Step II, Hints and Answers
June 2005

STEP MATHEMATICS PAPER 2: 9470: JULY 2005: HINTS AND ANSWERS

Q1 Differentiation leads to $f'(x) = 2xe^{-x^2} - 2x^3e^{-x^2}$. Since $e^{-x^2} \neq 0$ for any finite x then $f'(x) = 0 \Rightarrow x - x^3 = 0 \Rightarrow x = 0, 1, -1$

For the rest of the question, observe first that $P'(x) - 2xP(x) \equiv x(x^2 - a^2)(x^2 - b^2)\,(*)$ within a multiplicative non - zero constant. Thus $P(x)$ can take the form $-x^4/2 + px^2 + q$ and hence substitution into (*) plus equating the coefficient of x^2 and constant terms leads to a possible result for $P(x)$.

A similar argument based on setting $P(x) = \sum_{i=0}^{4} c_i x^i$ is feasible, but it involves more working and so is correspondingly more error prone.

Alternatively, one can multiply (*) by e^{-x^2} and then integrate with respect to x to obtain $P(x)e^{-x^2} = \int x(x^2-a^2)(x^2-b^2)e^{-x^2}\,dx$. From $\int xe^{-x^2}\,dx = -(1/2)e^{-x^2}$ the integrals $\int x^3e^{-x^2}\,dx$ and $\int x^5e^{-x^2}\,dx$ can be evaluated by use of the integration by parts rule. It then only remains to cancel out the factor e^{-x^2} to obtain $P(x) = -x^4/2 + (a^2/2 + b^2/2 - 1)x^2 - 1 + a^2/2 + b^2/2 - a^2b^2/2$.

Q2 (a) (i) Following the definition of $f(N)$, it is immediate that $f(12) = 12(1 - 1/2)(1 - 1/3) = 4$, and $f(180) = 180(1 - 1/2)(1 - 1/3)(1 - 1/5) = 48$.

(ii) The result may seem obvious but care must be taken in order to construct a complete proof. For example, $N = p_1^{\alpha_1} \dots p_k^{\alpha_k} \Rightarrow f(N) = p_1^{\alpha_1 - 1} \dots p_k^{\alpha_k - 1}(p_1 - 1)\dots(p_k - 1)$. Thus as p_i is a positive integer and $\alpha_i - 1$ is a non-negative integer for $1 \leq i \leq k$, then $f(N)$ is an integer.

(b) In each of (i),(ii),(iii), the conclusion must be made clear.

(i) As $f(3)f(9) = 2 \times 6 = 12 \neq f(27) = 18$, then the statement is false.

(ii)For any two primes p and q, $f(p)f(q) = p(1 - 1/p)q(1 - 1/q) = pq(1 - 1/p)(1 - 1/q) = f(pq)$. Hence the statement is true.

(iii) Consider $f(5) = 4$, $f(6) = 2$, $f(30) = 8 = 2 \times 4$. Then as 6 is not a prime it is clear that the statement is false.

(c)Start with $p^{m-1}(p - 1) = 146410$, then without difficulty it will tbe found that $p = 11$ and $m = 5$ (not 4).

Q3 Here $dy/dx = x \sin x$ which is zero at $x = 0$ and is positive for $0 < x \leq \pi/2$. A further differentiation will show that $d^2y/dx^2 = 0$ at $x = 0$ and positive for $0 < x \leq \pi/2$. Also, as $y(0) = 0$ and $y(\pi/2) = 1$, then $0 \leq y \leq 1$ for $0 \leq x \leq \pi/2$, and the sketch can now be completed consistently with the above conclusions.

(i) $\int_0^{\pi/2} \sin x\,dx = \dots = 1$ and use of the integration by parts rule will show that $\int_0^{\pi/2} x\cos x\,dx = \pi/2 - 1$. The displayed result for $\int_0^{\pi/2} y\,dx$ then follows immediately.

(ii) Start with $\int_0^{\pi/2} y^2\,dx = \int_0^{\pi/2} \sin^2 x\,dx - \int_0^{\pi/2} x\sin 2x\,dx + \int_0^{\pi/2} x^2\cos^2 x\,dx$.

Next, express $\sin^2 x$ and $\cos^2 x$ in terms of $\cos 2x$ so that now there are only two essentially different integrals involving trigonometric terms, namely, $\int_0^{\pi/2} x\sin 2x\,dx$ and $\int_0^{\pi/2} x^2\cos 2x\,dx$. The second of these can be obtained from the first, again by application of the integration by parts rule. A correct application of this rule to the first integral and the careful collection of terms will lead to the displayed result.

