Modelagem no Domínio da Frequência

Fundamentos de Controle

Revisão da Transformada de Laplace

$$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

$$\boxed{\mathscr{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds = f(t)u(t)}$$

Revisão da Transformada de Laplace

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s \to \infty} sF(s)$	Initial value theorem[2]

Exemplo 2.1

Transformada de Laplace de uma Função do Tempo

PROBLEMA: Obter a transformada de Laplace de $f(t) = Ae^{-at}u(t)$.

$$F(s) = \int_0^\infty f(t)e^{-st}\ dt = \int_0^\infty Ae^{-at}e^{-st}\ dt = A\int_0^\infty e^{-(s+a)t}\ dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{A}{s+a}$$

Exemplo 2.2

Transformada Inversa de Laplace

PROBLEMA: Obter a transformada inversa de Laplace de $F_1(s) = 1/(s+3)^2$.

$$F(s) = 1/s^2 \qquad tu(t)$$

$$F(s+a) = 1/(s+a)^2 \qquad e^{-at}tu(t)$$

$$f_1(t) = e^{-3t}tu(t)$$

1. Deduza a transformada de Laplace para as seguintes funções do tempo: [Seção: 2.2]

a. $u(t)$

b. $tu(t)$

c. sen $\omega t\ u(t)$

d. cos $\omega t\ u(t)$

Expansão em Frações Parciais

$$F_1(s) = \frac{s^3 + 2s^2 + 6s + 7}{s^2 + s + 5}$$

$$F_1(s) = s + 1 + \frac{2}{s^2 + s + 5}$$

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathscr{L}^{-1}\left[\frac{2}{s^2 + s + 5}\right]$$

Caso1. As Raízes do Denominador de *F*(*s*) São Reais e Distintas

$$F(s) = \frac{2}{(s+1)(s+2)}$$

$$F(s) = \frac{2}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

$$\frac{2}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)}$$

$$f(t) = (2e^{-t} - 2e^{-2t})u(t)$$

Caso 2. As Raízes do Denominador de $F(s)$ São Reais e Repetidas

$$F(s) = \frac{2}{(s+1)(s+2)^2}$$

$$F(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)}$$

$$\frac{2}{s+1} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3$$

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3$$

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

Caso 3. As Raízes no Denominador de *F*(*s*) São Complexas ou Imaginárias

$$F(s) = \frac{3}{s(s^2 + 2s + 5)}$$

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2s + 5}$$

$$F(s) = \frac{3}{s(s^2 + 2s + 5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s + 2}{s^2 + 2s + 5}$$

6.	$\sin \omega t u(t)$	$\dfrac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\dfrac{s}{s^2 + \omega^2}$

4.	$\mathscr{L}[e^{-at} f(t)]$	$= F(s + a)$	Frequency shift theorem

$$\mathscr{L}[Ae^{-at} \cos \omega t] = \frac{A(s + a)}{(s + a)^2 + \omega^2} \qquad \mathscr{L}[Be^{-at} \sin \omega t] = \frac{B\omega}{(s + a)^2 + \omega^2}$$

$$\mathscr{L}[Ae^{-at} \cos \omega t + Be^{-at} \sin \omega t] = \frac{A(s + a) + B\omega}{(s + a)^2 + \omega^2}$$

$$\mathscr{L}[Ae^{-at}\cos\omega t + Be^{-at}\sin\omega t] = \frac{A(s+a)+B\omega}{(s+a)^2+\omega^2}$$

$$F(s) = \frac{3}{s(s^2+2s+5)} = \frac{3/5}{s} - \frac{3}{5}\frac{s+2}{s^2+2s+5}$$

$$F(s) = \frac{3/5}{s} - \frac{3}{5}\frac{(s+1)+(1/2)(2)}{(s+1)^2+2^2}$$

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Exercício 2.2

PROBLEMA: Obtenha a transformada de Laplace inversa de $F(s) = 10/[s(s+2)(s+3)^2]$.

RESPOSTA: $f(t) = \frac{5}{9} - 5e^{-2t} + \frac{10}{3}te^{-3t} + \frac{40}{9}e^{-3t}$

A solução completa está disponível no GEN-IO, Ambiente de Aprendizagem do Grupo GEN.

Exemplo 2.3

Solução Via Transformada de Laplace de uma Equação Diferencial

PROBLEMA: Dada a equação diferencial a seguir, obter a solução para $y(t)$ considerando que todas as condições iniciais são iguais a zero. Utilize a transformada de Laplace.

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 32y = 32u(t) \tag{2.14}$$

$$s^2Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)}$$

$$Y(s) = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{(s+4)} + \frac{K_3}{(s+8)}$$

$$K_1 = \left.\frac{32}{(s+4)(s+8)}\right|_{s \to 0} = 1$$

$$K_2 = \left.\frac{32}{s(s+8)}\right|_{s \to -4} = -2$$

$$K_3 = \left.\frac{32}{s(s+4)}\right|_{s \to -8} = 1$$

$$Y(s) = \frac{1}{s} - \frac{2}{(s+4)} + \frac{1}{(s+8)}$$

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$