Q1 This question is all about using substitutions to simplify the working required to solve various increasingly complicated looking equations. To begin with, you are led gently by the hand in (i), where the initial substitution has been given directly to you. You are also reminded that y must be non-negative, since $\sqrt{x}$ denotes the *positive* square-root of x (positive unless x is 0, of course). The result is obviously a quadratic equation, $y^2 + 3y - \frac{1}{2} = 0$, and is solvable by (for instance) use of the quadratic formula. However, only one of the apparent solutions, $y = \dfrac{-3 \pm \sqrt{11}}{2}$, is positive, so the other is rejected and we proceed to find that $x = \left(\dfrac{\sqrt{11}-3}{2}\right)^2 = 5 - \frac{3}{2}\sqrt{11}$.

In (ii) (a), the approach used in (i) should lead you to consider the substitution $y = \sqrt{x+2}$, which gives the quadratic equation $y^2 + 10y - 24 = 0$. This, in turn, yields $y = \sqrt{x+2} = -12$ or 2. Again, this must be non-negative, so we find that $\sqrt{x+2} = 2$ and $x = 2$.

In (ii) (b), it should not now be too great a leap of faith to set $y = \sqrt{2x^2 - 8x - 3}$ which, with a bit of modest tinkering, yields up $y^2 + 2y - 15 = 0 \Rightarrow y = \sqrt{2x^2 - 8x - 3} = -5,\ 3$. Again, $y \ge 0$, so $\sqrt{2x^2 - 8x - 3} = 3 \Rightarrow$ (with cancelling) $x^2 - 4x - 6 = 0 \Rightarrow x = 2 \pm \sqrt{10} \Rightarrow x^2 = 14 \pm 4\sqrt{10}$. The final step here is to check that both apparent solutions work in the original equation, since the squaring process usually creates invalid solutions. [Note that it is actually quite easy to see that both solutions are indeed valid, but this still needs to be shown, or otherwise explained. Longer methods involving much squaring usually generated four solutions, two of which were *not* valid.]

Q2 The "leading actor" throughout this question is the *integer-part* function, $\lfloor x \rfloor$, often referred to as the *floor-function*. Purely as an aside, future STEP candidates may find it beneficial to play around with such strange, possibly artificial, kinds of functions as part of their preparation because, although they clearly go beyond the scope of standard syllabuses at this level, they *are* within reach and require little more than a willingness to be challenged. [Note that some care needs to be taken when exploring such things. In the case of this *floor-function*, at least a couple of function plotting software programs that recognise the "INT" function do so incorrectly for $x < 0$: for instance, interpreting INT(–2.7) as –2 rather than –3.]

The key elements of the sketch in (i) are as follows. The jump in the value of $\lfloor x \rfloor$ whenever x hits an integer value means that the graph is composed of lots of "unit" segments, the LH end of which is included but not the RH end. The usual convention for signalling these properties is that the LH endpoint has a filled-in dot while the RH endpoint has an open dot. Then, in-between integer values of x, each segment of the curve is of the form $\dfrac{n}{x}$ and thus appears to be a portion of a reciprocal curve.

The purpose of parts (ii) and (iii) is to see if you can use your graph to decide how to solve some otherwise fairly simple equations: the key is to have a clear idea as to where the various portions of the graph exist. The analysis looks complicated, but candidates were actually only required to pick the appropriate n's and write down the relevant answers (so the working only needed to reflect what was going on "inside one's head"). Note that, for $n \le x < n+1$, $\lfloor x \rfloor = n$ so $f(x) = \frac{n}{x}$. Also, $\frac{n}{n+1} < f(x) \le 1$ for $x > 0$, and $f(x) \ge 1$ for $x < 0$, so $f(x) = \frac{7}{12}$ only in [1, 2), yielding the equation $\frac{1}{x} = \frac{7}{12}$ in (ii) $\Rightarrow x = \frac{12}{7}$.

