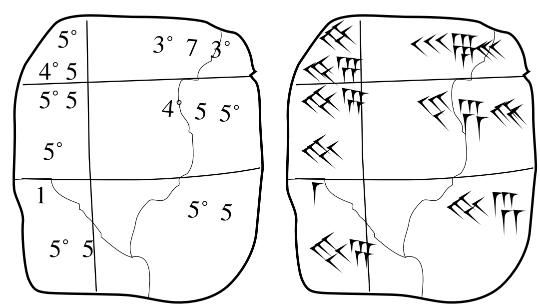
A Remarkable Collection of Babylonian Mathematical Texts

Jöran Friberg

About 120 new Mesopotamian mathematical cuneiform texts, all from the Norwegian Schøyen Collection, are published in the author's book *A Remarkable Collection of Babylonian Mathematical Texts*, Springer (2007). Most of the texts are Old Babylonian (1900–1600 BC), but some are older (Sumerian), or younger (Kassite). In addition to the presentation and discussion of these new texts, the book contains a broad and fairly thorough account of important aspects of Mesopotamian mathematics in general, from its beginnings in the late 4th millennium BC till its last manifestations in the late 1st millennium BC.[1]

The present paper contains brief presentations of a selection of interesting mathematical cuneiform texts in the Schøyen Collection (tablet numbers beginning with MS). Actually, the subcollection of mathematical cuneiform texts in the Schøyen Collection is so extensive that it is possible to use examples from it to follow in detail the progress of Old Babylonian scribe school students in handwriting and computational ability from the first year student's elementary multiplication exercises written with large and clumsy number signs to the accomplished model student's advanced mathematical problem texts written in a sure hand and with almost microscopically small cuneiform signs.

A beginner's multiplication exercises with large number signs. Figure 1 shows an example of a young student's *multiplication exercises*, beginning with the computation, *in terms of sexagesimal numbers in floating place value notation*, of the product $50 \times 45 = 37\ 30$. (The corresponding result in decimal numbers is, of course, $50 \times 45 = 2,250 = 37 \times 60 + 30$.) Note that there was no cuneiform number sign for zero, but instead there were separate number signs for the ones, from 1 to 9 (upright wedges), and for the tens, from 10 to 50 (oblique wedges). In the transliteration to modern number signs (to the left in Figure 1) the tens are written with a little circle in the upper right corner.



conform transliteration hand copy

Figure 1. MS 2728. An Old Babylonian clay tablet with three multiplication exercises.

Arithmetical table texts with sexagesimal numbers. For computations of this kind, Babylonian scribe school students could make use of *sexagesimal multiplication tables*, which they first copied from the teacher's master copy and then, at best, learned by heart. Figure 2 shows what such sexagesimal multiplication tables could look like.

This is a multiplication table listing multiples of the *head number* 12 (here in transliteration to modern number signs and letters). The repeatedly occurring word a.rá is a Sumerian loan word in the Babylonian text, meaning "times".

In the table are listed the products of 12 and all integers from 1 to 19, as well as all the tens from 20 to 50. The last two lines of the table assert that

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[1] *More about Mesopotamian/Babylonian mathematics and about its apparent impact on Egyptian and Greek mathematics can be found in the author's* Unexpected Links Between Egyptian and Babylonian Mathematics, *World Scientific (2005), and in the sequel* Amazing Traces of a Babylonian Origin in Greek Mathematics, *World Scientific (2007).*

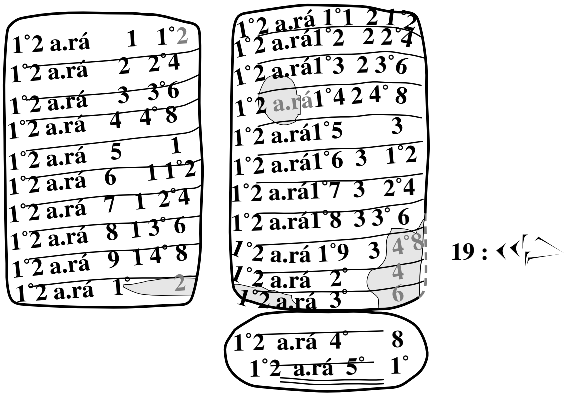

Figure 2. MS 2184/3. An Old Babylonian sexagesimal multiplication table, head number 12.

$12 \times 40 = 8$ and $12 \times 50 = 10$, which has to be understood as meaning that $12 \times 40 = 8 \times 60$ (= 480) and $12 \times 50 = 10 \times 60$ (= 600). The reason why the Babylonians operated in this way with floating values for their sexagesimal numbers was, of course, that they had no zeros (neither final nor initial) to indicate multiplication with positive or negative powers of the base 60.

Curiously, each student seems to have used a squiggle of his own design to write the number 19 in his multiplication tables!

In addition to multiplication tables, Babylonian scribe school students had to learn to work also with several other kinds of arithmetical tables, among them *tables of squares*, which went from "$1 \times 1 = 1$" to "$19 \times 19 = 6\ 01$" (361), continuing with "$20 \times 20 = 6\ 40$", "$30 \times 30 = 15$" (meaning 15 00), "$40 \times 40 = 26\ 40$", "$50 \times 50 = 41\ 40$", and "$1 \times 1 = 1$" (meaning $1\ 00 \times 1\ 00 = 1\ 00\ 00$).

Tables of square sides (square roots) went from "1 has the side 1" all the way to "58 01 has the side 59" and "1 has the side 1".

For some unknown reason there were no tables of cubes, but *tables of cube sides* went from "1 has the side 1" all the way to "57 02 59 has the side 59", and again "1 has the side 1". Figure 3 shows a quite small clay tablet with a brief excerpt of only 5 lines from such a table of cube sides. The table goes from "36 37 13" to "1 21 53 17", where 13 and 17 are the cube sides.

The word íb.si-*tam* written on the right edge is a combination of a Sumerian word íb.si with the approximate meaning "equal-sided" (a cube being equalsided) and a Babylonian accusative ending -*tam*. As a matter of fact, the role played by Sumerian loan words in Babylonian mathematics was just as important as the role played by Greek and Latin loan words in modern mathematics.

A characteristic feature of Old Babylonian mathematics was the use of *tables of reciprocals*, actually a kind of division tables. The example in Figure 4 is a transliteration of such a table of reciprocals, apparently with so many errors that

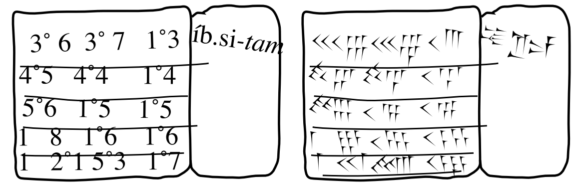

Figure 3. MS 3966. A brief excerpt of 5 lines from an Old Babylonian table of cube sides.

an irate teacher has crossed over the text on both sides of the clay tablet.

