Step II, Hints and Answers
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HINTS AND ANSWERS

Q1 If you read through at least part (i) of the question, you will see that it is necessary to work with u_1, u_2, u_3 and u_5 (and hence, presumably – as the sequence is defined recursively – with u_4 also). Although it is not the only way to go about the problem, it makes sense to work each of these terms out first. Each will be an expression involving k and should ideally be simplified as you go. Thus, $u_1 = 2$ gives $u_2 = k - 18$, $u_3 = k - \dfrac{36}{k-18} = \dfrac{k^2 - 18k - 36}{k-18}$, etc. Then, for (a), $u_2 = 2$; for (b), $u_3 = 2$; and, for (c), $u_5 = 2$. Each result leads to a polynomial equation (of increasing orders) to be solved. Finally, you need to remember that, in the case of (c) for instance, of the four solutions given by the resulting equation, two of them must have arisen already in parts (a) and (b) – you'll see why if you think about it for a moment. Ideally, you would see this beforehand, and then this fact will help you factorise the quartic polynomial by the *factor theorem*.

A simple line of reasoning can be employed to establish the first result in (ii) without the need for a formal inductive proof. If $u_n \ge 2$, then $u_{n+1} = 37 - \dfrac{36}{u_n} \ge 37 - \dfrac{36}{2} = 19 > 2$. Since $u_1 = 2$, it follows that all terms of the sequence are ≥ 2. In fact, most of them are much bigger than this. Then, for the final part of the question, the informal observation that, eventually, all terms effectively become equal is all that is required. Setting $u_{n+1} = u_n = l$ (say) leads to a quadratic, with two roots, one of which is obviously less than 2 and can therefore be rejected.

Answers: (i) $k =$ (a) 20; (b) 0; (c) $\pm 6\sqrt{2}$. (ii) 36.

Q2 The formula books give a series for e^x. Setting $x = 1$ then gives you e as the limit of an infinite sum of positive terms, and the sum of the first four of these will then provide a lower bound to its value.

In the next part, you (again) can provide a perfectly sound argument for the required result without having to resort to a formally inductive one (although one would be perfectly valid, of course). Noting firstly that $4! = 24 > 16 = 2^4$, $(n + 4)!$ consists of the product of $4!$ and n positive integers, each greater than 2; while 2^{n+4} consists of 16 and a further n factors of 2. Since each term in the first number is greater than the corresponding term in the second, the result follows. [Alternatively, $4! > 2^4$ and $n! > 2^n \Rightarrow (n + 1)! = (n + 1) \times n! > 2 \times n!$ (since $n > 4$) $> 2 \times 2^n$ (by hypothesis) $= 2^{n+1}$, and proof follows by induction.] Now, adding the terms in the expansion for e **beyond** the cubed one, and noting that each is less than a corresponding power of $\frac{1}{2}$ using the result just established, gives $e < \frac{8}{3}$ + the sum-to-infinity of a convergent GP.

There are two common methods for showing that a stationary value of a curve is a max. or a min. One involves the second derivative evaluated at the point in question.

There are several drawbacks involved with this approach. One is that you have to differentiate twice (which is ok with simple functions). A second is that you need to know the exact value(s) of the variable being substituted (which isn't the case here). Another is that the sign of $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ doesn't necessarily tell you what is happening to the curve. (Think of the graph of $y = x^4$, which has $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0$ at the origin, yet the stationary point here ***is*** a minimum!)

Thus, it is the other approach that you are clearly intended to use on this occasion. This examines the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ slightly to each side of the point in question. When $x = \frac{1}{2}$, using $e < \frac{67}{24}$ shows …..; at $x = 1$, using $e > \frac{8}{3}$ shows …..; and at $x = \frac{5}{4}$, we can use any suitable bound for e, such as $e < 3$ for instance, to show that …..

Finally, since the answers are given in the question, it is important to state carefully the reasoning that supports these answers.

