# STEP MATHEMATICS 3

# 2018

# Mark Scheme

1. (i) $f(\beta) = \beta - \frac{1}{\beta} - \frac{1}{\beta^2}$

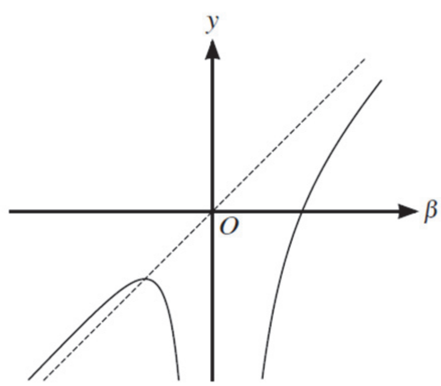
$f'(\beta) = 1 + \frac{1}{\beta^2} + \frac{2}{\beta^3} = \frac{\beta^3+\beta+2}{\beta^3}$

$f'(\beta) = 0 \Rightarrow \beta^3 + \beta + 2 = 0$ **M1**

$\beta^3 + \beta + 2 = (\beta + 1)(\beta^2 - \beta + 2)$

$\beta^2 - \beta + 2 \neq 0$ as $\beta^2 - \beta + 2 = \left(\beta - \frac{1}{2}\right)^2 + \frac{7}{4} \geq \frac{7}{4} > 0$ or discriminant $= -7$ **E1**

So the only stationary point is $(-1, -1)$ **A1**




**G2 (5)**

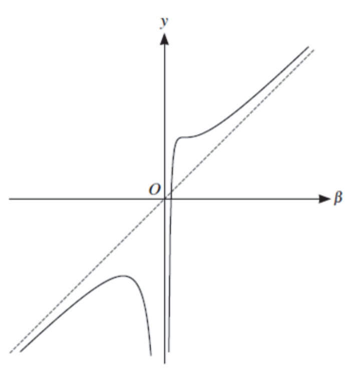
$g(\beta) = \beta + \frac{3}{\beta} - \frac{1}{\beta^2}$

Then $g'(\beta) = 1 - \frac{3}{\beta^2} + \frac{2}{\beta^3}$

$$1 - \frac{3}{\beta^2} + \frac{2}{\beta^3} = \frac{\beta^3 - 3\beta + 2}{\beta^3} = \frac{(\beta - 1)^2(\beta + 2)}{\beta^3}$$

**M1**

So the stationary points are $\left(-2, -\frac{15}{4}\right)$ and $(1,3)$ **A1**




**G2 (4)**

(ii) $u+v=-\alpha$ and $uv=\beta$ so $u+v+\frac{1}{uv}=-\alpha+\frac{1}{\beta}$

and $\frac{1}{u}+\frac{1}{v}+uv=\frac{u+v}{uv}+uv=\frac{-\alpha}{\beta}+\beta$ **B1 (1)**

(iii)

$u+v+\frac{1}{uv}=-1 \;\Rightarrow\; -\alpha+\frac{1}{\beta}=-1$ so $\alpha=1+\frac{1}{\beta}$

Thus $\frac{1}{u}+\frac{1}{v}+uv=\frac{-\alpha}{\beta}+\beta=\beta-\frac{1}{\beta}-\frac{1}{\beta^2}$ **M1**

$u,v$ real $\Leftrightarrow \alpha^2-4\beta\geq 0$

As $\alpha^2-4\beta\geq 0$, $\left(1+\frac{1}{\beta}\right)^2-4\beta\geq 0$ and thus $4\beta^3-\beta^2-2\beta-1\leq 0$

$(\beta-1)(4\beta^2+3\beta+1)\leq 0$ **M1 A1**

$4\beta^2+3\beta+1=\left(2\beta+\frac{3}{4}\right)^2+\frac{7}{16}\geq\frac{7}{16}>0$ or discriminant $=-7$ and positive quadratic so

$4\beta^2+3\beta+1>0$ , **E1**

and so $\beta\leq 1$ **B1**

Hence, from the sketch in part (i), $f(\beta)=\beta-\frac{1}{\beta}-\frac{1}{\beta^2}=\frac{1}{u}+\frac{1}{v}+uv\leq -1$ as required. **E1 (6)**

(iv) If $u+v+\frac{1}{uv}=3 \;\Rightarrow\; -\alpha+\frac{1}{\beta}=3$ so $\alpha=\frac{1}{\beta}-3$

Thus $\frac{1}{u}+\frac{1}{v}+uv=\frac{-\alpha}{\beta}+\beta=\beta+\frac{3}{\beta}-\frac{1}{\beta^2}$ **M1**

$u,v$ real $\Leftrightarrow \alpha^2-4\beta\geq 0$ , so $\left(\frac{1}{\beta}-3\right)^2-4\beta\geq 0$ and thus $4\beta^3-9\beta^2+6\beta-1\leq 0$

Therefore $(\beta-1)^2(4\beta-1)\leq 0$ and so $\beta=1$ or $\beta\leq\frac{1}{4}$ **M1 A1**

From the graph of $g(\beta)$ we can deduce that the greatest value of $\frac{1}{u}+\frac{1}{v}+uv$ is 3 **E1 (4)**

2. (i)

$$\frac{dy_n}{dx} = \frac{d}{dx}\left((-1)^n \frac{1}{z}\frac{d^n z}{dx^n}\right) = (-1)^n \left[2xe^{x^2}\frac{d^n z}{dx^n} + \frac{1}{z}\frac{d^{n+1}z}{dx^{n+1}}\right]$$

**M1**

$$= 2x(-1)^n \frac{1}{z}\frac{d^n z}{dx^n} - (-1)^{n+1}\frac{1}{z}\frac{d^{n+1}z}{dx^{n+1}}$$

**M1**

$$= 2xy_n - y_{n+1}$$

as required. **A1* (3)**

(ii) Suppose $y_{k+1} = 2xy_k - 2ky_{k-1}$ for some $k$ **B1**

$$\frac{dy_{k+1}}{dx} = 2x\frac{dy_k}{dx} + 2y_k - 2k\frac{dy_{k-1}}{dx}$$

**M1**

So using (i)

$$2xy_{k+1} - y_{k+2} = 2x(2xy_k - y_{k+1}) + 2y_k - 2k(2xy_{k-1} - y_k)$$

**M1**

$$= 4x^2y_k - 2xy_{k+1} + 2y_k + 2ky_k - 2x(2xy_k - y_{k+1})$$

**M1**

$$= 4x^2y_k - 2xy_{k+1} + 2y_k + 2ky_k - 4x^2y_k + 2xy_{k+1}$$

$$= 2y_k + 2ky_k$$

Thus $y_{k+2} = 2xy_{k+1} - 2(k+1)y_k$ which is the required result for $k+1$ **A1**

$$y_0 = 1$$

$$y_1 = (-1)\frac{1}{z}\frac{dz}{dx} = -1e^{x^2}\frac{d}{dx}\left(e^{-x^2}\right) = -e^{x^2}.-2xe^{-x^2} = 2x$$

