2.3.3 Group of N individuals (Games with N>1)

2.3.3.1 Non-cooperative games

2.3.3.1.1 Static games

The players:

- Who is involved?

The rules:

- Who moves when?
- What do they know when they move?
- What can they do?

The outcomes:

- For each possible set of actions by the players, what is the outcome of the game?

The payoffs:

- What are the players’ preferences (i.e. utilities) over the possible outcomes?

Assumption:

- Pure strategies – there is no randomization when players take an action

2.3.3.1.1.1 Constant sum game (zero-sum)

2.3.3.1.1.1.a Pure Strategy

Example 9: Boris and Sophie divide the surplus

a) Situation in words

Imagine Sophie and Boris decide to finalize a price in the following manner:

- They sit in different rooms, don't communicate with each other and announce their decision simultaneously: *A* or *B*
- If both announce A they trade without concessions (i.e. each gets zero additional surplus)
- If Boris announces B and Sophie – A : The price will shift towards Sophie by CHF 5
- If Boris announces A and Sophie – B: The price will shift towards Boris by CHF 4
- If both announce B: Sophie will make a concession towards Boris by CHF 1

b) Game

i) Players: 2 (Boris, Sophie)
ii) Actions: 2 (A, B)
iii) Preferences

Boris:

$$(A,B) \succ (B,B) \succ (A,A) \succ (B,A)$$

Sophie:

$$(A,B) \prec (B,B) \prec (A,A) \prec (B,A)$$

i) Payoffs are presented in the *normal form* below

		Sophie	
		A	B
Boris	A	*0,0*	*4,-4*
	B	*-5,5*	*1,-1*

$\pi_S(a_B, a_S) = -\pi_B(a_B, a_S) \rightarrow$ *zero-sum game!*

c) Solution

Solution concept – Maxmin analysis

Maximize the minimum you can get!

		Sophie	
		A	B
Boris	A	*0,0*	*4,-4*
	B	*-5,5*	*1,-1*

Maxmin analysis: Boris

Sophie

Boris		A	B	*Min payoff to Boris*
	A	*0,0*	*4,-4*	*0*
	B	*-5,5*	*1,-1*	*-5*

← Maxmin payoff to Boris

Maxmin analysis: Sophie

		Sophie		
		A	B	*Min payoff to Boris*
Boris	A	*0,0*	*4,-4*	*0*
	B	*-5,5*	*1,-1*	*-5*
	Min payoff to Sophie	*0*	*-4*	

Maxmin payoff to Boris

Maxmin payoff to Sophie

Maxmin analysis: Saddle point

		Sophie		
		A	B	*Min payoff to Boris*
Boris	A	*0,0*	*4,-4*	*0*
	B	*-5,5*	*1,-1*	*-5*
	Min payoff to Sophie	*0*	*-4*	

Maxmin payoff to Boris

Maxmin payoff to Sophie

Strategies (A,A) form equilibrium of the zero-sum game (often called *saddle point*). Observe:

(maxmin payoff to Boris [=0]) **= -** *(maxmin payoff to Sophie [=0]*

Equilibrium

We say pure actions (a_1^*, a_2^*) (strategies) form an equilibrium if

$\pi_1(a_1^*, a_2^*) \geq \pi_1(a_1, a_2^*)$, for $\forall a_1 \in A_1$ and

$\pi_2(a_1^*, a_2^*) \geq \pi_2(a_1^*, a_2)$, for $\forall a_2 \in A_2$.

In maxmin equilibrium (saddle point) we have:

$$\pi_1(a_1^*, a_2^*) = \max_{a_1 \in A_1} \min_{a_2 \in A_2} \pi_1(a_1, a_2) = \min_{a_2 \in A_2} \max_{a_1 \in A_1} \pi_1(a_1, a_2)$$

Theorem: Maxmin (John von Neumann)

Every zero-sum game with two players and finite number of strategies has a solution. That is, there is a unique number v, called the value of the game, and there are optimal strategies* for Player 1 and Player 2 such that:

i. If Player 1 plays his optimal strategy, his expected gain will be larger or equal to v, no matter what Player 2 does, and

ii. If Player 2 plays his optimal strategy, his expected loss will be smaller or equal to v, no matter what Player 1 does.

