Federigo Enriques's Quest to Prove the "Completeness Theorem"

Donald Babbitt and Judith Goodstein

The golden age of the Italian school of algebraic geometry began with Antonio Luigi Gaudenzio Giuseppe Cremona and included among its main contributors Enrico Castelnuovo, Federigo Enriques, and Francesco Severi. The Italian school spanned nearly a century, from the unification of Italy in 1861 to Enriques's posthumously published post-World War II monograph on algebraic surfaces [Enrq 49]. In the 1890s Enriques, a mathematician who once quipped that "intuition is the aristocratic way of discovery, rigour the plebian way" [Hodge 48], and his colleague and future brother-in-law Castelnuovo began their monumental work on the birational theory of algebraic surfaces over the complex numbers $\mathbf{C}$.[1] Severi joined them in this effort a few years later.

Broadly speaking, the aspect of algebraic surfaces that will concern us here can be traced back to Rudolf Friedrich Alfred Clebsch, Arthur Cayley, and Max Noether, who in 1868–1875 introduced two different genera for characterizing an algebraic surface F. The first (proposed by Clebsch) was the dimension of the space of algebraic regular (i.e., without poles or, over $\mathbf{C}$, holomorphic) 2-forms on F, a direct analog of *the genus of an algebraic curve C*, i.e., the dimension of the space of algebraic regular 1-forms on C. Shortly after, Arthur Cayley, seeking an easier way to calculate Clebsch's genus, introduced an expression in the degree and characteristics of the singularities of a generic projection of the surface to $\mathbf{P}^3$ which he hoped gave the same number. But, in fact, it turned out that the second could be negative, unlike the

first. The first number is now called the *geometric genus* and is denoted by p_g, and the second one is called the *arithmetic genus* and is denoted by p_a. The difference $p_g - p_a$ is called the *irregularity* of F and is always ≥ 0. If $p_g = p_a$ we say the surface is *regular*. Otherwise it is said to be *irregular*. In modern language: $p_a := (a_F - 1)$ where a_F is the Euler characteristic of the structure sheaf or the constant term of the Hilbert polynomial of F [Dieu 74].[2]

In 1885 Émile Picard initiated the study of "Picard integrals of the first kind" on an algebraic surface F, i.e., integrals whose integrands are closed algebraic regular 1-forms on an algebraic surface F. Subsequently he showed that there are no Picard integrals of the first kind on smooth surfaces $F \subset \mathbf{P}^3(\mathbf{C})$ which were known to be regular. In 1901, Enriques showed that when there are such integrals, the irregularity of F is > 0.

These results indicate that there is an interesting relation between the irregularity of F and the space of Picard integrals of the first kind on F.

The work of these mathematicians paved the way for the Fundamental Theorem of Irregular Surfaces—one of the triumphs of early-twentieth-century algebraic geometry. The theorem states that:

$$p_g - p_a = \dim P(F) = b_1/2$$

where $\dim P(F)$ is the dimension of the linear space of Picard integrals of the first kind on F and b_1 is the first Betti number of F.[3] The Fundamental Theorem and its proof came about, in turn, through the combined efforts of Picard, Enriques, Castelnuovo, Severi, and Henri Poincaré. In the modern language going back to Hodge, this is the assertion that $h^{0,1} = h^{1,0} = b_1/2$.

In his landmark paper, "Sulla proprietà caratteristica delle superficie algebriche irregolari",

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[1]*See Appendix A for the definitions of some of the key terms of birational algebraic geometry used in this paper.*

[2]*See [Mum 76], Ch. 6.C, for details on the Hilbert polynomial of F. Additional discussion of the arithmetic genus and its history can be found in [Kl 05], p. 243.*

[3]*More of this story can be found in [Kl 05].*

published in 1905 in the *Rendiconti dell'Accademia delle Scienze dell'Istituto di Bologna*, Enriques provided a key piece of the proof of the Fundamental Theorem [Enrq 05]. Generally referred to as the Completeness Theorem, Enriques's result and proof were of an algebro-geometric nature—i.e., not involving transcendental techniques such as complex analysis and topology. Francesco Severi offered his own algebro-geometric proof in 1905 also [Sev 05]. Five years later, in 1910, Poincaré gave an independent transcendental proof of both the Fundamental Theorem and the Completeness Theorem [Poin 10]. Although both Enriques's and Severi's proofs turned out to be fatally flawed, as Severi himself showed in 1921 [Sev 21],[4] neither mathematician gave up trying to find a satisfactory algebro- geometric proof of the Completeness Theorem. From 1921 on, Enriques and Severi carried on an open feud, which spilled over into other areas of their professional lives. By the time the English geometer Leonard Roth arrived in Rome in 1930 to study with them and Castelnuovo, relations between Enriques and Severi had completely soured. "Either he [Severi] had just taken offence or else he was in the process of giving it," Roth recalled [Roth63]. Perhaps testifying to their pugnacity, both ultimately joined the Fascist Party, Severi in 1932, Enriques in 1933.

