Example 7: Secretary problem

There exist different versions of the problem. Here is an easy version:

- There is one secretarial position available
- The number of n of applicants is known
- The applicants are interviewed sequentially in random order
- Each time you see an applicant you have to decide if you take him/her or not. The rejection of an applicant cannot be revoked later on. If you decide on taking one applicant you can not see any other applicant thereafter
- You compare each candidate with the ones you have already seen

$\rightarrow$ **How should you proceed in order to maximize the probability of selecting the best candidate?**

The surprising answer is:

You should let $\frac{n}{e}$ applicants go by and then select the first one whose value exceeds those of all the (earlier) rejected ones

The probability of getting the best one with this strategy is $\frac{1}{e}$, i.e. 37 percent

Formulation of the problem

- n applicants (n is known) are to be presented in a randomized sequential order
- When n applicant is in front of you, you must either choose him, and the game is over, or you reject him, and you can't go back
- The probability to choose the best is at each stage is $\frac{1}{n}$

- Suppose you reject the first candidate, then there is always a $\frac{1}{n}$ chance that this one was the best and you failed to select the best
- The second now presents himself and you compare him to the first. If he is not as good as the first you will obviously not select him; but even if the second is better than the first, you might still want to pass up on that person because you think the best is among the remaining ones
- The best of the rejected ones serves you as the standard for judging the ones to come. So the best of the first x (x: rejected ones) will represent a standard against which to judge the remainder

How many should you let go before making your choice? What chance do you have to pick the best?

Answer:

Idea: Divide the universal sequence of all candidates into two groups; the first group - called the rejected ones or the “standard-setting group” - consisting of a proportion t of candidates, is used only to identify the best in that group; we then sequentially observe the candidates in the second group - called the selection group - and choose the first who beats the best in the standard-setting group; this is our **Strategy S**

$P(t)$: Probability of finding the best candidate when the standard-setting group comprises a proportion of t candidates

Strategy S will result in the choice of the best, provided that:

- the second best falls in the first group (proportion $t, \ 0 \leq t \leq 1$) and the first best in the second (proportion $(1-t)$). This has the probability $p = t(1-t)$.

  Or

- the third best falls in the first group (probability $\mathrm{p} = t$) and the first in the second group ($\mathrm{p} = (1-t)$) and the second best also in the second group ($p = (1-t)$, and the first comes before the second ($\mathrm{p} = ½$). This has probability of $p = t\frac{(1-t)^2}{2}$.

or

- the fourth best falls in the first segment and the first, second, and third best in the second segment and the first comes before the second or the third:

$$p = \frac{t(1-t)^3}{3}$$

or

- and so on ….

Hence:

$$P(t) = t(1-t) + \frac{t(1-t)^2}{2} + \frac{t(1-t)^3}{3} + \ ... \quad \text{(assumption: n} \to \infty\text{)}$$

$$= t\underbrace{\left[(1-t) + \frac{(1-t)^2}{2} + \frac{(1-t)^3}{3} + \ ...\right]}_{-\ln(t)}$$

see side note hereafter

Therefore $P(t) = -t\ln(t)$

To find the optimal proportion for the standard-setting group, we differentiate $P(t)$ and set $P'(t)$ to 0.

$$\frac{dP(t)}{dt} = P'(t) = -\ln(t) - \frac{t}{t} = -\ln(t) - 1$$

$$P'(t) = 0 = -\ln(t) - 1 \ \Rightarrow \ \ln(t) = -1$$

$$\Rightarrow t = e^{-1}; \qquad t = \frac{1}{e} \leftarrow Optimum$$

$$P(e^{-1}) = -e^{-1}\underbrace{\ln(e^{-1})}_{-1} = e^{-1}; \quad P(e^{-1}) = \frac{1}{e} \ \leftarrow \ Optimum$$

Side note

Proof that [...] is equal to –ln(t) :

Let $S(x) = 1 + x + x^2 + x^3 + \ldots$ for $0 < x < 1$

Then multiplying both sides by x we get

$xS(x) = x + x^2 + x^3 + \ldots$

and subtracting the second from the first we have

$S(x) - xS(x) = 1;$

meaning $(1 - x)S(x) = 1,$

or $S(x) = (1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

Taking the integral of both sides we get

$$-\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} \ldots$$

Letting $x = 1 - t$, we establish: [...] = $-\ln(t)$

q.e.d.

Secretary Problem: Concrete Case, n small

- You want to select the best candidate, only the best
- Random order, each order being equally likely
- You can rank all applicants from worst to best; the decision to accept or reject is based only on the relative rank
- Let n=4
- Best candidate: rank 4, least best 1
- Ranks are ordinal numbers
- n! permutations possible, i.e. 24
- Strategy: Let t ($0 \leq t < 4$) applicants pass by, then select the first one that is better that the rejected one(s)

1 2 3 4	2 1 3 4	3 1 2 4	4 1 2 3
1 2 4 3	2 1 4 3	3 1 4 2	4 1 3 2
1 3 2 4	2 3 1 4	3 2 1 4	4 2 1 3
1 3 4 2	2 3 4 1	3 2 4 1	4 2 3 1
1 4 2 3	2 4 1 3	3 4 1 2	4 3 1 2
1 4 3 2	2 4 3 1	3 4 2 1	4 3 2 1

Strategy sucessful: selecting the best one, i.e. applicant with rank 4

1.Pass 0 by, then take the first one -> 6/24 p=0,25

2.Pass 1 by, then take the first one that is better -> 11/24 p=0,4583

3.Pass 2 by, then take the first one that is better -> 10/24 p=0,416

4.Pass 3 by, then take the first one that is better [4] -> 6/24 p=0,25

- Best strategy -> «pass 1 by, then take the first one that is better». $\Rightarrow t = 1$
- Check with the case n → ∞ where $\boldsymbol{t}$ optimal is $\boxed{\boldsymbol{t} = \dfrac{\boldsymbol{n}}{\boldsymbol{e}}}$

same result

- $\boldsymbol{n} = 4 : \boldsymbol{t} \approx 1{,}471 \Rightarrow t = 1$ *(rounded)*