* This applies for pure and mixed strategies.
Pure strategy is a strategy in which there is no randomization.
Mixed strategy is a randomization over a combination of pure strategies (choosing according to certain probability distribution)

d) Comments

- Game theory begun with studies of zero-sum games
- In zero-sum (also called constant-sum) games:
  - The sum of payoffs in each cell is zero (or constant)
  - The interests of the players are strictly opposite
- Maxmin strategy enables a player to calculate the maximum of the minimum payoff he can achieve. This strategy guarantees him a security level – the minimum payoff for a player, when he plays non-cooperatively.

Example 10: 2 players, 4 strategies

a) Situation in words

Sophie and Boris play a game where they chose an action and depending on their decisions the payoff of both is determined

b) Game

i) Players: 2 (Boris, Sophie)
ii) Actions: 4 (A, B, C, D)
iii) Payoffs are presented in the *normal form*

Sophie

Boris

	A	B	C	D
A	*12,-12*	*-1,1*	*1,-1*	*0,0*
B	*5,-5*	*1,-1*	*7,-7*	*-20,20*
C	*3,-3*	*2,-2*	*4,-4*	*3,-3*
D	*-16,16*	*0,0*	*0,0*	*16,-16*

c1) Solution A (maxmin: max the min gain)

		Sophie				
		A	B	C	D	Min payoff to Boris
Boris	A	*12,-12*	*-1,1*	*1,-1*	*0,0*	*-1*
	B	*5,-5*	*1,-1*	*7,-7*	*-20,20*	*-20*
	C	*3,-3*	*2,-2*	*4,-4*	*3,-3*	*2*
	D	*-16,16*	*0,0*	*0,0*	*16,-16*	*-16*

← Maxmin to Boris

Sophie

		A	B	C	D	Min payoff to Boris
Boris	A	*12,-12*	*-1,1*	*1,-1*	*0,0*	*-1*
	B	*5,-5*	*1,-1*	*7,-7*	*-20,20*	*-20*
	C	*3,-3*	*2,-2*	*4,-4*	*3,-3*	*2*
	D	*-16,16*	*0,0*	*0,0*	*16,-16*	*-16*
	Min payoff to Sophie	*-12*	*-2*	*-7*	*-16*	

Maxmin to Boris

Maxmin to Sophie

Strategies (C,B) form equilibrium of the zero-sum game.
Observe:
(maxmin payoff to Boris) **=** **-***(maxmin payoff to Sophie)*

Boris and Sophie maximize their minimal gains (payoffs).

Illustration of the saddle point

Sophie

Boris		A	B	C	D
	A	12,-12	-1,1	1,-1	0,0
	B	5,-5	1,-1	7,-7	-20,20
	C	3,-3	2,-2	4,-4	3,-3
	D	-16,16	0,0	0,0	16,-16