An Old Babylonian table of reciprocals always begins with two lines saying that "2/3 of 60 is 40" and "half (of 60) is 30". Then it continues with "1/3 is 20", "1/4 is 15" and so on, all the way to "1/54 is 1 06 40" and "1/1 is 1", "1/1 04 (1/64) is 56 15", and finally "1/1 21 (1/81) is 44 26 40".

Note that an Old Babylonian table of reciprocals lists only reciprocals of "regular" sexagesimal integers between 2 and 1 21. An integer is called (sexagesimally) regular if it is a divisor of "1", where "1" can mean any power of 60.

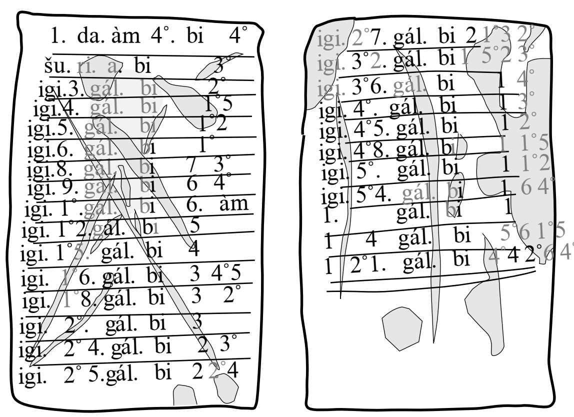

Figure 4. MS 3890. A crossed-over Old Babylonian table of reciprocals.

There are, for instance, no regular integers between 54 and 60, and 7 is, of course, not a regular integer. The reason why the Old Babylonian table of reciprocals ends with the reciprocals of 1 04 and 1 21 is, probably, that $1\ 04 = 64 = 2^6$ and $1\ 21 = 81 = 3^4$. (Indeed, one recently published atypical Old Babylonian (or Sumerian?) table of reciprocals ends with the reciprocal of 2 05, where $2\ 05 = 125 = 5^3$.)

Arithmetical exercises. Counting with long "many-place regular sexagesimal numbers" played an important role in Babylonian mathematics. A beautiful example is the "descending table of powers" in Figure 5, which begins with

$$46\ 20\ 54\ 51\ 30\ 14\ 03\ 45 = (3\ 45)^6 = 15^{12}.$$

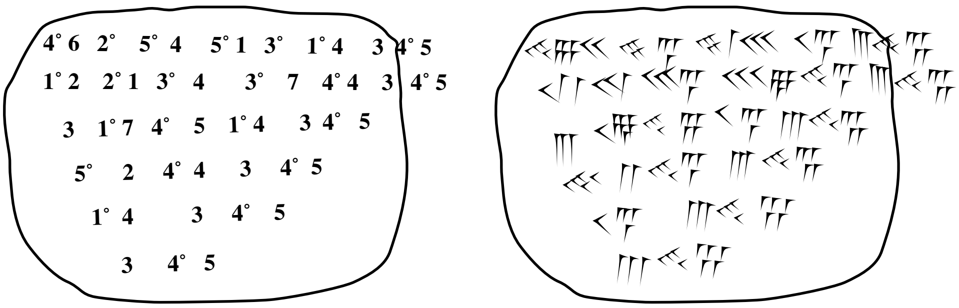

Figure 5. MS 2242. A descending table of powers, with powers of the regular number 3 45 (= sq. 15).

Then, one at a time, a factor 3 45 is removed until at the end

$$(3\ 45)^2 = 14\ 03\ 45 \text{ and } (3\ 45)^1 = 3\ 45.$$

Note that 3 45 (225) is the square of 15 and the reciprocal of 16, the square of 4.

Figure 6. The sexagesimal number 13 22 50 54 59 09 29 58 26 43 17 31 51 06 40 = 20^5. Written in two lines on the obverse of the tablet and continued onto the reverse.

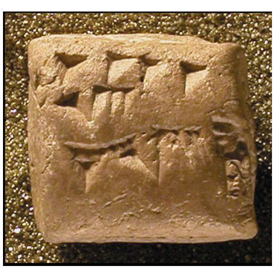

Figure 7a. A division exercise: 1 01 01 01 divided by 13 = 4 41 37.

Two different division methods can be observed in Babylonian mathematics. The first method was used when a sexagesimal number a should be divided by a *regular* sexagesimal number b. Then the first step was to compute the reciprocal of b, called in Sumerian igi b, probably meaning "the *opposite* number to b", and the second step was to compute a/b as $a \times \text{igi}\, b (= a \times 1/b)$. However, if b was a *non*-regular sexagesimal number, another method had to be used. One case of this kind appears in the next example (Figures 7a and 7b). There three numbers recorded below each other at first sight look like 4, 13, and 4 41 37. A closer look reveals that the first number really is 1 01 01 01. (The zeros, indicating missing tens, are inserted here for clarity.) The small clay tablet is probably an *assignment*, a problem that a student was required to take home with him, and to return to the teacher the next morning with the correct solution. What the student had to do in this case was to show that 1 01 01 01 divided by 13 is 4 41 37.

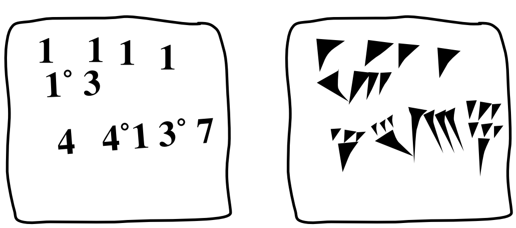

Figure 7b. MS 2317. A sexagesimal division problem with the non-regular divisor 13.