The University of Rome geometer Fabio Conforto, who served as Castelnuovo's assistant in the early 1930s, later organized and edited Enriques's lectures on rational surfaces for publication, an exercise that gave him some insight into Enriques's character. At a gathering of mathematicians to commemorate Enriques, held in Rome in February of 1947, he noted: "The problem of conferring integrity upon the theorem of the characteristic series of a continuous complete system tormented Enriques for his entire life" [Conf 47]. Unfortunately, no mathematician would find such a proof during Enriques's lifetime. For reasons possibly aesthetic, the Italians chafed at Poincaré's transcendental proof.[5] In 1945, nine months before his death, Enriques sent a plaintive letter to the Italian algebraic geometer Beniamino Segre, in which he lamented the lack of a proof and suggested that his idea of infinitely close curves of higher order—an idea he had advanced in earlier lectures and elaborated on in several papers between 1936 and 1938—might lead to one. Enriques was right (!), although he did not have the mathematical tools to convert his idea into an acceptable algebro-geometric proof. It is the story
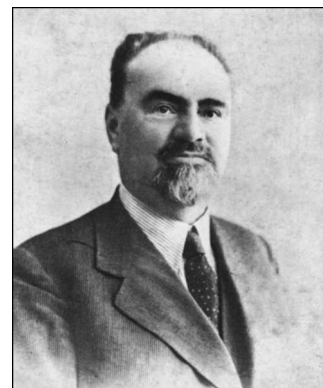
of Enriques's relentless quest that we would like to tell here.

Who Was Federigo Enriques?

Expelled by Spain in 1492, many Sephardic Jews found a safe haven in Tuscany, including the ancestors of Abramo Giulio Umberto Federigo Enriques, who was born in Livorno on January 5, 1871, and died in Rome on June 14, 1946. One of three children of Giacomo Enriques and Matilde Coriat, he moved to Pisa with his family when he was seven, displaying at an early age a taste for logic and numbers. Bored by a homework assignment set by his tutor that involved computing the squares of numbers from 1 to 30, Federigo, then age eleven, figured out that he could generate the squares by adding successive odd numbers: 1, 1 + 3, 1 + 3 + 5, and so on. Buoyed by his discovery, he went on to calculate the squares of numbers from 1 to 1,000, publishing his results in a small pamphlet, which cost him his entire savings (seven

Federigo Enriques, 1930s.
Photo courtesy of Lorenzo Enriques.

lira). When Enriques's daughter asked him, many years later, whether his parents had been pleased with his enterprise, he flashed a smile and replied, "They never knew about it." The eleven-year-old had shown even then a streak of independence he never lost [Enrq 47].

Enriques's mother, a Tunisian by birth, lavished attention on the education of her children. The family was well off, thanks in part to her substantial dowry, which allowed her husband, a wealthy rug merchant, to stop working altogether after their marriage. Federigo and his siblings were raised and remained in academic circles: Paolo Enriques, his younger brother, became a biologist; Elbina, their older sister, married Guido Castelnuovo, who wrote and published two papers on the geometry of algebraic surfaces in 1891, the first two papers on the subject in Italy.

When he turned thirteen Federigo Enriques discovered geometry, although his mother assumed he would lose interest quickly. "You know how this boy is," she once wrote to her sister, "every day his head has a new idea that lasts for about the space of a morning" [Enrq 47]. Despite his mother's skepticism, at the Liceo di Pisa Enriques honed his taste not just for mathematics but also for a number of other subjects—logic, epistemology, pedagogy, and the history of science—displaying an intellectual curiosity that he later told a colleague could be traced back to "a philosophical infection" contracted in high school [ScoDra 53].

