathrm{d}x\right)\left(\int_0^1 [\mathrm{g}(x)]^2\,\mathrm{d}x\right)$$ **B1** Clearly stated

Choosing g(x) = $\mathrm{e}^{-\frac{1}{4}x^2}$ gives **M1**

$$\left(\int_0^1 x\mathrm{e}^{-\frac{1}{4}x^2}\,\mathrm{d}x\right)^2 \le \tfrac{1}{3}\left(1^3 - 0^3\right)\left(\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x\right)$$

$$\left(\left[-2\mathrm{e}^{-\frac{1}{4}x^2}\right]_0^1\right)^2 \le \tfrac{1}{3}\left(\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x\right)$$ **A1 A1** LHS, RHS correct

$$\Rightarrow\ \int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \ge 3\left(-2\left[-\mathrm{e}^{-\frac{1}{4}} + 1\right]\right)^2$$

i.e. $\displaystyle\int_0^1 \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x \ge 12\left(1 - \mathrm{e}^{-\frac{1}{4}}\right)^2$ **A1** **Given Answer** legitimately obtained **5**

(iii) With f(x) = 1, g(x) = $\sqrt{\sin x}$, $a = 0$, $b = \tfrac{1}{2}\pi$, **M1** Correct choice for f, g (or v.v.)

(*) becomes **M1** Any sensible f, g used in **(*)**

$$\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,\mathrm{d}x\right)^2 \le \tfrac{1}{2}\pi\left(\int_0^{\frac{1}{2}\pi} \sin x\,\mathrm{d}x\right)$$ **A1**

RHS is $\tfrac{1}{2}\pi\,\left[-\cos x\right]_0^1 = \tfrac{1}{2}\pi$

(and since LHS is positive) we have $\displaystyle\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,\mathrm{d}x \le \sqrt{\frac{\pi}{2}}$

A1 RH half of **Given** inequality obtained from fully correct working **4**

With $f(x) = \cos x$, $g(x) = \sqrt[4]{\sin x}$, $a = 0$, $b = \frac{1}{2}\pi$, **M1** Correct choice for f, g (or v.v.)

(*) gives

$$\left(\int_0^{\frac{1}{2}\pi} \cos x.(\sin x)^{\frac{1}{4}}\,dx\right)^2 \le \left(\int_0^{\frac{1}{2}\pi} \cos^2 x\,dx\right)\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx\right)$$ **A1**

$$\text{LHS} = \left(\left[\tfrac{4}{5}(\sin x)^{\frac{5}{4}}\right]_0^{\frac{1}{2}\pi}\right)^2 = \tfrac{16}{25}$$ **M1 A1** By recognition/substitution integration

and $$\int_0^{\frac{1}{2}\pi} \cos^2 x\,dx = \int_0^{\frac{1}{2}\pi} \left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x\right)dx$$ **M1**

$$= \left(\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\frac{1}{2}\pi}\right)^2 = \tfrac{1}{4}\pi$$ **A1**

Giving the required LH half of the **Given** inequality:

$$\tfrac{16}{25} \le \tfrac{1}{4}\pi\left(\int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx\right) \quad \text{i.e.} \quad \int_0^{\frac{1}{2}\pi} \sqrt{\sin x}\,dx \ge \frac{64}{25\pi}$$ **6**

Withhold the last A mark if final result is not arrived at

Question 5

(i)	$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{2a}{2at} = \dfrac{1}{t}$	**M1** Finding gradt. of tgt. (or by implicit diffn.)
	$\Rightarrow$ Grad. nml. at P is $-p$	**A1**
	$\Rightarrow$ Eqn. nml. to C at P is $x - 2ap = -p(x - ap^2)$	**B1 FT** any form, e.g. $y = -px + ap(2 + p^2)$
	Nml. meets C again when $x = an^2$, $y = 2an$	**M1** Substd. into nml. eqn.
	$\Rightarrow$ $2an = -pan^2 + ap(2 + p^2)$	
	$\Rightarrow$ $0 = pn^2 + 2n - p(2 + p^2)$	
	$\Rightarrow$ $0 = (n - p)(pn + [2 + p^2])$	**M1** Solving attempt
	Since $n = p$ at P, it follows that $n = -\dfrac{2 + p^2}{p}$ at N	
	i.e. $n = -\left(p + \dfrac{2}{p}\right)$	**A1 Given Answer** legitimately obtained **6**

