Question 1

(i) $\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r} = \frac{(n-1)!\,r!}{(n+r-1)!} - \frac{n!r!}{(n+r)!}$ **M1**

$= \frac{(n-1)!\,r!\,[(n+r)-n]}{(n+r)!}$

$= \frac{(n-1)!\,r!\,r}{(n+r)!}$ **M1**

$\therefore \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right) = \frac{r+1}{r}\frac{(n-1)!\ r!\ r}{(n+r)!}$

$= \frac{(n-1)!\,(r+1)!}{(n+r)!} = \frac{1}{{}^{n+r}C_{r+1}}$ **A1*** **(3)**

$\sum_{n=1}^{\infty} \frac{1}{{}^{n+r}C_{r+1}} = \sum_{n=1}^{\infty} \frac{r+1}{r}\left(\frac{1}{{}^{n+r-1}C_r} - \frac{1}{{}^{n+r}C_r}\right)$ **M1**

$= \frac{r+1}{r}\left(\frac{1}{{}^{r}C_r} - \frac{1}{{}^{r+1}C_r} + \frac{1}{{}^{r+1}C_r} - \frac{1}{{}^{r+2}C_r} + \frac{1}{{}^{r+2}C_r} - \frac{1}{{}^{r+3}C_r} + \cdots\right)$ **M1**

$= \frac{r+1}{r}\frac{1}{{}^{r}C_r}$ because ${}^{n+r}C_r \to \infty$ as $n \to \infty$ **E1**

$= \frac{r+1}{r}$ **A1** **(4)**

$\sum_{n=2}^{\infty} \frac{1}{{}^{n+2}C_{2+1}} = \frac{2+1}{2} - \frac{1}{{}^{1+2}C_{2+1}} = \frac{3}{2} - 1 = \frac{1}{2}$ **M1 M1 (2)**

(ii) ${}^{n+1}C_3 = \frac{(n+1)!}{(n-2)!3!} = \frac{(n+1)n(n-1)}{3!} = \frac{n^3-n}{3!} < \frac{n^3}{3!}$ **M1**

So $\frac{3!}{n^3} < \frac{1}{{}^{n+1}C_3}$ **A1*** **(2)**

$$\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} - \frac{5!}{n^3} = \frac{120}{n(n^2-1)} - \frac{120}{n(n^2-1)(n^2-4)} - \frac{120}{n^3}$$

$= \frac{120}{n^3(n^2-1)(n^2-4)}\left(n^2(n^2-4) - n^2 - (n^2-1)(n^2-4)\right)$ **M1**

$= \frac{-480}{n^3(n^2-1)(n^2-4)} < 0$

as $n \geq 3$ and so denominator is positive. **E1** **(2)**

Hence, $\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} < \frac{5!}{n^3}$

Alternatively,

$$\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} = \frac{5!}{n(n^2-1)} - \frac{5!}{n(n^2-1)(n^2-4)} = \frac{5!}{n(n^2-1)(n^2-4)} \times \left((n^2-4)-1\right)$$

$$= \frac{5!}{n^3} \times \frac{n^4-5n^2}{n^4-5n^2+4} < \frac{5!}{n^3}$$

as $n \geq 3$ and so $n^2 > 5$

$$\sum_{n=3}^{\infty} \frac{3!}{n^3} < \sum_{n=3}^{\infty} \frac{1}{{}^{n+1}C_3} = \sum_{n=2}^{\infty} \frac{1}{{}^{n+2}C_3} = \frac{1}{2}$$

M1

So $\sum_{n=1}^{\infty} \frac{3!}{n^3} < \frac{3!}{1} + \frac{3!}{8} + \frac{1}{2} = \frac{29}{4}$

M1

And therefore $\sum_{n=1}^{\infty} \frac{1}{n^3} < \frac{29}{24} = \frac{116}{96}$

A1* (3)

$$\sum_{n=3}^{\infty} \frac{5!}{n^3} > \sum_{n=3}^{\infty} \left(\frac{20}{{}^{n+1}C_3} - \frac{1}{{}^{n+2}C_5} \right) = 20 \times \frac{1}{2} - \left(\sum_{n=1}^{\infty} \frac{1}{{}^{n+4}C_5} \right) = 10 - \frac{5}{4}$$

M1 **M1**

Therefore

$$\sum_{n=3}^{\infty} \frac{5!}{n^3} > \frac{35}{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3} > \frac{35}{4 \times 5!} + \frac{1}{1} + \frac{1}{8} = \frac{7}{96} + \frac{96}{96} + \frac{12}{96} = \frac{115}{96}$$

M1 **A1* (4)**

Question 2

(i) $z' - a = e^{i\theta}(z - a)$ **M1**

Thus $z' = a + e^{i\theta}z - e^{i\theta}a = e^{i\theta}z + a\left(1 - e^{i\theta}\right)$ **A1* (2)**

(ii) $z'' = e^{i\varphi}z' + b\left(1 - e^{i\varphi}\right)$

$= e^{i\varphi}\left(e^{i\theta}z + a\left(1 - e^{i\theta}\right)\right) + b\left(1 - e^{i\varphi}\right)$ **M1 A1**

So $z'' = e^{i(\varphi+\theta)}z + \left(ae^{i\varphi} - ae^{i(\varphi+\theta)} + b - be^{i\varphi}\right)$

This is a rotation about c if

$c\left(1 - e^{i(\varphi+\theta)}\right) = ae^{i\varphi} - ae^{i(\varphi+\theta)} + b - be^{i\varphi}$ **M1**

If $\varphi + \theta = 2n\pi$, $\left(1 - e^{i(\varphi+\theta)}\right) = 0$, so c cannot be found. **E1**

Otherwise, multiplying by $-e^{\frac{-i(\varphi+\theta)}{2}}$,

$c\left(e^{\frac{i(\varphi+\theta)}{2}} - e^{\frac{-i(\varphi+\theta)}{2}}\right) = a\left(e^{\frac{i(\varphi+\theta)}{2}} - e^{\frac{i(\varphi-\theta)}{2}}\right) + b\left(e^{\frac{i(\varphi-\theta)}{2}} - e^{\frac{-i(\varphi+\theta)}{2}}\right)$

$2ci\sin\frac{1}{2}(\varphi + \theta) = 2aie^{i\varphi/2}\sin\frac{1}{2}\theta + 2bie^{-i\theta/2}\sin\frac{1}{2}\varphi$ **M1**

$c\sin\frac{1}{2}(\varphi + \theta) = ae^{i\varphi/2}\sin\frac{1}{2}\theta + be^{-i\theta/2}\sin\frac{1}{2}\varphi$ **A1* (6)**

If $\varphi + \theta = 2n\pi$, $z'' = z + \left(ae^{i\varphi} - a + b - be^{i\varphi}\right)$ **M1**