Similarly, $\frac{n}{n+1} > \frac{17}{24} \Rightarrow 24n > 17n + 17 \Rightarrow n > 2\frac{3}{7}$, i.e. $n \ge 3$; so $f(x) = \frac{17}{24}$ only in [1, 2) and [2, 3). In [1, 2), $f(x) = \frac{1}{x} = \frac{17}{24} \Rightarrow x = \frac{24}{17}$ and in [2, 3), $f(x) = \frac{2}{x} = \frac{17}{24} \Rightarrow x = \frac{48}{17}$. Next, for $x < 0$, $1 \le f(x) < \frac{n}{n+1}$, and $\frac{-n}{-n-1} < \frac{4}{3} \Rightarrow -4n - 4 > -3n \Rightarrow n < -4$; so $f(x) = \frac{4}{3}$ only in [–4, –3), [–3, –2), [–2, –1) and [–1, 0). However, since $f(-3) = 1$ there is no solution in [–4, –3). Otherwise, in [–3, –2), $f(x) = \frac{-3}{x} = \frac{4}{3} \Rightarrow x = -\frac{9}{4}$; in [–2, –1), $f(x) = \frac{-2}{x} = \frac{4}{3} \Rightarrow x = -\frac{3}{2}$; and in [–1, 0), $f(x) = \frac{-1}{x} = \frac{4}{3} \Rightarrow x = -\frac{3}{4}$.

For (iii), $\frac{n}{n+1} > \frac{9}{10}$ for $n > 9$ so $f(x_{max}) = \frac{9}{10}$ in [8, 9) and $f(x) = \frac{8}{x} = \frac{9}{10} \Rightarrow x = \frac{80}{9}$.

Only the very last part required any great depth of insight, and the ability to hold one's nerve. The equation $f(x) = c$ has exactly n roots when the horizontal line $y = c$ cuts the curve that number of times. That is …

… when $x > 0$: $\frac{n}{n+1} < c \le \frac{n+1}{n+2}$; … when $x < 0$: $\frac{n+1}{n} \le c < \frac{n}{n-1}$, $n \ge 2$; … and $c \ge 2$ for $n = 1$.

Q3 This vector question is tied up with the geometric understanding that, for distinct points with position vectors $\mathbf{x}$ and $\mathbf{y}$, the point with p.v. $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$ cuts XY in the ratio $(1-\lambda):\lambda$ (though it is important to realise that this point is only between X and Y if $0 < \lambda < 1$). Part (i) tests (algebraically) the property of *commutativity* (whether the composition yields different results if the order of the application of the operation is changed):

$$X*Y = Y*X \quad \Leftrightarrow \lambda\mathbf{x} + (1-\lambda)\mathbf{y} = \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \Leftrightarrow (2\lambda - 1)(\mathbf{x} - \mathbf{y}) = \mathbf{0} \Leftrightarrow (\text{since } \mathbf{x} \ne \mathbf{y})\ \lambda = \tfrac{1}{2}.$$

Part (ii) then explores the property of *associativity* (whether the outcome is changed when the order of the elements involved in two successive operations remains the same but the pairings within those successive operations is different). Here we have

$$(X*Y)*Z = \lambda(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) + (1-\lambda)\mathbf{z} = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$$

and $$X*(Y*Z) = \lambda\mathbf{x} + (1-\lambda)[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}] = \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$$

Thus, $(X*Y)*Z - X*(Y*Z) = \lambda(1-\lambda)(\mathbf{x} - \mathbf{z})$ and the two are distinct provided $\lambda \ne 0$, 1 or $X \ne Z$.

Part (iii) now explores a version of the property of *distributivity* (although usually referring to two distinct operations): $(X*Y)*Z = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$, and

$$(X*Z)*(Y*Z) = [\lambda\mathbf{x} + (1-\lambda)\mathbf{z}]*[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}] = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{z} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$$

$$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}, \text{ and the two are always equal.}$$

Next, $X*(Y*Z) = \lambda\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)^2\mathbf{z}$, and

$$(X*Y)*(X*Z) = [\lambda\mathbf{x} + (1-\lambda)\mathbf{y}]*[\lambda\mathbf{x} + (1-\lambda)\mathbf{z}]$$
$$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + \lambda(1-\lambda)\mathbf{x} + (1-\lambda)^2\mathbf{z}$$
$$= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}.$$

Hence $X*(Y*Z) = (X*Y)*(X*Z)$ also.