Q3 If you fail to notice that $\frac{1}{5+\sqrt{24}} = 5 - \sqrt{24}$, then this question is going to be a bit of a non-starter for you. The idea of conjugates, from the use of the *difference of two squares*, should be a familiar one. As is the *binomial theorem*, which you can now use to expand both $(5 + \sqrt{24})^4$ and $(5 - \sqrt{24})^4$. When you do this, you will see that all the $\sqrt{24}$ bits cancel out, to leave you with an integer. For the next part, some fairly simple inequality observations, such as

$$20.25 < 24 < 25 \Rightarrow 4.5 < \sqrt{24} < 5 \text{ and } 2 \times 100 = 200 < 208 = 11 \times 19 \Rightarrow \frac{2}{19} < \frac{11}{100}$$

help to establish the required results. It follows that $0.1^4 < (5 - \sqrt{24})^4 < 0.11^4$ and the difference between the integer and $(5 + \sqrt{24})^4$ is this small number, which lies between …..

For part (ii), it is simply necessary to mimic the work of part (i) but in a general setting, again starting with the key observations that $\frac{1}{N+\sqrt{N^2-1}} = N - \sqrt{N^2-1}$ and that the binomial expansions for $\left(N+\sqrt{N^2-1}\right)^k + \left(N-\sqrt{N^2-1}\right)^k$ will lead to the cancelling of all surd terms, to give an integer, M say. Now $\left(N-\sqrt{N^2-1}\right)^k$ is positive, and the reciprocal of a number > 1, so $\left(N-\sqrt{N^2-1}\right)^k \to 0+$ as $k \to \infty$. Also,

$$2N - \tfrac{1}{2} < N + \sqrt{N^2-1} < 2N \Rightarrow \frac{1}{2N-\frac{1}{2}} > N - \sqrt{N^2-1} > \frac{1}{2N}.$$

Thus $\left(N+\sqrt{N^2-1}\right)^k = M - \left(N-\sqrt{N^2-1}\right)^k$ differs from an integer (M) by less than $\left(\dfrac{1}{2N-\frac{1}{2}}\right)^k = (2N-\tfrac{1}{2})^{-k}$.

Answers: (i) 9601.9999

Q4 Using the given substitution, the initial result is established by splitting the integral into its two parts, and then making the simple observation that $\int_0^{\pi} x\,\mathrm{f}(\sin x)\ \mathrm{d}x = \int_0^{\pi} t\,\mathrm{f}(\sin t)\,\mathrm{d}t$.

This result is now used directly in (i), along with a substitution (such as $c = \cos x$). The resulting integration can be avoided by referring to your formula book, or done by using partial fractions. In (ii), the integral can be split into two; one from 0 to π, the second from π to 2π. The first of these is just (i)'s integral, and the second can be determined by using a substitution such as $y = x - \pi$ (the key here is that the limits will then match those of the initial result, which you should be looking to make use of as much as possible). In part (iii), the use of the double-angle formula for $\sin 2x$ gives an integral involving sines and cosines, but this must also count as a function of $\sin x$, since $\cos x = \sqrt{1-\sin^2 x}$. Thus the initial result may be applied here also. Once again, the substitution $c = \cos x$ reduces the integration to a standard one.

Answers: (i) $\frac{1}{4}\pi\ln 3$; (ii) $-\frac{1}{2}\pi\ln 3$; (iii) $\pi\ln\frac{4}{3}$.

Q5 The crucial observation here is that the integer-part (or *INT* or "floor") function is a whole number. Thus, when drawing the graphs, the two curves must coincide at the left-hand (integer) endpoints of each unit range, with the second curve slowly falling behind in the first instance, and remaining at the integer level in the second. Note that the curves with the *INT* function-bits in them will jump at integer values, and you should not therefore join them up at the right-hand ends (to form a continuous curve).