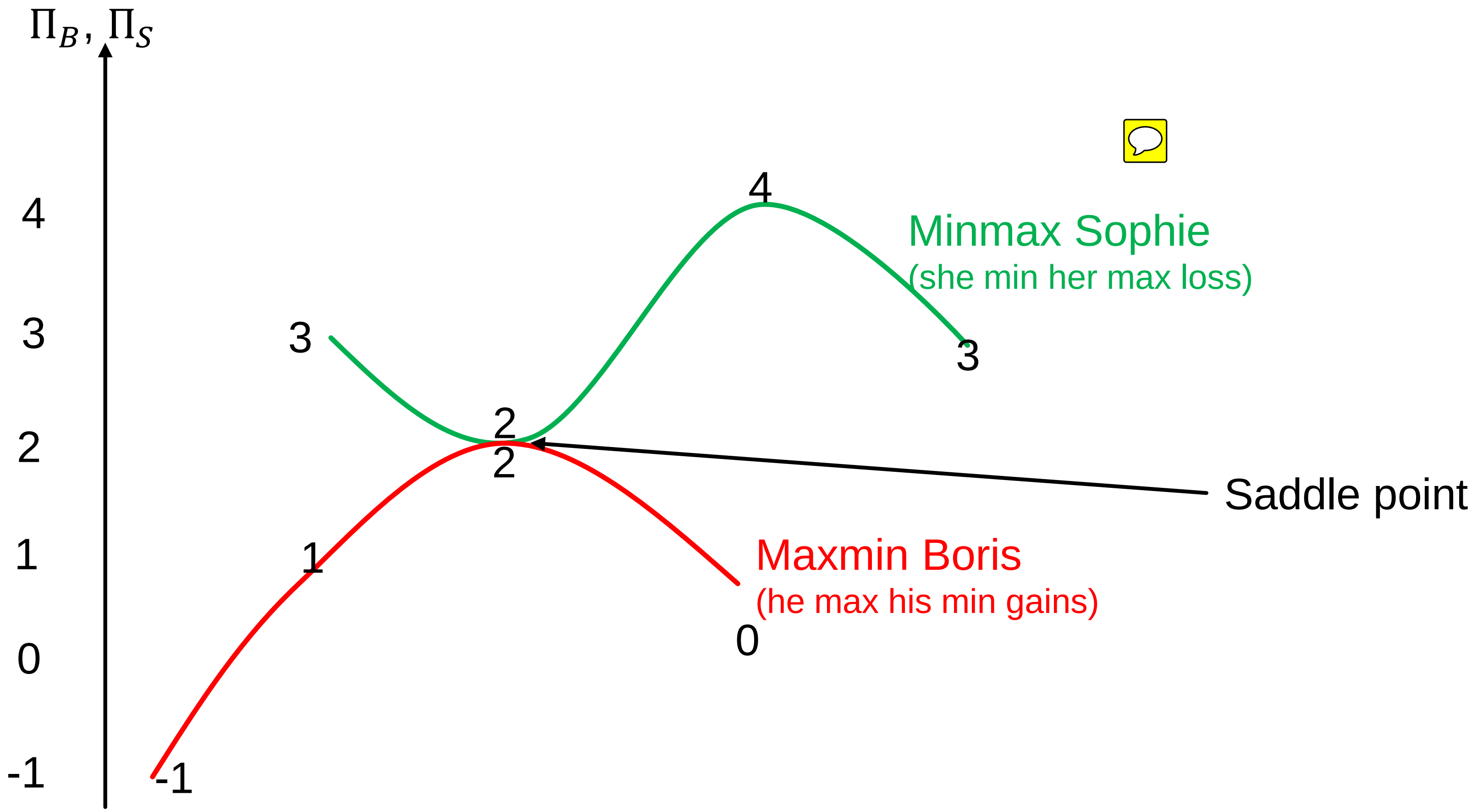
Sophie's payoffs are here presented as gains
Now, let's put her gains as losses

Sophie

		A	B	C	D
Boris	A	12,12	-1,-1	1,1	0,0
	B	5,5	1,1	7,7	-20,-20
	C	3,3	2,2	4,4	3,3
	D	-16,-16	0,0	0,0	-16,-16

Sophie’s payoffs are now presented as losses

Sketch of the saddle point: Payoff Boris, Payoff Sophie (Π_B, Π_S)