In (iv), you will notice that the condition $0 < \lambda < 1$ comes into play, so that P_1 cuts XY *internally* in the ratio $(1-\lambda)\!:\!\lambda$. Following this process through a couple more steps shows us that P_n cuts XY in the ratio $(1-\lambda^n)\!:\!\lambda^n$, which is easily established inductively.

Q4 The first part to this question involved two integrals which can readily be integrated by "recognition", upon spotting that

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(\mathrm{f}(\tan x)\big) = \mathrm{f}'(\tan x)\times\sec^2 x \text{ and } \frac{\mathrm{d}}{\mathrm{d}x}\big(\mathrm{f}(\sec x)\big) = \mathrm{f}'(\sec x)\times\sec x\tan x\,.$$

(They can, of course be integrated using suitable substitutions, etc.) Thus, we have

$$\int \tan^n x.\sec^2 x \;\mathrm{d}x = \left[\frac{1}{n+1}\tan^{n+1} x\right] = \frac{1}{n+1}$$

and $\displaystyle\int \sec^n x.\tan x \;\mathrm{d}x = \int \sec^{n-1} x.\sec x\tan x \;\mathrm{d}x = \left[\frac{1}{n}\sec^n x\right] = \frac{\left(\sqrt{2}\right)^n - 1}{n}$.

The two integrals in part (ii) can be approached in many different ways – the examiners worked out more than 25 slightly different approaches, depending upon how, and when, one used the identity $\sec^2 x = 1 + \tan^2 x$, how one split the "parts" in the process of "integrating by parts", and even whether one approached the various secondary integrals that arose as a function of $\sec x$ or $\tan x$. Only one of these approaches appears below for each of these two integrals..

$$\int_0^{\pi/4} x\sec^4 x\tan x\,\mathrm{d}x = \left[x.\frac{\sec^4 x}{4}\right]_0^{\pi/4} - \int_0^{\pi/4}\frac{\sec^4 x}{4}\,\mathrm{d}x \quad (\text{by } \textit{parts}) = \frac{\pi}{4} - \frac{1}{4}J\text{, where } J = \int_0^{\pi/4}\sec^4 x \;\mathrm{d}x.$$

Then, $\displaystyle J = \int_0^{\pi/4}\sec^2 x \;\mathrm{d}x + \int_0^{\pi/4}\sec^2 x\tan^2 x \;\mathrm{d}x = \left[\tan x + \tfrac{1}{3}\tan^3 x\right] = \frac{4}{3}$, and our integral is $\dfrac{\pi}{4} - \dfrac{1}{3}$.

Next, $\displaystyle\int x^2\left(\sec^2 x.\tan x\right)\mathrm{d}x = \left[x^2.\tfrac{1}{2}\tan^2 x\right] - \int 2x.\tfrac{1}{2}\tan^2 x\,\mathrm{d}x \;(\text{by } \textit{parts}) = \frac{\pi^2}{32} - \int x\left(\sec^2 x - 1\right)$

$$= \frac{\pi^2}{32} - K + \int x\,\mathrm{d}x,\;\text{ where } K = \int_0^{\pi/4} x\sec^2 x \;\mathrm{d}x.$$

Then, $\displaystyle K = \left[x.\tan x\right] - \int\tan x \;\mathrm{d}x = x\tan x - \ln(\sec x) = \frac{\pi}{4} - \frac{1}{2}\ln 2$, so that this last integral is

$$\frac{\pi^2}{32} - \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) + \frac{\pi^2}{32} \text{ or } \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln 2\,.$$

Q5 This question simply explores the different possibilities that arise when considering curves of a particular quadratic form. In (i), with a zero product term, we have equal amounts of x^2 and y^2, and this is symptomatic of a circle's equation: $x^2+3x+y^2+y=0 \Leftrightarrow \left(x+\frac{3}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=\left(\frac{1}{2}\sqrt{10}\right)^2$, which is a circle with centre $\left(-\frac{3}{2},-\frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{10}$. This circle passes through the points (0, 0), (0, –1) & (–3, 0).

In (ii), with $k=\frac{10}{3}$, we have $(3x+y)(x+3y+3)=0$. This factorisation tells us that we have the line-pair $y=-3x$ & $x+3y=-3$; the first line passing through the origin with negative gradient, while the second cuts the coordinate axes at (0, –1) and (–3, 0).

