STEP 2 2013 Hints and Solutions

Question 1.

The gradient of a line from a general point on the curve to the origin can be calculated easily and the gradient of the curve at a general point can be found by differentiation. Setting these two things to be equal will then lead to the correct value of m. A similar consideration of gradients to the origin will establish the second result and if the line intersects the curve twice then a sketch will illustrate that there must be one intersection on each side of the point of contact found in the first case. A similar process will establish the result for part (ii).

For part (iii) the gradient of the line must be smaller than the gradient of the line through the origin which touches the curve, so the intersection with the y-axis must be at a positive value. This means that the conditions of part (ii) are met, which allows for the comparison between π^e and e^π to be made.

The condition given in part (iv) is equivalent to stating that the line is parallel to the one found at the very beginning of the question. This implies that the intersection with the y-axis is at a negative value and so an adjustment to the steps taken in part (ii) will establish the required result.

Question 2.

The obvious substitution in the first part leads easily to the required result. It should then be easy to establish the second result by making the integral into the sum of two integrals and noting that taking out a common factor leaves $(1-x)+x$ to be simplified. Integration by parts will lead to the next result after which taking out one of the factors of $(1-x)$ will allow the integral to be split into a difference of two integrals.

The result in part (ii) is most easily proved by induction. It is necessary to fill in the gap in the factorial on the denominator by multiplying both the numerator and denominator by the missing even number. In alternative approaches, it needs to be remembered that the product of the even numbers up to and including $2n$ can be written as $2^n n!$

The final part is a straightforward substitution, although care needs to be taken with the signs. The final result can be obtained using the relationship established in part (i) as none of the reasoning requires n to be an integer.

Question 3.

For it to be possible for the cubic to have three real roots it must have two stationary points. Since the coefficient of x^3 is positive it must have a specific shape. A sketch will show that only the two cases given will result in an intercept with the y-axis at a negative value.

In order for the cubic in part (ii) to have three positive roots, both of the turning points must be at positive values of x. Differentiation will allow most of the results to be established. The condition that $c < 0$ is needed to ensure that the leftmost root is also positive.

The condition $ab < 0$ implies that there must be a turning point at a positive value of x. The shape of the graph is as in part (i), but this time the intersection with the y-axis is at a positive value. This is sufficient to deduce the signs of the roots.

For part (iv) it is easiest to note that changing the value of c does not (as long as c remains negative) change whether or not the conditions of (*) are met. As this represents a vertical translation of the graph any example of a case satisfying (*) can be used to create an answer for this part by translating the graph sufficiently far downwards.

Question 4.

The equations of the line and circle are easily found and so the second point of intersection (and so the coordinates of M) can be easily found. The two parts of this question then involve regarding the coordinates of M as parametric equations.

In part (i) a is the parameter and is restricted so that the point that the line passes through is inside the circle. This gives a straight line between the points which result from the cases $a = -1$ and $a = 1$. The length of this line can be determined easily from the coordinates of its endpoints.

In part (ii) it is again quite easy to eliminate the parameter from the pair of equations and the shapes of the loci should be easily recognised. In part (b) however, the restriction on the values of b need to be considered as the locus is not the whole shape that would be identified from the equation.

Question5.

Simple applications of the chain rule lead to relationships that will allow the three cases of zero gradients to be identified in part (i).

In part (ii) the relationships follow easily from substitution and therefore the three stationary points identified in part (i) must all exist. By considering the denominator there are clearly two vertical asymptotes and the numerator is clearly always positive. Additionally, the numerator is much larger than the denominator for large values of x. Given this information there is only one possible shape for the graph.

In part (iii) the solutions of the first equation will already have been discovered when the coordinates of the stationary points in part (ii) were calculated. The range of values satisfying the first inequality should therefore be straightforward. One of the solutions of the second equation should be easy to spot, and consideration of the graph shows that there must be a total of six roots. Applying the two relationships about the values of f will allow these other roots to be found. The solution set for the inequality then follows easily from consideration of the graph.

Question 6.

The definition of the sequence can be used to find a relationship between u_{n+2} and u_n and therefore also a relationship between u_n and u_{n-2}. Taking the difference of these then leads to the required result.

It is clear from the definition of the sequence that, if one term is between 1 and 2, then the next term will also be between 1 and 2. This is then easy to present in the form of a proof by induction for part (ii).

The result of part (i) shows that the sequence in part (iii) is increasing and the result proved in part (ii) shows that it is bounded above. The theorem provided at the start of the question therefore shows that the sequence converges. Similarly the second sequence is bounded below and decreasing (and therefore if the terms are all multiplied by -1 a sequence will be generated which is bounded above and increasing). Therefore the second sequence also converges to a limit.