Sketch of the saddle point

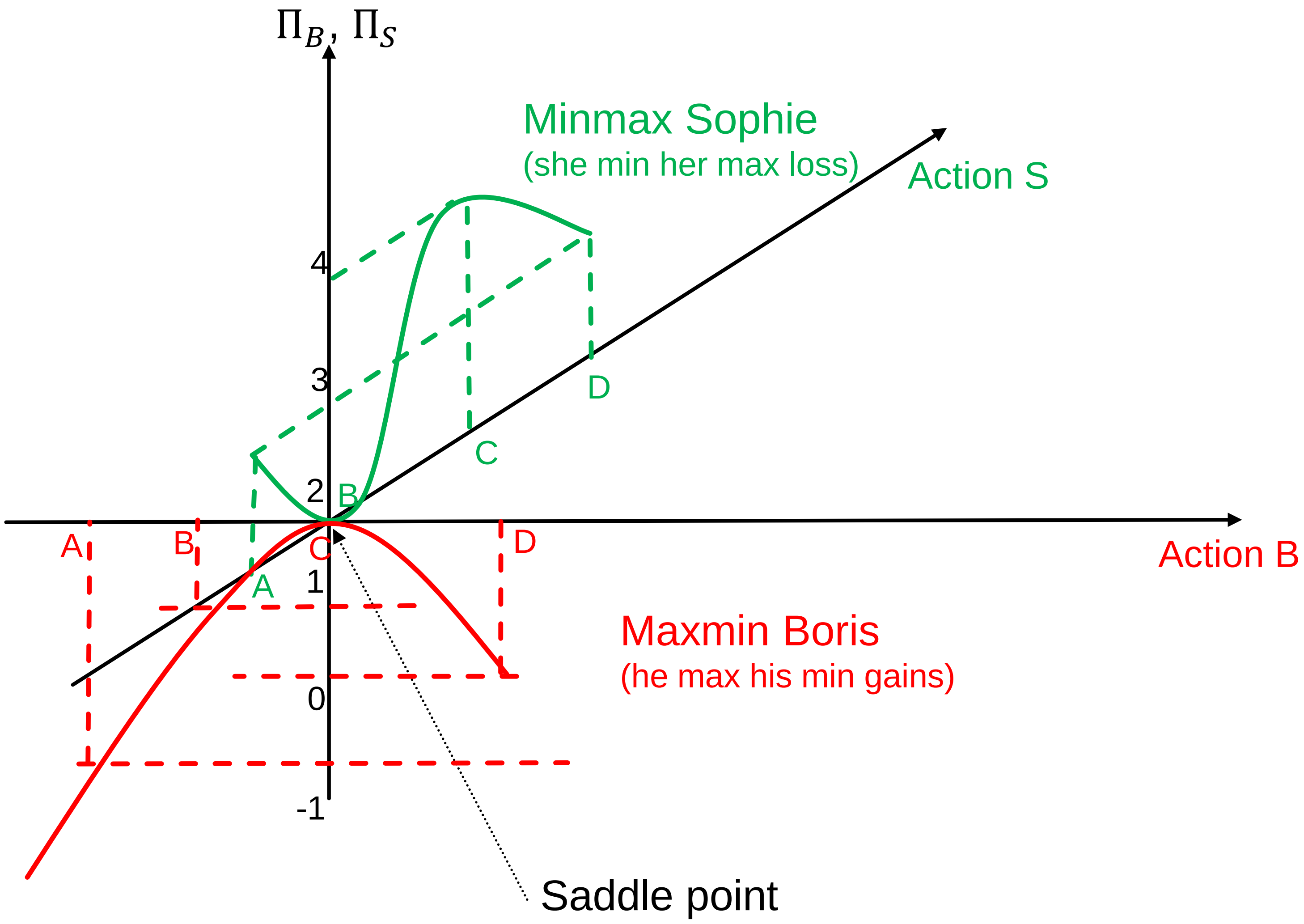
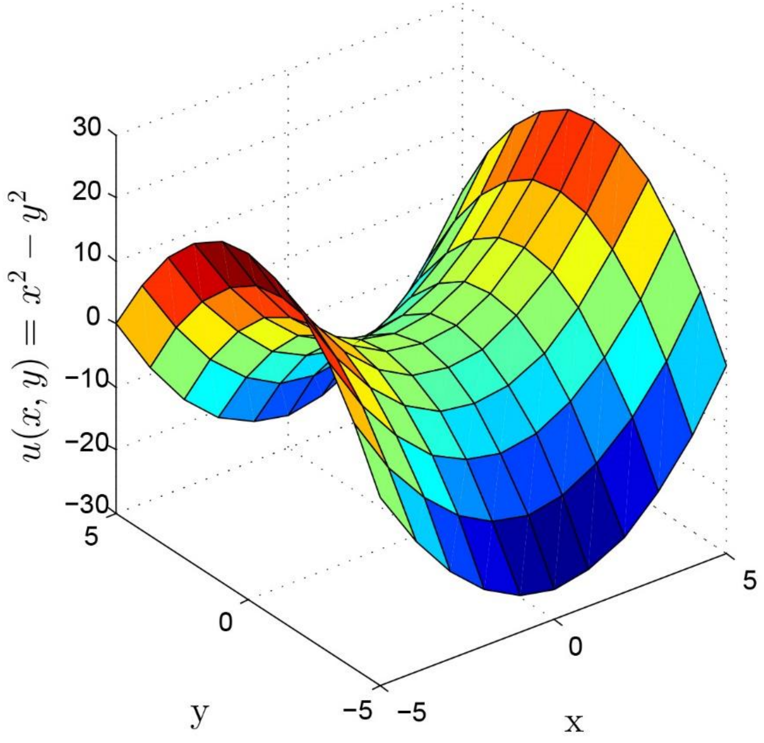


Illustration of a saddle point in 3d space



c2) “Solution” B (maxmax: max the max gain)