For the final result, begin with the observation that $y^2 < y$ for $0 < x < \pi/2$. From this, it is immediate that $\int_0^{\pi/2} y^2\,dx < \int_0^{\pi/2} y\,dx$ and hence that $\pi^3/48 - \pi/8 < 2 - \pi/2$. The final displayed result can then be obtained without difficulty.

Q4 The first two parts of this question depend on the identity $\tan^{-1} A + \tan^{-1} B = \tan^{-1}[(A+B)/(1-AB)]$ which is simply another way of writing $\tan(\alpha+\beta) = [\tan\alpha + \tan\beta]/[1 - \tan\alpha\tan\beta]$.

In the second part, it follows from the data that

$$\tan^{-1}[1/(p+q+s)] + \tan^{-1}[1/(p+q+t)] = \tan^{-1}[1/(p+q)]$$

and

$$\tan^{-1}[1/(p+r+u)] + \tan^{-1}[1/(p+r+v)] = \tan^{-1}[1/(p+r)].$$

As also from the data,

$$\tan^{-1}[1/(p+q)] + \tan^{-1}[1/(p+r)] = \tan^{-1}(1/p),$$

then the proof is complete.

For the final part, it is clear that $p = 7$ and this leads to $st = (7+q)^2 + 1$, $uv = (7+r)^2$, $qr = 50$. From the second displayed result it is obvious that $q + s = 6$, $q + t = 14$, so that $(7+q)^2 + 1 = (6-q)(14-q) \Rightarrow q = 1$, and hence $s = 5$, $t = 13$. The values $r = 50$, $u = 25$, $v = 130$ can be obtained by a similar strategy.

The solution given above is not unique. Moreover, other plausible strategies may lead to incorrect solutions. It is important, therefore, to check that the solution obtained not only satisfies the displayed identity, but also the given conditions.

Q5 At the outset it should be emphasised that a large, well annotated diagram will enable insight into this question.

The first result may be obtained expeditiously by observing that if S_1 touches the sides BC, CA, AB at P, Q, R, respectively, then $AQ = AR = r \Rightarrow BR = c - r$, $CQ = b - r$. Thus $b - r + c - r = a \Rightarrow 2r = b + c - a$.

This result leads to $r = a(q-1)/2$ which is the key to the remainder of the question. In fact $R = [2bc - \pi a^2(q-1)^2]/\pi a^2$ $(*)$.

From the data $a^2 = b^2 + c^2 \Rightarrow 2bc = (a+b+c)(b+c-a) \Rightarrow bc/a^2 = (q^2-1)/2$ which together with (*) leads to the second displayed result.

The obtaining of the turning value of a quadratic function is routine. In this context the method of completion of the square is to be preferred to the use of the calculus. Where the critical value of q is obtained from $dR/dq = 0$, it is important to give a reason as to why this defines an upper bound for R.

Q6 (i) The power series representations of $(1+x)^{-k}$ for $k = 1, 2, 3$ are standard and should be well known by any candidate for this examination. In fact the general terms of these three series are x^n, $(n+1)x^n$, $(1/2)(n+1)(n+2)x^n$, respectively.

The displayed series may be summed by using the general terms obtained. Thus,

$\sum_{n=1}^{\infty} n2^{-n} = (1/2)(1-1/2)^{-2} = 2$,

and as $\sum_{n=1}^{\infty} n(n+1)2^{-n} = 8$, then $\sum_{n=1}^{\infty} n^2 2^{-n} = 8 - 2 = 6$.

(ii) The obtaining of the general term of the power series (*) for $(1-x)^{-1/2}$ is a straightforward application of the binomial series for a general exponent.

To sum the penultimate series, put $x = 1/3$ in (*) and to sum the final series, first differentiate (*) with respect to x and then put $x = 1/3$. The sums will be found to be $\sqrt{3/2}$ and $(1/4)\sqrt{3/2}$, respectively.

Q7 (i) The absence of a $\mathbf{k}$ component in the specification of the locus of P, shows immediately that its motion takes place in the plane $z = 0$, i.e. in the $x - y$ plane. Also, it is obvious that $x^2 + y^2 = 1$. Hence P describes a circle centre O and radius 1 in the $x - y$ plane.

For the locus of Q, it is helpful to write $x = (3/2)\cos(t+\pi/4)$, $y = 3\sin(t+\pi/4)$, $z = (3\sqrt{3}/2)\cos(t+\pi/4)$. It is then evident that $\sqrt{3}x - z = 0$ and so this defines the plane in which the motion of Q takes place. Furthermore, it is clear that $x^2 + y^2 + z^2 = 9$ which shows that the distance of Q from O is constant and equal to 3. Hence Q describes the circle centre O and radius 3 in the plane $\sqrt{3}x - z = 0$.

(ii) Use of the scalar product leads to $\cos\theta = |(1/2)\cos t\cos(t+\pi/4) + \sin t\sin(t+\pi/4)| = \ldots = |3/4\sqrt{2} - (1/4)\cos(2t+\pi/4)|$.

