

## Section A: Pure Mathematics

- 1** (i) Express  $(3 + 2\sqrt{5})^3$  in the form  $a + b\sqrt{5}$  where  $a$  and  $b$  are integers.
- (ii) Find the positive integers  $c$  and  $d$  such that  $\sqrt[3]{99 - 70\sqrt{2}} = c - d\sqrt{2}$ .
- (iii) Find the two real solutions of  $x^6 - 198x^3 + 1 = 0$ .

- 2** The square bracket notation  $[x]$  means the greatest integer less than or equal to  $x$ . For example,  $[\pi] = 3$ ,  $[\sqrt{24}] = 4$  and  $[5] = 5$ .

- (i) Sketch the graph of  $y = \sqrt{[x]}$  and show that

$$\int_0^a \sqrt{[x]} \, dx = \sum_{r=0}^{a-1} \sqrt{r}$$

when  $a$  is a positive integer.

- (ii) Show that  $\int_0^a 2^{[x]} \, dx = 2^a - 1$  when  $a$  is a positive integer.
- (iii) Determine an expression for  $\int_0^a 2^{[x]} \, dx$  when  $a$  is positive but not an integer.

- 3** (i) Show that  $x - 3$  is a factor of

$$x^3 - 5x^2 + 2x^2y + xy^2 - 8xy - 3y^2 + 6x + 6y. \quad (*)$$

Express  $(*)$  in the form  $(x - 3)(x + ay + b)(x + cy + d)$  where  $a$ ,  $b$ ,  $c$  and  $d$  are integers to be determined.

- (ii) Factorise  $6y^3 - y^2 - 21y + 2x^2 + 12x - 4xy + x^2y - 5xy^2 + 10$  into three linear factors.

4 Differentiate  $\sec t$  with respect to  $t$ .

(i) Use the substitution  $x = \sec t$  to show that  $\int_{\sqrt{2}}^2 \frac{1}{x^3 \sqrt{x^2 - 1}} dx = \frac{\sqrt{3} - 2}{8} + \frac{\pi}{24}$ .

(ii) Determine  $\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}} dx$ .

(iii) Determine  $\int \frac{1}{(x+2)\sqrt{x^2 + 4x - 5}} dx$ .

5 The positive integers can be split into five distinct arithmetic progressions, as shown:

$A$ : 1, 6, 11, 16, ...

$B$ : 2, 7, 12, 17, ...

$C$ : 3, 8, 13, 18, ...

$D$ : 4, 9, 14, 19, ...

$E$ : 5, 10, 15, 20, ...

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in  $B$  and any term in  $C$  is a term in  $E$ .

Prove also that the square of every term in  $B$  is a term in  $D$ . State and prove a similar claim about the square of every term in  $C$ .

(i) Prove that there are no positive integers  $x$  and  $y$  such that

$$x^2 + 5y = 243\,723.$$

(ii) Prove also that there are no positive integers  $x$  and  $y$  such that

$$x^4 + 2y^4 = 26\,081\,974.$$

- 6 The three points  $A$ ,  $B$  and  $C$  have coordinates  $(p_1, q_1)$ ,  $(p_2, q_2)$  and  $(p_3, q_3)$ , respectively. Find the point of intersection of the line joining  $A$  to the midpoint of  $BC$ , and the line joining  $B$  to the midpoint of  $AC$ . Verify that this point lies on the line joining  $C$  to the midpoint of  $AB$ .

The point  $H$  has coordinates  $(p_1 + p_2 + p_3, q_1 + q_2 + q_3)$ . Show that if the line  $AH$  intersects the line  $BC$  at right angles, then  $p_2^2 + q_2^2 = p_3^2 + q_3^2$ , and write down a similar result if the line  $BH$  intersects the line  $AC$  at right angles.

Deduce that if  $AH$  is perpendicular to  $BC$  and also  $BH$  is perpendicular to  $AC$ , then  $CH$  is perpendicular to  $AB$ .

- 7 (i) The function  $f(x)$  is defined for  $|x| < \frac{1}{5}$  by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where  $a_0 = 2$ ,  $a_1 = 7$  and  $a_n = 7a_{n-1} - 10a_{n-2}$  for  $n \geq 2$ .

Simplify  $f(x) - 7xf(x) + 10x^2f(x)$ , and hence show that  $f(x) = \frac{1}{1-2x} + \frac{1}{1-5x}$ .

Hence show that  $a_n = 2^n + 5^n$ .

- (ii) The function  $g(x)$  is defined for  $|x| < \frac{1}{3}$  by

$$g(x) = \sum_{n=0}^{\infty} b_n x^n,$$

where  $b_0 = 5$ ,  $b_1 = 10$ ,  $b_2 = 40$ ,  $b_3 = 100$  and  $b_n = pb_{n-1} + qb_{n-2}$  for  $n \geq 2$ . Obtain an expression for  $g(x)$  as the sum of two algebraic fractions and determine  $b_n$  in terms of  $n$ .

- 8 A sequence  $t_0, t_1, t_2, \dots$  is said to be *strictly increasing* if  $t_{n+1} > t_n$  for all  $n \geq 0$ .