(ii)	Distance $P(ap^2, 2ap)$ to $N(an^2, 2an)$ is given by	
	$PN^2 = \left[a(p^2 - n^2)\right]^2 + \left[2a(p - n)\right]^2$	**M1**
	$= a^2(p - n)^2\left\{(p + n)^2 + 4\right\}$	
	$= a^2\left(2p + \dfrac{2}{p}\right)^2\left\{\left(\dfrac{-2}{p}\right)^2 + 4\right\}$	**M1** Substituting for n
	$= 16a^2\left(\dfrac{p^2 + 1}{p}\right)^2\left\{\dfrac{1 + p^2}{p}\right\} = 16a^2\dfrac{(p^2 + 1)^3}{p^4}$	**A1 Given Answer** legitimately obtained **3**

$\dfrac{d(PN^2)}{dp} = 16a^2\dfrac{d(p^2 + 3 + 3p^{-2} + p^{-4})}{dp}$	**M1** Differentiation directly, or by the Quotient Rule
$= 16a^2(2p - 6p^{-3} - 4p^{-5})$	**A1** Correct, unsimplified
$= 32a^2\dfrac{p^6 - 3p^2 - 2}{p^5}$	
$= \dfrac{32a^2}{p^5}(p^2 + 1)^2\left[p^2 - 2\right]$	

Note that $\dfrac{d(PN^2)}{dp} = 16a^2\left\{\dfrac{p^4.3(p^2 + 1)^2.2p - (p^2 + 1)^3.4p^3}{p^8}\right\}$

$= \dfrac{32a^2}{p^8}.p^3(p^2 + 1)^2\left[3p^2 - 2(p^2 + 1)\right]$ by the Quotient Rule

$\dfrac{d(PN^2)}{dp} = 0$ only when $p^2 = 2$	**A1 Given Answer** fully shown
Justification that it is a minimum	**E1**
(either by examining the sign of $\dfrac{d(PN^2)}{dp}$ or by explaining that PN^2 cannot be maximised	**4**

(iii) Grad. PQ is $\dfrac{2}{p+q}$ **B1**

Grad. NQ is $\dfrac{2}{n+q}$ or $\dfrac{2}{q-p-\dfrac{2}{p}}$ **B1**

Since $\angle PQN = 90°$ (by "$\angle$ in a semi-circle"; i.e. *Thales Theorem*)

$$\frac{2}{p+q}\times\frac{2}{q-p-\dfrac{2}{p}}=-1$$ **M1**

$$\Rightarrow\ 4=(p+q)\left(p-q+\frac{2}{p}\right)=p^2-q^2+2+\frac{2q}{p}$$

$$\Rightarrow\ 2=p^2-q^2+\frac{2q}{p}$$ **A1** **Given Answer** legitimately obtained **4**

PN minimised when $p^2=2\ \Rightarrow\ q^2=\dfrac{2q}{p}$ **M1** Substituted into given expression

$\Rightarrow\ q=0$ or $q=\dfrac{2}{p}=\pm\sqrt{2}$ **A1**

But $q=\pm\sqrt{2}\ \Rightarrow q=p$ (which is not the case) **E1** Other cases must be ruled out

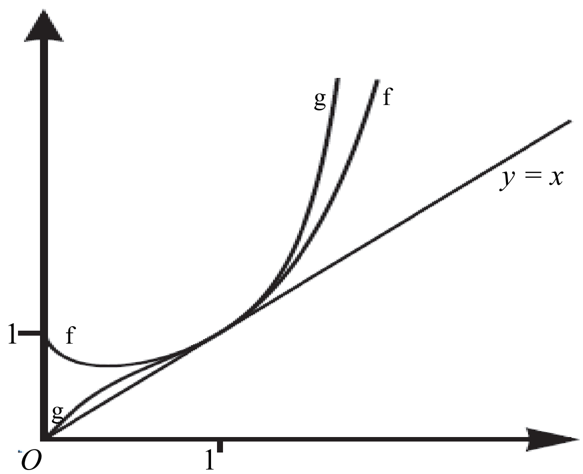
<u>Special Case:</u> 1/3 for substg. $q=0$ and verifying that $p^2=2$ **3**