**B1**

$$y_2 = (-1)^2\frac{1}{z}\frac{d^2z}{dx^2} = e^{x^2}\frac{d^2\left(e^{-x^2}\right)}{dx^2} = e^{x^2}\left[\frac{d}{dx}\left(-2xe^{-x^2}\right)\right] = e^{x^2}\left(-2e^{-x^2} + 4x^2e^{-x^2}\right)$$

$$= -2 + 4x^2$$

**M1A1**

$2xy_1 - 2 \times 1y_0 = 2x(2x) - 2 = 4x^2 - 2$ so result true for $n = 1$ **A1**

Hence $y_{n+1} = 2xy_n - 2ny_{n-1}$ for $n \geq 1$ by induction. **A1 (10)**

As $y_{n+1} = 2xy_n - 2ny_{n-1}$ , $y_{n+2} = 2xy_{n+1} - 2(n+1)y_n$ **M1**

Eliminating $x$,

$$(y_{n+1})^2 - y_n y_{n+2} = 2(n+1)(y_n)^2 - 2ny_{n-1}y_{n+1}$$

**M1**

Thus

$$y_{n+1}(y_{n+1} + 2ny_{n-1}) = y_n(y_{n+2} + 2(n+1)y_n)$$

So

$$y_{n+1}{}^2 - y_n y_{n+2} = 2n(y_n{}^2 - y_{n-1}y_{n+1}) + 2y_n{}^2$$

**A1 (3)**

(iii) Suppose $y_k{}^2 - y_{k-1}y_{k+1} > 0$ for some $k \geq 1$ **B1**

Then, as $2y_k{}^2 \geq 0$ , $2k(y_k{}^2 - y_{k-1}y_{k+1}) + 2y_k{}^2 > 0$ , i.e. by (ii) $y_{k+1}{}^2 - y_k y_{k+2} > 0$ **E1**

Consider $k = 1$

$$y_1{}^2 - y_0 y_2 = (2x)^2 - 1 \times (-2 + 4x^2) = 4x^2 + 2 - 4x^2 = 2 > 0$$

**B1**

Hence the result $y_n{}^2 - y_{n-1}y_{n+1} > 0$ for $n \geq 1$

**B1 (4)**

3.

$$x^a\left(x^b(x^c y)'\right)' = x^a[x^b(x^c y' + cx^{c-1}y)]'$$

**M1**

$$= x^a[x^{b+c}y' + cx^{b+c-1}y]'$$

$$= x^a[x^{b+c}y'' + cx^{b+c-1}y' + (b+c)x^{b+c-1}y' + c(b+c-1)x^{b+c-2}y]$$

$$= x^{a+b+c}y'' + (b+2c)x^{a+b+c-1}y' \ + c(b+c-1)x^{a+b+c-2}y$$

**M1A1**

This is of the required form if

$$a+b+c=2$$

$$b+2c=1-2p$$

$$c(b+c-1)=p^2-q^2$$

**M1**

Thus

$$c(1-2p-2c+c-1)=p^2-q^2$$

$$c^2+2pc+p^2=q^2$$

$$c+p=\pm q$$

**M1**

Thus it is possible if

$c=q-p$ , $b=1-2q$ , $a=1+p+q$

or

$c=-q-p$ , $b=1+2q$ , $a=1+p-q$

**A1 (6)**

(i) $x^2y''+(1-2p)xy'+(p^2-q^2)y=0$

So

$$x^a\left(x^b(x^c y)'\right)'=0$$

$$\left(x^b(x^c y)'\right)'=0$$

$$x^b(x^c y)'=A$$

**M1**

$$(x^c y)'=Ax^{-b}$$

$x^c y = \frac{Ax^{-b+1}}{-b+1} + B$ unless $b=1$ **B1**

$b \neq 1 \Rightarrow q \neq 0$ in which case

$$y = \frac{Ax^{-b-c+1}}{-b+1} + Bx^{-c}$$

$$y = \frac{Ax^{p\pm q}}{\pm 2q} + Bx^{p\pm q}$$

That is

$$y = Cx^{p+q} + Dx^{p-q}$$

**M1A1**

However, if $b = 1$ , $x^c y = A\ln x + B$ **M1**

so $y = Ax^{-c}\ln x + Bx^{-c}$

So if $q = 0$ , $y = Ax^p \ln x + Bx^p$ **M1A1 (7)**

(ii) $x^2 y'' + (1-2p)xy' + p^2 y = x^n$

Thus $q = 0$ , and $c = -p$ , $b = 1$ , $a = 1 + p$ **B1**

$$x^a(x(x^c y)')' = x^n$$

$$(x(x^c y)')' = x^{n-a}$$

$x(x^c y)' = \frac{x^{n+1-a}}{n+1-a} + A$ for $n+1-a \neq 0$ or $x(x^c y)' = \ln x + A$ for $n+1-a = 0$ **M1 B1**

$$n+1-a = 0 \Rightarrow n = p$$

Thus $(x^c y)' = \frac{x^{n-a}}{n+1-a} + Ax^{-1}$ or $(x^c y)' = x^{-1}\ln x + Ax^{-1}$

So $x^c y = \frac{x^{n+1-a}}{(n+1-a)^2} + A\ln x + B$ or $x^c y = \frac{(\ln x)^2}{2} + A\ln x + B$ **M1 M1**

So for $n \neq p$ , $y = \frac{x^n}{(n-p)^2} + A\,x^p \ln x + Bx^p$ **A1**

and for $n = p$ , $y = x^n \frac{(\ln x)^2}{2} + Ax^n \ln x + Bx^n$ **A1 (7)**

4.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

**M1**

(Alternatively,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$$

also earns **M1**)

Thus at P

$$\frac{dy}{dx} = \frac{a\sec\theta}{a^2}\frac{b^2}{b\tan\theta} = \frac{b}{a\sin\theta}$$

**A1**

So the tangent at P is

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

**M1**

Hence

$$ay\sin\theta - ab\frac{\sin^2\theta}{\cos\theta} = bx - ab\frac{1}{\cos\theta}$$

So

$$bx - ay\sin\theta = ab\left(\frac{1}{\cos\theta} - \frac{\sin^2\theta}{\cos\theta}\right) = ab\frac{\cos^2\theta}{\cos\theta} = ab\cos\theta$$

as required. **A1* (4)**

(i) S is the intersection of $bx - ay\sin\theta = ab\cos\theta$ and $\frac{x}{a} = \frac{y}{b}$