Sophie

		A	B	C	D	Max payoff to Boris
Boris	A	*12,-12*	*-1,1*	*1,-1*	*0,0*	*12*
	B	*5,-5*	*1,-1*	*7,-7*	*-20,20*	*7*
	C	*3,-3*	*2,-2*	*4,-4*	*3,-3*	*4*
	D	*-16,16*	*0,0*	*0,0*	*16,-16*	*16*
	Max payoff to Sophie	*16*	*1*	*0*	*20*	

Maxmax to Boris

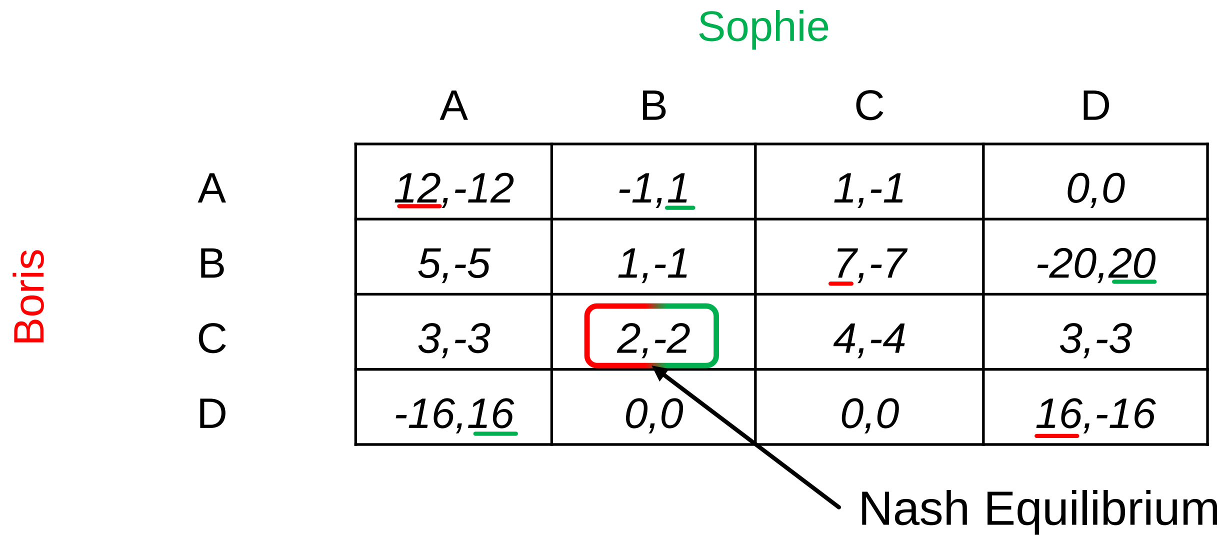
Maxmax to Sophie

If both play D then S will get only **-16**; if S would know that Boris plays D she would play A and get **+16**.
But if he would know that she plays A then he would play A and get **+12** and she would get **-12**
=> Maxmax: **unsafe game**!

Strategies (D,D) form **no** equilibrium of the game. Observe:
maxmax payoff to Boris [16] **≠ -** *maxmax payoff to Sophie[20]*

c3) Solution C (best response)

Meaning: "Given the strategies of the other players, decide what maximizes your payoff!"



		Sophie			
		A	B	C	D
Boris	A	12,-12	-1,1	1,-1	0,0
	B	5,-5	1,-1	7,-7	-20,20
	C	3,-3	2,-2	4,-4	3,-3
	D	-16,16	0,0	0,0	16,-16

Nash Equilibrium

Theoretical concept

(1928 - 2015)

We call a strategy a_i', is a best response for Player i to some strategy combination of the other players, denoted by a_{-i}, if

$$\pi_i(a_i', a_{-i}) \geq \pi_i(a_i, a_{-i}), \quad \text{for } \forall a_i \in A_i.$$

Meaning: "Given the strategies of the other players, decide what maximizes your payoff!"

The action profile $a^* = (a_1^*, \cdots, a_i^*, \cdots a_N^*)$ is a Nash Equilibrium in pure strategies of a game with N players if and only if every player's action is a best response to the other player's actions. No one has an incentive to change his strategy **unilaterally!**

Remark 1: In static games there is no difference between terms 'strategy' and 'action'. However, in dynamic game this is not true.