So $z'' = z + (b - a)\left(1 - e^{i\varphi}\right)$ **A1**

This is a translation by $(b - a)\left(1 - e^{i\varphi}\right)$ **A1 (3)**

(iii) If RS = SR, and if $\varphi + \theta = 2n\pi$, then

$$(b - a)\left(1 - e^{i\varphi}\right) = (a - b)\left(1 - e^{i\theta}\right)$$

M1

$$(a - b)\left(e^{i\theta} + e^{i\varphi} - 2\right) = 0$$

So $a = b$, or if $a \neq b$, $e^{i\theta} + e^{i(2n\pi-\theta)} - 2 = 0$

A1

$2\cos\theta - 2 = 0$ **M1**

Thus $\theta = 2n\pi$ **A1 (4)**

If $\varphi + \theta \neq 2n\pi$

$ae^{i\varphi/2}\sin\frac{1}{2}\theta + be^{-i\theta/2}\sin\frac{1}{2}\varphi = be^{i\theta/2}\sin\frac{1}{2}\varphi + ae^{-i\varphi/2}\sin\frac{1}{2}\theta$ **M1**

$2i(a - b)\sin\frac{1}{2}\varphi\sin\frac{1}{2}\theta = 0$ **A1**

So $a = b$, $\theta = 2n\pi$, or $\varphi = 2n\pi$

A1 **A1** **A1 (5)**

Question 3

$\alpha\beta + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta + \beta\gamma = -A$ **M1**

$A = -q$ **A1 (2)**

(i) $y^3 - 3y^2 - 40y + 84 = 0$ **M1 A1**
$(y-2)(y^2 - y - 42) = 0$ **M1**

$(y-2)(y-7)(y+6) = 0$ **M1 A1**

So $\alpha\beta + \gamma\delta = 7$ **A1 (6)**

(ii) $(\alpha+\beta)(\gamma+\delta) = \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta = 3 - \alpha\beta - \gamma\delta = -4$

M1 **M1** **A1 (3)**

$(\alpha+\beta) + (\gamma+\delta) = 0$ **M1**

Thus $(\alpha+\beta)$ is a root of $t^2 - 4 = 0$ **M1**

So $\alpha + \beta = \pm 2$ **A1**

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = 6$$

$$\alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = 6$$

$2(\alpha\beta - \gamma\delta) = \pm 6$ **M1**

$\alpha\beta - \gamma\delta = 3$ as $\alpha\beta > \gamma\delta$ (and so $\alpha + \beta = -2$)

So $\alpha\beta = 5$ **A1 (5)**

Alternatively, $\alpha\beta\gamma\delta = 10$, **M1 A1** so $\alpha\beta$ and $\gamma\delta$ are the roots of

$t^2 - 7t + 10 = 0$ **M1 A1** and as $\alpha\beta > \gamma\delta$, $\alpha\beta = 5$ (and $\gamma\delta = 2$). **A1 (5)**

(iii) Thus α and β are the roots of $t^2 + 2t + 5 = 0$ and γ and δ are the roots of
$t^2 - 2t + 2 = 0$ **M1 A1**

So $x = 1 \pm i$, $-1 \pm 2i$ **A1 A1 (4)**

Question 4

(i) $e^{x\ln a} = a^x$ (formula book)

So if $\log_a f(x) = z$

$f(x) = a^z = e^{z\ln a}$ **E1**

and so $\ln f(x) = z\ln a = \ln a\ \log_a f(x)$ **B1**

Therefore,

$$e^{\frac{1}{y}\int_0^y \ln f(x)dx} = e^{\frac{1}{y}\int_0^y \ln a\ \log_a f(x)\,dx} = e^{\frac{1}{y}\ln a\int_0^y \log_a f(x)\,dx}$$

M1 **M1**

Thus, $F(y) = a^{\frac{1}{y}\int_0^y \log_a f(x)\,dx}$ **A1* (5)**

(ii) $H(y) = e^{\frac{1}{y}\int_0^y \ln h(x)dx} = e^{\frac{1}{y}\int_0^y \ln(f(x)g(x))dx}$ **M1**

$$= e^{\frac{1}{y}\int_0^y \ln f(x)+\ln g(x)dx}$$

$= e^{\frac{1}{y}\left(\int_0^y \ln f(x)dx+\int_0^y \ln g(x)dx\right)}$ **M1**

$= e^{\frac{1}{y}\int_0^y \ln f(x)dx}\, e^{\frac{1}{y}\int_0^y \ln g(x)dx} = F(y)G(y)$

M1 **A1* (4)**

(iii) Let $f(x) = b^x$,

Then $F(y) = e^{\frac{1}{y}\int_0^y \ln b^x dx} = e^{\frac{1}{y}\int_0^y x\ln b\,dx} = e^{\frac{1}{y}\ln b\int_0^y x dx}$

M1 **M1**

$= e^{\frac{1}{y}\ln b\left[\frac{1}{2}x^2\right]_0^y} = e^{\frac{1}{y}\ln b\,\frac{1}{2}y^2} = e^{\frac{1}{2}y\ln b} = b^{\frac{1}{2}y} = \sqrt{b^y}$

A1 **M1** **A1* (5)**

(iv) $e^{\frac{1}{y}\int_0^y \ln f(x)dx} = \sqrt{f(y)}$

$\frac{1}{y}\int_0^y \ln f(x)dx = \ln\sqrt{f(y)} = \frac{1}{2}\ln f(y)$

$\int_0^y \ln f(x)dx = \frac{y}{2}\ln f(y)$ **M1**

$\ln f(y) = \frac{1}{2}\ln f(y) + \frac{yf'(y)}{2f(y)}$ **M1**

$\frac{yf'(y)}{f(y)} = \ln f(y)$ so $\frac{f'(y)}{f(y)\ln f(y)} = \frac{1}{y}$ **M1**

Integrating $\ln\ln f(y) = \ln y + c = \ln y + \ln k = \ln ky$ **M1 A1**

$\ln f(y) = ky$

$f(y) = e^{ky} = e^{y\ln b} = b^y$

$f(x) = b^x$ **A1* (6)**

Question 5

$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + \frac{dr}{d\theta}\sin\theta = f\cos\theta + f'\sin\theta$ **M1**

$x = r\cos\theta$

$\frac{dx}{d\theta} = -r\sin\theta + \frac{dr}{d\theta}\cos\theta = -f\sin\theta + f'\cos\theta$ **M1**

$\frac{dy}{dx} = \frac{f\cos\theta + f'\sin\theta}{-f\sin\theta + f'\cos\theta} = \frac{f + f'\tan\theta}{-f\tan\theta + f'}$ **M1 A1 (4)**