Question 6

(i)		
When $n = 1$ $$S_1 = 1 \leq 2\sqrt{1} - 1$$	**B1**	Clear verification.
Assume that the statement is true when $n = k$: $$S_k \leq 2\sqrt{k} - 1$$	**B1**	Must be clear that this is assumed.
Then $$S_{k+1} = S_k + \frac{1}{\sqrt{k+1}}$$	**M1**	Linking S_{k+1} and S_k
$$\leq 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}}$$	**M1**	Using assumed result
Sufficient to prove: $$2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \leq 2\sqrt{k+1} - 1$$	**M1**	
i.e. $2\sqrt{k(k+1)} + 1 \leq 2(k+1)$	**A1**	Multiplying by $\sqrt{k+1}$ or putting over a common denominator
i.e. $2\sqrt{k(k+1)} \leq 2k + 1$		
i.e. $4k^2 + 4k \leq 4k^2 + 4k + 1$	**A1**	
Which is clearly true. Therefore by induction the statement is true for all $n \geq 1$.	**B1**	Clear conclusion showing logic of induction.
	[8]	
(ii)		
Required to prove: $$(4k+1)^2(k+1) > (4k+3)^2 k$$	**M2**	Squaring given inequality
i.e. $16k^3 + 24k^2 + 9k + 1 > 16k^3 + 24k^2 + 9k$ which is clearly true.	**A1**	
	[3]	
When $n = 1$: $$S_1 = 1 \geq 2 + \frac{1}{2} - c$$	**M1**	
So we need $c \geq \frac{3}{2}$	**A1**	
Prove $c = \frac{3}{2}$ works using induction	**M1**	
Assume holds when $n = k$: $$S_k \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} - \frac{3}{2}$$	**M1**	Allow a general c.
Then $$S_{k+1} = S_k + \frac{1}{\sqrt{k+1}} \geq 2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - c$$	**M1**	
Sufficient to prove: $$2\sqrt{k} + \frac{1}{2\sqrt{k}} + \frac{1}{\sqrt{k+1}} - c \geq 2\sqrt{k+1} + \frac{1}{2\sqrt{k+1}} - c$$	**A1**	
i.e. $4k\sqrt{k+1} + \sqrt{k+1} + 2\sqrt{k} \geq 4\sqrt{k}(k+1) + \sqrt{k}$	**A1A1**	
Which simplifies to the previously proved inequality. No further restrictions on c, so the minimum value is $c = \frac{3}{2}$	**B1**	
	[9]	

Question 7

(i)	For $0 < x < 1$, x is positive and $\ln x$ is negative		
	so $0 > x \ln x > \ln x$		
	$\Rightarrow$ $e^0 > e^{x \ln x} > e^{\ln x}$ or $\ln 1 > \ln x^x > \ln x$		
	$\Rightarrow$ $(1 >)$ $f(x) > x$ since ln is a strictly increasing fn.	**B1**	
	Again, since $\ln x < 0$, it follows that		
	$\ln x < f(x) \ln x < x \ln x$		
	$\Rightarrow$ $\ln x < \ln\{g(x)\} < \ln\{f(x)\}$	**M1** Suitably coherent justification	
	$\Rightarrow$ $x < g(x) < f(x)$	**A1** **Given Answer** legitimately obtained	
	For $x > 1$, $\ln x > 0$ and so $x < f(x) < g(x)$	**B1** No justification required	**4**
(ii)	$\ln\{f(x)\} = x \ln x$	**M1** Taking logs and attempting implicit diffn. **Alt.** Writing $y = e^{x \ln x}$ and diffg.	
	$\frac{1}{f(x)}.f'(x) = x.\frac{1}{x} + 1.\ln x$ i.e. $f'(x) = (1 + \ln x)f(x)$	**A1**	
	$f'(x) = 0$ when $1 + \ln x = 0$, $\ln x = -1$, $x = e^{-1}$	**A1**	**3**
(iii)	$\lim_{x \to 0} (f(x)) = \lim_{x \to 0} (e^{x \ln x}) = \lim_{x \to 0} (e^0) = 1$	**B1** Suitably justified	
	$\lim_{x \to 0} (g(x)) = \lim_{x \to 0} (x^{f(x)}) = \lim_{x \to 0} (x^1) = 0$	**B1** May just be stated	
	Alt. $\lim_{x \to 0} (g(x)) = \lim_{x \to 0} (e^{f(x) \ln x}) = \lim_{x \to 0} (e^{\ln x}) = \lim_{x \to 0} (x) = 0$		**2**
(iv)	For $y = \frac{1}{x} + \ln x$ $(x > 0)$,		
	$\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{x}$ or $\frac{x-1}{x^2} = 0$ …	**M1** Diffg. and equating to zero	
	… when $x = 1$	**A1** From correct derivative	
	For $x = 1-$, $\frac{dy}{dx} < 0$ and for $x = 1+$, $\frac{dy}{dx} > 0$	**M1** Method for deciding	
	$(1, 1)$ is a MINIMUM of $y = \frac{1}{x} + \ln x$	**A1**	
	(Since there are no other TPs or discontinuities)		
	$y \geq 1$ for all $x > 0$	Conclusion must be made for all 4 marks	**4**
	$\ln(g(x)) = f(x) \ln x$	**M1** Taking logs and attempting implicit diffn.	
	$\frac{1}{g(x)}.g'(x) = f(x).\frac{1}{x} + \ln x\{f(x)(1 + \ln x)\}$	**A1** using $f'(x)$ from **(ii)**	
	$\Rightarrow$ $g'(x) = f(x).g(x)\left\{\frac{1}{x} + \ln x + (\ln x)^2\right\}$		
	$\geq f(x).g(x)\{1 + (\ln x)^2\}$	**M1** using previous result of **(iv)**	
	> 0 since f, g > 0 from **(i)**		
	and $1 + (\ln x)^2 \geq 1 > 0$	**A1** **Given Answer** fully justified	**4**