The easiest approach in (i) is not to consider $\int y_1\ \mathrm{d}x - \int y_2\ \mathrm{d}x$ (i.e. separately), but rather $\int (y_1 - y_2)\mathrm{d}x$. This gives a multiple of $x - [x]$ to consider at each step, and this simply gives a series of "unit" right-angled triangles of area $\frac{1}{2}$ to be summed.

In (ii), several possible approaches can be used, depending upon how you approached (i). If you again focus on the difference in area across a representative integer range, then you end up having to sum $k + \frac{11}{6}$ from $k = 1$ to $k = n - 1$. Otherwise, there is some integration (for the continuous curves) and some summation (for the integer-part lines) to be done, which may require the use of standard summation results for $\sum k$ and $\sum k^2$.

Answers: (i) $\frac{3}{2}n(n-1)$.

Q6 The two vectors to be used are clearly $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$**.** The inequality arises when you note that $\cos^2\theta \le 1$. The statement is an equality (equation) when $\cos\theta = \pm 1$, in which case the two vectors must be parallel, so that one is a (non-zero) multiple of the other. [The question cites an example of a result widely known as the *Cauchy-Schwarz Inequality*.] The equality case of the inequality is then used in the two following parts; simply in (i) – since we must have $y = z = \ldots..$, from which it follows that $x = \frac{1}{2}$ this. In (ii), you should check that this is indeed an equality case of the inequality when the two vectors are ….. and ….. The parallel condition (one being a multiple of the other) now gives p, q and r in terms of some parameter (say λ), and you can substitute them into the linear equation (of the two given this is clearly the more straightforward one to use), find λ, and then deduce p, q and r; these values actually being unique.

Answers: $x = \lambda a$, $y = \lambda b$ and $z = \lambda c$; (i) $x = 7$; (ii) $p = 24$, $q = 6$, $r = 1$.

Q7 This is a reasonably routine question to begin with. The general gradient to the curve can be found by differentiating either implicitly or parametrically. Finding the gradient and equation of line AP is also standard enough; as is setting $y = b$ in order to find the coordinates of Q: $\left(\frac{(1-k)a}{(1+k)}, b\right)$. The equation of line PQ follows a similar line of working, to get $y = \left(\frac{-(1-k^2)b}{2ka}\right)x + \frac{b(1+k^2)}{2k}$. If you are not familiar with the $t = \tan\frac{1}{2}$-angle identities, the next part should still not prove too taxing, as you should be able to quote, or derive (from the formula for $\tan(A + B)$ in the formula books), the formula for $\tan 2A$ soon enough; and the widely known, "*Pythagorean*", identity $\operatorname{cosec}^2 A = 1 + \cot^2 A$ will help you sort out the gradient and intercept of PQ to show that the two forms of this line are indeed the same when $k = \tan(\frac{1}{2}\alpha)$.

A sketch of the ellipse, though not explicitly asked-for, should be made (at least once) so that you can draw on the lines PQ in the cases $k = 0$ and $k = 1$.

Answers: Yes; PQ is the vertical tangent to the ellipse.
Yes; PQ is the horizontal tangent to the ellipse.

Q8 I'm afraid that this question involves but a single idea: namely, that of intersecting lines. The first two parts are simple "bookwork" tasks, requiring nothing more than an explanation of the vector form of a line equation as $\mathbf{r}$ = p.v. of any point on the line + some scalar multiple of any vector (such as $\mathbf{y} - \mathbf{x}$, in this case) parallel to the line; then the basic observation that $CB \parallel OA \Rightarrow \overrightarrow{CB} = \lambda\mathbf{a}$ to justify the second result.

Thereafter, it is simply a case, with (admittedly) increasingly complicated looking position vectors coming into play, of equating **a**'s and **c**'s in pairs of lines to find out the position vector of the point where they intersect. If the final part is to be answered numerically, then the parameter λ must cancel somewhere before the end.