So $bx - bx\sin\theta = ab\cos\theta$ **M1**

Thus, S is $\left(\frac{a\cos\theta}{1-\sin\theta}, \frac{b\cos\theta}{1-\sin\theta}\right)$ **A1**

Similarly, T is the intersection of $bx - ay\sin\theta = ab\cos\theta$ and $\frac{x}{a} = -\frac{y}{b}$

So T is $\left(\frac{a\cos\theta}{1+\sin\theta}, -\frac{b\cos\theta}{1+\sin\theta}\right)$ **M1A1**

The midpoint of ST is therefore $\left(\frac{1}{2}\left(\frac{a\cos\theta}{1-\sin\theta} + \frac{a\cos\theta}{1+\sin\theta}\right), \frac{1}{2}\left(\frac{b\cos\theta}{1-\sin\theta} - \frac{b\cos\theta}{1+\sin\theta}\right)\right)$ **M1**

$$\frac{1}{2}\left(\frac{a\cos\theta}{1-\sin\theta} + \frac{a\cos\theta}{1+\sin\theta}\right) = \frac{1}{2}a\cos\theta\frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)} = a\sec\theta$$

$$\frac{1}{2}\left(\frac{b\cos\theta}{1-\sin\theta}-\frac{b\cos\theta}{1+\sin\theta}\right)=\frac{1}{2}b\cos\theta\frac{1+\sin\theta-1+\sin\theta}{(1-\sin\theta)(1+\sin\theta)}=b\tan\theta$$

**M1**

which means it is P. **A1 (7)**

(ii) As the tangents at P and Q are perpendicular,

$$\frac{b}{a\sin\theta}\times\frac{b}{a\sin\varphi}=-1$$

**B1**

(This is possible because $a>b$ )

That is

$$a^2\sin\theta\ \sin\varphi+b^2=0$$

The intersection of the tangents is given by the solution of

$$bx-ay\sin\theta=ab\cos\theta$$

$$bx-ay\sin\varphi=ab\cos\varphi$$

Thus

$$x=a\frac{(\sin\varphi\cos\theta-\sin\theta\cos\varphi)}{(\sin\varphi-\sin\theta)}$$

**M1**

and

$$y=b\frac{(\cos\theta-\cos\varphi)}{(\sin\varphi-\sin\theta)}$$

$$x^2=\left[a\frac{(\sin\varphi\cos\theta-\sin\theta\cos\varphi)}{(\sin\varphi-\sin\theta)}\right]^2$$

**A1**

$$y^2=-a^2\sin\theta\ \sin\varphi\left[\frac{(\cos\theta-\cos\varphi)}{(\sin\varphi-\sin\theta)}\right]^2$$

**A1**

Note $\sin\theta\ \sin\varphi=-\frac{b^2}{a^2}<0$ so $\sin\varphi\neq\sin\theta$ and so $\sin\varphi-\sin\theta\neq0$ **E1**

So

$$x^2+y^2=\frac{a^2}{(\sin\varphi-\sin\theta)^2}[(\sin\varphi\cos\theta-\sin\theta\cos\varphi)^2-\sin\theta\ \sin\varphi\,(\cos\theta-\cos\varphi)^2]$$

$$=\frac{a^2}{(\sin\varphi-\sin\theta)^2}[\sin^2\varphi\cos^2\theta+\sin^2\theta\cos^2\varphi-\sin\theta\ \sin\varphi\cos^2\theta-\sin\theta\ \sin\varphi\ \cos^2\varphi]$$

**M1**

$= \frac{a^2}{(\sin\varphi - \sin\theta)^2}[\sin^2\varphi\,(1-\sin^2\theta) + \sin^2\theta\,(1-\sin^2\varphi) - 2\sin\varphi\sin\theta + 2\sin\varphi\sin\theta -$
$\sin\theta\;\sin\varphi\cos^2\theta - \sin\theta\;\sin\varphi\;\cos^2\varphi]$ **M1**

$$= \frac{a^2}{(\sin\varphi - \sin\theta)^2}[(\sin\varphi - \sin\theta)^2 + \sin\varphi\sin\theta(2 - 2\sin\varphi\sin\theta - \cos^2\theta - \cos^2\varphi)]$$

$$= \frac{a^2}{(\sin\varphi - \sin\theta)^2}[(\sin\varphi - \sin\theta)^2 + \sin\varphi\sin\theta\;(\sin^2\varphi - 2\sin\varphi\sin\theta + \sin^2\theta)]$$

$$= a^2 + a^2\sin\theta\;\sin\varphi = a^2 - b^2$$

as required. **M1A1* (9)**

5. (i)

$(k+1)(\ A_{k+1}-G_{k+1})-k(A_k-G_k)\geq 0$

$\Leftrightarrow\ \ (a_1+a_2+\cdots+a_k+a_{k+1})-(k+1)G_{k+1}-(a_1+a_2+\cdots+a_k)+kG_k\geq 0$ **M1**

$\Leftrightarrow\ \ a_{k+1}+kG_k\geq(k+1)G_{k+1}$

$\Leftrightarrow\ \ \frac{a_{k+1}}{G_k}+k\geq(k+1)\frac{G_{k+1}}{G_k}$ as $G_k>0$ **M1 E1**

$\frac{G_{k+1}}{G_k}=\frac{\left(G_k{}^k a_{k+1}\right)^{1/k+1}}{G_k}=\left(\frac{a_{k+1}}{G_k}\right)^{1/k+1}=\lambda_k$ **B1**

So

$\frac{a_{k+1}}{G_k}+k\geq(k+1)\frac{G_{k+1}}{G_k}\ \Leftrightarrow\ \lambda_k{}^{k+1}+k\geq(k+1)\lambda_k$

$\Leftrightarrow\ \lambda_k{}^{k+1}-(k+1)\lambda_k+k\geq 0$ as required. **M1A1 (6)**

(ii) $f(x)=x^{k+1}-(k+1)x+k$

So $f'(x)=(k+1)x^k-(k+1)=(k+1)\left(x^k-1\right)$ and $f''(x)=(k+1)kx^{k-1}$ **M1M1**

Thus, if $x$ is positive, there is a single stationary point for $x=1$ and it is a minimum. **E1**

$f(1)=0$

and so $f(x)=x^{k+1}-(k+1)x+k\geq 0$ **E1* (4)**

(iii) (a) Assume $A_k\geq G_k$ for some particular $k$ **B1**

then by (ii), the condition for (i) is met and so $A_{k+1}-G_{k+1}\geq\frac{k}{k+1}(A_k-G_k)$ **E1**

and thus $A_{k+1}\geq G_{k+1}$

$A_1=a_1$ and $G_1=a_1$ and so $A_1\geq G_1$ (in fact $A_1=G_1$) **B1**

Thus, by the principle of mathematical induction, $A_n\geq G_n$ for all $n$ **B1 (4)**