Remark 2: We have already given a definition of an equilibrium before. This definition just uses the terms of the best responses. Notion of best responses is helpful when one solves a particular problem or tries to prove the existence of an equilibrium.

d) Comments

- We applied two solution concepts A and B: Maxmin by von Neumann and Nash equilibrium
- Both concepts led to the same solution
- The Maxmin concept was developed first and is applicable to constant sum games.
- Nash's best response is more widely applicable and we will use it from now on.

Example 11: Nash equilibrium in pure strategies*

a) Situation in words

Sophie and Boris play now a game with two strategies each

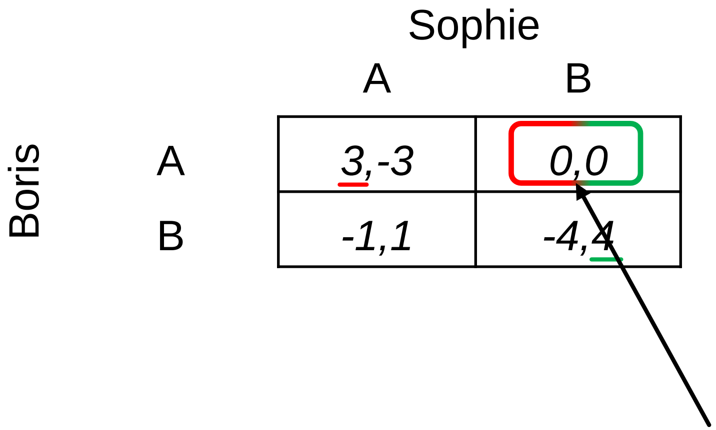
b) Game

i) Players: 2 (Boris, Sophie)
ii) Actions: 2 (A, B)
iii) Payoffs are presented in the *normal form*

		Sophie	
		A	B
Boris	A	*3,-3*	*0,0*
	B	*-1,1*	*-4,4*

*no randomization over actions

c) Solution (Nash)



		Sophie	
		A	B
Boris	A	*3,-3*	*0,0*
	B	*-1,1*	*-4,4*

Nash Equilibrium
No one has an incentive to change his strategy **unilaterally!**

d) Comments

In this easy game we have a Nash equilibrium in pure strategy: The combination of the strategies, where Boris plays B and Sophie plays B

Example 12: No equilibrium in pure strategies

a) Situation in words

Sophie and Boris play now again a game with two strategies each

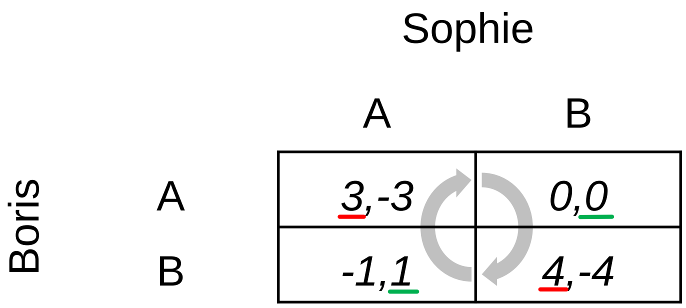
b) Game

i) Players: 2 (Boris, Sophie)
ii) Actions: 2 (A, B)
iii) Payoffs are presented in the *normal form*

		Sophie	
		A	B
Boris	A	*3,-3*	*0,0*
	B	*-1,1*	*4,-4*

NB: numbers in Exp 12 are different from Exp 11

c) Solution (Nash)