B1 One of f, g correct …
B1 Both correct …
… relative to $y = x$

B1 All three passing thro' (1, 1)

3

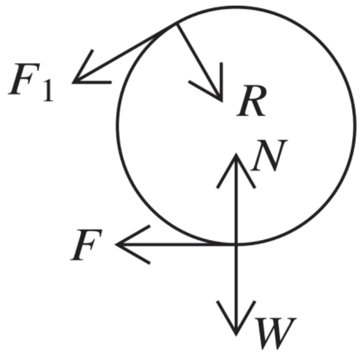
Question 8

Line thro' A perpr. to BC is $\mathbf{r} = \mathbf{a} + \lambda\,\mathbf{u}$	**B1**
Line thro' B perpr. to CA is $\mathbf{r} = \mathbf{b} + \mu\,\mathbf{v}$	**B1**
Lines meet when $(\mathbf{r} = \mathbf{p} =)\ \mathbf{a} + \lambda\,\mathbf{u} = \mathbf{b} + \mu\,\mathbf{v}$	**M1** Equated
$\Rightarrow\ \mathbf{v} = \dfrac{1}{\mu}(\mathbf{a} - \mathbf{b} + \lambda\,\mathbf{u})$	**A1**
Since $\mathbf{v}$ is perpr. to CA, $(\mathbf{a} - \mathbf{b} + \lambda\,\mathbf{u}) \bullet (\mathbf{a} - \mathbf{c}) = 0$	**M1**
$\Rightarrow\ (\mathbf{a} - \mathbf{b}) \bullet (\mathbf{a} - \mathbf{c}) + \lambda\,\mathbf{u} \bullet (\mathbf{a} - \mathbf{c}) = 0$	**A1** Correctly multiplied out
$\Rightarrow\ \lambda = \dfrac{(\mathbf{b} - \mathbf{a}) \bullet (\mathbf{a} - \mathbf{c})}{\mathbf{u} \bullet (\mathbf{a} - \mathbf{c})}$	**M1** Re-arranging for λ
	A1 Correct (any sensible form)
$\Rightarrow\ \mathbf{p} = \mathbf{a} + \left(\dfrac{(\mathbf{b} - \mathbf{a}) \bullet (\mathbf{a} - \mathbf{c})}{\mathbf{u} \bullet (\mathbf{a} - \mathbf{c})}\right)\mathbf{u}$	**A1** **FT** their λ (if only **a**, **b**, **c**, **u** involved)
	9
$\overrightarrow{CP} = \mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} + \lambda\mathbf{u}$	**B1** **FT** their λ
Attempt at $\overrightarrow{CP} \bullet \overrightarrow{AB}$	**M1**
$= (\mathbf{a} - \mathbf{c} + \lambda\mathbf{u}) \bullet (\mathbf{b} - \mathbf{a})$	**A1** Correct to here
$= (\mathbf{a} - \mathbf{c}) \bullet (\mathbf{b} - \mathbf{a}) + \lambda\mathbf{u} \bullet (\mathbf{b} - \mathbf{a})$	
Now $\mathbf{u} \bullet (\mathbf{b} - \mathbf{c}) = 0$ since $\mathbf{u}$ perpr. to BC	**M1**
$\Rightarrow\ \mathbf{u} \bullet \mathbf{b} = \mathbf{u} \bullet \mathbf{c}$	**A1**
so that $\overrightarrow{CP} \bullet \overrightarrow{AB} = (\mathbf{a} - \mathbf{c}) \bullet (\mathbf{b} - \mathbf{a}) + \lambda\mathbf{u} \bullet (\mathbf{c} - \mathbf{a})$	**M1** Substituted in
$= (\mathbf{a} - \mathbf{c}) \bullet (\mathbf{a} - \mathbf{b} + \lambda\mathbf{u})$	**M1 A1** Factorisation attempt; correct
$= 0$ from boxed line above	**A1** **E1** Statement; justified
$\Rightarrow\ CP$ is perpr. to AB	**E1** For final, justified statement **11**

Notice that the “value” of   is never actually required

Any candidate who states the result is true because P is the *orthocentre* of ΔABC may be awarded **B2** for actually knowing something about triangle-geometry, but only in addition to any of the first 3 marks earned in the above solution: i.e. a maximum of 5/11 for the second part of the question.

