María González Cabrera (in collaboration with Germán Sánchez)

A simple wealth model

I. Solving the ABHI Model

1. The recursive formulation.

The maximization problem of the households, given prices, is given by the following program:

$$max \sum_{t=0} \sum_{yt} \square^{t} \pi_t(y^t)\, U(c_t(y^t))$$

s.t.

$$a_{t+1}(y^t) \geqslant -\, y_{min} \sum_{s=0}^{T-(t+1)} (1+r)^{-s}$$

$$c_t(y^t) + a_{t+1}(y^t) = y_t(y^t) + (1+r)\, a_t(y^{t-1})$$

Substituting consumption from the budget constraint into the utility function, we end up to a maximization problem over $a_{t+1}(y^t)$. The value function will be given by:

$$V(a, y) = max\ U(y + (1+r)a - a') + \square \sum_{y'} \pi_{y'|y} v(a', y')$$

s.t.

$$a' \geqslant -\, y_{min} \sum_{s=0}^{T-(t+1)} (1+r)^{-s}$$

Solving for a’, the resulting first order condition is given by:

$$U'(c) = \square \sum_{y'} \pi_{y'|y} v_1(a', y')$$

By the Envelope Theorem we have that:

$$v_1(a, y) = U'(c)*(1+r) \rightarrow v_1(a', y') = U'(c')*(1+r)$$

So finally, the Euler Equation written in a general way results as follows:

$$U'(y + (1+r)a - a') = \square \sum_{y'} \pi_{y'|y} U'(y' + (1+r)a' - a'')(1+r)$$

Quadratic utility case: $cbar - (y + (1+r)a - a') = \square \sum_{y'} \pi_{y'|y}[cbar - (y' + (1+r)a' - a'')](1+r)$

CRRA utility case: $(y + (1+r)a - a')^{-\sigma} = \square \sum_{y'} \pi_{y'|y}(y' + (1+r)a' - a'')^{-\sigma}(1+r)$

Now, we will discuss points 2 and 3 for each of these utilities. First, we will consider the Quadratic case and finally the CRRA utility.

Quadratic utility case

2. The infinitely-lived households economy.

For doing this part of the exercise, we used the value function iteration method. First of all, we parametrized the model as follows:

Parameter	Value
ρ	0.06
$\square$	$1/(1+\rho)$
γ	0.5
σ_y	0.2
cbar	100

Given that, we can generate the transition matrix π for the shocks y1 and y2, which are given by the values $(1-\sigma_y)$ and $(1+\sigma_y)$ respectively.

With that, the first that we have done is to generate the matrix Y×A×A', in order to have all the possible combinations of assets from today and tomorrow, given that the shock was y_i (with i = 1, 2). Now, we can generate the matrix M, the guess for W and the Value Function from the first iteration as follows:

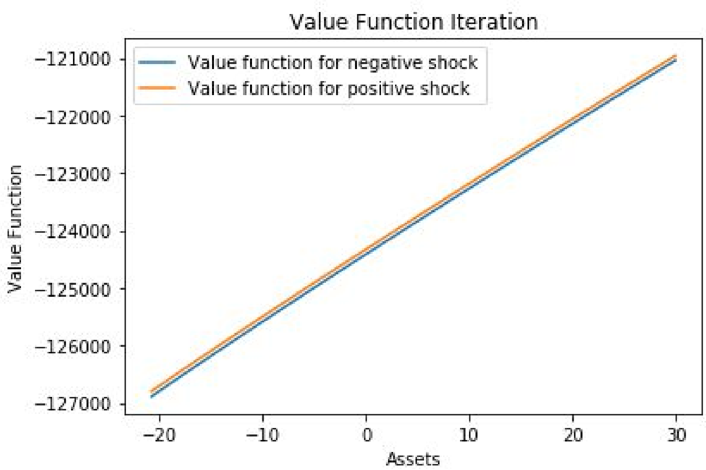
```
c = y+(1+r)*ai-aj

@vectorize

def M(c):

    return -0.5*(c-cbar)**2

M = M(c)
M = np.reshape(M, (1, 12800))
M = np.reshape(M, (160, 80))

# Initial guess for the value function is a vector of zeros:

Vs = np.zeros(160)

# Compute the matrix W:

def W1(A):

    return pi[0, 0]*(-0.5*(Y[0] + (1+r)*A - A - cbar)**2)/(1-beta) + pi[0, 1]*(-0.5*(Y[1] + (1+r)*A
- A - cbar)**2)/(1-beta)
```

```
def W2(A):

    return pi[1, 0]*(-0.5*(Y[0] + (1+r)*A - A - cbar)**2)/(1-beta) + pi[1, 1]*(-0.5*(Y[1] + (1+r)*A
- A - cbar)**2)/(1-beta)


W1 = W1(A)
W1 = np.reshape(W1, (80,1))
W1 = np.tile(W1, 80)
W1 = np.transpose(W1)

W2 = W2(A)
W2 = np.reshape(W2, (80,1))
W2 = np.tile(W2, 80)
W2 = np.transpose(W2)

W = [W1, W2]
W = np.reshape(W, (160,80))

# Compute the matrix X:

X = M + beta*W

Vs1 = np.amax(X, axis = 1)

diffVs = Vs - Vs1
```

After doing 7999 iterations, we obtain convergence of the value function, so now we can create the vector of policy function, with which we can obtain the optimal decision for assets and therefore, the optimal decision for consumption, given the shock y_i (with i = 1, 2).

We need to take into account that for all the Problem Set we set the minimum level of assets to the natural borrowing limit, which give the agents the possibility of borrow but they'll always can return it, even though they'll be in the worst possible scenario.

These solutions give us the following graphs:



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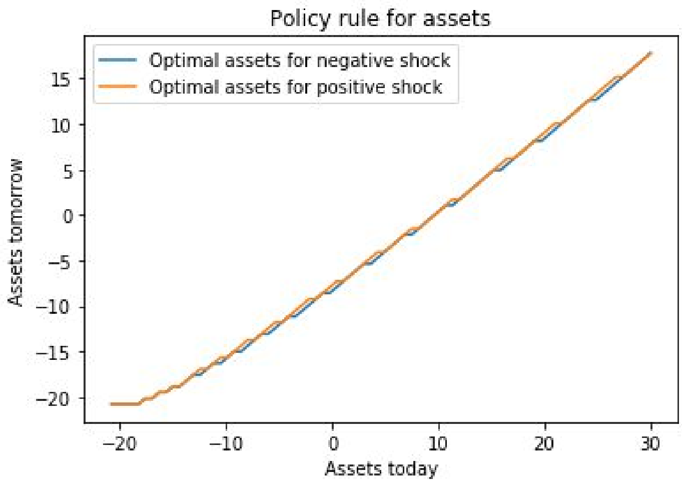
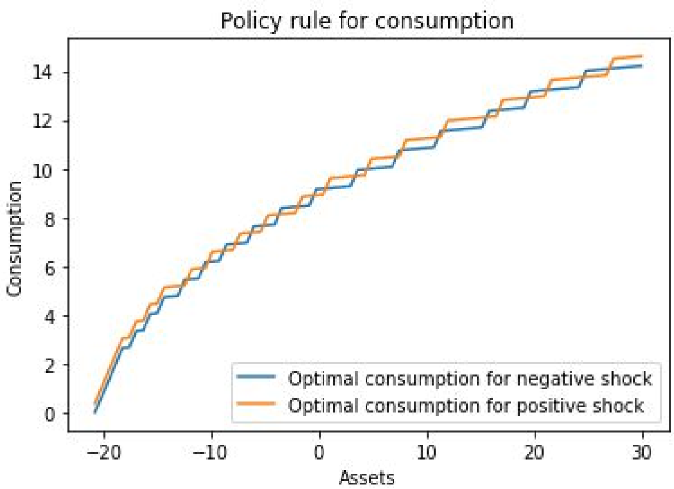