$\frac{f + f'\tan\theta}{-f\tan\theta + f'} \times \frac{g + g'\tan\theta}{-g\tan\theta + g'} = -1$ **M1**

$fg + f'g\tan\theta + fg'\tan\theta + f'g'\tan^2\theta = -fg\tan^2\theta + f'g\tan\theta + fg'\tan\theta - f'g'$

$(fg + f'g')\sec^2\theta = 0$ **M1**

$fg + f'g' = 0$ **A1* (3)**

$g(\theta) = a(1 + \sin\theta)$

$g'(\theta) = a\cos\theta$

So $f'a\cos\theta + fa(1 + \sin\theta) = 0$ **M1**

$\frac{f'}{f} = -\frac{(1+\sin\theta)}{\cos\theta} = -\sec\theta - \tan\theta$ **A1**

$\ln f = -\ln(\sec\theta + \tan\theta) + \ln\cos\theta + c = \ln\left(\frac{k\cos\theta}{\sec\theta + \tan\theta}\right) = \ln\left(\frac{k\cos^2\theta}{1+\sin\theta}\right)$ **M1 A1**

$f(\theta) = \left(\frac{k\cos^2\theta}{1+\sin\theta}\right) = \frac{k(1-\sin^2\theta)}{1+\sin\theta} = k(1 - \sin\theta)$ **M1 A1)**

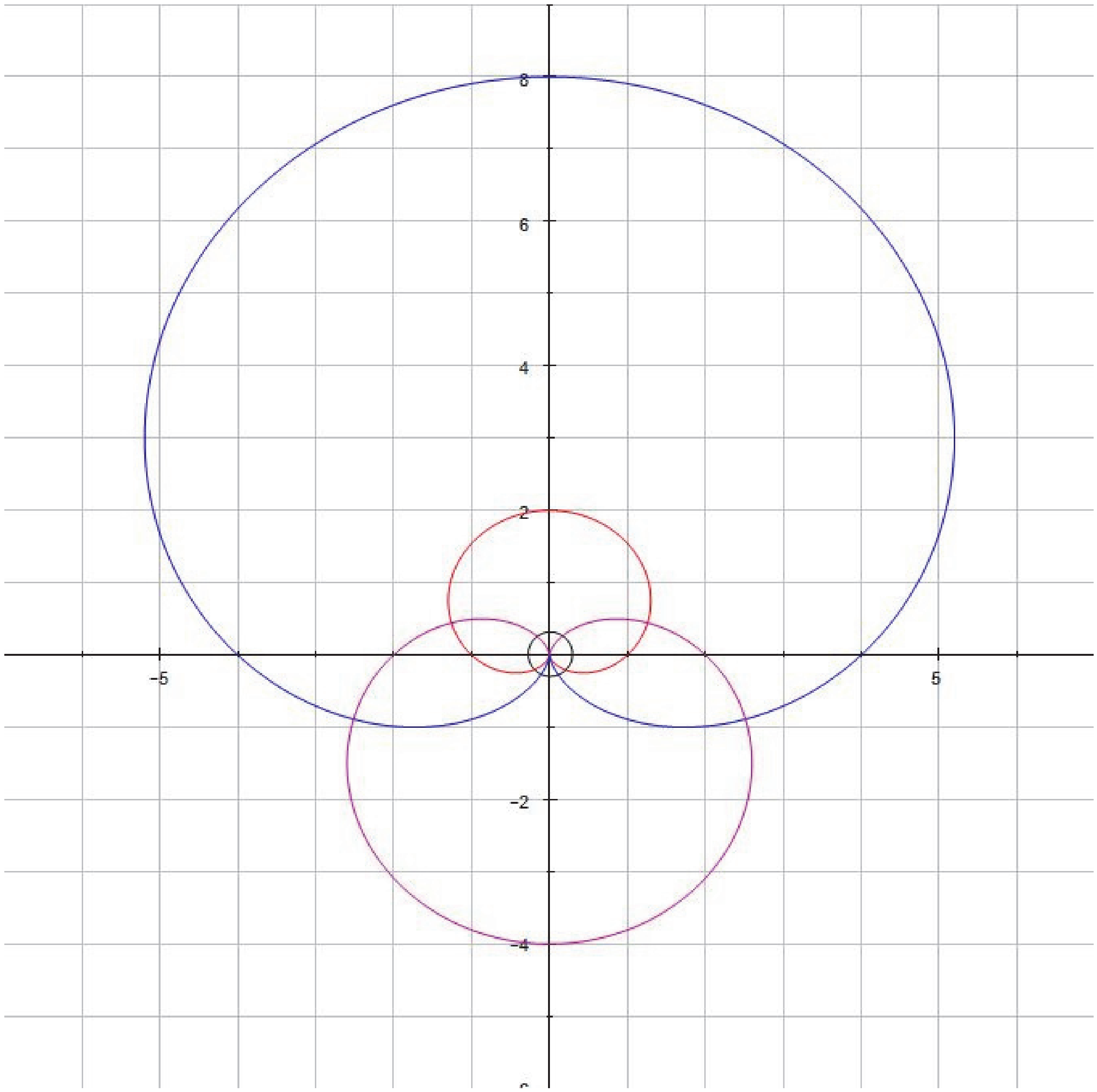
Alternatively, $\frac{f'}{f} = -\frac{(1+\sin\theta)}{\cos\theta} = -\frac{\cos\theta}{(1-\sin\theta)}$ **M1 A1**

$\ln f = \ln\big((1 - \sin\theta)\big) + c = \ln\big(k(1 - \sin\theta)\big)$ **M1**

and hence $f(\theta) = k(1 - \sin\theta)$ **A1**

$r = 4, \theta = -\frac{1}{2}\pi$ so $4 = 2k$ **M1**

Thus $f(\theta) = 2(1 - \sin\theta)$ **A1 (8)**



G1 G1 dG1 G1 G1(5)

Question 6

(i) $T(x)=\int_0^x \frac{1}{1+u^2}du$

Let $u=v^{-1}$, $\frac{du}{dv}=-v^{-2}$

B1

So

$$T(x)=\int_{\infty}^{x^{-1}}\frac{1}{1+v^{-2}}\times -v^{-2}dv=\int_{x^{-1}}^{\infty}\frac{1}{v^2+1}dv=\int_0^{\infty}\frac{1}{1+u^2}du-\int_0^{x^{-1}}\frac{1}{1+u^2}du$$

M1 **M1**

$T(x)=T(\infty)-T(x^{-1})$ **A1* (4)**

(ii) $v=\frac{u+a}{1-au}\Leftrightarrow v-auv=u+a\Leftrightarrow v-u=a(1+uv)\Leftrightarrow a=\frac{v-u}{1+uv}$

M1

$0=\frac{(1+uv)\left(\frac{dv}{du}-1\right)-(v-u)\left(u\frac{dv}{du}+v\right)}{(1+uv)^2}$ **M1**