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[4] *Severi [Sev 41] claimed to have discovered a gap in both proofs while reconstructing and simplifying Poincaré's proof of the Completeness Theorem.*
[5] *See the discussion at the end of Chapter 2.4 in [Brig-Cil 95] on this point.*

In the summer of 1887 Enriques took and passed the *esame di licenza liceale*, the final high school examination. That fall, he entered the University of Pisa and the Scuola Normale Superiore, the elite teacher-training school connected to the university, graduating with a doctor's degree in mathematics, with honors, in June 1891. His mathematics teachers there included Enrico Betti, Luigi Bianchi, Vito Volterra, and Riccardo De Paolis, professor of higher geometry, who also served as his thesis advisor.

Enriques's postgraduate training began with a year of studies and teaching at the Scuola Normale (1891–1892) and a second year (1892–1893) in Rome, where Guido Castelnuovo, that university's newly minted associate professor of analytic and projective geometry, took him under his wing. "Enriques was a mediocre reader," Castelnuovo later recalled, a defect that providentially led to their celebrated "peripatetic conversations"; the future direction of the Italian school of algebraic geometry evolved as the two young mathematicians crisscrossed the streets of the capital [Cast 47]. In June 1893, Enriques published his first memoir dealing with the theory of algebraic surfaces, *Ricerche di geometria sulle superficie algebriche* [Enrq 93], followed by a more comprehensive general theory three years later [Enrq 96]. In spring 1894, after a failed bid for a professorial appointment at the University of Turin, Enriques petitioned the University of Pisa for a *libera docenza*, which would allow him to teach projective geometry there. In its report granting the request, the committee of Pisan mathematicians hailed his 1893 memoir as an example of a keen mind and an expansive talent—although noting that "the precision and the rigor of the exposition leave something to be desired" [UPisa 94]. Shortly afterward, Enriques passed his *Abilitazione* exams with honors, another prerequisite for scaling Italy's academic ladder.

In 1894 Enriques joined the Bologna faculty as a temporary instructor for the projective and descriptive geometry courses, thanks in part to Vito Volterra's intervention on his behalf. Two years later, he took first place in a national competition for the vacant chair at Bologna; that university remained his home until 1922, when he was called to Rome to teach complementary mathematics, a new course designed for high school mathematics

After the liberation of Rome, Guido Castelnuovo played a key role in the rebirth of the Lincei, whose president he became in 1945.
Photograph courtesy of the Accademia Nazionale dei Lincei.

teachers. In 1923 Enriques became professor of higher geometry at Rome, a position he maintained until the 1938 anti-Jewish racial laws deprived him—despite his membership in the Fascist Party, which was summarily rescinded—of his university chair. The Fascist Severi, who had been Enriques's assistant in Bologna, took it over; Enriques would not regain his chair until 1944, following the liberation of Rome. Castelnuovo, who was also Jewish, had retired from teaching at Rome in 1935, but, like his other Jewish university colleagues, was barred from using the department's mathematics library.

Unlike Castelnuovo, who had switched fields after World War I and lectured, wrote, and published about relativity theory and the theory of probability, Enriques continued to work actively in algebraic geometry until the end of his life, often collaborating with colleagues, students, and assistants in publishing a steady stream of papers (nine with Castelnuovo, four with Severi[6]), school textbooks, voluminous university-level treatises, lecture notes, and monographs on that subject alone.

However, Enriques's interests, as they had at the Liceo di Pisa, ranged far and wide—from the foundations of mathematics and physiological psychology to the philosophy and history of science and Einstein's theory of relativity. He founded journals (*Scientia* and *Mathesis*) and organizations (the Italian Philosophical Society and the National Institute for the History of Science); edited, annotated, and published in Italian Euclid's *Elements*; and contributed many articles to the *Enciclopedia Italiana*, a massive multivolume reference set.

As a teacher, Enriques loved nothing better than to engage in his own leisurely peripatetic conversations with students, in the public gardens in Bologna or under its arcades after class. When he moved to Rome, the labyrinthine network of paths in the Villa Borghese became his favorite destination; he would stop there every so often, one student at that time recalls, "to trace mysterious figures on the ground, with the tip of his inseparable walking stick" [Camp 47]. In remarks made shortly after Enriques's death, Fabio Conforto described his colleague and coauthor's "powerful intuitive spirit" and unalterable belief in "an algebraic world that exists in and of itself, independent and outside of us"—a world in which "seeing" was the most important implement in a mathematician's toolbox: Enriques "did not feel the need of a logical demonstration of some property, because he 'saw'; and that provided the assurance about the truth of the proposition in question and satisfied him completely." Conforto

[6]*In 1907, Enriques and Severi won the prestigious Bordin Prize of the French Academy of Sciences for their work on hyperelliptical surfaces.*

recalled that on one memorable occasion, when he had failed

> to see the truth of a statement that [Enriques] considered obvious but that we had tried in vain to demonstrate logically, he stopped suddenly (we were in the course of one of the habitual walks), and instead of trying one final demonstration he flourished his walking stick and, pointing to a puppy on a windowsill, said, "You don't see it? For me, it is as if I said to myself, 'I don't see that puppy!'" [Conf 47].

