Analytic Number Theory: Modular Forms

Maria Baumgartner

Rutgers University

December 11, 2025

Motivation

- Prove the functional equation for L-functions associated to Hecke eigenforms

Outline

Transforms

Definition (Fourier Transform)

For integrable function $f : \mathbb{R} \to \mathbb{C}$, the Fourier Transform is defined by

$$\tilde{f}(s) = \int_{\mathbb{R}} f(t)e^{-2\pi ist} \; dt.$$

Definition (Mellin Transform)

For integrable function $f : \mathbb{R} \to \mathbb{C}$, the Mellin Transform is defined by

$$\left(\mathcal{M}f\right)(s) = \int_0^\infty t^{s-1} f(t) \; dt.$$

Riemann Zeta Function

Definition

For $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$
$$= \prod_{p} \frac{1}{1 - 1/p^s}, \text{ by Lemma (Euler Product).}$$

Riemann Zeta Function

Definition

For $\mathrm{Re}(s) > 1$,

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \prod_{p} \frac{1}{1 - 1/p^s}, \text{ by Lemma (Euler Product).}\end{aligned}$$

- May be used to show infinitude of primes.

Definition (Dirichlet Characters)

The function $\chi : \mathbb{Z} \to \mathbb{C}$ with associated modulus q satisfying

- $\chi(ab) = \chi(a)\chi(b)$ (multiplicativity)

Dirichlet Characters

Definition (Dirichlet Characters)

The function $\chi : \mathbb{Z} \to \mathbb{C}$ with associated modulus q satisfying

- $\chi(ab) = \chi(a)\chi(b)$ (multiplicativity)
- $\chi(a) = 0$ iff $(a, q) \neq 1$

Dirichlet Characters

Definition (Dirichlet Characters)

The function $\chi : \mathbb{Z} \to \mathbb{C}$ with associated modulus q satisfying

- $\chi(ab) = \chi(a)\chi(b)$ (multiplicativity)
- $\chi(a) = 0$ iff $(a, q) \neq 1$
- $\chi(a + q) = \chi(a)$ (i.e. periodic mod q)

Dirichlet Characters

Definition (Dirichlet Characters)

The function $\chi : \mathbb{Z} \to \mathbb{C}$ with associated modulus q satisfying

- $\chi(ab) = \chi(a)\chi(b)$ (multiplicativity)
- $\chi(a) = 0$ iff $(a, q) \neq 1$
- $\chi(a + q) = \chi(a)$ (i.e. periodic mod q)

Definition (Primitive Dirichlet Characters)

Note that χ has *quasiperiod* d if $\chi(m) = \chi(n)$ for all m and n coprime to q such that $m \equiv n \mod d$.

Dirichlet Characters

Definition (Dirichlet Characters)

The function $\chi : \mathbb{Z} \to \mathbb{C}$ with associated modulus q satisfying

- $\chi(ab) = \chi(a)\chi(b)$ (multiplicativity)
- $\chi(a) = 0$ iff $(a, q) \neq 1$
- $\chi(a + q) = \chi(a)$ (i.e. periodic mod q)

Definition (Primitive Dirichlet Characters)

Note that χ has *quasiperiod* d if $\chi(m) = \chi(n)$ for all m and n coprime to q such that $m \equiv n \mod d$.

The smallest such d is the *conductor*.

Dirichlet Characters

Definition (Dirichlet Characters)

The function $\chi : \mathbb{Z} \to \mathbb{C}$ with associated modulus q satisfying

- $\chi(ab) = \chi(a)\chi(b)$ (multiplicativity)
- $\chi(a) = 0$ iff $(a, q) \neq 1$
- $\chi(a + q) = \chi(a)$ (i.e. periodic mod q)

Definition (Primitive Dirichlet Characters)

Note that χ has *quasiperiod* d if $\chi(m) = \chi(n)$ for all m and n coprime to q such that $m \equiv n \mod d$.

The smallest such d is the *conductor*.

If the conductor equals q, then χ is primitive.