How a division problem of this kind could be solved is shown by a similar but 500-years-older division exercise in a mathematical cuneiform text from the city Ebla. The method used in the Ebla text is that a number of successively more complicated division problems are solved, one after the other, until the desired result is reached. Thus, if one wants to divide, for instance, 1 01 01 01 by 13, the first step is to divide 1 00 by 13. The result is that

$$1\ 00 = 13 \times 4 + 8$$
$$(1\ 00/13 = 4 \text{ with the remainder } 8).$$

(Double zeros, indicating missing sixties, are inserted here for clarity.) In the next couple of steps, one sees that

$$1\ 00\ 00 = 13 \times 4\ 36 + 12, \text{and}$$
$$1\ 00\ 00\ 00 = 13 \times \ \ 4\ 36\ 55 + 5.$$

If the results are added together one gets the final result that

$$\begin{aligned} 1\ 01\ 01\ 01 &= 1\ 00\ 00\ 00 + 1\ 00\ 00 + 1\ 00 + 1 \\ &= 13 \times (4\ 36\ 55 + 4\ 36 + 4) + (5 + 12 + 8 + 1) \\ &= 13 \times 4\ 41\ 35 + 26 = 13 \times 4\ 41\ 37. \end{aligned}$$

Therefore, the answer to the division problem is that 1 01 01 01 divided by 13 is 4 41 37. From a modern point of view the method can be explained as follows, in terms of *infinite sexagesimal fractions* and *common fractions*:

$$1/13 = ;04\ 36\ 55\ 23\ldots$$
$$(= 4/60 + 36/(60 \cdot 60) + \cdots),$$

and

$$1\ 01\ 01\ 01 \times 1/13 = 4\ 36\ 55\ \ 5/13 +$$
$$4\ 36\ \ 12/13 + 4\ \ 8/13 + 1/13 =$$
$$4\ 41\ 37.$$

Solutions in table form to combined market rate problems. The clay tablet in Figure 8 contains two computations, both of the same kind. In the first computation, four different commodities (wares) have the "market rates" 1, 2, 3, and 4, all listed in column 1 of table a. What that means is that 1 weight or volume unit of the first commodity, or 2 of the second, or 3 of the third, or 4 of

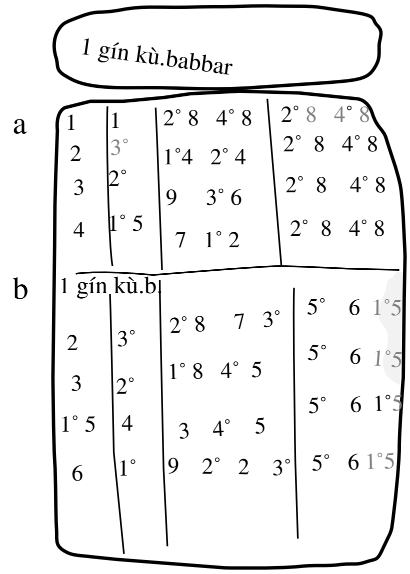

Figure 8. MS 2830. Two computations of combined market rates.

the fourth, can be bought for 1 shekel of silver (in Sumerian 1 gín kù.babbar). Apparently, the question is how much a person can buy for 1 shekel of silver if *equal amounts* are bought of all four commodities.

The first step of the solution algorithm is the computation of the "unit price" for each commodity. It is 1 shekel for 1 unit of the first commodity, 1/2 (= ;30) shekel for 1 unit of the second commodity, and so on. These values are listed in column 2 as the numbers

$$1, 30(= ;30 = 1/2), 20(= ;20 = 1/3),$$

and

$$15(= ;15 = 1/4).$$

Consequently, the "combined unit price" for 1 unit of each commodity is

$$(1 + ;30 + ;20 + ;15) \text{ shekels} = 2;05 \text{ shekels.}$$

Now, the reciprocal of

$$2;05 = 2 \quad 1/12 = 25/12$$

is

$$12/25 = (12/5)/5 = (2;24)/5 = \ ;28\ 48$$
$$= 28/60 + 48/60^2.$$

Therefore, *;28 48 units of each commodity can be bought for 1 shekel of silver, as noted in column 4 of table a.* The prices paid for ;28 48 units of each kind are listed in column 3. They are ;28 48 shekel for the first commodity, ;14 24 shekel for the second, and so on. It is easy to check that the sum of the prices recorded in column 3 is precisely 1 shekel. Indeed,

$$(;28\ 48 + ;14\ 24 + ;09\ 36 + ;07\ 12) \text{ shekels}$$
$$= 1 \text{ shekel.}$$

Clearly, then, the object of the exercise was to compute the "combined market rate" ;28 48.

Geometric exercises. Old Babylonian round or square "hand tablets", hastily inscribed with numbers or geometric figures, seem to have been a kind of brief notes, written down by scribe school students more or less attentively listening to a teacher's explanation of how a mathematical problem should be solved. It is likely that, precisely as in the case of the small clay tablet in Figure 7a, each student was supposed to go home with his hand tablet and spend part of the evening writing down a detailed version of the solution procedure, to be brought back to school the next day.

In Figure 9 is shown an example of such a hand tablet, with a diagram showing a parallel trapezoid divided in three parts by two transversals parallel to the base and the top of the trapezoid. A reasonable interpretation of this text is that the student was expected to find the lengths of the four parallel straight lines in the case when it is known that the two transversals divide the long side (or, rather, the height) of the trapezoid in three parts of length 10, 20, and 30 "rods" (1 rod = 6 meters), while the whole trapezoid is divided in three parts of which the first and the third have the same area, 1 40 square rods. If the lengths of the four parallel straight lines are called p, q, r, s, then a problem of this kind can be replaced by *four linear equations for four unknowns:*

$$(p + q)/2 \times 10 = 1\ 40,$$
$$(r + s)/2 \times 30 = 1\ 40,$$
$$(p - q)/10 = (q - r)/20 = (r - s)/30$$

The solution to this system of equations is indicated in the diagram:

$$p = 10\ 50, q = 9\ 10, r = 5\ 50, s = 50.$$

The example demonstrates a clear distinction between Babylonian and Greek mathematics: While Greek geometry was *abstract and reasoning*, Babylonian geometry was *concrete and numerical*.

This distinction is also demonstrated by the next example, an elaborate diagram on another hand tablet (Figure 10). Here three parallel trapezoids of the same form and size are joined to

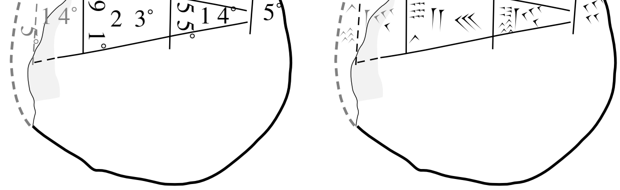

Figure 9. MS 3908. A parallel trapezoid divided by two parallel transversals.