Part (iii) is the genuinely tough part of the question, but help is given to point you in the right direction. When k = 2, we have $(x+y)^2+3x+y=0$, and using θ = 45° in the given substitution gives $x+y=X\sqrt{2}$ and $y-x=Y\sqrt{2} \Rightarrow x=\dfrac{X-Y}{\sqrt{2}}$ and $y=\dfrac{X+Y}{\sqrt{2}}$. Then

$$(x+y)^2+3x+y=0 \text{ becomes } 2X^2+\frac{3X-3Y}{\sqrt{2}}+\frac{X+Y}{\sqrt{2}}=0 \Rightarrow 2X^2+2\sqrt{2}X=Y\sqrt{2} \text{ or}$$

$\left(\sqrt{2}X+1\right)^2-1=Y\sqrt{2}$. This is now in what should be a familiar form for a parabola, with axis of symmetry $X=\dfrac{-1}{\sqrt{2}} \Rightarrow$ (substituting back) $\dfrac{x+y}{\sqrt{2}}=\dfrac{-1}{\sqrt{2}}$ i.e. $x+y=-1$. For the sketch, we must rotate the standard parabola through 45° *anticlockwise* about O in order to get the original parabola $x^2+2xy+y^2+3x+y=0$.

For those who have encountered such things, all three curves here are examples of *conic sections*.

Q6 It should be pretty clear that this question is all about binomial coefficients. The opening result – the well-known *Pascal Triangle* formula for generating one row's entries from those of the previous row – is reasonably standard and can be established in any one of several ways. The one intended here was as follows: the coefficient of x^r in $(1+x)^{n+1}$ is $\binom{n+1}{r}$, and this is obtained from $(1+x)(1+x)^n$, where the coefficient of x^r comes from

$$(1+x)\left(\ \ldots\ldots+\binom{n}{r-1}x^{r-1}+\binom{n}{r}x^r+\ldots\ldots\ \right)$$

and the required result follows.

In the next stage, for n even, write $n=2m$ so that

$$B_{2m}+B_{2m+1}=\binom{2m}{0}+\binom{2m-1}{1}+\binom{2m-2}{2}+\ldots\ldots+\binom{m+1}{m-1}+\binom{m}{m}$$
$$+\binom{2m+1}{0}+\binom{2m}{1}+\binom{2m-1}{2}+\binom{2m-2}{3}+\ldots\ldots+\binom{m+1}{m}.$$

and, pairing these terms suitably, this is

$$\binom{2m+1}{0}+\left[\binom{2m}{0}+\binom{2m}{1}\right]+\left[\binom{2m-1}{1}+\binom{2m-1}{2}\right]+\ldots\ldots+\left[\binom{m+1}{m-1}+\binom{m+1}{m}\right]+\binom{m}{m}.$$

Now, using the opening result, and the fact that $\binom{2m+1}{0}=\binom{2m+2}{0}=\binom{m}{m}=\binom{m+1}{m+1}=1$, we have

$$\binom{2m+2}{0} + \left[\binom{2m+1}{1}\right] + \left[\binom{2m}{2}\right] + \ldots\ldots + \left[\binom{m+2}{m}\right] + \binom{m+1}{m+1},$$

and this is just $\sum_{j=0}^{m+1} \binom{2(m+1)-j}{j} = B_{2m+2}$, as required.

In the case n odd, write $n = 2m + 1$, so that

$$B_{2m+1} + B_{2m+2} = \binom{2m+1}{0} + \binom{2m}{1} + \binom{2m-1}{2} + \ldots\ldots + \binom{m+2}{m-1} + \binom{m+1}{m}$$

$$+ \binom{2m+2}{0} + \binom{2m+1}{1} + \binom{2m}{2} + \binom{2m-1}{3} + \ldots\ldots + \binom{m+2}{m} + \binom{m+1}{m+1}$$

which gives, upon pairing terms suitably,

$$\binom{2m+2}{0} + \left[\binom{2m+1}{0} + \binom{2m+1}{1}\right] + \left[\binom{2m}{1} + \binom{2m}{2}\right] + \ldots\ldots + \left[\binom{m+2}{m-1} + \binom{m+2}{m}\right] + \left[\binom{m+1}{m} + \binom{m+1}{m+1}\right]$$