Orthogonality of Dirichlet Characters

Definition (χ_0)

$$\chi_0(n) = \begin{cases} 1 & (n,q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Orthogonality of Dirichlet Characters

Definition (χ_0)

$$\chi_0(n) = \begin{cases} 1 & (n, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma (Orthogonality of Dirichlet Characters)

For a character χ with modulo q,

$$\sum_{m=0}^{q-1} \chi(m) = \begin{cases} \phi(q) & \chi = \chi_0, \\ 0 & \chi \neq \chi_0. \end{cases}$$

Dirichlet L-function

Definition (Dirichlet L-function)

For Dirichlet Character χ, we define $L(\chi, s)$ for $\mathrm{Re}(s) > 1$ as

$$
\begin{aligned}
L\left(\chi, s\right) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\
&= \prod_p \frac{1}{1 - \chi(p)/p^s}, \text{ by Lemma (Euler product).}
\end{aligned}
$$

Nonvanishing of $L(\chi, 1)$

Theorem

If $L(\chi, 1) \neq 0$ for all $\chi \neq \chi_0$, then there exist infinitely many primes.

Nonvanishing of $L(\chi, 1)$

Theorem

If $L(\chi, 1) \neq 0$ for all $\chi \neq \chi_0$, then there exist infinitely many primes.

Proof.

Proof Sketch.

- We start with

$$L(\chi, s) = \prod_p \frac{1}{1 - \chi(p)/p^s}$$

Nonvanishing of $L(\chi, 1)$

Theorem

If $L(\chi, 1) \neq 0$ for all $\chi \neq \chi_0$, then there exist infinitely many primes.

Proof.

Proof Sketch.

- We start with

$$L(\chi, s) = \prod_p \frac{1}{1 - \chi(p)/p^s}$$

- Then take the $\log$ of both sides,

$$\log L(\chi, s) = \sum_p \sum_{1 \leq n} \frac{\chi(p^n)}{np^{ns}}.$$

Proof ct.

Proof.

Proof Sketch ct.

- Considering

$$\frac{1}{\phi(q)} \sum_{\chi \in \chi_q} \bar{\chi}(a) \log L(\chi, s)$$

Proof.

Proof Sketch ct.

- Considering
$$\frac{1}{\phi(q)} \sum_{\chi \in \chi_q} \bar{\chi}(a) \log L(\chi, s)$$
we plug in what we got for $\log L(\chi, s)$, getting
$$\sum_{p} \sum_{1 \leq n} \frac{1}{np^{ns}} \left(\frac{1}{\phi(q)} \sum_{\chi \in \chi_q} \bar{\chi}(a) \chi(p^n) \right).$$

Proof ct.

Proof.

Proof Sketch ct.

- We then break up the summation

$$\underbrace{\sum_{\substack{p \\ p\equiv a \pmod q}} \frac{1}{p^s}}_{(1)} + \underbrace{\sum_{2\leq n}\left(\sum_{\substack{p \\ p^n\equiv a \pmod q}} \frac{1}{np^{ns}}\right)}_{(2)}.$$

Proof ct.

Proof.

Proof Sketch ct.

- We then break up the summation

$$\underbrace{\sum_{p \equiv a \pmod q} \frac{1}{p^s}}_{(1)} + \underbrace{\sum_{2 \leq n} \left(\sum_{p^n \equiv a \pmod q} \frac{1}{np^{ns}} \right)}_{(2)}.$$

- We are able to bound (2). Thus, we must show our original expression is unbounded to prove (1) is unbounded.

Proof ct.

Proof.

Proof Sketch ct.

- In particular, we show

$$\lim_{s \to 1^+} \frac{1}{\phi(q)} \sum_{\chi \in \chi_q} \bar{\chi}(a) \log L(\chi, s) = \infty$$

  since this would imply (1) diverges to ∞.

Proof ct.

Proof.

Proof Sketch ct.

- In particular, we show

$$\lim_{s\to 1^+} \frac{1}{\phi(q)} \sum_{\chi\in\chi_q} \bar{\chi}(a)\log L(\chi,s) = \infty$$

  since this would imply (1) diverges to ∞.
- Breaking up the sum, we observe χ_0 produces a $\log(\zeta(s))$ which goes to ∞ as $s \to 1^+$. All other $\chi \neq \chi_0$ contribute only bounded terms if $L(\chi,1) \neq 0$.