(b) If $A_k=G_k$ for some $k$ , then as $A_n\geq G_n$ for all $n$ , $A_{k-1}\geq G_{k-1}$ **E1**

and by (i) and (ii) $A_{k-1}=G_{k-1}$ and **E1**

$$\left(\frac{a_k}{G_{k-1}}\right)^{1/k}=1$$

in which case $a_k=G_{k-1}$ . **B1**

But as $A_n = G_n$, $A_k = G_k$ and $a_k = G_{k-1}$ for all $k$, for $k = 1$ to $n$ **E1**

But, $A_1 = G_1 = a_1$ **B1** and so $a_2 = G_1 = a_1$ and thus $A_2 = G_2 = a_1$ and $a_3 = G_2 = a_1$ and so on up to $A_n = G_n = a_1$

Hence $a_1 = a_2 = a_3 = \cdots = a_n$ **E1 (6)**

6. (i)

$\overrightarrow{AQ}$ is parallel to $\overrightarrow{AC}$

So $q - a = \lambda(c - a)$ where $\lambda$ is real.

Therefore $\frac{q-a}{c-a} = \lambda$ which is real as required. **E1 (1)**

Hence,

$$\frac{q-a}{c-a} = \left(\frac{q-a}{c-a}\right)^*$$

**M1**

$$\left(\frac{q-a}{c-a}\right)^* = \frac{(q-a)^*}{(c-a)^*} = \frac{q^* - a^*}{c^* - a^*}$$

**M1**

So as

$$\frac{q-a}{c-a} = \frac{q^* - a^*}{c^* - a^*}$$

$$(c-a)(q^* - a^*) = (c^* - a^*)(q - a)$$

**A1* (3)**

$$(c-a)\left(q^* - \frac{1}{a}\right) = \left(\frac{1}{c} - \frac{1}{a}\right)(q - a)$$

**M1**

Multiplying by $ac$ ,

$$(c-a)(acq^* - c) = -(c-a)(q-a)$$

**M1**

Thus

$$acq^* - c = -(q-a)$$

as $c - a \neq 0$ **E1**

and so

$$q + acq^* = a + c$$

**A1* (4)**

(ii) Q lies on AC, so from (i)

$$q + acq^* = a + c$$

Also Q lies on BD, so similarly

$$q + bdq^* = b + d$$

**M1**

Subtracting $(ac - bd)q^* = (a + c) - (b + d)$ **M1***

Multiplying the AC equation by $bd$ and the BD one by $ac$ and subtracting,

$$(ac - bd)q = ac(b + d) - bd(a + c)$$

**M1**

So adding these two equations

$$(ac - bd)(q + q^*) = ac(b + d) - bd(a + c) + (a + c) - (b + d)$$

**M1**

Rearranging

$$(ac - bd)(q + q^*) = (a - b)(1 + cd) + (c - d)(1 + ab)$$

as required. **A1* (5)**

(iii) P lies on AB, so from (i)

$$p + abp^* = a + b$$

**M1**

But as $p$ is real, $p = p^*$ , and so

$$p + abp = a + b$$

**M1***

That is

$$p(1 + ab) = a + b$$

Similarly, as P lies on CD

$$p(1 + cd) = c + d$$

**M1**

Multiplying the final result of (ii) by $p$ we have

$$(ac - bd)p(q + q^*) = (a - b)p(1 + cd) + (c - d)p(1 + ab)$$

**M1**

Thus

$$(ac - bd)p(q + q^*) = (a - b)(c + d) + (c - d)(a + b)$$

**M1**

So, simplifying

$$(ac - bd)p(q + q^*) = 2ac - 2bd = 2(ac - bd)$$

And as $ac - bd \neq 0$ , $p(q + q^*) = 2$ **E1 A1* (7)**

7. (i)

$$\frac{(\cot\theta+i)^{2n+1}-(\cot\theta-1)^{2n+1}}{2i}=\frac{(\cos\theta+i\sin\theta)^{2n+1}-(\cos\theta-i\sin\theta)^{2n+1}}{2i\sin^{2n+1}\theta}$$

**M1**

$$=\frac{(\cos(2n+1)\theta+i\sin(2n+1)\theta)-(\cos(2n+1)\theta-i\sin(2n+1)\theta)}{2i\sin^{2n+1}\theta}$$

**M1**

$$=\frac{2i\sin(2n+1)\theta}{2i\sin^{2n+1}\theta}=\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta}$$

**A1* (3)**

Let $y=\cot\theta$ , then

$$\frac{(y+i)^{2n+1}-(y-i)^{2n+1}}{2i}$$

**M1**

$$=\frac{\left(y^{2n+1}+\binom{2n+1}{1}y^{2n}i+\binom{2n+1}{2}y^{2n-1}i^2+\cdots\right)-\left(y^{2n+1}-\binom{2n+1}{1}y^{2n}i+\binom{2n+1}{2}y^{2n-1}i^2-\cdots\right)}{2i}$$

**M1**

$$=\binom{2n+1}{1}y^{2n}-\binom{2n+1}{3}y^{2n-2}+\cdots+(i)^{2n}$$

**M1**

So if $(2n+1)\theta=m\pi$ where $m=1,2,\ldots,n$ , $\sin\theta\neq 0$ and $\sin(2n+1)\theta=0$ **E1**

So

$$\binom{2n+1}{1}\cot^{2n}\theta-\binom{2n+1}{3}\cot^{2n-2}\theta+\cdots+(-1)^n=0$$

Thus if $x=\cot^2\theta$ , $\binom{2n+1}{1}x^n-\binom{2n+1}{3}x^{n-1}+\cdots+(-1)^n=0$ which gives the required result. **A1* (5)**

(ii)

$$\sum_{m=1}^{n}\cot^2\left(\frac{m\pi}{2n+1}\right)$$

is the sum of the roots of the equation in part (i) , and so is equal to **M1**

$$\frac{\binom{2n+1}{3}}{\binom{2n+1}{1}}=\frac{(2n+1)!}{(2n-2)!\,3!}\times\frac{(2n)!\,1!}{(2n+1)!}=\frac{(2n)!}{(2n-2)!\,3!}=\frac{2n\times(2n-1)}{3\times 2}=\frac{n(2n-1)}{3}$$

**A1** **A1* (3)**

(iii) As $0 < \sin\theta < \theta < \tan\theta$ , $\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{1}{\tan\theta}$ as these are all positive, and

$$\frac{1}{\sin^2\theta} > \frac{1}{\theta^2} > \frac{1}{\tan^2\theta}$$

**M1**

Thus

$$\csc^2\theta > \frac{1}{\theta^2} > \cot^2\theta$$

**M1**

That is

$$1 + \cot^2\theta > \frac{1}{\theta^2} > \cot^2\theta$$

**M1**

or as required

$$\cot^2\theta < \frac{1}{\theta^2} < 1 + \cot^2\theta$$

$$\sum_{m=1}^{n} \cot^2\left(\frac{m\pi}{2n+1}\right) < \sum_{m=1}^{n} \left(\frac{2n+1}{m\pi}\right)^2 < \sum_{m=1}^{n} \left(1 + \cot^2\left(\frac{m\pi}{2n+1}\right)\right)$$