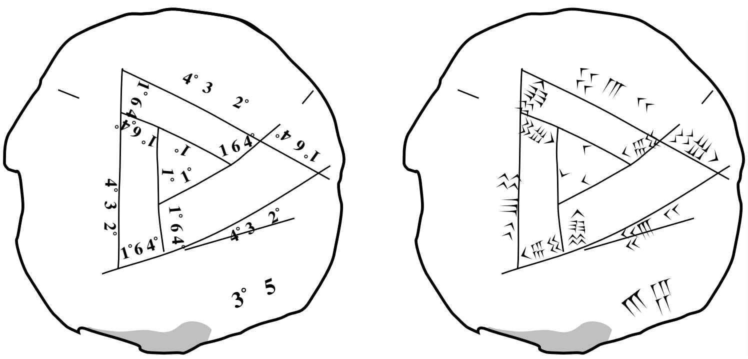

Figure 10. MS 2192. A triangular band divided into a ring of three trapezoids.

form a "triangular band" bounded on the inside and the outside by parallel and concentric equilateral triangles. A reasonable interpretation is that the diagram illustrates the problem to find the area of the triangular band when it is known only that the two equilateral triangles have the sides 10 and 1 00, respectively.

This problem can be solved in the following way: Let a be the length of one of the oblique sides in one of the trapezoids. Then the lengths of the two parallel sides in the trapezoid are $10 + a$ and $10 + 2a$. At the same time, the length of the side of the outer equilateral triangle is $10 + 3\,a = 1\ 00$. Therefore,

$$a = 50/3 = 16;40, 10 + a = 26;40,$$

and

$$10 + 2a = 43;20,$$

as indicated in the diagram.

Knowing this, the area of the triangular band can be computed in two ways. One way is to compute the combined area of the three equal trapezoids. It is

$$3 \times (43;20 + 26;40)/2 \times h = 3 \times 35 \times h,$$

where h is the height of the trapezoid.

Another way is to observe that since the side of the outer equilateral triangle is 6 times as long as the side of the inner equilateral triangle, its area is 36 times as large. Consequently,

> the area of the triangular band is 35 times the area of the inner equilateral triangle.

Thus, in both cases, the number 35 recorded near the lower edge of the lentil can be explained as a number closely associated with the computation of the area of the triangular band!

Another, considerably less elegant, example of a schoolboy's brief note of a geometric problem, this time on a small square clay tablet with rounded corners, is shown in Figure 11. The diagram on the clay tablet shows a circle inscribed in the middle of a square. The meaning of the numbers surrounding various parts of the diagram is far from obvious. Fortunately, however, there is another text that may explain what is going on here, namely an Old Babylonian mathematical problem text from the city Susa in Western Iran, east of Mesopotamia itself. In that text, a so-called "concave square" is inscribed symmetrically around the middle of a square, and the explicitly stated problem is to find the side of the square when both the area of the region between the square and the concave square and the shortest distance from the concentric square to the sides of the square are known. This problem is reduced to a quadratic equation. And so on. (The correct interpretation of the problem was found by Kazuo Muroi.)

If the problem associated with the diagram in Figure 11 was of the same kind as the mentioned problem in the text from Susa, it can be explained as follows: Given are the distance b from the circle to the sides of the square and the area B of the region between the circle and the square. What are then the circumference a of the circle and the side s of the square?

The side of the square is $;20a + 2b$, and the area of the circle is $;05 \times$ sq. a. (Here sq. a means the square on a, while $;20 = 1/3$ and $;05 = 1/12$

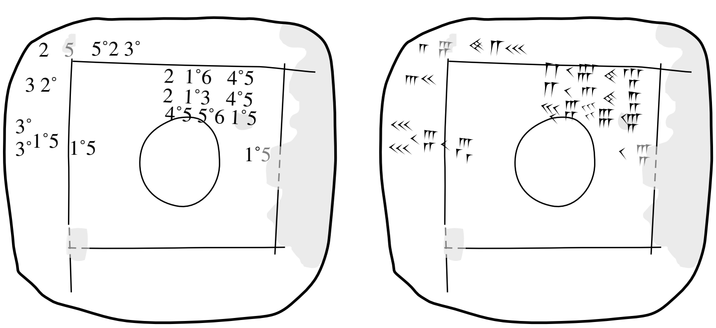

Figure 11. MS 2985. A circle in the middle of a square.

are the Babylonian approximations to what we call $1/\pi$ and $1/(4\pi)$.) Therefore the circumference a of the circle can be computed as the solution to the quadratic equation

$$\text{sq. } (;20a + 2b) - ;05 \times \text{sq. } a = B,$$

or, since sq.;$20 - ;05 = ;06\,40 - ;05 = ;01\,40$ = sq.;10, and $4 \times ;20 = 1;20$,

$$\text{sq. } (;10a) + 1;20\, b \times a + \text{sq. } (2\, b) = B$$

Apparently, as indicated in two places in the diagram, $b = 15$, and, as indicated by the notations near the left edge of the clay tablet, sq. $(2b) =$ sq. $30 = 15\ 00$. Therefore, the equation above for a can be reduced to

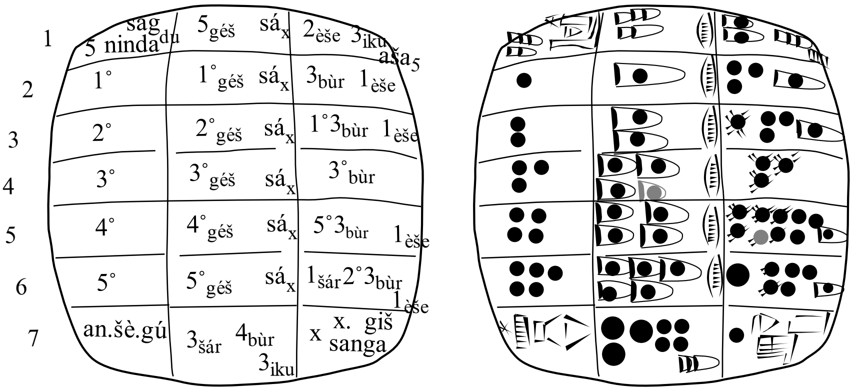

Figure 12. MS 3047. An Old Sumerian metric table of rectangle sides and areas.

$$\text{sq. } (;10a) + 20 \times a + 15\ 00 = B, \text{ or}$$
$$\text{sq. } (;10a + 1\ 00) = B + 45\ 00.$$

Counting backwards from the most likely solution, one finds that the given value for B probably was 39 36;33 45. With this value for B, the equation for a becomes

$$\text{sq. } (;10a + 1\ 00) = 39\ 36;33\ 45 + 45\ 00 = 1\ 24\ 36;33\ 45 = \text{sq. } 1\ 11;15.$$

Consequently,

$$;10a + 1\ 00 = 1\ 11;15, \text{ so that}$$
$$a = 1\ 07;30, \quad ;20a = 22;30, \text{ and } s = 52;30.$$

Note that the value $s = 52;30$ is recorded along the side of the square, and that the corresponding value sq. $s = 45\ 56;15$ is recorded near the middle of the square.