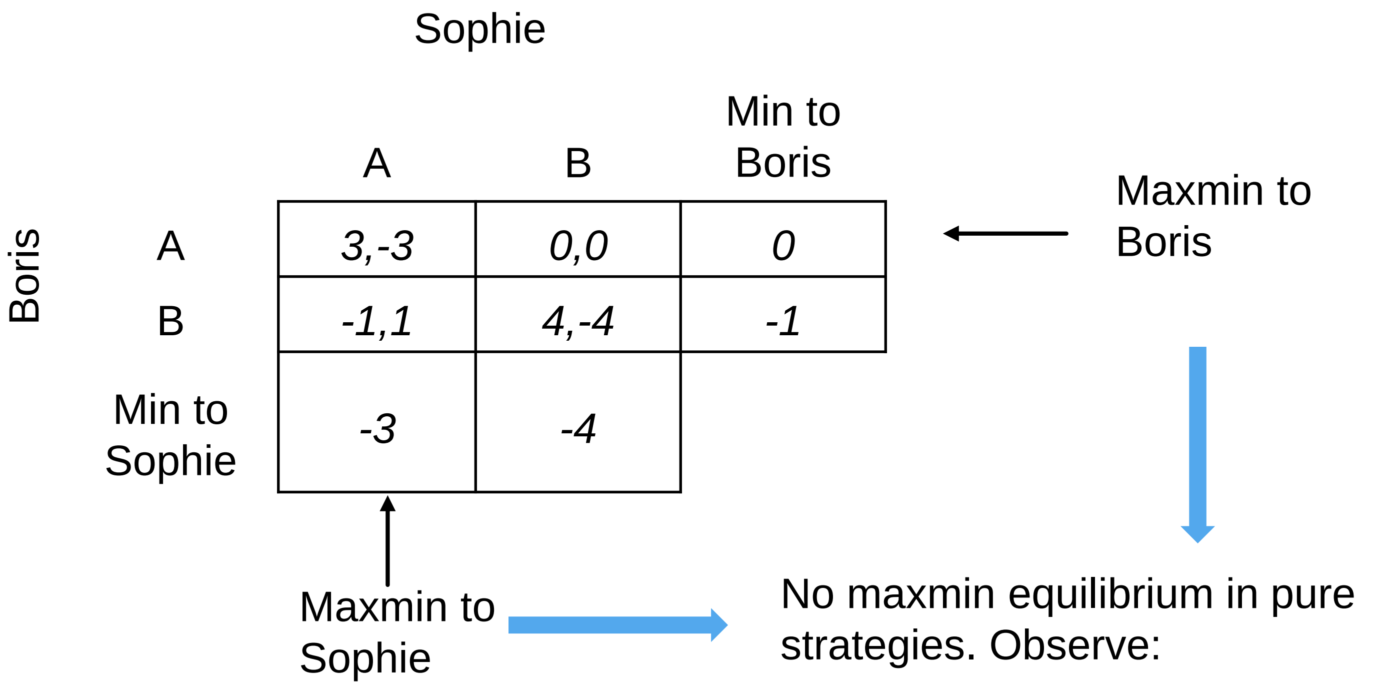


		Sophie	
		A	B
Boris	A	3,-3	0,0
	B	-1,1	4,-4

No Nash Equilibrium

d) Comments

As an **illustration** we use the maxmin (von Neumann) solution method for comparison
→ as there is no Nash there is also no saddle point



		Sophie		
		A	B	Min to Boris
Boris	A	*3,-3*	*0,0*	*0*
	B	*-1,1*	*4,-4*	*-1*
	Min to Sophie	*-3*	*-4*	

Maxmin to Boris

Maxmin to Sophie

No maxmin equilibrium in pure strategies. Observe:

(maxmin payoff to Boris) **≠ -** *(maxmin payoff to Sophie)*

2.3.3.1.1.1.b Mixed Strategy

Theoretical concept of mixed strategy

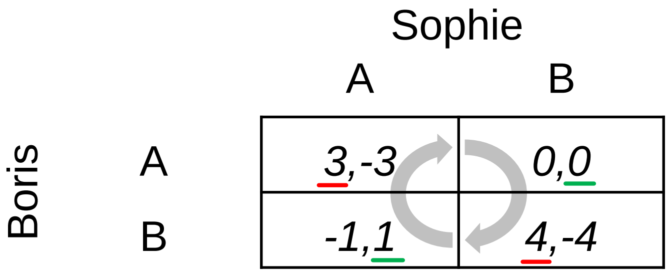
- So far, we have analyzed games in *pure strategies*, i.e. in which there is no randomization over actions available to players
- A *mixed strategy* specifies the probability with which each of the pure strategies is used
- Example:
  Suppose the set of pure strategies to player i is $\boldsymbol{S}_i = \{s_a, s_b, s_c, \dots\}$
  Then the mixed strategy is a vector of probabilities

$$\sigma_i = (p(s_a), p(s_b), p(s_c), \dots), \text{s.t.} \sum_{s \in \boldsymbol{S}_i} p(s) = 1$$

- Note: a pure strategy s_b, for example, can be represented as

$$\sigma_i = (0,1,0, \dots)$$

Back to Example 12



		Sophie	
		A	B
Boris	A	3,-3	0,0
	B	-1,1	4,-4

No Nash Equilibrium

According to the definition on the previous slide, the *mixed strategy* is a randomization of a combination of pure strategies, i.e. you choose your action according to a certain probability distribution.