Proof ct.

Proof.

Proof Sketch ct.

- In particular, we show

$$\lim_{s \to 1^+} \frac{1}{\phi(q)} \sum_{\chi \in \chi_q} \bar{\chi}(a) \log L(\chi, s) = \infty$$

  since this would imply (1) diverges to ∞.
- Breaking up the sum, we observe χ_0 produces a $\log(\zeta(s))$ which goes to ∞ as $s \to 1^+$. All other $\chi \neq \chi_0$ contribute only bounded terms if $L(\chi, 1) \neq 0$.
- Using orthogonality and Euler products, we can conclude with the hypothesis that there are infinitely many primes.

□

Further Results

- For Complex χ, $L(\chi, 1) \neq 0$ (Lemma).
- For Real χ, $L(\chi, 1) \neq 0$ (Lemma).

Modular Forms/Motivation

We've built up some analytic tools.

Why analyze Modular Forms?

- We can choose matrices in $SL_2(\mathbb{Z})$ to yield interesting results,
- Possess Fourier expansions $f(z) = \sum a_n e^{2\pi i n z}$.

Definition (Slash Operator)

For $\gamma \in \mathsf{GL}_2(\mathbb{R})$,

$$f|_k\gamma(z) := \det\gamma^{k/2} j(\gamma, z)^{-k} f(\gamma z)$$

Note: $\mathsf{GL}_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) \neq 0\}$.

Modular Forms

Definition (Slash Operator)

For $\gamma \in \mathsf{GL}_2(\mathbb{R})$,

$$f|_k\gamma(z) := \det\gamma^{k/2} j(\gamma, z)^{-k} f(\gamma z)$$

Note: $\mathsf{GL}_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) \neq 0\}$.

Definition (Modular Form)

Note that $\mathcal{H} = \{z \in \mathbb{C} \mid z \text{ has positive imaginary part}\}$. A Modular Form is a function $f : \mathcal{H} \to \mathbb{C}$ associated to a weight k such that

$$f|_k\gamma(z) = f(z)$$

where $\gamma \in \Gamma$ and Γ is a subgroup of SL_2.

Modular Forms

Definition (Slash Operator)

For $\gamma \in \mathsf{GL}_2(\mathbb{R})$,

$$f|_k\gamma(z) := \det\gamma^{k/2} j(\gamma, z)^{-k} f(\gamma z)$$

Note: $\mathsf{GL}_2(\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) \neq 0\}$.

Definition (Modular Form)

Note that $\mathcal{H} = \{z \in \mathbb{C} \mid z$ has positive imaginary part$\}$. A Modular Form is a function $f : \mathcal{H} \to \mathbb{C}$ associated to a weight k such that

$$f|_k\gamma(z) = f(z)$$

where $\gamma \in \Gamma$ and Γ is a subgroup of SL_2. **Note:**

- $\mathsf{SL}_2(\mathbb{Z}) = \{A \in M_{2\times 2}(\mathbb{Z}) \mid \det(A) = 1\}$.
- $\gamma z = (az + b)/(cz + d)$ and $j(\gamma, z) = cz + d$.

Cusp Forms

Definition (Cusp Form)

Function $f : \mathcal{H} \to \mathbb{C}$ in modular form and vanishes at $i\infty$.

- Vanish at every cusp

Cusp Forms

Definition (Cusp Form)

Function $f : \mathcal{H} \to \mathbb{C}$ in modular form and vanishes at $i\infty$.

- Vanish at every cusp

Further:

- If f has weight k, then f is a weight k cusp form iff $f(\infty) = 0$ (Lemma).

Cusp Forms

Definition (Cusp Form)

Function $f : \mathcal{H} \to \mathbb{C}$ in modular form and vanishes at $i\infty$.

- Vanish at every cusp

Further:

- If f has weight k, then f is a weight k cusp form iff $f(\infty) = 0$ (Lemma).
- If f has fourier expansion:

$$f(z) = \sum_{0 \leq m} a_m e^{2\pi i m z}.$$

then $a_0 = 0$.