An Old Sumerian metric table of rectangles. The oldest mathematical text in the Schøyen Collection (Figure 12) is an Old Sumerian table text from ED III (the Early Dynastic III period, c. 2600–2350 BC). It is so old that the sexagesimal numbers appearing in it are written without the use of place value notation, with special number signs not only for 1 and 10 but also for 60 and $10 \cdot 60$. The sign for 60 is a larger variant of the sign for 1, and the sign for $10 \cdot 60$ is the sign for 60 with the sign for 10 inside. In addition, the text is so old that numbers are not written with cuneiform (wedge-like) number signs but with rounded signs, punched into the clay of the tablet with a round stylus used only for numbers.

The first six lines of this text can be interpreted as a "metric table of rectangle sides and areas". In each line are recorded first the length of the short side of a rectangle (Sum.: sag), then the length of the long side, and finally the area of the rectangle (Sum.: aša$_5$). In all the six lines, the long side is 60 times as long as the short side. This artificially imposed condition is enough to show that the text is mathematical rather than "practical" (a surveyor's work notes). In the transliteration below of the six lines of the metric table of rectangles, the following Sumerian terms are used:

> 1 geš = sixty, 1 iku = sq. (10 rods), 1 èše = 6 iku, 1 bùr = 3 èše, 1 šár = sixty bùr.

The seventh line gives the sum (Sumerian: an.šè.gú) of the six computed areas. This line may have been added to the six lines of the metric multiplication table in order to artificially give the exercise the appearance of a practical text!

The sum can be computed as follows:

5 rods×5 geš (rods)=2 èše 3 iku (2 1/2 èše)
10×10 geš= 3 bùr 1 èše (10 èše)
20×20 geš= 13 bùr 1 èše (40 èše)
30×30 geš= 30 bùr (90 èše)
40×40 geš= 53 bùr 1 èše (160 èše)
50×50 geš=1 šár 23 bùr 1èše (250 èše)
sum 3 šár 4 bùr 3 iku (552 1/2 èše)

Babylonian labyrinths of a previously unknown type. Two clay tablets in the Schøyen Collection are inscribed with labyrinths. Both labyrinths are of completely new types, which is greatly surprising, since up till now all known ancient drawings or depictions of labyrinths have been either simple and uninteresting or diverse variants of the classical "Greek" or "Mycenaean" labyrinth. One of the new Babylonian labyrinths is the one in Figures 13 and 13a. Unlike the classical labyrinth it is (fairly) symmetrical, and there are two openings into the labyrinth. If you enter through one of the openings, you will ultimately exit through the other one. (Due to a misinterpretation of an unclear photo, an unsymmetrical version of the labyrinth with two dead ends at the center was proposed in section 8.3a of my book *A Remarkable Collection*. I am very grateful to Tony Phillips for pointing out the mistake. The same easily corrected mistake was made in section 8.3c of the book in the case of a more complicated Babylonian rectangular labyrinth with two open and eight closed gates.) The labyrinth is skillfully and exactly drawn, which

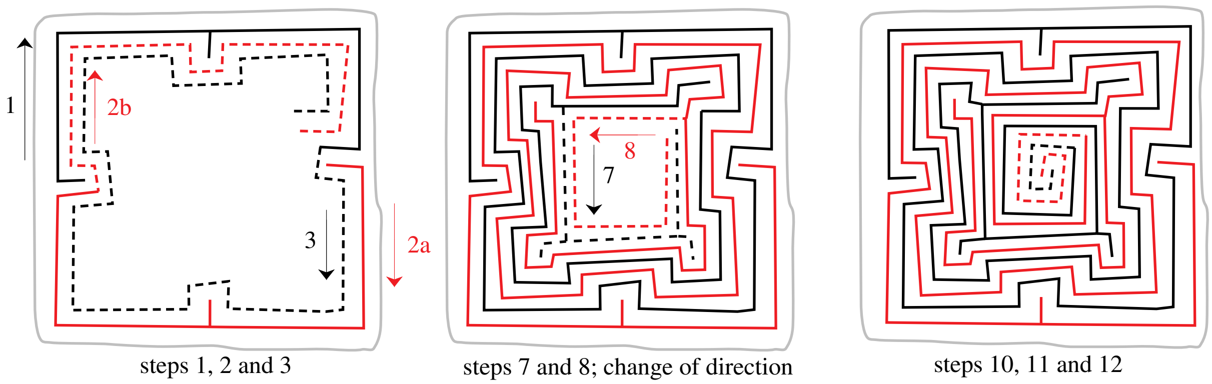

Figure 13. MS 4515. The step-wise algorithm for the construction of the Babylonian square labyrinth. In steps 1 and 2a, the outer walls are constructed, with two open and two closed gates. After step 2a, alternatingly red follows black, at a constant distance, or black follows red, at the same constant distance.

Figure 13a. A Babylonian square labyrinth.

would hardly have been possible if the one who drew it had not used a cleverly devised algorithm for the drawing of the labyrinth. Moreover, that algorithm must have been totally different from the well known algorithm for the construction of a classical Greek labyrinth.

Generating rules for rational solutions to the "Pythagorean equation". There are six larger "problem texts" in the Schøyen Collection, with more or less explicit questions, solution procedures, and answers. In Figure 14 is exhibited an example of an Old Babylonian problem text. One of the surprises in this new text is the set of five exercises in § 3. In each one of the five exercises, a pair of *reciprocal sexagesimal numbers* is given, called igi and igi.bi. In § 3 e, for instance,

igi = 1 12 and igi.bi = 50 (floating values).

It is easy to check that (with appropriately chosen absolute values) igi × igi.bi = 1;12 × ;50 = 1. With these given values, the first step of the procedure is the computation of the half-sum

(igi + igi.bi)/2 = (1;12 + ;50)/2 = 1;01.

Then are computed, in succession,

sq. 1;01 = 1;02 01, sq. 1;01 – 1 = ;02 01, and the square side sqs. ;02 01 = ;11.

In the last line of the exercise, ;11 is called "the 5th short side" (Sum.: sag ki.5).

Similarly in § 3 d, where igi = 1;20, igi.bi = ;45, so that (igi + igi.bi)/2 =1;02 30, sq. 1:02 30 = 1;05 06 15, sq. 1;02 30 – 1 = ;05 06 15, and the "4th short side" = sqs. ;05 06 15 = ;17 30. It is clear that the purpose of each one of the five exercises is to construct a rectangle with rational sides and rational diagonal, "normalized" in the sense that the long side = 1. The basic idea is to let the rational diameter be a *half-sum* of the form (igi + igi.bi)/2, with igi × igi.bi = 1, at the same time as the long side is required to be 1. Then the short side will automatically be a *half-difference* of the form (igi – igi.bi)/2, and therefore rational. It is easy to check that in § 3 e, for instance, (1;12 – ;50)/2 = ;11 = the 5th short side.