(iii) From the result just obtained, it is immediate that $\theta \geq \pi/4 \Rightarrow -1/\sqrt{2} \leq 3/4\sqrt{2} - c/4 \leq 1/\sqrt{2}$ $\quad (c \equiv \cos(2t+\pi/4) \Rightarrow -1/\sqrt{2} \leq c \leq 7/\sqrt{2}$,

and as $c \leq 1$, then c is restricted to the interval $-1/\sqrt{2} \leq c \leq 1$ so that $t \notin [\pi/4, \pi/2]$ and $t \notin [5\pi/4, 3\pi/2]$ are required. Hence $T = 3\pi/2$

Q8 Separation of variables will lead to

$$A - 1/y = \int x^3(1+x^2)^{-5/2}\,dx,$$

where A is a constant. There are several possible strategies for the evaluation of the integral on the right; by parts, or by any of such substitutions as $w = 1 + x^2$, $x = \tan t$, $x = \sinh v$. One way or another, the result of this integration correctly carried out will lead to the equivalent of

$$A - 1/y = -(1/3)x^2(1+x^2)^{-3/2} - (2/3)(1+x^2)^{-1/2}.$$

Use of the initial condition $y(0) = 1$ will then show that $A = 1/3$ and the required result follows at once.

$1/y = 1/3 + (2+3x^2)/[3(1+x^2)^{3/2}]$.

To obtain the required approximation for y for large postive x, first write

$$1/y \approx 1/3 + (2+3x^2)(1 - 3/2x^2)/3x^3,$$

from which it follows that $1/y = 1/3 + 1/x + O(1/x^3)$ and this can easily be worked to the displayed approximation for y.

For the sketch, the main features are that it has a zero gradient at $x = 0$ and that for $x > 0$, it is monotonically increasing and has exactly one point of inflexion and that it is asymptotic to the line $y = 3$.

The two given differential equations are related by $y = z^2$. Thus it is unnecessary to solve the second differential equation independently of the first. In any case, the question does not require a formal solution for z. Nevertheless, it is helpful to obtain the approximation $z \approx \pm\sqrt{3} \mp 3\sqrt{3}/2x$ from $y \approx 3 - 9/x$, for large positive x.

All the main features of the $x - z$ sketch may be derived from those listed above for the $x - y$ sketch. The two curves which make up the sketch of z are reflections of each other in the $x-$ axis. They start at $(0, \pm 1)$ and are asymptotic to the lines $z = \pm\sqrt{3}$.

Q9(i) To begin with it is essential to draw a complete force diagram without omission or duplication of forces. In this respect, no particular direction, such as the horizontal or up the plane, should be assumed for the action of P. A convenient specification for the direction of P is $\theta + \pi/6$ with the horizontal, where at this stage, $0 < \theta < \pi/2$, but otherwise is general.

For motion up the slope it is necessary that

$$P\cos\theta \geq mg + mg/2 + (1/2\sqrt{3})(mg\sqrt{3}/2) + (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta).$$

From this inequality it follows that $P\cos(\theta-\pi/6) \geq (11\sqrt{3}/8)mg$ so that P_{min} is defined by $\theta = \pi/6$ and hence is equal to $(11\sqrt{3}/8)mg$.

Thus P_{min} acts at a direction making an angle of $\pi/3$ above the horizontal.

(ii) In order to clarify ideas in the second part of this question, it is advisable to draw a separate diagram. The friction, which again is limiting, acts up the slope so that now

$$P\cos\theta \geq mg + mg/2 - (1/2\sqrt{3})(mg\sqrt{3}/2) - (1/\sqrt{3})(mg\sqrt{3} - P\sin\theta)$$

which implies $P\cos(\theta+\pi/6) \geq (\sqrt{3}/8)mg$. Thus in this case, $P_{min} = (\sqrt{3}/8)mg$ and is achieved when $\theta = -\pi/6$, so that P_{min} acts horizontally.

Q10 Take horizontal and vertical axes with origin at A and denote the position of the missile projected from A at time t by (x_1, y_1). Then prior to the collision, at time t_c,

$$x_1 = 80t,\ y_1 = 60t - 5t^2.$$

If (x_2, y_2) is the position of the anti-missile missile at time t, where $t_c \leq t \leq T$, then

$$x_2 = 180 - 120(t-T),\ y_2 = 160(t-T) - 5(t-T)^2.$$

At the collision,

$x_1 = x_2 \Rightarrow 200t_c = 120T + 180$ (1), $y_1 = y_2 \Rightarrow 60t_c - 5t_c^2 = 160(t_c - T) - 5(t_c - T)^2$ (2).