Question 9

(i)  		
$\underline{O \circlearrowleft}$ $F\,.\,r = F_1\,.\,r$ $\Rightarrow F = F_1$	**B1**	For correct moment equation.
Res. $\leftrightarrow$ $F + F_1\cos\theta = R\sin\theta$	**B1**	For resolving horizontally for one cylinder.
Together give $R\sin\theta = F(1+\cos\theta)$	**AG**	
Since $F_1 \le \mu R$, with $\mu = \frac{1}{2}$,	**M1**	Use of the Friction law
it follows that $\dfrac{F}{R} \le \frac{1}{2} \Rightarrow \dfrac{\sin\theta}{1+\cos\theta} \le \frac{1}{2}$	**A1**	Combining with previous answer
i.e. $2\sin\theta \le 1+\cos\theta$	**AG** **Subtotal: 4**	
(ii) Res. $\uparrow$ for RH cylinder $W = N - R\cos\theta - F_1\sin\theta$	**B1**	F_1 might correctly be replaced with F.
Res. $\uparrow$ for plank $kW = 2R\cos\theta + 2F\sin\theta$	**B1**	
Eliminating W: $k(N - R\cos\theta - F\sin\theta) = 2R\cos\theta + 2F\sin\theta$	**M1**	For eliminating W
$N = R\cos\theta\left(\dfrac{2}{k}+1\right) + F\sin\theta\left(\dfrac{2}{k}+1\right)$	**A1**	For correct rearrangement for N
$N = \left(\dfrac{2}{k}+1\right)\left(\dfrac{1+\cos\theta}{\sin\theta}.\cos\theta + \sin\theta\right)F$	**M1**	For use of $R = \left(\dfrac{1+\cos\theta}{\sin\theta}\right)F$
$N = \left(\dfrac{2}{k}+1\right)\left(\dfrac{\cos\theta + \cos^2\theta + \sin^2\theta}{\sin\theta}\right)F$	**B1**	Obtaining $\cos^2\theta + \sin^2\theta$
$= \left(\dfrac{2}{k}+1\right)\left(\dfrac{\cos\theta+1}{\sin\theta}\right)F$	**AG**	
For no slipping at the ground, $F \le \mu N$	**M1**	Using Friction equation
$\Rightarrow F \le \frac{1}{2}\left(\dfrac{2}{k}+1\right)\left(\dfrac{\cos\theta+1}{\sin\theta}\right)F$	**A1**	Using previous part
ie. $2k\sin\theta \le (k+2)(1+\cos\theta)$	**A1**	Rearranging into a "useful" form.

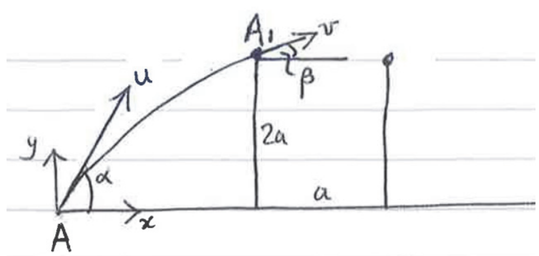
However, we already have that $2k\sin\theta \le k(1+\cos\theta) \le (k+2)(1+\cos\theta)$ so there are no extra restrictions on θ.	**E1** **Subtotal: 10**	Properly justified
(iii) $4\sin^2\theta \le 1 + 2\cos\theta + cos^2\theta$ $4(1-\cos^2\theta) \le 1 + 2\cos\theta + \cos^2\theta$ $0 \le 5\cos^2\theta + 2\cos\theta - 3$ $0 \le (5\cos\theta - 3)(\cos\theta + 1)$ Since $\cos\theta \ge 0$ we have $\cos\theta \ge \frac{3}{5}$ For appropriate angles $\cos\theta$ is decreasing and $\sin\theta$ is increasing. Therefore $\sin\theta \le \frac{4}{5}$ $\sin\theta = \dfrac{r-a}{r}$ So $5r - 5a \le 4r$ $r \le 5a$	**M1** **M1** **A1** **E1** **AG** **B1** **M1** **AG** **Subtotal: 6**	Squaring up an appropriate trig inequality Creating and simplifying quadratic inequality in one trig ratio A graphical argument is perfectly acceptable here. N.b It is possible that inequalities like $2s - 1 \le c$ are squared. If this is done without justifying that both sides are positive then withhold this final **E1**. Combining with previous result