$\frac{dv}{du}(1+uv-uv+u^2)=1+uv+v^2-uv$

$\frac{dv}{du}=\frac{1+v^2}{1+u^2}$ **A1* (3)**

Alternatively,

$$v=\frac{u+a}{1-au}\Leftrightarrow\frac{dv}{du}=\frac{(1-au)+a(u+a)}{(1-au)^2}=\frac{1+a^2}{(1-au)^2}=\frac{(1+a^2)(1+u^2)}{(1-au)^2(1+u^2)}$$

M1

$$=\frac{(1-au)^2+(u+a)^2}{(1-au)^2(1+u^2)}=\frac{1+v^2}{1+u^2}$$

M1 **A1**

$$T(x)=\int_0^x\frac{1}{1+u^2}du=\int_a^{\frac{x+a}{1-ax}}\frac{1}{1+u^2}\frac{1+u^2}{1+v^2}dv=\int_a^{\frac{x+a}{1-ax}}\frac{1}{1+v^2}dv=\int_0^{\frac{x+a}{1-ax}}\frac{1}{1+v^2}dv-\int_0^a\frac{1}{1+v^2}dv$$

M1 **M1**

$T(x)=T\left(\frac{x+a}{1-ax}\right)-T(a)$ **A1* (3)**

As $T(x)=T(\infty)-T(x^{-1})$, $T(a)=T(\infty)-T(a^{-1})$

So

$$T(x^{-1})=T(\infty)-T(x)=T(\infty)-\left(T\left(\frac{x+a}{1-ax}\right)-T(a)\right)=T(\infty)-\left(T\left(\frac{x+a}{1-ax}\right)-\left(T(\infty)-T(a^{-1})\right)\right)$$

M1 **M1**

Thus

$T(x^{-1}) = 2T(\infty) - T\left(\frac{x+a}{1-ax}\right) - T(a^{-1})$ **A1* (3)**

Let $y = x^{-1}$, $= a^{-1}$, then $x < \frac{1}{a}$ implies $\frac{1}{y} < b$ which is $y > \frac{1}{b}$ **M1**

$T(y) = 2T(\infty) - T\left(\frac{y^{-1}+b^{-1}}{1-b^{-1}y^{-1}}\right) - T(b) = 2T(\infty) - T\left(\frac{b+y}{by-1}\right) - T(b)$ **A1* (2)**

(iii) Using $T(y) = 2T(\infty) - T\left(\frac{b+y}{by-1}\right) - T(b)$ with $y = b = \sqrt{3}$ **M1**

$T(\sqrt{3}) = 2T(\infty) - T\left(\frac{\sqrt{3}+\sqrt{3}}{\sqrt{3}\sqrt{3}-1}\right) - T(\sqrt{3})$

$T(\sqrt{3}) = 2T(\infty) - T(\sqrt{3}) - T(\sqrt{3})$

$3T(\sqrt{3}) = 2T(\infty) \Leftrightarrow T(\sqrt{3}) = \frac{2}{3}T(\infty)$ **A1* (2)**

Using $T(x) = T(\infty) - T(x^{-1})$ with $x = 1$,

$T(1) = T(\infty) - T(1)$ and so $T(1) = \frac{1}{2}T(\infty)$ **B1**

Using $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ with $x = \sqrt{2} - 1$ and $a = 1$ **M1**

$T(\sqrt{2} - 1\ ) = T\left(\frac{\sqrt{2}-1\ +1}{1-(\sqrt{2}-1\ )}\right) - T(1)$

$T(\sqrt{2} - 1\ ) = T\left(\frac{\sqrt{2}}{2-\sqrt{2}}\right) - T(1) = T\left(\frac{1}{\sqrt{2}-1}\right) - T(1) = T(\sqrt{2} + 1) - T(1)$

Using $T(x) = T(\infty) - T(x^{-1})$, $T(\sqrt{2} + 1\ ) = T(\infty) - T(\sqrt{2} - 1)$

So $T(\sqrt{2} - 1\ ) = T(\infty) - T(\sqrt{2} - 1) - T(1)$

$2T(\sqrt{2} - 1\ ) = T(\infty) - T(1) = T(\infty) - \frac{1}{2}T(\infty)$

$T(\sqrt{2} - 1\ ) = \frac{1}{4}T(\infty)$ **A1* (3)**

Alternatively, using $T(x) = T\left(\frac{x+a}{1-ax}\right) - T(a)$ with $x = a = \sqrt{2} - 1$

$$T(\sqrt{2} - 1\ ) = T\left(\frac{2(\sqrt{2} - 1\ )}{1 - (\sqrt{2} - 1\ )^2}\right) - T(\sqrt{2} - 1\ ) = T\left(\frac{2(\sqrt{2} - 1\ )}{2(\sqrt{2} - 1\ )}\right) - T(\sqrt{2} - 1\ )$$

Therefore $2T(\sqrt{2} - 1\ ) = T(1)$ and so $T(\sqrt{2} - 1\ ) = \frac{1}{2}T(1) = \frac{1}{4}T(\infty)$

Question 7

$$\frac{\frac{a^2(1-t^2)^2}{(1+t^2)^2}}{a^2}+\frac{\frac{4b^2t^2}{(1+t^2)^2}}{b^2}=\frac{(1-t^2)^2+4t^2}{(1+t^2)^2}=\frac{1-2t^2+t^4+4t^2}{1+2t^2+t^4}=1 \quad \textbf{B1 (1)}$$

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}+\frac{2y\frac{dy}{dx}}{b^2}=0$ **M1**

$\frac{dy}{dx}=-\frac{b^2x}{a^2y}=-\frac{b^2a(1-t^2)(1+t^2)}{a^2(1+t^2)2bt}=-\frac{b(1-t^2)}{2at}$ **M1 A1**

So L is $y-\frac{2bt}{(1+t^2)}=-\frac{b(1-t^2)}{2at}\left(x-\frac{a(1-t^2)}{(1+t^2)}\right)$ **M1**

$2at(1+t^2)y-4abt^2=-bx(1-t^2)(1+t^2)+ab(1-t^2)^2$

$2at(1+t^2)y+bx(1-t^2)(1+t^2)=ab(1-t^2)^2+4abt^2=ab(1+t^2)^2$

Thus $2aty+bx(1-t^2)=ab(1+t^2)$ **M1**

and as (X,Y) lies on this line $2atY+bX(1-t^2)=ab(1+t^2)$

$0=(a+X)bt^2-2atY+b(a-X)$ **A1* (6)**

For there to be two distinct lines, there need to be two values of t .