Eisenstein Series and Delta Function

Definition (Eisenstein series)

$$E_k(z) = \sum \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$$
$$= 1 - \frac{2k}{B_k} \sum_{1\leq m} \sigma_{k-1}(m) e^{2\pi i m z}, \text{ by Lemma (Fourier Expansion).}$$

Eisenstein Series and Delta Function

Definition (Δ-Function)

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}.$$

Eisenstein Series and Delta Function

Definition (Δ-Function)

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}.$$

Note:

- Δ is a weight 12 cusp form.

Eisenstein Series and Delta Function

Definition (Δ-Function)

$$\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728}.$$

Note:

- Δ is a weight 12 cusp form.
- Non-vanishing of Δ on $\mathcal{H}$ except for a simple zero at $i\infty$ can be shown via the Valence Formula (Lemma).

Structure of M_k

In the space M_k of modular forms of weight k, the following properties hold:

- If $k < 0$ or odd, $M_k = \emptyset$.
- If $k = 0$, $M_k = \mathbb{C}$.
- If $k = 2$, then $M_k = \emptyset$.
- If $4 \leq k \leq 10$ is even, $M_k = \mathbb{C} \cdot E_k$.
- If $k \geq 12$, then $M_k = \Delta M_{k-12} \oplus \mathbb{C} E_k$

Structure of M_k

In the space M_k of modular forms of weight k, the following properties hold:

- If $k < 0$ or odd, $M_k = \emptyset$.
- If $k = 0$, $M_k = \mathbb{C}$.
- If $k = 2$, then $M_k = \emptyset$.
- If $4 \leq k \leq 10$ is even, $M_k = \mathbb{C} \cdot E_k$.
- If $k \geq 12$, then $M_k = \Delta M_{k-12} \oplus \mathbb{C} E_k$

Further,

$$M_k = \Delta M_{k-12} \ \oplus \ \mathbb{C} E_k.$$

Structure of M_k

In the space M_k of modular forms of weight k, the following properties hold:

- If $k < 0$ or odd, $M_k = \emptyset$.
- If $k = 0$, $M_k = \mathbb{C}$.
- If $k = 2$, then $M_k = \emptyset$.
- If $4 \leq k \leq 10$ is even, $M_k = \mathbb{C} \cdot E_k$.
- If $k \geq 12$, then $M_k = \Delta M_{k-12} \oplus \mathbb{C} E_k$

Further,

$$M_k = \Delta M_{k-12} \ \oplus \ \mathbb{C} E_k.$$

Consequences:

- All modular forms are built from E_4, E_6, and Δ.
- $M_{k+12} \in \mathbb{C}[E_4, E_6]$ (Proven via induction.)

Hecke Operators

Definition (Hecke Operator)

Acts on function $f \in M_k(\Gamma_0(N))$ as

$$(T_n f)(z) = n^{k/2-1} \sum_{\delta \in \Delta_n^N} f|_k \delta(z).$$

Remark: can think of Hecke operators as an averaging operator.

Hecke Operators

Definition (Hecke Operator)

Acts on function $f \in M_k(\Gamma_0(N))$ as

$$(T_n f)(z) = n^{k/2-1} \sum_{\delta \in \Delta_n^N} f|_k \delta(z).$$

Remark: can think of Hecke operators as an averaging operator.

Hecke operators have the property:

$$T_m T_n = T_{mn} \quad \text{if } (m,n) = 1.$$

Hecke Operators

Definition (Hecke Operator)

Acts on function $f \in M_k(\Gamma_0(N))$ as

$$(T_n f)(z) = n^{k/2-1} \sum_{\delta \in \Delta_n^N} f|_k \delta(z).$$

Remark: can think of Hecke operators as an averaging operator.