Answers: (ii) $\mathbf{d} = \left(\dfrac{1}{1-\lambda}\right)\mathbf{c}$; $\mathbf{e} = \frac{1}{3}\mathbf{a}$; $\mathbf{f} = \mathbf{c} + \frac{1}{2}\lambda\mathbf{a}$; $\mathbf{g} = \left(\dfrac{2\lambda}{2+3\lambda}\right)\mathbf{a} + \left(\dfrac{2}{2+3\lambda}\right)\mathbf{c}$; $\mathbf{h} = \dfrac{2}{5}\mathbf{a}$.

Thus $OH : HA = 2 : 3$ (as H lies two-fifths of the way along the line OA).

Q9 The most important thing you can do on a question like this, is to draw a good, decent-sized diagram first, marking on it all the relevant forces. In fact, since a lot of extra forces come into play in the second part of the question, a completely new diagram here is pretty much essential. It is also helpful to have the painter, P, in a general position on the ladder; say, a distance xa from its base up along it. [Note that xa is so much better than x, so that – since all distances are now multiples of a – these will cancel in the moments equation and make things *look* simpler.] Now resolve twice and take moments (easier about the base of the ladder), and use the *Friction Law* (in its inequality form, since we don't need to know when it attains its maximum). And then sort out the remaining algebra. On this occasion, it is not unreasonable to assume that P is at the top of the ladder when slipping is most likely, and go from there.

In (ii), the extra forces involved are the weight of the table, the reaction forces between its legs and the ground **and** the reaction of the ladder's base on the table (previously ignored when the ladder was on the ground). The standard approach now is to assume that the system is rotationally stable and see when slipping occurs; then to assume that the system is translationally stable and see when tilting occurs. Again, this involves resolving twice and taking moments; using the *Friction Law* – with equilibrium broken when one of the reactions between table and ground is zero – and deciding which, if any, happens first.

Answers: Table slips on ground when P is distance $5a$ up the ladder. Table turns about edge furthest from the wall when P is distance $\frac{11}{3}a$ up the ladder. Thus, tilting occurs first.

Q10 The first two collisions, between A and B and then between B and C, each require the application of the principles of *conservation of linear momentum* (CLM) and *Newton's experimental law of restitution* (NEL or NLR). This will give the intermediate and final velocities of B along with the final velocities of A and C (although the latter is not needed anywhere) in terms of u. [It is simplest to take all velocities in the same direction, along AB, so that "opposite" directions will then be accounted for entirely (and consistently) by signs alone.] For a second collision between A and B, $V_A > V_B$ (irrespective of their signs!) and this leads to a quadratic equation in k. Note that any negative solutions are inappropriate here.

Using $k = 1$ (which presumably MUST lie in the range found previously), the velocities of all particles can now be noted less algebraically. The time between contacts is in two parts: the time for B to reach C, and then the time for A to catch up with B (from its new position when B & C collide). After B leaves C, it is only the relative speed of A and B that matters, and this simplifies the working considerably.

Answers: (i) $0 < k < \frac{3}{2}$.

Q11 The equations of motion in the x- and y-directions can be found by integrating up from accelerations, or by using the *constant-acceleration formulae*. Setting $y = 0$ gives $t = 0$ or $t = \ldots..$ (as usual). Substituting this into the expression for x then gives the distance OA.

In (i), the time when $\dot{x} = 0$ must occur before the time found above. This gives an inequality involving sine and cosine, which can be simplified to give the tangent of the angle required.

In (ii), OB is just OA with $\theta = 45^\circ$. Then OA is maximised either by calculus (a little trickier here) or by using the double-angle formulae for sines and cosines and then working with an expression of the form $a \cos 2\theta + b \sin 2\theta + c$, for which there is a standard piece of work to yield the form $R \cos(2\theta - \varnothing) + c$, which has an obvious maximum of $R + c$ (with R here being in terms of f and g).