Question 10

$ma = F - (Av^2 + R)$ $\text{WD} = \int_0^d F \, dx$ $= \int_0^d \left(ma + Av^2 + R\right) dx$ Since $a = v\frac{dv}{dx}$ $\text{WD} = \int_{x=0}^{x=d} \left(ma + Av^2 + R\right)\frac{dx}{dv} dv$ $= \int_{x=0}^{x=d} \left(ma + Av^2 + R\right)\frac{v}{a} dv$ Using $v^2 = u^2 + 2as$ with $v = w, u = 0, s = d \Rightarrow$ $w = \sqrt{2ad}$ Therefore: $\text{WD} = \int_{v=0}^{v=w} \frac{\left(ma + Av^2 + R\right)v}{a} \, dv$	**B1** **M1** **AG** **B1** **M1** **B1** **AG** **[5]**	Clear use of N2L Attempting to change variable of integration. Justifying limits. Ignore absence of ±
(i) $\text{WD} = \left[\left(m + \frac{R}{a}\right)\frac{v^2}{2} + \frac{Av^4}{4a}\right]_0^{\sqrt{2ad}}$ $= \left(m + \frac{R}{a}\right)ad + Aad^2$ For second half-journey, $\text{WD} = \int_w^0 \frac{\left(-ma + Av^2 + R\right)v}{-a} \, dv$ $= -mad + Rd + Aad^2$ Summing gives $2dR + 2Aad^2$ $R > ma \Rightarrow F = Av^2 + R - ma > 0$ always	**M1** **A1** **B1B1** **A1** **AG** **E1** **[6]**	Performing integration Correct answer in terms of d. B1 for correct limits B1 for correct integrand N.b. integrals may be combined to get to the same result.

(ii) If $R < ma$ then F is zero when $Av^2 = ma - R$ i.e. when $v = V = \sqrt{\dfrac{ma-R}{A}}$	**B1**	Finding an expression for the critical speed.
For F to fall to zero during motion, $V < w$	**E1**	
i.e. when $\dfrac{ma-R}{A} < 2ad$ i.e. $R > ma - 2Aad$	**E1**	
In this case, WD = $mad + Rd + Aad^2$, as before, for the first half-journey	**B1**	
For the second half WD = $\displaystyle\int_w^V \frac{\left(-ma + Av^2 + R\right)v}{-a}\,\mathrm{d}v$	**M2**	
$\left[(ma-R)\dfrac{v^2}{2a} - \dfrac{Av^4}{4a}\right]_w^V$	**A1**	
$=\dfrac{1}{2a}(ma-R)\left(\dfrac{ma-R}{A}\right) - \dfrac{A}{4a}\left(\dfrac{ma-R}{A}\right)^2 -$ $\dfrac{1}{2a}(ma-R)(2ad) + \dfrac{A}{4a}\left(4a^2d^2\right)$ $= \dfrac{1}{2Aa}(ma-R)^2 - \dfrac{1}{4Aa}(ma-R)^2 - (ma-R)d +$ Aad^2	**M1**	Substituting expressions for V and w.
$= \dfrac{1}{4Aa}(ma-R)^2 - mad + Rd + Aad^2$	**A1 CAO**	Without wrong working
So total WD = $\dfrac{1}{4Aa}(ma-R)^2 + 2Rd + 2Aad^2$	**AG** **[9]**	

Question 11



(i)		
At A, $KE = \frac{1}{2}mu^2 = \frac{5}{2}mag, PE = 0$	**B1**	
At $A_1, K = \frac{1}{2}mv^2, PE = 2mag$	**B1**	
Conservation of energy: $$\frac{5}{2}mag = \frac{1}{2}mv^2 + 2mag$$	**M1**	
$$v^2 = ga$$		
$$v = \sqrt{ga}$$	**A1**	
	[4]	
If angle at A_1 is β and it just passes the second wall then we have:		
$$0 = v\sin\theta\, t - \frac{1}{2}gt^2$$	**M1**	Using $s = ut + \frac{1}{2}at^2$
So $t = \frac{2v}{g}\sin\beta$	**A1**	Solving for t at second wall.
Also, $a = v\cos\beta\, t$	**M1**	Considering horizontal distance
$$= \frac{2v^2\sin\beta\cos\beta}{g}$$		N.b. Some candidates may just quote this (or equivalent). Give full credit.
$$= 2a\sin\beta\cos\beta$$	**A1**	Combining previous results.
So $\sin(2\beta) = 1$	**A1**	
Therefore $\beta = 45°$	**AG**	Condone absence of domain considerations.
	[5]	
x velocity is constant so		
$$u\cos\alpha = vcos\,\beta$$	**M1**	Comparing x velocities
$$\sqrt{5ag}\cos\alpha = \sqrt{ag}\frac{1}{\sqrt{2}}$$ $$\cos\alpha = \frac{1}{\sqrt{10}}$$	**A1**	
$$\sin\alpha = \frac{3}{\sqrt{10}}, \tan\alpha = 3$$	**A1**	Converting to a more useful ratio.