**M1**

$$\frac{n(2n-1)}{3} < \left(\frac{2n+1}{\pi}\right)^2 \sum_{m=1}^{n} \frac{1}{m^2} < n + \frac{n(2n-1)}{3}$$

**M1**

$$\frac{n(2n-1)}{3(2n+1)^2} \times \pi^2 < \sum_{m=1}^{n} \frac{1}{m^2} < \frac{n(2n+2)}{3(2n+1)^2} \times \pi^2$$

**M1**

Letting $n \to \infty$ ,

$$\frac{n(2n-1)}{3(2n+1)^2} \times \pi^2 \to \frac{\pi^2}{6}$$

**M1**

and

$$\frac{n(2n+2)}{3(2n+1)^2} \times \pi^2 \to \frac{\pi^2}{6}$$

**M1**

and so $\sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$ **A1* (9)**

8. (i)

$$\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{y(1+y)}\,dy=\int_{1}^{\infty}\frac{f(y)}{y(1+y)}\,dy$$

**M1**

Making a change of variable, $y=x^{-1}$ , $\frac{dy}{dx}=-x^{-2}$ **M1**

so

$$\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{y(1+y)}\,dy=\int_{1}^{\infty}\frac{f(y)}{y(1+y)}\,dy=\int_{1}^{0}\frac{f(x^{-1})}{x^{-1}(1+x^{-1})}.-x^{-2}\,dx$$

**M1**

$$=\int_{1}^{0}\frac{-f(x^{-1})}{(x+1)}.dx=\int_{0}^{1}\frac{f(x^{-1})}{(1+x)}.dx=I$$

**A1* (4)**

$$I=\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{y(1+y)}\,dy=\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{y}-\frac{f(y)}{1+y}\,dy$$

**M1**

$$=\sum_{n=1}^{\infty}\int_{x=n-1}^{n}\frac{f(x+1)}{x+1}dx-\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{1+y}\,dy$$

**M1**

$$=\sum_{n=1}^{\infty}\int_{x=n-1}^{n}\frac{f(x)}{x+1}dx-\sum_{n=1}^{\infty}\int_{y=n}^{n+1}\frac{f(y)}{1+y}\,dy$$

**M1**

$$=\int_{x=0}^{\infty}\frac{f(x)}{x+1}dx-\int_{x=1}^{\infty}\frac{f(x)}{x+1}dx$$

**M1**

$$=\int_{x=0}^{1}\frac{f(x)}{1+x}dx$$

as required. **A1* (5)**

(ii) From (i)

$$\int_{x=0}^{1} \frac{\{x^{-1}\}}{x+1}\,dx = \int_{x=0}^{1} \frac{\{x\}}{x+1}\,dx$$

**M1**

$$= \int_{x=0}^{1} \frac{x}{x+1}\,dx = \int_{x=0}^{1} 1 - \frac{1}{x+1}\,dx = [x - \ln(x+1)]_0^1 = 1 - \ln 2$$

**M1** **M1** **A1 (4)**

$$\int_{x=0}^{1} \frac{\{2x^{-1}\}}{x+1}\,dx$$

$\{2(x+1)\} = \{2x+2\} = \{2x\}$ **E1**

and so we can once again use the result from part (i), and thus

$$\int_{x=0}^{1} \frac{\{2x^{-1}\}}{x+1}\,dx = \int_{x=0}^{1} \frac{\{2x\}}{x+1}\,dx = \int_{x=0}^{\frac{1}{2}} \frac{2x}{x+1}\,dx + \int_{x=\frac{1}{2}}^{1} \frac{2x-1}{x+1}\,dx$$

**M1** **M1A1**

$$= 2[x - \ln(x+1)]_0^{\frac{1}{2}} + [2x - 3\ln(x+1)]_{\frac{1}{2}}^{1}$$

**dM1**

$$= 1 - 2\ln\frac{3}{2} + 2 - 3\ln 2 - 1 + 3\ln\frac{3}{2}$$

$$= 2 + \ln\frac{3}{16} = 2 - \ln\frac{16}{3}$$

**M1A1 (7)**

9. (i) NELI for the n-1 th collision between P and Q gives

$$v_{n-1} - u_{n-1} = e(v_{n-2} + u_{n-2})$$

**M1**

and conserving momentum

$$kmv_{n-1} + mu_{n-1} = mu_{n-2} - kmv_{n-2}$$

**M1**

which simplifies to

$$kv_{n-1} + u_{n-1} = u_{n-2} - kv_{n-2}$$

Eliminating $v_{n-2}$ between the two equations by multiplying the first by $k$ and the second by $e$ and adding gives **M1**

$$k(1+e)v_{n-1} + (e-k)u_{n-1} = e(1+k)u_{n-2}$$

**A1**

Similarly, the nth collision gives

$$v_n - u_n = e(v_{n-1} + u_{n-1})$$

and

$$kv_n + u_n = u_{n-1} - kv_{n-1}$$

**M1**

Eliminating $v_n$ between these two equations by multiplying the first by k and subtracting from the second **M1**

$(1+k)u_n = (1-ke)u_{n-1} - k(1+e)v_{n-1}$ (**) **A1**

Adding the left hand side of the equation from the n-1 th collision to the right hand side of that just obtained (and vice versa)

$$(1+k)u_n + e(1+k)u_{n-2} = (1-ke)u_{n-1} + (e-k)u_{n-1}$$

**M1**

Thus

$$(1+k)u_n + e(1+k)u_{n-2} = (1+e-k-ke)u_{n-1} = (1-k)(1+e)u_{n-1}$$

**M1**

Giving

$$(1+k)u_n - (1-k)(1+e)u_{n-1} + e(1+k)u_{n-2} = 0$$

**A1* (10)**

(ii) The first impact gives using (**)

$$\left(1+\frac{1}{34}\right)u_1 = \left(1-\frac{1}{34}\times\frac{1}{2}\right)u_0 - \frac{1}{34}\left(1+\frac{1}{2}\right)v_0$$

**M1**

Thus

$$70u_1 = 67u_0 - 3v_0$$

**A1**

Letting $n = 0$

$$u_0 = A + B$$

**M1**

and letting $n = 1$

$$A\left(\frac{7}{10}\right) + B\left(\frac{5}{7}\right) = u_1 = \frac{67u_0 - 3v_0}{70}$$