$$= \binom{2m+3}{0} + \left[\binom{2m+2}{1}\right] + \left[\binom{2m+1}{2}\right] + \ldots\ldots + \left[\binom{m+3}{m}\right] + \binom{m+2}{m+1}$$

using the opening result and the fact that $\binom{2m+2}{0} = \binom{2m+3}{0} = 1$

$$= \sum_{j=0}^{m+1} \binom{2(m+1)+1-j}{j} = B_{2m+3}.$$

To complete an inductive proof, we must also check that the starting terms match up properly, but this is fairly straightforward. We conclude that, since $B_0 = F_1$, $B_1 = F_2$ and B_n & F_n satisfy the same recurrence relation, we must have $B_n = F_{n+1}$ for all n.

Q7 In (i), there is a generous tip given to help you on your way with this question. Starting from $y = ux$ we have $\frac{dy}{dx} = u + x\frac{du}{dx}$, so that the given differential equation becomes $u + x\frac{du}{dx} = \frac{1}{u} + u$ or $\int u \, du = \int \frac{1}{x} \, dx$ upon separation of variables. You are now in much more familiar territory and may proceed in the standard way: $\tfrac{1}{2}u^2 = \frac{y^2}{2x^2} = \ln x + C \Rightarrow y^2 = x^2(2\ln x + 2C)$. Using the given conditions $x = 1$, $y = 2$ to determine $C = 2$ then gives the required answer $y = x\sqrt{2\ln x + 4}$. However, there is one small detail still required, namely to justify the taking of the *positive* square-root, which follows from the fact that $y > 0$ when $x = 1$. (Note that you were given $x > e^{-2}$, for the validity of the square-rooting to stand, so it is not necessary to justify this. However, it should serve as a hint that a similar justification may be required in the later parts of the question.)

In (ii), either of the substitutions $y = ux$ or $y = ux^2$ could be used to solve this second differential equation. In each case, the method then follows that of part (i)'s solution very closely indeed; separating variables, integrating, eliminating u and substituting in the condition $x = 1$, $y = 2$ to evaluate the arbitrary constant. The final steps require a justifying of the taking of the positive square-root and a statement of the appropriate condition on x in order to render the square-rooting a valid thing to do. The answer is $y = x\sqrt{5x^2 - 1}$ for $x > \frac{1}{\sqrt{5}}$.

In (iii), only the substitution $y = ux^2$ can be used to get a variable-separable differential equation, which boils down to $\int u \, \mathrm{d}u = \int \frac{1}{x^2} \, \mathrm{d}x \Rightarrow \frac{1}{2}u^2 = \frac{-1}{x} + D$. Using $x = 1$, $y = 2$ ($u = 2$) to evaluate the constant D leads to the answer $y = x\sqrt{6x^2 - 2x}$ for $x > \frac{1}{3}$.

Q8 This question is all about composition of functions and their associated domains and ranges. Whilst being essentially a very simple question, there is a lot of scope for minor oversights. One particular pitfall is to think that $\sqrt{x^2} = x$, when it is actually $|\, x \,|$. Also, when considering the composite function fg, it is essential that the domain of g (the function that "acts" first on x) is chosen so that the output values from it are suitable input values for f. You should check that this is so for the four composites required in part (i).

In (ii), the functions fg and gf *look* the same (both are $|\, x \,|$) but their domains and ranges are different: fg has domain $\mathbb{R}$ and range $y \geq 0$, while the second has domain $|\, x \,| \geq 1$ and range $y \geq 1$.

In (iii), the essentials of the graph of h are: it starts from (1, 1) and increases. Since $\sqrt{x^2 - 1}$ is just (x – a tiny bit) after a while, the graph of h approaches $y = 2x$ from below. Using similar reasoning, the graph of k for $x \geq 1$, also starts at (1, 1) but decreases asymptotically to zero. (It is well worth noting that $x - \sqrt{x^2 - 1}$ is the reciprocal of $x + \sqrt{x^2 - 1}$, since their product is 1.) However, this second graph has a second branch for $x \leq -1$, which is easily seen to be a rotation about O (through 180°) of the single branch of h, this time approaching $y = 2x$ from above. Finally, note that the domain of kh is $x \geq 1$, and since the range of h is $y \geq 1$, the range of kh is $0 < y \leq 1$.