Method 1: $$2a = \sqrt{5ag}\frac{3}{\sqrt{10}}t - \frac{1}{2}gt^2$$ $$= \frac{3\sqrt{ag}}{\sqrt{2}}t - \frac{1}{2}gt^2$$ So $$t^2 - \frac{3\sqrt{2a}}{\sqrt{g}}t + \frac{4a}{g} = 0$$	**M1**	Using $s = ut + \frac{1}{2}at^2$
$$\left(t - \sqrt{\frac{2a}{g}}\right)\left(t - 2\sqrt{\frac{2a}{g}}\right) = 0$$		
First time over the wall means that $t = \sqrt{\frac{2a}{g}}$	**A1**	
So $d = ucos\,\theta\, t = \sqrt{5ag} \times \frac{1}{\sqrt{10}} \times \sqrt{\frac{2a}{g}} = a$	**A1**	
Method 2: $$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2u^2}$$	**M1**	Using trajectory equation
$$2a = 3x - \frac{x^2}{a}$$	**A1**	Combining with previous results
$$(x-a)(x-2a) = 0$$		
$$x = a$$	**A1**	
	[6]	
If the speed at h above first wall is v then by conserving energy, $$\frac{1}{2}5ag = \frac{1}{2}v^2 + (2a+h)g$$	**M1**	
$$v^2 = ag - 2gh$$	**B1**	
Using trajectory equation with origin at top of first wall and angle β as particle moves over first wall: $$y = h + x\ \tan\beta - \frac{gx^2(1+\tan^2\beta)}{2v^2}$$ When $x = a$ we need $y = 0$: $$0 = h + a\ \tan\beta - \frac{ga^2(1+\tan^2\beta)}{2v^2}$$	**M1**	Use of trajectory equation (might be several kinematics equations effectively leading to the same thing)
Treating this as a quadratic in $\tan\beta$: $$-\frac{ga^2}{2v^2}\tan^2\beta + a\tan\beta + h - \frac{ga^2}{2v^2} = 0$$ $$-ga^2\tan^2\beta + 2av^2\tan\beta + 2hv^2 - ga^2 = 0$$ The discriminant is: $$4a^2v^4 + 4ga^2(2hv^2 - ga^2)$$	**M1**	Considering the quadratic (or equivalently differentiating to find the max)
$$= 4a^2(g^2(a^2 - 4ah + 4h^2) + 2g^2h(a-2h) - g^2a^2))$$ $$= 4a^2g^2(a^2 - 4ah + 4h^2 + 2ah - 4h^2 - a^2)$$ $$= -8a^3g^2h$$ $$< 0$$ Therefore no solution.	**A1**	Obtaining a clearly negative discriminant – this might take many alternative forms.
	[5]	