**M1**

So $49A + 50B = 67u_0 - 3v_0$

Thus

$$A = -17u_0 + 3v_0$$

and

$$B = 18u_0 - 3v_0$$

**M1 A1 (6)**

Thus

$$u_n = (-17u_0 + 3v_0)\left(\frac{7}{10}\right)^n + (18u_0 - 3v_0)\left(\frac{5}{7}\right)^n$$

**M1**

$$= \left(\frac{5}{7}\right)^n \left[(-17u_0 + 3v_0)\left(\frac{49}{50}\right)^n + (18u_0 - 3v_0)\right]$$

If $v_0 > 6u_0$ , ( $-17u_0 + 3v_0 > u_0$ ) and $18u_0 - 3v_0 < 0$ **M1**

For large $n$, the term $\left(\frac{49}{50}\right)^n \to 0$

**E1**

$u_n \to \left(\frac{5}{7}\right)^n (18u_0 - 3v_0) < 0$

**E1* (4)**

10. If G is the centre of mass of the combined disc and particle, then

$$(M+m) \times OG = M \times 0 + m \times a$$

**M1**

In equilibrium, G is vertically below A, so $\sin\beta = \frac{OG}{OA} = \frac{m}{M+m}$ **A1 (2)**

Applying the cosine rule to triangle OAP,

$$AP^2 = a^2 + a^2 - 2a^2 \cos\left(\frac{\pi}{2} - \beta\right) = 2a^2(1 - \sin\beta)$$

**M1** **M1**

Thus

$$\frac{AP}{a} = \sqrt{2(1-\sin\beta)} = \sqrt{2\left(1 - \frac{m}{M+m}\right)} = \sqrt{\frac{2M}{M+m}}$$

**M1** **A1* (4)**

The kinetic energy of the disc about L is $\frac{1}{2}I\dot{\theta}^2$ **B1** and the kinetic energy of the particle about L is $\frac{1}{2}m\left(AP\dot{\theta}\right)^2 = \frac{1}{2}m2a^2(1-\sin\beta)\dot{\theta}^2 = (1-\sin\beta)ma^2\dot{\theta}^2$

**B1** **M1**

The potential energy of the system relative to the zero level of the point G in equilibrium is $(M+m)gAG(1-\cos\theta) = (M+m)ga\cos\beta(1-\cos\theta)$

**B1** **M1**

So, during the motion, conserving energy

$$\frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (M+m)ga\cos\beta(1-\cos\theta)$$

Is constant. **E1 (6)**

$$\frac{1}{2}I\dot{\theta}^2 + (1-\sin\beta)ma^2\dot{\theta}^2 + (M+m)ga\cos\beta(1-\cos\theta) = c$$

Differentiating with respect to time,

$$I\dot{\theta}\ddot{\theta} + 2(1-\sin\beta)ma^2\dot{\theta}\ddot{\theta} + (M+m)ga\cos\beta\sin\theta\,\dot{\theta} = 0$$

**M1 A1**

Thus, as $m = \frac{3}{2}M$, $\sin\beta = \frac{m}{M+m} = \frac{3}{5}$ and $\cos\beta = \frac{4}{5}$ ($\cos\beta$ is positive as $\beta$ is acute) **B1**

and because $I = \frac{3}{2}Ma^2$,

$$\left(\frac{3}{2}Ma^2 + 2 \times \frac{2}{5} \times \frac{3}{2}Ma^2\right)\ddot{\theta} + \left(M + \frac{3}{2}M\right)ga \times \frac{4}{5}\sin\theta = 0$$

**M1**

For small oscillations, $\sin\theta \approx \theta$ , **M1**

so

$$\frac{27}{10}a\ddot{\theta} + 2g\theta \approx 0$$

That is

$$\ddot{\theta} \approx -\frac{20}{27}\frac{g}{a}\theta$$

**A1**

and hence the period of small oscillations is

$$2\pi\sqrt{\frac{27a}{20g}} = 3\pi\sqrt{\frac{3a}{5g}}$$

**M1** **A1* (8)**

11. At a general moment in the motion, when the acute angle between the string and the upward vertical is $\theta$ , and the speed of the particle is $v$ , resolving towards O

$$T' + mg\cos\theta = m\frac{v^2}{b}$$

where $T'$ is the tension in the string and $m$ is the mass of the particle. **M1**

So at the point when the string becomes slack,

$$g\cos\alpha = \frac{V^2}{b}$$

**M1**

i.e. $V^2 = bg\cos\alpha$ **A1 (3)**

If $x$ is the horizontal displacement of the particle from O at time $t$, and $y$ the vertical. Then

$$x = b\sin\alpha - Vt\cos\alpha$$

**M1**

and

$$y = b\cos\alpha + Vt\sin\alpha - \frac{1}{2}gt^2$$

**M1**

The string is taut when $x^2 + y^2 = b^2$ **M1**

So

$$(b\sin\alpha - VT\cos\alpha)^2 + \left(b\cos\alpha + VT\sin\alpha - \frac{1}{2}gT^2\right)^2 = b^2$$

**A1**

Thus

$$\begin{aligned} b^2\sin^2\alpha - 2b\sin\alpha VT\cos\alpha + V^2T^2\cos^2\alpha \\ + b^2\cos^2\alpha + 2b\sin\alpha VT\cos\alpha + V^2T^2\sin^2\alpha - bgT^2\cos\alpha - gVT^3\sin\alpha \\ + \frac{1}{4}g^2T^4 = b^2 \end{aligned}$$

**M1**

So

$$V^2T^2 - bgT^2\cos\alpha - gVT^3\sin\alpha + \frac{1}{4}g^2T^4 = 0$$

But as $V^2 = bg\cos\alpha$ ,

$$V^2T^2 - V^2T^2 - gVT^3\sin\alpha + \frac{1}{4}g^2T^4 = 0$$

**M1**

So

$$g^2T^4 = 4gVT^3 \sin\alpha$$

and as $T \neq 0$ ,

$$gT = 4V \sin\alpha$$

**A1* (7)**

$$\dot{x} = -V \cos\alpha$$

**B1**

$$\dot{y} = V \sin\alpha - gt$$

**B1**

Thus

$$\tan\beta = \frac{V\sin\alpha - gT}{-V\cos\alpha} = \frac{V\sin\alpha - 4V\sin\alpha}{-V\cos\alpha} = 3\tan\alpha$$

**M1 A1* (4)**

The particle comes instantaneously to rest if and only if its motion is radial at the point of impact.