3. The life-cycle economy.

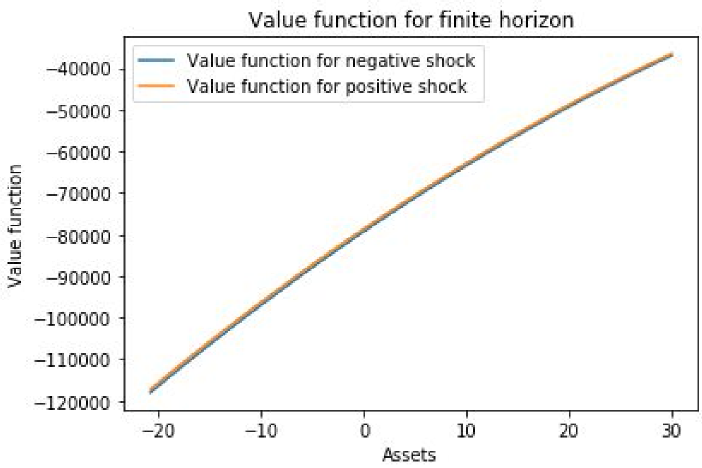
For solving the finite problem of the households what we need to do is to solve the program with backwards induction. Imposing a T limit of 45, we set the value function at T+1 to zero, so we know that the value function at the final period of the economy is the utility function (given by the matrix M).

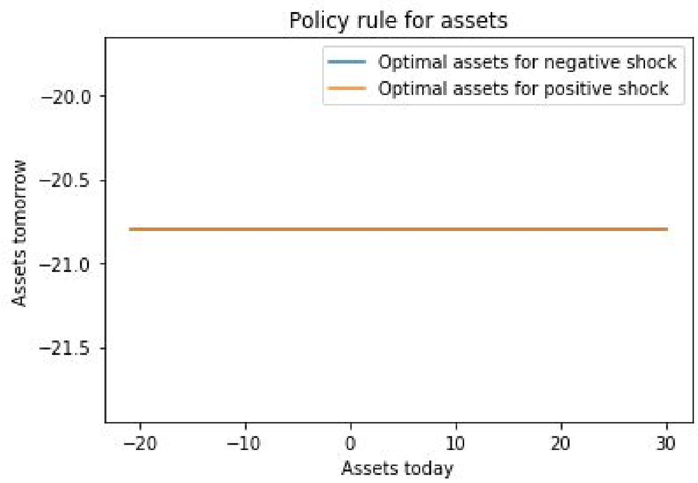
So, knowing the final value of the value function, we can iterate backwards as is shown below:

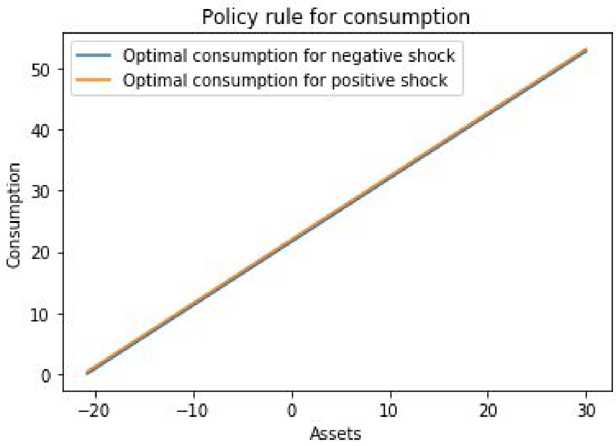
```
W = np.zeros((160, 80))

count = 0

while count < 45:

    W = np.amax((M + beta*W), axis = 1)
    W = np.reshape(W,(160, 1))
    W = W*np.ones((160, 80))

    count += 1

X = M + beta*W
```

```
g = np.argmax(X, axis = 1)

aopt_y1 = A[g[0:80]]     # optimal decision of assets given y1
aopt_y2 = A[g[80:]]      # optimal decision of assets given y2

c_y1 = Y[0]*np.ones(80) + (1+r)*A - aopt_y1

c_y2 = Y[1]*np.ones(80) + (1+r)*A - aopt_y2

for i in range(0, 80):

    if c_y1[i] < 0:

        c_y1[i] = 0

    if c_y2[i] < 0:

        c_y2[i] = 0
```

And the results from this backward iterations are as follows:







Take into account that this graphs are only defined for the first period. That is, the value function that we have shown before is showing the present value for the first 45 periods of this economy.

This is for simplicity of the graphs, but we didn't forget that each period has a value function and a policy rule for consumption and assets associated.

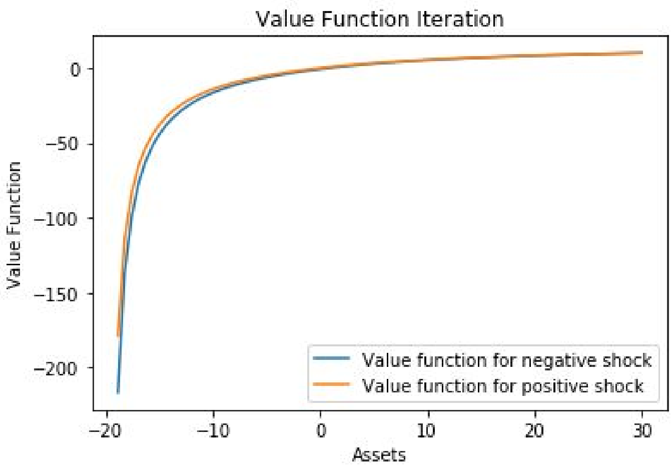
CRRA utility case

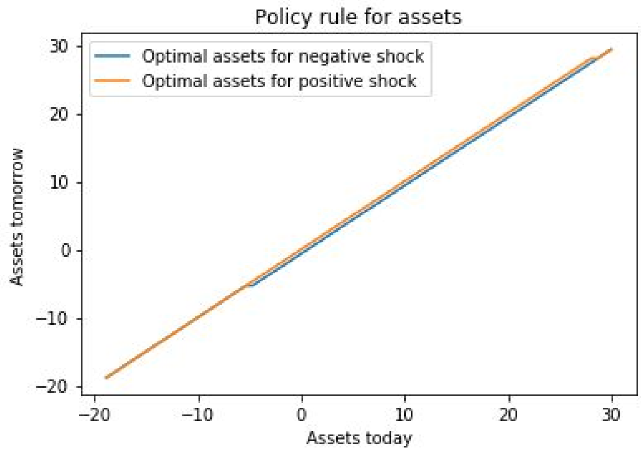
2. The infinitely-lived households economy.

