A Note on the *Geometry* and Descartes's Mathematical Work

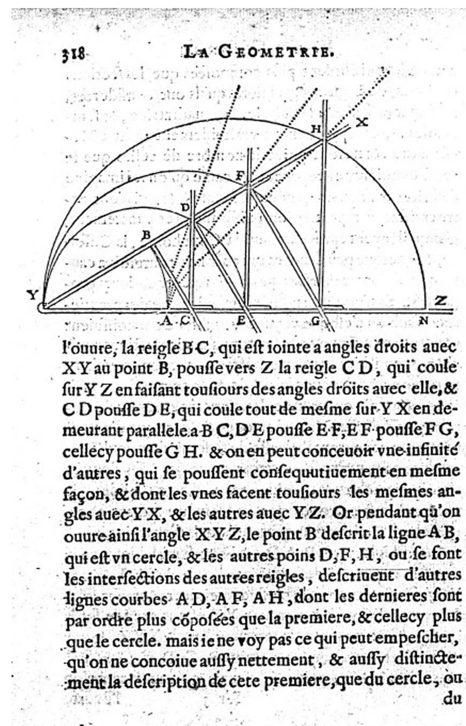
Michel Serfati

The *Geometry* is the only mathematical book written by Descartes. One can find some mathematical studies from his youth—constructions of proportional means and solutions of equations of third and fourth degrees—in the *Regulae*, the *Excerpta Mathematica*, and above all in the letters, mainly those to Beeckman, Mersenne, and Golius (Serfati2002, 56–72 and 95–104). All these subjects are treated in the *Geometry*. Note also the *De Solidorum Elementis* (1639?), an unpublished treatise on (convex) polyhedra that includes the equivalent of Euler's formula (1751), namely $F - E + V = 2$ (cf. Descartes 1987).

Descartes finished the *Geometry* at the very last moment, as the *Meteorology* was in print. It is divided into three Books but is difficult to read since it does not follow a coherent scheme, a paradox upon which Descartes insisted. I will distinguish three "threads of Ariadne" to lead us through the text: the acceptability of curves, the problem of Pappus, and the construction of roots "using curves".

Acceptable Curves, Cartesian Compasses, and Pappus's Problem

The first thread of Ariadne is to delineate the frontier between those curves that are acceptable in geometry, which Descartes called "geometric", and the rest, which he called "mechanical"; the modern terms "algebraic" and "transcendental" are due to Leibniz. Since antiquity (from Plato to Pappus), constructibility by ruler and compass usually served to define this boundary. Descartes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l'ouure, la reigle BC, qui est iointe a angles droits auec XY au point B, pousse vers Z la reigle CD, qui coule sur YZ en faisant tousiours des angles droits auec elle, & CD pousse DE, qui coule tout de mesme sur YX en demeurant parallele a BC, DE pousse EF, EF pousse FG, cellecy pousse GH. & on en peut conceuoir vne infinité d'autres, qui se poussent consequutiuement en mesme façon, & dont les vnes facent tousiours les mesmes angles auec YX, & les autres auec YZ. Or pendant qu'on ouure ainsi l'angle XYZ, le point B descrit la ligne AB, qui est vn cercle, & les autres poins D, F, H, ou se font les intersections des autres reigles, descriuent d'autres lignes courbes AD, AF, AH, dont les dernieres sont par ordre plus cõposées que la premiere, & cellecy plus que le cercle. mais ie ne voy pas ce qui peut empescher, qu'on ne concoiue aussy nettement, & aussy distinctement la description de cete premiere, que du cercle, ou du

Figure 1. "Cartesian compasses". Reproduction from page 318 of the original edition = ATvi, 391 = Smith & Latham, 46.[1] The author proposed a study of cartesian compasses in Serfati1993.

Michel Serfati is Professeur Honoraire de Mathématiques à l'IREM, Université Paris VII. His email address is `serfati@math.jussieu.fr`.