In other words,

$$\frac{y}{x} = \tan\beta$$

when $t = T$ . **M1**

Thus

$$x = b\sin\alpha - VT\cos\alpha = b\sin\alpha - V\frac{4V\sin\alpha}{g}\cos\alpha = b\sin\alpha - 4b\sin\alpha\cos^2\alpha$$

**M1**

and

$$y = b\cos\alpha + VT\sin\alpha - \frac{1}{2}gT^2 = b\cos\alpha + V\frac{4V\sin\alpha}{g}\sin\alpha - \frac{1}{2}g\left(\frac{4V\sin\alpha}{g}\right)^2$$

$$= b\cos\alpha + 4b\cos\alpha\sin^2\alpha - 8\,b\cos\alpha\sin^2\alpha = b\cos\alpha - 4b\cos\alpha\sin^2\alpha$$

**M1**

So

$$\tan\beta = 3\tan\alpha = \frac{b\cos\alpha - 4b\cos\alpha\sin^2\alpha}{b\sin\alpha - 4b\sin\alpha\cos^2\alpha}$$

**M1**

Rewritten. this is

$$\frac{3\sin\alpha}{\cos\alpha} = \frac{\cos\alpha\,(1-4\sin^2\alpha)}{\sin\alpha\,(1-4\cos^2\alpha)}$$

$$3\sin^2\alpha\left(1-4(1-\sin^2\alpha)\right) = (1-\sin^2\alpha)(1-4\sin^2\alpha)$$

**M1**

Thus

$$8(\sin^2\alpha)^2 - 4\sin^2\alpha - 1 = 0$$

So

$$\sin^2\alpha = \frac{4\pm4\sqrt{3}}{16} = \frac{1\pm\sqrt{3}}{4}$$

However, $\sin^2\alpha > 0$ , so $\sin^2\alpha = \frac{1+\sqrt{3}}{4}$ **A1* (6)**

12. (i) $P(Y_k \le y)$ is the probability that at least $k$ numbers are less than or equal to $y$ **E1**

The probability that exactly $k$ are smaller than or equal to $y$ is given by

$$\binom{n}{k} y^k (1-y)^{n-k}$$

**M1**

So

$$P(Y_k \le y)$$

$$= \binom{n}{k} y^k (1-y)^{n-k} + \binom{n}{k+1} y^{k+1} (1-y)^{n-k-1} + \binom{n}{k+2} y^{k+2} (1-y)^{n-k-2} + \cdots \binom{n}{n} y^n$$

$$= \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}$$

as required. **A1* (3)**

(ii)

$$m\binom{n}{m} = \frac{m \times n!}{(n-m)!\,m!} = \frac{n!}{(n-m)!\,(m-1)!} = \frac{n \times (n-1)!}{\big((n-1)-(m-1)\big)!\,(m-1)!} = n\binom{n-1}{m-1}$$

**B1** **M1** **A1***

$$(n-m)\binom{n}{m} = \frac{(n-m) \times n!}{(n-m)!\,m!} = \frac{n!}{(n-m-1)!\,m!} = \frac{n \times (n-1)!}{\big((n-1)-m\big)!\,m!} = n\binom{n-1}{m}$$

**M1** **A1 (5)**

$$F(y) = \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}$$

so

$$f(y) = \frac{d}{dy} \sum_{m=k}^{n} \binom{n}{m} y^m (1-y)^{n-m}$$

**M1**

$$= \sum_{m=k}^{n} m\binom{n}{m} y^{m-1} (1-y)^{n-m} + \sum_{m=k}^{n} -(n-m)\binom{n}{m} y^m (1-y)^{n-m-1}$$

**M1**

$$= \sum_{m=k}^{n} m\binom{n}{m} y^{m-1} (1-y)^{n-m} + \sum_{m=k}^{n-1} -(n-m)\binom{n}{m} y^m (1-y)^{n-m-1}$$

**M1**

$$= \sum_{m=k}^{n} n\binom{n-1}{m-1} y^{m-1} (1-y)^{n-m} + \sum_{m=k}^{n-1} -n\binom{n-1}{m} y^m (1-y)^{n-m-1}$$

**M1**

$$= \sum_{m=k}^{n} n\binom{n-1}{m-1} y^{m-1}(1-y)^{n-m} + \sum_{m=k+1}^{n} -n\binom{n-1}{m-1} y^{m-1}(1-y)^{n-m}$$

**M1**

$$= n\binom{n-1}{k-1} y^{k-1}(1-y)^{n-k}$$

as required. **A1* (6)**

Because $f(y)$ is a probability density function, $\int_0^1 f(y)dy = 1$ **E1**

Thus

$$\int_0^1 y^{k-1}(1-y)^{n-k}dy = \frac{1}{n\binom{n-1}{k-1}}$$

**B1 (2)**

(iii)

$$E(Y_k) = n\binom{n-1}{k-1}\int_0^1 y \times y^{k-1}(1-y)^{n-k}dy = n\binom{n-1}{k-1}\int_0^1 y^k(1-y)^{n-k}dy$$

**M1**

$$= n\binom{n-1}{k-1}\int_0^1 y^{k+1-1}(1-y)^{n+1-(k+1)}dy = n\binom{n-1}{k-1}\frac{1}{(n+1)\binom{n+1-1}{k+1-1}}$$

**M1** **M1**

$$= n\binom{n-1}{k-1}\frac{1}{(n+1)\binom{n}{k}} = \frac{n(n-1)!}{(n-k)!\,(k-1)!}\frac{(n-k)!\,k!}{(n+1)n!} = \frac{k}{n+1}$$

**A1 (4)**

13.

$$G(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \cdots$$

$$G(1) = p_0 + p_1 + p_2 + p_3 + \cdots$$

$$G(-1) = p_0 - p_1 + p_2 - p_3 + \cdots$$

**M1**

$$G(1) + G(-1) = 2p_0 + 2p_2 + \cdots = 2(p_0 + p_2 + p_4 + \cdots) = 2P(X = 0 \; or \; 2 \; or \; 4 \; \cdots)$$

**M1**

Thus

$$P(X = 0 \; or \; 2 \; or \; 4 \; \cdots) = \frac{1}{2}\big(G(1) + G(-1)\big)$$

as required. **A1* (3)**

$$P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!}$$

$$G(t) = \sum_{r=0}^{\infty} t^r e^{-\lambda}\frac{\lambda^r}{r!} = e^{-\lambda}\sum_{r=0}^{\infty}\frac{(\lambda t)^r}{r!} = e^{-\lambda}e^{\lambda t} = e^{-\lambda(1-t)}$$

**M1** **A1* (2)**

(i)

$$\sum_{r=0}^{\infty} P(Y = r) = k\,P(X = 0 \; or \; 2 \; or \; 4 \; \cdots) = k \times \frac{1}{2}\big(G(1) + G(-1)\big)$$

**M1**

$$= \frac{k}{2}\left(1 + e^{-2\lambda}\right) = k\frac{e^{\lambda} + e^{-\lambda}}{2e^{\lambda}} = k\frac{\cosh\lambda}{e^{\lambda}}$$