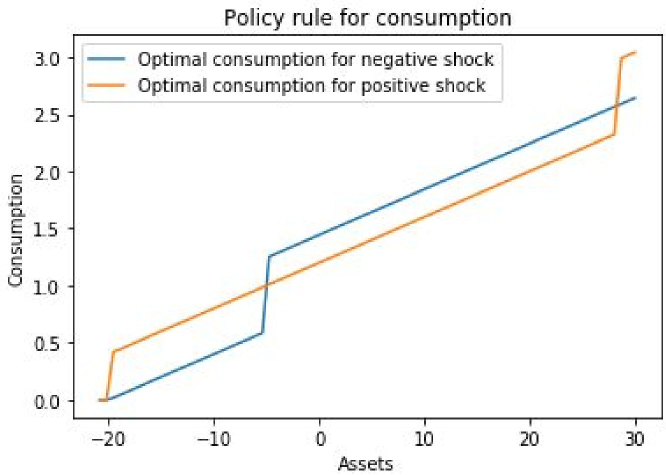
First, we need to set the new values for the utility parameter of this utility function:

Parameter	Value
σ	2

And then, following the same steps as in the previous part for the quadratic case, we obtain the following results:

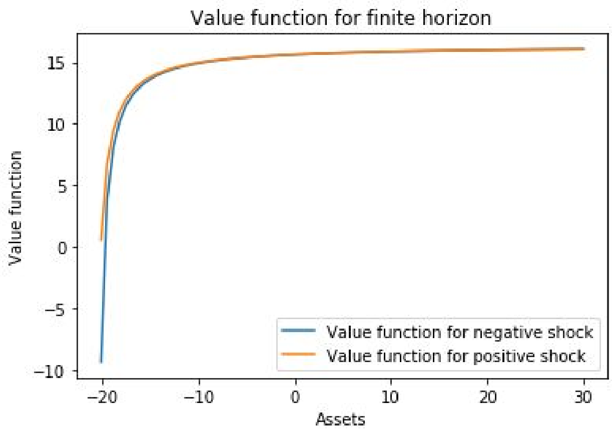


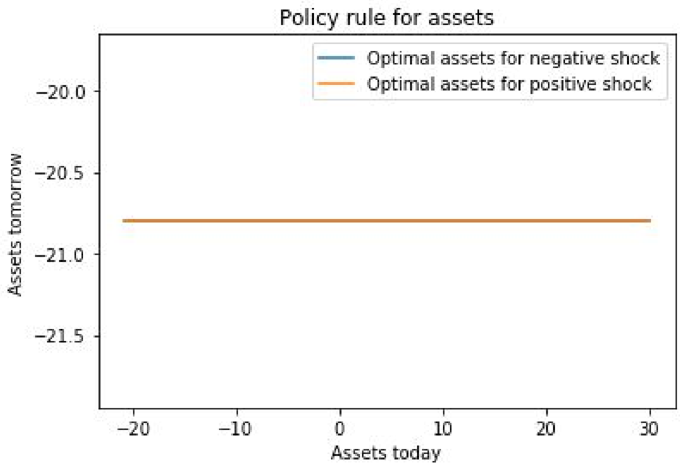


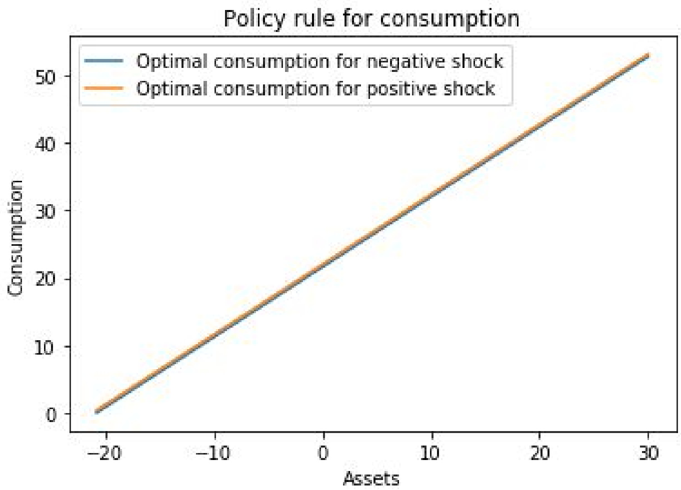


3. The life-cycle economy.

Our results for the finitely lived agents are shown below:







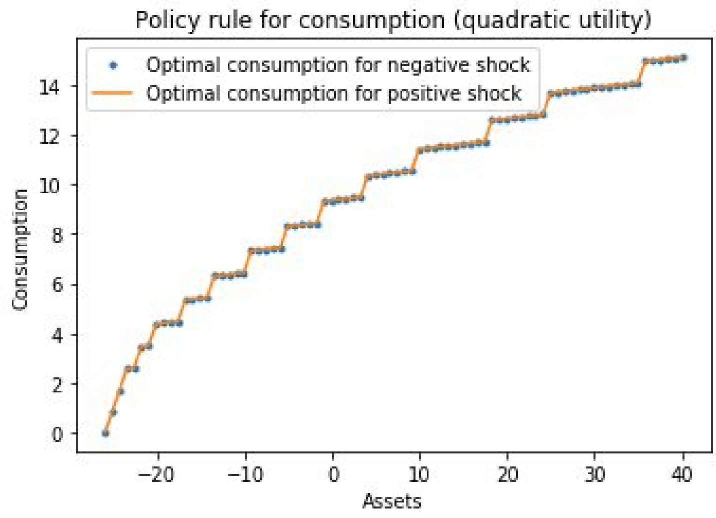
4. Partial equilibrium.

Quadratic utility case

a. With certainty

Suppose that there is no uncertainty in the economy, which means that σ_y is zero, and the shock y is always equals to one. Moreover, we are assuming ɣ equals to zero.

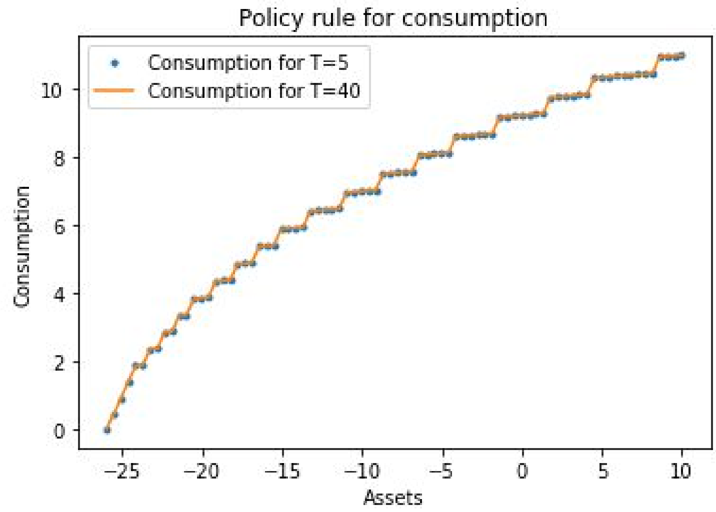
Proceeding as in the previous part, we end up with the following result for the policy rule of consumption. What we can see in the following graph, is that there are no more differences in the policy rule for consumption (because there are no differences in the value functions defined "for each of the shocks").



As we have said in the previous part of this problem set, we know that each period has associated a value function and a policy rule for assets and consumption.

Taking this into account, and solving for a finitely lived agents economy of 45 periods, we proceed as in the previous section and store each of the value functions of the backwards induction loop. Notice that this store thing can only be done for the finite economy, because solving by backwards induction implies that the value function for which we are solving is already the optimal one.

Given that, we can see in the following graph that the optimal consumption for periods 5 and 40 remains the same.