A much more extensive version of this note is my paper "René Descartes, Géométrie, Latin edition (1649), French edition (1637)" in Landmark Writings in Western Mathematics 1640–1940, *(I. Grattan-Guiness, ed.), Elsevier, Amsterdam, Boston, etc., 2005.*

[1] *Hereafter [ATz] will denote Adam-Tannery edition, where 'z' is the Roman number of the volume.*

extended this classification by introducing his set-square compass. This is a mechanism of sliding set squares that "push" each other (Figure 1). A specifically Cartesian invention, this (theoretical) machine served him from his youth (Serfati2002, 77–80) in the solution of cubic equations, the insertion of proportional means, and the construction of geometrically acceptable curves[2] where "acceptable" is used in its first Cartesian sense: curves that are at the end of some articulated chain, a conception that commentators have called the "criterion of continuous movements" (Serfati2005b, 7–8).

Following this criterion, Descartes rejected the quadratrix and spiral as mechanical since they are the result of separate motions, both circular and rectilinear, without any connection "that one could measure exactly" (ATvi, 390).

In 1631, Jacob Golius, a professor of mathematics from Leiden, sent Descartes an ancient geometrical problem, that "of Pappus on four lines" (for "line" here, read "straight line"). The ancient solution was unknown in the seventeenth century, so the problem became a test-case for Descartes. In modern terminology, the "four lines" problem can be stated as follows. Let k be a positive real number, L_1, L_2, L_3, L_4 four lines in the plane, and $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ four angles. Through a point C in the plane, draw four lines $\delta_1, \delta_2, \delta_3, \delta_4$ where δ_i meets L_i at a point, H_i say, and at the angle α_i. It is required to find the locus of points C such that

$$CH_1CH_2 = kCH_3CH_4.$$

This is generalized to "the problem of Pappus on $2n$ lines" as follows: given two sets of n lines in the plane, two sets of angular magnitudes, and a number k, what is the locus of the points in the plane the ratio of the products of whose distances from one set of lines to those from the other, all measured at the given angles, is equal to k? Descartes showed *how to solve the general problem analytically*. His method required the use of algebraic symbolism in Vieta's style and (implicitly) of a coordinate system in a nonorthogonal frame of reference; it employs similar triangles instead of calculations with distances. For four lines, he finds an equation of the second degree for one of the variables

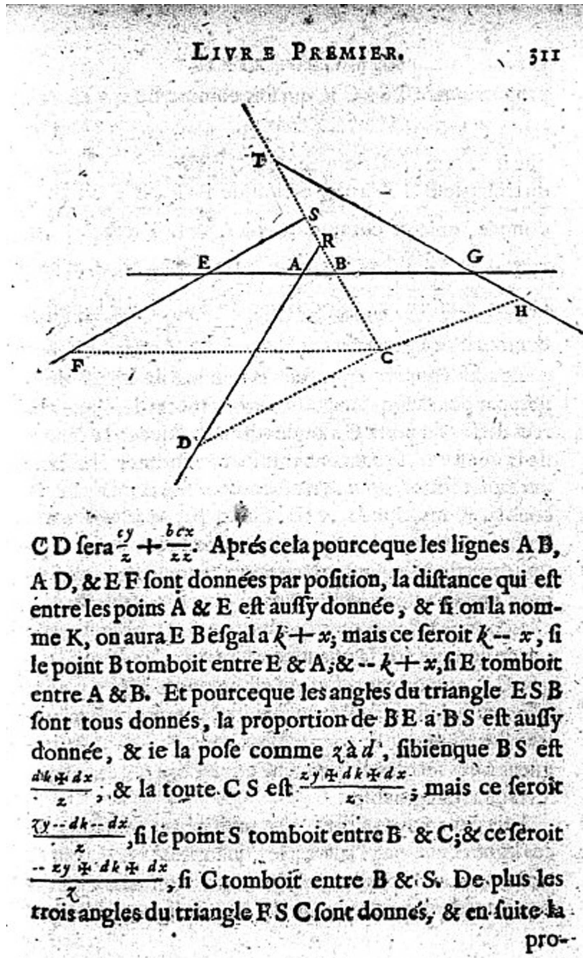
$$y(a_1x + b_1y + c_1) =$$
$$k(a_2x + b_2y + c_2)(a_3x + b_3y + c_3),$$

or,

$$x^2P_0(y) + xP_1(y) + P_2(y) = 0.$$

It was the first case in history that a geometric locus was expressed as an equation. The formulae for the solution contain, as well as the "four operations", only extractions of roots, which are