Q9 This question incorporates the topics of collisions and projectiles, each of which consists of several well-known and oft quoted results. However, much of the algebraic processing can be shortcut by a few insightful observations. To begin with, if the two particles start together at ground level and also meet at their highest points, then they must have the same vertical components of velocity. Thus $u \sin \alpha = v \sin \beta$. Then, if they both return to their respective points of projection, the collision must have ensured that they both left the collision with the same horizontal velocity as when they arrived; giving, by *Conservation of Linear Momentum*, that $mu\cos\alpha = Mv\cos\beta$. Dividing these two results gives the required answer, $m \cot\alpha = M \cot\beta$.

The collision occurs when $t = \dfrac{u \sin \alpha}{g} = \dfrac{v \sin \beta}{g}$ (from the standard constant-acceleration formulae) and at the point when A has travelled a distance $b = \dfrac{u^2 \sin \alpha \cos \alpha}{g}$ and B has travelled a distance $\dfrac{v^2 \sin \beta \cos \beta}{g} = \dfrac{1}{g}(v \sin \beta)(v \cos \beta)$, the sum of these two distances being denoted d. Substituting for the brackets using the two initial results then gives $b = \dfrac{Md}{m + M}$, as required. Moreover, the height of the two particles at the collision is given by $y = \dfrac{u^2 \sin^2 \alpha}{2g}$, so that

$$h = \frac{1}{2} \times \frac{u^2 \sin\alpha\cos\alpha}{g} \times \frac{\sin\alpha}{\cos\alpha} = \tfrac{1}{2} b \tan\alpha \,.$$

Q10 This question makes more obvious use of the standard results for collisions, but also ties them up with *Newton's 2nd Law* (N2L) of motion and, implicitly, the *Friction Law* and resolution of forces (in the simplest possible form). Thus, if R is the normal contact reaction force of floor on puck, F the frictional resistance between floor and puck, we have (in very quick order) the results $R = mg$, $F = \mu R = \mu mg$ and, by N2L, $\mu mg = -ma$, where a is the puck's acceleration. The constant-acceleration formula $v^2 = u^2 - 2as$ then gives $w_{i+1}{}^2 = v_i{}^2 - 2\mu gd$ (where v_i is the speed of the puck when leaving the i-th barrier, for $i = 0, 1, 2, \ldots$, and w_i is its speed when arriving at the i-th barrier, for $i = 1, 2, 3, \ldots$). Also, *Newton's (Experimental) Law of Restitution* (NEL or NLR) gives $v_{i+1} = r.w_{i+1}$, from which it follows that $v_{i+1}{}^2 = r^2 v_i{}^2 - 2r^2\mu gd$, as required.

Iterating with this result, starting with $v_1{}^2 = r^2 v^2 - 2r^2\mu gd$, then leads to the general result $v_n{}^2 = r^{2n}v^2 - 2r^2\mu gd\left\{1 + r^2 + r^4 + \ldots + r^{2n-2}\right\}$. The large bracket is the sum-to-n-terms of a GP, namely $\frac{1-r^{2n}}{1-r^2}$, and you simply need to set $v_n = 0$ and tidy up in order to obtain the result

$$\frac{v^2}{2\mu gd} r^{2n} = \frac{\left(1-r^{2n}\right)r^2}{1-r^2}.$$

Replacing $\frac{v^2}{2\mu gd}$ by k and re-writing the expression for r^{2n} $\left(= \frac{r^2}{r^2 + k\left(1-r^2\right)}\right)$, we simply have to take logs and solve for n to get $n = \frac{\ln\left(\frac{r^2}{r^2 + k\left(1-r^2\right)}\right)}{2\ln r}$. Setting $r = \mathrm{e}^{-1}$ in this result, and tidying up, then gives $n = \tfrac{1}{2}\ln\left(1 + k\left(\mathrm{e}^2 - 1\right)\right)$.

When $r = 1$, the distance travelled is just nd, and we have $v^2 = 2\mu gnd$ and $n = \frac{v^2}{2\mu gd} = k$.