Their disagreement didn't stop Enriques and Conforto from inserting that particular unproved statement in their 1939 book, *Le superficie razionali.* Because the racial laws banned Italian Jews from publishing, Conforto's name is the only one on the title page.

In 1941 Castelnuovo organized a clandestine university for Jewish students in Rome, operated in conjunction with the Fribourg Institut Technique Supérieure, a private school in Switzerland. Enriques taught a course there in the history of mathematics. Impeccably dressed (one student recalled that he invariably wore gloves), soft-spoken, and regal in his carriage and gestures, Enriques apparently kept his students on the edge of their seats. The Nazi occupation of Rome in fall 1943 forced the school's closure, and both Castelnuovo and Enriques went into hiding.

Severi, on the other hand, was well regarded by the Fascist regime and had been elected to its Academy of Italy in 1929 (Enriques was in the running for the position until the last moment). He was also the founding president of the Italian National Institute for Higher Mathematics during World War II. Briefly suspended from his university post after the liberation of Rome in June 1944, Severi was subsequently absolved of any criminal activity, and he resumed teaching, retiring in 1950. Beniamino Segre then moved from Bologna to Rome to take Severi's chair of geometry.

Although the Fascist racial laws had forced Enriques to step down as the longtime editor of *Periodico di Matematiche* (an influential and well-thumbed magazine that published historical and didactic articles for secondary school teachers), banished him from the university, and denied him the right to publish under his own name in Italy, they did not prevent him from publishing in Italy under the pseudonym "Adriano Giovannini" (derived from the names of his two children, Adriana and Giovanni). In 1942 he published an article in *Periodico di Matematiche* on errors in mathematics and a piece in *Archivio della cultura italiana* on the ideas of Galileo. After he went into hiding, however, even this publishing ceased.

Federigo Enriques with family and friends, circa 1907. *Photo courtesy of Lorenzo Enriques.*

Publishing abroad proved equally problematic. In 1940, he had submitted an article to the Madrid Academy of Sciences, of which he was a member, but at the end of the war, because of disruptions in the mail service, he still didn't know whether or not it had appeared.

In 1942 Enriques finished writing up his lectures at Rome on the theory of algebraic surfaces and set the manuscript aside until the Germans surrendered to the Allies in northern Italy in 1945. That September, Enriques handed the eleven chapters of *Le Superficie Algebriche* to the typesetters, reserving the right to make any necessary additions or changes while the book was in press. However, nine months later he was dead from a heart attack, leaving *Le Superficie Algebriche's* fate up in the air. Giuseppe Pompilj and Alfredo Franchetta, Enriques's last students, and Castelnuovo came to the rescue, volunteering to read the galleys, figuring that it would take only a few months of work. It took much longer, because of "several obscure points", Castelnuovo recalled—particularly in Chapter 9, which covered continuous systems of algebraic curves on irregular surfaces, including an exhaustive discussion of the status of the Completeness Theorem, the author's famous 1905 proof. Indeed, more than a hint of the difficulty the trio faced can be gleaned from the following comment, inserted as a footnote in the chapter:

> The part that follows [that is, the status of Enriques's theorem] has been left incomplete by the author, and the arguments developed there present many gaps; above all, it seems opportune to bring them back, because they contain ideas that perhaps, suitably completed, could furnish the start of a systematic account of the theory.[7]

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[7] *Chapter 9, p. 333.*

Several pages of Chapter 9 are then devoted to infinitely close higher-order curves, an idea that first appeared in *Sulla classificazione delle superficie algebriche particolarmente di genere zero*, a

With his appointment to Mussolini's new Academy of Italy in 1929, Francesco Severi became the regime's spokesman for Italian mathematics. *Photograph courtesy of the Lincei.*

volume of lectures on a particular category of surfaces that Enriques compiled and published in collaboration with Luigi Campedelli in 1934. Two years later, Enriques published a paper on infinitely close curves of higher order in the *Rendiconti dell'Accademia Nazionale dei Lincei* [Enrq 36-1] and a second paper on this theme in the pages of the *Rendiconti del Seminario matematico dell'Università di Roma* [Enrq 36-2], which later appeared in abridged form in the Lincei's journal [Enrq 37], presumably so that it might reach a wider audience. Beniamino Segre challenged Enriques's new proof in [Seg 38], which brought a quick response in June 1938 from Enriques [Enrq 38], along with a different proof.[8] And although even Castelnuovo came close to doubting, in the end, the theorem's provability, Enriques's "life's work", as Patrick DuVal later characterized it [DuVal 49], finally appeared between covers three years after he died.