It is explicitly shown by these five exercises that Old Babylonian mathematicians were aware of the fact that arbitrarily many rectangles with rational sides u, s, and diagonal d can be constructed by use of the "generating rule"

d, u, s = (igi + igi.bi)/2, 1, (igi – igi.bi)/2,

where igi, igi.bi are arbitrarily chosen regular sexagesimal numbers with igi × igi.bi = 1. Now, there are always integers p and q such that

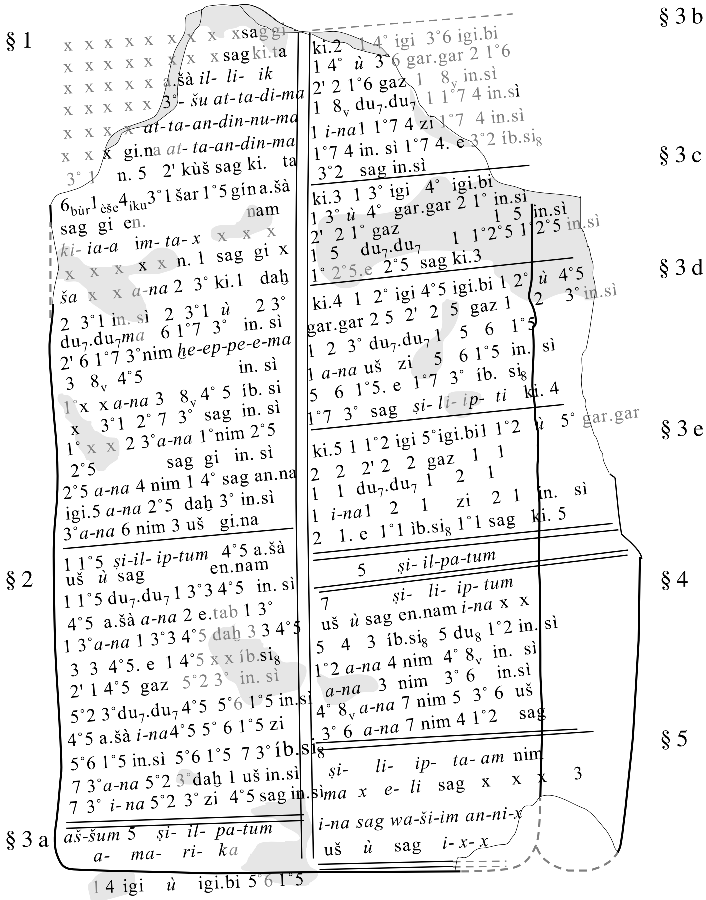

Figure 14. MS 3971. Transliteration of an Old Babylonian mathematical problem text.

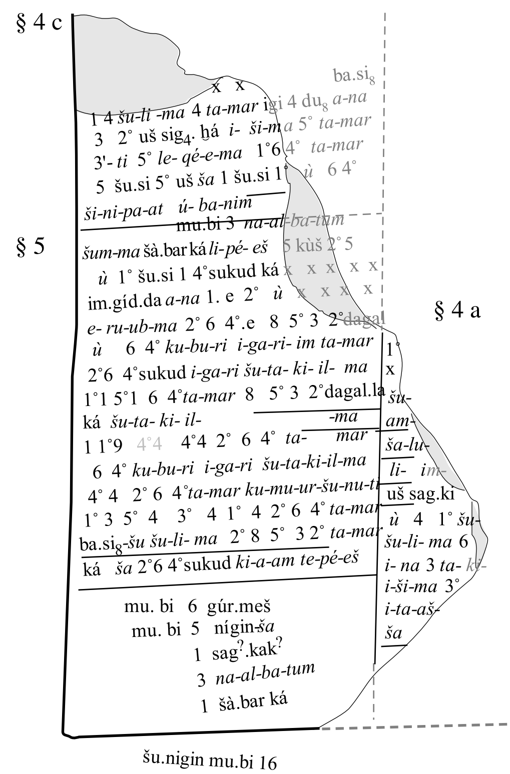

Figure 15. MS 3049, reverse. A problem for an "inner diagonal", and a summary.

igi = p/q and igi.bi = q/p, so that
(igi + igi.bi)/2, 1, (igi – igi.bi)/2
= {(sq. p + sq. q), 2 $p \times q$, (sq. p – sq. q)}/(2 $p \times$ q).

This Old Babylonian generating rule for the rational sides and diagonal of a rectangle is essentially identical to the modern generating rule, of Greek origin, for the rational sides of a right-angled triangle. (Note, however, that according to Old Babylonian conventions, p/q must be a regular sexagesimal number and (igi – igi.bi)/2 < 1, in other words sq. p – sq. q < 2 $p \times q$.)

It is interesting that the examples in § 3 of the new text in Figure 14 strongly support the interpretation of the famous table text Plimpton 322 suggested by the present author already in a paper in *Historia Mathematica* in 1981!

The Babylonian diagonal rule in 3 dimensions. In Figure 15 is shown the reverse of a small fragment in the Schøyen Collection of a large mathematical text with mixed problems. The text is possibly Kassite (post-Old Babylonian). By luck, the fragment contains a perfectly preserved summary of the various kinds of problems appearing in the text. No other known large Babylonian problem text, except a second text in the Schøyen Collection (see Figure 17), contains a similar summary.

According to the summary, the whole text originally contained 16 exercises, or, more precisely, 6 problems for "circles" (Sum.: gúr.meš), 5 problems for "squares" (Sum.: nígin), 1 problem for a "triangle" (Sum.: sag.kak), 3 problems for "brick molds" (trapezoids?) (Bab.: *na-al-ba-tum*), and 1 problem for an "inner diagonal of a gate" (Sum.: šà.bar ká). The problem for an inner diagonal (perfectly preserved on the reverse of the fragment) is a totally unexpected surprise in a Babylonian mathematical text (see Figure 16).

The object of the exercise is a gate in a city wall. The height h of the gate is "5 cubits and 10 fingers" = 5 1/3 cubits = ;26 40 (rods), the width w is ;08 53 20, and the thickness t of the city wall is ;06 40. The "inner diagonal" d of the gate is computed as follows, by use of a three-dimensional version of the Old Babylonian "diagonal rule" (incorrectly known as the "Pythagorean equation"):

sq. d = sq. h + sq. w + sq. t = ;11 51 06 40 + ;01 19 00 44 26 40 + ;00 44 26 40 = ;13 54 34 14 26 40, d = sqs. ;13 54 34 14 26 40 = ;28 53 20.