[2]*The curves are drawn as dotted lines on the figure.*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CD ſera $\frac{cy}{z} + \frac{bcx}{zz}$. Aprés cela pourceque les lignes AB, AD, & EF ſont données par poſition, la diſtance qui eſt entre les poins A & E eſt auſſy donnée, & ſi on la nomme K, on aura EB eſgal a $k + x$; mais ce ſeroit $k - x$, ſi le point B tomboit entre E & A; & $-k + x$, ſi E tomboit entre A & B. Et pourceque les angles du triangle ESB ſont tous donnés, la proportion de BE a BS eſt auſſy donnée, & ie la poſe comme z à d, ſibienque BS eſt $\frac{dk + dx}{z}$, & la toute CS eſt $\frac{zy + dk + dx}{z}$; mais ce ſeroit $\frac{zy - dk - dx}{z}$, ſi le point S tomboit entre B & C; & ce ſeroit $\frac{-zy + dk + dx}{z}$, ſi C tomboit entre B & S. De plus les trois angles du triangle FSC ſont donnés, & en ſuite la pro-

Figure 2. The Problem of Pappus on four lines. The points B, D, F, H are our H_i's. The figure is a reproduction from page 311 of the original edition = ATvi, 382 = Smith & Latham, 31.

compass-and-ruler constructible. For the general case with $2n$ lines, $x^sP_0(y) + x^{s-1}P_1(y) + \ldots + P_s(y) = 0$, Descartes tried to construct solutions in x for each fixed y. Thus, for a given y, the construction of an *arbitrary* point of the curve is the key step.[3] Solving equations therefore became a central problem (Serfati2005b, 11–12).

For Descartes, the Pappus curves served as models, in that they could generate all the curves he would henceforth regard as geometrically acceptable, in accordance with the (then new) second criterion,[4] *algebraicity*, that is, a curve is acceptable if and only if it has an equation. Posterity has

[3]*The quadratrix and spiral were again rejected by this criterion, since it was not possible to construct an arbitrary point; only points with certain rational abscissae, such as $k/2^n$, could be constructed.*

[4]*On the conflict between the criteria, cf. Bos and Serfati1993.*

clearly retained the algebraic criterion for classifying curves, being both simple and practical. Moreover, the criterion enables one to envisage "all the (geometric) curves", i.e., to extend the class of acceptable curves in a fashion inconceivable to the Ancients, who knew only a small number of curves occurring individually "on the ground".

Note that the resolution of the Pappus problem would have been unworkable without analytical tools and symbolic notation. The use of exponents (Descartes's invention), of a specific sign for equality, of letters to represent constants: these were conclusive for the advent of the new symbolism. The ease that Descartes displays in handling symbolic notation and the familiarity we have with it today must not obscure its profound novelty at the time. The statement in modern terms of Cardano's formula in one line in Book III proved to the reader the clear superiority of this symbolism over the laborious rhetoric of Cardano. Contemporary mathematicians were not mistaken in using the *Geometry* as a "Rosetta stone" for deciphering symbolism (Serfati2005a, 382). The new symbolism turned out to be an essential specific element of the Scientific Revolution (*idem*, 384–406).

Algebraic Equations. Constructions of Roots

The algebraic part of the *Geometry* is also remarkable (Serfati2005b, 13–14). The exposition is clear and systematic, and expressed in modern notation for the first time in history. Descartes first states (without proof) that the maximum number of roots that an equation "can have" is equal to its "dimension" (degree). When the total number of "true" (positive) and "false" (negative) roots is less than the dimension, one can, according to Descartes, artificially add "imaginary" roots, a naive term coined by him but not defined. He also proves that for a polynomial P to be divisible by $(X - a)$ it is necessary and sufficient that $P(a)$ be zero. He then uses his indeterminate coefficients to describe the division of a polynomial by $(X - a)$. So it was important for him to know at least one root. For an equation with rational coefficients, he studies the rational roots, if any.