After Enriques's death, Severi recalled [Sev 57] that the personality characteristics they shared ("vivacious, pugnacious, and sometimes impulsive") were what had driven them apart. Their colleague, Tullio Levi-Civita, on the other hand, claimed that their quarrel stemmed from competing textbooks aimed at the same scholastic market. Castelnuovo, the eldest of the trio (born in 1865), and by nature austere and level-headed, assiduously avoided taking sides, insisting instead that "it was the good fortune of the Italian school of algebraic geometry to have this disinterested collaboration between 1890 and 1910" [B-G 09]. In private, Severi lashed out at his Jewish colleagues, claiming that his work was underrated and theirs overrated. However, Beniamino Segre, Severi's protégé and assistant in 1927—and himself a victim of the 1938 racial laws—vigorously defended Severi against charges of anti-Semitism after World War II.

A towering figure in the world of algebraic geometry from the end of the nineteenth century until his death in 1946, Enriques made a number of major contributions to the field. To put it another way, his quest to rescue the proof of his Completeness Theorem represents a small (though important) fraction of his mathematical activity during his lifetime. Early in his career (1893), he constructed an (unexpected) example of an algebraic surface F for which $p_g = p_a = 0$ but F was not birationally equivalent to $\mathbf{P}^2(\mathbf{C})$. The existence of such a surface indicated at the very minimum that more birational invariants than just p_g and p_a would be needed to classify algebraic surfaces in the same way that mathematicians employed the geometric genus to classify algebraic curves. (See Appendix A for the definition of a birational invariant.) It also sent a signal that the classification itself would be more complicated. Three years later, in 1896, Enriques defined what he called the *plurigenera* of an algebraic surface, an infinite sequence of birational invariants associated with F. Over the next eighteen years, he and Castelnuovo, often together, used the *plurigenera* to distinguish various birational classes of algebraic surfaces. In 1914 their investigations culminated in the classification of algebraic surfaces into four natural classes defined in terms of the behavior of their plurigenera [Enrq 14], [Cast-Enrq 14].[9] Like the Fundamental Theorem of Irregular Surfaces, the classification of algebraic surfaces remains one of the great accomplishments of early-twentieth-century algebraic geometry.

The Completeness Theorem and the Fundamental Theorem

The 1905 Completeness Theorem of Enriques, as has been noted in the first part of this article, furnished one of the key ingredients in the proof of the Fundamental Theorem of Irregular Surfaces. Simply put, it stated that: The characteristic series of a "good" complete continuous system of curves on a smooth algebraic surface F is complete. (Here "good" in the old Italian terminology means not superabundant, or, in modern terms, the first cohomology group of divisor class should be zero.[10] See Appendix A for the definition of the terms used in the statement of the theorem. Interestingly, Severi had given an appropriate condition in 1944 [Sev44].)

The completion of the proof of the Fundamental Theorem followed quickly on the heels of that of the Completeness Theorem. In 1905, with $q := \dim P(F)$, the dimension of the Picard integrals

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[8]*The anti-Jewish decrees in September 1938 ended publishing in Italian journals for Enriques and Segre, but Enriques and Severi would continue to battle it out in 1942 and 1943 in the pages of the* Swiss Commentarii Mathematici Helvetici.

[9]*See [Gray 99] for an accessible account of plurigenera and the classification of algebraic surfaces.*

[10]*It was always clear to both Enriques and Severi that the system should be sufficiently ample in some sense. Enriques mentions this several times in his 1936 article.*

of the first kind, Severi proved that $p_g - p_a \geq q$ and $p_g - p_a = b_1 - q$; the latter equality was also proved shortly after by Picard. A few months later, using the Completeness Theorem in an essential way, Castelnuovo carried out the final step in the proof by establishing that $p_g - p_a \leq q$. These two results established the Fundamental Theorem, i.e., $p_g - p_a = q = b/2$.[11]

In 1921, as noted, Severi showed that both his and Enriques's 1905 algebro-geometric proofs of this theorem were fatally flawed. Forty years later, Alexander Grothendieck [Groth 61] introduced the use of higher order nilpotents in the structure sheaf to define nonreduced schemes and, using them, proved a very strong existence theorem for the Picard "scheme", the space of divisors mod linear equivalence. When he visited Harvard, Mumford pointed out to him that, as an immediate corollary, this gave the long-sought algebro-geometric proof of a conjecture he had never heard of!