So the discriminant must be positive, $(-2aY)^2-4(a+X)bb(a-X)>0$ **M1**

$4a^2Y^2>4b^2(a^2-X^2)$

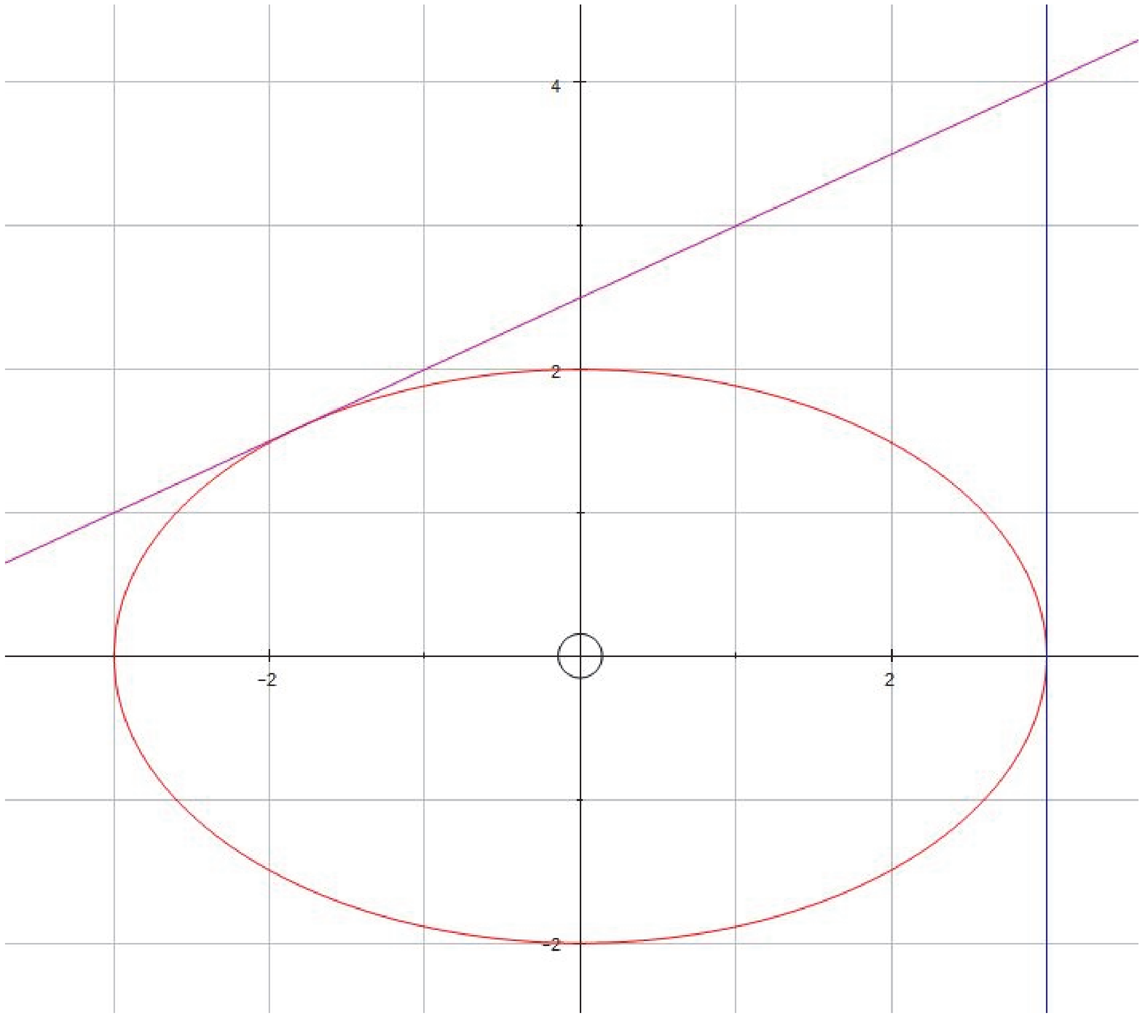
$a^2Y^2>(a^2-X^2)b^2$ **A1***

$\frac{Y^2}{b^2}>1-\frac{X^2}{a^2}$

$\frac{X^2}{a^2}+\frac{Y^2}{b^2}>1$ so (X,Y) lies outside the ellipse. **B1 (3)**

However, if $X^2=a^2$, $=\pm a$, one tangent is at $t=0$ or $t=\infty$, a vertical line. **E1**

If $\frac{X^2}{a^2}+\frac{Y^2}{b^2}>1$, then $Y\neq 0$. **E1**



G1 (3)

(ii) p and q are the roots of $0 = (a+X)bt^2 - 2atY + b(a-X)$

So $p+q = \frac{2aY}{(a+X)b}$ and $pq = \frac{b(a-X)}{(a+X)b}$ **M1**

Thus $(a+X)pq = a - X$ and $(a+X)(p+q)b = 2aY$ **A1 A1 (3)**

Without loss of generality $(0, y_1)$ lies on $(a+x)bp^2 - 2apy + b(a-x) = 0$

and $(0, y_2)$ lies on $(a+x)bq^2 - 2aqy + b(a-x) = 0$

So $abp^2 - 2apy_1 + ab = 0$, that is $bp^2 - 2py_1 + b = 0$ **M1**

and $bq^2 - 2qy_2 + b = 0$

As $y_1 + y_2 = 2b$, $\frac{bp^2+b}{2p} + \frac{bq^2+b}{2q} = 2b$ **M1**

$\frac{p^2+1}{p} + \frac{q^2+1}{q} = 4$

$p + q + \frac{p+q}{pq} = 4$

$\frac{2aY}{(a+X)b} + \frac{\frac{2aY}{(a+X)b}}{\frac{a-X}{a+X}} = 4$ **M1**

$\frac{2aY}{a+X} + \frac{2aY}{a-X} = 4b$

$2aY(a - X + a + X) = 4(a - X)(a + X)b$

$4a^2Y = 4(a^2 - X^2)b$

$\frac{Y}{b} = 1 - \frac{X^2}{a^2}$

$\frac{X^2}{a^2} + \frac{Y}{b} = 1$ **A1* (4)**

Question 8

$$\sum_{m=1}^{n} a_m(b_{m+1} - b_m) + \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m)$$

$$= \sum_{m=1}^{n} (a_m b_{m+1} - a_m b_m + b_{m+1}a_{m+1} - b_{m+1}a_m)$$

M1

$$= \sum_{m=1}^{n} (-a_m b_m + b_{m+1}a_{m+1}) = a_{n+1}b_{n+1} - a_1 b_1$$

M1

Hence,

$$\sum_{m=1}^{n} a_m(b_{m+1} - b_m) = a_{n+1}b_{n+1} - a_1 b_1 - \sum_{m=1}^{n} b_{m+1}(a_{m+1} - a_m)$$

A1* (3)