Q11 Since we are told that $\alpha + \beta < \tfrac{1}{2}\pi$, the two tensions $T\sin\beta$ and $T\cos\beta$ are effectively components of a notional force (***T***), inclined slightly to the right of the normal contact reaction force R, say. Hence, if there is motion, it will take place to the right. Then, resolving vertically and horizontally for the block, calling the frictional force (acting to the left) F, we have

$$R + T\sin\beta\cos\alpha + T\cos\beta\sin\alpha = W \text{ and } F + T\sin\beta\sin\alpha = T\cos\beta\cos\alpha$$

or, using trig. identities, $W = R + T\sin(\alpha + \beta)$ and $F = T\cos(\alpha + \beta)$. Since $W > T\sin(\alpha + \beta)$, it follows that $R > 0$ so the block does not rise. Otherwise, $F \le \mu R$ and using $\mu = \tan\lambda$ for equilibrium, we have $T\cos(\alpha + \beta) \le \tan\lambda\,\{W - T\sin(\alpha + \beta)\}$ i.e. $W\tan\lambda \ge T\tan\lambda\sin(\alpha + \beta) + T\cos(\alpha + \beta)$ $\Rightarrow W\sin\lambda \ge T\sin\lambda\sin(\alpha + \beta) + T\cos(\alpha + \beta)\cos\lambda = T\cos(\alpha + \beta - \lambda)$.

In the next part, $W = T\tan\tfrac{1}{2}(\alpha + \beta) \Rightarrow R = T\sin\tfrac{1}{2}(\alpha + \beta) - T\sin(\alpha + \beta) < 0$ so $R = 0$ (and $F = 0$) as the block lifts from ground. Taking unit vectors **i** and **j** in the directions $\rightarrow$ and $\uparrow$

respectively: $\boldsymbol{T_A} = \begin{pmatrix} -T\sin\beta\sin\alpha \\ T\sin\beta\cos\alpha \end{pmatrix}$, $\boldsymbol{T_B} = \begin{pmatrix} -T\cos\beta\cos\alpha \\ T\cos\beta\sin\alpha \end{pmatrix}$, $\boldsymbol{W} = \begin{pmatrix} 0 \\ -T\tan\frac{1}{2}(\alpha+\beta) \end{pmatrix}$, and the resultant force on the block is $\boldsymbol{T_A} + \boldsymbol{T_B} + \boldsymbol{W} = \begin{pmatrix} T\cos(\alpha+\beta) \\ T\sin(\alpha+\beta) - T\tan\frac{1}{2}(\alpha+\beta) \end{pmatrix}$, which is in the direction $\tan^{-1}\left(\dfrac{\sin(\alpha+\beta)-\tan\frac{1}{2}(\alpha+\beta)}{\cos(\alpha+\beta)}\right)$ (relative to **i**). Since $\alpha + \beta = 2\phi$, this is the direction

$$\tan^{-1}\left(\frac{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha+\beta) - \dfrac{\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha+\beta)}}{\cos(\alpha+\beta)}\right) = \tan^{-1}\left(\frac{\sin\frac{1}{2}(\alpha+\beta)\left\{2\cos^2\frac{1}{2}(\alpha+\beta)-1\right\}}{\cos(\alpha+\beta).\cos\frac{1}{2}(\alpha+\beta)}\right)$$

$= \tan^{-1}\left(\tan\frac{1}{2}(\alpha+\beta)\right) = \frac{1}{2}(\alpha+\beta) = \phi$, noting that $2\cos^2\frac{1}{2}(\alpha+\beta) - 1 = \cos(\alpha+\beta)$.

Q12 As with all such probability questions, there are many ways to approach the problem. The one shown here for part (i) is one that generalises well to later parts of the question. Suppose the container has 3R, 3B, 3G tablets. Then the probability is $\frac{3}{9}\times\frac{3}{8}\times\frac{3}{7} = \frac{3}{56}$ for one specified order (e.g. RBG). We then multiply by 3! = 6 for the number of permutations of the 3 colours to get $\frac{9}{28}$. The final part of (i) is really a test of whether you realiuse that this is the same situation viewed "in reverse", so the answer is the same.