Descartes was also interested in the number of real roots, and asserted without proof that the maximum number of positive roots of an equation is equal to the number of alternations of the signs "+" and "–" between consecutive nonzero coefficients, while the maximum number of negative roots is equal to the number of times the signs do not alternate in the same sequence. This is the celebrated "rule of signs", which earned unfounded criticism for Descartes. The result was proved in the eighteenth century, in particular by de Gua and Segner, and led to the "final" theorem of Sturm (1829).

The utility of *solving algebraic equations* arose from Pappus's problem. Constructing the roots *geometrically* was a consequence of Descartes's constructivist conception of knowledge. By intersecting a circle and a parabola (*auxiliary curves*), Descartes could solve the trisection of the angle and the duplication of the cube (two ancient problems leading to cubic equations) and, from there, general equations of degrees 3 and 4. For higher degrees, Descartes introduced more complicated algebraic auxiliary curves, including a specific cubic. In modern terms, the method regarded an algebraic equation $H(x) = 0$ as the *resultant* of eliminating y between $F(x, y) = 0$ and $G(x, y) = 0$. To construct the solutions of $H(x) = 0$, it suffices to make a suitable choice of F and G and then to study graphically the abscissae of the points of intersection of the curves for $F = 0$ and $G = 0$, the skill of the geometer lying in the "most simple" choice of F and G (Bos, 358).

Descartes also described some ovals, algebraic curves linked to the *Optics*. He also determined the tangent to an arbitrary point C of a curve by an algebraic method of computing the normal to the curve at that point. He wrote that, at the intersection of the curve with a variable circle containing C, there are two points coincident with C, so that the equation for their common ordinate has a fixed double root. To find them, Descartes described a Euclidean division of polynomials, using indeterminate coefficients for the first time in history. Immediately after the *Geometry* was published, the question of tangents became the subject of a quarrel, stirred up by Fermat into a famous dispute (cf. correspondence between Descartes, Mersenne, Fermat, and Roberval at the turn of 1637). In a short manuscript, *Methodus ad Disquirendam Maximam et Minimam*, Fermat proposed another method, of "differential" inspiration, while Descartes's had been algebraic (Boyer, 74–102).

After the *Géométrie*

Chasles famously described the *Geometry* as "*proles sine matre creata*"[5] (a child without a parent). It is true that Descartes borrowed little from his predecessors. It was to a large extent independently of his contemporaries, and guided by a specific "physical" view of the world, that Descartes wrote the *Geometry*, a work as exceptional for its contents (analytic geometry and the analytical method) as for its style (symbolic notation), which organized the final escape from the ancient and mediaeval pre-symbolic mathematical world. This treatise is actually the first in history to be directly accessible to modern-day mathematicians.

As early as 1630, Descartes declared to Mersenne that he was "tired of mathematics" (ATi,

[5] *Chasles1875, 91.*

139). But in 1637 he decided that he had to give a mathematical application of his Method, so he returned to the subject for a while. Afterwards, he turned away from all theoretical work in mathematics, even the new questions he had raised in the *Geometry* (except under duress, as in the case of Fermat and the tangents). He took no interest in the classification of cubics, which was accomplished by Newton in the *Enumeratio Linearum Tertii Ordinis* (1676?). The *Geometry* thus incontestably represents the culmination of Descartes's mathematical work and not an opening towards the future.

After 1649 and van Schooten's Latin translation, the *Geometry* became a long-lasting object of study for European mathematicians and a veritable bedside reader for geometers. Over many decades, a considerable number of treatises made essential reference to Descartes and the *Geometry*. A trail-blazing work of the modern mathematical era, the *Geometry* cleared the way for Leibniz and Newton.

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