Hecke operators have the property:

$$T_m T_n = T_{mn} \quad \text{if } (m,n) = 1.$$

If f has fourier expansion

$$f(z) = \sum_{1 \leq m} c_m e^{2\pi i m z}$$

with $c_1 = 1$ and there exists λ_n s.t. $T_n f = \lambda_n f$ for all n, then

$$c_{mn} = c_m c_n \quad \forall (m,n) = 1.$$

How Hecke Operators Act

Lemma

For f with Fourier expansion $f(z) = \sum_{0 \le m} c_m e^{2\pi i m z}$ then,

$$T_n f(z) = \sum_{0 \le m} \left(\sum_{d | (m,n)} \chi_0^N(d) d^{k-1} c_{mn/d^2} \right) e^{2\pi i m z}.$$

Lemma

For f with Fourier expansion $f(z) = \sum_{0 \leq m} c_m e^{2\pi imz}$ then,

$$T_n f(z) = \sum_{0 \leq m} \left(\sum_{d|(m,n)} \chi_0^N(d) d^{k-1} c_{mn/d^2} \right) e^{2\pi imz}.$$

Key consequences:

- T_n preserves cusp forms.
- Hecke operators commute and are self-adjoint under the Petersson inner product.

Thus $S_k(\Gamma_0(N))$ has a basis of simultaneous eigenforms.

Hecke Bound (Used in L-function Analysis)

Note that, for a cusp form of weight k:

$$|a_n| \ll n^{k/2}.$$

Hecke Bound (Used in L-function Analysis)

Note that, for a cusp form of weight k:

$$|a_n| \ll n^{k/2}.$$

Why this matters:

- Ensures L-function for cusp forms, $L(f, s) = \sum a_n n^{-s}$, converges absolutely for $\mathrm{Re}(s) > 1 + \frac{k}{2}$.
- Key part of deriving the functional equation for the L-functions associated with cusp forms.

Newforms

- For level N, cusp forms decompose into S_k^{old} and S_k^{new}
- A level N **newform** is an $f \in S_k^{\text{new}}$ such that f is a normalized simultaneous eigenform for all T_n with $(n, N) = 1$.

Newforms

- For level N, cusp forms decompose into S_k^{old} and S_k^{new}
- A level N **newform** is an $f \in S_k^{\mathrm{new}}$ such that f is a normalized simultaneous eigenform for all T_n with $(n, N) = 1$.
- Key property: **multiplicity one**.

$$f, g \in S_k^{\mathrm{new}}(\Gamma_0(N)),\ \lambda_n(f) = \lambda_n(g) \quad \forall (n, N) = 1 \quad \Rightarrow f = g.$$

Newforms

Theorem

A level N newform f is a normalized simultaneous eigenform of all Hecke operators T_n.

- If f is a newform and $T_n f = \lambda_n f$ for $(n, N) = 1$, then $T_m f = \lambda_m f \quad$ for all m.
- Note that proof makes use of multiplicity one principle.

The Fricke Involution

Define by letting $Wf = f|_k\omega$ where

$$\omega = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}.$$

The Fricke Involution

Define by letting $Wf = f|_k\omega$ where

$$\omega = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}.$$

Key facts:

- For $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$,

  $$\omega\gamma = \gamma'\omega$$

  for some $\gamma' \in \Gamma_0(N)$.

The Fricke Involution

Define by letting $Wf = f|_k\omega$ where

$$\omega = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}.$$

Key facts:

- For $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$,
$$\omega\gamma = \gamma'\omega$$
for some $\gamma' \in \Gamma_0(N)$.
- Fricke Involution preserves modular and cusp forms, i.e. for $f \in M_k(\Gamma_0(N))$ or $\in S_k(\Gamma_0(N))$,
$$(Wf)|_k\gamma = Wf.$$

The Fricke Involution

Define by letting $Wf = f|_k\omega$ where

$$\omega = \begin{pmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{pmatrix}.$$

Key facts:

- For $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$,

  $$\omega\gamma = \gamma'\omega$$

  for some $\gamma' \in \Gamma_0(N)$.
- Fricke Involution preserves modular and cusp forms, i.e. for $f \in M_k(\Gamma_0(N))$ or $\in S_k(\Gamma_0(N))$,

  $$(Wf)|_k\gamma = Wf.$$

- Fricke involution commutes with Hecke operators

  $$WT_n = T_nW$$

Fricke and Hecke Commute

The operators commute:

$$T_n W_N = W_N T_n.$$

Consequences:

- If $f \in S_k^{\mathsf{new}}(\gamma_0(N))$ is a Hecke eigenform, then $f = cg$ where g is a newform.