(i) Let $a_m = 1$ (or any constant) and $b_m = \sin mx$, **M1**

then

$$\sum_{m=1}^{n} (\sin(m+1)x - \sin mx) = \sin(n+1)x - \sin x - \sum_{m=1}^{n} \sin(m+1)x \quad (1-1)$$

M1 A1

So

$$\sum_{m=1}^{n} 2\cos\left(m + \frac{1}{2}\right)x \ \sin\frac{1}{2}x = \ (\sin(n+1)x - \sin x)$$

M1 A1

and therefore

$$\sum_{m=1}^{n} \cos\left(m + \frac{1}{2}\right)x = \frac{1}{2}(\sin(n+1)x - \sin x)\csc\frac{1}{2}x$$

A1* (6)

(ii) Let $a_m = m$ and $b_m = \sin(m-1)x - \sin mx$, **M1**

then

$$b_{m+1} - b_m = (\sin mx - \sin(m+1)x) - (\sin(m-1)x - \sin mx)$$

$$= -2\cos\left(m + \frac{1}{2}\right)x \ \sin\frac{1}{2}x + \ 2\cos\left(m - \frac{1}{2}\right)x \ \sin\frac{1}{2}x$$

M1 A1

$= 4\sin mx\ \sin\frac{1}{2}x\sin\frac{1}{2}x$ **M1 A1**

Thus, using the stem

$$\sum_{m=1}^{n} m \times 4\sin mx\ \sin^2\frac{1}{2}x$$
$$= (n+1)(\sin nx - \sin(n+1)x) - 1 \times (\sin(0 \times x) - \sin x)$$
$$- \sum_{m=1}^{n}(\sin mx - \sin(m+1)x)$$

M1 A1

So

$$4\sin^2\frac{1}{2}x\sum_{m=1}^{n} m\sin mx\ = (n+1)(\sin nx - \sin(n+1)x) + \sin x - \sin x + \sin(n+1)x$$

M1 A1

$$4\sin^2\frac{1}{2}x\sum_{m=1}^{n} m\sin mx\ = (n+1)\sin nx - n\sin(n+1)x$$

Thus

$$\sum_{m=1}^{n} m\sin mx\ = (p\sin nx + q\sin(n+1)x)\csc^2\frac{1}{2}x$$

where

$$p = -\frac{1}{4}n$$

A1

and

$$q = \frac{1}{4}(n+1)$$

A1 (11)

Alternatively, let $a_m = m$ and $b_m = \cos\left(m - \frac{1}{2}\right)x$, using stem, **M1**

$$\sum_{m=1}^{n} m\left(\cos\left(m + \frac{1}{2}\right)x - \cos\left(m - \frac{1}{2}\right)x\right)$$
$$= (n+1)\cos\left(n + \frac{1}{2}\right)x - \cos\frac{1}{2}x - \sum_{m=1}^{n}\cos\left(m + \frac{1}{2}\right)x$$

M1 A1

So,

$$\sum_{m=1}^{n} -2m \sin mx \ \sin\frac{1}{2}x$$

$$= (n+1)\cos\left(n+\frac{1}{2}\right)x - \cos\frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x)\csc\frac{1}{2}x$$

M1 A1

$$= \csc\frac{1}{2}x\left((n+1)\cos\left(n+\frac{1}{2}\right)x\sin\frac{1}{2}x - \sin\frac{1}{2}x\cos\frac{1}{2}x - \frac{1}{2}(\sin(n+1)x - \sin x)\right)$$

M1 A1

$$= \frac{1}{2}\csc\frac{1}{2}x\left(2(n+1)\cos\left(n+\frac{1}{2}\right)x\sin\frac{1}{2}x - 2\sin\frac{1}{2}x\cos\frac{1}{2}x - (\sin(n+1)x - \sin x)\right)$$

$$= \frac{1}{2}\csc\frac{1}{2}x\left((n+1)(\sin(n+1)x - \sin nx) - \sin x - \sin(n+1)x + \sin x\right)$$

M1 A1

$$= \frac{1}{2}\csc\frac{1}{2}x\,(n\sin(n+1)x - (n+1)\sin nx)$$

giving result as before.

Question 9

For A, $mg - Z = m\ddot{y}$ and for B, $Z = 2m\ddot{x}$ where Z is tension. **M1 A1 A1**

Adding, $\ddot{y} + 2\ddot{x} = g$ **M1**

Integrating with respect to time, $\dot{y} + 2\dot{x} = gt + c$

Initially, $t = 0$, $\dot{x} = 0$, $\dot{y} = 0 \Rightarrow c = 0$

Integrating with respect to time, $y + 2x = \frac{1}{2}gt^2 + c'$ **M1 M1**

Initially, $t = 0$, $x = 0$, $y = 0 \Rightarrow c' = 0$

So $y + 2x = \frac{1}{2}gt^2$ **A1* (7)**

When $x = a$, $t = T = \sqrt{\frac{6a}{g}}$ so $y = a$ **M1 A1**

Conserving energy, at time T we have shown there is no elastic potential energy, so

$$0 = \frac{1}{2}2m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mga$$

M1 A1 A1 A1 (6)

That is

$$2\dot{x}^2 + \dot{y}^2 = 2ga$$

B1

But also $\dot{y} + 2\dot{x} = gT$ and so $\dot{y} + 2\dot{x} = \sqrt{6ga}$ **M1 A1**

Thus $2\dot{x}^2 + \left(\sqrt{6ga} - 2\dot{x}\right)^2 = 2ga$ **M1 A1**

$$6\dot{x}^2 - 4\dot{x}\sqrt{6ga} + 4ga = 0$$

$$\dot{x}^2 - 2\dot{x}\sqrt{\frac{2ga}{3}} + \frac{2ga}{3} = 0$$

$$\left(\dot{x} - \sqrt{\frac{2ga}{3}}\right)^2 = 0$$

M1

and so $\dot{x} = \sqrt{\frac{2ga}{3}}$ **A1* (7)**

Alternatively,

$$Z = \frac{\lambda(y - x)}{a}$$

M1

Subtracting,

$$2mg - 3Z = 2m(\ddot{y} - \ddot{x})$$

$$\ddot{y} - \ddot{x} = -\frac{3\lambda(y-x)}{2ma} + g$$

M1

So,

$$y - x = \frac{2mga}{3\lambda}(1 - \cos\omega t)$$

M1

where

$$\omega^2 = \frac{3\lambda}{2ma}$$

As $y + 2x = \frac{1}{2}gt^2$, $3x = \frac{1}{2}gt^2 - \frac{2mga}{3\lambda}(1 - \cos\omega t)$ **M1**