For the very last part, $f = g$ with $\theta = 45^\circ$ gives $x = y$ for B's motion, and the particle moves up, and then down, a straight line inclined at 45° to the horizontal, to land at its original point of projection.

Answers: (i) $\alpha = \arctan\left(\dfrac{g}{2f}\right)$; (ii) answer as above.

Q12 In (i), the probability that one wicket is taken is

$$p(\text{A1} \cap \text{B0} \cap \text{C0}) + p(\text{A0} \cap \text{B1} \cap \text{C0}) + p(\text{A0} \cap \text{B0} \cap \text{C1}),$$

each of which is a product of three terms from a binomial distribution. The probability that it was Arthur who took the wicket is then the conditional probability

$$\frac{p(1,0,0)}{p(1,0,0) + p(0,1,0) + p(0,0,1)} .$$

Although this looks a pretty ferocious creature with all its terms in it, in fact almost all of them cancel in the fraction, and you are left with a few products to deal with (most involving further cancellable terms).

Part (ii) is a "quickie" – $30 \times \left(\dfrac{1}{36} + \dfrac{1}{25} + \dfrac{1}{41}\right)$ – to point you towards the use of the simple value of 3 in the next part. In (iii), since n is large and p is small, the Binomial can be approximated by the Poisson; and $p(W \geq 5) = 1 - \{p_0 + p_1 + p_2 + p_3 + p_4\}$. From here, you can use either the approximation $e^3 = 20$ (as given) and work with Poisson terms directly, or just resort to the use of the Poisson tables in your formula books.

Answers: (i) $\frac{3}{10}$.

Q13 To be honest, this was more of a counting question than anything, at least to begin with. Although it is possible to attack (i) by multiplying and adding various probabilities, it is most easily approached by examining the 24 permutations of {1, 2, 3, 4} individually, and seeing what choice is made in each case. To make life easy for yourself, be systematic in listing these possibilities.

This example should point you in the right direction, but don't be tempted to just write down the answer that you've spotted without any justification for how it arises ***in the general case***. To begin with, deal with what happens when the largest cone is offered first; then the second-largest being first; then the third. By this stage it should be easy to justify the general case as to what happens when the r^{th} largest cone is the first to be offered – then the largest is chosen if it appears first of the remaining (r – 1) cones that are bigger than the r^{th}. With probability …..

Answers: (i) $P_4(2) = \dfrac{7}{24}$; $P_4(3) = \dfrac{4}{24}$ or $\dfrac{1}{6}$; $P_4(1) = \dfrac{2}{24}$ or $\dfrac{1}{12}$;

(ii) $\dfrac{1}{n}\left\{0 + 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots.. + \dfrac{1}{n-1}\right\}$ or $\frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{r}$

Q14 For $y = \dfrac{1}{x\ln x}$, $y \to -\infty$ as $x \to 0$ and $y \to 0$ (+ve) as $x \to \infty$ are the obvious asymptotic tendencies of the graph. Since $\ln 1 = 0$, there is also a discontinuity at x = 1, and you must decide what happens to the graph either side of this point.

For the rest of the question, its is essential to be able to integrate $\dfrac{1}{x\ln x}$. This can be done either by the sneaky observation that it can be written in the form $\dfrac{1/x}{\ln x}$, so that the numerator is exactly the derivative of the denominator – a standard log. integral form – or by using a substitution such as $u = \ln x$.

In (i) and (ii), it is now just a case of substituting in the limits and sorting out the log. work. Having gained the answer for (ii), in log. form, the numerical approximation arises from using the first few terms of the series, given in the formula books, for $\ln(1 + x)$ with $x =$ …..

In the very final part, a range is given that turns out to be outside the non-zero part of the *pdf*. A little bit of work needs to be done to justify this, and then you can write down the answer.

Answers: (i) $\lambda = \dfrac{1}{\ln\frac{1}{2}}$ or $-\dfrac{1}{\ln 2}$; (iv) 0.