The relationship between u_n and u_{n-2} established in part (i) can then be used to find the value of this limit and, as it is the same for both the odd terms and the even terms, the sequence must tend to the same limit as well.

Finally, starting the sequence at 3 will still lead to the same conclusion as the next term will be between 1 and 2 and all further terms will also be within that range, so all of the arguments will still hold for this new sequence.

Question 7.

A solution of the equation should be easy to spot and a simple substitution will establish the new solution that can be generated from an existing one. This therefore allows two further solutions to be found easily by repeated application of this result.

In part (ii) write $x = 2m + 1$ and $y = 2n$ and then substitute into (*). With some simplification the required relationship will be established.

Since b is a prime number there is only two ways in which it can be split into a product of two numbers ($1 \times b^3$ and $b \times b^2$). The right hand side of the equation is clearly a difference of two squares and therefore a pair of simultaneous equations can be solved to give expressions for a and c^2. Finally, the expression for c^2 is similar to the relationship established in part (ii), so solutions to the original equation can be used to generate values of a, b and c which satisfy this equation.

Question 8.

Begin by calculating the largest area of a rectangle with a given width and then maximize this function as the width of the rectangle is varied. The definition of x_0 can be reached by setting the derivative of the area function to 0.

The definition of g involves the differentiation of an integral of f which uses the variable t as the upper limit. The derivative of $tg(t)$ is therefore $f(t)$. The next statement relates the area bounded by the curve and the line $y = f(t)$ with the area of the largest rectangle with edges parallel to the axes that can fit into that space, so the first area must be greater and since that integral is equal to $tg(t) - tf(t)$ the result that follows is easily deduced.

The final part of the question involves finding expressions for $A_0(t)$ and $g(t)$ and then simplifying the relationship established at the end of part (ii).

Question 9.

Resolving the forces vertically will establish the first result. For the second part of the question it can be established that all of the frictional forces are equal in magnitude by taking moments about the centre of one of the discs. Resolving forces vertically and horizontally for the discs individually will then lead to simultaneous equations that can be solved for the magnitudes of the reaction and frictional forces.

Since the discs cannot overlap there is a minimum value that θ can take and the value of $\frac{\sin\theta}{1+\cos\theta}$ is increasing as θ increases. This allows the smallest possible value of the frictional force between the discs to be calculated and therefore it can be deduced that no equilibrium is possible if the coefficient of friction is below this minimum value.

Question 10.

Following the usual methods of considering horizontal and vertical parts of the motion will lead to the first result (some additional variables will need to be used, but they will cancel out to reach the final result.

If B and C are the same point then the result in part (i) can be applied for this point which will give an equation which is easily solved to give $\alpha = 60°$ once the double angle formula has been applied.

For the final part it is possible to find the times at which the particle reaches each of the two points. The two equations reached can then be used to find an expression for the difference between the time at which the particle reaches each of the two points and then it can easily be deduced whether this is positive or negative, which will show which point is reached first.

Question 11.

The standard methods of conservation of momentum and the law of restitution will allow the speeds after the second collision to be deduced. A third collision would have to be between the first and second particles and this will only happen if the velocity of the first particle is greater than that of the second one.

Providing a good notation is chosen to avoid too much confusion, it is possible to find the velocities after the third collision and then consider the velocities of the second and third particles to determine whether or not there is a fourth collision.

Question 12.

The formula for the expectation of a random variable should be well known and both of the expectations can easily be written in terms of α and β.

Similarly, the formula for variance should be well known and so it is a matter of rearranging the sums in such a way as to reach the forms given in the question. Note that the definitions of α and β are such that $e^{\lambda} = \alpha + \beta$.

Since the $Var(X + Y) = Var(U)$ the equation in the final part of the question can be rewritten in terms of the variables defined at the start of the question. It can then be shown that this is not possible for any non-zero value of λ.

Question 13.

An alternating run of length 1 must be two results showing the same side of the coin. It is then easy to show that the probability is as given. Similarly a straight run of length 1 must be two different results (in either order) and so the probability can again be calculated easily. The terms involved are those in the expansion of $(p \pm q)^2$ and so starting with the statement that $(p - q)^2 \geq 0$ then relationship between the two probabilities can be established.

An alternating run of length 2 must be one result followed by the other one twice, while a straight run of length 2 must be two identical results followed by the other one. They will therefore be calculated by the same sums (with the products in a different order each time) so the probabilities must be equal. By considering the ways in which runs of length 3 can be obtained it is clear that these two probabilities must also be equal.

An alternating run of length $2n$ must be n of each of the two possibilities followed by a repeat of whichever came last. A straight run of length $2n$ must be $2n$ of one of the possibilities followed by 1 of the other. Taking the difference between these two probabilities gives an expression which can be seen to always have the same sign, which will determine which probability is greater. A similar method will also work for the final case.