When $x = a$, $t = T = \sqrt{\frac{6a}{g}}$

so $3a = 3a - \frac{2mga}{3\lambda}\left(1 - \cos\omega\sqrt{\frac{6a}{g}}\right)$ and thus $\frac{3\lambda}{2ma}\frac{6a}{g} = 4n^2\pi^2$, $\lambda = \frac{4n^2\pi^2 mg}{9}$ **M1**

$$3\dot{x} = gt - \frac{2mga\omega\sin\omega t}{3\lambda} = g\sqrt{\frac{6a}{g}} - 0$$

$$\dot{x} = \sqrt{\frac{2ga}{3}}$$

M1 A1* (7)

Question 10

Moment of inertia of PQ about axis through P is $\frac{1}{3}m(3a)^2 = 3ma^2$ **B1**

Conserving energy, $0 = \frac{1}{2}3ma^2\dot{\theta}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - mg\frac{3}{2}a\sin\theta - mgl\sin\theta$ **M1 A1 A1 A1**

Thus $(3a^2 + l^2)\dot{\theta}^2 = g(3a + 2l)\sin\theta$ **A1* (6)**

Differentiating with respect to time,

$$2(3a^2 + l^2)\dot{\theta}\ddot{\theta} = g(3a + 2l)\cos\theta\dot{\theta}$$

M1

So

$$2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$$

A1 (2)

Alternatively, taking moments about axis through P

$$m(3a^2 + l^2)\ddot{\theta} = mg\left(\frac{3}{2}a + l\right)\cos\theta$$

M1

So

$$2(3a^2 + l^2)\ddot{\theta} = g(3a + 2l)\cos\theta$$

A1 (2)

Resolving perpendicular to the rod for the particle,

$$mg\cos\theta - R = ml\ddot{\theta}$$

M1 A1

Thus

$$R = mg\cos\theta - ml\ddot{\theta} = mg\cos\theta\left(1 - \frac{l(3a + 2l)}{2(3a^2 + l^2)}\right)$$

M1 A1

$$1 - \frac{l(3a + 2l)}{2(3a^2 + l^2)} = \frac{6a^2 + 2l^2 - 3al - 2l^2}{2(3a^2 + l^2)} = \frac{3a(2a - l)}{2(3a^2 + l^2)} > 0$$

because $l < 2a$ **A1 (5)**

Resolving along the rod towards P for the particle,

$$F - mg\sin\theta = ml\dot{\theta}^2$$

M1 A1

Thus

$$F = mg\sin\theta + ml\dot{\theta}^2 = mg\sin\theta\left(1 + \frac{l(3a+2l)}{(3a^2+l^2)}\right) = mg\sin\theta\left(\frac{3(a^2+al+l^2)}{(3a^2+l^2)}\right)$$

M1

On the point of slipping $F = \mu R$, so **B1**

$$mg\sin\theta\left(\frac{3(a^2+al+l^2)}{(3a^2+l^2)}\right) = \mu mg\cos\theta\left(\frac{3a(2a-l)}{2(3a^2+l^2)}\right)$$

Thus

$$\tan\theta = \frac{\mu a(2a-l)}{2(a^2+al+l^2)}$$

A1* (5)

At the instant of release, the equation of rotational motion for the rod ignoring the particle is

$$mg\frac{3a}{2} = 3ma^2\ddot{\theta}$$

and thus

$$\ddot{\theta} = \frac{g}{2a}$$

M1

Therefore the acceleration of the point on the rod where the particle rests equals $l\ddot{\theta} = \frac{lg}{2a} > g$ if $l > 2a$, and so the rod drops away from the particle faster than the particle accelerates and the particle immediately loses contact. **A1 (2)**

(Alternatively, for particle to accelerate with rod from previous working $R < 0$, **M1** meaning that it would have to be attached to so accelerate, and as it is only placed on the rod, this cannot happen.) **A1 (2)**

Question 11

(i) Conserving (linear) momentum

$$Mu - nmv = 0$$

M1

$$u = \frac{nmv}{M}$$

A1

$$K = \frac{1}{2}Mu^2 + n \times \frac{1}{2}mv^2 = \frac{1}{2}M\left(\frac{nmv}{M}\right)^2 + \frac{1}{2}nmv^2 = \frac{1}{2}nmv^2\left(\frac{nm}{M} + 1\right)$$

M1 **M1** **A1* (5)**

as required.

(ii) Conserving momentum before and after r th gun fired

$$\big(M + \big(n - (r-1)\big)m\big)u_{r-1} = (M + (n-r)m)u_r - m(v - u_{r-1})$$

M1 A1

Therefore

$$(M + (n-r)m)(u_r - u_{r-1}) = mv$$

M1

and so

$$u_r - u_{r-1} = \frac{mv}{M + (n-r)m}$$

A1* (4)

Summing this result for $r = 1$ to $r = n$,

$$u_n - u_0 = \frac{mv}{M + (n-1)m} + \frac{mv}{M + (n-2)m} + \frac{mv}{M + (n-3)m} + \cdots + \frac{mv}{M + (n-n)m}$$

M1

Because

$$0 \le n - r \le n - 1$$

$$M \le M + (n-r)m \le M + (n-1)m$$

$$\frac{mv}{M + (n-1)m} \le \frac{mv}{M + (n-r)m} \le \frac{mv}{M}$$

with equality only for the term $r = n$

Thus

$$\frac{mv}{M + (n-1)m} + \frac{mv}{M + (n-2)m} + \frac{mv}{M + (n-3)m} + \cdots + \frac{mv}{M + (n-n)m} < \frac{nmv}{M}$$

E1

As $u_0 = 0$, $u_n < \frac{nmv}{M} = u$ **A1* (3)**

(iii) Considering the energy of the truck and the $(n-(r-1))$ projectiles before and after the r^{th} projectile is fired (the other $(r-1)$ already fired do not change their kinetic energy at this time),

$$K_r - K_{r-1} = \frac{1}{2}(M+(n-r)m){u_r}^2 + \frac{1}{2}m(v-u_{r-1})^2 - \frac{1}{2}\big(M+\big(n-(r-1)\big)m\big){u_{r-1}}^2$$

M1 A1

$$= \frac{1}{2}(M+(n-r)m)({u_r}^2 - {u_{r-1}}^2) + \frac{1}{2}m(v-u_{r-1})^2 - \frac{1}{2}m{u_{r-1}}^2$$

$$= \frac{1}{2}(M+(n-r)m)(u_r - u_{r-1})(u_r + u_{r-1}) + \frac{1}{2}mv^2 - mvu_{r-1}$$

$$= \frac{1}{2}mv(u_r + u_{r-1}) + \frac{1}{2}mv^2 - mvu_{r-1}$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}mv(u_r - u_{r-1})$$

M1

Summing this result for $r=1$ to $r=n$,

$$K_n - K_0 = \frac{1}{2}nmv^2 + \frac{1}{2}mv(u_n - u_0)$$

M1

So

$$K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n$$

A1* (5)