Example 8:
Casino (Game against nature)

- You have four actions: A, B, C, and D
- Choices of nature: W, X, Y, and Z

Payoff table:

		Nature			
		W	X	Y	Z
You	A	*2*	*2*	*0*	*1*
	B	*1*	*1*	*1*	*1*
	C	*0*	*4*	*0*	*0*
	D	*1*	*3*	*0*	*0*

Philip D. Straffin (1993): "Game Theory and Strategy", Mathematical

Method 1: If prior probabilities are known, choose action to maximize your expected payoff

Suppose:

$$\mathrm{p}(W) = 20\%, \mathrm{p}(X) = 40\%, \mathrm{p}(Y) = 30\%, p(Z) = 10\%$$

Expected utilities:

$\pi(A) = 0.2 \cdot 2 + 0.4 \cdot 2 + 0.3 \cdot 0 + 0.1 \cdot 1 = 1.3$

$\pi(B) = 0.2 \cdot 1 + 0.4 \cdot 1 + 0.3 \cdot 1 + 0.1 \cdot 1 = 1.0$

$\pi(C) = 0.2 \cdot 0 + 0.4 \cdot 4 + 0.3 \cdot 0 + 0.1 \cdot 0 = 1.6$ ← take action C

$\pi(D) = 0.2 \cdot 1 + 0.4 \cdot 3 + 0.3 \cdot 0 + 0.1 \cdot 0 = 1.4$

Method 2: (Laplace 1749 – 1827) If you do not have any information on the probability distribution, then assume choices of nature are equally possible. Then choose the action (in our table) which gives you the highest average value.

Equal probability:

$$\mathrm{p}(W) = 25\%, \mathrm{p}(X) = 25\%, \mathrm{p}(Y) = 25\%, p(Z) = 25\%$$

Average utilities:

$\pi(A) = 0.25 \cdot 2 + 0.25 \cdot 2 + 0.25 \cdot 0 + 0.25 \cdot 1 = 1.25$ ← take action A

$\pi(B) = 0.25 \cdot 1 + 0.25 \cdot 1 + 0.25 \cdot 1 + 0.25 \cdot 1 = 1$

$\pi(C) = 0.25 \cdot 0 + 0.25 \cdot 4 + 0.25 \cdot 0 + 0.25 \cdot 0 = 1$

$\pi(D) = 0.25 \cdot 1 + 0.25 \cdot 3 + 0.25 \cdot 0 + 0.25 \cdot 0 = 1$

Method 3: (maxmin strategy, Wald 1902 – 1950) Write down the minimum entry in each row. Choose the row with the largest minimum. →"Pessimist's strategy"

Nature

You		W	X	Y	Z	Min payoff
	A	*2*	*2*	*0*	*1*	*0*
	B	*1*	*1*	*1*	*1*	*1*
	C	*0*	*4*	*0*	*0*	*0*
	D	*1*	*3*	*0*	*0*	*0*

Maximum of the minimum

Take action B

Method 4: (maxmax strategy) Write down the maximum entry in each row. Choose the row with the largest maximum. →"Optimist's strategy"

Nature

You		W	X	Y	Z	Max payoff
	A	*2*	*2*	*0*	*1*	*2*
	B	*1*	*1*	*1*	*1*	*1*
	C	*0*	*4*	*0*	*0*	*4*
	D	*1*	*3*	*0*	*0*	*3*

Take action C

Maximum of the maximum

Method 5: (Leonid Hurwicz 1917 – 2008) Choose a coefficient of optimism, $\theta \in (0,1)$. For each row, compute:

$$\theta(\text{row maximum}) + (1-\theta)(\text{row minimum})$$

Choose the action for which weighted average is the highest. (= mixture between maxmax $[\theta = 1]$ and maxmin $[\theta = 0]$)

Suppose $\theta = 0.75$

Nature

You		W	X	Y	Z	Weighted sum
	A	*2*	*2*	*0*	*1*	$0.75 \cdot 2 + 0.25 \cdot 0 = 1.5$
	B	*1*	*1*	*1*	*1*	$0.75 \cdot 1 + 0.25 \cdot 1 = 1.0$
	C	*0*	*4*	*0*	*0*	$0.75 \cdot 4 + 0.25 \cdot 0 = 3.0$
	D	*1*	*3*	*0*	*0*	$0.75 \cdot 3 + 0.25 \cdot 0 = 2.25$

Take action C

<u>Method 6</u>: (Leonard Jimmie Savage 1917 –1971) Compute the regret matrix. Write down the largest entry in each row. Choose the row for which this largest entry is smallest. (min maxregret)

The regret matrix is computed the following way: take the original matrix (table) with payoffs. Find what is the maximum in each column. The result is what would be the best for you if you knew the decision of the nature. Compute the difference between this largest entry in the column and corresponding entry in the original matrix.

Example of regret table:

You	Nature: W	X	Y	Z	Row maximum
A	*0*	*2*	*1*	*0*	*2*
B	*1*	*3*	*0*	*0*	*3*
C	*2*	*0*	*1*	*1*	*2*
D	*1*	*1*	*1*	*1*	*1* ← Minimum of the maximum regret

Take action D

Summary of methods

METHOD	ACTION
Method 1: Expected utility optimization	C
Method 2: Laplace	A
Method 3: Maxmin	B
Method 4: Maxmax	C
Method 5: Hurwicz optimism coefficient	C
Method 6: Savage minimize regret	D