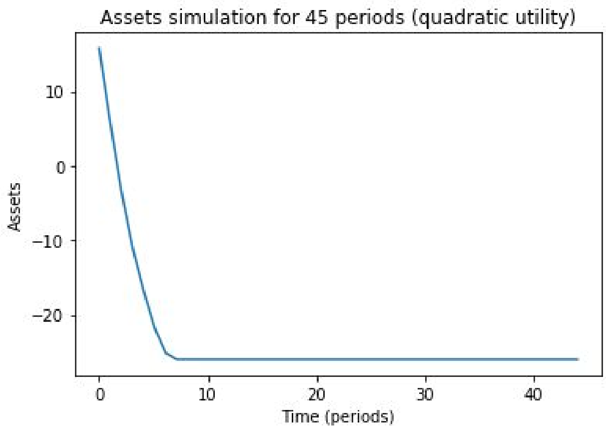


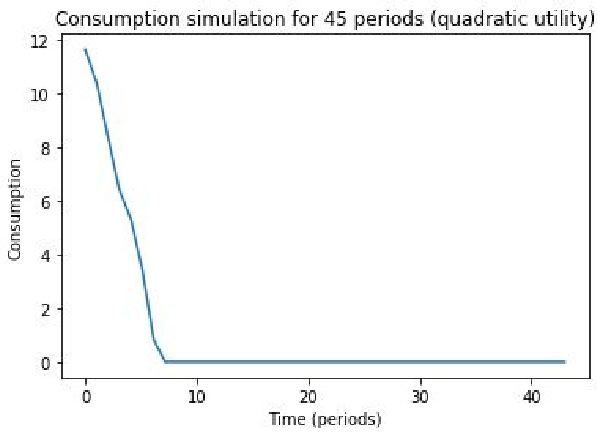
After doing that, we are asked to do a simulation of 45 periods of this economy. The code we have used is defined below:

```
# Simulate the shocks of an economy for T = 45

y = np.zeros([1, 80])
```

```
for i in range(0, 80):
    y[0, i] = 1  # In all cases, since gamma = 0, our y is going to be 1
# Simulation and plot for assets:
simulation = np.zeros(45)
aopt_y1 = A[g[0:80]]     # optimal decision of assets
g_y1 = g[0:80]
simulation[0]  = g_y1[79] # our initial guess of assets  (a0)
for i in range(1, 45):
        simulation[i] = g_y1[int(simulation[i-1])]
for i in range(0, 44):
        simulation[i] = aopt_y1[int(simulation[i])]

t = np.linspace(0, 44, 44)
```

Then, given that our initial assets is 15.7722, we obtain the following simulation for assets and consumption:



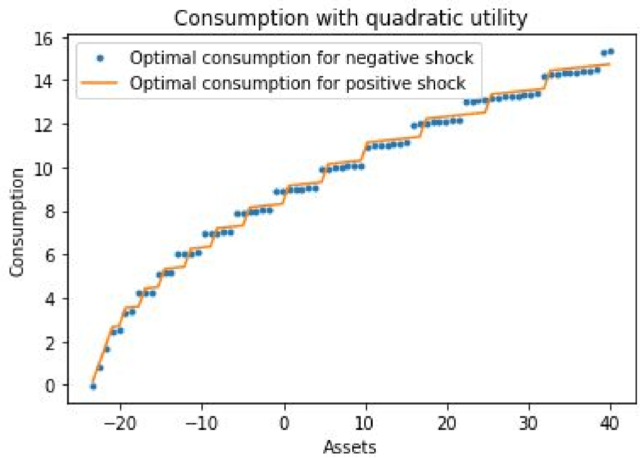


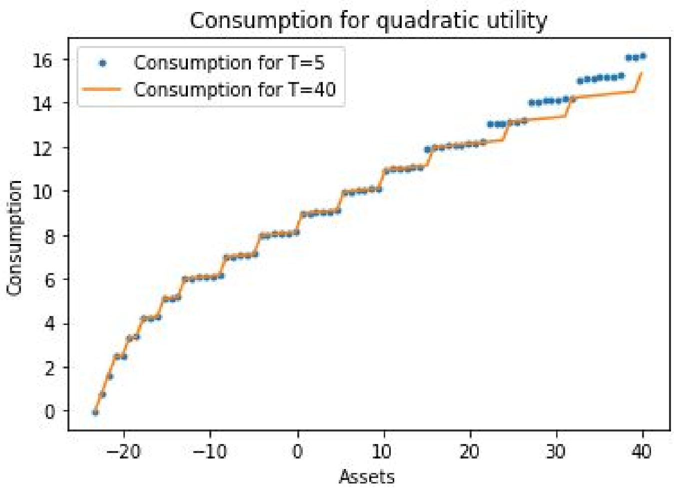
We can see that this simulation is showing a case of indebtedness, where the agent starts with a large amount of assets, that consume in the first periods and then, he starts borrowing and consuming zero in order to return his debt.

This result could change if we set the grid of assets such that agents can't borrow (so, set the minimum point of assets grid to zero), or starting with another starting value for assets.

b. With uncertainty

Introducing uncertainty in the model (σ_y is 0.1) the results that we have obtained are the following:





We can see that once we take into account different shocks in the income, the value functions, and therefore the policy functions for assets and consumption are not the same anymore, even though we see at the infinitely-lived households or the finite ones, as we can see in the previous graphs.

If we do a simulation for the first 45 periods of the economy, we end up to the following results:

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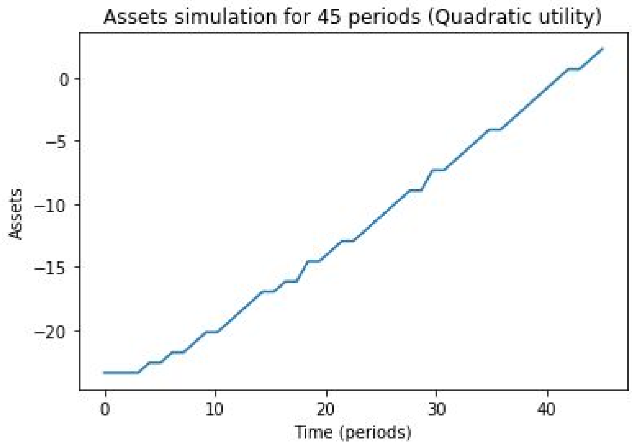
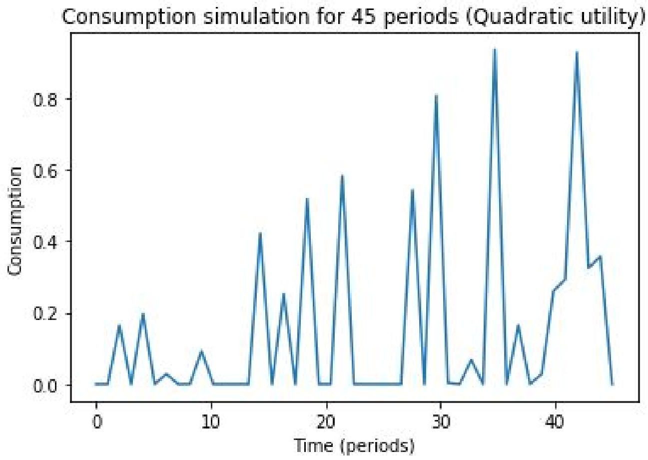


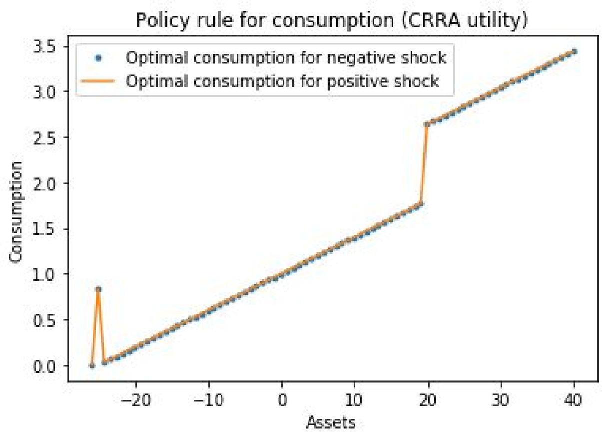


We can see that introducing uncertainty in the model and taking into account the borrowing limit, agents save and their consumption is not zero for a long time as in the certainty case.