Question 12

(i) $$P(X+Y=n)=\sum_{r=0}^{n}P(X=r)(P(Y=n-r)$$	**B2**	
$$=\sum_{r=0}^{n}\frac{e^{-\lambda}\lambda^{r}}{r!}\times\frac{e^{-\mu}\mu^{n-r}}{(n-r)!}$$	**B1**	
$$=\frac{e^{-\lambda}e^{-\mu}}{n!}\sum_{r=0}^{n}\frac{n!}{r!\,(n-r)!}\lambda^{r}\mu^{n-r}$$	**M1**	Attempting to manipulate factorials towards a binomial coefficient
$$=\frac{e^{-\lambda}e^{-\mu}}{n!}\sum_{r=0}^{n}\binom{n}{r}\lambda^{r}\mu^{n-r}$$	**B1**	Identifying correct binomial coefficient
$$=\frac{e^{-(\lambda+\mu)}}{n!}(\lambda+\mu)^{n}$$	**B1**	
Which is the the formula for $Po(\lambda+\mu)$	**E1**	Recognising result. Must state parameters
	[7]	
(ii)		
$$P(X=r\mid X+Y=k)=\frac{P(X=r)\times P(Y=k-r)}{P(X+Y=k)}$$	**M2**	(may be implied by following line)
$$=\frac{\frac{e^{-\lambda}\lambda^{r}}{r!}\times\frac{e^{-\mu}\mu^{k-r}}{(k-r)!}}{\frac{e^{-(\lambda+\mu)}}{k!}(\lambda+\mu)^{k}}$$	**A1**	
$$=\frac{k!}{r!\,(k-r)!}\left(\frac{\lambda}{\lambda+\mu}\right)^{r}\left(\frac{\mu}{\lambda+\mu}\right)^{k-r}$$	**A1**	
Which is a $B\left(k,\frac{\lambda}{\lambda+\mu}\right)$ distribution.	**E1**	Parameters must be stated.
	[5]	
(iii) This corresponds to r=1, k=1 from (ii)	**M2**	Can be implied by correct answer.
So probability is $\frac{\lambda}{\lambda+\mu}$.	**A1**	
(iv)	**[3]**	
Expected waiting time given that Adam is first is waiting time for first fish plus waiting time for Eve $\left(=\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)$	**B2**	Also accept waiting time given Eve is first. Must be clearly identified.
Expected waiting time is: E(Waiting time|Adam first)P(Adam first)+E(Waiting time|Eve first)P(Eve first)	**M2**	
=$\left(\frac{1}{\lambda+\mu}+\frac{1}{\mu}\right)\times\frac{\lambda}{\lambda+\mu}+\left(\frac{1}{\lambda+\mu}+\frac{1}{\lambda}\right)\times\frac{\mu}{\lambda+\mu}$	**A1**	
$$=\frac{1}{\lambda}+\frac{1}{\mu}-\frac{1}{\lambda+\mu}$$		No need for this algebraic simplification.
	[5]	

Question 13

(i)		
$P(\text{correct key on } k^{\text{th}}\text{attempt}) = \frac{1}{n}\left(1-\frac{1}{n}\right)^{k-1}$	**M1A1**	M1 for any attempt relating to the geometric distribution – e.g. missing first factor or power slightly wrong.
$= pq^{k-1}$ Where $p = \frac{1}{n}, q = 1 - \frac{1}{n}$		Although not strictly necessary, you may see this substitution frequently
Expected number of attempts is given by $p + 2pq + 3pq^2 \ldots$	**M1**	May be written in sigma notation
$= p(1 + 2q + 3q^2 \ldots)$		
$= p(1-q)^{-2}$	**M1**	Linking to binomial expansion
$= \frac{p}{p^2} = \frac{1}{p}$		
$= n$	**A1**	
	[5]	
(ii)		
$P(\text{correct key on } k^{\text{th}}\text{attempt}) = \frac{1}{n}$ for $k = 1 \ldots n$	**B1**	
Expected number of attempts is given by $\frac{1}{n} + \frac{2}{n} + \frac{3}{n} \ldots + \frac{n}{n}$	**M1**	
$= \frac{n+1}{2}$	**M1A1**	M1 for clearly recognising sum of integers / arithmetic series.
	[4]	
(iii)		
$P(\text{correct key on } k^{\text{th}}\text{attempt})$ $= \frac{n-1}{n} \times \frac{n}{n+1} \times \frac{n+1}{n+2} \ldots \times \frac{1}{n+k-1}$	**M1** **A1**	M1 for an attempt at this, possibly by pattern spotting the first few cases. Condone absence of checking $k = 1$ case explicitly.
$= \frac{n-1}{(n+k-2)(n+k-1)}$	**M1** **AG**	M1 for attempting telescoping (may be written as an induction)
$= (n-1)\left(\frac{-1}{n+k-1} + \frac{1}{n+k-2}\right)$	**M2** **A1**	Attempting partial fractions (This may be seen later)
	[6]	
Expected number of attempts is given by $(n-1)\sum_{1}^{\infty}\left(\frac{k}{n+k-2} - \frac{k}{n+k-1}\right)$	**M1**	
$= (n-1)\left[\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{2}{n} - \frac{2}{n+1}\right) + \left(\frac{3}{n+1} - \frac{3}{n+2}\right) \ldots\right]$		
$= (n-1)\left[\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} \ldots\right]$	**M1A1**	M1 for attempting telescoping
$= (n-1)\left(\sum_{r=1}^{\infty}\frac{1}{r} - \sum_{r=1}^{n-2}\frac{1}{r}\right)$	**B1**	
In the brackets there is an infinite sum minus a finite sum, so the result is infinite.	**E1**	
	[5]	