From (2) it follows that $T^2 + (32-2t)T - 20t = 0$ (3) and elimination of t from (1) and (3) yields $T^2 + [(151-6T)/5]T - 12T - 18 = 0$. Thus $T^2 - 91T + 90 = 0 \Rightarrow T = 1, 90$. However, in the absence of the collision, the flight time of the missile, would be 12 seconds, so that without ambiguity it may be concluded that $T = 1$.

Q11 Again, good supporting diagrams will enhance success with this question. The first result is standard. Beyond that, the motion of A up and down the slope need to be considered separately.

Let u be the velocity when the string breaks and T_1 be the time from this instant to when the particle A reaches its highest point. Thus $u = \lambda gT$, where $\lambda = (m_2 - m_1)/(m_2 + m_1)$, and as the deceleration of A during the time T_1 is g, then $T_1 = \lambda T$. Hence the total time taken by P to reach the highest point is $(1+\lambda)T$.

For the downward motion of A, the acceleration is $g/10$, so that the data given in the penultimate sentence of the question implies

$$(g/10)(1+\lambda)^2T^2 = (\lambda g/2)T^2 + (\lambda^2 g/2)T^2.$$

From this it follows that $\lambda = 1/4$ and hence that $m_1/m_2 = 3/5$.

Q12 It is important to adopt an effective notation. Thus, for example, let $\alpha \sim$ heads , $\beta \sim$ tails , $T \sim$ true , $F \sim$ false .

(i) Use of the multiplication and addition laws of probability leads immediately to $P(\alpha) = ap + bq$. The coin is given to be fair so that $P(\alpha) = 1/2$. Hence $2(ap + bq) = 1$.

(ii) Write $G = ap + bq$, then,

$P(\alpha) = P(\alpha TF)/[P(\alpha TF) + P(\beta FT)] = [G(1-G)/2]/[G(1-G)/2 + (1-G)G/2] = 1/2$.

independently of the value of G.

(iii) Here, it is given that $G = 1/2$. Although the argument below has some similarities with the previous working, there are important differences in the fine detail. Thus now,

$P(\alpha) = P(\alpha TT)/[P(\alpha TT) + P(\beta FF)] = [G^2/2]/[G^2/2 + (1-G)^2/2] = 1/2$.

Q13 The introductory result at the end of the first paragraph is standard. For the approximations exhibited in (i),(ii) and (iii) it is important to ensure that enough terms are taken in the relevant expansions.

(i) $q = 1 - (1+\lambda)e^{-\lambda} = 1 - (1+\lambda)[1 - \lambda + \lambda^2/2 + O(\lambda^3)]$, as $\lambda \to 0$.

Thus $q = \lambda^2/2 + O(\lambda^3) \approx \lambda^2/2$, as $\lambda \to 0$.

(ii) $P(Y = n) = p^n > 1 - \lambda \Rightarrow e^{-n\lambda}(1+\lambda)^n > 1 - \lambda$

$\Rightarrow [1 - n\lambda + n^2\lambda^2/2][1 + n\lambda + n(n-1)\lambda^2/2] + O(\lambda^3) > 1 - \lambda$

which leads to $n < 2/\lambda$ within $O(\lambda^3)$

(iii) Write $P(Y > 1 | Y > 0) = P(Y > 1)/P(Y > 0)$

$= (1 - q^n - npq^{n-1})/(1 - q^n) \approx 1 - np(\lambda^2/2)^{n-1}/[1 - (\lambda^2/2)^n]$,

from which follows the required result.

Q14 Standard methods lead to $k = 1/(1+\lambda)$ and $\mu = \lambda^2/[2(1+\lambda]$. respectively.

Also $E(X^2) = k + k\lambda^3/3 = (\lambda^3+3)/[3(1+\lambda)]$, so that $\sigma^2 = (\lambda^3+3)/[3(1+\lambda)] - \lambda^4/[4(1+\lambda)^2)] \Rightarrow \ldots \Rightarrow$ required result.

For the remainder of the question $k = 1/3$.

(i) The graph is made up of three segments corresponding to $x < 0$, $0 \le x \le 2$ and $x > 2$. In particular, the middle segment is a translation of the curve $y = \phi(x)/3$ upwards through $1/3$ parallel to the $f(x)$ axis.

(ii) If $F(x)$ is the CDF of X, then

$F(x) = \Phi(x)/3$ for $x \le 0$,

$F(x) = \Phi(x)/3 + x/3$ for $0 < x \le 2$,

$F(x) = \Phi(x)/3 + 2/3$ for $x > 2$.

(iii) Begin with $\mu = 2/3$, $\sigma^2 = 7/9$. then from the information given it is clear that

$P(0 < X < \mu + 2\sigma) = .9921/3 + 2/3 - 1/6 = 0.8307$.