The opening salvo of the last round of papers by Enriques and Severi on their attempted proofs of the Completeness Theorem took place against the backdrop of World War II. On November 23, 1940, on the occasion of the fortieth anniversary of the publication of his first original scientific paper, Severi submitted a ninety-three-page paper to Mussolini's Academy of Italy (the Lincei had ceased to exist in 1939, following its formal annexation by the Fascist Academy) on the general theory of continuous systems of curves on an algebraic surface. The paper, which included a fresh attempt by Severi to impose his own proof of Enriques's 1905 theorem [Sev 41], appeared in *Memorie della R. Accademia d'Italia* a year later. Now enjoying an enforced retirement, Enriques read it and wrote a brief rejoinder pointing out an error in Severi's proof, which he submitted in March 1942 to the Swiss journal *Commentarii Mathematici Helvetici*. The journal published it [Enrq 43] in 1943, but not before sending it on to Severi, who quickly countered with an article of his own [Sev 43]. Much to Enriques's annoyance, the journal's editors (belying Swiss neutrality) denied his request to rebut Severi's arguments.[12] In his 1943 paper, Severi once more shot down Enriques's proof and this time claimed to have proved an even more general theorem. As it happens, Severi's effort turned out to be wishful thinking, which brings us to Enriques's letter to Beniamino Segre, composed several months after German troops abandoned their hold on northern Italy and surrendered to the Allies.

[11]*Here we are using the notation introduced earlier. See also [Kl 05], p. 244.*

[12]*Efforts by one of us (JG) to examine relevant documents in the journal's archives failed.*

The first two mathematicians (l. to r.) are Beniamino Segre and Beppo Levi. The others are unidentified Italian mathematicians. *Courtesy of Sergio Segre.*

The Remarkable Letter from Federigo Enriques to Beniamino Segre in 1945

On September 11, 1945, writing from Rome, Enriques sent a letter to Beniamino Segre in response to a letter Segre had sent to Enriques from Manchester, U.K., on August 2nd. Most of Enriques's letter deals with his ongoing quest to vindicate his 1905 theorem.

Rome, September 11, 1945

> Dear Segre,
> I thank you for your nice letter of August 2 (I received it the other day), and I congratulate you on your scientific activity…. [After a brief discussion on the work of another Italian mathematician, Beppo Levi, Enriques comes straight to the point.] Personally, I am especially interested in the problem of a continuous system on irregular surfaces which was discussed in my 1942 lectures on surfaces, which is now in press [Enrq 49]. This question is extremely delicate. I was unable at that time to reconstruct the proof that you had indicated on the basis of the information that you had given me before my departure for Paris [1937?]. Severi, with whom you have had more interaction, believed he had finally succeeded in giving a proof [Sev 41]. His exposition seems obscure to me and therefore dubious; I believed (in the *Commentarii Helvetici* paper [Enrq 43]) to have overcome the difficulty: In reality, my proof

was erroneous, but this realization also pointed out the error in Severi's proof. At that time, I was not allowed to add anything to my paper in the *Commentarii Helvetici* although Severi was allowed to write a note to my paper [Sev 43] in which he says that he has obtained a more general theorem.... But shortly afterward, Severi himself, who was expounding that theory in his lectures at the Institute of Higher Mathematics, realized that his proposed proof was flawed due to a radical error. *I really wish that this thing could be settled. I have reexamined my earlier proof based on infinitely close curves of various orders and I believe it is substantially right, even if it is not rigorously complete* [our italics].

In closing, Enriques asks Segre, whose August 2nd letter must have been written in English, a language he became proficient in while living in England during the war, to give him more details in Italian.[13] In particular, he asks Segre to limit himself to a very specific type of surface and continuous system, where Enriques apparently expects there will be the need for infinitesimally close curves of higher order. [See Appendix B for a copy of the original letter.]