- (i) The terms of the sequence  $x_0, x_1, x_2, \dots$  satisfy

$$x_{n+1} = \frac{x_n^2 + 6}{5}$$

for  $n \geq 0$ . Prove that if  $x_0 > 3$  then the sequence is strictly increasing.

- (ii) The terms of the sequence  $y_0, y_1, y_2, \dots$  satisfy

$$y_{n+1} = 5 - \frac{6}{y_n}$$

for  $n \geq 0$ . Prove that if  $2 < y_0 < 3$  then the sequence is strictly increasing but that  $y_n < 3$  for all  $n$ .

## Section B: Mechanics

- 9 A particle is projected over level ground with a speed  $u$  at an angle  $\theta$  above the horizontal. Derive an expression for the greatest height of the particle in terms of  $u$ ,  $\theta$  and  $g$ .

A particle is projected from the floor of a horizontal tunnel of height  $\frac{9}{10}d$ . Point  $P$  is  $\frac{1}{2}d$  metres vertically and  $d$  metres horizontally along the tunnel from the point of projection. The particle passes through point  $P$  and lands inside the tunnel without hitting the roof. Show that

$$\arctan \frac{3}{5} < \theta < \arctan 3.$$

- 10 A particle is travelling in a straight line. It accelerates from its initial velocity  $u$  to velocity  $v$ , where  $v > |u| > 0$ , travelling a distance  $d_1$  with uniform acceleration of magnitude  $3a$ . It then comes to rest after travelling a further distance  $d_2$  with uniform deceleration of magnitude  $a$ . Show that

(i) if  $u > 0$  then  $3d_1 < d_2$ ;

(ii) if  $u < 0$  then  $d_2 < 3d_1 < 2d_2$ .

Show also that the average speed of the particle (that is, the total distance travelled divided by the total time) is greater in the case  $u > 0$  than in the case  $u < 0$ .

**Note:** In this question  $d_1$  and  $d_2$  are distances travelled by the particle which are not the same, in the second case, as displacements from the starting point.

- 11 Two uniform ladders  $AB$  and  $BC$  of equal length are hinged smoothly at  $B$ . The weight of  $\widehat{AB}$  is  $W$  and the weight of  $BC$  is  $4W$ . The ladders stand on rough horizontal ground with  $\widehat{ABC} = 60^\circ$ . The coefficient of friction between each ladder and the ground is  $\mu$ .

A decorator of weight  $7W$  begins to climb the ladder  $AB$  slowly. When she has climbed up  $\frac{1}{3}$  of the ladder, one of the ladders slips. Which ladder slips, and what is the value of  $\mu$ ?

## Section C: Probability and Statistics

- 12** In a certain factory, microchips are made by two machines. Machine A makes a proportion  $\lambda$  of the chips, where  $0 < \lambda < 1$ , and machine B makes the rest. A proportion  $p$  of the chips made by machine A are perfect, and a proportion  $q$  of those made by machine B are perfect, where  $0 < p < 1$  and  $0 < q < 1$ . The chips are sorted into two groups: group 1 contains those that are perfect and group 2 contains those that are imperfect.

In a large random sample taken from group 1, it is found that  $\frac{2}{5}$  were made by machine A. Show that  $\lambda$  can be estimated as

$$\frac{2q}{3p + 2q}.$$

Subsequently, it is discovered that the sorting process is faulty: there is a probability of  $\frac{1}{4}$  that a perfect chip is assigned to group 2 and a probability of  $\frac{1}{4}$  that an imperfect chip is assigned to group 1. Taking into account this additional information, obtain a new estimate of  $\lambda$ .

- 13** (i) Three real numbers are drawn independently from the continuous rectangular distribution on  $[0, 1]$ . The random variable  $X$  is the maximum of the three numbers. Show that the probability that  $X \leq 0.8$  is 0.512, and calculate the expectation of  $X$ .
- (ii)  $N$  real numbers are drawn independently from a continuous rectangular distribution on  $[0, a]$ . The random variable  $X$  is the maximum of the  $N$  numbers. A hypothesis test with a significance level of 5% is carried out using the value,  $x$ , of  $X$ . The null hypothesis is that  $a = 1$  and the alternative hypothesis is that  $a < 1$ . The form of the test is such that  $H_0$  is rejected if  $x < c$ , for some chosen number  $c$ .

Using the approximation  $2^{10} \approx 10^3$ , determine the smallest integer value of  $N$  such that if  $x \leq 0.8$  the null hypothesis will be rejected.

With this value of  $N$ , write down the probability that the null hypothesis is rejected if  $a = 0.8$ , and find the probability that the null hypothesis is rejected if  $a = 0.9$ .

- 14** Three pirates are sharing out the contents of a treasure chest containing  $n$  gold coins and 2 lead coins. The first pirate takes out coins one at a time until he takes out one of the lead coins. The second pirate then takes out coins one at a time until she draws the second lead coin. The third pirate takes out all the gold coins remaining in the chest.

Find:

- (i) the probability that the first pirate will have some gold coins;
- (ii) the probability that the second pirate will have some gold coins;
- (iii) the probability that all three pirates will have some gold coins.