Now

$$u_n < \frac{nmv}{M}$$

so

$$\frac{1}{2}mvu_n < \frac{1}{2}\frac{nm^2v^2}{M}$$

M1

and thus

$$\underbrace{K_n = \frac{1}{2}nmv^2 + \frac{1}{2}mvu_n}_{=K} < \frac{1}{2}nmv^2 + \frac{1}{2}\frac{nm^2v^2}{M} = \frac{1}{2}nmv^2\left(1+\frac{m}{M}\right) < \frac{1}{2}nmv^2\left(\frac{nm}{M}+1\right)$$

M1

as $n>1$ **E1 (3)**

Question 12

(i)

$$\sum_{y=1}^{n}\sum_{x=1}^{n} P(X=x, Y=y) = 1$$

$$\sum_{y=1}^{n}\sum_{x=1}^{n} k(x+y) = 1$$

M1

$$k\sum_{y=1}^{n}\left(\frac{1}{2}n(n+1)+ny\right) = 1$$

M1 A1

$$k\left(\frac{1}{2}n^2(n+1)+\frac{1}{2}n^2(n+1)\right) = 1$$

M1

Therefore,

$$k = \frac{1}{n^2(n+1)}$$

A1 (5)

$$P(X=x) = \sum_{y=1}^{n} k(x+y) = k\left(nx+\frac{1}{2}n(n+1)\right) = \frac{\left(2nx+n(n+1)\right)}{2n^2(n+1)} = \frac{n+1+2x}{2n(n+1)}$$

M1 A1 (2)

$$P(Y=y) = \frac{n+1+2y}{2n(n+1)}$$

B1

For X and Y to be independent, $P(X=x, Y=y) = P(X=x)\times P(Y=y)$ **M1**

So

$$\frac{n+1+2x}{2n(n+1)}\times\frac{n+1+2y}{2n(n+1)} = \frac{(x+y)}{n^2(n+1)}$$

M1

$$(n+1+2x)(n+1+2y) = 4(n+1)(x+y)$$

$$(n+1)^2 - 2(n+1)(x+y) + 4xy = 0$$

$$\left((n+1)-(x+y)\right)^2 - (x-y)^2 = 0$$

M1

which does not happen for e.g. $x = n$, $y = 1$. (Many equally valid examples possible.)

X and Y are not independent. **E1 (5)**

(ii)

$$E(XY) = \sum_{y=1}^{n}\sum_{x=1}^{n} kxy(x+y) = k\sum_{y=1}^{n}\left(y\frac{n(n+1)(2n+1)}{6} + y^2\frac{n(n+1)}{2}\right)$$

M1

$$= k\frac{n^2(n+1)^2(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

M1 A1 (3)

$$E(X) = E(Y) = \sum_{x=1}^{n} x\frac{n+1+2x}{2n(n+1)} = \frac{\frac{1}{2}n(n+1)^2 + \frac{2}{6}n(n+1)(2n+1)}{2n(n+1)}$$

M1 A1

$$= \frac{(n+1)}{4} + \frac{(2n+1)}{6} = \frac{(7n+5)}{12}$$

A1 (3)

Thus

$$Cov(X,Y) = \frac{(n+1)(2n+1)}{6} - \left(\frac{(7n+5)}{12}\right)^2 = \frac{-n^2+2n-1}{144} = \frac{-(n-1)^2}{144} < 0$$

M1 **E1 (2)**

Question 13

$$V(x) = E((X-x)^2) = E(X^2) - 2xE(X) + x^2 = \sigma^2 + \mu^2 - 2x\mu + x^2 = \sigma^2 + (x-\mu)^2$$

M1 M1 M1 A1 (4)

$$E(Y) = E\big(V(X)\big) = E(\sigma^2 + (X-\mu)^2) = \sigma^2 + \sigma^2 = 2\sigma^2$$

M1 A1* (2)

If $X \sim U(0,1)$, then $\mu = \frac{1}{2}$ and $\sigma^2 = \frac{1}{12}$, so $V(x) = \frac{1}{12} + \left(x - \frac{1}{2}\right)^2 = x^2 - x + \frac{1}{3}$

B1 B1 M1 A1 (4)

$$Y = V(X) = X^2 - X + \frac{1}{3} = \frac{1}{12} + \left(X - \frac{1}{2}\right)^2$$

$$Y \in \left[\frac{1}{12}, \frac{1}{3}\right]$$

$$P(Y < y) = P\left(\frac{1}{12} + \left(X - \frac{1}{2}\right)^2 < y\right) = P\left(\frac{1}{2} - \sqrt{y - \frac{1}{12}} < X < \frac{1}{2} + \sqrt{y - \frac{1}{12}}\right)$$

$$= 2\sqrt{y - \frac{1}{12}}$$

M1 M1 A1

$f(y) = \frac{d}{dy}\big(F(y)\big) = \frac{d}{dy}\left(2\sqrt{y - \frac{1}{12}}\right) = \left(y - \frac{1}{12}\right)^{-\frac{1}{2}}$, $\frac{1}{12} \le y \le \frac{1}{3}$ and 0 otherwise.

M1 A1 A1 (6)

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy = \int_{\frac{1}{12}}^{\frac{1}{3}} \left(y - \frac{1}{12}\right)\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} + \frac{1}{12}\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy$$

M1 M1

$$= \left[\frac{2}{3}\left(y - \frac{1}{12}\right)^{\frac{3}{2}} + \frac{1}{6}\left(y - \frac{1}{12}\right)^{\frac{1}{2}}\right]_{\frac{1}{12}}^{\frac{1}{3}} = \frac{1}{12} + \frac{1}{12} = 2 \times \frac{1}{12}$$

M1 A1 (4)

as required.

Alternatively, for final integral,

let $u^2 = y - \frac{1}{12}$,

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy = \int_0^{\frac{1}{2}} \frac{u^2 + \frac{1}{12}}{u}\, 2u\,du = \left[\frac{2}{3}u^3 + \frac{1}{6}u\right]_0^{\frac{1}{2}} = 2 \times \frac{1}{12}$$

M1 **M1** **M1** **A1 (4)**

or further

let $u = y - \frac{1}{12}$,

$$E(Y) = \int_{\frac{1}{12}}^{\frac{1}{3}} y\left(y - \frac{1}{12}\right)^{-\frac{1}{2}} dy = \int_0^{\frac{1}{4}} \frac{u + \frac{1}{12}}{u^{\frac{1}{2}}}\, du = \left[\frac{2}{3}u^{\frac{3}{2}} + \frac{1}{6}u^{\frac{1}{2}}\right]_0^{\frac{1}{4}} = 2 \times \frac{1}{12}$$

M1 **M1** **M1** **A1 (4)**

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