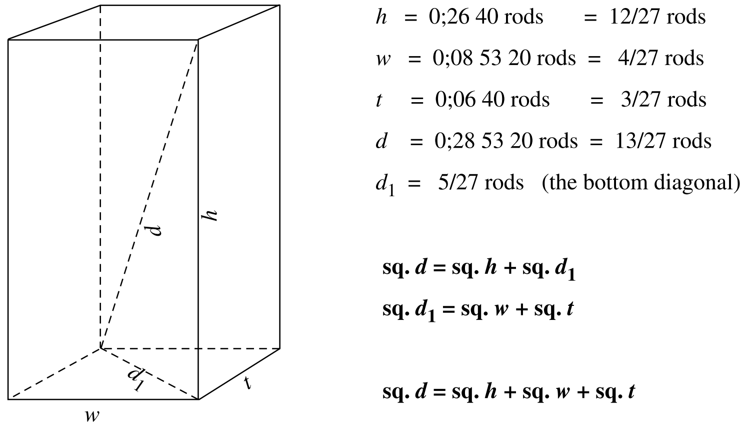

Figure 16. The construction of the data in the problem for an "inner diagonal of a gate".

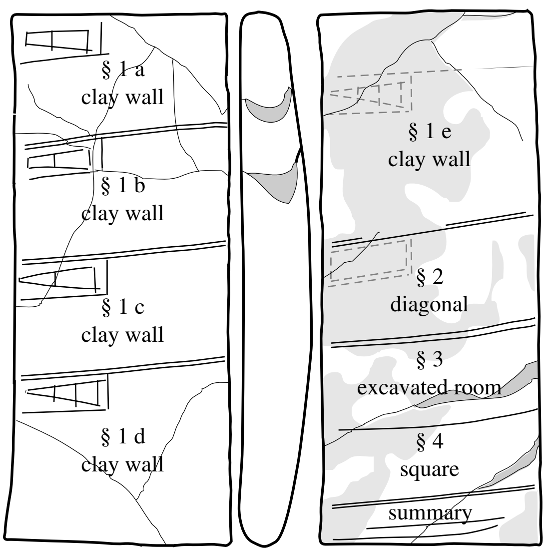

Figure 17. MS 3052. A mathematical problem text with 4 themes and a summary.

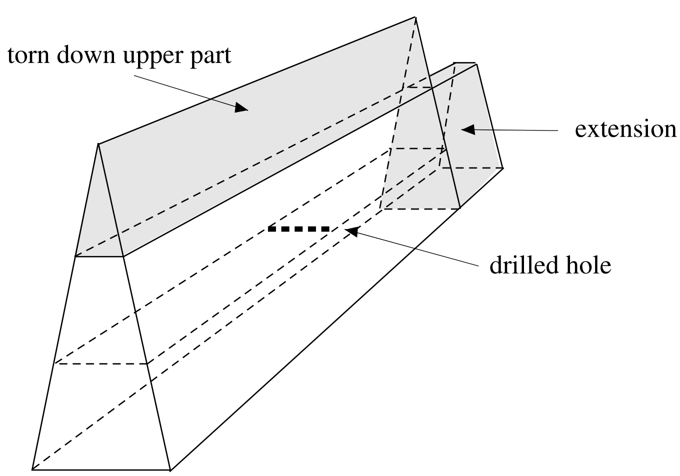

Figure 18. MS 3052 § 1 d. A clay wall with an extension and a hole.

An explanation for the complicated form of the data is easy to find, since

h = ;26 40 = 12/27, w = ;08 53 20 = 4/27, t = ;06 40 = 3/27, and d = ;28 53 20 = 13/27.

Therefore, the given values of the parameters for the gate can be understood as

$$d, h, w, t = 1/27 \cdot (13, 12, 4, 3).$$

The "diagonal quartet" (13, 12, 4, 3) is a solution in integers to the equation

$$\text{sq. } d = \text{sq. } h + \text{sq. } w + \text{sq. } t.$$

It is obvious that it was constructed through a combination of the two "diagonal triples" (13, 12, 5) and (5, 4, 3). The idea must have been that if d_1 is the diagonal of the bottom rectangle of the gate, then

$$\text{sq. } d = \text{sq. } h + \text{sq. } d_1 \text{ with sq. } d_1 = \text{sq. } w + \text{sq. } t.$$

Therefore, $d, h, d_1 = 1/27 \cdot (13, 12, 5)$ and $d_1, w, t = 1/27 \cdot (5, 4, 3)$.

Complicated stereometric problems disguised as problems for clay walls. The clay tablet shown in outline in Figure 17 is a problem text in the Schøyen Collection with a well preserved obverse but a much less well preserved reverse. Just like the text in Figure 15, this text ends with a summary of the four themes appearing in it.

According to the summary, § 2 is a problem for a "diagonal". It is, actually, an igi-igi.bi problem of the same kind as the exercises in § 3 of MS 3971 (Figure 14). The theme of § 3 is an "excavated room", and the theme of § 4 a "square". The theme of § 1, which contains 5 exercises, is a "clay wall". The 5 associated diagrams, looking like depictions of trapezoids and triangles, are actually depictions of the cross sections of 5 clay walls. Some of the problems in this paragraph are quite complicated and lead to equations that are solved by use of really surprising solution procedures. No Babylonian mathematical texts with similar problems or solution procedures have been published earlier.

The problem in § 1 a is about a clay wall with a trapezoidal cross section. The trapezoid has a given height, 6 cubits, a given base, 3 cubits, and a given top, 1/3 cubit. The length of the clay wall is given, too, 3 00 rods. Now the clay wall has to be extended by 20 rods. The material for the extension is obtained through tearing down the upper part of the original clay wall. The question is how much lower the new wall will become. The answer is 1 1/2 cubit lower.

The clay wall in § 1 b has again a trapezoidal cross section. The length, height, and volume of the clay wall are given. At a certain height over the ground a hole drilled through the clay wall has a given length. The question is how wide the clay wall is at the base and at the top.

In § 1 c, which is similar to § 1 b, the clay wall has a triangular cross section, and the problem in

§ 1 d (Figure 18) is a combination of the problems in §§ 1 a and 1 c. It is likely that the problem in the damaged § 1 e was a combination of the problems in §§ 1 a and 1 b.

Apparently, § 1 of the text in Figure 17 is an excerpt from some large and systematically organized Old Babylonian "theme text" with elegantly devised problems.