Using the method above, with a suitable notation, in part (ii) we have

$$\mathrm{P}_3(n) = \frac{n}{3n}\times\frac{n}{3n-1}\times\frac{n}{3n-2}\times 3! = \frac{2n^2}{(3n-1)(3n-2)} \text{ or } \frac{n^3}{\binom{3n}{3}}.$$

Then in (iii), P(correct tablet on each of the n days) = $\mathrm{P}_2(n)\times\mathrm{P}_2(n-1)\times\mathrm{P}_2(n-2)\times \ldots \mathrm{P}_2(2)\times\mathrm{P}_2(1)$

$= \left[\dfrac{n^2.2!}{2n(2n-1)}\right]\times\left[\dfrac{(n-1)^2.2!}{(2n-2)(2n-3)}\right]\times\left[\dfrac{(n-2)^2.2!}{(2n-4)(2n-5)}\right]\times\ldots\times\left[\dfrac{2^2.2!}{4.3}\right]\times\left[\dfrac{1^2.2!}{2.1}\right]$

$= \dfrac{(n!)^2 2^n}{(2n)!}$ or $\dfrac{2^n}{\binom{2n}{n}}$ .Then, using $n! \approx \sqrt{2n\pi}\left(\dfrac{n}{\mathrm{e}}\right)^n$ and $(2n)! \approx \sqrt{4n\pi}\left(\dfrac{2n}{\mathrm{e}}\right)^{2n}$, we have

$$\text{Prob.} = \frac{2n\pi\times\dfrac{n^{2n}}{\mathrm{e}^{2n}}\times 2^n}{\sqrt{4n\pi}\times\dfrac{2^{2n}n^{2n}}{\mathrm{e}^{2n}}} = \frac{2n\pi\times 2^n}{2\sqrt{n\pi}\times 2^{2n}} = \frac{\sqrt{n\pi}}{2^n}.$$

Q13 First, note that $x \in \{0, 1, 2, 3\}$, so there are four probabilities to work out (actually, three, and the fourth follows by subtraction from 1). The easier ones to calculate are $X = 0$ and $X = 3$, so let us find these first.

For P($X = 0$): the 7 pairs from which a singleton can be chosen can be done in ${}^{26}\mathrm{C}_7$ ways; then, we can choose one from each pair in 2^7 ways; so that P($X = 0$) = $\dfrac{{}^{26}\mathrm{C}_7\times 2^7}{{}^{52}\mathrm{C}_7}$.

Now ${}^{26}C_7 = \dfrac{26.25.24.23.22.21.20}{7.6.5.4.3.2.1}$ & ${}^{52}C_7 = \dfrac{52.51.50.49.48.47.46}{7.6.5.4.3.2.1} \Rightarrow P(X = 0) = \dfrac{3520}{5593}$.

P(X = 3): the 3 pairs from 26 can be chosen in ${}^{26}C_3$ ways; then the one singleton from the remaining 23 pairs can be chosen in ${}^{23}C_1 = 23$ ways, and the one from that pair in 2^1 ways; so that $P(X = 3) = \dfrac{{}^{26}C_3 \times {}^{23}C_1 \times 2^1}{{}^{52}C_7} = \dfrac{5}{5593}$ (similarly for calculation).

For P(X = 1): the 1 pair can be chosen in ${}^{26}C_1 = 26$ ways; the 5 pairs from which a singleton is chosen can be done in ${}^{25}C_5$ ways; and the singletons from those pairs in 2^5 ways; so that $P(X = 1) = \dfrac{26 \times {}^{25}C_5 \times 2^5}{{}^{52}C_7} = \dfrac{1848}{5593}$.

For P(X = 2): the 2 pairs can be chosen in ${}^{26}C_2$ ways; then the 3 pairs from which a singleton is chosen can be done in ${}^{24}C_3$ ways; and the singletons from those pairs can be chosen in 2^3 ways; so that $P(X = 2) = \dfrac{{}^{26}C_2 \times {}^{24}C_3 \times 2^3}{{}^{52}C_7} = \dfrac{220}{5593}$.

Then $E(X) = \sum x \, . \, p(x) = \dfrac{1}{5593}\left(0 \times 3520 + 1 \times 1848 + 2 \times 220 + 3 \times 5\right)$

$$= \frac{2303}{5593} = \frac{7 \times 7 \times 47}{7 \times 17 \times 47} = \frac{7}{17}.$$