Later that year, Castelnuovo also urged Segre to put his mind to the problem. He wrote many letters to Segre in 1945, mainly dealing with conditions in postwar Italy and urging him not to turn his back on his homeland, but on this occasion he tried to enlist Segre's help in bringing closure to Enriques's 1905 theorem. "Given the delicacy of the matter, a work aiming *exclusively* at settling the interesting question about the characteristic series of a continuous system of curves would seem appropriate," Castelnuovo wrote [his italics]. "Under what conditions," he asked,

> can one use the theorem of the completeness of the characteristic series of a complete continuous system? The theorem has been used in fundamental investigations on surfaces, and we would need to have a fully satisfying geometrical demonstration [Cast 45].

It doesn't appear that Segre, who turned from algebraic geometry to combinatorial geometry within a few years, took the bait.

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[13]*While organizing the Beniamino Segre papers, JG found Enriques's letter. There is no copy of Segre's reply in his papers.*

Conclusion

Contrary to the received wisdom on the subject, Enriques came tantalizingly close to realizing his quest. In the final paragraph of the last paper he would publish in Italy on the Completeness Theorem, Enriques summed up his novel approach, reiterated his deep-felt belief that the history of failed attempts in the history of science is instructive for future researchers, and issued a gentle rebuke to his naysayers, declaring

> But the use of infinitely close points and curves and the corresponding *language*, even if not rigorous, is fruitful and even succeeds in giving correct and coherent results for those who can learn to understand and adopt them as I have succeeded in doing in a clear and definitive manner in other parts of my work. A skeptical attitude toward these ideas is easy to have but is not very productive. Instead, those who are more trusting of what these concepts can yield, will, I am sure, discover new results in other fields. [Enrq 38]

Indeed, Enriques himself realized that he lacked the tools to make his infinitely close points and curves "rigorously complete". For Enriques, the important thing was always to make discoveries; the necessary rigorous proofs, he used to tell students, would follow in due time. It is a pity that he did not live to see that he had had the right intuitive idea in the 1930s—an idea that eventually would lead to a rigorous proof of his Completeness Theorem.

In an unpublished paper prepared in 2008 about Grothendieck's work and how it affected his own understanding of the algebraic geometry of Enriques's generation, the American mathematician David Mumford wrote:

> Although Enriques's 1905 paper on the Completeness Theorem missed the key issue, this paper [Enrq 38] does have the right idea. He speaks of the exponential map in the Picard variety and asserts that analogously higher order infinitely near curves can be generated from first order infinitely near ones. Unfortunately, he possesses no tools whatsoever for going beyond an intuitive description of why the method of higher order infinitesimals should work: the theory of schemes is what he lacked. [Mum 08]

See the accompanying article by David Mumford, "Intuition and Rigor and Enriques's Quests", for the end of this story.

Appendix A
Birational Equivalence and Birational Invariants[14]

Much of Enriques's work dealt with the birational geometry of projective algebraic surfaces $F \subset \mathbf{P}^n(\mathbf{C})$, where $\mathbf{P}^n(\mathbf{C})$ is complex n-dimensional projective space. By birational geometry we mean the study of birational mappings $T : F \text{ "} \to \text{"} F'$ of projective algebraic surfaces and the corresponding notions of birational equivalence and birational invariants.

Perhaps the most familiar smooth algebraic surfaces to nonalgebraic geometers are those given by the locus of zeros of an irreducible homogeneous polynomial f on $\mathbf{P}^3(\mathbf{C})$, i.e.,

$$F_f := \{(z_0 : z_1 : z_2 : z_3)$$
$$\in \mathbf{P}^3(\mathbf{C}) \mid f(z_0 : z_1 : z_2 : z_3) = 0\},$$

where $(z_0 : z_1 : z_2 : z_3)$ denotes the standard homogeneous coordinates in $\mathbf{P}^3(\mathbf{C})$ and f is an irreducible homogeneous polynomial whose differential never vanishes at points of F_f. It turns out that these surfaces are always *regular*. However, it is the smooth *irregular* surfaces, i.e., when $p_g - p_a > 0$, that are most relevant for us. Irregular surfaces will thus, of necessity, be embedded in $\mathbf{P}^n(\mathbf{C})$, $n > 3$. They will be given by the locus of zeros of some set of homogeneous polynomials on $\mathbf{P}^n(\mathbf{C})$ (1) with conditions on the polynomials that make sure they define an irreducible algebraic set of dimension 2 and (2) that are locally transverse intersections of hypersurfaces with locally independent differentials. From now on, we will suppress the reference to "complex smooth algebraic" and refer simply to curves and surfaces.