The weight of a colossal icosahedron made of 20 finger-thick copper plates. Perhaps the most interesting mathematical cuneiform text in the Schøyen Collection is the one shown in Figure 19. It is a small clay tablet with a minute script and an unusual terminology. It is probably Kassite, that is from the time *after* the Old Babylonian period in Mesopotamia (the second half of the second millennium BC). Only one mathematical problem text from the Kassite period was published earlier.

The text begins with a curious computation of the number of certain "gaming-piece fields". If this translation of the corresponding Sumerian term in the text is correct, then "gaming-piece fields" must be geometric figures looking, in some way, like gaming-pieces. The number N of such figures is computed in the following way:

> Given is a "constant" 6.
>
> This constant is diminished by 1, and the remainder is multiplied by 4.
>
> The result is that $N = (6 - 1) \cdot 4 = 20$.

The next step of the procedure is the computation, in the following way, of the area A of a "gaming-piece field" with the side 3 cubits:

> 3 cubits = ;15 rods, 1/2 · 3 cubits = ;07 30 rods,
>
> 1/2 · 3 cubits × 3 cubits = ;01 52 30 sq. rods,
>
> 1/8 · ;01 52 30 sq. rods = ;00 14 03 45 sq. rods,
>
> A = (;01 52 30 – ;00 14 03 45) sq. rods = ;01 38 26 15 sq. rods.

The computation shows that "gaming-piece field" is a term with the meaning "equilateral triangle". Indeed, if s is the side of an equilateral triangle, then the area of the triangle is

$$A = 1/2 \cdot \text{sq. } s \cdot \sqrt{3}/2.$$

The same equation for the area of an equilateral triangle is used in the text in Figure 19, but with $\sqrt{3}/2$ replaced by the approximation $1 - 1/8 = 7/8$. This means that the approximation used for $\sqrt{3}$ was 7/4, which is a good approximation, since sq. 7/4 = 49/16 = 3 1/16 (= 3;03 45).

Figure 19. MS 3876. A computation of the weight of a colossal icosahedron.

In the last part of the text is computed the weight of a "horn figure" composed of $(6 - 1) \cdot 4 = 20$ gaming-piece figures (equilateral triangles) made of 1 finger thick copper plates. The first step of this part of the procedure is the computation of the combined area of the 20 gaming-piece figures:

$$20\,A = 20 \cdot \text{;01 38 26 15 sq. rods} = \text{;32 48 45 sq. rods.}$$

Next is computed the combined volume V of the 20 triangular copper plates, all 1 finger thick. To understand the computation, one must know that 1 finger = 1/30 cubit = ;02 cubit, and that the Babylonian basic volume unit was 1 sq. rod × 1 cubit. Therefore,

$$V = \text{;32 48 45 sq. rods} \times \text{;02 cubit} = \text{;01 05 37 30 sq. rods} \times 1 \text{ cubit.}$$

The last step in the procedure is a multiplication by the number 1 12, called "the constant for copper". What this means can be explained as follows:

According to a Babylonian conventional computation rule,

> 1 talent is the weight of a square copper plate with the side 1 cubit and 1 finger thick, where 1 talent = 1 00 minas = 1 00 00 shekels (approximately 30 kg).

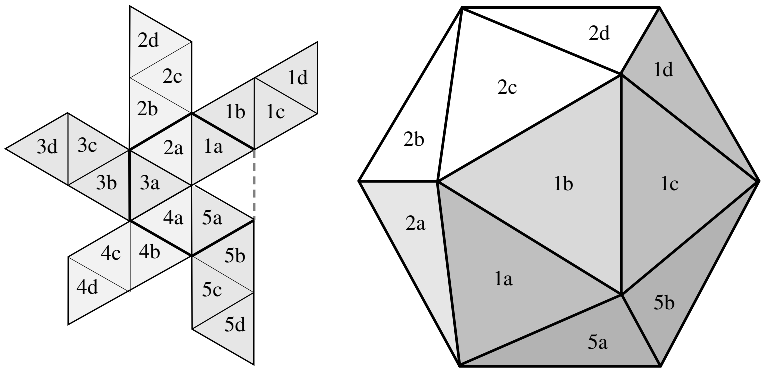

Figure 20. How an icosahedron can be formed by folding a plane figure.

Alternatively, since 1 rod = 12 cubits and 1 cubit = 30 fingers,

1 12 00 talents is the weight of 1 volume unit (1 sq. rod $\times$ 1 cubit) of copper.

Consequently, the weight of the "horn figure" is

$$W = ;01\ 05\ 37\ 30 \text{ sq. rods} \times 1 \text{ cubit} \cdot 1\ 12\ 00 \text{ talents/1 sq.rod} \times 1 \text{ cubit} = 1\ 18;45 \text{ talents (c. 2360 kg)}.$$

After this explanation of all the computations in the Kassite text in Figure 19, it remains only to explain the meaning of the term "horn figure": *Which figure can be constructed by use of 20 equilateral triangles, where 20 is computed as (6 – 1) · 4?*

The only possible answer seems to be that "horn figure" was the name for a regular polyhedron with 20 faces, more precisely what we call an *icosahedron*, a term of Greek origin. The explanation for the curious computation of the number 20 can then be that an icosahedron can be constructed by folding together in an appropriate way a plane figure composed of (6 – 1) · 4 equilateral triangles joined together in the way shown in Figure 20, left. The first step in the construction of that plane figure is to remove an equilateral triangle from a regular hexagon, leaving 6 – 1 equilateral triangles. The next step is to form 6 – 1 chains of four equilateral triangles, in Figure 20, left called 1a – 1d, 2a - 2d, etc. The result of the folding together of the figure composed of 6 – 1 chains is shown in Figure 20, right.

Many surprises in Babylonian mathematics. The six Babylonian problem texts in the Schøyen Collection amply confirm the empirical rule that *all new Babylonian problem texts tend to contain surprises*. What this means is that still very little is known about the true extent of Babylonian mathematics. Why that is so is probably because *very few of the known Babylonian cuneiform texts are well organized original theme texts* produced by some of the extraordinarily talented but anonymous mathematicians who laid the foundation for Babylonian mathematics. What is known, so far, is mainly a large number of excerpts from table texts, and various simple exercises, written by scribe school students at a relatively elementary level, often full of errors. A much smaller number of known advanced exercises were probably copied by older students from the teachers' treasured original theme texts. There are also several known mathematical "recombination texts", large clay tablets apparently produced by enterprising teachers who more or less systematically collected and wrote down together such copies of parts of the original theme texts. For all these reasons, the picture we presently have of Babylonian mathematics at the most advanced level is probably far from complete.

Other examples of color photographs like the ones in this article can be found at `http://cdli.ucla.edu` by searching for author Friberg. More information can be found at the author's homepage `http://www.geocities.com/jranfrib`.