CRRA utility case

a. With certainty

The results for this utility function are shown below:



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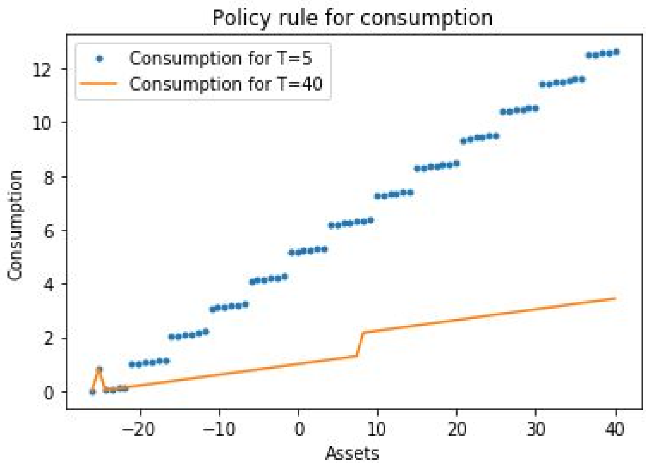

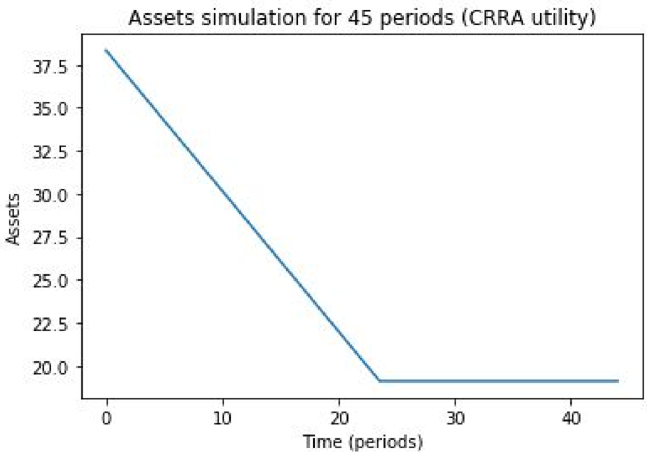

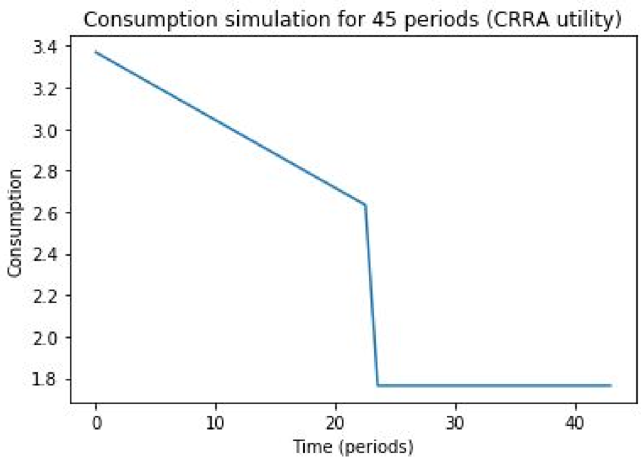

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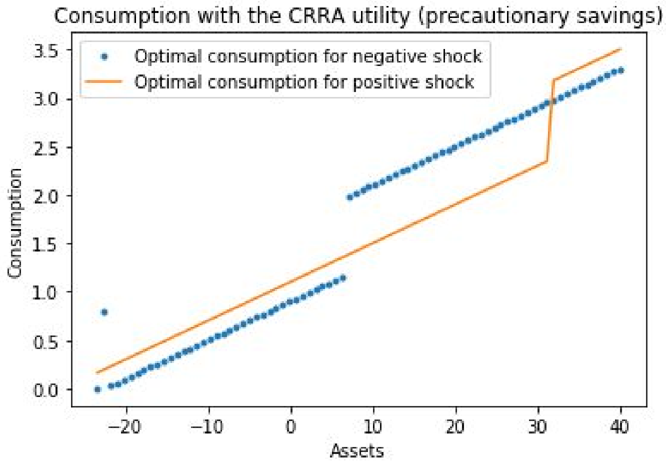
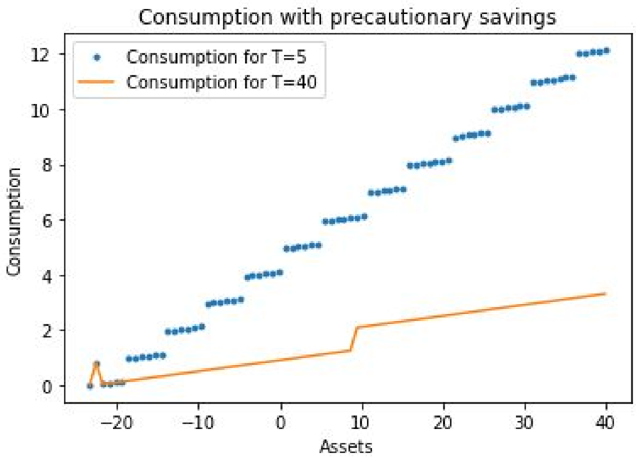
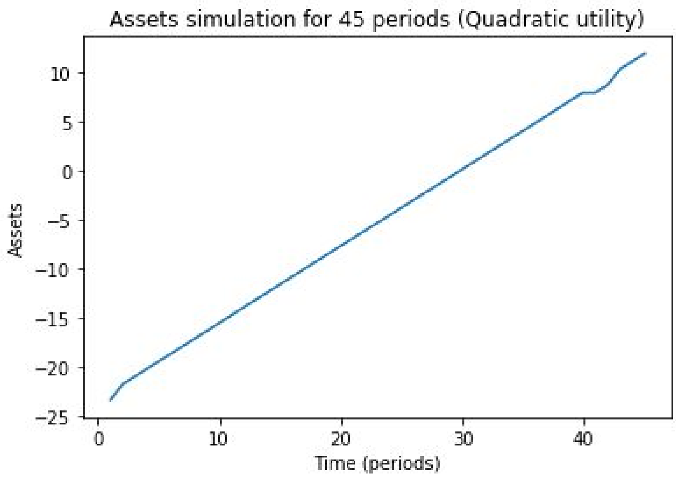
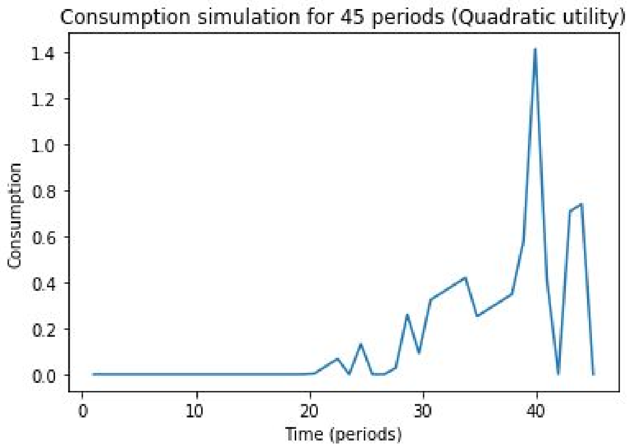
b. With uncertainty