As

$$\sum_{r=0}^{\infty} P(Y = r) = 1$$

$$k = \frac{e^{\lambda}}{\cosh\lambda}$$

**A1**

$$G_Y(t) = \sum_{r=0}^{\infty} P(Y = r)\,t^r = k\left(e^{-\lambda} + e^{-\lambda}\frac{\lambda^2}{2!}t^2 + e^{-\lambda}\frac{\lambda^4}{4!}t^4 + \cdots\right)$$

**M1**

$$= ke^{-\lambda}\left(1 + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^4}{4!} + \cdots\right)$$

$$= \frac{ke^{-\lambda}}{2}\left[\left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^4}{4!} + \cdots\right) + \left(1 - \lambda t + \frac{(\lambda t)^2}{2!} - \frac{(\lambda t)^3}{3!} + \frac{(\lambda t)^4}{4!} - \cdots\right)\right]$$

$$= \frac{e^{\lambda}e^{-\lambda}}{\cosh\lambda}\frac{e^{\lambda t}+e^{-\lambda t}}{2} = \frac{\cosh\lambda t}{\cosh\lambda}$$

**M1** **A1* (5)**

as required.

$$E(Y) = G'_Y(1)$$

$$G'_Y(t) = \frac{\lambda\sinh\lambda t}{\cosh\lambda}$$

**M1**

Thus

$$E(Y) = \frac{\lambda\sinh\lambda}{\cosh\lambda} = \lambda\tanh\lambda < \lambda$$

**M1** **A1* (3)**

for $\lambda > 0$

Alternatively,

$$E(Y) = \frac{\lambda\sinh\lambda}{\cosh\lambda} = \lambda\frac{e^{\lambda}-e^{-\lambda}}{e^{\lambda}+e^{-\lambda}} = \lambda\frac{1-e^{-2\lambda}}{1+e^{-2\lambda}} < \lambda$$

(ii)

$$G(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 + \cdots$$

$$G(1) = p_0 + p_1 + p_2 + p_3 + \cdots$$

$$G(-1) = p_0 - p_1 + p_2 - p_3 + \cdots$$

$$G(i) = p_0 + ip_1 - p_2 - ip_3 + \cdots$$

$$G(-i) = p_0 - ip_1 - p_2 + ip_3 + \cdots$$

$$G(1)+G(-1)+G(i)+G(-i) = 4p_0 + 4p_4 + \cdots = 4P(X = 0 \text{ or } 4 \cdots)$$

$$\sum_{r=0}^{\infty} P(Z=r) = c\,P(X = 0 \text{ or } 4 \cdots) = c \times \frac{1}{4}\big(G(1)+G(-1)+G(i)+G(-i)\big)$$

$$= \frac{c}{4}\big(1 + e^{-2\lambda} + e^{-\lambda(1-i)} + e^{-\lambda(1+i)}\big)$$

$$= \frac{c}{4}\big(1 + e^{-2\lambda} + e^{-\lambda}(\cos\lambda + i\sin\lambda + \cos\lambda - i\sin\lambda)\big)$$

$$= \frac{c}{4}\frac{\left(e^{\lambda}+e^{-\lambda}+2\cos\lambda\right)}{e^{\lambda}} = \frac{c(\cosh\lambda+\cos\lambda)}{2e^{\lambda}}$$

**M1**

As $\sum_{r=0}^{\infty} P(Z=r) = 1$ , $c = \frac{2e^{\lambda}}{\cosh\lambda+\cos\lambda}$ **A1**

$$G_Z(t) = \sum_{r=0}^{\infty} P(Z = r)\, t^r = c\left(e^{-\lambda} + e^{-\lambda}\frac{\lambda^4}{4!}t^2 + e^{-\lambda}\frac{\lambda^8}{8!}t^8 + \cdots\right)$$

$$= ce^{-\lambda}\left(1 + \frac{(\lambda t)^4}{4!} + \frac{(\lambda t)^8}{8!} + \cdots\right)$$

$$= \frac{ce^{-\lambda}}{4}\left[\left(1 + \lambda t + \frac{(\lambda t)^2}{2!} + \cdots\right) + \left(1 - \lambda t + \frac{(\lambda t)^2}{2!} - \cdots\right) + \left(1 + \lambda it - \frac{(\lambda t)^2}{2!} + \cdots\right) + \left(1 - \lambda it - \frac{(\lambda t)^2}{2!} + \cdots\right)\right]$$

$$= \frac{ce^{-\lambda}}{4}\left(e^{\lambda t} + e^{-\lambda t} + e^{i\lambda t} + e^{-i\lambda t}\right)$$

$$= \frac{ce^{-\lambda}}{2}(\cosh \lambda t + \cos \lambda t)$$

**M1**

As $G_Z(1) = 1$ , $c = \frac{2e^{\lambda}}{\cosh\lambda + \cos\lambda}$

So

$$G_Z(t) = \frac{(\cosh \lambda t + \cos \lambda t)}{(\cosh \lambda + \cos \lambda)}$$

**A1ft**

$$G'_Z(t) = \frac{\lambda(\sinh \lambda t - \sin \lambda t)}{(\cosh \lambda + \cos \lambda)}$$

And thus

$$E(Z) = G'_Z(1) = \frac{\lambda(\sinh \lambda - \sin \lambda)}{(\cosh \lambda + \cos \lambda)}$$

**M1**

If $\lambda = \frac{3\pi}{2}$, **M1**

$$E(Z) = \frac{3\pi}{2}\frac{\left(\sinh\frac{3\pi}{2} + 1\right)}{\cosh\frac{3\pi}{2}}$$

$$= \frac{3\pi}{2} \times \frac{\frac{e^{\frac{3\pi}{2}} - e^{\frac{-3\pi}{2}}}{2} + 1}{\frac{e^{\frac{3\pi}{2}} + e^{\frac{-3\pi}{2}}}{2}} = \frac{3\pi}{2} \times \frac{\frac{e^{\frac{3\pi}{2}} + e^{\frac{-3\pi}{2}}}{2} + \left(1 - e^{\frac{-3\pi}{2}}\right)}{\frac{e^{\frac{3\pi}{2}} + e^{\frac{-3\pi}{2}}}{2}}$$

As $e^{\frac{-3\pi}{2}} < 1$ , in this case, $E(Z) > \lambda$ and so no, $E(Z)$ is not less than $\lambda$ for all positive values of $\lambda$

**A1 (7)**

Cambridge Assessment Admissions Testing offers a range of tests to support selection and recruitment for higher education, professional organisations and governments around the world. Underpinned by robust and rigorous research, our assessments include:

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Cambridge Assessment
Admissions Testing
The Triangle Building
Shaftesbury Road
Cambridge
CB2 8EA
United Kingdom

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