Fricke and Hecke Commute

The operators commute:

$$T_n W_N = W_N T_n.$$

Consequences:

- If $f \in S_k^{\text{new}}(\gamma_0(N))$ is a Hecke eigenform, then $f = cg$ where g is a newform.
- We can evaluate,
  $$T_n W g = \lambda_n W g.$$
  So $W g$ is a Hecke eigenform.

Fricke and Hecke Commute

The operators commute:

$$T_n W_N = W_N T_n.$$

Consequences:

- If $f \in S_k^{\text{new}}(\gamma_0(N))$ is a Hecke eigenform, then $f = cg$ where g is a newform.
- We can evaluate,

  $$T_n W g = \lambda_n W g.$$

  So Wg is a Hecke eigenform.
- Since $Wg = wh$ where h is a newform, we get

  $$T_n h = \lambda_n h,$$

Fricke and Hecke Commute

The operators commute:

$$T_n W_N = W_N T_n.$$

Consequences:

- If $f \in S_k^{\mathsf{new}}(\gamma_0(N))$ is a Hecke eigenform, then $f = cg$ where g is a newform.
- We can evaluate,
  $$T_n W g = \lambda_n W g.$$
  So Wg is a Hecke eigenform.
- Since $Wg = wh$ where h is a newform, we get
  $$T_n h = \lambda_n h,$$
- So g, h have same Hecke eigenvalues so $g = h$. Thus, $Wf = wf$ for some $w \in \mathbb{C}$.

Fricke and Hecke Commute

Lemma

If f is a Hecke eigenform then f is an eigenform of the Fricke involution. Specifically $Wf = \pm f$.

Setting Up the Functional Equation

For a cusp form $f(z) = \sum a_n e^{2\pi i n z}$, define its L-function:

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s}.$$

Completed L-function (scaled by $N^{s/2}$):

$$\Lambda(f, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f, s).$$

Setting Up the Functional Equation

For a cusp form $f(z) = \sum a_n e^{2\pi i n z}$, define its L-function:

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s}.$$

Completed L-function (scaled by $N^{s/2}$):

$$\Lambda(f, s) = N^{s/2} (2\pi)^{-s} \Gamma(s) L(f, s).$$

Use Mellin transform:

$$\Lambda(f, s) = N^{s/2} \int_0^\infty f(it)\, t^s \frac{dt}{t}.$$

Setting Up the Functional Equation

For a cusp form $f(z) = \sum a_n e^{2\pi i n z}$, define its L-function:

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s}.$$

Completed L-function (scaled by $N^{s/2}$):

$$\Lambda(f, s) = N^{s/2} (2\pi)^{-s} \Gamma(s) L(f, s).$$

Use Mellin transform:

$$\Lambda(f, s) = N^{s/2} \int_0^\infty f(it)\, t^s \frac{dt}{t}.$$

Split integral at $t = 1/\sqrt{N}$ (allowed by Fricke),

$$N^{s/2} \left(\int_0^{1/\sqrt{N}} f(it)\, t^s \frac{dt}{t} + \int_{1/\sqrt{N}}^\infty f(it)\, t^s \frac{dt}{t} \right).$$

Using Fricke and Final Equation

Compute:

$$\int_0^{1/\sqrt{N}} f(it)\, t^s \frac{dt}{t} = \pm i^{\,k} N^{\frac{k}{2}-s} \int_{1/\sqrt{N}}^{\infty} f(it)\, t^{k-s} \frac{dt}{t}.$$

Using Fricke and Final Equation

Compute:

$$\int_0^{1/\sqrt{N}} f(it)\, t^s \frac{dt}{t} = \pm i^{\,k} N^{\frac{k}{2}-s} \int_{1/\sqrt{N}}^{\infty} f(it)\, t^{k-s} \frac{dt}{t}.$$

Combining with the second piece yields:

$$\boxed{\Lambda(f,s) = \pm\, i^k\, \Lambda(f, k-s).}$$

This is the functional equation for L-functions associated with Hecke eigenforms.

Acknowledgements

Much thanks to the DRP team and to my wonderful mentor Preston Tranbarger!

Click here for the notes used!

Thank you!