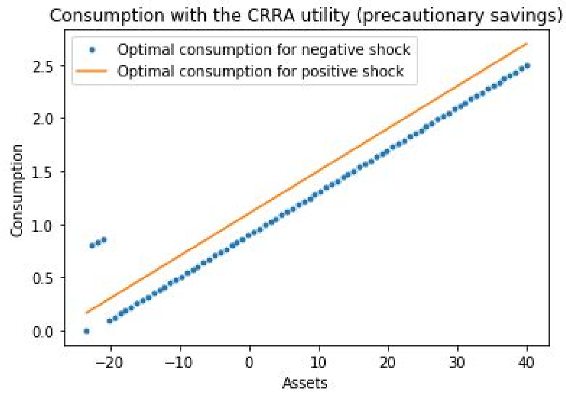
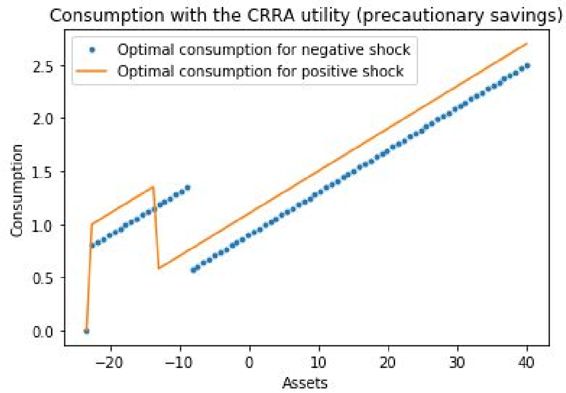
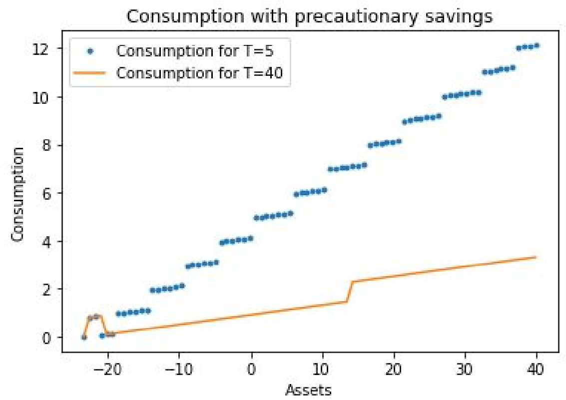
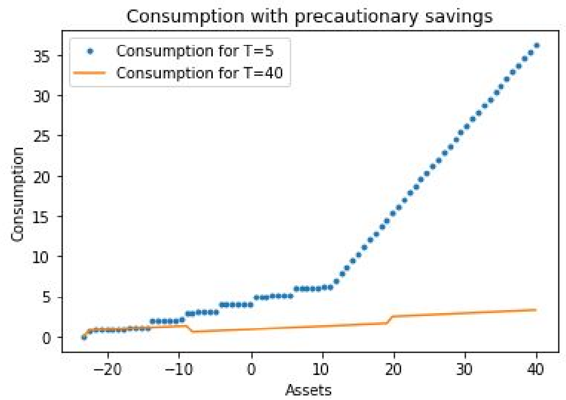


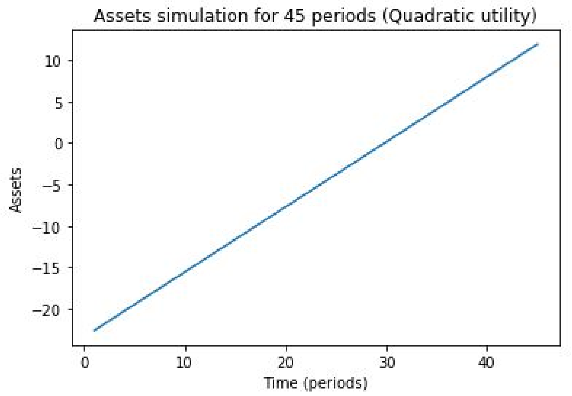


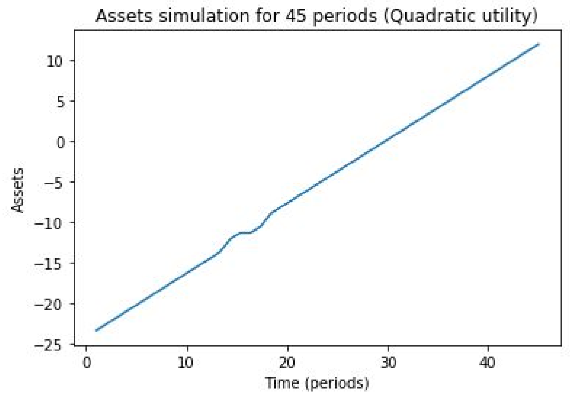


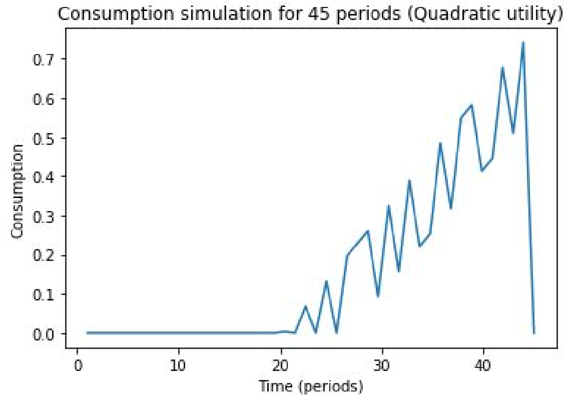


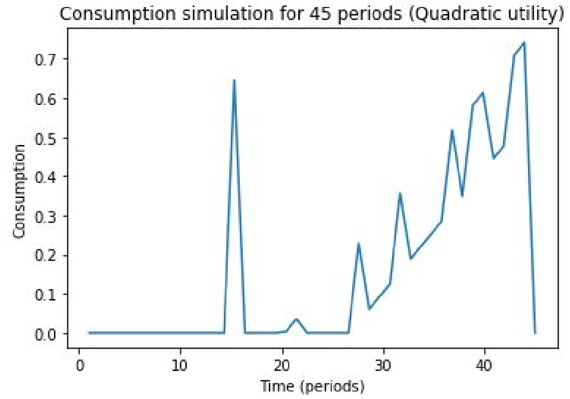
- Now, what happens if we change the value of σ?