To define the notion of a birational mapping between surfaces F and F', we first need to define the notions of a Zariski open subset in F and rational mapping from F to F'. A *Zariski open set* in F is just the complement $F - \bigcup_i W_i$ of a finite set of subvarieties $W_i \subset F$. A *rational mapping* of F to F' consists of a Zariski open subset F_1 of F and a mapping $T : F_1 \to F'$, given by rational functions f of the coordinates of F. T is *birational* if T is injective and T^{-1} is rational. F and F' are *birationally equivalent* if there exists a birational mapping between them.

Example. Let $F = F' = \mathbf{P}^2(\mathbf{C})$[15] and $F_1 = \mathbf{P}^2(\mathbf{C}) - \{z_0 z_1 z_3 = 0\}$ and define T by $(z_0 : z_1 : z_2) \to (1/z_0 :$

$1/z_1 : 1/z_2)(= (y_0 : y_1 : y_2))$. Note that the domain of T^{-1} is $\mathbf{P}^2(\mathbf{C}) - \{y_0 y_1 y_3 = 0\}$ and $T^{-1}((y_0 : y_1 : y_2)) = (1/y_0 : 1/y_1 : 1/y_2)$.

A *birational invariant* is an integer-valued function $\mathbf{I}$ from the set (category) of algebraic surfaces to the integers $\mathbf{Z}$ such that $\mathbf{I}(F) = \mathbf{I}(F')$ if F and F' are birationally equivalent. (There are similar definitions for algebraic curves.)

In 1870 M. Noether proved that the geometric genus p_g was a birational invariant, while Zeuthen, in 1871, proved that the arithmetic genus p_a was also a birational invariant. The *plurigenera* of Enriques are also birational invariants.

Remark. Birational invariants that are also topological invariants: An algebraic surface F over $\mathbf{C}$ is also a topological space. As a topological space p_g and p_a are also topological invariants, i.e., if algebraic surfaces F and F' are homeomorphic, then $p_g = p'_g$ and $p_a = p'_a$.

The Definition of the Terms in the Completeness Theorem

A *good algebraic curve C on F* usually means it is sufficiently ample, that is, it is a sufficiently high multiple of a hyperplane section in some projective embedding of F. In the case of the Completeness Theorem, it is enough that it has no higher cohomology groups. In simplest terms, this means $\dim L(D)$ equals its self-intersection number (D^2) minus the genus of D plus the arithmetic genus of F plus two and that there are no regular 2-forms on F zero along D.

A *complete algebraic (sometimes called a continuous or nonlinear) system of curves C_a, $a \in S$ on F*, where S is an algebraic variety, is maximal in the set of algebraic systems of curves on F.[16]

The algebraic system is *good* if it consists of good algebraic curves.

The *characteristic linear system at a curve C_0 of an algebraic system, C_a, $a \in S$ on F* is the set of divisors on C_0 which are the limits of the intersections $C_0 \cap C_a$ as a tends to 0. It is not hard to show that this defines a linear system on C_0.

The linear system on C_0 is "complete" as a linear system if it is maximal in the set of linear systems on C_0.[17]

Acknowledgments

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[14]*A very nice discussion of birational algebraic geometry can be found in the paper by A. Grassi in the* Bulletin of the AMS *(2009), pp. 99–123.*

[15]*Here we think of $\mathbf{P}^2(\mathbf{C})$ embedded in $\mathbf{P}^3(\mathbf{C})$ by $(z_0 : z_1 : z_2) \to (z_0 : z_1 : z_2 : 0)$.*

[16]*See [Shaf 74], Ch. I.3.5, for the definition of an algebraic system of curves on a surface.*

[17]*See [Mum 76], Ch. 6A, for a discussion of linear systems.*

Appendix B. Text of a letter from F. Enriques to Beniamino Segre, 11 September 1945, recounting the recent history of his efforts to prove the "Completeness Theorem". The war had officially ended. But the censor's stamp in the upper left-hand corner reminds us that postal authorities still opened and read letters. *Courtesy of California Institute of Technology.*

to publish his grandfather's letter and information about his family; Sergio Segre for opening Beniamino Segre's papers to scholars; Elisa Piccio, Michele Vallisneri, and Annalisa Capristo for discussions; Francesca Rosa for archival research in Rome and Pisa; and Sara Lippincott for editorial suggestions.

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