Suppose now Sophie chooses A with probability *p* and B with *1-p*

Boris' expected payoff:

$$3 \cdot p + 0 \cdot (1 - p) \qquad (-1) \cdot p + 4 \cdot (1 - p)$$

If Boris plays A *If Boris plays B*

Boris is indifferent between playing A and B if Sophie's probability *p* satisfies:

$$3 \cdot p + 0 \cdot (1 - p) = (-1) \cdot p + 4 \cdot (1 - p)$$
$$p^* = \frac{1}{2}$$

Suppose now, Boris chooses A with probability *q* and B with *1-q*

Sophie is indifferent between A and B if Boris' probability *q* satisfies:

$$-3 \cdot q + 1 \cdot (1 - q) = 0 \cdot q + (-4) \cdot (1 - q)$$
$$q^* = \frac{5}{8}$$

Mixed strategy profile:
$$\left(q^* = \frac{5}{8}, p^* = \frac{1}{2}\right)$$
is equilibrium of the game.

Observe:
$\pi_S(q^*, p^*) = -1.5$
$\pi_B(q^*, p^*) = 1.5$

2.3.3.1.1.2 Non constant sum game
Example 13: ‘Maroni’ game (“Prisoners Dilemma”)

a) Situation in words

- There are 2 maroni sellers, competing for 100 clients
- Sellers set one of the following prices simultaneously (e.g. without knowing the decision of the counterpart):
  - High price: 6 SFr per packet
  - Low price: 5 SFr per packet
- If both sellers choose the same price, each buyer will make his decision with 50% probability
- Whenever the chosen prices are different, the one who sets the smallest price will get all 100 clients.
- What will be the outcome of the competition?

b) Game

i. Players: 2 (seller 1 and seller 2)
ii. Actions of players: $a_i \in \{\text{high}, \text{low}\}, \text{for } i \in \{\text{seller } 1, \text{seller } 2\}$
iii. Payoffs: See monetary payoffs presented in the *normal form* below

		Seller 2	
		High	Low
Seller 1	High	300, 300	0, 500
	Low	500, 0	250, 250

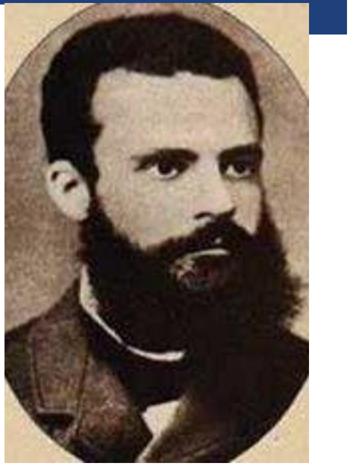
c1) Solution: Nash Equilibrium

		Seller 2	
		High	Low
Seller 1	High	300, 300	0, 500
	Low	500, 0	250, 250

Nash
Equilibrium

c2) Optimality considerations

We call an outcome Pareto Optimal (named after Vilfredo Pareto) if there is no other outcome that increases payoff to one player without decreasing payoff to another player.



(1848 – 1923)

Observe:
If both sellers choose “high” prices, both sellers would be better off; we call this strategy combination a Pareto improvement compared to the strategy combination “low/low” (Nash Equilibrium)

		Seller 2	
		High	Low
Seller 1	High	300, 300	0, 500
	Low	500, 0	250, 250

Nash Equilibrium in this case Pareto-inferior

d2) Comments

- Game illustrates conflict in “economics”
- For both sellers strategy “low”-pricing **dominates** strategy “high”-pricing
- Nash Equilibrium is not always Pareto efficient outcome
- This is a typical illustration of the Prisoners’ Dilemma

Classical Prisoners’ Dilemma

(Name proposed by: Albert Tucker in 1950)

Situation in words

Two partners in a crime are imprisoned in two separate prison cells. They are both offered the same deal:

- If one of them confesses – and the other doesn’t – he will be set free (**0 years** in prison), the other will serve **3 years** in prison
- If both confess, each of them serves **2 years** in prison
- If both don’t confess, both of them will only serve **1 year** in prison

Game in the normal form

		P2	
		Don’t confess	Confess
P1	Don’t confess	-1 , -1 Pareto** (-improvement)	-3 , 0
	Confess	0 , -3	-2 , -2 Nash*

* None of the players has an incentive to change his strategy ***unilaterally***

** None of the players can improve his situation *without* hurting the other