σ = 5	σ = 20
	
	

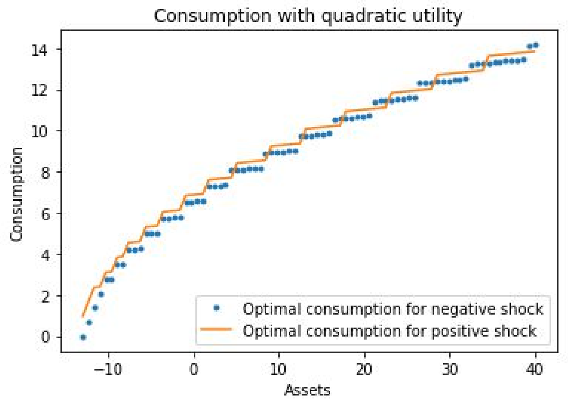
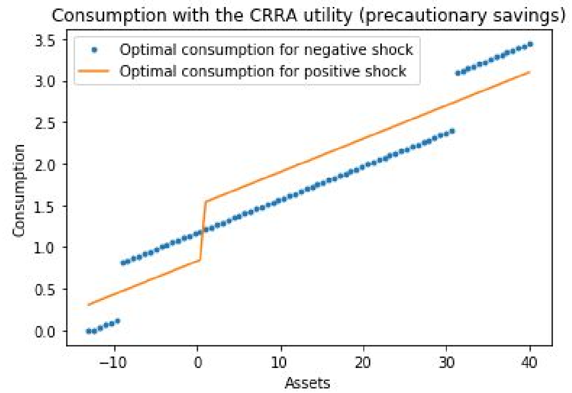


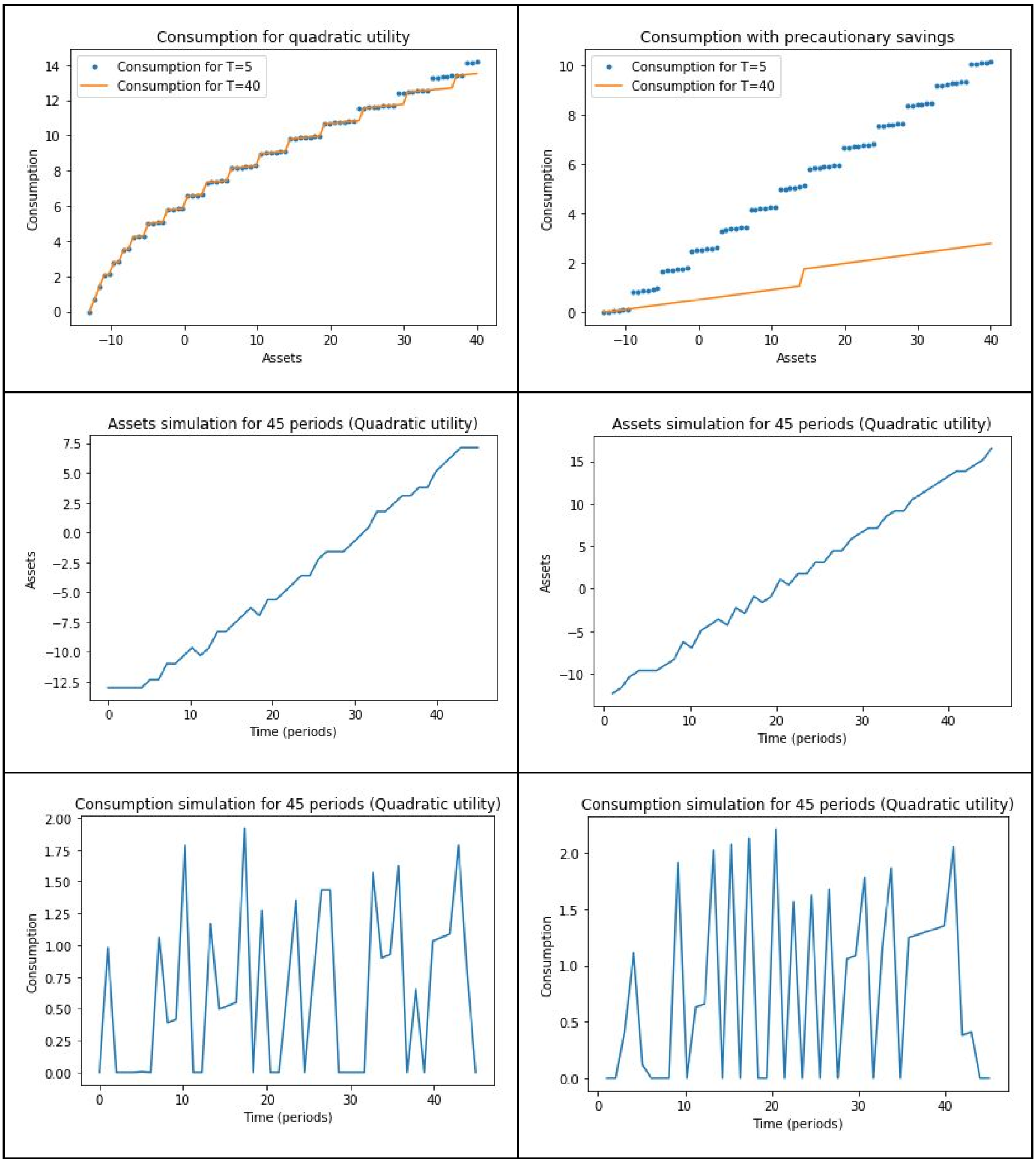






- Now, what happens if we change the value of σ_y to 0.5?

Quadratic utility	CRRA utility
	


Consumption for quadratic utility
Consumption for T=5
Consumption for T=40
Consumption
Assets
Consumption with precautionary savings
Consumption for T=5
Consumption for T=40
Consumption
Assets
Assets simulation for 45 periods (Quadratic utility)
Assets
Time (periods)
Assets simulation for 45 periods (Quadratic utility)
Assets
Time (periods)
Consumption simulation for 45 periods (Quadratic utility)
Consumption
Time (periods)
Consumption simulation for 45 periods (Quadratic utility)
Consumption
Time (periods)

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- Now, what happens if we change the value of ɣ to 0.95?

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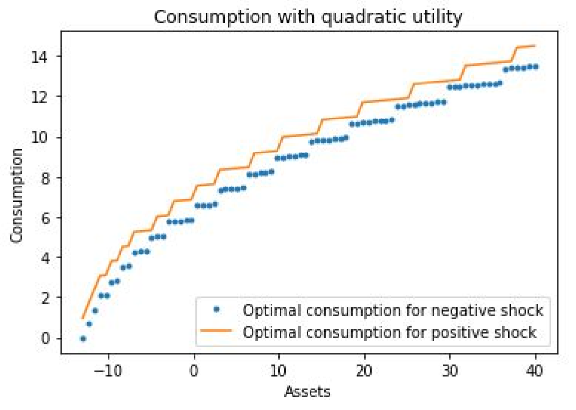
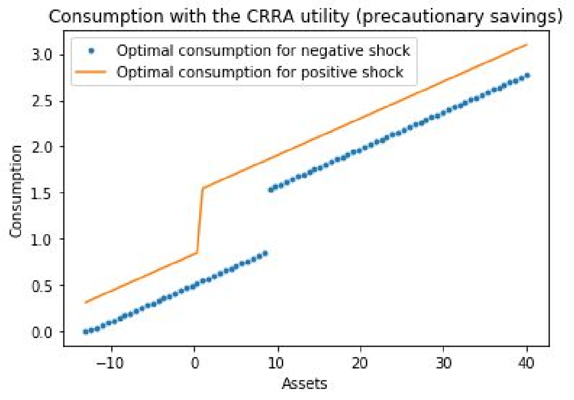
<tr><td>
</td><td>
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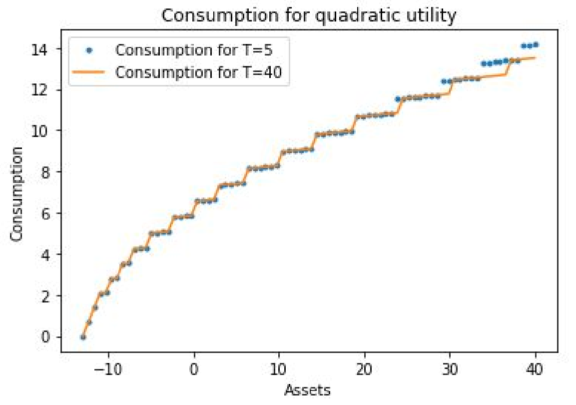
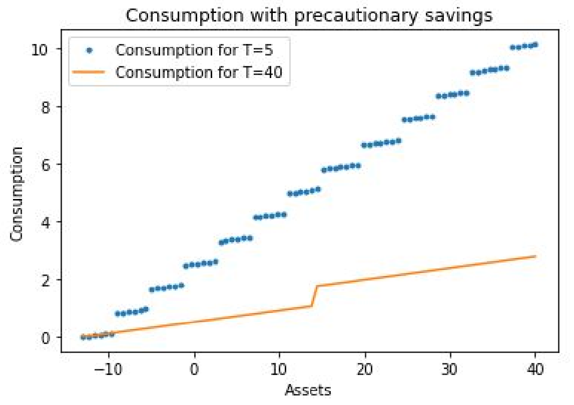
<tr><td>
</td><td>
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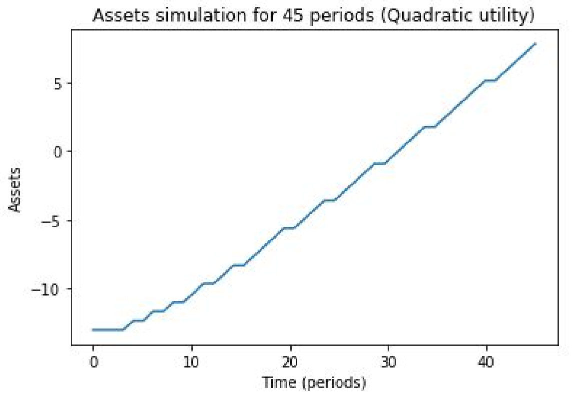
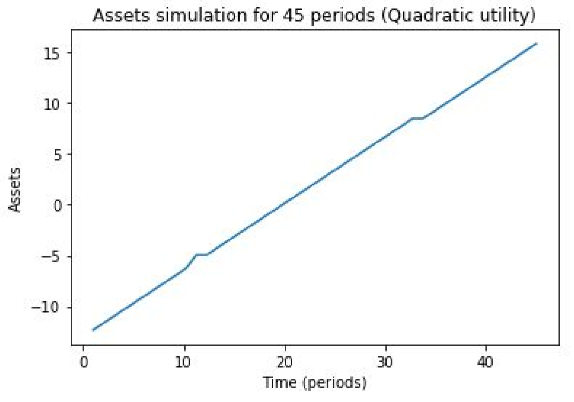
<tr><td>
</td><td>
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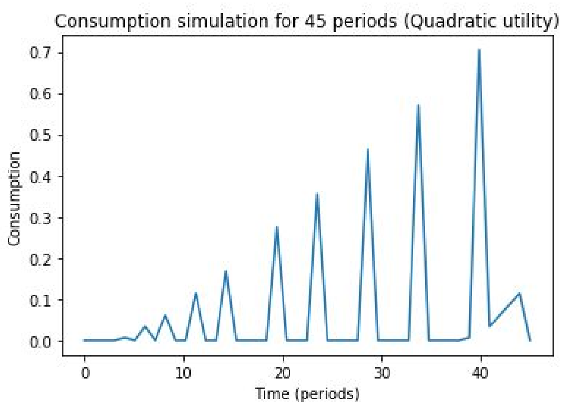

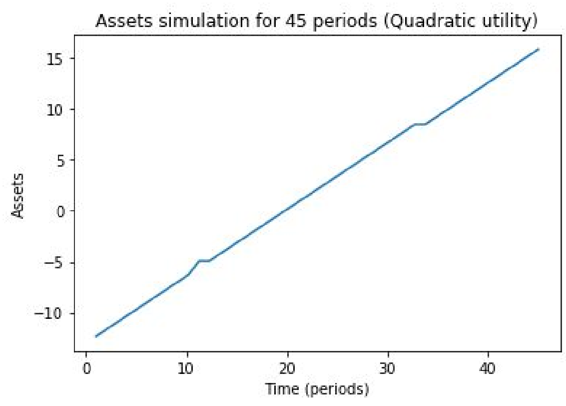

5. General equilibrium

1. Parameters used

In this case we used a code created by Sargent with Quantecon. Then, there are some parts that, as he used prebuilt functions, we do not completely understand so as to create the distribution for consumption and income in equilibrium.

Parameter	**Value**
ρ	0.03
β	1/(1+ρ)
α	0.33
δ	0.05
A	1

And the transition matrix is given by: *Π=[[0.6, 0.4], [0.4, 0.6]]*

In this graph we can check the supply and demand for capital with respect to the interest rate. When both curves intersect we have the equilibrium:

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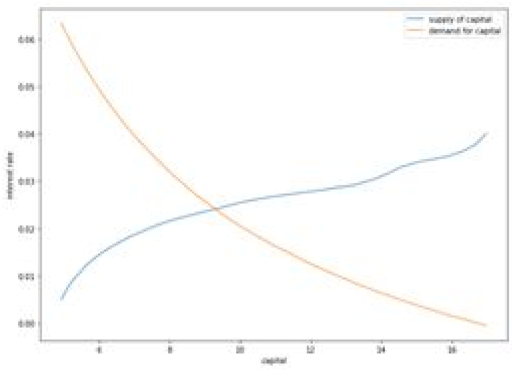
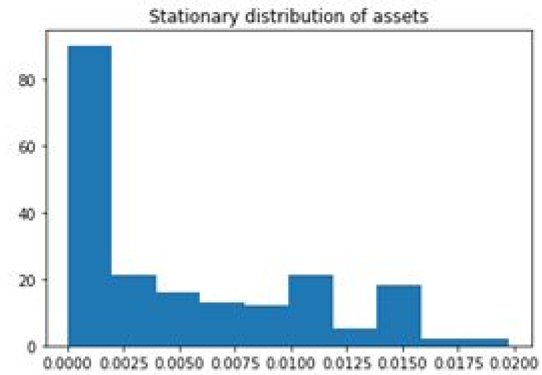


The following graph shows the stationary distribution for assets.



As we can observe, most of the people is poor in the stationary equilibrium. The frequency of people observed in the graph is decreasing when assets increase.
The Gini's coefficient is relatively similar to that used by Krueger, Mitman and Perri. In their case 0.77 and in our model it is 0.61.

2. Parameters used (based on Aiyagari)

Parameter	Value
ρ	0.04
□	0.96
α	0.36
δ	0.08
A	1

Now our transition matrix has 7 different states:

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$$\Pi = \begin{array}{l}[[0.5, 0.2, 0.1, 0.075, 0.075, 0.025, 0.025],\\ [0.1, 0.5, 0.1, 0.1, 0.1, 0.05, 0.05],\\ [0.1, 0.1, 0.5, 0.1, 0.1, 0.05, 0.05],\\ [0.05, 0.1, 0.1, 0.5, 0.1, 0.1, 0.05],\\ [0.05, 0.05, 0.1, 0.1, 0.5, 0.1, 0.1],\\ [0.05, 0.05, 0.1, 0.1, 0.1, 0.5, 0.1],\\ [0.05, 0.05, 0.1, 0.1, 0.1, 0.1, 0.5]],\end{array}$$